Depth and Stanley Depth of Powers of Edge Ideals Associated to some classes of Caterpillar graph.

Tooba Zahid
Regn.#0203212

A thesis submitted in partial fulfillment of the requirements
for the degree of **Master of Science**
in
Mathematics

Supervised by: Dr Muhammad Ishaq

Department of Mathematics

School of Natural Sciences
National University of Sciences and Technology
H-12, Islamabad, Pakistan

2019

FORM TH-4

National University of Sciences & Technology

MS THESIS WORK

We hereby recommend that the dissertation prepared under our supervision by: TOOBA ZAHID, Regn No. 00000203212 Titled: Depth and Stanley Depth of Powers of Edge Ideals Associated to Some Classes of Caterpillar Graph be accepted in partial fulfillment of the requirements for the award of **MS** degree.

Examination Committee Members

1. Name: DR. MUJEEB UR REHMAN Signature:

2. Name: DR. MUHAMMAD ASIF FAROOQ Signature:

External Examiner: DR. WAQAS MAHMOOD Signature:

Supervisor's Name DR. MUHAMMAD ISHAQ Signature:

Head of Department

31-7-19
Date

COUNTERSINGED

Date: 31/7/2019

Dean/Principal

THESIS ACCEPTANCE CERTIFICATE

Certified that final copy of MS thesis written by **Ms. Tooba Zahid,** (Registration No. **00000203212**), of **School of Natural Sciences** has been vetted by undersigned, found complete in all respects as per NUST statutes/regulations, is free of plagiarism, errors, and mistakes and is accepted as partial fulfillment for award of MS/M.Phil degree. It is further certified that necessary amendments as pointed out by GEC members and external examiner of the scholar have also been incorporated in the said thesis.

Signature: ______________________

Name of Supervisor: Dr. Muhammad Ishaq

Date: 31/7/2019

Signature (HoD): ______________________

Date: 31-7-19

Signature (Dean/Principal): ______________________

Date: 31/7/2019

Dedication

To my parents and teachers

Acknowledgements

Completion of this thesis was not entirely an effort of a single person rather it is the effort of number of individuals. I would like to acknowledge some significant people for their support in the completion of this thesis.

All of me and my efforts are for Allah to whom I am thankful. The foremost among all individuals is my supervisor Dr.Muhammad Ishaq who guided me on every step and helped me overcome every hurdle that stood before me. I owe him alot for converting my mere dream into reality. I would not have been able to do this task if it had not been for sir Ishaq who took time out of his busy schedule.

I would like to thank my family for their sincere concern and devoted prayers throughout my research phase. A special thanks to my mother who was always there to encourage me at times of need.

Finally I would like to thank whole faculty and staff of my department for providing best possible environment to the students.

Abstract

This thesis comprises of bounds for the depth and Stanley depth of edge ideal of three different types of trees and some other graphs containing cycles which turns out to be sharper than the existing one. General but a weaker bound, as compared to our, for the depth and Stanley depth of power of edge ideal of any graph is given by Morey and Fouiza. Their bound depends on the number of components of the graph q, power p of the edge ideal and total number of vertices of the graph. Our work presents a much sharper bound for the depth and Stanley depth of powers of edge ideal associated to some classes of caterpillar graph. This bound being dependent on the power p and number of the pendant vertices becomes much sharper than the previous one.

Contents

List of Figures

Preface

In the field of commutative algebra two totally different invariants gained much focus and a conjecture linking the two was studied by many mathematicians. First one is the depth of a module, an algebraic invariant, which gives the bound for other algebraic invariants. Depth of a module N over a ring R is the length of maximal m-sequence in N. Second one is the Stanley depth of a graded module over a graded ring, named after R. P. Stanley who defined it in 1982. Stanley depth of a module N is defined to be the maximum of Stanley depth of all possible Stanley decompositions of module N.

Our goal is to investigate the values and bounds for the depth and Stanley depth of some powers of edge ideals associated to caterpillar graph and other graphs.

This thesis consists of four chapters. The first chapter has three sections. The first section covers the ring theory where ring R is taken to be commutative ring with unity. Second section covers the module theory with N as an R- module. Third section covers the basic definitions and concepts of graph theory required for the further understanding of this thesis.

Second chapter is the literature review. In this chapter we have defined the concept of depth and Stanley depth of a module. Some useful results from papers [1],[7] and [11] are reviewed. In this chapter we have seen that for a graph H with q connected components and edge ideal $L = L(H)$, $\operatorname{depth} U/L^p \geq \lceil \frac{l-3}{3} \rceil + q - 1$ where l is the diameter of the graph. Using some lemmas from paper by Morey we have analyzed the bounds for the depth and Stanley depth given by Alipour for L the edge ideal of the special type star graph defined by him.

In chapter three we have worked on the depth and Stanley depth of different types of trees and other graphs containing one cycle.

The last chapter includes the determination of bound for the depth and Stanley depth of U/L^p where L is edge ideal corresponding to subclass of caterpillar graph. The bound depends on the order of path in the graph, number of pendent vertices and power $p \geq 1$.

Chapter 1

Preliminaries

In this chapter we will recall some basic definitions and results from ring theory, module theory and graph theory.

1.1 Ring Theory

Definition 1.1.1. A set R together with two binary operations of $'+'$ and $'\times'$ called addition and multiplication satisfying the following axioms is called a ring

1. $(R, +)$ is an abelian group,
2. $'\times'$ is associative i.e, for all $n_1, n_2, n_3 \in R$
$$n_1 \times (n_2 \times n_3) = (n_1 \times n_2) \times n_3,$$
3. distributive law holds i,e: for all $n_1, n_2, n_3 \in R$
$$(n_1 + n_2) \times n_3 = (n_1 \times n_3) + (n_2 \times n_3) \qquad n_1 \times (n_2 + n_3) = (n_1 \times n_2) + (n_1 \times n_3).$$

1.1.1 Polynomial Rings

Definition 1.1.2. The polynomial ring $T[x]$ in an indeterminate x with coefficients from the field T is a set of all the formal sums of the form $a_m x^m + a_{m-1} x^{m-1} + \cdots +$

$a_1x + a_0$ with $m \geq 0$ and each $a_j \in T$. Addition in the polynomial ring is defined component wise that is

$$\sum_{j=0}^{m} a_j x^j + \sum_{j=0}^{m} b_j x^j = \sum_{j=0}^{m} (a_j + b_j) x^j$$

and multiplication is defined as

$$\Big(\sum_{j=0}^{m} a_j x^j\Big) \times \Big(\sum_{j=0}^{n} b_j x^j\Big) = \sum_{i=0}^{m+n} \Big(\sum_{j=0}^{i} a_j b_{i-j}\Big) x^i.$$

The polynomial ring $T[x_1][x_2]$ is ring of polynomials with coefficients from $T[x_1]$ and inderterminate x_2. That is

$$T[x_1][x_2] = T[x_1, x_2].$$

So polynomial ring can be defined in any number of indeterminates as

$$T[x_1, x_2, \dots, x_{m-1}][x_m] = T[x_1, x_2, \dots, x_{m-1}, x_m].$$

Let $U := T[x_1, x_2, \dots, x_m]$ be the polynomial ring with m indeterminates and coefficients from the field T. For all nonnegative integers a_i, any product of the form $x_1^{a_1} x_2^{a_2} \dots x_m^{a_m}$ is called a monomial.

Definition 1.1.3. An subring J of the ring R is called an ideal of R if we have $xJ \subset J$ for all $x \in R$. An ideal in a polynomial ring U which is generated by monomials is called a monomial ideal.

For example $U = T[x_1, x_2, x_3]$ be the polynomial ring then $L = (x_1^2, x_2^2, x_3^2)$, $L = (x_1^2, x_2x_3)$ are some monomial ideals of U with generating sets $\{x_1^2, x_2^2, x_3^2\}$ and $\{x_1^2, x_2x_3\}$ respectively. Similarly for $U = \mathbb{R}[x_1, x_2, \dots, x_m]$ some of its monomial ideals are $L = (x_1)$, $L = (x_1, x_2, x_3)$, $L = (x_1, x_2, \dots, x_m)$.

Definition 1.1.1. For the ring R, the intersection of all maximal ideals of R is called Jacobson radical of ring R denoted by $\mathfrak{R}$.

1.1.2 Operations on Ideal

The operation of sum, product and intersection of ideals are commutative and associative.

1. ***Sum of ideals***

   Let L and L' be the ideals of the ring R, then $L + L'$ is the set of all the elements of the form $l + l'$ where $l \in L$ and $l' \in L'$. The set $L + L'$ is an ideal and is the smallest ideal which contains L and L'. Sum of any number of ideals can be defined in the same way. For the family of ideals $\{L_i\}$ where $i \in I$, their sum $\sum_{i \in I} L_i$ consists of the elements of the form $\sum_{i \in I} x_i$ with $x_i \in L_i$ with only finite many $x_i's$ nonzero.

2. ***Product of ideals***

   Product of two ideals L and L' of the ring R is an ideal LL' which is generated by all products of the form ll', where $l \in L$ and $l' \in L'$. The ideal LL' is the set of all finite sums $\sum l_k l'_k$ with each $l_k \in L$ and $l'_k \in L'$. Product of finite number of ideals can also be defined. In particular for $(n > 0)$, the ideal L^n is generated by all products $l_1 l_2 \ldots l_n$ with each $l_i \in L$. Conventionally we take $L^0 = (1)$.

3. ***Intersection of ideals***

   Intersection of ideals is again an ideal. For example for the two ideals $6\mathbb{Z}$ and $10\mathbb{Z}$, $6\mathbb{Z} \cap 10\mathbb{Z} = 30\mathbb{Z}$.

4. ***Colon of ideals***

   For the ideals L and L' of ring R, their colon is again an ideal and is defined to be the following subset of R,

$$(L : L') = \{x \in R \mid xL' \subseteq L\}.$$

For example for the ring $R = \mathbb{Z}$ and ideals $L = (16)$ and $L' = (4)$ we have

$$\begin{aligned}(L : L') &= \{x \in \mathbb{Z} : x(4) \subseteq (16)\} \\ &= (4).\end{aligned}$$

$(0 : L)$ is an ideal called the annihilator of L denoted by $\mathrm{Ann}(L)$ given by

$$\mathrm{Ann}(L) = \{x \in R \mid xL = 0\}.$$

5. ***Radical of an ideal***

   For ideal L of the ring R, radical of L is again an ideal of R given by

$$\sqrt{L} = \{x \in R \mid x^n \in L, \ \ for \ \ some \ \ n > 0\}.$$

   If $\sqrt{L} = L$, where R is a commutative ring, then L is called radical ideal. Also all squarefree Prime ideals of a ring are radical ideals. Radical of an ideal is whole ring if and only if ideal itself is a ring. We also have a result which says that $\sqrt{L}$ is the intersection of all primes ideals containing L.

***Direct product and direct sum of family of rings*:**

Let $R_1, R_2, \ldots$ be the family of rings. The direct product of the family of rings

$$R = \prod_i R_i,$$

is the set of all sequences $r = (r_1, r_2, \ldots)$ where $r_i \in R_i$ with componentwise addition and multiplication. Direct product of rings is again a ring. The direct sum

$$R' = \bigoplus R_i \subseteq \prod_i R_i$$

consists of elements of the form $r = (r_1, r_2, \ldots,)$ where finite many $r_i's$ are non-zero. When the number of rings is finite then $\bigoplus R_i = \prod R_i$.

1.1.3 Primary Decomposition and Primary Ideals

Let $L \subset U$ be a monomial ideal. Then for each K_i having generating set consisting of pure powers of variables that is K_i is of the form $(x_1^{a_1}, x_2^{a_2}, \ldots, x_m^{a_m})$ we have,

$$L = \bigcap_{i=1}^{n} K_i.$$

The presentation of ideal L in the form of intersection of these primary ideals is unique. If a monomial ideal is already generated by the pure power of variables then the ideal L is irreducible that is it cannot be expressed as a proper intersection of two other monomial ideals. For a squarefree monomial ideal L, these $K_i's$ are the monomial prime ideal of L. A prime ideal P containing L with no prime ideal containing L properly contained in P is called minimal prime ideal of L. Set of minimal prime ideals of L is denoted by $\mathrm{Min}(L)$. For a squarefree monomial ideal L we have

$$L = \bigcap_{P \in Min(L)} P$$

and each P is monomial prime ideal. Let U be a Noetherian ring and N be a finitely generated U-module. If there exists an element $a \in N$ such that $\mathrm{Ann}(a) = P$ where P is a prime ideal of U then this P is called associated prime ideal of N. The set of all associated prime ideals of N is denoted by $\mathrm{Ass}(N)$. Following is an example of primary decomposition of square free monomial ideal L.

Example 1. Let $L = (x_1x_2, x_3x_5, x_2x_3, x_2x_4, x_3x_4, x_1x_4)$ be an ideal, then

$$\begin{aligned}
L &= (x_1x_2, x_3x_5, x_2x_3, x_2x_4, x_3x_4, x_1x_4) \\
&= (x_2, x_4, x_5) \cap (x_3, x_1x_2, x_2x_4, x_1x_4) \\
&= (x_2, x_4, x_5) \cap (x_1, x_3, x_2) \cap (x_1x_2, x_3, x_4) \\
&= (x_2, x_4, x_5) \cap (x_2, x_4, x_3) \cap (x_1, x_3, x_4) \cap (x_1, x_2, x_3) \\
&= (x_2, x_4, x_5) \cap (x_1, x_3, x_4) \cap (x_2, x_4, x_3) \cap (x_1, x_2, x_3).
\end{aligned}$$

Since L is a squarefree monomial ideal so it can be seen that (x_2, x_4, x_5), (x_1, x_3, x_4), (x_1, x_2, x_3) and (x_2, x_4, x_3) are minimal prime ideals of L.

The primary decomposition is known as irredundant primary decomposition if none of the $K_i's$ can be omitted in this intersection and for all $i \neq j$ we have $P_i \neq P_j$ where $\text{Ass}(K_i) = \{P_i\}$. If $L = \cap_{i=1}^{n} K_i$ is an irredundant primary decomposition of L then the K_i is known as the P_i-primary component of L and $\text{Ass}(L) = \{P_1, P_2, \ldots, P_n\}$. One may not always have a decomposition of an ideal which is also irredundant decomposition. However in such a situation one can construct an irredundant primary decomposition by letting P-primary components of L be the intersection of all $K_i's$ with $\text{Ass}(K_i) = \{P\}$. For example for ideal $L = (x_1^3, x_3^3, x_1^2 x_2^2, x_1 x_3 x_2^2, x_2^2 x_3^2)$ the primary decomposition is as follows

$$L = (x_1^3, x_2^2, x_3^3) \cap (x_1^2, x_3) \cap (x_1, x_3^2).$$

As $\text{Ass}(x_1^2, x_3) = \text{Ass}(x_1, x_3^2) = \{(x_1, x_3)\}$, so intersecting (x_1^2, x_3) and (x_1, x_3^2) we get $(x_1^2, x_1 x_3, x_3^2)$. So the irredundant primary decomposition is

$$L = (x_1^3, x_2^2, x_3^3) \cap (x_1^2, x_1 x_3, x_3^2).$$

1.1.4 Dimension and Height of a Ring

Dimension of a ring is commonly known as Krull dimension after Wolfgang Krull. Consider the chain of prime ideal with respect to inclusion

$$P_0 \subset P_1 \subset P_2 \subset \ldots \subset P_m.$$

The supremum of lengths of chains of prime ideals is called the Krull dimension of R. If there is no upper bound for such chains in R then we have $\dim(R) = \infty$. All the PIDs have dimension one since all ideals are principle in PIDs. All fields have dimension equal to zero. The polynomial ring U has dimension m for the chain of prime ideals

$$(x_1) \subset (x_1, x_2) \subset (x_1, x_2, x_3) \subset \ldots \subset (x_1, x_2, \ldots, x_m).$$

For the polynomial ring with infinite many indeterminates the dimension of the ring is infinity. Height of a prime ideal P is defined to be the maximum of lengths of chains of prime ideals contained in P with respect to inclusion. The height of any ideal L is the minimum of the heights of prime ideals containing L. For the ideal $L = (x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_1x_3, x_2x_4, x_3x_5)$, we have the following primary decomposition

$$L = (x_1, x_2, x_3, x_4) \cap (x_1, x_2, x_3, x_5) \cap (x_1, x_2, x_4, x_5),$$

then $\mathrm{ht}\,(L) = 4$.

1.1.5 H-Grading

Let H be an abelian group and R a commutative ring. A family $\{R_h\}_{h\in H}$ of abelain subgroups of $(R, +)$ satisfying the following conditions for all $h, f \in H$:

1. $R = \bigoplus_{h\in H} R_h$,
2. $R_hR_f \subseteq R_{h+f}$,

is called H-grading of R. The elements of R_h are known as homogeneous elements of degree h. If $a \in R_h$ we write the degree of a as $\deg(a) = h$ or $|a|= h$. Every ring is a graded ring by letting $R_0 = R$ and $R_i = 0$ for all $i \neq 0$.
For example if $H = \mathbb{Z}$, $U = T[x]$ then for $h \in \mathbb{Z}$, we have

$$U_h = \begin{cases} Tx^h, & \text{if} \quad h \geq 0; \\ 0, & \text{if} \quad h < 0. \end{cases}$$

This is an $\mathbb{Z}$-grading.

1.2 Module Theory

In this section, we will define another algebraic structure called module. We will give some of its properties and results. We will also see their behavior in exact sequences.

1.2.1 Modules

An abelian group N is called R-module, where R is a commutative ring, if there is a map

$$\star : R \times N \longrightarrow N$$

defined as

$$x \star n = xn, \qquad\qquad for\ x \in R, n \in N$$

such that the following conditions are satisfied for all $x, y \in R$ and $n, n_1, n_2 \in N$

1. $x(n_1 + n_2) = xn_1 + xn_2$,
2. $(x + y)n = xn + yn$,
3. $(xy)n = x(yn)$,
4. $1n = n$.

Set of integers is an example of $\mathbb{R}$-module. Every ring R is also a module over itself. For the ring R and ideal L , R/L is an R-module.

A multiplicatively closed (by the elements of R) subgroup N' of additive group N is called submodule of N. The dimension of an R-module N is defined in the following way

$$\dim(N) = \dim(R/ann(N)).$$

Let N be an R-module and $B \subseteq N$. The module N is called free module on B if for $0 \neq n \in N$ there exists unique elements $x_1, x_2, \ldots, x_n$ in N and unique elements $b_1, b_2, \ldots, b_n$ in B such that

$$n = x_1 b_1 + x_2 b_2 + \cdots + x_n b_n, \qquad \text{for some } \ n \in \mathbb{Z}^+.$$

In this case B is called the basis set or set of free generators for N.

Lemma 1.2.1. *(**Nakayama's Lemma.**) Let R be a ring and let N be a finitely generated R-module. Suppose that L is an ideal of R such that $L \subseteq \mathfrak{R}$ where $\mathfrak{R}$ denotes the Jacobson radical. If $LN = N$ then $N = 0$.*

1.2.2 Graded Module

Let H be an abelian group, R be a graded ring with $R = \bigoplus_{h \in H} R_h$ and N be an R-module. N is called graded R-module if there exist subgroups N_h of N for all $h \in H$

such that $N = \bigoplus_{h\in H} N_h$ and $R_h N_f \subseteq N_{h+f}$ for all $h, f \in H$. Clearly each N_h is an R_0-module since $R_0 N_h \subseteq N_h$. Also R is graded R-module. For example if $U = T[x,y]$, $H = \mathbb{Z}$, then

$$U = T \oplus (Tx + Ty) \oplus (Tx^2 + Txy + Ty^2) \oplus (Tx^3 + Tx^2y + Txy^2 + Ty^3) \oplus \ldots.$$

1.2.3 Exact Sequences

A sequence of R-homomorphisms and R-modules

$$\ldots \xrightarrow{g_{i-1}} B_{i-1} \xrightarrow{g_i} B_i \xrightarrow{g_{i+1}} B_{i+1} \xrightarrow{g_{i+2}} \ldots$$

is said to be exact at B_i if,

$$Im(g_i) = Ker(g_{i+1}).$$

The sequence

$$0 \to B' \xrightarrow{g'} B$$

is called exact sequence at B' iff g' is injective and the sequence. The sequence

$$B \xrightarrow{g''} B'' \to 0$$

is called exact sequence at B'' iff g'' is surjective. The sequence

$$0 \to B \xrightarrow{g} B' \xrightarrow{g'} B'' \to 0,$$

is called short exact sequence iff

$$Ker(g') = Im(g).$$

Let $B = \mathbb{Z}$ and $B'' = \mathbb{Z}/n\mathbb{Z}$ are $\mathbb{Z}$-modules,

$$0 \to \mathbb{Z} \xrightarrow{i} \mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/n\mathbb{Z} \to 0,$$

where i is inclusion map defined as $i(a) = (a, 0)$ and π is projection map defined as $\pi(a, b) = b$. Then the above sequence is a short exact sequence.

1.2.4 Noetherian Ring

Let $\sum$ be a poset with respect to $\leq$. Then the following two conditions are equivalent

1. Every increasing sequence

$$a_1 \leq a_2 \leq \ldots \leq a_n \leq \ldots,$$

   in $\sum$ is stationary that is there exists $n \in \mathbb{N}$ such that $a_n = a_m$ for all $m \geq n$,

2. All the nonempty subsets of $\sum$ have a maximal element.

For the case when $\sum$ is a set of submodules of N with relation of containment, then first condition mentioned above is called ascending chain condition and the second one is called the maximal condition.

Definition 1.2.2. If every ascending chain of R-submodules of R-module N is stationary then N is called Noetherian. Ring R is called Noetherian if R is Noetherian as an R-module.

Theorem 1.2.3. ***(Hilbert Basis Theorem.)***
If R is Noetherian ring, then $R[x]$ is also Noetherian.

Following is an immediate corollary that follows from the above theorem.

Corollary 1.2.4. *If R is Noetherian, then so does the ring $R[x_1, x_2, \ldots, x_m]$.*

Proposition 1.2.5. *Let $0 \to B \xrightarrow{g} B' \xrightarrow{g'} B'' \to 0$, be a short exact sequence, then B' is Noetherain $\Longleftrightarrow$ B and B'' are Noetherian.*

Corollary 1.2.6. *[2, Corollary 6.4] If $N_1, N_2, \ldots, N_n$ are Noetherian R-modules, then $\bigoplus_{k=1}^{n} N_k$ is Noetherian.*

1.3 Graph Theory

A graph H is a pair of sets (V, E) where $V = \{v_1, v_2, \ldots, v_n\}$ is th set of vertices and E is the set of edges. For example for $V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ and $E = \{(v_1, v_1), (v_1, v_2), (v_2, v_3), (v_3, v_4), (v_3, v_4), (v_3, v_5), (v_4, v_5),$
$(v_5, v_6)\}$ we have the following graph.

