# Approximate Solution of Burgers' Equation

**Tayyaba Zafar**

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**Examination Committee Members**

1. Name: Dr. Yousaf Habib ______ Signature: ______
2. Name: Dr. Muhammad Asif Farooq ______ Signature: ______
3. Name: Dr. Adnan Maqsood ______ Signature: ______
4. Name: Dr. Muhammad Amer Qureshi ______ Signature: ______
5. Co-Supervisor: Prof. Azad A. Siddiqui ______ Signature: ______

Supervisor's Name: Dr. Tooba Feroze ______ Signature: ______

______ Head of Department

07-05-2015 Date

**COUNTERSINGED**

Date: 07/05/15

______ Dean/Principal

**Dedicated to**

***My Loving Parents***

***&***

***Husband***

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# Abstract

In this thesis we have given a systematic procedure to find the approximate solution of non linear partial differential equation. We reduced Burgers' equation into ordinary differential equation using similarity method, which is solved numerically. We approximated numerical solution by a function. Further more, we used inverse similarity transformation to get approximate solution of Burgers' equation in the form of function. To check the validity of above mentioned process we calculated the exact solution of Burgers' equation and compared it with approximate solution.

# Contents

# Chapter 1

# Introduction

We live in a universe of interrelated changing entities. The position of the earth changes with time, velocity of a falling body changes with distance, area of a circle changes with size of the radius, bending of a beam changes with weight of the load placed on it, path of a projectile changes with velocity and angle at which it is fired, in the same way we can mention a lot of examples like this. In the language of mathematics, changing entities are called variables and the rate of change of one variable with respect to another is a derivative. Equations which express a relationship among these variables and their derivatives are called differential equations.

Generally two types of differential equations can be seen in literature, one is ordinary differential equation in which we take the derivative of dependent variable with respect to one independent variable, the other type of differential equation is partial differential equation in which the derivative of dependent variable(s) are with respect to more than one dependent variables. Another case is the order of differential equation which is basically the highest derivative of the dependent variable involved in that differential equation. Both ordinary and partial differential equations are broadly classified as linear and nonlinear. If the dependent variable(s) and its derivatives occur only to the first degree and also the product of dependent variable(s) and its derivatives is not present then we call it a linear differential equation. In the case any one of the condition of linearity is deviated then the differential equation is called a nonlinear differential equation. Absence of forcing function i.e. function of independent variable makes an equation homogeneous and (forcing function could be a constant as well). Otherwise its presence makes it nonhomogeneous.

Usually the quantities are dependent on various variables, such as temperature, velocity, position etc and are generally represented through partial differential equations. Scientist like Euler, Langrange, Laplace and DAlembert [1] did the initial studies about partial differential equations in eighteenth century. Scientist used partial differential equations in analytical study of models in science and engineering. During nineteenth century, use of partial differential equations became common, specifically by Riemann in certain fields of mathematics [1].

Modeling of the theories regarding physics and engineering are used in applications of partial differential equations. For example, one is able to derive the whole theory of magnetism and electricity using the Maxwell equations [1].

A general form of the first order partial differential equation in two independent variables is,

$$G(u, v, w(u, v), w_u(u, v), w_v(u, v)) = G(u, v, w, w_u, w_v) = 0.$$

Similarly,

$$G(u, v, w, w_u, w_v, w_{uu}, w_{uv}, w_{vv}) = 0,$$

is a second order partial differential equation. Following are examples of linear partial differential equations of first and second order respectively,

$$3v^2 \frac{\partial w}{\partial u} + \frac{\partial w}{\partial v} = 2w, \tag{1.0.1}$$

$$\frac{\partial^2 w}{\partial u^2} + g(u, v) \frac{\partial^2 w}{\partial v^2} = 0. \tag{1.0.2}$$

The following equation

$$w \frac{\partial^2 w}{\partial u^2} + \left( \frac{\partial w}{\partial v} \right)^2 = w^2,$$

is an example of a nonlinear second order partial differential equation.

Solution of a partial differential equation is a function that satisfies the equation identically. If the analytical solution of a partial differential equation can not be obtained then we have numerical methods to approach the solution which is commonly known as approximate solution.

In previous mention paper by [2] and thesis of [3], [4] and [5] a method to find an approximate (closed form) solution of partial differential equations was developed. In this method a partial differential equation is first converted to an ordinary differential equation by symmetry transformation, and then find its numerical solution and approximate it by a function. And then compare the both solutions and analyze the errors. In this thesis we have adopted this procedure to find approximate closed form solution. The Lie symmetry method is applied to the one-parameter Burgers' equation. With the help of Lie symmetry partial differential equation is reduced to an ordinary differential equation. After that we find numerical solution of this reduced ordinary differential equation. And then approximate it by a function. Compare both numerical solution and approximate solution. To check that how we get accurate result through this procedure we calculate exact solution of Burgers' equation and compare it with approximate function.

## 1.1 History of Burgers' Equation

The Burgers' equation is given as

$$W_{xx} + WW_x = W_t. \tag{1.1.1}$$

It is a second order partial differential equation which is taken into consideration in many aspects of science, especially, in fundamentals of mathematical calculation of continuum models in engineering, chemistry, physics and finance. The Burgers' equation was first recognized by Forsyth [6] in 1906. In view of its various applications many analytic solutions have been calculated. In 1915, Burgers' equation was introduced in a physical context by Bateman [7] who mentioned two of the essentially steady solutions. Later, between 1940's and 1950's, Burgers started highlighting its significance and came with special solution and also referred its form as theory of turbulence. The vast work by Burgers is the main cause of naming this equation as Burgers' equation.

In 1949, Lagerstorm [8] was able to transform Burgers' equation to linear heat equation. In the context of gas dynamics, Burgers' equation was studied by Hopf [8] in 1950 after the discovery of a coordinate transformation that maps Burgers' equation to the heat equation. In 1951 Cole [9] studied the Burgers' equation and gave a theoretical solution, based on Fourier series, using the appropriate initial and boundary conditions. Another theoretical solution which was based on the test and trial method using the proper initial and boundary conditions was given by Madsen and Sincovec [10]. In 1956 and 1964, Burgers' equation was used in study of propagation of one dimensional acoustic of limited amplitude by Lighthill [11] and Blackstock [12] respectively. Hayes, in 1958, used these equations to discuss shock structure in Navier-Stokes fluid. In 1970, Rodin [13] form a Riccati solution from Burgers' equation without using several extra conditions. In 1972, thirty five distinct solutions were found by Benton and Platzman [14], for the problems faced regarding Burgers' equation in infinite domain and also defined exact solutions of one dimensional Burgers' equation. Later in 1972 Armes [15] discovered the way to apply Morgan-Michal method to Burgers' equation for determining proper groups.

Painlevé property of the partial differential equations were defined by Weiss et al. [16] in 1983, which helped in determining various things namely integrability, Backlund transforms, Lax pairs of Burgers' equation and the KdV equation with its modified form. In 1989, Vorus [17] studied the perturbed vortex plate against its steady-flow state by sinusoidal excitation so that it was tested by the separation point. The calculated differential equation is Burgers' equation at dynamic medium. He applied Cole-Hopf [18] transformation in achieving the analytical solution. Shtelen [19] was able to find out this transformation theoretically.

There are studies regarding strength and bifurcation of solution of Burgers' equation in controlled conditions when an integral term is added. This type of study was done by Peralta-Fabi and Plaschko [20].

In 1966, Ozis and Ozdes discovered a new technique [21] to find out the exact solution. They generated an approximate sequence which converges to the exact solution through direct variational method. In 1997, the transverse scheme was implemented along with the combination of boundary value methods, by Mazzia and Mazzia [22], to convert Burgers' equation into an ordinary differential equation.

There are some nonlinear partial differential equations that may be thought of as a formal generalization of Burgers' equation. For example, Burgers' Huxley equation

$$W_t + \alpha W W_x = \beta W_{xx} + \beta W + \omega W_2 - \delta W_3.$$

Some generalizations are the Kolmogorov-Petrovsky-Piskunov equation

$$W_t = \beta W_{xx} + \beta W + \omega W_2,$$

Korteveg-de Vries Burgers' equation

$$W_t + \alpha W W_x + W_{xxx} = \beta W_{xx},$$

and

Kuramoto-Sivashinky equation

$$W_t + W W_x + \alpha W_{xx} + \beta W_{xxx} + \gamma W_{xxxx} = 0.$$

### 1.1.1 Applications of Burgers' Equation

Burgers' equation is a good model for certain reasons. Many authors have considered this equation both for the theoretical purposes for class of physics and for testing various numerical methods. It applies in different areas of mathematics. This equation is frequently used in boundary layer estimation for flow from a viscous fluid. It helps mathematicians to solve standard test problems of partial differential equations. It is also important in the process of standard problem for numerical methods.

The main use of Burgers' equation is in fluid dynamics and practically as a model for turbulence, gas dynamics, acoustics, longitudinal elastic waves in an isotropic solid, heat conduction, dispersive water waves, number theory, shock wave formation, boundary layer behavior, traffic, continuous stochastic processes, and mass transport. Numerical algorithms can be verified through it. It has diverted attention of several researchers to its solution due to its broad range of applicability. So far they are able to find an analytical solution of Burgers' equation for a limited set of arbitrary initial and boundary conditions.

### 1.1.2 Numerical Solution of Burgers' Equation

In most of the physical problems judgment made by scientists is biased or more precisely incomplete and inaccurate. These limitations and phenomenological models employed are to be taken into considerations while solving such problems numerically. Errors can be calculated using different means while uncertainties can not be recognized due to lack of knowledge. Hence, errors can be categorized as deterministic quantities, uncertainties are estimated within a probabilistic framework and are referred to as stochastic in nature. Uncertainty is approximated in numerical simulations in model parameter values, boundary or initial conditions and also in the geometry of the domain of the problem referred to as input uncertainty. For numerical methods validation, certification and uncertainty quantification are evaluated and it is really useful for making critical decisions. Following are the applications of uncertainty quantifications but are not limited to fluid mixing, turbulence, turbulent combustion, climatology, flow in porous media and computational electromagnetics.

To approximate Burgers' equation solution, numerous methods have been presented and implemented. Various techniques have been used based on finite element, differences and boundary element methods in order to evaluate the solution of equation. Kadalbajoo [23] was the first one to solve time dependent Burgers' equation using a uniform parameter implicit difference method. Bhattacharya [24] was the first one to introduce the exponential finite difference method explicitly for solving heat equation. He along with Handschuh and Keith used this method in order to solve Burgers' equation. The connection of exact solutions to Burgers' equation makes it attractive for numerical analysis as a model problem to test many numerical methods like spectral methods, finite differences and finite element method.

Numerical solution of Burgers' equation using finite element method was first evaluated by Varoglu and Finn [25]. Cubic spline finite element and finite difference used by Caldwell and Smith [26], Evans and Abdullah [27] introduced group explicit methods to solve Burgers' equation. The generalized boundary approach was given by Kakuda and Tosaka [28]. A linear space-time finite element method was discovered by Nguyen and Rynen [29] based on least square approach. To generate an estimated solution Ozis and Ozdes [21] used a direct variational method in the form of sequence solution. Kutluay [30] proposed the least-squares quadratic B-spline finite element method for solution of Burgers' equation. Abd-el-Malek and El-Mansi [31] applied the one parameter group of transformations to the Burgers' equation. Under these transformations, the given partial differential equation is reduced to an ordinary differential equation with the appropriate corresponding conditions.

## 1.2 Lie Symmetry

In partial differential equations the difference between linear and nonlinear equations is really important. Linear partial differential equations can be solved easily by using particular methods like superposition, separation of variables, Fourier series, Laplace transform e.t.c. But the exact solution of nonlinear partial differential equations are difficult to solve. That's why naturally, nonlinear partial differential equations are often much harder to understand than the linear ones. So nonlinear partial differential equations have always a great attention because of their wide range of applications in engineering and science. To find the exact solution of the nonlinear partial differential equations there is no organized theory. Mostly, each individual equation has to be studied as a unique problem. It is well known that the symmetry group method plays an important role in the analysis of differential equations. Now a days symmetry method is one of the most powerful tools for solving nonlinear partial differential equations.

Similarity method is a powerful technique for determining transformations that reduces partial differential equations to ordinary differential equations. This method takes advantage of the natural symmetries in a partial differential equations and allows us to define special variables that give rise to reduction. Algebraic symmetries were first used for the study of diffusion equations, a hundred years ago by an Austrian physicist Boltzman, though he did not explicitly reveal the symmetries or the transformation groups that he used. Later, in 1950s, his work was reviewed by an American scientist Garrett's Brikhoff who generalized his procedure for non-linear partial differential equations. He proved that if a reduced ordinary differential equation is solved then it is possible to find the solution of the related partial differential equation.

The theory of Lie symmetry groups of differential equations was developed by Sophus Lie [32]. These Lie groups are invertible point transformations in the differential equations. The analysis of equations is of appreciable significance and can be obtained by symmetry group methods. It helps to understand and to reach a final solution of differential equation amongst their several applications regarding theory of differential equations. The most important ones, including the minimization of the order of differential equations, are approaching invariant solutions, mapping of one solution to another and detecting linearizing transformations. Similarity transformation methods also convert the original partial differential equation to an ordinary differential equation or sometime a partial differential equation with a reduced number of independent variables. After applying this method higher order ordinary differential equation gives result in the form lower order differential equation.

In ordinary differential equations, the general solution contains arbitrary constant while in case of partial differential equations, general solutions have one or several variables. In order to determine the arbitrary constant in the general solution we need some conditions known as initial and boundary conditions. The solutions rely frequently on the constant parameters having special characteristics. The general classes of solutions can be written using the special solutions only if there are linear differential equations and by linear superposition. Symmetries enable us to find out special solutions such as closed form solution methods of partial differential equations.

let us consider the general nonlinear system of partial differential equations of order kth in r independent and s dependent variables is given as a system of equations as

$$\triangle_v(x, \mu^{(n)}) = 0, \qquad v = 1, 2, ...., l, \tag{1.2.1}$$

involving $x = (x^1, x^2, ..., x^r)$, $\mu = (\mu^1, \mu^2, ..., \mu^s)$ and the derivative of $\mu$ with respect to x up to k, where $\mu^{(k)}$ represents all the derivatives of $\mu$ of all order from 0 to k.

A one-parameter Lie group of infinitesimal transformations acting on the independent and dependent variables of the system Eqs. (1.2.1) are

$$\bar{x^m} = x^m + \epsilon\xi^m(x, \mu) + O(\epsilon^2), \qquad m = 1, 2, ..., r, \tag{1.2.2}$$

$$\bar{\mu^n} = \mu^n + \epsilon\zeta^n(x, \mu) + O(\epsilon^2), \qquad n = 1, 2, ..., s \tag{1.2.3}$$

where $\epsilon$ is a parameter of transformation and $\xi^m$, $\zeta^n$ are the infinitesimals transformations for the dependent and independent variables, respectively.

Symmetry of system of differential equations is transformation that maps any solution to another solution of the system. These transformations are groups that depend on continuous parameters and consist of point transformations acting on the system's space of independent and dependent variables. The contact transformations are acting on independent and dependent variables as well as on all first derivatives of the dependent variables. Lie groups and their infinitesimals generators, can be prolonged to act on the space of dependent variables, independent variables, and derivatives of the dependent variables. In simple words, the symmetry group of a system of differential equation is a group of transformations of dependent and independent variables leaving the set of all solution invariant.

### 1.2.1 Group

A group $M$ is defined to be a set of elements which hold law of composition $\theta$ between elements satisfying following principles [33],

- Closure property: for any $m_1$,$m_2 \in M$, such that

$$\theta(m_1, m_2) \in M.$$

- Associative property: for any $m_1, m_2, m_3 \in M$, such that

$$\theta(m_1, \theta(m_2, m_3)) = \theta(\theta(m_1, m_2), m_3).$$

- Identity: an element $e \in M$ such that for all $m \in M$, such that

$$\theta(m, e) = \theta(e, m) = e.$$

- Inverse: for any $m \in M$ there exist a unique $m^{-1} \in M$ such that

$$\theta(m, m^{-1}) = \theta(m^{-1}, m) = e.$$

A group M is referred as an **Abelian group** if for all $m_1, m_2 \in M$,

$$\theta(m_1, m_2) = \theta(m_2, m_1).$$

A **subgroup** of $M$ is a subset of $M$ whose elements form a group under the same law of composition $\theta$.

### 1.2.2 Transformation Group

Suppose $\mathbf{m} = (m_1, m_2, m_3, ..., m_n)$ which is in region B $\subset \mathbf{R}^n$. Then set of transformation

$$\mathbf{m}^* = \mathbf{M}(\mathbf{m}; \epsilon), \tag{1.2.4}$$

which is defined for each $\mathbf{m} \in$ B and parameter $\epsilon \in \mathbf{V} \subset \mathbf{R}$ and $\theta(\epsilon, \delta)$ defines law of composition of parameters $\epsilon, \delta \in \mathbf{V}$, forms a 1-parameter group of transformations in B if the following condition is satisfied [33],

- For each $\epsilon \in \mathbf{V}$, the transformation is one-to-one in B.(Hence $\mathbf{m}^* \in$ B.)
- $\mathbf{V}$ forms a group with law of composition $\theta$.
- For each $\mathbf{m} \in$ B, we have $\mathbf{m}^* = \mathbf{m}$, when $\epsilon = \epsilon_0$ corresponds to identity element $e$.
- If $\mathbf{m}^* = M(\mathbf{m}; \epsilon)$ and $\mathbf{m}^{**} = M(\mathbf{m}^*; \delta)$ then, we have

$$\mathbf{m}^{**} = M(\mathbf{m}; \theta(\epsilon, \delta)). \tag{1.2.5}$$

### 1.2.3 Lie Group of Transformation in One Parameter

If a transformation group satisfies the following axioms, then it is known as one parameter Lie group of transformation [33],

- Parameters to be continuous, such that $\mathbf{V}$ is an interval in $\mathbf{R}$. Without loss of generality, $\epsilon = 0$ correspond to identity $e$ .

• Differentiability of **M** should be infinite w.r.t. $x \in \mathrm{B}$ and is an analytic function of $\epsilon \in \mathbf{V}$.

• $\theta(\epsilon, \delta)$ is an analytic function of $\delta$ and $\epsilon$, where $\delta, \epsilon \in \mathbf{V}$.

### 1.2.4 Algorithm for Finding Point Symmetries of Partial Differential Equation

Consider a partial differential equation of kth order

$$G(x, t, \mu, \mu_x, \mu_t, ....) = 0, \tag{1.2.6}$$

$$\bar{x} = x + \epsilon\xi(x, t, \mu),$$

$$\bar{t} = t + \epsilon\delta(x, t, \mu),$$

$$\bar{\mu} = \mu + \epsilon\zeta(x, t, \mu),$$

$$\bar{\mu}_i = \mu_i + \epsilon\zeta_i(x, t, \mu, \mu_1),$$

$$\vdots$$

$$\bar{\mu}_{i_1 i_2 \ldots i_n} = \mu_{i_1 i_2 \ldots i_n} + \epsilon\zeta_{i_1 i_2 \ldots i_n}(x, t, \mu, \mu_1, ..., \mu_n),$$

where $\epsilon$ is group parameter. Corresponding generator is

$$I = \xi(x, t, \mu)\frac{\partial}{\partial x} + \delta(x, t, \mu)\frac{\partial}{\partial t} + \zeta(x, t, \mu)\frac{\partial}{\partial \mu}, \tag{1.2.7}$$

where,

$\xi(x, t, \mu) = (\frac{\partial \bar{x}}{\partial \epsilon})_{\epsilon=0}$, $\delta(x, t, \mu) = (\frac{\partial \bar{t}}{\partial \epsilon})_{\epsilon=0}$, $\zeta(x, t, \mu) = (\frac{\partial \bar{\mu}}{\partial \epsilon})_{\epsilon=0}$.