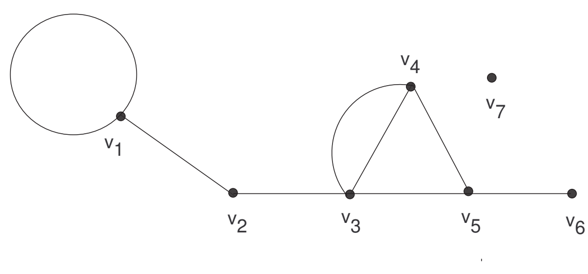


Figure 1.1: Graph H

Order of a graph is the cardinality of the vertex set of the graph whereas size of a graph is the cardinality of the edge set of the graph. Degree of a vertex v in a graph, denoted by $\deg(v)$, is number of the edges incident on it. For example in the above graph H, order of H is 7, size is 8, $\deg(v_1) = 3$, $\deg(v_2) = 2$ and so on. Loop is an edge which connects a vertex to itself for example, $\{v_1, v_1\}$ is the loop in the above graph H. Leaf is a vertex which is adjacent to only one vertex. In the above graph v_6 is the leaf. Isolated vertex is the one which is not adjacent to another vertex of the graph. In the graph H, v_7 is the isolated vertex. Neighborhood of a vertex v is a set of vertices adjacent to it in the graph denoted by $N(v)$. For example in the above graph $N(v_2) = \{v_1, v_3\}$, $N(v_5) = \{v_3, v_4, v_6\}$. A graph is said to be connected if every vertex of it is reachable from every other vertex. Graph H is disconnected since v_7 is not reachable from any other vertex of the graph.

1.3.1 Subgraphs

A graph $H_1 = (V_1, E_1)$ is said to be the subgraph of $H = (V, E)$ if

1. $V_1 \subseteq V$,

2. Every edge of H_1 is also an edge of H.

For example, below given graphs H_1 and H_2 are two subgraphs of H.

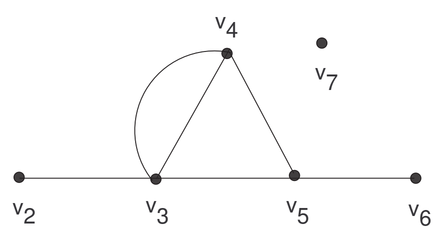


Figure 1.2: Graph H_1

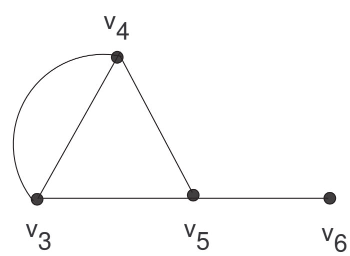


Figure 1.3: Graph H_2

1.3.2 Components of a Graph

A component H' of graph H is its subgraph such that

1. H' is connected and

2. Either H' is an induced graph from those edges of H which have an end vertex in H' or H' is an isolated vertex of H.

Following graphs H' and H'' are the components of graph H.

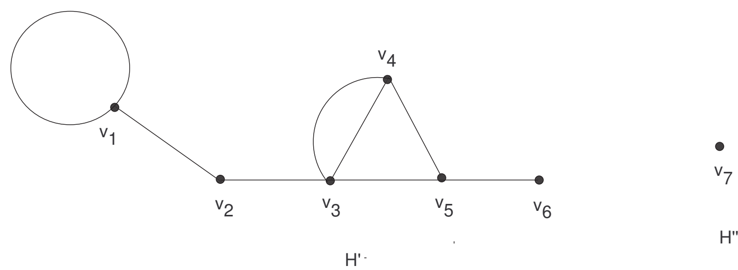


Figure 1.4: Graph H' and H'' are components of H

1.3.3 Minor of Graph

A subgraph of the graph H which is formed by successive deletion of edges is called the minor of graph H. For example following is a minor of the graph H given above

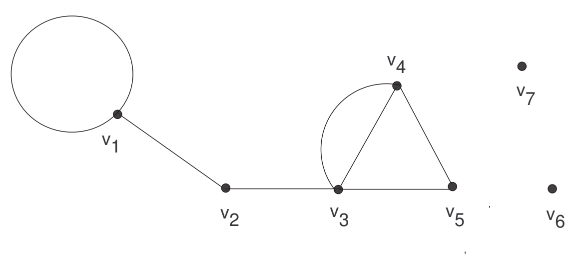


Figure 1.5: minor of H

1.3.4 Some common graphs

Here we will define some of the common graphs in graph theory.

Path graph

Path graph $P_n = (V, E)$ is a finite sequence of the form

$$v_1 e_1 v_2 e_2 \dots e_{j-1} v_j \dots e_{n-1} v_n,$$

where n is the order of graph P_n with $v_i' s \in V$ and $e_i' s \in E$.

Figure 1.6: P_8 Path on 8 vertices

Cycle graph

Cycle graph $C_n = (V, E)$ is a finite sequence of the form

$$v_1 e_1 v_2 e_2 \dots e_{j-1} v_j \dots e_{n-1} v_n e_n v_1,$$

where n is the order of graph C_n with $v_i' s \in V$ and $e_i' s \in E$.

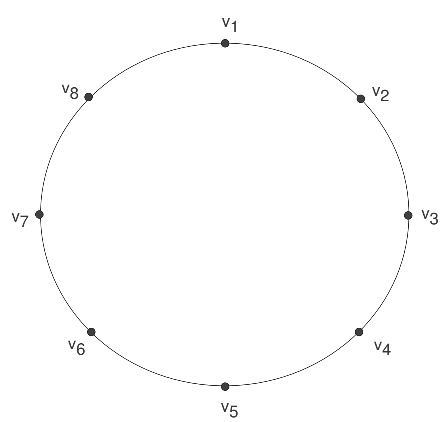


Figure 1.7: C_8 Cycle on 8 vertices

Bipartite graph

If the vertex set V of the graph can be partitioned into two subsets V_1 and V_2 such that each edge has one end in V_1 and other in V_2, then such a partition (V_1, V_2) is called the bipartition of graph. V_1 and V_2 are known as partite sets of the graph. Bipartite graph have a special property which is that it contains no odd cycle. Following is an example of bipartite graph.

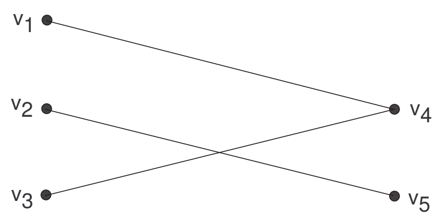


Figure 1.8: $(3, 2) - bigraph$

Tree

Tree is a connected graph with no cycle. Since tree has no cycle so it is also a bipartite graph. The vertices in the tree with degree equal to one are called leaves or pendant vertices. Following is an example of tree graph.

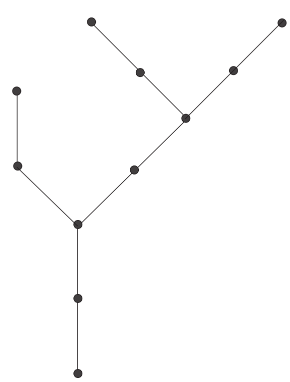


Figure 1.9: Tree graph

Caterpillar graph

A caterpillar graph is a tree in which the removal of all pendant vertices results in a chordless path, P_n.

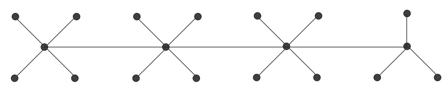


Figure 1.10: caterpillar

Star graph

A complete bipartite graph with one partite set having cardinality 1 is called a star graph. Star graph on n vertices is denoted by S_n.

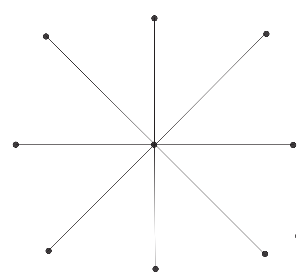


Figure 1.11: S_9 Star on 9 vertices.

Complete graph

A simple graph in which every vertex is adjacent to every other vertex is called a complete graph. K_n represents a complete graph on n vertices.

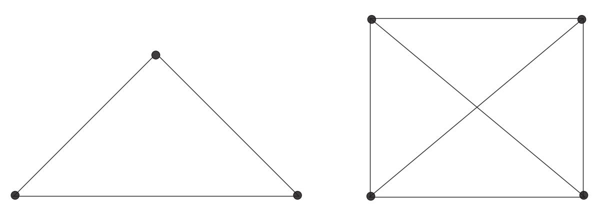


Figure 1.12: K_3 and K_4 on 3 and 4 vertices respectively.

Chapter 2

Depth and Stanley depth of $\mathbb{Z}^n$-graded modules.

2.1 Depth and Stanley depth

Corresponding to the graph H with the vertex set $V = \{1, 2, \ldots, m\}$ and edge set E, we define edge ideal L to be an ideal generated by the monomials of the form $x_i x_j$ such that $\{i, j\} \in E$.

Definition 2.1.1. Let R be a ring and N be a module over R. Then an element $x \neq 0$ of R is called a zero divisor of module N if $xn = 0$ for some nonzero n. An element which is not a zero divisor is called a regular element.

Definition 2.1.2. A sequence $x = x_1, x_2, \ldots, x_m$ of elements of R satisfying the following conditions:

1. x_i is $N/(x_1, x_2, \ldots, x_{i-1})N$ regular for any i,

2. $N \neq (x)N$,

is called N-regular sequence or N-sequence where N is a R-module.

Since our ring is local noetherain ring so has unique maximal ideal thus Jacobson radical $\mathfrak{R}$ is the maximal ideal. So by Nakayama's lemma since $N \neq 0$ thus $N \neq (x)N$. Thus the second condition for sequence to be regular is always satisfied in our case.

For the polynomial ring U and $N = U$ the sequence $x_1, x_2, \ldots, x_m$ is N- regular. Now we define what is depth of a module

Let R be a local Noetherian ring with unique maximal ideal m and N be a finitely generated R-module. The common length of all maximal N-sequences in m is called the depth of N and is denoted by depth(N). For example depth of ring R over itself as an R-module is equal to the dimension of the ring. A striking conjecture was proposed by Richard Stanley in 1982 in his article [17] giving an upper bound in terms of an invariant for the depth of multigraded module. This invariant is known as Stanley depth of a module.

Let U br the ring, N be a finitely generated $\mathbb{Z}^n$-graded U-module, w a homogeneous element in N and $Z \subset \{x_1, x_2, \ldots, x_m\}$. Here $wT[Z]$ denotes the T-subspace of N generated by monomials of the form wv where v is a monomial in $T[Z]$. Then T-subspace $wT[Z]$ is called a Stanley space of dimension $|Z|$ if it is free $T[Z]$-module. A Stanley decomposition of N is a presentation of T-vector space N as a finite direct sum of Stanley spaces

$$\mathcal{D} : N = \bigoplus_{i=1}^{s} w_i T[Z_i].$$

The Stanley depth of a decomposition $\mathcal{D}$ is

$$\operatorname{sdepth} \mathcal{D} = \min\{|Z_i|, i = 1, \ldots, s\}.$$

The Stanley depth of N is

$$\operatorname{sdepth}(N) = \max\{\operatorname{sdepth} \mathcal{D} : \mathcal{D} \text{ is the Stanley decomposition of } N\}.$$

For example, the Stanley decomposition of the module $T[x, y]/(xy, y^2) = T[x] \bigoplus yT$, thus $\operatorname{sdepth}(T[x, y]/(xy, y^2)) = 0$. Since a module can have infinite many decompositions which obviously makes the determination of Stanley depth almost impossible in general. However Herzog et al. in [9] shortened the problem to find a partition of a finite ordered set. In [16] Rinaldo gave a computer implementation for this algorithm, in a computer algebra system CoCoA.

2.2 Computation of Stanley Depth of monomial ideal

An algorithm was given by Herzog, et al. in [9] to compute the Stanley depth of module L/I where $I \subset L \subset U$ are monomial ideals of U. In this case L/I is a $\mathbb{Z}^n$-graded U-module. For this they fixed an integer vector $g \in \mathbb{Z}^n$ with the property that $a \leq g$ for all $a \in \mathbb{Z}^n$ with $v^a \in L/I$, where $\leq$ represents partial order in $\mathbb{Z}$ given by component wise comparison. With this data they defined characteristic poset $P^g_{L/I}$ of L/I with respect to g,

$$P^g_{L/I} = \{a \in \mathbb{Z}^n : v^a \in L/I, a \leq g\}.$$

They showed that the partition of this poset induces a Stanley decomposition of the module L/I. They also showed that corresponding to any Stanley decomposition of L/I there exists one induced by the partition of $P^g_{L/I}$ whose Stanley depth is greater than or equal to the actual one. This method is usually quite laborious as one cannot find all possible decompositions of a module. Consider the following graph

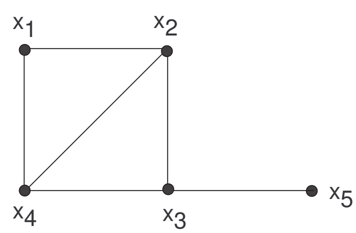


Figure 2.1: H

with the corresponding edge ideal $L = L(H) = (x_1x_2, x_3x_5, x_2x_3, x_2x_4, x_3x_4, x_1x_4)$ and $I = (0)$, then we get the following poset $P^g_{U/L}$

$P^g_{U/L} = \{(0,0,0,0,0), (1,0,1,0,0), (1,0,0,0,0), (0,1,0,0,0), (0,1,0,0,1), (0,0,1,0,0), (0,0,0,1,0), (0,0,0,0,1), (0,0,0,1,1), (1,0,0,0,1)\}$.

There are many partitions possible corresponding to this poset. One of the partition $\mathcal{P}$ of $P^g_{U/L}$ is given by

$$[00000, 10100] \cup [01000, 01001] \cup [00010, 00011] \cup [00001, 10001].$$

Corresponding Stanley Decomposition is

$$U/L = T[x_1, x_3] \bigoplus x_1T[x_1, x_2] \bigoplus x_2T[x_2, x_5] \bigoplus x_4T[x_4, x_5] \bigoplus x_5T[x_1, x_5],$$

hence

$$\operatorname{sdepth}(U/L) = 2.$$

Using CoCoA we get

$$\operatorname{sdepth}(U/L) = 2.$$

So above given partition is one which gives the Stanley depth of U/L. Another examples for a non-squarefree monomial ideal is given below.
$L = (x_1^2x_2^2, x_1x_2^2x_3, x_1x_2^2x_5, x_1x_2x_3x_4, x_2^2x_3^2, x_2^2x_3x_5, x_2x_3^2x_4, x_2^2x_5^2, x_2x_3x_4x_5, x_3^2x_4^2)$ and $I = (0)$ then following is the partition $\mathcal{P}$ of corresponding poset

$$\begin{aligned}
&[00000, 20200] \cup [01000, 02020] \cup [00010, 20020] \cup [00001, 20002] \\
&\cup [11000, 21200] \cup [01100, 02120] \cup [01001, 21002] \cup [00110, 20210] \\
&\cup [00101, 00202] \cup [00011, 00022] \cup [12000, 12020] \cup [11010, 21020] \\
&\cup [10101, 20201] \cup [10011, 20021] \cup [02001, 02021] \cup [01200, 01202] \\
&\cup [01101, 21102] \cup [01011, 01022] \cup [00120, 20120] \cup [00111, 00212] \\
&\cup [11011, 21021] \cup [10111, 20211] \cup [10102, 20102] \cup [10012, 20012] \\
&\cup [00121, 00122] \cup [11201, 21201] \cup [11012, 21012] \cup [10112, 20112].
\end{aligned}$$

Corresponding Stanley decomposition is

$$
\begin{aligned}
U/L =& T[x_1^2,x_3^2] \bigoplus x_2T[x_2^2,x_4^2] \bigoplus x_4T[x_1^2,x_4^2] \bigoplus x_5T[x_1^2,x_5^2] \\
&\bigoplus x_1x_2T[x_1^2,x_2,x_3^2] \bigoplus x_2x_3T[x_2^2,x_3,x_4^2] \bigoplus x_2x_5T[x_1^2,x_2,x_5^2] \\
&\bigoplus x_3x_4T[x_1^2,x_3^2,x_4] \bigoplus x_3x_5T[x_3^2,x_5^2] \bigoplus x_4x_5T[x_4^2,x_5^2] \\
&\bigoplus x_1x_2^2T[x_1,x_2^2,x_4^2] \bigoplus x_1x_2x_4T[x_1^2,x_2,x_4] \bigoplus x_1x_3x_5T[x_1^2,x_3^2,x_5] \\
&\bigoplus x_1x_4x_5T[x_1^2,x_4^2,x_5] \bigoplus x_2^2x_5T[x_2^2,x_4^2,x_5] \bigoplus x_2x_3^2T[x_1,x_2^2,x_5^2] \\
&\bigoplus x_2x_3x_5T[x_1^2,x_2,x_3,x_5^2] \bigoplus x_2x_4x_5T[x_2,x_4^2,x_5^2] \bigoplus x_3x_4^2T[x_1^2,x_3,x_4^2] \\
&\bigoplus x_3x_4x_5T[x_3^2,x_4,x_5^2] \bigoplus x_1x_2x_4x_5T[x_1^2,x_2,x_4^2,x_5] \bigoplus x_1x_3x_4x_5T[x_1^2,x_3^2,x_4,x_5] \\
&\bigoplus x_1x_3x_5^2T[x_1^2,x_3,x_4^2] \bigoplus x_1x_4x_5^2T[x_1^2,x_4,x_5^2] \bigoplus x_3x_4^2x_5T[x_3,x_4^2,x_5^2] \\
&\bigoplus x_1x_2x_3^2x_5T[x_1^2,x_2,x_3^2,x_5] \bigoplus x_1x_2x_4x_5^2T[x_1^2,x_2,x_4,x_5^2] \\
&\bigoplus x_1x_3x_4x_5^2T[x_1^2,x_3,x_4,x_5^2].
\end{aligned}
$$

So by method given by Herzog we get sdepth$(U/L) = 2$. The above decomposition is one of the all possible decompositions of the module U/L.
Following is another example for the monomial ideal

$L = (x_2x_3x_6, x_2^2x_5, x_1x_2x_4, x_1x_2x_3, x_2^2x_3, x_1x_2^2, x_3^2x_6^2, x_1x_3x_4x_6, x_1x_3^2x_6, x_1^2x_4^2, x_1^2x_3x_4, x_1^2x_3^2)$.

and $I = (0)$, then we get the following partition $\mathcal{P}$ of corresponding poset

$$
\begin{aligned}
&[000000, 200010] \cup [010000, 210010] \cup [001000, 002010] \cup [000100, 000210] \\
&\cup [000001, 000012] \cup [101000, 201010] \cup [100100, 200110] \cup [100001, 200011] \\
&\cup [020000, 020200] \cup [011000, 012010] \cup [010100, 010210] \cup [010001, 020002] \\
&\cup [001100, 002200] \cup [001001, 002011] \cup [000101, 000211] \cup [110001, 210011] \\
&\cup [012000, 102010] \cup [101100, 102200] \cup [101001, 201011] \cup [100200, 100210] \\
&\cup [100101, 200111] \cup [100002, 200002] \cup [011100, 012200] \cup [010101, 020201] \\
&\cup [010011, 010012] \cup [001110, 002110] \cup [001101, 002201] \cup [001002, 001012] \\
&\cup [000102, 000202] \cup [110002, 210002] \cup [101110, 102110] \cup [101002, 201002] \\
&\cup [100201, 100211] \cup [100102, 200102] \cup [011110, 012110] \cup [010111, 010211] \\
&\cup [010102, 020102] \cup [001111, 002111] \cup [001102, 001202].
\end{aligned}
$$

Corresponding Stanley decomposition is

$$
\begin{aligned}
U/L =& T[x_1^2, x_5] \bigoplus x_2T[x_1^2, x_2, x_4] \bigoplus x_3T[x_3^2, x_5] \bigoplus x_4T[x_4^2, x_5] \\
&\bigoplus x_6T[x_5, x_6^2] \bigoplus x_1x_3T[x_1^2, x_3, x_5] \bigoplus x_1x_4T[x_1^2, x_4, x_5] \\
&\bigoplus x_1x_6T[x_1^2, x_5, x_6] \bigoplus x_2^2T[x_2^2, x_4^2] \bigoplus x_2x_3T[x_2, x_3^2, x_5] \\
&\bigoplus x_2x_4T[x_2, x_4^2, x_5] \bigoplus x_2x_6T[x_2^2, x_6^2] \bigoplus x_3x_4T[x_3^2, x_4^2] \\
&\bigoplus x_3x_6T[x_3^2, x_5, x_6] \bigoplus x_4x_6T[x_4^2, x_6, x_5] \bigoplus x_1x_2x_6T[x_1^2, x_2, x_5, x_6] \\
&\bigoplus x_2x_3^2T[x_1, x_3^2, x_5] \bigoplus x_1x_3x_4T[x_1, x_3^2, x_4^2] \bigoplus x_1x_3x_6T[x_1^2, x_3, x_5, x_6] \\
&\bigoplus x_1x_4^2T[x_1, x_4^2, x_5] \bigoplus x_1x_4x_6T[x_1^2, x_4, x_5, x_6] \bigoplus x_1x_6^2T[x_1^2, x_6^2] \\
&\bigoplus x_2x_3x_4T[x_2, x_3^2, x_4^2] \bigoplus x_2x_4x_6T[x_2^2, x_4^2, x_6] \bigoplus x_2x_5x_6T[x_2, x_5, x_6^2] \\
&\bigoplus x_3x_4x_5T[x_3^2, x_4, x_5] \bigoplus x_3x_4x_6T[x_3^2, x_4^2, x_6] \\
&\bigoplus x_3x_6^2T[x_3, x_5, x_6^2] \bigoplus x_4x_6^2T[x_4^2, x_6^2] \bigoplus x_1x_2x_6^2T[x_1^2, x_2, x_6] \\
&\bigoplus x_1x_3x_4x_5T[x_1, x_3^2, x_4, x_5] \bigoplus x_1x_3x_6T[x_1^2, x_3, x_6^2] \\
&\bigoplus x_1x_4^2x_6T[x_1, x_4^2, x_5, x_6] \bigoplus x_1x_4x_6T[x_1^2, x_4, x_6^2] \bigoplus x_2x_3x_4x_5T[x_2, x_3^2, x_4, x_5] \\
&\bigoplus x_2x_4x_5x_6T[x_2, x_4^2, x_5, x_6] \bigoplus x_2x_4x_6^2T[x_1^2, x_4, x_6^2] \bigoplus x_3x_4x_5x_6T[x_3^2, x_4, x_5, x_6] \\
&\bigoplus x_3x_4x_6^2T[x_3, x_4^2, x_6^2].
\end{aligned}
$$

Thus by the Herzog's method we get $\operatorname{sdepth}(U/L) = 2$. The above decomposition is one of the all possible decompositions of the module U/L which gives the Stanley depth.