Now we are in need of kth extension of I to apply to kth order partial differential equation

$$I^{[k]}F \mid_{F=0}= 0 \tag{1.2.8}$$

where

$$I^{[k]} = I + \zeta_i(x, t, \mu, \mu_1)\frac{\partial}{\partial \mu_i} + ...... + \zeta_{i_1,i_2,...,i_n}(x, t, \mu, \mu_1, \mu_2, ...\mu_n)\frac{\partial}{\partial \mu_{i_1,i_2...i_n}},$$

is the kth order prolongation of the generator $I$ and $\zeta_i, ...\zeta_{i_1}...\zeta_{i_1,i_2,...i_n}$ are called prolongation coefficient and defined as

$$\zeta_i = D_i\zeta - (D_i\xi)\frac{\partial}{\partial \mu_x} - (D_i\delta)\frac{\partial}{\partial \mu_t}, \tag{1.2.9}$$

$$\zeta_{i_1,i_2,...i_n} = D_{i_n}\zeta_{i_1,i_2,...i_{n-1}} - (D_{i_n}\xi)\mu_{i_1,i_2,...i_{n-1}x} - (D_{ik}\delta)\mu_{i_1,i_2,...i_{n-1}t} \tag{1.2.10}$$

and

$$D_x = \frac{\partial}{\partial x} + \mu_x\frac{\partial}{\partial \mu} + \mu_{xx}\frac{\partial}{\partial \mu_x} + \mu_{xt}\frac{\partial}{\partial \mu_t} + ...,$$
$$D_t = \frac{\partial}{\partial t} + \mu_t\frac{\partial}{\partial \mu} + \mu_{xt}\frac{\partial}{\partial \mu_x} + \mu_{tt}\frac{\partial}{\partial \mu_t} + ...,$$

for k=2 the form of $I^{[2]}$ is given as

$$I^{[2]} = \xi(x, t, \mu)\frac{\partial}{\partial x}+\delta(x, t, \mu)\frac{\partial}{\partial t}+\zeta(x, t, \mu)\frac{\partial}{\partial \mu}+\zeta_1\frac{\partial}{\partial \mu_x}+\zeta_2\frac{\partial}{\partial \mu_t}+\zeta_{11}\frac{\partial}{\partial \mu_{xx}}+\zeta_{12}\frac{\partial}{\partial \mu_{xt}}+\zeta_{22}\frac{\partial}{\partial \mu_{tt}}, \tag{1.2.11}$$

where

$$
\begin{aligned}
\zeta_1 &= D_x(\zeta) - \mu_t D_x(\delta) - \mu_x D_x(\xi) \\
&= \zeta_x + (\zeta_\mu - \xi_x)\mu_x - \delta_x\mu_t - \xi_\mu(\mu_x)^2 - \delta_\mu\mu_t\mu_x, \\
\zeta_2 &= D_t(\zeta) - \mu_t D_t(\delta) - \mu_x D_t(\xi) \\
&= \zeta_t + (\zeta_\mu - \delta_t)\mu_t - \xi_t\mu_x - \delta_\mu(\mu_t)^2 - \xi_\mu\mu_t\mu_x, \\
\zeta_{11} &= D_x(\zeta_1) - \mu_{tx} D_x(\delta) - \mu_{xx} D_x(\xi) \\
&= \zeta_{xx} + (2\zeta_{x\mu} - \xi_{xx})(\mu_x) - \delta_{xx}\mu_t + (\zeta_\mu - 2\xi_x)(\mu_{xx}) - 2\delta_x\mu_{tx} + (\zeta_{\mu\mu} - 2\xi_{x\mu})(\mu_x)^2 - 2\delta_{x\mu}\mu_t\mu_x \\
&\quad - \xi_{\mu\mu}(\mu_x)^3 - \delta_{\mu\mu}\mu_t(\mu_x)^2 - 3\xi_\mu\mu_x\mu_{xx} - \delta_\mu\mu_t\mu_{xx} - 2\delta_\mu\mu_x\mu_{tx},
\end{aligned}
$$

$$
\begin{aligned}
\zeta_{12} =& D_x(\zeta_2) - \mu_{tt} D_x(\delta) - \mu_{tx} D_x(\xi) \\
=& \zeta_{tx} + (2\zeta_{t\mu} - \xi_{tx})(\mu_x) + (\zeta_{\mu x} - \delta_{tx})(\mu_t) + (\zeta_{\mu\mu} - \delta_{t\mu} - \xi_{\mu x})(\mu_t\mu_x) + (\zeta_\mu - \delta_t - \xi_x)(\mu_{tx}) \\
& - \delta_{\mu x}(\mu_t)^2 - \delta_{\mu\mu}\mu_x(\mu_t)^2 - \xi_\mu\mu_t\mu_{xx} - \xi_{t\mu}(\mu_x)^2 - \xi_t\mu_{xx} - \delta_x\mu_{tt} - \xi_{\mu\mu}\mu_t(\mu_x)^2 - 2\xi_\mu\mu_x\mu_{tx} \\
& - \delta_\mu\mu_x\mu_{xt} - 2\delta_\mu\mu_t\mu_{xt},
\end{aligned}
$$

$$
\begin{aligned}
\zeta_{22} =& D_t(\zeta_2) - \mu_{tt} D_t(\delta) - \mu_{tx} D_t(\xi) \\
=& \zeta_{tt} + (2\zeta_{t\mu} - \delta_{tt})\mu_t + (\zeta_{\mu\mu} - 2\delta_{t\mu})(\mu_t)^2 - \delta_{\mu\mu}(\mu_t)^3 - 3\delta_\mu\mu_t\mu_{tt} - \xi_{tt}\mu_x - 2\xi_{\mu t}\mu_t\mu_x - 2\xi_t\mu_{tx} \\
& - \xi_{\mu\mu}\mu_x(\mu_t)^2 - \xi_\mu\mu_x\mu_{tt} - 2\xi_\mu\mu_t\mu_{xt}.
\end{aligned}
$$

Condition given by Eq. (1.2.8) provides a nonlinear partial differential equation in $\xi,\ \delta$ and $\zeta$. Here, $\xi,\ \delta$ and $\zeta$ must depend only on $x, t$ and $\mu$ since we are dealing with point symmetries. Therefore, a system of partial differential equations is obtained by comparing the coefficients of powers of derivatives of $\mu$. The solution of system yields the values of $\xi,\ \delta$ and $\zeta$. Corresponds to a each infinitesimal constant generator. These infinitesimal generators form a Lie algebra. Hence involving some constant dimension of the Lie algebra is determined by the number of constants in $I$.

Through the obtained symmetries, point transformations are generated which leaves the differential equation invariant. To obtain these point transformations we evaluate

$$\frac{\partial \bar{x}}{\partial \epsilon} = \xi(\bar{x}, \bar{t}, \bar{\mu}), \qquad \frac{\partial \bar{t}}{\partial \epsilon} = \delta(\bar{x}, \bar{t}, \bar{\mu}), \qquad \frac{\partial \bar{\mu}}{\partial \epsilon} = \zeta(\bar{x}, \bar{t}, \bar{\mu}), \tag{1.2.12}$$

subject to the initial conditions

$$\bar{x} \mid_{\epsilon=o} = x, \qquad \bar{t} \mid_{\epsilon=o} = t, \qquad \bar{\mu} \mid_{\epsilon=o} = \mu. \tag{1.2.13}$$

### 1.2.5 Example

To elaborate above method we consider heat equation

$$\mu_t = \mu_{xx}. \tag{1.2.14}$$

Now applying second extension of I to Eq. (1.2.14), we have

$$\zeta_2 = \zeta_{11}.$$

Also we get the following system of equations from the algorithm described in section (1.2.4)

$$\delta_{\mu} = 0,$$

$$\delta_{x} = 0,$$

$$\xi_{\mu} = 0,$$

$$-2\xi_{x} + \delta_{t} = 0,$$

$$\xi_{t} + 2\zeta_{x\mu} - \xi_{xx} = 0,$$

$$\zeta_{\mu\mu} = 0,$$

$$-\zeta_{xx} + \zeta_{t} = 0.$$

After solving this system of equations, we get

$$\delta = c_1 + c_3 t + \frac{1}{4} c_4 t^2,$$

$$\xi = c_2 + \frac{1}{2}(c_3 + c_4 t)x + c_6 t,$$

$$\zeta = (-\frac{1}{8} c_4 x^2 - \frac{1}{4} c_4 t + \frac{1}{2} c_6 x + c_5)\mu + U(x,t),$$

where $U(x,t)$ is function of integration.

By taking $c_1 = 1$ and the other five constants to be zero then we get the symmetry $I_1$ of the form,

$$I_1 = \frac{\partial}{\partial x},$$

similarly, taking $c_2 = 1$ and others to be zero we have,

$$I_2 = \frac{\partial}{\partial t}.$$

In the same manner we get the following symmetries

$$I_3 = \frac{1}{2}x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t},$$

$$I_4 = \frac{1}{2}tx\frac{\partial}{\partial x} + \frac{1}{2}t^2\frac{\partial}{\partial t} - (\frac{1}{8}x^2\mu + \frac{1}{2}t\mu)\frac{\partial}{\partial \mu},$$

$$I_5 = \mu\frac{\partial}{\partial \mu},$$

$$I_6 = t\frac{\partial}{\partial x} - \frac{1}{2}x\mu\frac{\partial}{\partial \mu},$$

$$I_7 = U((t,x)\frac{\partial}{\partial \mu}.$$

### 1.2.6 Point Transformation and Group Reduction

Now we are able to evaluate the following invariant surface condition to reduce partial differential equations to ordinary differential equations for each symmetry once we obtain the generators,

$$\frac{dt}{\delta(x,t,\mu)} = \frac{dx}{\xi(x,t,\mu)} = \frac{d\mu}{\zeta(x,t,\mu)}, \tag{1.2.15}$$

by solving Eq.(1.2.15) we get two invariant namely $v$ and $z$. Therefore,the reduction variables is,

$$v = \omega(z). \tag{1.2.16}$$

After substituting Eq.(1.2.16) in Eq.(1.2.15), an ordinary differential equation is obtained. Solving the ordinary differential equation leads us to the value of $\omega$.