As we have to find all possible Stanley decompositions of the module which is obviously a tiring work and even sometime impossible to determine all so the method given by Herzog is not sufficient. Bounds determined for the depth and Stanley depth thus have an importance in the field. Many mathematicians worked on bounds for the two invariants. Following are the two lemmas which are widely used in the literature.

Lemma 2.2.1. *[1, Lemma 2.2](Depth Lemma)*
If

$$0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow X_3 \longrightarrow 0,$$

is a short exact sequence of $\mathbb{Z}^n$-graded R-module, then

1. $\operatorname{depth} X_2 \geq \min\{\operatorname{depth} X_1, \operatorname{depth} X_3\}$,

2. $\operatorname{depth} X_1 \geq \min\{\operatorname{depth} X_2, \operatorname{depth} X_3 + 1\}$,

3. $\operatorname{depth} X_3 \geq \min\{\operatorname{depth} X_2, \operatorname{depth} X_1 - 1\}$.

and a similar lemma for Stanley depth.

Lemma 2.2.2. *[1, Lemma 2.4] Let*

$$0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow X_3 \longrightarrow 0,$$

is a short exact sequence of $\mathbb{Z}^n$-graded R-module, then

$$\operatorname{sdepth} X_2 \geq \min\{\operatorname{sdepth} X_1, \operatorname{sdepth} X_3\}.$$

2.3 Stanley's Conjecture

The conjecture proposed by Stanley giving the upper bound for the depth of a module gained much importance due to its property of comparing two totally different invariants.

Conjecture 2.3.1. (***Stanley Conjecture.***) *Let U be a polynomial ring over the field T and N be the finitely generated $\mathbb{Z}^n$-graded U-module then Stanley conjectured that*

$$\operatorname{sdepth}(N) \geq \operatorname{depth}(N).$$

Apel was the first person, upto known information, who studied this conjecture in detail and also proved it in some cases in 2003. Popescu in [13] gave a far reaching result in this direction that R/L satisfies the conjecture for the polynomial ring in atmost five variable. Cimpoeas gave an affirmative answer to the conjecture for L and R/L, where L is the monomial ideal generated by atmost 3 elements. Later a counterexample for the conjecture was given by Duval, Klivans, Goechneker and Martine in 2015.

2.4 Some important results

In [4, Corrollary 1.8], by Cimpeaos we see that if the ideals L and I satisfy the Stanley's conjecture then so does their intersection. Similarly he also showed that if for ideals $L \subset U'$ and $I \subset U''$ with $U' = T[x_1, x_2, \dots, x_t]$ and $U'' = T[x_{t+1}, \dots, x_m]$, modules U'/L and U''/I satisfy the Stanley's conjecture then so does the module $U/(LU'+IU'')$. In the same paper Cimpeaos showed that for $L \subset U$ a monomial ideal and $x \in U$ a monomial, then

1. $\operatorname{sdepth}(L \cap (x)) \geq \operatorname{sdepth}(L)$,

2. $\operatorname{sdepth}(L, x) \geq \min\{\operatorname{sdepth}(L), \operatorname{sdepth}(U/L)\}$,

3. $\operatorname{sdepth}(U/(L, x)) \geq \operatorname{sdepth}(U/L) - 1$,

4. $\operatorname{sdepth}(U/(L \cap (x))) \geq \operatorname{sdepth}(U/L)$.

In the same paper he gave following important proposition.

Proposition 2.4.1. *For a monomial ideal L in U and $x \in U$ a monomial such that $x \notin L$ then*

$$\operatorname{sdepth}(U/(L : x)) \geq \operatorname{sdepth}(U/L).$$

A similar proposition was given by D.Popescu in [13] which is,

Proposition 2.4.2. *For a monomial ideal L in U and $x \in U$ a monomial such that $x \notin L$ then*

$$\operatorname{sdepth}(L : x) \geq \operatorname{sdepth}(L).$$

For the depth Asia Rauf in [15] gave the following result,

Proposition 2.4.3. *For a monomial ideal L in U and $x \in U$ a monomial such that $x \notin L$ then*

$$\operatorname{depth}(U/(L : x)) \geq \operatorname{depth}(U/L).$$

L. Fouli and S. Morey, in [7] gave the following lemma,

Lemma 2.4.4. *[7, Lemma 2.2] Let U be a polynomial ring and L be its ideal. Let x be an indeterminant over ring U and $U' = U[x]$. Then*

$$\operatorname{depth}(U'/LU') = \operatorname{depth}(U/L) + 1.$$

Lemma 2.4.5. *[9, Lemma 3.6] Let U be a polynomial ring and L be its ideal. Let x be an indeterminant over ring U and $U' = U[x]$. Then*

$$\operatorname{sdepth}(U'/LU') = \operatorname{sdepth}(U/L) + 1.$$

2.5 Bounds for the depth and Stanley depth.

We will review two papers, one by A. Alipour [1] and the other by L. Fouli [7]. In [1], Alipour covers the bounds for depth and Stanley depth of modules U/L^p with p greater than equals to 1 and L the edge ideal of the special type of star graph defined by him whereas in [7] L. Fouli and Morey covers the bounds for the depth and Stanley depth of modules U/L^p with p greater than equals to 1 and L the edge ideal of any graph H. These bounds depend upon the variants like power of edge ideal, number of variables and number of components of the graph. Some terms which we will use in the text are defined below,

1. For H be a graph and $x \in V(H)$, deleting x from H yields a minor of H with edge ideal denoted by L_x. Given a collection of vertices $s_1, s_2, \ldots, s_l$, define $L_0 = L$ and we define minor L_i of L where $1 \leq i \leq l$ which we get as a result of deletion of vertices $s_1, s_2, \ldots, s_i$.,

2. We define U_i as $U_i = U/(s_1, s_2, \ldots, s_i)$,

3. When a vertex x is fixed in graph H, the connected component of H containing x is denoted by ${}_xH$. Thus for the edge ideal L and fixed x in H, diameter of ${}_xH(L)$ is denoted by $d({}_xH(L))$,

4. $\mathcal{L}(H)$ denotes the number of leaves in H.

First we will see the bounds for the depth and Stanley depth of the module U/L that is for the first power.

Theorem 2.5.1. *[11, Theorem 3.1] Let H be a connected graph and $L(H)$ be the edge ideal. For $u, v \in V(H)$ with $d(u, v) = l$, then*

$$\operatorname{depth} U/L \geq \left\lceil \frac{l+1}{3} \right\rceil.$$

Proof. We will prove this result by induction on the total number of variables m in graph H. It is clear that for any fixed l we have $m \geq l + 1$.
Now since L is squarefree so $m \notin \operatorname{Ass}(U/L)$, hence the $\operatorname{depth}(U/L) \geq 1$. Now if $l \leq 2$ then we have $\operatorname{depth}(U/L) \geq 1$, hence the result is true for $l \leq 2$. Now assume that $l \geq 3$. We have two cases:
Case 1: When

$$m = l + 1.$$

In this case graph is a path graph and we have

$$\begin{aligned}\operatorname{depth}(U/L) &\geq \left\lceil \frac{l+1}{3} \right\rceil \\ &= \left\lceil \frac{m}{3} \right\rceil,\end{aligned}$$

hence the result is true by [27 lemma 2.8].
Case 2: When

$$m > l + 1 \geq 4.$$

Let us consider a short exact sequence for $a \in V(H)$

$$0 \longrightarrow U/(L:a) \longrightarrow U/L \longrightarrow U/(L,a) \longrightarrow 0.$$

For this let $a \in N(v)$ such that a lies on the path between u and v which has length l. Notice that $(L : a) = (I, N(a))$, where I is an edge ideal of the minor of H formed as a result of deletion of the variable in $N(a)$. As $l \geq 3$ so there exist $z \in V(H)$

such that $d(u,z)=l-3$. Also since no generator of $(I,N(a))$ is a multiple of a, thus $(I,N(a))\subset U'[N(a)]$ where $U'=U\backslash(a\cup N(a))$. So we have

$$\begin{aligned}\operatorname{depth}(U/L:a) &= \operatorname{depth}(U'[a,N(a)]/(I,N(a))\\ &= \operatorname{depth}(U'[a]/I).\end{aligned}$$

By using Lemma 2.4.4 we have

$$\begin{aligned}\operatorname{depth}(U/L:a) &= \operatorname{depth}(U'/I)+1\\ &\geq \left\lceil\frac{l-3+1}{3}\right\rceil+1,\end{aligned}$$

then by induction on m we have

$$\operatorname{depth}(U/(L:a)) = \left\lceil\frac{l+1}{3}\right\rceil.$$

Now consider (L,a), we can see that $(L,a)=(K,a)$. Here K is an edge ideal of the minor of H formed as a result of deletion of a. Now there are two cases:

Case 1: H'' is connected.

In this case after deletion of a if H'' is connected, then we can say that

$$d(u,v)\geq l.$$

Then by induction on m we have,

$$\begin{aligned}\operatorname{depth}(U/(L,a)) &= \operatorname{depth}(U/(K,a))\\ &\geq \left\lceil\frac{l+1}{3}\right\rceil.\end{aligned}$$

Case 2: H'' is not connected.

If deletion of a from H yields a disconnected graph, then there is a vertex $z\in {}_uH''$ with $d(u,z)\geq l-2$ and $v\notin {}_uH''$. Now if v is an isolated vertex, then by using Lemma 2.4.4 and by induction on m we get

$$\begin{aligned}\operatorname{depth}(U/(K,a)) &\geq \left\lceil\frac{l-2+1}{3}\right\rceil+1\\ &\geq \left\lceil\frac{l+1}{3}\right\rceil.\end{aligned}$$

Otherwise if v is not an isolated vertex, then v is in a connected component of H''. This component is also squarefree hence has depth atleast one. So we have by [36 lemma 6.2.7]

$$\begin{aligned}\operatorname{depth}(U/(K,a)) &\geq \left\lceil\frac{l-2+1}{3}\right\rceil+1\\ &\geq \left\lceil\frac{l+1}{3}\right\rceil.\end{aligned}$$

Thus in both the cases

$$\operatorname{depth}(U/(L,a)) \geq \left\lceil\frac{l+1}{3}\right\rceil.$$

□

Corollary 2.5.2. *[11, Corollary 3.4] Let H be a graph and let $L = L(H)$ and fix $u \in V(H)$. Let $e \in V(H)$ and $d(u,e) = t$ for some $0 \leq t$. Then*

$$\operatorname{depth}(U/(L:e)) \geq \left\lceil\frac{t+2}{3}\right\rceil.$$

Proof. Let $e \in V(H)$ such that $d(u,e) = t$. Clearly $(L:e) = (I, N(e))$ where J is an edge ideal corresponding to the minor of H formed by deleting variables in $N(e)$. For $U' = U\backslash(e \cup N(e))$, we have

$$\operatorname{depth}(U/(L:e)) = \operatorname{depth}(U'[e,N(e)]/(I,N(e)).$$

Then by using Lemma 2.4.4 we have

$$\operatorname{depth}(U/(L:e)) = \operatorname{depth}(U'/I)+1.$$

Now if $t < 2$ then the result holds as in this case $\operatorname{depth}(U/(L:e)) \geq 1$. So we will assume that $t \geq 2$. Since deletion of variables in $N(e)$ may result in change of diameter of H' such that $d(H') \geq t-2$, then by applying Theorem 2.5.1 we have

$$\begin{aligned}\operatorname{depth}(U'/L)+1 &\geq \left\lceil\frac{t-2+1}{3}\right\rceil+1\\ &= \left\lceil\frac{t+2}{3}\right\rceil.\end{aligned}$$

Therefore we have

$$\operatorname{depth}(U'/(L:e)) = \left\lceil\frac{t+2}{3}\right\rceil.$$

□

Corollary 2.5.3. *[11, Corollary 3.2] For a connected graph H with diameter $l \geq 1$ and $L = L(H)$ we have* $\operatorname{depth} U/L \geq \left\lceil \frac{l+1}{3} \right\rceil$.

Corollary 2.5.4. *[11, Corollary 3.3] For the graph H with q connected components, $L = L(H)$. Let l_i be the diameter of ith connected component. Then* $\operatorname{depth} U/L \geq \sum_{i=1}^{q} \left\lceil \frac{l_1+1}{3} \right\rceil$. *In particular ,* $\operatorname{depth} U/L \geq \left\lceil \frac{l+1}{3} \right\rceil + q - 1$.

Proposition 2.5.5. *[11, Proposition 3.5] For connected graph H with loops and let $L = L(H)$. If there exists $x \in V(H)$ and $d(x,u) \geq t$ for all u such that $\{u,u\} \in E(H)$, then*

$$\operatorname{depth} U/L \geq \left\lceil \frac{t-1}{3} \right\rceil.$$

Proof. If $t = 1$, then $\operatorname{depth} U/L \geq 0$, which is a trivial bound, Now assume that $t \geq 2$. we will use induction on the number of loops in the graph. Let $\{u,u\} \in E(H)$ i.e there is loop on vertex u. Consider the short exact sequence:

$$0 \longrightarrow\ U/(L:u) \longrightarrow\ U/L \longrightarrow\ U/(L,u) \longrightarrow\ 0.$$

Clearly

$$(L:u) = (I, N(u)),$$

where I is the edge ideal of the minor of H formed as a result of deletion of variables in $N(u)$. As $u \in N(u)$ so the graph induced by edge ideal ideal I i.e H' has less number of loops then $H(L)$. Since all the vertices in $N(u)$ are atleast $t-1$ distant apart from x, so

$$d({}_xH'(I)) \geq t-2,$$

with $d(x,v) \geq t$ for all loops at v. Now if ${}_xH'(I)$ is loopless, then by theorem 3.1

$$\operatorname{depth} U/(L:u) \geq \left\lceil \frac{d(H')+1}{3} \right\rceil$$

which implies,

$$\operatorname{depth} U/(L:u) \geq \left\lceil \frac{(t-2)+1}{3} \right\rceil = \left\lceil \frac{t-1}{3} \right\rceil.$$

Otherwise if ${}_xH'(I)$ has a loop at f, then by supposition

$$d(x,f) \geq t.$$

So by induction on the number of loops we have

$$\operatorname{depth} U/(L : u) \geq \left\lceil \frac{t-1}{3} \right\rceil.$$

Clearly $(L, u) = (K, u)$, with K is the edge ideal of minor of H formed as a result of deletion of u. Then $d({}_xH'') \geq t-1$. Now if ${}_xH''$ is loopless, then by using theorem 3.1

$$\operatorname{depth} U/(L, u) \geq \left\lceil \frac{t}{3} \right\rceil \geq \left\lceil \frac{t-1}{3} \right\rceil$$

and if there is a loop in ${}_xH''$, then by induction

$$\operatorname{depth} U/(L : u) \geq \left\lceil \frac{t-1}{3} \right\rceil.$$

Hence by depth lemma we have

$$\operatorname{depth} U/L \geq \left\lceil \frac{t-1}{3} \right\rceil.$$

□

Similarly for Stanley depth we have the following proposition,

Proposition 2.5.6. *Let we have a graph H which is connected having loops and let $L = L(H)$. If there exists $x \in V(H)$ and $d(x, u) \geq t$ for all u such that $\{u, u\} \in E(H)$, then*

$$\operatorname{sdepth} U/L \geq \left\lceil \frac{t-1}{3} \right\rceil.$$

Following is the star defined by Alipour and Tehranian in [1].

Definition 2.5.7. Consider that $t, m_1, m_2, \ldots, m_t$ be positive integers. Let $U = T[x, x_{i,j} : 1 \leq i \leq m_j, 1 \leq j \leq t]$ be a polynomial ring over T. For every j with $1 \leq j \leq t$, let P_{m_j+1} be a path of lengths m_j and

$$L_j = (xx_{1,j}, x_{i,j}x_{i+1,j} : 1 \leq i \leq m_j - 1)$$

be the edge edge ideal of this path in U. Let

$$L = \sum_{j=1}^{t} L_j.$$

A graph H with vertices as variables from polynomial ring U and edge ideal L is called $(t; m_1, m_2, \ldots, m_t) - star$ graph.

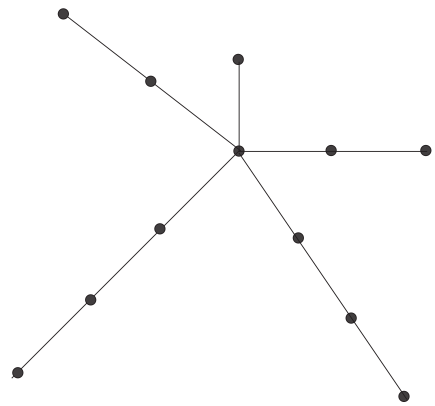


Figure 2.2: $(5; 2, 1, 2, 3, 3) - star\ \ graph$

Theorem 2.5.8. *[1, Theorem 2.7] Let H be a $(t; m_1, m_2, \ldots, m_t)$-star graph and $L = L(H)$ be its edge ideal.*

1. *If $m_j \equiv 0\,or\,2\,(mod\ \ 3)$ for all $1 \leq j \leq t$, then*

$$\sum_{j=1}^{t} \left\lceil \frac{m_j}{3} \right\rceil \leq \operatorname{depth}(U/L) \leq 1 + \sum_{j=1}^{t} \left\lceil \frac{m_j}{3} \right\rceil.$$

2. *If $m_j \equiv 1(mod\ \ 3)$ for some $1 \leq j \leq t$, then*

$$\operatorname{depth}(U/L) = 1 + \sum_{j=1}^{t} \left\lceil \frac{m_j - 1}{3} \right\rceil.$$

Theorem 2.5.9. *[1, Theorem 2.6] Let H be a $(t; m_1, m_2, \ldots, m_t)$-star graph and $L = L(H)$ be its edge ideal.*

1. *If $m_j \equiv 0\,or\,2\,(mod\ \ 3)$ for all $1 \leq j \leq t$, then*

$$\sum_{j=1}^{t} \left\lceil \frac{m_j}{3} \right\rceil \leq \operatorname{sdepth}(U/L) \leq 1 + \sum_{j=1}^{t} \left\lceil \frac{m_j}{3} \right\rceil.$$

2. *If $m_j \equiv 1(mod\ \ 3)$ for some $1 \leq j \leq t$, then*

$$\operatorname{sdepth}(U/L) = 1 + \sum_{j=1}^{t} \left\lceil \frac{m_j - 1}{3} \right\rceil.$$

Now stepping forward to the bounds for the depth and Stanley depth of cyclic module U/L^p for $p > 1$. For this we need some other results given in [11].

Lemma 2.5.10. *[7, Lemma 2.5] Let $\mathcal{N}$ and L be a monomial and an ideal in a polynomial ring U respectively. If we have a variable that z does not divide $\mathcal{N}$ and $\mathcal{K}$ is the extension in U of the image of L in U/z, then*

$$((L^p : \mathcal{N}), z) = ((\mathcal{K}^p : \mathcal{N}), z).$$

Lemma 2.5.11. *[7, Lemma 2.10] Let we have a graph H with edge ideal $L = L(H)$. Let u be a leaf of H and v be the only neighbor of u. Then $(L^p : uv) = L^{p-1}$ for any $p \geq 2$.*

Proof. Since u is a leaf and v is its unique neighbor so we have $\{u, v\} \in E(H)$ which means $uv \in G(J)$. One inclusion is clear which is $L^{p-1} \subset (L^p : uv)$. Now for the reverse inclusion, let x be the monomial generator of $(L^p : uv)$ then we have $xuv = e_1e_2 \dots e_ph$ where $e_i's$ are the monomials of degree two corresponding to the edges of the graph H. Since u is leaf with v its unique neighbor so if a monomial $x \in L^{p-1}$ contains u then its must be in the form of an edge uv so for $x \notin L^{p-1}$ we can say that u divides e_j and v divides e_k for some $j \neq k$. Assume that $j = p$. Since u is a leaf so, we have $e_j = e_p = uv$ and thus $a = e_1e_2 \dots e_{p-1}h \in L^{p-1}$. □

Lemma 2.5.12. *[11, Lemma 2.6] For the graph H and edge ideal $L = L(H)$ and $\{u, v\} \in E(H)$. Then $(L^2 : uv) = (L, E)$, where $E =< u_iv_j | u_i \in N(u), v_j \in N(v) >$. More generally, if $u_1u_2 \dots u_{2p} \in L^p$, then $(L^{p+1} : u_1u_2 \dots u_{2p}) = (L, E)$, where E is the ideal generated by all monomials of degree two v_1v_2 supported on $\cup_{i=1}^{2p} N(u_i)$ satisfying $v_1v_2u_1 \dots u_{2p} \in L^{p+1}$.*

Proof. Let a be a monomial generator of (L, E). Now if $a \in J$, then $a \in (L^2 : uv)$ since $\{u, v\} \in E(L)$. If $a \notin L$, then $a = u_iv_j \in E$ where $u_i \in N(u)$ and $v_j \in N(v)$. So we get $auv = u_iv_juv \in L^2$ which means $a \in (L^2 : uv)$. Thus we have $(L, E) \subseteq (L^2 : uv)$. Now conversely, suppose that $b \in (L^2 : uv)$ but $b \notin L$. Now suppose that b is a monomial since $(L^2 : uv)$ is a monomial ideal. Thus $buv \in L^2$. so $buv = e'_1e'_2h$ where e'_1 and e'_2

be the monomials of degree two corresponding to edges in H. Since $b \notin L$ so neither e'_1 divides b nor e'_2 divides b. So without loss of generality u divides e'_1 and v divides e'_2 thus $e'_1 = uu_i$ and $e'_2 = vv_j$ for some $u_i \in N(u)$ and $v_j \in N(v)$. Thus u_iv_j divides b so $b \in E \subset (L, E)$.
Generalized statement can be proved on the same outline keeping in view that all x_i need not all be distinct. $\square$

Lemma 2.5.13. *[11, Lemma 4.2] Let L be an ideal of a polynomial ring U and let $\mathcal{N}$ be the monomial in U. Let a set of variables $\{s_1, s_2, \dots, s_t\}$ where for all i, $\mathcal{N}$ is not a multiple of any $s_i's$. Let x, y be the two nonnegative integers such that* $\operatorname{depth} U_{i-1}/(L^p_{i-1} : \mathcal{N}s_i) \geq x$ *for all $i \geq 1$ and* $\operatorname{depth} U_t/(L^p_t : \mathcal{N}) \geq y$ *then* $\operatorname{depth} U_i/(L^p_i : \mathcal{N}) \geq \min\{x, y\}$ *for each $i \geq 0$. In particular,* $\operatorname{depth} U/(L^p : \mathcal{N}) \geq \min\{x, y\}$.

Proof. For the proof of this theorem we will repeatedly make use of short exact sequences. Consider the following short exact sequence:

$$0 \longrightarrow U/(L^p : \mathcal{N}s_1) \longrightarrow U/(L^p : \mathcal{N}) \longrightarrow U/((L^p : \mathcal{N}), s_1) \longrightarrow 0.$$

By hypothesis

$$\operatorname{depth} U/(L^p : \mathcal{N}s_1) \geq x.$$

Now for the $\operatorname{depth} U/((L^p : \mathcal{N}), s_1)$ which is equal to $\operatorname{depth} U_1/(L^p_1 : \mathcal{N})$, consider the following short exact sequence:

$$0 \longrightarrow U_1/(L^p_1 : \mathcal{N}s_2) \longrightarrow U_1/(L^p_1 : \mathcal{N}) \longrightarrow U_1/((L^p_1 : \mathcal{N}), s_2) \longrightarrow 0.$$

again by hypothesis

$$\operatorname{depth} U_1/(L^p_1 : \mathcal{N}s_2) \geq x.$$

Continuing in the same way we reach at the last short exact sequence:

$$0 \longrightarrow U_{t-1}/(L^p_{t-1} : \mathcal{N}s_t) \longrightarrow U_{t-1}/(L^p_{t-1} : \mathcal{N}) \longrightarrow U_{t-1}/((L^p_{t-1} : \mathcal{N}), s_t) \longrightarrow 0$$

By hypothesis

$$\operatorname{depth} U_{t-1}/(L^p_{t-1} : \mathcal{N}s_t) \geq x,$$

and

$$\begin{aligned} \operatorname{depth} U_{t-1}/((L^p_{t-1} : \mathcal{N}), s_t) &= \operatorname{depth} U_t/(L^p_t : \mathcal{N}) \\ &\geq y. \end{aligned}$$

So using depth lemma starting from last and working up to first sequence yields $\operatorname{depth} U_i/(L^p_i : \mathcal{N}) \geq \min\{x, y\}$ for each $0 \leq i \leq t-1$. Finally we have

$$\operatorname{depth} U/(L^p : \mathcal{N}) \geq \min\{x, y\}.$$

□

The same lemma is valid for Stanley depth as well that is,

Lemma 2.5.14. *Let L be an ideal in a polynomial ring U and let $\mathcal{N}$ be the monomial in U. Let a set of variables $\{s_1, s_2, \ldots, s_t\}$ where for all i, $\mathcal{N}$ is not a multiple of any $s_i's$. Let x, y be the two nonnegative integers such that* $\operatorname{sdepth} U_{i-1}/(L^p_{i-1} : \mathcal{N}s_i) \geq x$ *for all $i \geq 1$ and* $\operatorname{sdepth} U_t/(L^p_t : \mathcal{N}) \geq y$ *then* $\operatorname{sdepth} U_i/L^p_i : \mathcal{N}) \geq \min\{x, y\}$ *for each $i \geq 0$. In particular,* $\operatorname{sdepth} U/(L^p : \mathcal{N}) \geq \min\{x, y\}$.

With the help of above mentioned results L. Fouli and S. Morey gave the following bounds.

Proposition 2.5.15. *[7, Proposition 4.3] For a graph H with q components and edge ideal $L = L(H)$,*

$$\operatorname{depth} U/L^p \geq q - p.$$

where $p \geq 1$,

Theorem 2.5.16. *[11, Theorem 4.4] For a graph H with edge ideal $L = L(H)$. Let H has q connected component and $l = d(H)$ be the diameter of H. Then*

$$\operatorname{depth} U/L^2 \geq \left\lceil \frac{l-3}{3} \right\rceil + q - 1.$$

Proof. The proof is done using induction on order of H. If $m \leq 4$ then we have $l \leq 3$ and $q \leq 2$ as isolated vertices are not considered as components. If $q = 1$, then

$$\begin{aligned} \operatorname{depth} U/L^2 &\geq \left\lceil \frac{l-3}{3} \right\rceil \\ &= 0. \end{aligned}$$

So the bound is trivial.

Now if $q = 2$, then H must contain two disjoint edges so by Theorem 3.4 of [11]

$$\begin{aligned} \operatorname{depth} U/L^2 &\geq \max\{\left\lceil \frac{(l-2)+2}{3} \right\rceil + q - 1, q\} \\ &= \max\{\left\lceil \frac{l}{3} \right\rceil + q - 1, q\} \\ &= \max\{\left\lceil \frac{1}{3} \right\rceil + 2 - 1, 2\}, \end{aligned}$$

so

$$\operatorname{depth} U/L^2 \geq 2.$$

Hence the bound is satisfied.

Now for the case when $q \geq 2$ and $m \geq 5$ we will repeatedly make use of depth lemma. The $\operatorname{depth} U/L^2 \geq 1$ by using Lemma 2.1 of [3]. Now consider the case $m \geq 5$. For this let P be the path with end points x, y which determines the diameter l of the graph H. Now let $s \in N(y)$ and let $N(s) = \{s_1, s_2, \ldots, s_t\}$ be ordered as in lemma 2.5.13, such a way that $d(x, s)$ is finite in L_i for $i < t$. Clearly $L_0 = L$ so for each $1 \leq i \leq t$ we have,

$$(L_{i-1}^2 : ss_i) = (L_{i-1}, E_{i-1}),$$

where E_{i-1} is an edge ideal as in Lemma 2.5.12. Now (L_{i-1}, E_{i-1}) is an edge ideal of a graph H', may be having loops, of diameter atleast $l - 1$ as $d(x, s) \geq l - 1$. So if the ideal (L_{i-1}, E_{i-1}) is square free, then using corollary 3.3 we have

$$\begin{aligned} \operatorname{depth} U_{i-1}/(L_{i-1}, E_{i-1}) &\geq \left\lceil \frac{d(H')+1}{3} \right\rceil + q - 1 \\ &\geq \left\lceil \frac{(l-1)+1}{3} \right\rceil + q - 1 \\ &= \left\lceil \frac{l}{3} \right\rceil + q - 1 \\ &\geq \left\lceil \frac{l-3}{3} \right\rceil + q - 1. \end{aligned}$$

Now if ideal (L_{i-1}, E_{i-1}) is non-square free, then there exists $u \in V(H)$ such that $u^2 \in E_{i-1}$, obviously $u \in N(s)$ so,

$$d(x, u) \geq l - 2.$$

Every other component of $H((L_{i-1}, E_{i-1}))$ will be square free other than ${}_xH((L_{i-1}, E_{i-1}))$ so, will have depth atleast one. Then by using Lemma 6.2.7 of [19] with proposition 2.5.5 we have By using Lemma 2.5.13 we have

$$\operatorname{depth} U_{i-1}/(L_{i-1}, E_{i-1}) \geq \left\lceil \frac{(l-2)-1}{3} \right\rceil + q - 1,$$

since $d(x, u) \geq l - 2$. Now s is isolated in L_t so,

$$(L_t^2 : s) = L_t^2,$$

and s will be a free variable in U_t/L_t^2. As

$$d({}_xH(L_t)) \geq l - 3.$$

Then by the use of induction and Lemma 2.4.4 we have

$$\operatorname{depth} U_t/(L_t^2 : s) \geq \left\lceil \frac{d({}_xH(L_t)) - 3}{3} \right\rceil + q - 1 + 1.$$

Thus

$$\operatorname{depth} U_t/(L_t^2 : s) \geq \left\lceil \frac{l-3}{3} \right\rceil + q - 1,$$

hence

$$\operatorname{depth} U/(L^2 : s) \geq \left\lceil \frac{l-3}{3} \right\rceil + q - 1.$$

Now we will work on the depth of $U/(L_s^2, s)$. For this we clearly know that

$$(L^2, s) = (L_s^2, s).$$

If $y \in {}_xH(L_s)$, then $d({}_xH(L_s)) \geq d$ and by induction on m we have

$$\operatorname{depth} U/(L^2, s) \geq \left\lceil \frac{l-3}{3} \right\rceil + q - 1.$$

For the case when $y \notin {}_xH(L_s)$

$$d({}_xH(L_s)) \geq l-2,$$

with $H(L_s)$ containing additional components or an isolated vertex. Hence we have

$$\begin{aligned} \operatorname{depth} U/(L^2, s) &= \operatorname{depth} U/(L_s^2, w) \\ &\geq \left\lceil \frac{(l-2)-3}{3} \right\rceil + (q+1) - 1 \\ &= \left\lceil \frac{l-2}{3} \right\rceil + q - 1. \end{aligned}$$

By depth lemma we have

$$\operatorname{depth} U/L^2 \geq \left\lceil \frac{l-3}{3} \right\rceil + q - 1.$$

□

Similarly for Stanley depth we have the following result,

Theorem 2.5.17. *For a graph H with edge ideal $L = L(H)$. Let H has q connected component and $l = d(H)$ be the diameter of H. Then*

$$\operatorname{sdepth} U/L^2 \geq \left\lceil \frac{l-3}{3} \right\rceil + q - 1.$$

Lemma 2.5.18. *[7, Lemma 4.7] Let H be a graph with edge ideal $L = L(H)$ then for a fixed $x \in V(H)$ and for a vertex w of H such that $w \notin {}_xH$, we have*

$$\operatorname{depth}(U/L^2 : w) \geq \left\lceil \frac{h}{3} \right\rceil,$$

where $h = d({}_xH)$.

Proof. Let $L = (M, N)$, where $M = L({}_wH)$. Let $N(w) = \{s_1, s_2, \ldots, s_t\}$ be ordered as in Lemma 2.5.13. Notice that $L_i = (M_i, N_i)$ and $M_i = M$ for all i. From Lemma 2.5.12 we see that $(L_{i-1}^2 : ws_i) = (L_{i-1}, E_{i-1})$ with endpoints of all the edges of E_{i-1} in $V(-xH)$. Let U'_{i-1} be a polynomial ring with variables corresponding to $V(H(M_{i-1}))$, so

$$\operatorname{depth} U_{i-1}/(L_{i-1}, E_{i-1}) \geq \operatorname{depth} U'_{i-1}/M_{i-1} \geq \left\lceil \frac{h+1}{3} \right\rceil.$$

Since w is isolated vertex in L_t so, $(L_t^2 : w) = L_t^2$ with w a free variable. Thus we have

$$\begin{aligned}
\operatorname{depth} U_t/(L_t^2 : w) &\geq \operatorname{depth} U_t/L_t^2 + 1 && (2.1)\\
&\geq \left\lceil \frac{h-3}{3} \right\rceil + 1 && (2.2)\\
&= \left\lceil \frac{h}{3} \right\rceil. && (2.3)
\end{aligned}$$

Using Lemma 2.5.13 we have,

$$\operatorname{depth}(U/L^2 : w) \geq \left\lceil \frac{h}{3} \right\rceil.$$

□

Lemma 2.5.19. *[7, Lemma 4.12] For the graph H with edge ideal $L = L(H)$, a fixed $x \in V(H)$ and an edge $uv \in E(H)$ with $d(u,x) = t$ for some $t \leq l$ we have,* $\operatorname{depth} U/(L^3 : uv) \geq \left\lceil \frac{t-6}{3} \right\rceil$.

Theorem 2.5.20. *[7, Theorem 4.13] Let H be the graph and q be its number of components. Let $L = L(H)$ and l be the diameter of the graph. Then*

$$\operatorname{depth} U/L^3 \geq \left\lceil \frac{l-7}{3} \right\rceil + q - 1.$$

Proof. We will prove this result by induction on m, the total number of variables. For the case when $m \leq 8$, we have $l \leq 7$ and $q \leq 4$. It is clear that when $q = 4$, the bound is trivial by Proposition 2.5.15. For $q = 3$, we have $\operatorname{depth} U/L^p \geq 1$ for all $p \leq q$ by Lemma 2.1 of [3]. Now if $q = 1$ or $q = 2$ with $l \leq 4$, then we get a trivial bound. If $q = 2$ and $l = 5$, graph consists of two path graphs which are disconnected. Hence the result is true from Theorem 3.4 of [11].

Now we may assume $m \geq 9$. Consider the path, with end points x, y, that realizes the graph's diameter. Let $w \in N(y)$ such that w lies on the path which realizes the diameter of H. Notice that $(L^3, w) = (K^3, w)$. Now if x and y are in the same component of K, then $d(K) \geq l$ and $q(K) \geq q$. Using induction on m we have

$$\begin{aligned}
\operatorname{depth} U/(L^3, w) &\geq \left\lceil \frac{d(K)-7}{3} \right\rceil + q(K) - 1\\
&\geq \left\lceil \frac{l-7}{3} \right\rceil + q - 1.
\end{aligned}$$

Now if x and y are two different components of the graph corresponding to edge ideal K, then $d(k) \geq l-2$ and $q(K) \geq q+1$ or if y is isolated vertex then by induction we have

$$\begin{aligned} \operatorname{depth} U/(L^3, w) &\geq \left\lceil \frac{d(K)-7}{3} \right\rceil + q + 1 - 1 \\ &\geq \left\lceil \frac{l-9}{3} \right\rceil + q + 1 - 1 \\ &\geq \left\lceil \frac{l-7}{3} \right\rceil + q - 1. \end{aligned}$$

Now in order to calculate depth of $U_{i-1}/(L_{i-1}^3 : ws_i)$ where $N(w) = \{s_1, s_2, \ldots, s_t\}$, we will use Lemma 2.5.13. Ordering $\{s_1, s_2, \ldots, s_t\}$ as in Lemma 2.5.13 and using Lemma 2.5.19 we get

$$\operatorname{depth} U_{i-1}/(L_{i-1}^3 : ws_i) \geq \left\lceil \frac{l-7}{3} \right\rceil.$$

Notice that w is an isolated vertex in L_t so, $(L_t^3 : w) = L_t^3$. Thus by using induction on m we have,

$$\begin{aligned} \operatorname{depth} U_t/(L_t^3 : w) &= \operatorname{depth} U_t/L_t^3 + 1 \\ &\geq \left\lceil \frac{d(L_t)-7}{3} \right\rceil + q(L_t) - 1 + 1 \\ &\geq \left\lceil \frac{l-3-7}{3} \right\rceil + q - 1 + 1 \\ &= \left\lceil \frac{l-7}{3} \right\rceil + q - 1. \end{aligned}$$

Thus

$$\operatorname{depth} U/(L^3 : w) \geq \left\lceil \frac{l-7}{3} \right\rceil + q - 1.$$

Hence by using depth lemma we have

$$\operatorname{depth} U/L^3 \geq \left\lceil \frac{l-7}{3} \right\rceil + q - 1.$$

$\square$

Similar result holds for Stanley depth of the module U/L^3, that is

Theorem 2.5.21. *[7, Theorem 4.13] Let H be the graph and q be its number of components. Let $L = L(H)$ and l be the diameter of the graph. Then*

$$\operatorname{sdepth} U/L^3 \geq \left\lceil \frac{l-7}{3} \right\rceil + q - 1.$$

Corollary 2.5.22. *[7, Corollary 4.14] Let H be a graph with edge ideal $L = L(H)$. For a fixed vertex x and a vertex u such that $d(u,x) = h$, the*

$$\operatorname{depth} U/(L^3 : u) \geq \left\lceil \frac{h-6}{3} \right\rceil.$$

Proof. For $h \leq 6$ the bound is trivial. Assume that $h \geq 7$. Ordering the set $N(u) = \{s_1, s_2, \ldots, s_t\}$ as in Lemma 2.5.13 and by using Lemma 2.5.19 we have

$$\operatorname{depth} U_{i-1}/(L^3_{i-1} : us_i) \geq \left\lceil \frac{h-6}{3} \right\rceil.$$

Since u is isolated vertex in L_t, thus $(L^3_t : u) = L^3_t$ and $d(L_t) \geq h - 2$. Using Theorem 2.5.20 we have

$$\begin{aligned} \operatorname{depth} U_t/(L^3_t : u) &\geq \left\lceil \frac{d(L_t)-7}{3} \right\rceil + 1 \\ &\geq \left\lceil \frac{h-9}{3} \right\rceil + 1 \\ &= \left\lceil \frac{h-6}{3} \right\rceil. \end{aligned}$$

Hence by using Lemma 2.5.13 we have

$$\operatorname{depth} U/(L^3 : u) \geq \left\lceil \frac{h-6}{3} \right\rceil.$$

□

Similarly Corollary 2.5.22 and Lemma 2.5.19 are also true for Stanley depth also, that is

Lemma 2.5.23. *For a graph H with edge ideal $L = L(H)$, a fixed $x \in V(H)$ and an edge $uv \in E(H)$ with $d(u,x) = t$ for some $t \leq l$ we have,*

$$\operatorname{sdepth} U/(L^3 : uv) \geq \left\lceil \frac{t-6}{3} \right\rceil.$$

and

Corollary 2.5.1. *Let H be a graph with edge ideal $L = L(H)$. For a fixed vertex x and a vertex e such that $d(u, e) = h$, then*

$$\operatorname{sdepth} U/(L^3 : e) \geq \left\lceil \frac{h-6}{3} \right\rceil.$$

Theorem 2.5.24. *[1, Theorem 2.8] Let H be a $(t; m_1, m_2, \ldots, m_t)$-star graph and $L = L(H)$ be its edge ideal with $d(H) = l$, then*

$$\operatorname{sdepth} U/L^p \geq \max\{1, \sum_{j=1}^{t} \left\lceil \frac{m_j - p}{3} \right\rceil\},$$

for every $p \geq 1$.

Proof. Clearly Theorem 2.7 of [14] we have

$$\operatorname{sdepth} U/L^p \geq \max\{1, \left\lceil \frac{l-p+2}{3} \right\rceil\}.$$

Now if $\left\lceil \frac{l-p+2}{3} \right\rceil \geq 1$, then

$$\operatorname{sdepth} U/L^p \geq \left\lceil \frac{l-p+2}{3} \right\rceil \geq 1,$$

so we have

$$\operatorname{sdepth} U/L^p \geq 1.$$

Now we are left with to show that

$$\operatorname{sdepth} U/L^p \geq \sum_{j=1}^{t} \left\lceil \frac{m_j - p}{3} \right\rceil.$$

The proof is done by induction on p and m where m is the total number of vertices in graph H. If $m \leq 3$ and $t \leq 2$, then H is a path and by Theorem 6 of [18] we have

$$\operatorname{sdepth} U/L^p = \max\{\left\lceil \frac{m-p+1}{3} \right\rceil, 1\},$$

which implies that

$$\operatorname{sdepth} U/L^p = 1.$$

Now for $m \geq 4$ and $t \geq 3$, when $p = 1$ then by Theorem 2.5.9 we have for $m_j \equiv 0 \ or \ 2(mod \ 3)$ for all $1 \leq j \leq t$,

$$\text{sdepth}(U/L) \geq \sum_{j=1}^{t} \left\lceil \frac{m_j}{3} \right\rceil \geq \sum_{j=1}^{t} \left\lceil \frac{m_j - 1}{3} \right\rceil$$

and for $m_j \equiv 1(mod \ 3)$ for some $1 \leq j \leq t$,

$$\text{sdepth}(U/L) \geq 1 + \sum_{j=1}^{t} \left\lceil \frac{m_j - 1}{3} \right\rceil \ > \sum_{j=1}^{t} \left\lceil \frac{m_j - 1}{3} \right\rceil.$$

Hence the bound is satisfied for $p = 1$. Now further assume that $p \geq 2$ we will use some short exact sequences and depth lemma to prove the result. Consider the short exact sequence:

$$0 \longrightarrow \ U/(L^p : x_{m_i-1,i}) \longrightarrow \ U/L^p \longrightarrow \ U/(L^p, x_{m_i-1,i}) \longrightarrow \ 0.$$

We see that

$$(L^p, x_{m_i-1,i}) = (K^p, x_{m_i-1,i}),$$

where K is the edge ideal associated to the minor of H formed as a result of deletion of $x_{m_i-1,i}$. Hence

$$\operatorname{sdepth} U/(L^p, x_{m_i-1,i}) = \operatorname{sdepth} U_1[x_{m_i,i}, x_{m_i-1,i}]/(L^p, x_{m_i-1,i}).$$

Using Lemma 2.4.4 of [11] we have,

$$\begin{aligned}
\operatorname{sdepth} U/(L^p, x_{m_i-1,i}) &= \operatorname{sdepth} U_1[x_{m_i-1,i}]/(L^p, x_{m_i-1,i}) + 1 \\
&= \operatorname{sdepth} U_1/L^p + 1 \\
&\geq \sum_{i \neq j=1}^{t} \left\lceil \frac{m_j - p}{3} \right\rceil + \left\lceil \frac{(m_i - 2) - p}{3} \right\rceil + 1 \\
&\geq \sum_{j=1}^{t} \left\lceil \frac{m_j - p}{3} \right\rceil,
\end{aligned}$$

where U_1 is the polynomial ring obtained by deleting $x_{m_i,i}$ and $x_{m_i-1,i}$ from U. For determining the Stanley depth of $U/(L^p : x_{m_i-1,i})$, consider the following short exact sequence:

$$0 \longrightarrow \ U/(L^p : x_{m_i-1,i}x_{m_i,i}) \longrightarrow \ U/L^p \longrightarrow \ U/((L^p : x_{m_i-1,i}), x_{m_i,i}) \longrightarrow \ 0.$$

Without loss of generality we may have that $m_i \geq 3$ for some $1 \leq i \leq t$. As $x_{m_i,i}$ is a leaf of H and $x_{m_i-1,i}$ is its unique neighbour so, using Lemma 2.5.11 we have

$$(L^p : x_{m_i,i}x_{m_i-1,i}) = L^{p-1}.$$

So, by induction on p

$$\begin{aligned}
\operatorname{sdepth} U/(L^p : x_{m_i,i}x_{m_i-1,i}) &= \operatorname{sdepth} U/L^{p-1} \\
&\geq \sum_{j=1}^{t} \left\lceil \frac{m_j - (p-1)}{3} \right\rceil \\
&\geq \sum_{j=1}^{t} \left\lceil \frac{m_j - p}{3} \right\rceil.
\end{aligned}$$