### 1.2.7 Example 1

In the section Eq.(1.2.5) the invariant surface condition for $I_3$ gives us,

$$\frac{dt}{t} = 2\frac{dx}{x} + \frac{d\mu}{0}.$$

Now

$$2\frac{dx}{x} = \frac{dt}{t} \implies z = \frac{x^2}{t},$$
$$\frac{d\mu}{0} = 2\frac{dx}{x} \implies \mu = u.$$

Using Eq.(1.2.16) the reduction variables can be written as,

$$\mu = \omega\left(\frac{x^2}{t}\right), \tag{1.2.17}$$

and after substituting in Eq.(1.2.14), we get

$$4zw'' + (2+z)w' = 0.$$

The solution of ordinary differential equation given by Eq.(1.2.17) is

$$w(z) = 2\sqrt{\pi}b_1 erf\left(\frac{1}{2}\sqrt{z}\right) + b_2, \tag{1.2.18}$$

where $b_1$ and $b_2$ are arbitrary constants and **erf** represent error function. Substituting Eq.(1.2.18) in Eq.(1.2.17) we get exact solution that the form,

$$\mu(x,t) = 2\sqrt{\pi}b_1 erf\left(\frac{1}{2}\sqrt{\frac{x^2}{t}}\right)b_2.$$

It is sometimes difficult to find out exact solutions of the reduced ordinary differential equations. Therefore, numerical approach is considered in such cases. Once solution of reduced ordinary differential equation is obtained, then through reciprocal bijection of point transformation we can obtain the solution of original partial differential equation.

### 1.2.8 Example 2

Now we find the point transformation for $I_3$ in Example (1.2.5). Eq.(1.2.12) implies

$$\frac{\partial \bar{x}}{\partial \epsilon} = \frac{1}{2}\bar{x}, \qquad \frac{\partial \bar{t}}{\partial \epsilon} = \bar{t}, \qquad \frac{\partial \bar{\mu}}{\partial \epsilon} = 0,$$

which gives,

$$\bar{x} = e^{\frac{1}{2}\epsilon}k_1, \qquad \bar{t} = e^{\epsilon}k_2, \qquad \bar{\mu} = k_3,$$

where $k_1, k_2, k_3$ are arbitrary constants. The initial conditions given by Eq. (1.2.13) yields

$$\bar{x} = e^{\frac{\epsilon}{2}}x, \qquad \bar{t} = e^{\epsilon}t, \qquad \bar{\mu} = \mu.$$

Thus, Lie scaling group is obtained by the point transformation acquired from $I_3$. Symmetries of heat equation are found by using the illustrated algorithm and then one of these symmetries is used to find exact solution of heat equation. Symmetry is then used to find point transformation leaving differential equation invariant.

# Chapter 2

# Approximate Solutions of Burgers' Equation

The mathematical modeling of most of the physical processes i.e. fusion, chemical kinetics, wave mechanics, general transport and fluid mechanics leads to such nonlinear partial differential equations whose exact solution is difficult to find. A powerful technique to analyze such partial differential equations and their reduction to ordinary differential equations is given by Lie symmetry method. In this method, time and again, a partial differential equation is first reduced to an ordinary differential equation. The solution of this reduced ordinary differential equation then leads to the solution of the partial differential equation. The limitation of this process is that most often, the reduced ordinary differential equation may not be solved analytically. In such situations, a procedure used is to first find numerical solution of the reduced ordinary differential equation and then approximate it by a function. In this thesis we solve Burgers' equation using this approach.

The general form of nonlinear Burgers' equation is

$$\frac{\partial^2 C(x,t)}{\partial x^2} + C(x,t)\frac{\partial C(x,t)}{\partial x} = \frac{\partial C(x,t)}{\partial t}. \tag{2.0.1}$$

Applying second extension of generator I and Eq. (1.2.8) to above Burgers' equation (2.0.1), we have

$$\zeta_{,xx} + \zeta C_x + C\zeta_{,x} = \zeta_{,t}.$$

Following the procedure, explained in section (1.2.4) we get the following system of partial differential equations

$$2\zeta_{xc} - \xi_{xx} + C\xi_x = -\xi_t, \tag{2.0.2}$$

$$-\delta xx - C\delta_x - 2\xi_x = -\delta_t, \tag{2.0.3}$$

$$-2\delta_x = 0, \tag{2.0.4}$$

$$\zeta_{cc} - 2\xi_{xc} + 3\xi_c C - C\xi_c = 0, \tag{2.0.5}$$

$$-\xi_{cc} = 0, \tag{2.0.6}$$

$$-2\delta_{xc} - 2\xi_c = 0, \tag{2.0.7}$$

$$-\delta_{cc} = 0, \tag{2.0.8}$$

$$-\delta_c = 0, \tag{2.0.9}$$

$$-2\delta_c = 0. \tag{2.0.10}$$

After solving the above equations, we get

$$\left.\begin{aligned}\delta &= a_1 t^2 + 2a_2 t + a_3,\\ \xi &= a_1 t x + a_2 x + a_4 t + a_5,\\ \zeta &= -(a_1 t + a_2)C - a_1 x - a_4.\end{aligned}\right\}, \tag{2.0.11}$$

which leads to the following symmetries

$$I_1 = \frac{\partial}{\partial t}, \tag{2.0.12}$$

$$I_2 = \frac{\partial}{\partial x}, \tag{2.0.13}$$

$$I_3 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - C\frac{\partial}{\partial C}, \tag{2.0.14}$$

$$I_4 = t\frac{\partial}{\partial x} - \frac{\partial}{\partial C}, \tag{2.0.15}$$

$$I_5 = t^2\frac{\partial}{\partial t^2} + tx\frac{\partial}{\partial x} - (x + tC)\frac{\partial}{\partial C}. \tag{2.0.16}$$

## 2.1 Solution of Burgers' Equation using Symmetry $I_1$

Using the $I_1$ symmetry, the characteristic equation is

$$\frac{dt}{1} = \frac{dx}{0} = \frac{dC}{0}. \tag{2.1.1}$$

Solution of the characteristic equation (2.1.1) leads to

$$x = z,$$

and

$$C = \mu(z). \tag{2.1.2}$$

The reduction variable is $\mu = \omega(z)$. So we have

$$C = \omega \tag{2.1.3}$$

Taking partial derivatives of Eq.(2.1.3) w.r.t $x$ and $t$ respectively, we get

$$C_x = \omega', \quad (where \text{ "}'\text{" } is\ the\ derivative\ w.r.t\ z) \tag{2.1.4}$$

$$C_{xx} = \omega'', \tag{2.1.5}$$

$$C_t = 0. \tag{2.1.6}$$

Now using Eq.(2.1.4), Eq.(2.1.5) and Eq.(2.1.6) in Eq.(2.0.1) we get

$$\omega'' + \omega\omega' = 0. \tag{2.1.7}$$

### 2.1.1 Numerical Solution of the Reduced Ordinary Differential Equation

The resulting ordinary differential equation (2.1.7) is nonlinear. We have used Computer Algebra System particularly MATLAB built in function *bvp4c* in order to solve Eq. (2.1.7) numerically, with boundary conditions $\omega(0) = 0$ , $\omega(\infty) = 1$. and is shown in Figure (2.1).

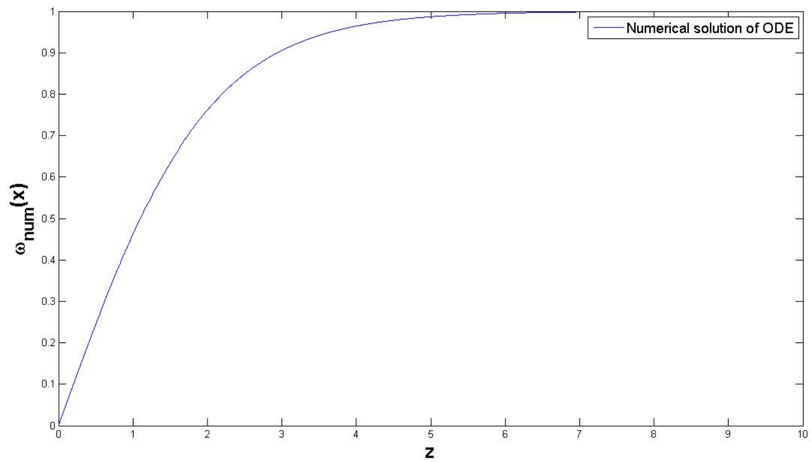


Figure 2.1: Graph of $\omega_{num}$(z)

### 2.1.2 Approximation of Numerical Solution of the Reduced Ordinary Differential Equation

The curve in Figure (2.1) resembles to the graph of exponential function,

$$g(z) = 1 - exp(-z), \tag{2.1.8}$$

shown in Figure (2.2).

The numerical solution of $\omega_{num}$, and function $g(z)$ are illustrated in Figure (2.3).

An error exist, between $\omega_{num}$ and $g(z)$ and is shown in Figure (2.4).