Now consider the ideal

$$((L^p : x_{m_i-1,i}), x_{m_i,i}) = ((J^p : x_{m_i-1,i}), x_{m_i,i}).$$

Here J is the edge ideal associated to the H'' minor of H formed as a result of deletion of variable $x_{m_i,i}$. Thus

$$\operatorname{sdepth} U/((L^p : x_{m_i-1,i}), x_{m_i,i}) = \operatorname{sdepth} U/(J^p : x_{m_i-1,i}),$$

where U_2 is the polynomial ring formed from U by deleting $x_{m_i,i}$ from U. In order to determine the Stanley depth of $U_2/(J^p : x_{m_i-1,i})$, consider the following short exact sequence:

$$0 \longrightarrow U_2/(J^p : x_{m_i-1,i}x_{m_i-2,i}) \longrightarrow U_2/(J^p : x_{m_i-1,i}) \longrightarrow U_2/((J^p : x_{m_i-1,i}), x_{m_i-2,i}) \longrightarrow 0.$$

Now since $x_{m_i-1,i}$ is the leaf in U_2 with its unique neighbor $x_{m_i-2,i}$. So again Lemma 2.5.11 we have

$$\begin{aligned}
\operatorname{sdepth} U/(J^p : x_{m_i-1,i}x_{m_i-2,i}) &= \operatorname{sdepth} U_2/J^{p-1} \\
&\geq \sum_{i\neq j=1}^{t} \left\lceil \frac{m_j - (p-1)}{3} \right\rceil + \left\lceil \frac{(m_i - 1) - (p-1)}{3} \right\rceil \\
&= \sum_{i\neq j=1}^{t} \left\lceil \frac{m_j - p + 1}{3} \right\rceil + \left\lceil \frac{m_i - p}{3} \right\rceil \\
&\geq \sum_{j=1}^{t} \left\lceil \frac{m_j - p}{3} \right\rceil.
\end{aligned}$$

Also

$$\begin{aligned}((J^p : x_{m_i-1,i}), x_{m_i-2,i}) &= ((I^p : x_{m_i-1,i}), x_{m_i-2,i}) \\ &= (I^p, x_{m_i-2,i}),\end{aligned}$$

where I is the edge ideal associated to the minor of H formed as a result of deletion of variables $x_{m_i,i}$ and $x_{m_i-2,i}$. Let U_3 be the polynomial ring formed by deleting $x_{m_i-2,i}$ and $x_{m_i-1,i}$ from U_2. Since the number of vertices in the remaining graph has decreased so by using induction on m and Lemma 2.4.4 we have,

$$\begin{aligned}\operatorname{sdepth} U_2/(I^p, x_{m_i-2,i}) &= \operatorname{sdepth} U_3/I^p + 1 \\ &\geq \sum_{i\neq j=1}^{t} \left\lceil \frac{m_j - p}{3} \right\rceil + \left\lceil \frac{(m_i-3)-p}{3} \right\rceil + 1 \\ &= \sum_{j=1}^{t} \left\lceil \frac{m_j - p}{3} \right\rceil.\end{aligned}$$

Combining the results in the proof we have

$$\operatorname{sdepth} U/(L^p : x_{m_i,i}x_{m_i-1,i}) \geq \sum_{j=1}^{t} \left\lceil \frac{m_j - p}{3} \right\rceil,$$

$$\operatorname{sdepth} U/(L^p, x_{m_i-1,i}) \geq \sum_{j=1}^{t} \left\lceil \frac{m_j - p}{3} \right\rceil$$

and

$$\operatorname{sdepth} U/((L^p : x_{m_i-1,i}), x_{m_i,i}) \geq \sum_{j=1}^{t} \left\lceil \frac{m_j - p}{3} \right\rceil.$$

Hence by using Lemma 2.2.2 we have

$$\operatorname{sdepth} U/L^p \geq \sum_{j=1}^{t} \left\lceil \frac{m_j - p}{3} \right\rceil.$$

$\square$

Same result holds for the depth of the module U/L^p.

Theorem 2.5.25. *[1, Theorem 2.9] Let H be a $(t; m_1, m_2, \ldots, m_t)$-star graph and $L = L(H)$ be its edge ideal with $d(H) = l$, then*

$$\operatorname{depth} U/L^p \geq \max\{1, \sum_{j=1}^{t} \left\lceil \frac{m_j - p}{3} \right\rceil\},$$

for every $p \geq 1$.

Chapter 3

Depth and Stanley depth of some special trees and other graphs.

First section of this chapter covers the bounds for the depth and Stanley depth for module U/L where L is edge ideal corresponding to caterpillar graph. Second section covers the bound for the depth and Stanley depth for the firecracker graphs. Third section includes the bound for depth and Stanley depth of U/L where L is the edge ideal corresponding to the graphs, one formed by the rooted product of P_n and P_l path graph and the other formed by the rooted product of C_n and P_l graphs. The last section covers the bounds for the depth and Stanley depth of edge ideals associated to the graph $B(m,n)$ formed by linking n number of C_m cycle graphs to a vertex each through an edge.

3.1 Depth and Stanley depth of caterpillar graph.

Let H be a caterpillar graph denoted by $S_{n;k_1,k_2,\dots,k_n}$ where k_i denotes the number of pendant vertices of vertex x_i of P_n path in caterpillar with $1 \leq i \leq n$ and edge ideal corresponding to this graph is denoted by $L = L(H)$.

Theorem 3.1.1. *Let $L = L(S_{n;k_1,k_2,\dots,k_n})$. For $n \geq 2$ and $k_i \geq 3$,*

$$\operatorname{depth}(U/L) \geq \min\{k_i \,:\, 1 \leq i \leq n\}.$$

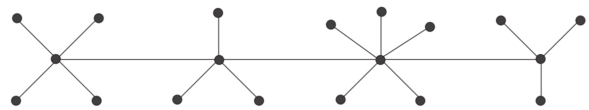


Figure 3.1: $S_{4;4,3,5,3}$

Proof. Let x_j be the vertices of the path of caterpillar, where $1 \leq j \leq n$. The proof is done by induction on n. Now for $n = 2$

$$(L : x_2) = (\mathcal{L}(S_{k_2}), x_1)$$

so,

$$\begin{aligned}
\operatorname{depth}(U/(L : x_2)) &= |\mathcal{L}(S_{k_1})| + 1 \\
&= (k_1 - 1) + 1 \\
&= k_1.
\end{aligned}$$

Clearly,

$$(L, x_2) = (L(S_{k_1}), x_2)$$

so,

$$\begin{aligned}
\operatorname{depth}(U/(L, x_2)) &= |\mathcal{L}(S_{k_2})| + \operatorname{depth} U'/L(S_{k_1}) \\
&= (k_2 - 1) + 1 \\
&= k_2,
\end{aligned}$$

where $U' = U \backslash (\{x_2\} \cup \{\mathcal{L}(S_{k_2})\})$.
Then by Depth Lemma,

$$\operatorname{depth}(U/L) \geq \min\{k_1, k_2\}.$$

Now for $n = 3$, we have

$$(L : x_3) = (L(S_{k_1}), \mathcal{L}(S_{k_3}), x_2)$$

so,

$$\begin{aligned}
\operatorname{depth}(U/(L:x_3)) &= \operatorname{depth}(U''/L(S_{k_1})) + |\mathcal{L}(S_{k_2})|+1 \\
&= 1 + (k_2 - 1) + 1 \\
&= k_2 + 1 \\
&\geq \min\{k_1, k_2, k_3\},
\end{aligned}$$

where $U'' = U\backslash\{V(S_{k_2}) \cup V(S_{k_3})\}$.
Also

$$(L, x_3) = (L(S_{2;k_1,k_2}), x_3)$$

and

$$\begin{aligned}
\operatorname{depth}(U/(L, x_3)) &\geq \min\{k_1, k_2\} + |\mathcal{L}(S_{k_3})| \\
&= \min\{k_1, k_2\} + (k_3 - 1) \\
&\geq \min\{k_1, k_2, k_3\}.
\end{aligned}$$

Depth Lemma implies

$$\operatorname{depth}(U/L) \geq \min\{k_1, k_2, k_3\}.$$

Assume $n \geq 4$, then

$$(L : x_n) = (L(S_{(n-2);k_1,k_2,\ldots,k_{(n-2)}}), \mathcal{L}(S_{k_n}), x_{n-1}).$$

Then by induction on n,

$$\begin{aligned}
\operatorname{depth}(U/(L:x_n)) &= \operatorname{depth}(U'/L(S_{(n-2);k_1,k_2,\ldots,k_{(n-2)}})) + |\mathcal{L}(S_{k_{n-1}})|+1 \\
&\geq \min\{k_1, \ldots, k_{n-2}\} + (k_{n-1} - 1) + 1 \\
&\geq \min\{k_i \;:\; 1 \leq i \leq n\},
\end{aligned}$$

where $U' = U\backslash(V(S_{k_{n-1}}), V(S_{k_n}))$. Clearly,

$$(L, x_n) = (L(S_{(n-1);k_1,k_2,\ldots,k_{(n-1)}}), x_n).$$

Now by using induction

$$\begin{aligned}\operatorname{depth}(U/(L,x_n)) &= \operatorname{depth}(U''/(S_{(n-1);k_1,k_2,\ldots,k_{(n-1)}})) + |\mathcal{L}(S_{k_n})| \\ &\geq \min\{k_1,\ldots,k_{n-1}\} + (k_n - 1) \\ &\geq \min\{k_i \ : \ 1 \leq i \leq n\},\end{aligned}$$

where $U'' = U\backslash(V(S_{k_n}))$.
Hence, by Depth Lemma

$$\operatorname{depth}(U/L) \geq \min\{k_i \ : \ 1 \leq i \leq n\}.$$

□

Thus the new bound is better than already given by Morey in [11, Theorem 3.1] when minimum of the $k_i's$ is greater than $\lceil \frac{l+1}{3} \rceil$. A similar result hold for the Stanley depth of the graph.

Proposition 3.1.2. *Let $L = L(S_{n;k_1,k_2,\ldots,k_n})$. For $n \geq 2$ and $k_i \geq 3$,*

$$\operatorname{sdepth}(U/L) \geq \min\{k_i \ : \ 1 \leq i \leq n\}.$$

Proof. By using [1, Lemma 2.4] and [1, Theorem 2.4] and working on the same lines we get the required result. □

Now we will find the bounds for the graph formed by adding an edge linking the first and last vertex of path P_n. This graph is denoted by $CS_{n;k_1,k_2,\ldots,k_n}$.

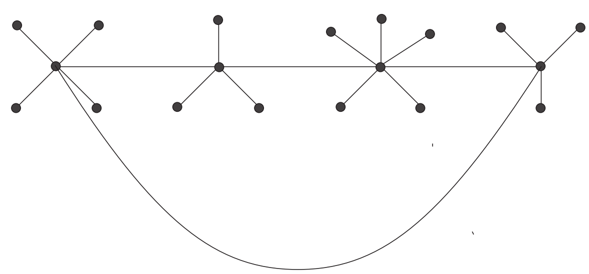


Figure 3.2: $CS_{4;4,3,5,3}$

Theorem 3.1.3. *Let $L = L(CS_{n;k_1,k_2,\ldots,k_n})$. For $n \geq 3$ and $k_i \geq 3$,*

$$\operatorname{depth}(U/L) \geq \min\{k_i \,:\, 1 \leq i \leq n\}.$$

Proof. Let x be any vertex in path of the caterpillar graph. For $n = 3$, let $x = x_3$

$$(L : x_3) = (x_1, \mathcal{L}(S_{k_3}), x_2),$$

so,

$$\begin{aligned}
\operatorname{depth}(U/(L : x_3)) &= |\mathcal{L}(S_{k_1})|+|\mathcal{L}(S_{k_2})|+1 \\
&= (k_1 - 1) + (k_2 - 1) + 1 \\
&= k_1 + k_2 - 1 \\
&\geq \min\{k_1, k_2, k_3\}.
\end{aligned}$$

Also

$$(L, x_3) = (L(S_{2;k_1,k_2}), x_3)$$

where $i \in \{1, 2\}$ so,

$$\begin{aligned}
\operatorname{depth}(U/(L, x_3)) &\geq \min\{k_1, k_2\} + |\mathcal{L}(S_{k_3})| \\
&= \min\{k_1, k_2\} + (k_3 - 1) \\
&\geq \min\{k_1, k_2, k_3\}.
\end{aligned}$$

Depth Lemma implies

$$\operatorname{depth}(U/L) \geq \min\{k_1, k_2, k_3\}.$$

Assume $n \geq 4$, and $x = x_n$

$$(L : x_n) = (L(S_{(n-3);k_1,k_2,\ldots,k_{(n-3)}}), \mathcal{L}(S_{k_n}), x_{n-1}, x_1),$$

where $i \in \{2, \ldots, n-2\}$. Then by using Theorem 3.1.1,

$$\begin{aligned}
\operatorname{depth}(U/(L : x_n)) &= \operatorname{depth}(U'/L(CS_{(n-3);k_1,k_2,\ldots,k_{(n-3)}})) + |\mathcal{L}(S_{k_{n-1}})|+|\mathcal{L}(S_{k_1})|+1 \\
&\geq \min\{k_2, \ldots, k_{n-2}\} + (k_{n-1} - 1) + (k_1 - 1) + 1 \\
&\geq \min\{k_i \,:\, 1 \leq i \leq n\},
\end{aligned}$$

where $U' = U\backslash(V(S_{k_{n-1}}), V(S_{k_n}, V(S_{k_1})))$. Notice that

$$(L, x_n) = (L(S_{(n-1);k_1,k_2,\ldots,k_{(n-1)}}), x_n)$$

where $i \in \{1, \ldots, n-1\}$. Now by using Theorem 3.1.1

$$\begin{aligned} \operatorname{depth}(U/(L, x_n)) &= \operatorname{depth}(U''/(S_{(n-1);k_1,k_2,\ldots,k_{(n-1)}})) + |\mathcal{L}(S_{k_n})| \\ &\geq \min\{k_1, \ldots, k_{n-1}\} + (k_n - 1) \\ &\geq \min\{k_i \,:\, 1 \leq i \leq n\}, \end{aligned}$$

where $U'' = U\backslash(V(S_{k_n}))$.

Hence, by Depth Lemma

$$\operatorname{depth}(U/L) \geq \min\{k_i \,:\, 1 \leq i \leq n\}.$$

□

In comparison to bound by Morey in [11, Theorem 3.1], the new bound is much improved when minimum of the $k_i's$ is enough larger. A similar result hold for the Stanley depth of the graph.

Proposition 3.1.4. *Let $L = L(CS_{n;k_1,k_2,\ldots,k_n})$. For $n \geq 3$ and $k_i \geq 3$,*

$$\operatorname{sdepth}(U/L) \geq \min\{k_i \,:\, 1 \leq i \leq n\}.$$

Proof. By using proposition 3.1.2 and [1, Lemma 2.4] we get the desired result working on the lines of Theorem 3.1.3. □

3.2 Depth and Stanley depth of a graph formed by joining m number of n-star graph to a vertex each through an edge.

This section covers the of bounds for depth and Stanley depth of edge ideal corresponding to a graph H, denoted by $S_m(k; n_1, n_2, \ldots, n_m)$, formed by linking m number of caterpillar graphs to a vertex each through an edge. Each vertex of the path in caterpillar graph has k number of leaves.

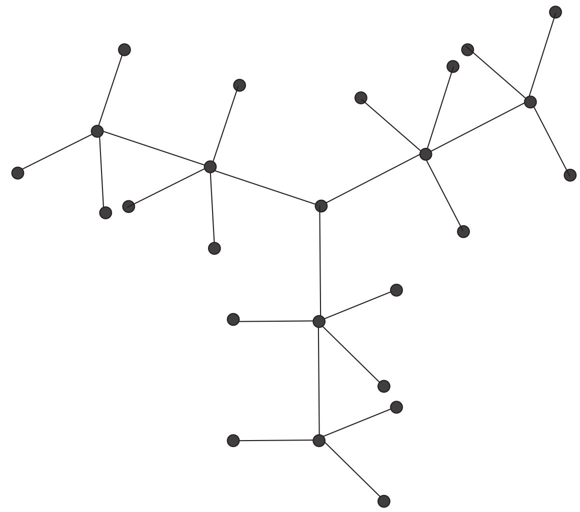


Figure 3.3: $S_3(4;2,2,2)$

Theorem 3.2.1. *Let* $L = L(S_m(k;n_1,n_2,\ldots,n_m))$*. For* $m \geq 2$ *and* $k_i \geq 3$*,*

$$\operatorname{depth}(U/L) \geq \operatorname{depth}(U_1''/(L(S_{n_1,k}))) + \cdots + \operatorname{depth}(U_m''/(L(S_{n_m,k})))$$

where $U_i'' = U\backslash(\{x\} \cup \{V(S_{n_1,k}), V(S_{n_2,k}), \ldots, V(S_{n_{i-1},k}), V(S_{n_{i+1},k}), \ldots, V(S_{n_m,k})\})$*.*

Proof. Let x be the vertex of the graph with which all caterpillars are linked each through an edge. Consider a short exact sequence:

$$0 \longrightarrow U/(L:x) \longrightarrow U/L \longrightarrow U/(L,x) \longrightarrow 0.$$

Notice that

$$(L:x) = (N(x), L(S_{n_1-1,k}), L(S_{n_2-1,k}), \ldots, L(S_{n_m-1,k})).$$

So,

$$\begin{aligned}
\operatorname{depth}(U/(L:x_2)) &= m(k-1) + 1 + \operatorname{depth}(U_1'/(L(S_{n_1-1,k}))) + \cdots + \operatorname{depth}(U_m'/(L(S_{n_m-1,k}))) \\
&= mk - m + 1 + \operatorname{depth}(U_1'/(L(S_{n_1-1,k}))) + \cdots + \operatorname{depth}(U_m'/(L(S_{n_m-1,k}))),
\end{aligned}$$

where $U_i' = U\backslash(\{N(x), x\} \cup \{V(S_{n_1-1,k}), V(S_{n_2-1,k}), \ldots, V(S_{n_{i-1}-1,k}), V(S_{n_{i+1}-1,k}), \ldots, V(S_{n_m-1,k})\})$. Clearly,

$$(L,x) = (x, L(S_{n_1,k}), L(S_{n_2,k}), \ldots, L(S_{n_m,k})).$$

So,

$$\text{depth}(U/(L,x)) \;=\; \text{depth}(U_1''/(L(S_{n_1,k}))) + \cdots + \text{depth}(U_m''/(L(S_{n_m,k}))),$$

where $U_i'' = U\backslash(\{x\} \cup \{V(S_{n_1,k}), V(S_{n_2,k}), \ldots, V(S_{n_{i-1},k}), V(S_{n_{i+1},k}), \ldots, V(S_{n_m,k})\})$. Then by Depth Lemma,

$$\text{depth}(U/L) \geq \min\{\text{depth}(U/(L:x_2)), \text{depth}(U/(L,x))\},$$

thus

$$\text{depth}(U/L) \geq \text{depth}(U_1''/(L(S_{n_1,k}))) + \cdots + \text{depth}(U_m''/(L(S_{n_m,k}))).$$

□

Similarly Using [1, Lemma 2.4] we conclude the following result,

Proposition 3.2.2. *Let $L = L(S_m(k; n_1, n_2, \ldots, n_m))$. For $m \geq 2$ and $k_i \geq 3$,*

$$\text{sdepth}(U/L) \geq \text{sdepth}(U_1''/(L(S_{n_1,k}))) + \cdots + \text{sdepth}(U_m''/(L(S_{n_m,k}))).$$

3.3 Depth and Stanley depth of Firecracker graph.

Firecracker graph $F^l_{n,k}$ is formed by concatenation of n number of star graphs. This is done by linking one leaf from each star. In the graph $n-1$ number of S_k stars with one S_l star at one end are linked. When $l = k$, then we denote the graph by $F_{n,k}$.

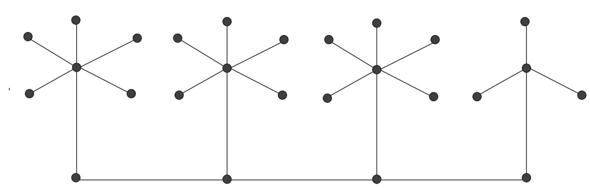


Figure 3.4: $F^5_{4,7}$

Theorem 3.3.1. *For a firecracker graph $F^l_{n,k}$, let $L = L(F^l_{n,k})$. Then for $n \geq 2$ and $k \geq 3$,*

$$\text{depth}(U/L) \geq \begin{cases} n\,, & \textit{when } l > 2; \\ n-1\,, & \textit{when } l \leq 2. \end{cases}$$

Proof. The proof is done by the induction on n. For $n = 2$ and $l \in \{1, 2\}$, the result holds by Theorem 2.5.8. Now for $n = 2$ and $l \geq 3$, let u be the central vertex of S_l star graph. Consider the short exact sequence:

$$0 \longrightarrow U/(L : u) \longrightarrow U/L \longrightarrow U/(L : u) \longrightarrow 0.$$

Notice that $(L : u) = (N(u), L(S_k))$, thus

$$\operatorname{depth} U/(L : u) = 1 + 1 = 2.$$

Now we notice that $(L, u) = (u, J)$, so,

$$\operatorname{depth} U/(L, u) = 2 + l - 2 = l.$$

So by depth lemma we have,

$$\operatorname{depth} U/L \geq n.$$

Now for $n \geq 3$, when $l = 1$, let u be the vertex of S_l. Consider the short exact sequence:

$$0 \longrightarrow U/(L : u) \longrightarrow U/L \longrightarrow U/(L : u) \longrightarrow 0.$$

We see that, $(L : u) = (N(u), L(S_{k-1}), L(S_k))$, thus

$$\operatorname{depth} U/(L : u) = 1 + 1 + 1 = 3.$$

Now notice that $(L, u) = (u, L(F_{n-1,k}))$, thus

$$\operatorname{depth} U/(L, u) = n - 1.$$

When $l = 2$, let u be the leaf of S_2. Consider the short exact sequence:

$$0 \longrightarrow U/(L : u) \longrightarrow U/L \longrightarrow U/(L : u) \longrightarrow 0.$$

Notice that $(L : u) = (N(u), L(F_{2,k}))$, thus

$$\operatorname{depth} U/(L : u) = 1 + 2 = 3.$$

Now we see that $(L, u) = (u, L(F^1_{3,k}))$, so

$$\operatorname{depth} U/(L, u) = 2 = n - 1.$$

Thus by depth lemma we have,

$$\operatorname{depth} U/L \geq n-1.$$

Now for $l \geq 3$, let u be the central vertex of S_l star. Consider the short exact sequence:

$$0 \longrightarrow U/(L:u) \longrightarrow U/L \longrightarrow U/(L:u) \longrightarrow 0.$$

Notice that $(L:u) = (N(u), L(F_{n-1,k})$, so

$$\operatorname{depth} U/(L:u) = 1+n-1 = n$$

and $(L,u) = (u, L(F^1_{n,k}))$, so

$$\operatorname{depth} U/(L,u) = l-2+n-1 = n+l-3.$$

Since $l \geq 3$, we get by depth lemma,

$$\operatorname{depth} U/L \geq n.$$

$\square$

Theorem 3.3.2. *For a firecracker graph $F^l_{n,k}$, let $L = L(F^l_{n,k})$. Then for $n \geq 2$ and $k \geq 3$,*

$$\operatorname{sdepth}(U/L) \geq \begin{cases} n\,, & \textit{when } l > 2; \\ n-1\,, & \textit{when } l \leq 2. \end{cases}$$

Proof. By using theorem 2.5.9 and Lemma 2.2.2 we get the required result working on the same lines. $\square$

3.4 Depth and Stanley depth of closed Firecracker graph.

Firecracker graph $CF^l_{n,k}$ is formed by adding an edge in $F^l_{n,k}$ linking the linked vertex of first star with the linked vertex of last star S_l.

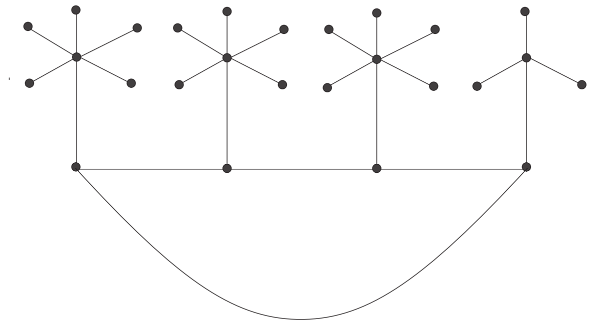


Figure 3.5: $CF^5_{4,7}$

Theorem 3.4.1. *For a graph $CF^l_{n,k}$, let $L = L(CF^l_{n,k})$. Then for $n \geq 3$ and $k \geq 3$,*

$$\operatorname{depth}(U/L) \geq \begin{cases} n\,, & \text{when } l > 2; \\ n-1\,, & \text{when } l \leq 2. \end{cases}$$

Proof. We will use induction on n to prove the result. For $n = 3$ and $l = 1$, consider the short exact sequence with u be the vertex of the last star S_1:

$$0 \longrightarrow U/(L:u) \longrightarrow U/L \longrightarrow U/(L:u) \longrightarrow 0.$$

Notice that $(L:u) = (N(u), 2(L(S_{k-1})))$, thus

$$\operatorname{depth} U/(L:u) = 1 + 2 = 3.$$

Now we notice that $(L, u) = (u, L(F_{2,k}))$, so,

$$\operatorname{depth} U/(L, u) = 2 = n - l.$$

So by depth lemma we have,

$$\operatorname{depth} U/L \geq n - 1.$$

For $l = 2$, consider the short exact sequence where u is the leaf of last star S_2:

$$0 \longrightarrow U/(L:u) \longrightarrow U/L \longrightarrow U/(L:u) \longrightarrow 0.$$

We see that, $(L:u) = (N(u), (L(F_{2,k})))$, thus

$$\operatorname{depth} U/(L:u) = 1 + 2 = 3.$$

Now we notice that $(L, u) = (u, L(CF^1_{3,k}))$, so

$$\operatorname{depth} U/(L, u) = 2 = n - 1.$$

So by depth lemma we have,

$$\operatorname{depth} U/L \geq n - 1.$$

For $l \geq 3$, consider the short exact sequence with u be the leaf of S_l which is linked with other stars:

$$0 \longrightarrow U/(L : u) \longrightarrow U/L \longrightarrow U/(L : u) \longrightarrow 0.$$

Notice that $(L : u) = (N(u), 2(L(S_{k-1})))$, thus

$$\operatorname{depth} U/(L : u) = 1 + 2 + (l - 2) = l + 1.$$

Now we notice that $(L, u) = (u, L(F_{2,k}), L(S_{l-1}))$ so,

$$\operatorname{depth} U/(L, u) = 2 + l = 3 = n.$$

So by depth lemma we have,

$$\operatorname{depth} U/L \geq n.$$

Now for $n = 4$ and $l = 1$, consider the short exact sequence with u be the vertex of the last star S_1:

$$0 \longrightarrow U/(L : u) \longrightarrow U/L \longrightarrow U/(L : u) \longrightarrow 0.$$

Notice that $(L : u) = (N(u), 2(L(S_{k-1})), (L(S_k)))$, thus

$$\operatorname{depth} U/(L : u) = 1 + 2 + 1 = 4.$$

Now we notice that $(L, u) = (u, L(F_{3,k}))$, so

$$\operatorname{depth} U/(L, u) = 3 = n - 1.$$

So by depth lemma we have,

$$\operatorname{depth} U/L \geq n - 1.$$

For $l = 2$, consider the short exact sequence where u is the leaf of last star S_2:

$$0 \longrightarrow U/(L:u) \longrightarrow U/L \longrightarrow U/(L:u) \longrightarrow 0.$$

We see that, $(L:u) = (N(u), (L(F_{3,k})))$, thus

$$\operatorname{depth} U/(L:u) = 1 + 3 = 4.$$

Now we notice that $(L, u) = (u, L(CF^1_{4,k}))$, so

$$\operatorname{depth} U/(L,u) = 3 = n - 1.$$

So by depth lemma we have,

$$\operatorname{depth} U/L \geq n - 1.$$

For $l \geq 3$, consider the short exact sequence with u be the leaf of S_l which is linked with other stars:

$$0 \longrightarrow U/(L:u) \longrightarrow U/L \longrightarrow U/(L:u) \longrightarrow 0.$$

Notice that $(L:u) = (N(u), L(S_k), 2(L(S_{k-1})))$, thus

$$\operatorname{depth} U/(L:u) = 1 + 1 + 2 + (l - 2) = l + 2.$$

Now we notice that $(L, u) = (u, L(F_{3,k}), L(S_{l-1}))$, so,

$$\operatorname{depth} U/(L,u) = 3 + l = 4 = n.$$

So by depth lemma we have,

$$\operatorname{depth} U/L \geq n$$

Now for $n \geq 5$ and $l = 1$, consider the short exact sequence with u be the vertex of S_l:

$$0 \longrightarrow U/(L:u) \longrightarrow U/L \longrightarrow U/(L:u) \longrightarrow 0.$$

Notice that $(L:u) = (N(u), 2(L(S_{k-1})), (L(F_{n-3,k})))$, thus

$$\operatorname{depth} U/(L:u) = 1 + 2 + (n - 3) = n.$$

Now we notice that $(L, u) = (u, L(F_{n-1,k}))$, so,

$$\operatorname{depth} U/(L, u) = n - 1.$$

So by depth lemma we have,

$$\operatorname{depth} U/L \geq n - 1.$$

For $l = 2$, consider the short exact sequence where u is the leaf of last star S_2:

$$0 \longrightarrow U/(L : u) \longrightarrow U/L \longrightarrow U/(L : u) \longrightarrow 0.$$

We see that, $(L : u) = (N(u), (L(F_{n-1,k})))$, thus

$$\operatorname{depth} U/(L : u) = 1 + n - 1 = n.$$

Now we notice that $(L, u) = (u, L(CF^1_{n,k}))$, so,

$$\operatorname{depth} U/(L, u) = n - 1.$$

So by depth lemma we have,

$$\operatorname{depth} U/L \geq n - 1.$$

Now for $l \geq 3$, consider the short exact sequence with u be the leaf of S_l which is linked with other stars:

$$0 \longrightarrow U/(L : u) \longrightarrow U/L \longrightarrow U/(L : u) \longrightarrow 0.$$

Notice that $(L : u) = (N(u), L(F_{n-3,k}), 2(L(S_{k-1})))$, thus

$$\operatorname{depth} U/(L : u) = 1 + (n - 3) + 2 + (l - 2) = n + l - 2.$$

Now we notice that $(L, u) = (u, L(F_{n-1,k}), L(S_{l-1}))$, so,

$$\operatorname{depth} U/(L, u) = (n - 1) + l = n.$$

So by depth lemma we have,

$$\operatorname{depth} U/L \geq n.$$

$\square$

Theorem 3.4.2. *For a closed firecracker graph $CF^l_{n,k}$, let $L = L(CF^l_{n,k})$. Then for $n \geq 3$ and $k \geq 3$,*

$$\operatorname{sdepth}(U/L) \geq \begin{cases} n\,, & \textit{when } l > 2; \\ n-1\,, & \textit{when } l \leq 2. \end{cases}$$

Proof. By using theorem 2.5.9, theorem 3.3.1 and Lemma 2.2.2 we get the required result by working on the same lines. □

3.5 Depth and Stanley depth of graphs formed by Rooted product of Paths and Cycles.

In this section two types of graphs are under consideration. First type of graph which is discussed here is the one formed by the rooted product of P_n and P_l path graphs, denoted by $P_{n,l}$. Second type of linked graph is obtained by the rooted product of C_n and P_l graphs, denoted by $CP_{n,l}$. For both types, bounds for the depth and Stanley depth depends on number of paths and the length of paths. Two lemmas which are widely used in this section tells us the depth and Stanley depth of the path graph P_n.

Lemma 3.5.1. *[11, Lemma 2.8] For $H = P_n$, $n \geq 2$ and $L = L(P_n)$,*

$$\operatorname{depth}(U/L) = \left\lceil \frac{n}{3} \right\rceil.$$

A similar lemma for the Stanley depth is

Lemma 3.5.2. *[18, Lemma 4] For $H = P_n$ with $n \geq 2$ and $L = L(P_n)$,*

$$\operatorname{sdepth}(U/L) = \left\lceil \frac{n}{3} \right\rceil.$$

Lemma 3.5.3. *[6, Proposition 1.3] Let $H = C_n$ with $n \geq 3$ and $L = L(C_n)$,*

$$\operatorname{depth} U/L = \left\lceil \frac{n-1}{3} \right\rceil.$$

Lemma 3.5.4. *[6, Proposition 1.8] Let $H = C_n$ with $n \geq 3$ and $L = L(C_n)$,*

$$\operatorname{sdepth} U/L = \left\lceil \frac{n-1}{3} \right\rceil.$$

Now we define the graph $P_{n,l}$.

Definition 3.5.5. Let $H = P_{n,l}$ be a graph obtained by the rooted product of P_n and P_m path graphs.

Below is an example of the graph formed by the rooted product of P_5 and P_6 path graphs.

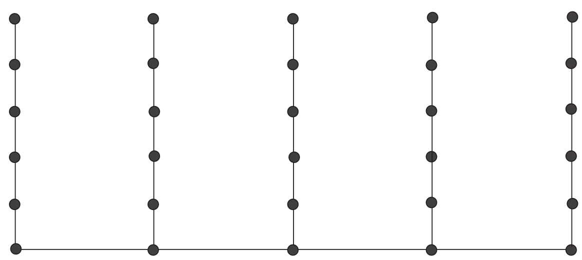


Figure 3.6: $P_{(5,6)}$

Following theorems give the bounds for the depth and Stanley depth of module $U/L(P_{n,l})$

Theorem 3.5.6. *For any $n \geq 3$ and $l \geq 3$,*

$$\operatorname{depth}(U/L) \geq \begin{cases} \frac{nl}{3}, & \textit{when } l \equiv 0 \ (mod\ 3); \\ \frac{nl}{3} - \frac{n}{3} + 1, & \textit{when } l \equiv 1 \ (mod\ 3); \\ \frac{nl}{3} + \frac{n}{3}, & \textit{when } l \equiv 2 \ (mod\ 3). \end{cases}$$

Proof. Let $L = L(P_{n,l})$ be an edge ideal. For $n = 3$, let u be the linked vertex of last attached path P_l. Consider the short exact sequence:

$$0 \longrightarrow U/(L:u) \longrightarrow U/L \longrightarrow U/(L,u) \longrightarrow 0.$$

By depth lemma,

$$\operatorname{depth}(U/L) \geq \min\{\operatorname{depth}(U/(L:u), \operatorname{depth}(U/(L,u)\}.$$

Notice that

$$(L:u) = (I, N(u)) = (L(P_l), L(P_{l-1}), L(P_{l-2}), N(u)),$$

where I is the edge ideal of minor formed as a result of deletion of neighbors of u and

$$(L,u) = (L(P_{2l}), L(P_{l-1}), u).$$

Lemma 2.4.4 and Lemma 3.5.1 implies,

$$\operatorname{depth}(U/(L:u)) = \left\lceil\frac{l}{3}\right\rceil + \left\lceil\frac{l-1}{3}\right\rceil + \left\lceil\frac{l-2}{3}\right\rceil + 1,$$

$$\operatorname{depth}(U/(L,u)) = \left\lceil\frac{2l}{3}\right\rceil + \left\lceil\frac{l-1}{3}\right\rceil.$$

When $l \equiv 0, 1 \pmod 3$

$$\operatorname{depth}(U/L:u) = l+1$$

and

$$\operatorname{depth}(U/L,u) = l.$$

So by depth lemma we have

$$\operatorname{depth}(U/L) \geq l.$$

For $l \equiv 2 \pmod 3$

$$\operatorname{depth}(U/(L:u)) = l+1,$$

$$\operatorname{depth}(U/(L,u)) = l+1,$$

so, by depth lemma we have

$$\operatorname{depth}(U/(L)) \geq l+1.$$

Now for $n = 4$, consider the same short exact sequence, then

$$(L:u) = (L(P_{2l}), L(P_{l-1}), L(P_{l-2}), N(u)),$$

and

$$(L,u) = (L(P_{3,l}), L(P_{l-1}), u).$$

So by Lemma 2.4.4, Lemma 3.5.1,

$$\operatorname{depth}(U/(L:u)) = \left\lceil\frac{2l}{3}\right\rceil + \left\lceil\frac{l-1}{3}\right\rceil + \left\lceil\frac{l-2}{3}\right\rceil + 1.$$

Using induction on n,

$$\operatorname{depth}(U/(L,u)) \geq \left\lceil \frac{l-1}{3} \right\rceil + l, \qquad \text{when } l \equiv 0,1 \pmod 3,$$

$$\operatorname{depth}(U/(L,u)) = \left\lceil \frac{l-1}{3} \right\rceil + l + 1, \quad \text{when } l \equiv 2 \pmod 3.$$

So for $l \equiv 0 \pmod 3$

$$\operatorname{depth}(U/(L,u)) = \frac{4l}{3},$$

$$\operatorname{depth}(U/(L:u)) = \frac{4l}{3} + 1.$$

Hence by using depth lemma we have

$$\operatorname{depth}(U/(L)) \geq \frac{4l}{3}.$$

For $l \equiv 1 \pmod 3$

$$\operatorname{depth}(U/(L,u)) = \frac{4l-1}{3},$$

$$\operatorname{depth}(U/(L:u)) = \frac{4l+2}{3},$$

so, by depth lemma we have

$$\operatorname{depth}(U/(L)) \geq \frac{4l-1}{3}.$$

For $l \equiv 2 \pmod 3$

$$\operatorname{depth}(U/(L,u)) = \frac{4l+4}{3},$$

$$\operatorname{depth}(U/(L:u)) = \frac{4l+4}{3},$$

hence by depth lemma we have

$$\operatorname{depth}(U/(L)) = \frac{4l+4}{3}.$$

Assume $n \geq 5$, notice that

$$(L:u) = (I, N(u)) = (L(P_{n-2,l}), L(P_{l-1}), L(P_{l-2}), N(u))$$

and

$$(L,u) = (L(P_{n-1,l}), L(P_{l-1}), u).$$

Case 1: $l \equiv 0$ (**mod** 3)

By Lemma 2.4.4, Lemma 3.5.1 and induction on n,

$$\operatorname{depth}(U/L:u) \geq \left\lceil \frac{l-1}{3} \right\rceil + \left\lceil \frac{l-2}{3} \right\rceil + \frac{(n-2)l}{3} + 1$$

$$\operatorname{depth}(U/L:u) \geq 1 + \frac{l}{3} + \frac{l}{3} + \frac{nl}{3} - \frac{2l}{3} = 1 + \frac{nl}{3}$$

$$\operatorname{depth}(U/L,u) \geq \left\lceil \frac{l-1}{3} \right\rceil + \frac{(n-1)l}{3}$$

$$\operatorname{depth}(U/L,u) \geq \frac{l}{3} - \frac{l}{3} + \frac{nl}{3} = \frac{nl}{3}.$$

Depth lemma implies,

$$\operatorname{depth}(U/L) \geq \frac{nl}{3}, \qquad l \equiv 0 \pmod 3).$$

Case 2: $l \equiv 1$ (**mod** 3)

Using Lemma 2.4.4, Lemma 3.5.1 and induction on n,

$$\begin{aligned}
\operatorname{depth}(U/L:u) &\geq \left\lceil \frac{l-1}{3} \right\rceil + \left\lceil \frac{l-2}{3} \right\rceil + \frac{(n-2)l-n+5}{3} + 1 \\
&= 1 + \frac{l-1}{3} + \frac{l-1}{3} + \frac{nl}{3} - \frac{2l}{3} - \frac{n}{3} + \frac{2}{3} + 1 \\
&= 1 + \frac{nl}{3} - \frac{n}{3},
\end{aligned}$$

$$\begin{aligned}
\operatorname{depth}(U/L,u) &\geq \left\lceil \frac{l-1}{3} \right\rceil + \frac{(n-1)l-n+1+3}{3} \\
&= \frac{l-1}{3} - \frac{l}{3} + \frac{nl}{3} - \frac{n}{3} + \frac{1}{3} + 1 \\
&= \frac{nl}{3} - \frac{n}{3} + 1.
\end{aligned}$$

So by depth lemma,

$$\operatorname{depth}(U/L) \geq \frac{nl}{3} - \frac{n}{3} + 1, \qquad l \equiv 1 \pmod 3).$$

Case 3: $l \equiv 2$ (**mod** 3)

Then Lemma 2.4.4, Lemma 3.5.1 and induction implies,

$$\begin{aligned}
\operatorname{depth}(U/L:u) &\geq \left\lceil \frac{l-1}{3} \right\rceil + \left\lceil \frac{l-2}{3} \right\rceil + \frac{(n-2)l+n-2}{3} + 1 \\
&= 1 + \frac{l+1}{3} + \frac{l-2}{3} + \frac{nl}{3} - \frac{2l}{3} + \frac{n}{3} - \frac{2}{3} \\
&= \frac{nl}{3} + \frac{n}{3},
\end{aligned}$$

$$\begin{aligned}
\operatorname{depth}(U/L,u) &\geq \left\lceil \frac{l-1}{3} \right\rceil + \frac{(n-1)l+n-1}{3} \\
&= \frac{l+1}{3} - \frac{l}{3} + \frac{nl}{3} + \frac{n}{3} - \frac{1}{3} \\
&= \frac{nl}{3} + \frac{n}{3} + 1.
\end{aligned}$$

Hence by depth lemma,

$$\operatorname{depth}(U/L) \geq \frac{nl+n}{3}, \qquad l \equiv 2 \pmod 3).$$

□

The new bound determined above is sharper than given by Morey as in Theorem [11, Theorem 3.1], since all new bounds contain terms which are product of n and l. Now using Lemma 3.5.2 and Lemma 2.2.2 we can compute following bound on the same lines,

Proposition 3.5.7. *For any $n \geq 3$ and $l \geq 3$,*

$$\operatorname{sdepth}(U/L) \geq \begin{cases} \frac{nl}{3}, & \text{when } l \equiv 0 \ (mod\ 3); \\ \frac{nl}{3} - \frac{n}{3} + 1, & \text{when } l \equiv 1 \ (mod\ 3); \\ \frac{nl}{3} + \frac{n}{3}, & \text{when } l \equiv 2 \ (mod\ 3). \end{cases}$$

Proof. By using [18, Lemma 2.4] and [1, Lemma 2.4] we get the desired result. □

Now considering the second type of graph $CP_{n,l}$.

Definition 3.5.8. Let $H = CP_{n,l}$ be a graph obtained by the rooted of C_n and P_m graphs .