The error curve resembles to the graph of the function

$$h(z) = rz^s e^{tz}. \tag{2.1.9}$$

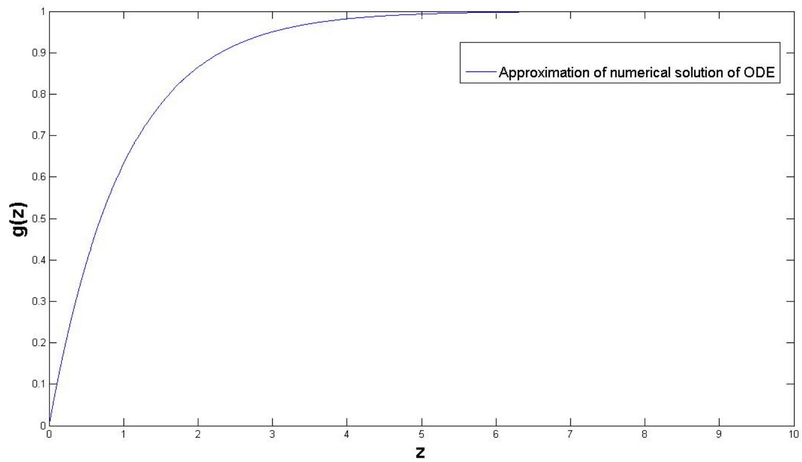


Figure 2.2: Graph of $g(z)$

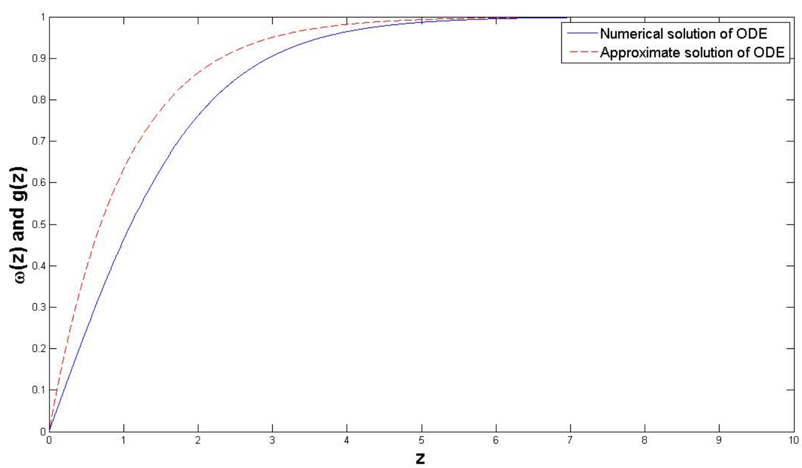


Figure 2.3: Graph of $\omega_{num}$ and $g(z)$

The three points $(z_1, \omega_1), (z_2, \omega_2)$ and $(z_3, \omega_3)$ are taken into consideration in order to find constant $r, s$ and $t$. Now we have following three equations

$$\omega_1 = r z_1^s e^{t z_1}, \tag{2.1.10}$$

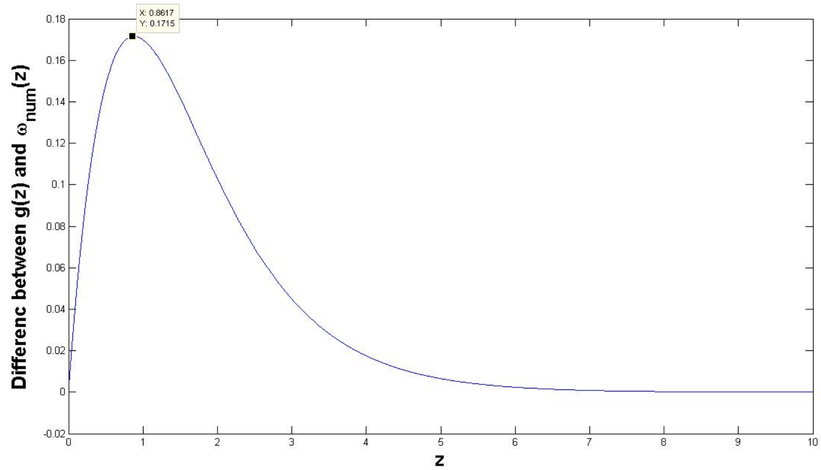


Figure 2.4: Graph of error between $g(z)$ and $\omega_{num}$(z)

$$\omega_2 = rz_2^s e^{tz_2}, \tag{2.1.11}$$

$$\omega_3 = rz_3^s e^{tz_3}. \tag{2.1.12}$$

Dividing Eq.(2.1.10) by Eq.(2.1.11) and then taking natural logarithm function on both sides to have

$$ln(\frac{\omega_1}{\omega_2}) = sln(\frac{z_1}{z_2}) + t(z_1 - z_2). \tag{2.1.13}$$

Similarly, dividing Eq.(2.1.10) by Eq.(2.1.12) and then again taking natural logarithm we get

$$ln(\frac{\omega_1}{\omega_3}) = sln(\frac{z_1}{z_3}) + t(z_1 - z_3). \tag{2.1.14}$$

Multiplying Eq.(2.1.13) by $(z_1 - z_3)$ and Eq.(2.1.14) by $(z_1 - z_3)$ and subtracting the two equations we get

$$s((z_1 - z_3)ln(\frac{z_1}{z_2}) - (z_1 - z_3)ln(\frac{z_1}{z_3})) = (z_1 - z_3)ln(\frac{\omega_1}{\omega_2}) - (z_1 - z_2)ln(\frac{\omega_1}{\omega_3}),$$

or

$$s = \frac{(z_1 - z_3)ln(\frac{\omega_1}{\omega_2}) - (z_1 - z_2)ln(\omega_1\omega_3)}{(z_1 - z_3)ln(\frac{z_1}{z_2}) - (z_1 - z_2)ln(\frac{z_1}{z_3})}. \tag{2.1.15}$$

Now from Eq.(2.1.13), we have

$$t = \frac{ln(\frac{\omega_1}{\omega_2}) - sln(\frac{z_1}{z_2})}{z_1 - z_2}, \tag{2.1.16}$$

and from Eq. (2.1.10), we get

$$r = \frac{\omega_1}{z_1^s e^{tz_1}}. \tag{2.1.17}$$

Now taking points $(0.00334, 0.001664)$, $(0.8617, 0.1715)$ and $(5.708, 0.003205)$, on the curve in Figure (2.4), and substituting in Eq.(2.1.15), Eq.(2.1.16) and Eq.(2.1.17), we get the values of unknowns $r, s$ and $t$ as

$$r = 0.5717, \qquad s = 1.0234, \qquad t = -1.2205.$$

Using these values in Eq.(2.1.9) we have

$$h(z) = 0.5717z^{1.0234}e^{-1.2205z}. \tag{2.1.18}$$

The approximate solution, $\omega_{approx}$ is

$$\omega_{approx} = g(z) + h(z),$$

or

$$\omega_{approx} = 1 - exp(-z) + 0.5717z^{1.0234}e^{-1.2205z}. \tag{2.1.19}$$

Figure (2.5) shows the error graph and its approximation in form of the function $h(z)$. Figure (2.6) shows the graph of error between the numerical solution and ini-

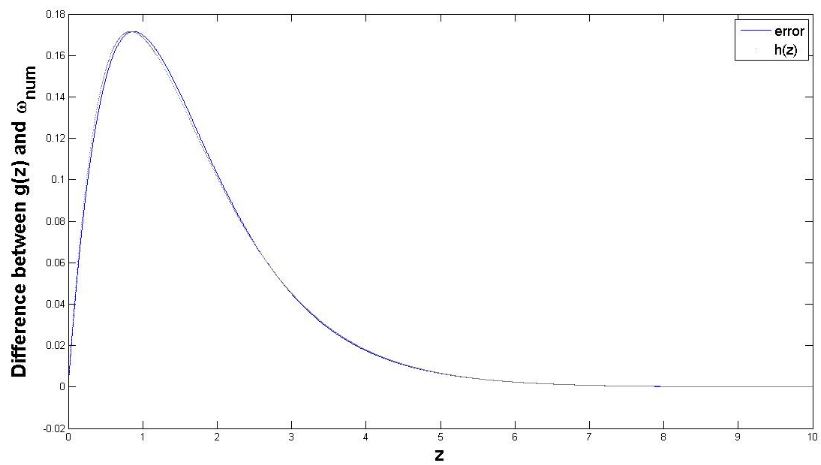


Figure 2.5: Graph of error and $h(z)$

tial approximation $g(z)$ and the error between the numerical and the approximate $\omega_{approx}$ solution.

The old maximum error is 0.1715, now we can see from Figure (2.6), maximum error is reduced to 0.0055.

Figure (2.7) represent the graph of $\omega_{num}, \omega_{approx}$ and $g(z)$.

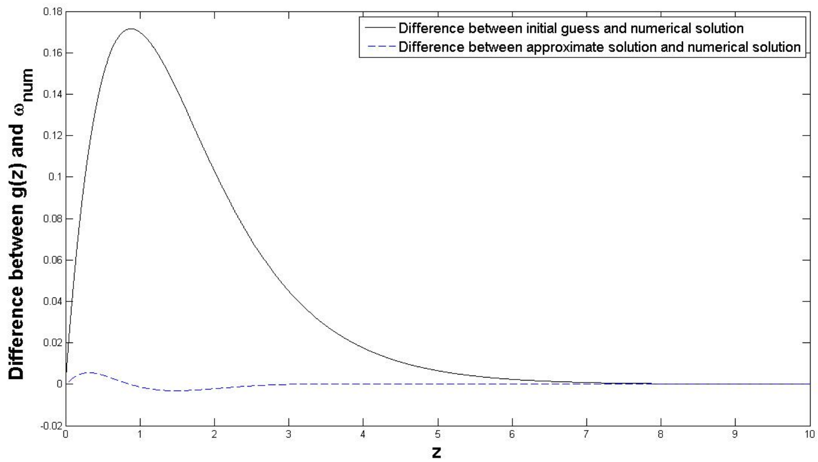


Figure 2.6: Textured line shows the graph of error between numerical solution and initial guess while the continuous line represents error between numerical solution and approximate solution

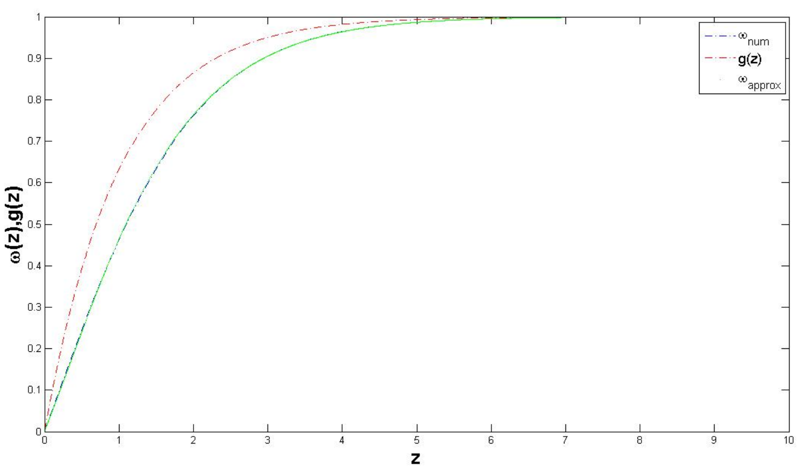


Figure 2.7: Graph of g(z), $\omega_{num}$ and $\omega_{approx}$

Again considering the similarity transformation given in Eq.(2.1.3) which is

$$C(x,t) = \mu(z), \qquad where \ \ x = z. \tag{2.1.20}$$

Substituting Eq. (2.1.19) in above equation, we get

$$C(x,t) = 1 - exp(-z) + 0.5717z^{1.0234}e^{-1.2205z}, \tag{2.1.21}$$

or

$$C(x,t) = 1 - exp(-x) + 0.5717x^{1.0234}e^{-1.2205x}, \tag{2.1.22}$$

which is closed form approximate solution of Burger's equation.