Theorem 3.5.9. *For $L = L(CP_{n,l})$ with $n \geq 3$ and $l \geq 3$,*

$$\operatorname{depth}(U/L) \geq \begin{cases} \frac{nl}{3}, & \text{when } l \equiv 0 \ (mod\ 3); \\ \frac{nl}{3} - \frac{n}{3} + 1, & \text{when } l \equiv 1 \ (mod\ 3); \\ \frac{nl}{3} + \frac{n}{3}, & \text{when } l \equiv 1 \ (mod\ 3). \end{cases}$$

Proof. Let $L = L(CP_{n,l})$ be an edge ideal. For $n = 3$, let u be the end vertex of any attached path P_l. Consider the short exact sequence:

$$0 \longrightarrow U/(L:u) \longrightarrow U/L \longrightarrow U/(L,u) \longrightarrow 0,$$

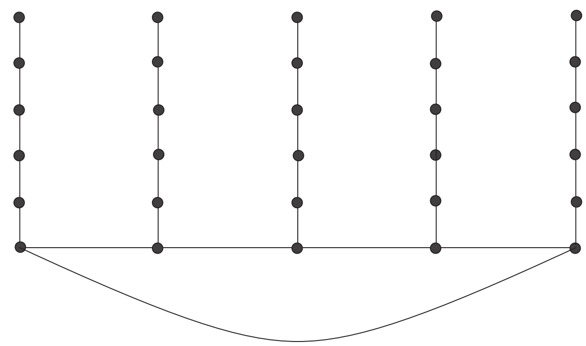


Figure 3.7: $CP(5,6)$

by depth lemma,

$$\operatorname{depth}(U/L) \geq \min\{\operatorname{depth}(U/(L:u), \operatorname{depth}(U/(L,u)\}.$$

Notice that

$$(L:u) = (I, N(u)) = (2L(P_{l-1}), L(P_{l-2}), N(u)),$$

where I is the edge ideal of minor formed by deleting neighbors of u and

$$(L,u) = (L(P_{2l}), L(P_{l-1}), u).$$

Lemma 2.4.4 and Lemma 3.5.1 implies,

$$\operatorname{depth}(U/(L:u)) = 2\left\lceil \frac{l-1}{3} \right\rceil + \left\lceil \frac{l-2}{3} \right\rceil + 1,$$

$$\operatorname{depth}(U/(L,u)) = \left\lceil \frac{2l}{3} \right\rceil + \left\lceil \frac{l-1}{3} \right\rceil.$$

When $l \equiv 0 \pmod 3$

$$\operatorname{depth}(U/L:u) = l+1,$$

and

$$\operatorname{depth}(U/L,u) = l,$$

so, by depth lemma we have

$$\operatorname{depth}(U/L) \geq l.$$

For $l \equiv 1 \pmod 3$

$$\operatorname{depth}(U/(L:u)) = l,$$

$$\operatorname{depth}(U/(L,u)) = l,$$

so, by depth lemma we have

$$\operatorname{depth}(U/(L)) = l.$$

For $l \equiv 2 \pmod 3$

$$\operatorname{depth}(U/(L:u)) = l+1,$$

$$\operatorname{depth}(U/(L,u)) = l+1,$$

so by depth lemma we have

$$\operatorname{depth}(U/(L)) = l+1.$$

Now for $n = 4$, consider the same short exact sequence. Notice that

$$(L:u) = (L(P_l), 2L(P_{l-1}), L(P_{l-2}), N(u))$$

and

$$(L,u) = (L(P_{3,l}), L(P_{l-1}), u).$$

So by Lemma 2.4.4, Lemma 3.5.1 and Theorem 3.5.6 we have,

$$\operatorname{depth}(U/(L:u)) = \left\lceil \frac{l}{3} \right\rceil + 2\left\lceil \frac{l-1}{3} \right\rceil + \left\lceil \frac{l-2}{3} \right\rceil$$

and

$$\operatorname{depth}(U/(L,u)) \geq \left\lceil \frac{l-1}{3} \right\rceil + l, \qquad \text{when } l \equiv 0,1 \pmod 3,$$
$$\operatorname{depth}(U/(L,u)) = \left\lceil \frac{l-1}{3} \right\rceil + l + 1, \quad \text{when } l \equiv 2 \pmod 3.$$

For $l \equiv 0 \pmod 3$

$$\operatorname{depth}(U/(L,u)) = \frac{4l}{3},$$

$$\operatorname{depth}(U/(L:u)) = \frac{4l}{3} + 1,$$

hence by using depth lemma we have

$$\operatorname{depth}(U/(L)) \geq \frac{4l}{3}.$$

For $l \equiv 1 \pmod 3$

$$\operatorname{depth}(U/(L,u)) = \frac{4l-1}{3},$$

$$\operatorname{depth}(U/(L:u)) = \frac{4l+2}{3},$$

so, by depth lemma we have

$$\operatorname{depth}(U/(L)) \geq \frac{4l-1}{3}.$$

For $l \equiv 2 \pmod 3$

$$\operatorname{depth}(U/(L,u)) = \frac{4l+4}{3},$$

$$\operatorname{depth}(U/(L:u)) = \frac{4l+4}{3},$$

hence by depth lemma we have

$$\operatorname{depth}(U/(L)) = \frac{4l+4}{3}.$$

Now for $n = 5$, consider the same short exact sequence. Notice that

$$(L:u) = (L(P_{2l}), 2L(P_{l-1}), L(P_{l-2}), N(u))$$

and

$$(L,u) = (L(P_{4,l}), L(P_{l-1}), u).$$

So by Lemma 2.4.4, Lemma 3.5.1 and Theorem 3.5.6, we get

$$\operatorname{depth}(U/(L:u)) = \left\lceil \frac{2l}{3} \right\rceil + 2\left\lceil \frac{l-1}{3} \right\rceil + \left\lceil \frac{l-2}{3} \right\rceil$$

and

$$\operatorname{depth}(U/(L,u)) \geq \left\lceil \frac{l-1}{3} \right\rceil + \frac{4l}{3}, \qquad \text{when } l \equiv 0 \pmod 3,$$

$$\operatorname{depth}(U/(L,u)) \geq \left\lceil \frac{l-1}{3} \right\rceil + \frac{4l}{3} - \frac{1}{3}, \qquad \text{when } l \equiv 1 \pmod 3,$$

$$\operatorname{depth}(U/(L,u)) = \left\lceil \frac{l-1}{3} \right\rceil + \frac{4l}{3} + \frac{4}{3}, \quad \text{when } l \equiv 2 \pmod 3.$$

For $l \equiv 0 \pmod 3$

$$\operatorname{depth}(U/(L,u)) = \frac{5l}{3},$$

$$\operatorname{depth}(U/(L:u)) = \frac{5l}{3} + 1,$$

hence by using depth lemma we have

$$\operatorname{depth}(U/(L)) \geq \frac{5l}{3}.$$

For $l \equiv 1 \pmod 3$

$$\operatorname{depth}(U/(L,u)) = \frac{5l}{3} - \frac{5}{3} + 1,$$

$$\operatorname{depth}(U/(L:u)) = \frac{5l+1}{3},$$

so by depth lemma we have

$$\operatorname{depth}(U/(L)) \geq \frac{5l}{3} - \frac{5}{3} + 1.$$

For $l \equiv 2 \pmod 3$

$$\operatorname{depth}(U/(L,u)) = \frac{5l+5}{3},$$

$$\operatorname{depth}(U/(L:u)) = \frac{5l+5}{3},$$

hence by depth lemma we have

$$\operatorname{depth}(U/(L)) = \frac{5l+5}{3}.$$

Assume $n \geq 6$, notice that

$$(L:u) = (L(P_{n-3,l}), 2L(P_{l-1}), L(P_{l-2}), N(u))$$

and

$$(L,u) = (L(P_{n-1,l}), L(P_{l-1}), u).$$

Case 1: $l \equiv 0 \pmod{3}$ **.**

By Lemma 2.4.4, Lemma 3.5.1 and Theorem 3.5.6,

$$\begin{aligned}
\operatorname{depth}(U/L:u) &\geq 2\left\lceil \frac{l-1}{3} \right\rceil + \left\lceil \frac{l-2}{3} \right\rceil + \frac{(n-3)l}{3} + 1 \\
&= 1 + \frac{2l}{3} + \frac{l}{3} + \frac{nl}{3} - \frac{3l}{3} \\
&= 1 + \frac{nl}{3},
\end{aligned}$$

$$\begin{aligned}
\operatorname{depth}(U/L,u) &\geq \left\lceil \frac{l-1}{3} \right\rceil + \frac{(n-1)l}{3} \\
&= \frac{l}{3} - \frac{l}{3} + \frac{nl}{3} \\
&= \frac{nl}{3}.
\end{aligned}$$

Depth lemma implies,

$$\operatorname{depth}(U/L) \geq \frac{nl}{3}, \qquad l \equiv 0 \pmod{3}.$$

Case 2: $l \equiv 1 \pmod{3}$**.**

Using Lemma 2.4.4, Lemma 3.5.1 and theorem 3.5.6,

$$\begin{aligned}
\operatorname{depth}(U/L:u) &\geq 2\left\lceil \frac{l-1}{3} \right\rceil + \left\lceil \frac{l-2}{3} \right\rceil + \frac{(n-3)l-n+6}{3} + 1 \\
&= 1 + \frac{2(l-1)}{3} + \frac{l-1}{3} + \frac{(n-3)l}{3} - \frac{(n-3)}{3} + 1 \\
&= 2 + \frac{nl}{3} - \frac{n}{3},
\end{aligned}$$

$$\begin{aligned}
\operatorname{depth}(U/L,u) &\geq \left\lceil \frac{l-1}{3} \right\rceil + \frac{(n-1)l-n+1+3}{3} \\
&= \frac{l-1}{3} - \frac{l}{3} + \frac{nl}{3} - \frac{n}{3} + \frac{1}{3} + 1 \\
&= \frac{nl}{3} - \frac{n}{3} + 1.
\end{aligned}$$

So by depth lemma,

$$\operatorname{depth}(U/L) \geq \frac{nl}{3} - \frac{n}{3} + 1, \qquad l \equiv 1 \pmod{3}.$$

Case 3: $l \equiv 2$ $(\mathbf{mod}\ 3)$.

Then Lemma 2.4.4, Lemma 3.5.1 and Theorem 3.5.6 implies,

$$\begin{aligned}\operatorname{depth}(U/L:u) &\geq 2\left\lceil\frac{l-1}{3}\right\rceil+\left\lceil\frac{l-2}{3}\right\rceil+\frac{(n-3)l+n-3}{3}+1\\ &= 1+\frac{2(l+1)}{3}+\frac{l-2}{3}+\frac{nl}{3}-\frac{3l}{3}+\frac{n}{3}-1\\ &= \frac{nl}{3}+\frac{n}{3},\end{aligned}$$

$$\begin{aligned}\operatorname{depth}(U/L,u) &\geq \left\lceil\frac{l-1}{3}\right\rceil+\frac{(n-1)l+n-1}{3}\\ &= \frac{l+1}{3}-\frac{l}{3}+\frac{nl}{3}+\frac{n}{3}-\frac{1}{3}\\ &= \frac{nl}{3}+\frac{n}{3}.\end{aligned}$$

Hence by depth lemma,

$$\operatorname{depth}(U/L) \geq \frac{nl+n}{3}, \qquad l \equiv 2 \ (\mathrm{mod}\ 3).$$

$\square$

Bound given above is improved from already given since new bound contains the term which is the product of n and l. Similarly using Lemma 3.5.2 and Lemma 2.2.2 we can compute following bound for Stanley depth on the same lines.

Proposition 3.5.10. *For $L = L(CP_{n,l})$ with $n \geq 3$ and $l \geq 3$,*

$$\operatorname{sdepth}(U/L) \geq \begin{cases} \frac{nl}{3}, & \text{when } l \equiv 0 \ (mod\ 3);\\ \frac{nl}{3}-\frac{n}{3}+1, & \text{when } l \equiv 1 \ (mod\ 3);\\ \frac{nl}{3}+\frac{n}{3}, & \text{when } l \equiv 1 \ (mod\ 3).\end{cases}$$

3.6 Depth and Stanley depth of Cycles on star graph.

In this section we will work on depth and Stanley depth of a graph $B(m,n)$ formed by linking n number of $C_m's$ graphs to a vertex each through an edge.

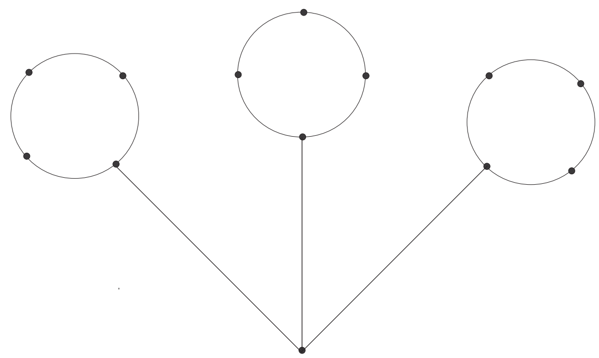


Figure 3.8: $B(4,3)$

Theorem 3.6.1. *For $n \geq 1$ and $m_i \geq 4$ for all $1 \leq i \leq n$, we have,*

1. *When $m_i \equiv 1, 2$ (mod 3) for some m_i's then,*

$$\operatorname{depth}(U/L) \geq \sum_{i=1}^{n} \left\lceil \frac{m_i - 1}{3} \right\rceil,$$

2. *When all $m_i \equiv 0$ (mod 3) for all m_i's then,*

$$\operatorname{depth}(U/L) \geq \sum_{i=1}^{n} \left\lceil \frac{m_i}{3} \right\rceil.$$

Proof. Let $W = \{x_1, x_2, \ldots, x_n\}$ be the vertices each in $C'_{m_i}s$ which are linked with vertex u through an edge. Consider the short exact sequence:

$$0 \longrightarrow U/(L:u) \longrightarrow U/L \longrightarrow U/(L,u) \longrightarrow 0.$$

By depth lemma we have,

$$\operatorname{depth} U/L \geq \min\{\operatorname{depth} U/(L:u), \operatorname{depth} U/(L,u)\}.$$

Notice that,

$$(L:u) = (x_1, x_2, \ldots, x_n, L(P_{m_1-1}), L(P_{m_2-1}), \ldots, L(P_{m_n-1}),$$

so, by using [11, Lemma 2.8] we have,

$$\operatorname{depth} U/(L:u) = 1 + \left\lceil \frac{m_1 - 1}{3} \right\rceil + \left\lceil \frac{m_2 - 1}{3} \right\rceil + \cdots + \left\lceil \frac{m_n - 1}{3} \right\rceil.$$

Now for (L, u) we see that,

$$(L, u) = (u, L(C_{m_1}), L(C_{m_2}), \ldots, L(C_{m_n})).$$

By using [6, Proposition 1.3] we have,

$$\operatorname{depth} U/(L, u) = \left\lceil \frac{m_1 - 1}{3} \right\rceil + \left\lceil \frac{m_2 - 1}{3} \right\rceil + \cdots + \left\lceil \frac{m_n - 1}{3} \right\rceil.$$

Thus using depth lemma we have

$$\operatorname{depth} U/L \geq \left\lceil \frac{m_1 - 1}{3} \right\rceil + \left\lceil \frac{m_2 - 1}{3} \right\rceil + \cdots + \left\lceil \frac{m_n - 1}{3} \right\rceil,$$

that is

$$\operatorname{depth} U/L \geq \sum_{i=1}^{n} \left\lceil \frac{m_i - 1}{3} \right\rceil.$$

Note that when all $m_i's$ are multiple of 3 then

$$\operatorname{depth} U/L = \left\lceil \frac{m_i}{3} \right\rceil.$$

□

For Stanley depth we have,

Theorem 3.6.2. *For $n \geq 1$ and $m_i \geq 4$ for all $1 \leq i \leq n$, we have,*

1. *When $m_i \equiv 1, 2 \pmod 3$ for some $m_i's$ then,*

$$\operatorname{sdepth}(U/L) \geq \sum_{i=1}^{n} \left\lceil \frac{m_i - 1}{3} \right\rceil,$$

2. *When all $m_i \equiv 0 \pmod 3$ for all $m_i's$ then,*

$$\operatorname{sdepth}(U/L) \geq \sum_{i=1}^{n} \left\lceil \frac{m_i}{3} \right\rceil.$$

Proof. By using proposition [6, Proposition 1.8] and [1, Lemma 2.4] we get the required result on the same lines. □

Chapter 4

Depth and Stanley depth of power of edge ideal associated to caterpillar graph.

Louiza Fouli and Susan Morey in [7] gave lower bounds for the powers of edge ideal that are not so sharp. The bounds in their paper are given in terms of diameter l, number of components q of the graph and the power p of edge ideals. Higher the power lower the bound is. We will focus our work on finding the bounds that are sharper then these already present. In this chapter we have determined the bounds which are sharper then previous one. These new bounds are in terms of total number of variables and number of components q.

Definition 4.0.3. The graph $S_{n,k}$ is a caterpillar graph with $k-1$ number of pendant vertices at each vertex of path P_n.

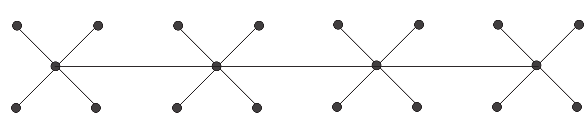


Figure 4.1: $S_{4,5}$

The bound which was given by Louiza Fouli and Morey for any graph H and power p is,

Proposition 4.0.4. *[7, Proposition 4.3] Let H be a graph having q connected components and let $L = L(H)$. Then, for every $p \geq 1$,*

$$\operatorname{depth} U/L^p \geq q - p.$$

We will work on lower bounds for depth and Stanley depth of cyclic module U/L^p such that the new bound is sharper then above.

Lemma 4.0.5. *[11, Lemma 2.6] Let $L = L(H)$ for H a bipartite graph. Then for all $p \geq 1$,*

$$\operatorname{depth}(U/L^p) \geq 1.$$

Similar result for the Stanley depth we have,

Theorem 4.0.6. *[5, Theorem 1.4] For a finitely generated $\mathbb{Z}^n$-graded U-module N, if* $\operatorname{sdepth}(N) = 0$ *then* $\operatorname{depth}(N) = 0$.

Definition 4.0.7. The graph denoted by $S_{n,k}^{(l)}$ is a subclass of caterpillar graph with first $n-1$ vertices of the path P_n having $k-1$ pendant vertices and last vertex of path P_n having $l-1$ pendant vertices.

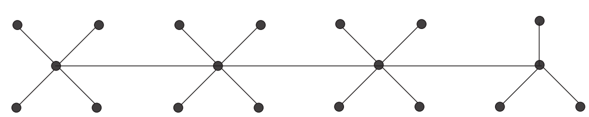
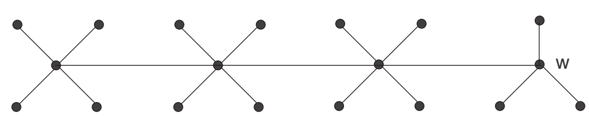


Figure 4.2: $S_{4,5}^{(4)}$

Lemma 4.0.8. *Let $L = L(S_{n,k}^{(l)})$ and w be the end vertex of path P_n in $S_{n,k}^{(l)}$ as shown in the following figure,*



then for $p \geq 1$,

$$(L^p, w) = (L^p(S_{n-1,k}), w).$$

Proof. When w is introduced as a single variable in L, for $p = 1$, all the edges containing w are deleted. As w is last vertex of the path P_n, all the edges incident to it are removed. Hence we are left with a graph consisting of caterpillar on path P_{n-1} with each of its vertex having $k-1$ pendant vertices denoted by $S_{n-1,k}$ and an isolated vertex w. Same logic applies for $p \geq 1$. $\square$

Lemma 4.0.9. *Let $L = L(S_{2,k}^{(l)})$, for $k \geq 3$ and $1 \leq l \leq k$,*

$$\operatorname{depth}(U/L) \geq l.$$

Proof. Let u and v be the vertices of path P_2 with u having $k-1$ and v having $l-1$ pendent vertices respectively. Consider the short exact sequence:

$$0 \longrightarrow U/(L:u) \longrightarrow U/L \longrightarrow U/(L,u) \longrightarrow 0.$$

Then

$$\begin{aligned} (L:u) &= (N(u)) \\ \operatorname{depth}(U/(L:u)) &= 1 + (l-1) = l \end{aligned}$$

and

$$\begin{aligned} (L,u) &= (L(S_l),u) \\ \operatorname{depth}(U/(L,u)) &= 1 + (k-1) = k. \end{aligned}$$

By Depth Lemma,

$$\operatorname{depth}(U/L) \geq \min\{\operatorname{depth}(U/(L:u)), \operatorname{depth}(U/(L,u))\}$$

$$\operatorname{depth}(U/L) \geq l.$$

$\square$

The above bound hold for Stanley depth also.

Theorem 4.0.10. *Let $L = L(S^{(l)}_{n,k})$, for $n \geq 3$, $l \leq k$ and $k \geq 3$,*

$$\operatorname{depth}(U/L) \geq \begin{cases} \left(\frac{n-2}{2}\right)k + l\,, & \textit{when } \; n \textit{ is even;} \\ \left(\frac{n-1}{2}\right)k + 1\,, & \textit{when } \; n \textit{ is odd.} \end{cases}$$

Proof. Let u be the last vertex of path P_n having $l-1$ pendant vertices. Notice that

$$(L : u) = (L(S_{n-2,k}), N(u)),$$

and

$$(L, u) = (L(S_{n-1,k}), u).$$

Case 1: When n is even.

By induction and Lemma 2.4.4,

$$\begin{aligned} \operatorname{depth}(U/(L : u)) &= \left(\frac{n-2}{2}\right)k + (k-1) + 1 \\ &= \left(\frac{n}{2}\right)k, \end{aligned}$$

and

$$\begin{aligned} \operatorname{depth}(U/(L, u)) &= \left(\frac{n-2}{2}\right)k + 1 + (l-1) \\ &= \left(\frac{n-2}{2}\right)k + l. \end{aligned}$$

Then by Depth Lemma,

$$\operatorname{depth}(U/L) \geq \left(\frac{n-2}{2}\right)k + l.$$

Case 2: When n is odd.

By induction and Lemma 2.4.4,

$$\begin{aligned} \operatorname{depth}(U/(L : u)) &= \left(\frac{n-3}{2}\right)k + 1 + (k-1) + 1 \\ &= \left(\frac{n-1}{2}\right)k + 1, \end{aligned}$$

and

$$\operatorname{depth}(U/(L, u)) = \left(\frac{n-1}{2}\right)k + (l-1),$$

Then by Depth Lemma,

$$\operatorname{depth}(U/L) \geq \Big(\frac{n-1}{2}\Big)k.$$

$\square$

Now using the results of Stanley depth we have

Theorem 4.0.11. *Let $L = L(S_{n,k}^{(l)})$, for $n \geq 3$, $l \leq k$ and $k \geq 3$,*

$$\operatorname{sdepth}(U/L) \geq \begin{cases} \Big(\frac{n-2}{2}\Big)k + l\,, & \text{when } n \text{ is even;} \\ \Big(\frac{n-1}{2}\Big)k + 1\,, & \text{when } n \text{ is odd.} \end{cases}$$

Now we will give the bounds for the depth and Stanley depth of edge ideal associated to a graph $CS_{n,k}^{(l)}$ formed as a result of adding an edge in $S_{n,k}^{(l)}$ linking the first vertex to the last vertex of path P_n.

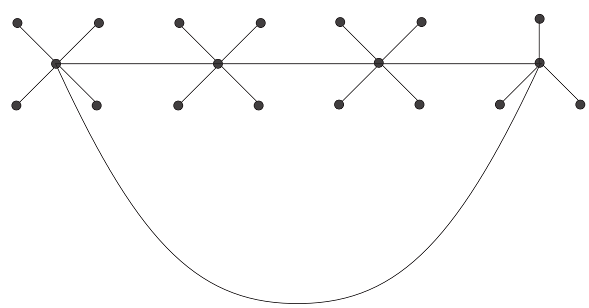


Figure 4.3: $CS_{4,5}^{(4)}$

Theorem 4.0.12. *Let $L = L(CS_{n,k}^{(l)})$, for $n \geq 3$, $l \geq 1$ and $k \geq 3$, then*

1. *when n is even,*

$$\operatorname{depth}(U/L) \geq \Big(\frac{n-2}{2}\Big)k + l,$$

2. *when n is odd,*

$$\operatorname{depth}(U/L) \geq \Big(\frac{n-1}{2}\Big)k + l - 1.$$

Proof. First we will prove for $n = 3$. Let u be the vertex of of path P_n having $l-1$ pendent vertices. Notice that

$$(L : u) = (N(u))$$

thus we have

$$\begin{aligned}\operatorname{depth} U/(L : u) &= (k-1) + (k-1) + 1 \\ &= 2k - 1.\end{aligned}$$

Now we see that

$$(L, u) = (u, J(S_{2,k}))$$

thus

$$\begin{aligned}\operatorname{depth} U/(L, u) &= (l-1) + k \\ &= k + l - 1.\end{aligned}$$

So by Depth lemma we have

$$\operatorname{depth} U/L \geq k + l - 1.$$

Assume $n \geq 4$ Let u be the non-leaf of S_l. Notice that

$$(L : u) = (L(S_{n-3,k}), N(u))$$

and

$$(L, u) = (L(S_{n-1,k}), u).$$

Case 1: When n is even.