## 2.2 Solution of Burgers' Equation using Symmetry $I_4$

In this section we find the solution of Burgers' equation as in the previous section, using similar approach but using the different symmetry namely $I_4$ given in Eq. (2.0.15) The $I_4$ symmetry, leads to the following characteristic equation

$$\frac{dx}{t} = \frac{dt}{0} = \frac{dC}{-1}. \tag{2.2.1}$$

Solving Eq. (2.2.1), to get two integrals denoted by $v$ and $z$ as

$$\frac{dx}{t} = \frac{dt}{0} \qquad \Rightarrow \qquad z = t^2, \tag{2.2.2}$$

and

$$\frac{dx}{t} = \frac{dC}{-1} \qquad \Rightarrow \qquad v = C + \frac{x}{t}. \tag{2.2.3}$$

The reduction variable can be written as

$$v = \omega(z) \tag{2.2.4}$$

and after substituting in Eq. (2.0.1), we get

$$2z\omega' + \omega = 0. \tag{2.2.5}$$

### 2.2.1 Numerical Solution of the Reduced Ordinary Differential Equation

In previous section using $I_1$ symmetry we get second order ordinary differential equation but using $I_4$ symmetry we have first order differential equation. The numerical solution of Eq. (2.2.5) using *ode45* which is shown in Figure (2.8).

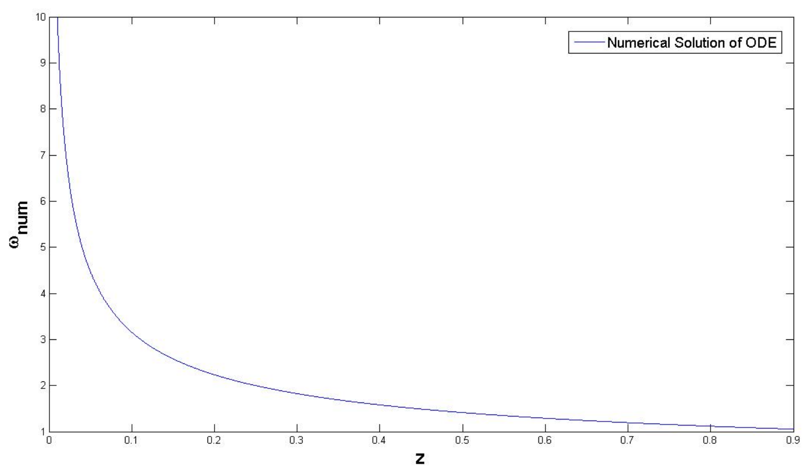


Figure 2.8: Graph of $\omega_{num}(z)$

### 2.2.2 Approximation of Numerical Solution of the Reduced Ordinary Differential Equation

In this section we review the procedure to obtain the required function as given in [3], [4] and [5] thesis. The numerical solution, $\omega_{num}$, in Figure (2.8) resembles to the graph of the function,

$$f(z) = \frac{e^{-0.1z}}{z}, \tag{2.2.6}$$

where Figure (2.9) shows the graph of $f(z)$.

Figure (2.10) shows the comparison of graph of $\omega_{num}$ with the function $f(z)$ which

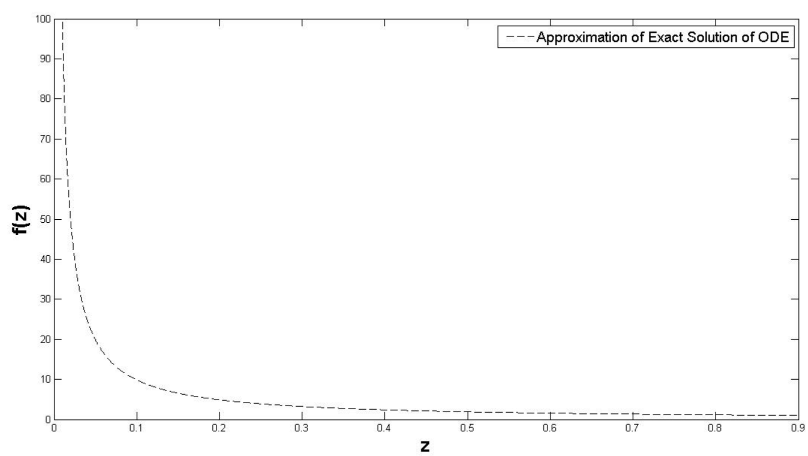


Figure 2.9: Graph of f(z)

is an approximation of exact solution of reduced ordinary differential equation.

The Figure (2.10) indicates that there is an error in approximation of exact solution of reduced ordinary differential equation. This error is plotted in Figure (2.11).

The graph indicates that the maximum error in approximation of exact solution is

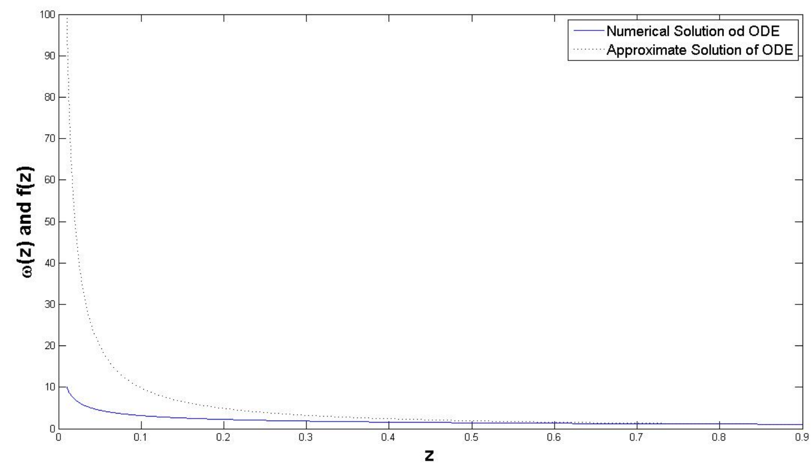


Figure 2.10: Comparison of $\omega_{num}$ and f(z)

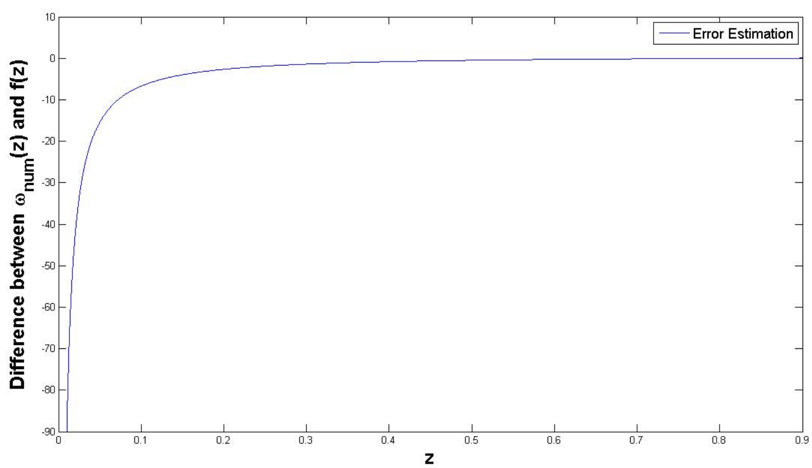


Figure 2.11: Error estimation of $\omega_{num}$(z) and f(z)

89.9. To minimize the maximum error a function $\varphi(z)$ can be found which approximates the error graph. This function is then added up to $f(z)$ in order to reduce the error. The error graph resembles the following function,

$$\varphi(z) = -ae^{bz^c}. \tag{2.2.7}$$

In this function $\varphi(z)$, $a, b$ and $c$ are unknown constant. In pursuit to find these unknowns, consider three points on the curve in figure 2.11 i.e $(z_1, \varphi_1)$, $(z_2, \varphi_2)$ and $(z_3, \varphi_3)$. By substituting these points, the following three equations are obtained

$$\varphi_1 = -ae^{bz_1^c}, \tag{2.2.8}$$

$$\varphi_2 = -ae^{bz_2^c}, \tag{2.2.9}$$

$$\varphi_3 = -ae^{bz_3^c}. \tag{2.2.10}$$

Dividing Eq.(2.2.9) by Eq.(2.2.8) and then applying the natural log on each side, we have

$$ln(\frac{\varphi_2}{\varphi_1}) = (z_1^c - z_2^c)b,$$

$$b = \frac{ln(\frac{\varphi_2}{\varphi_1})}{z_1^c - z_2^c}. \tag{2.2.11}$$

Again dividing Eq.(2.2.10) and Eq.(2.2.8) and taking natural log to get

$$ln(\frac{\varphi_3}{\varphi_1}) = b(z_1^c - z_3^c). \tag{2.2.12}$$

Substituting the value of $b$ from Eq.(2.2.11) into Eq.(2.2.12), we have

$$d = \frac{z_1^c - z_3^c}{z_1^c - z_2^c}, \qquad where \;\; d = \frac{ln(\frac{\varphi_3}{\varphi_1})}{ln(\frac{\varphi_2}{\varphi_1})}. \tag{2.2.13}$$

let,

$$\frac{z_2}{\alpha} = z_1 \qquad \Rightarrow \qquad z_2 = \alpha z_1, \tag{2.2.14}$$

$$\frac{z_3}{\beta} = z_1 \qquad \Rightarrow \qquad z_3 = \beta z_1. \tag{2.2.15}$$

Now Eq. (2.2.13) becomes

$$d = \frac{z_1^c - \beta^c z_1^c}{z_1^c - \alpha^c z_1^c}, \tag{2.2.16}$$

or

$$c = \frac{ln(d-1) - ln(d)}{ln(\alpha) - ln(\beta)}. \tag{2.2.17}$$

from Eq.(2.2.8) we have

$$a = -\varphi_1 e^{bz_1^c}. \tag{2.2.18}$$

Now we consider the three points on the graph given in the Figure (2.11) as

$$(0.01, -89.9), \qquad (0.1, -6.738), \quad \text{and} \quad (0.81, -0.0274).$$

On substituting these points in Eqs.(2.2.17), (2.2.11) and (2.2.18), the values of constants $a, b$ and $c$ as

$$a = 0.4097, \qquad b = -1.4544, \qquad c = -0.2845.$$

Using these values in Eq.(2.2.7), we have

$$\varphi(z) = -0.4097e^{1.4544z^{-0.2845}}. \tag{2.2.19}$$

The approximate solution $\omega_{approx}$ is

$$\omega_{approx} = f(z) + \varphi(z),$$

$$\omega_{approx} = \frac{e^{-0.1z}}{z} - 0.4097e^{1.4544z^{-0.2845}}. \tag{2.2.20}$$

Figure (2.12) indicates a comparison between error graph and approximate function $\varphi(z)$.

Figure (2.13) represents the graph of $\omega_{num}$ and $\omega_{approx}$.

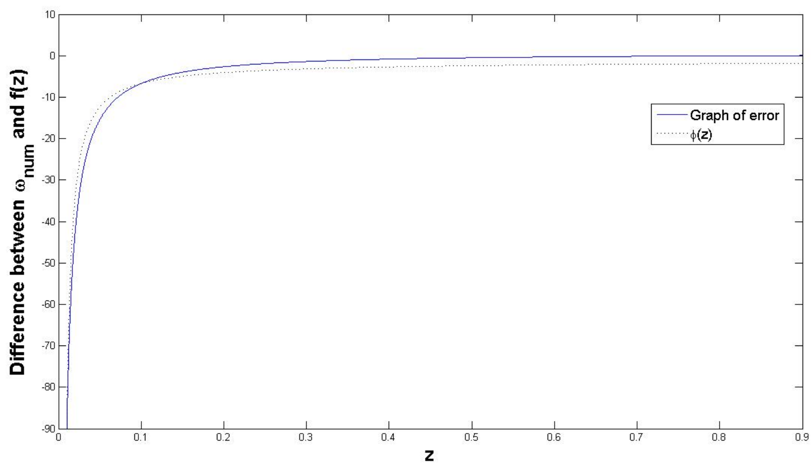


Figure 2.12: Graph of error and $\varphi(z)$

Error estimated between $\omega_{num}$ and initial approximation $f(z)$ and also error estimation between $\omega_{num}$ and $\omega_{approx}$ can be seen in figure 2.14.

Now we can see from the Figure (2.14), maximum error is reduced to 8.8223. Figure (2.15) represents graph of $\omega_{num}, f(z)$ and $\omega_{approx}$

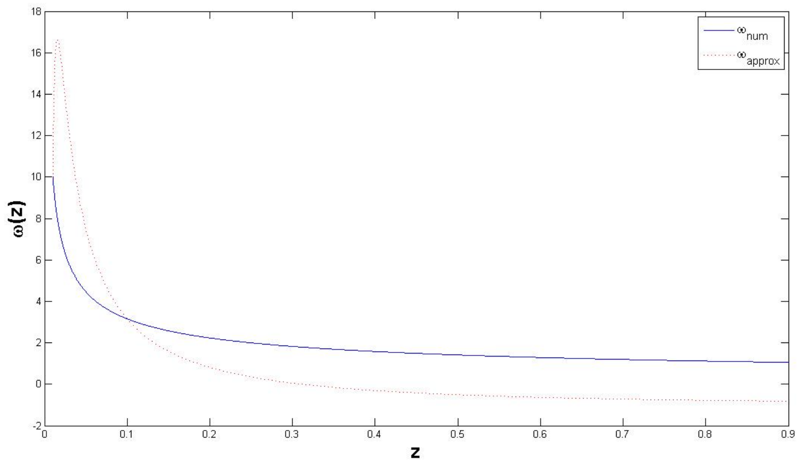


Figure 2.13: Graph of $\omega_{num}$ and $\omega_{approx}$

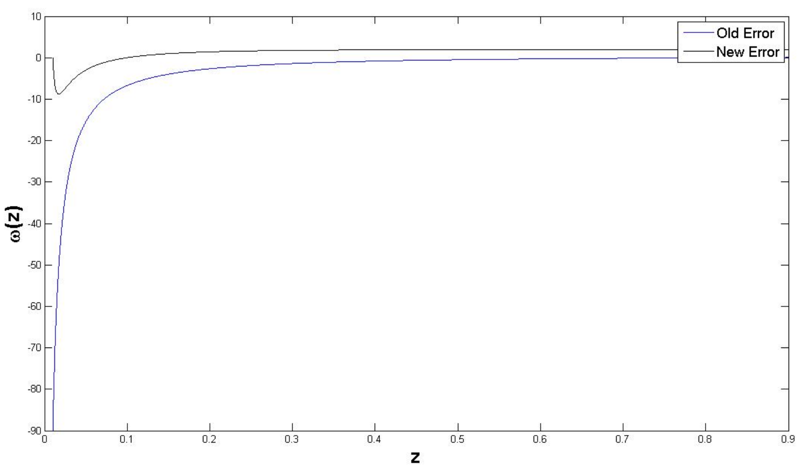


Figure 2.14: Error estimation between f(z) and $\omega_{exact}(z)$along with $\omega_{exact}$ and $\omega_{approx}$

Again considering the similarity transformation given in Eq.(2.2.3) which is

$$\frac{dx}{t} = \frac{dC}{-1} \qquad \Rightarrow \qquad v = C + \frac{x}{t}. \tag{2.2.21}$$

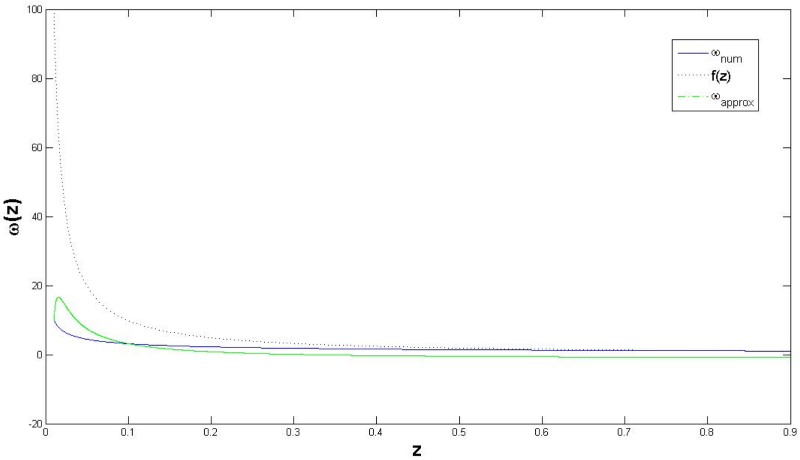


Figure 2.15: Graph of f(z), $\omega_{num}(z)$ and $\omega_{approx}$

Substituting Eq. (2.2.20) in above equation, we get

$$C(x,t) = \frac{e^{-0.1z}}{z} - 0.4097e^{1.4544z^{-0.2845}} - \frac{x}{t} \tag{2.2.22}$$

or

$$C(x,t) = \frac{e^{-0.1t^2}}{t^2} - 0.4097e^{1.4544t^{-0.569}} - \frac{x}{t}, \tag{2.2.23}$$

which is closed form approximate solution of Burger's equation.

# Chapter 3

# Summary

In this thesis a practical way is indicated to find the approximate solution of nonlinear Burgers' equation. We obtained approximate solution of Burgers' equation with the help of similarity methods.

In second chapter we have followed following steps to get the approximate solution of Burger's equation:

• Using similarity variables the non-linear Burgers' equation is reduced to an ordinary differential equation.

• The numerical solution of the reduced ordinary differential equation is calculated by using built in function define in *MATLAB*.

• Numerical solution of the reduced ordinary differential equation is approximated by a function.

• By applying the inverse similarity transformations to the approximated function, we get the approximate solution of the Burgers' equation in form of a function.

## 3.1 Comparison with Exact Solutions

In this section, we find the exact solution of ordinary differential equations of Eq. (2.1.7) and Eq. (2.2.5) and compare it with the approximate solutions given by Eq. (2.1.19) and Eq. (2.2.20) respectively.

### 3.1.1 Comparison Between the Exact and the Approximate Solutions of $\omega'' + \omega\omega' = 0$

Here we find the exact solution of Burgers' equation using the $I_4$ symmetry and then compare it with the approximate solution $\omega_{approx}$. The exact solution of Eq. (2.1.7) is

$$\omega = tanh(\frac{z}{2}) \tag{3.1.1}$$

which is shown in figure (3.1)

Now we compare the exact solution with approximate solution that is given by Eq. (2.1.19). The Figure (3.2) shows the comparison between $\omega_{exact}$ and $\omega_{approx}$. Figure (3.3) shows the error between $\omega_{exact}$ and $\omega_{approx}$.

The maximum error between $\omega_{exact}$ and $\omega_{approx}$ is 0.0055.

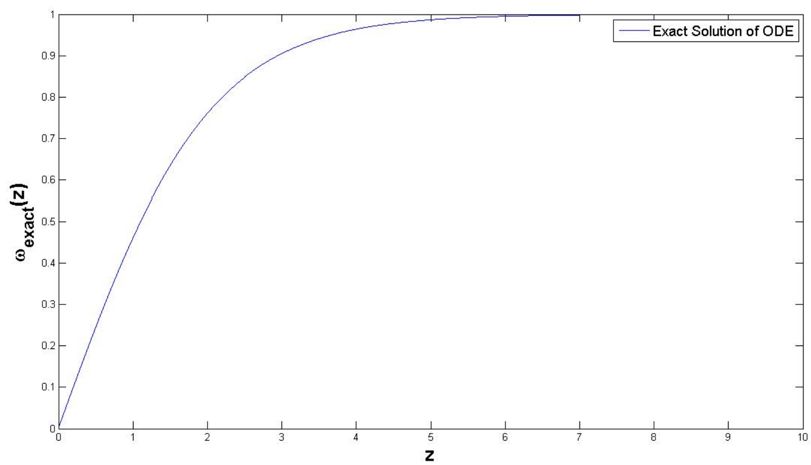


Figure 3.1: Graph of $\omega_(z)$

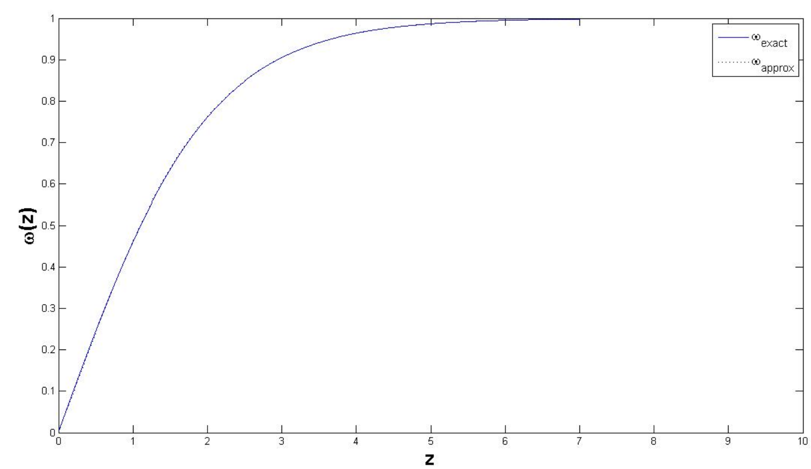


Figure 3.2: Graph of $\omega_{exact}(z)$ and $\omega_{approz}(z)$

### 3.1.2 Comparison Between the Exact and the Approximate Solutions of $2z\omega' + \omega = 0$

We find the exact solution of Burgers' equation using the $I_4$ symmetry and then compare it with the approximate solution $\omega_{approx}$. The exact solution of Eq. (2.2.5)

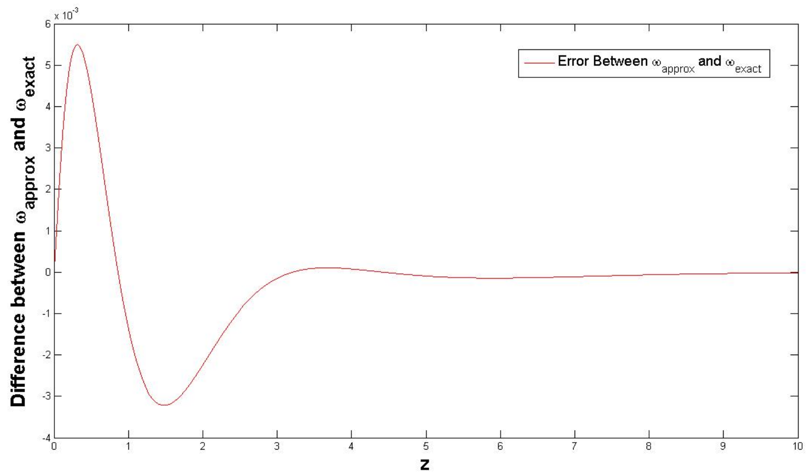


Figure 3.3: Graph of $\omega_{exact}(z)$ and $\omega_{approz}(z)$

is

$$\omega = \frac{a}{\sqrt{z}}, \tag{3.1.2}$$

which is shown in Figure (3.4).

We compare the exact solution with approximate solution that is given by Eq. (2.2.20). The Figure (3.5) shows the comparison between $\omega_{exact}$ and $\omega_{approx}$. Figure (3.6) shows the error between $\omega_{exact}$ and $\omega_{approx}$. The maximum error between $\omega_{exact}$ and $\omega_{approx}$ is 8.8195.

The approximate solution that we obtained by using symmetry $I_1$ given by Eq. (2.0.12) gives more accurate result as compare to the approximate solution obtained by using symmetry $I_4$ given by Eq. (2.0.15). By using $I_1$ symmetry the maximum error between exact and approximate solution is 0.0055 while by using the symmetry

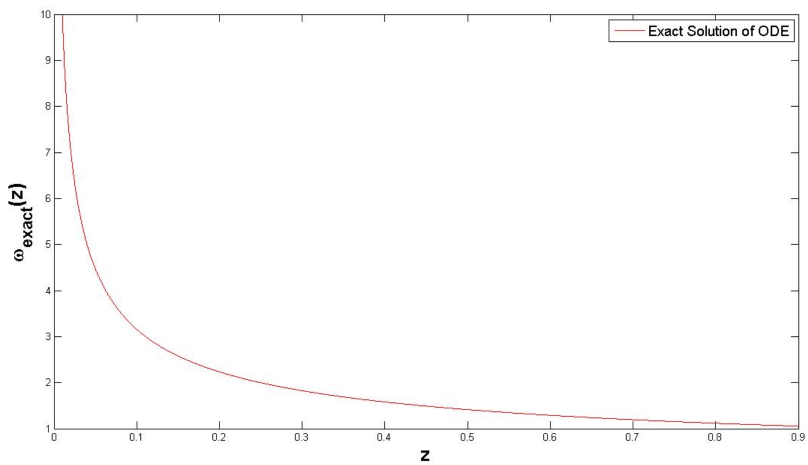


Figure 3.4: Graph of $\omega_{exact}(z)$

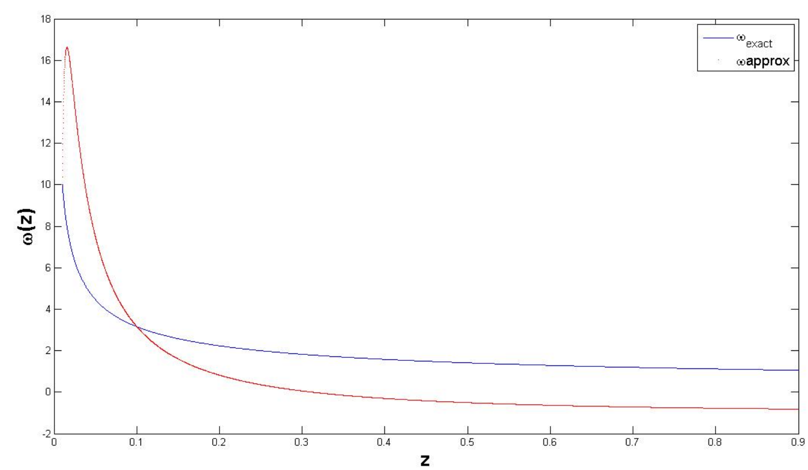


Figure 3.5: Graph of $\omega_{exact}(z)$ and $\omega_{approx}$

$I_4$ the maximum error is 8.8195.

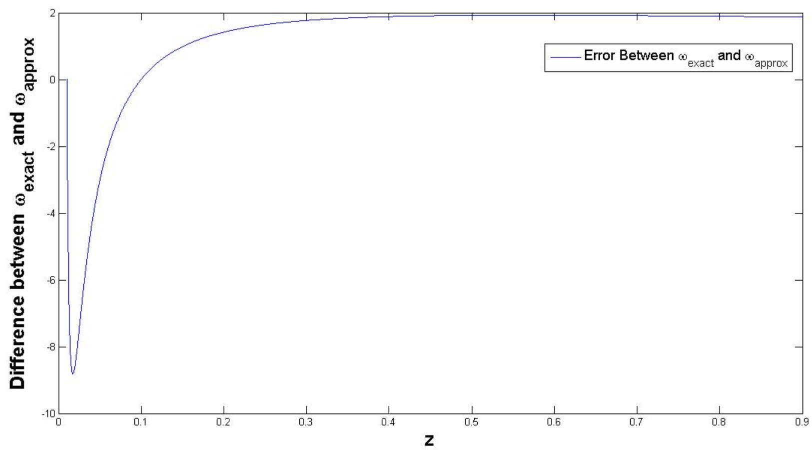


Figure 3.6: Error of $\omega_{exact}(z)$ and $\omega_{approx}$

### 3.1.3 Exact Solution of Burger's Equation

We find exact solution by considering the similarity transformation given in Eq.(2.1.3). Substituting Eq. (3.1.1) in similarity transformation, we get

$$C(x,t) = tanh(\frac{x}{2}), \tag{3.1.3}$$

Again considering the similarity transformation given in Eq.(2.2.3). Substituting Eq. (3.1.2) in Eq.(2.2.3), we get

$$C(x,t) = \frac{a-x}{t}. \tag{3.1.4}$$

In this thesis we generate a method to find approximate similarity solution of nonlinear partial differential equations. The method of solving partial differential equations via similarity transform can be results in ordinary differential equations whose exact solution is difficult to find. Then we implement numerical approach to find the solution of the reduced ordinary differential equation. So the idea of this thesis is to approximate the numerical solution of reduced ordinary differential equation by a function. Then by applying the inverse similarity transformations to the approximated function one may get the approximate solution of the nonlinear partial differential equation in the form of a function.

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