By Theorem 4.0.10 and Lemma 2.4.4,

$$\begin{aligned}\operatorname{depth}(U/(L : u)) &= \Big(\frac{n-4}{2}\Big)k + 1 + (k-1) + (k-1) + 1 \\ &= \Big(\frac{n}{2}\Big)k,\end{aligned}$$

and

$$\begin{aligned}\operatorname{depth}(U/(L, u)) &= \Big(\frac{n-2}{2}\Big)k + 1 + (l-1) \\ &= \Big(\frac{n-2}{2}\Big)k + l.\end{aligned}$$

Then by Depth Lemma,

$$\operatorname{depth}(U/L) \geq \Big(\frac{n-2}{2}\Big)k + l.$$

Case 2: When n is odd.

By Theorem 4.0.10 and Lemma 2.4.4,

$$\begin{aligned}\operatorname{depth}(U/(L:u)) &= \Big(\frac{n-3}{2}\Big)k + 1 + (k-1) + (k-1) \\ &= \Big(\frac{n+1}{2}\Big)k - 1,\end{aligned}$$

and

$$\operatorname{depth}(U/(L,u)) = \Big(\frac{n-1}{2}\Big)k + (l-1).$$

Then by Depth Lemma,

$$\operatorname{depth}(U/L) \geq \Big(\frac{n-1}{2}\Big)k + l - 1.$$

□

Bounds given in above theorem are sharper than that given by Morey in [7] since this new bound depends on k. Higher the value of k, higher the bound is. Similar result holds for the Stanley depth of the edge ideal associated to this graph.

Theorem 4.0.13. *Let $L = L(CS^{(l)}_{n,k})$, for $n \geq 3$, $l \geq 1$ and $k \geq 3$, then*

1. *when n is even,*
$$\operatorname{sdepth}(U/L) \geq \Big(\frac{n-2}{2}\Big)k + l,$$
2. *when n is odd,*
$$\operatorname{sdepth}(U/L) \geq \Big(\frac{n-1}{2}\Big)k + l - 1.$$

Proof. Using Theorem 4.0.11 and Lemma 2.4.5 we get the desired result. □

For the case when $p = 2$ and $l = k$ we get the following range.

4.0.1 Depth and Stanley depth of power of edge ideal associated to $S^{(l)}_{n,k}$-caterpillar graph.

Theorem 4.0.14. *Let $L = L(S^{(l)}_{n,k})$, for $n \geq 2$, $k \geq 3$,$1 \leq l \leq k$ and $p \geq 1$,*

$$\operatorname{depth}(U/L^p) \geq \begin{cases} \max\{1, \left(\frac{n-p-1}{2}\right)k + l - 1\}, & \textit{when } n \textit{ and } p \textit{ have opposite parity;} \\ \max\{1, \left(\frac{n-p}{2}\right)k + 1\}, & \textit{when } n \textit{ and } p \textit{ have same parity and } 2 < l \leq k; \\ \max\{1, \left(\frac{n-p}{2}\right)k - 1\}, & \textit{when } n \textit{ and } p \textit{ have same parity } l \in \{1,2\}. \end{cases}$$

Proof. Let x, y be the end point of the path realizing the diameter of graph. Let $u \in N(y)$. Let $N(u) = \{s_1, s_2, \ldots, s_{l-1}, s_l\}$. Since $S^{(l)}_{n,k}$ is a bipartite graph so $\operatorname{depth}(U/L^p) \geq 1$ for all p by Lemma 4.0.5.

The result will be proved by induction on n and p. For $n = 2$ and $p \geq 1$ the result is true by Lemma 4.0.5. For $n \geq 3$ and $p = 1$ the result follows from Theorem 4.0.10. Now assume that $n \geq 3$ and $p \geq 2$.

Case 1: When n and p have same parity.

In this case we have three subcases. first one is when $l = 1$, then consider a short exact sequence:

$$0 \longrightarrow U/(L^p : y) \longrightarrow U/L^p \longrightarrow U/(L^p, y) \longrightarrow 0.$$

By Depth lemma,

$$\operatorname{depth}(U/L^p) \geq \min\{\operatorname{depth}(U/(L^p : y)), \operatorname{depth}(U/(L^p, y))\}.$$

By Lemma 4.0.8

$$(L^p, y) = (y, L^p(S_{n-1,k})).$$

Here $n - 1$ and p have the opposite parity, so by induction on n,

$$\begin{aligned} \operatorname{depth}(U/L^p, y) &\geq \max\{1, \left(\frac{n-p-2}{2}\right)k + k - 1\} \\ &= \max\{1, \left(\frac{n-p}{2}\right)k - 1\}. \end{aligned}$$

For the depth of module $U/(L^p : y)$ we will consider another short exact sequence as follows:

$$0 \longrightarrow U/(L^p : yw) \longrightarrow U/L^p : y \longrightarrow U/((L^p : y), w) \longrightarrow 0.$$

Now by using lemma 2.5.11 we have,

$$\operatorname{depth}(U/L^p : yw) \;\geq\; \max\{1, \Big(\frac{n-p}{2}\Big)k\}$$

and

$$\begin{aligned}\operatorname{depth}(U/((L^p : y), w) &\geq 1 + (k-1) + \max\{1, \Big(\frac{n-p-2}{2}\Big)k + 1\} \\ &= k + \max\{1, \Big(\frac{n-p}{2}\Big)k - k + 1\} \\ &= \max\{1+k, \Big(\frac{n-p}{2}\Big)k + 1\}.\end{aligned}$$

Thus by depth lemma we have,

$$\operatorname{depth} U/L^p \geq \max\{1, \Big(\frac{n-p}{2}\Big)k - 1\}.$$

Now the second subcase is when $l = 2$. For this consider the short exact sequence:

$$0 \longrightarrow U/(L^p : w) \longrightarrow U/L^p \longrightarrow U/((L^p, w) \longrightarrow 0.$$

By Depth lemma,

$$\operatorname{depth}(U/L^p) \geq \min\{\operatorname{depth}(U/(L^p : w)), \operatorname{depth}(U/(L^p, w))\}.$$

By Lemma 4.0.8

$$(L^p, w) = (w, L^p(S_{n-1,k})).$$

Here $n-1$ and p have the opposite parity, so by induction on n,

$$\begin{aligned}\operatorname{depth}(U/L^p, w) &\geq 1 + \max\{1, \Big(\frac{n-p-2}{2}\Big)k + k - 1\} \\ &> \max\{1, \Big(\frac{n-p}{2}\Big)k - 1\}.\end{aligned}$$

For the depth of module $U/(L^p : w)$ we will consider another short exact sequence as follows:

$$0 \longrightarrow U/(L^p : wy) \longrightarrow U/L^p : w \longrightarrow U/((L^p : w), y) \longrightarrow 0.$$

Now by using lemma 2.5.11 we have,

$$\operatorname{depth}(U/L^p : yw) \;\geq\; \max\{1, \Big(\frac{n-p}{2}\Big)k + 1\}.$$

Now for depth of $((L^p : w), y)$ we will consider the following short exact sequence where x is the $n-1^{th}$ vertex of the path P_n.

$$0 \longrightarrow U_y/(L_y^p : wx) \longrightarrow U_y/L_y^p : w \longrightarrow U_y/((L_y^p : w), x) \longrightarrow 0.$$

$$\operatorname{depth}(U_y/((L_y^p : wx) \geq \max\{1, \Big(\frac{n-p+1}{2}\Big)k - 1\}$$

and

$$\begin{aligned}\operatorname{depth}(U_y/((L_y^p : w), x) &\geq 1 + (k-1) + \max\{1, \Big(\frac{n-p-2}{2}\Big)k + 1\} \\ &= k + \max\{1, \Big(\frac{n-p}{2}\Big)k - k + 1\} \\ &= \max\{1+k, \Big(\frac{n-p}{2}\Big)k + 1\}.\end{aligned}$$

Hence by depth lemma we have,

$$\operatorname{depth} U/L^p \geq \max\{1, \Big(\frac{n-p}{2}\Big)k - 1\}.$$

Now the third subcase is when $l \geq 3$, for this consider a short exact sequence:

$$0 \longrightarrow U/(L^p : u) \longrightarrow U/L^p \longrightarrow U/(L^p, u) \longrightarrow 0.$$

By Depth lemma,

$$\operatorname{depth}(U/L^p) \geq \min\{\operatorname{depth}(U/(L^p : u)), \operatorname{depth}(U/(L^p, u))\}.$$

By Lemma 4.0.8

$$(L^p, u) = (L^p(S_{n-1,k}), u),$$

here $n-1$ and p have opposite parity so, by using induction on n,

$$\begin{aligned}\operatorname{depth}(U/L^p, u) &= \operatorname{depth}(U''/L^p(S_{n-1,k})) + |\mathcal{L}(S_l)| \\ &\geq \Big(\frac{(n-1)-p-1}{2}\Big)k + k - 1 + l - 1 \\ &= \Big(\frac{n-p}{2}\Big)k + l - 2,\end{aligned}$$

where $U'' = U\backslash(\mathcal{L}(S_l) \cup \{u\})$.

Now for the depth of $U/(L^p : u)$ we will use Lemma 2.5.13. For this consider a short exact sequence:

$$0 \longrightarrow U/(L^p : us_1) \longrightarrow U/(L^p : u) \longrightarrow U/((L^p : u), s_1) \longrightarrow 0.$$

Then by Lemma 2.5.11

$$\operatorname{depth}(U/L^p : us_1) = \operatorname{depth}(U/L^{p-1}).$$

Here n and $p-1$ have the opposite parity so, by using induction on p,

$$\begin{aligned}
\operatorname{depth}(U/L^p : us_1) &\geq \Big(\frac{n-(p-1)-1}{2}\Big)k + l - 1 \\
&= \Big(\frac{n-p}{2}\Big)k + l - 1 \\
&> \Big(\frac{n-p}{2}\Big)k + 1.
\end{aligned}$$

For depth of $U/((L^p : u), s_1)$, consider another short exact sequence:

$$0 \longrightarrow U_1/(L_1^p : us_2) \longrightarrow U_1/(L_1^p : u) \longrightarrow U_1/((L_1^p : u), s_2) \longrightarrow 0.$$

Lemma 2.5.11 implies

$$\begin{aligned}
\operatorname{depth}(U_1/L_1^p : us_2) &\geq \Big(\frac{n-(p-1)-1}{2}\Big)k + l - 1 \\
&= \Big(\frac{n-p}{2}\Big)k + l - 1 \\
&\geq \Big(\frac{n-p}{2}\Big)k + 1.
\end{aligned}$$

Similarly, for the sequence:

$$0 \longrightarrow U_{l-1}/(L_{l-1}^p : us_l) \longrightarrow U_{l-1}/(L_{l-1}^p : u) \longrightarrow U_{l-1}/((L_{l-1}^p : u), s_l) \longrightarrow 0$$

$$\operatorname{depth}(U_{l-1}/L_{l-1}^p : us_l) = \operatorname{depth}(U_{l-1}/L_{l-1}^{p-1}).$$

By using induction on p,

$$\begin{aligned}
\operatorname{depth}(U_{l-1}/L_{l-1}^p : us_l) &\geq \Big(\frac{n-(p-1)-1}{2}\Big)k + l - 1 \\
&= \Big(\frac{n-p}{2}\Big)k + l - 1 \\
&\geq \Big(\frac{n-p}{2}\Big)k + 1.
\end{aligned}$$

Notice that

$$\operatorname{depth}(U_l/L_l^p : u) = \operatorname{depth}(U_l/L^p(S_{n-2,k})) + |\mathcal{L}(S_k)|+1.$$

Since $n-2$ and p have the same parity, so by induction on n,

$$\begin{aligned}\operatorname{depth}(U_l/L_l^p : u) &\geq \Big(\frac{(n-2)-p}{2}\Big)k+1+(k-1)+1\\ &= \Big(\frac{n-p}{2}\Big)k+1.\end{aligned}$$

So by using Lemma 2.4.4, we get

$$\operatorname{depth}(U/L^p : u) \geq \Big(\frac{n-p}{2}\Big)k+1.$$

Depth lemma implies,

$$\operatorname{depth}(U/L^p) \geq \Big(\frac{n-p}{2}\Big)k+1.$$

Case 2: When n and p have the opposite parity. In this case we also have three more sub-cases. First one is when $l=1$, , then consider a short exact sequence:

$$0 \longrightarrow\ U/(L^p : y) \longrightarrow\ U/L^p \longrightarrow\ U/(L^p, y) \longrightarrow\ 0.$$

By Depth lemma,

$$\operatorname{depth}(U/L^p) \geq \min\{\operatorname{depth}(U/(L^p : y)), \operatorname{depth}(U/(L^p, y))\}.$$

then by Lemma 4.0.8

$$(L^p, y) = (y, L^p(S_{n-1,k})).$$

Here $n-1$ and p have the same parity, so by induction on n,

$$\begin{aligned}\operatorname{depth}(U/L^p, y) &\geq \max\{1, \Big(\frac{n-p-1}{2}\Big)k+1\}\\ &= \max\{1, \Big(\frac{n-p-1}{2}\Big)k+1\}.\end{aligned}$$

For the depth of module $U/(L^p : y)$ we will consider another short exact sequence as follows:

$$0 \longrightarrow\ U/(L^p : yw) \longrightarrow\ U/L^p : y \longrightarrow\ U/((L^p : y), w) \longrightarrow\ 0.$$

Now by using lemma 2.5.11 we have,

$$\operatorname{depth}(U/L^p : yw) \;\geq\; \max\{1, \Big(\frac{n-p+1}{2}\Big)k - 1\}$$

and

$$\begin{aligned}\operatorname{depth}(U/((L^p : y), w) &\geq 1 + (k-1) + \max\{1, \Big(\frac{n-p-3}{2}\Big)k + k - 1\}\\ &= k + \max\{1, \Big(\frac{n-p-1}{2}\Big)k - 1\}\\ &= \max\{1+k, \Big(\frac{n-p-1}{2}\Big)k + k - 1\}.\end{aligned}$$

Thus by depth lemma we have,

$$\operatorname{depth} U/L^p \geq \max\{1, \Big(\frac{n-p}{2}\Big)k - 1\}.$$

Now the second sub-case is when $l = 2$. For this consider the short exact sequence:

$$0 \longrightarrow U/(L^p : w) \longrightarrow U/L^p \longrightarrow U/((L^p, w) \longrightarrow 0.$$

By Depth lemma,

$$\operatorname{depth}(U/L^p) \geq \min\{\operatorname{depth}(U/(L^p : w)), \operatorname{depth}(U/(L^p, w))\}.$$

then by Lemma 4.0.8

$$(L^p, w) = (w, L^p(S_{n-1,k})).$$

Here $n-1$ and p have the same parity, so by induction on n,

$$\begin{aligned}\operatorname{depth}(U/L^p, w) &\geq 1 + \max\{1, \Big(\frac{n-p-1}{2}\Big)k + 1\}\\ &> \max\{1, \Big(\frac{n-p-1}{2}\Big)k - 1\}.\end{aligned}$$

For the depth of module $U/(L^p : w)$ we will consider another short exact sequence as follows:

$$0 \longrightarrow U/(L^p : wy) \longrightarrow U/L^p : w \longrightarrow U/((L^p : w), y) \longrightarrow 0.$$

Now by using lemma 2.5.11 we have,

$$\operatorname{depth}(U/L^p : yw) \;>\; \max\{1, \Big(\frac{n-p-1}{2}\Big)k + 1\}.$$

Now for depth of $((L^p : w), y)$ we will consider the following short exact sequence where x is the $n-1^{th}$ vertex of the path P_n.

$$0 \longrightarrow U_y/(L_y^p : wx) \longrightarrow U_y/L_y^p : w \longrightarrow U_y/((L_y^p : w), x) \longrightarrow 0.$$

$$\operatorname{depth}(U_y/((L_y^p : wx) \geq \max\{1, \Big(\frac{n-p+1}{2}\Big)k+1\}$$

and

$$\begin{aligned}
\operatorname{depth}(U_y/((L_y^p : w), x) &\geq 1+(k-1)+\max\{1, \Big(\frac{n-p-3}{2}\Big)k+2-1\} \\
&= k+\max\{1, \Big(\frac{n-p-1}{2}\Big)k-k+1\} \\
&\geq \max\{1, \Big(\frac{n-p-1}{2}\Big)k+1\}.
\end{aligned}$$

Hence by depth lemma we have,

$$\operatorname{depth} U/L^p \geq \max\{1, \Big(\frac{n-p-1}{2}\Big)k-1\}.$$

Now the third subcase is when $l \geq 3$, for this consider a short exact sequence:

$$0 \longrightarrow U/(L^p : u) \longrightarrow U/L^p \longrightarrow U/(L^p, u) \longrightarrow 0.$$

By Depth lemma,

$$\operatorname{depth}(U/L^p) \geq \min\{\operatorname{depth}(U/(L^p : u)), \operatorname{depth}(U/(L^p, u))\}.$$

By Lemma 4.0.8

$$(L^p, u) = (L^p(S_{n-1,k}), u),$$

Here $n-1$ and p have the same parity, so by induction on n,

$$\begin{aligned}
\operatorname{depth}(U/L^p, u) &= \operatorname{depth} U'/L^p(S_{n-1,k})) + |\mathcal{L}(S_l)| \\
&\geq \Big(\frac{(n-1)-p}{2}\Big)k+1+l-1 \\
&> \Big(\frac{n-p-1}{2}\Big)k+l-1,
\end{aligned}$$

where $U' = U \backslash (\mathcal{L}(S_l) \cup \{u\})$.

Now for depth of $U/(L^p : u)$ we will use Lemma 2.5.13. For this consider a short exact sequence:

$$0 \longrightarrow U/(L^p : us_1) \longrightarrow U/(L^p : u) \longrightarrow U/((L^p : u), s_1) \longrightarrow 0.$$

Then by Lemma 2.5.11 and induction on p ,

$$\begin{aligned} \operatorname{depth}(U/L^p : us_1) &= \operatorname{depth}(U/L^{p-1}) \\ &\geq \Big(\frac{n-(p-1)}{2}\Big)k+1 \\ &= \Big(\frac{n-p-1}{2}\Big)k+k+1 \\ &> \Big(\frac{n-p-1}{2}\Big)k+l-1, \end{aligned}$$

where n and $p-1$ are having the same parity. Now for depth of $U/((L^p : u), s_1)$, consider another short exact sequence:

$$0 \longrightarrow U_1/(L_1^p : us_2) \longrightarrow U_1/(L_1^p : u) \longrightarrow U_1/((L_1^p : u), s_2) \longrightarrow 0.$$

Again by Lemma 2.5.11 and induction on p ,

$$\begin{aligned} \operatorname{depth}(U_1/L_1^p : us_2) &= \operatorname{depth}(U_1/L_1^{p-1}) \\ &\geq \Big(\frac{n-(p-1)}{2}\Big)k+1 \\ &= \Big(\frac{n-p-1}{2}\Big)k+k+1 \\ &> \Big(\frac{n-p-1}{2}\Big)k+l-1. \end{aligned}$$

Proceeding in the same manner we reach at the following sequence:

$$0 \longrightarrow U_{l-1}/(L_{l-1}^p : us_l) \longrightarrow U_{l-1}/(L_{l-1}^p : u) \longrightarrow U_{l-1}/((L_{l-1}^p : u), s_l) \longrightarrow 0.$$

Clearly,

$$\begin{aligned} \operatorname{depth}(U_{l-1}/L_{l-1}^p : us_l) &= \operatorname{depth}(U_{l-1}/L_{l-1}^{p-1}) \\ &\geq \Big(\frac{n-(p-1)}{2}\Big)k-1 \\ &= \Big(\frac{n-p-1}{2}\Big)k+k-1 \\ &\geq \Big(\frac{n-p-1}{2}\Big)k+l-1. \end{aligned}$$

Notice that

$$\operatorname{depth}(U_l/L_l^p : u) = \operatorname{depth}(U_l/L^p(S_{n-2,k})) + |\mathcal{L}(S_k)|+1.$$

Since $n-2$ and p have opposite parity, so by induction on n,

$$\begin{aligned}\operatorname{depth}(U_l/L_l^p : u) &\geq \Big(\frac{(n-2)-p-1}{2}\Big)k + k - 1 + (k-1) + 1 \\ &= \Big(\frac{n-p-1}{2}\Big)k + k - 1 \\ &\geq \Big(\frac{n-p-1}{2}\Big)k + l - 1.\end{aligned}$$

So by using Lemma 2.5.13 we have

$$\operatorname{depth}(U/L^p : u) > \Big(\frac{n-p-1}{2}\Big)k + l - 1,$$

and hence by Depth Lemma

$$\operatorname{depth}(U/L^p) > \Big(\frac{n-p-1}{2}\Big)k + l - 1.$$

□

So this new bound having a term as a multiple of k is much sharper than the one proposed by Morey in [11, Theorem 4.4], and for general power p as in [7, Proposition 4.3]. Similarly for the Stanley depth by using Lemma 4.0.6 and Lemma 2.5.14 we have,

Corollary 4.0.15. *Let $L = L(S_{n,k}^{(l)})$, for $n \geq 2$, $k \geq 3$,$1 \leq l \leq k$ and $p \geq 1$,*

$$\operatorname{sdepth}(U/L^p) \geq \begin{cases} \max\{1, \big(\frac{n-p-1}{2}\big)k + l - 1\}, & \textit{when } n \textit{ and } p \textit{ have opposite parity;} \\ \max\{1, \big(\frac{n-p}{2}\big)k + 1\}, & \textit{when } n \textit{ and } p \textit{ have same parity and } 2 < l \leq k; \\ \max\{1, \big(\frac{n-p}{2}\big)k - 1\}, & \textit{when } n \textit{ and } p \textit{ have same parity } l \in \{1,2\}. \end{cases}$$

Proof. Working on the lines as in above theorem and using Lemma 2.4.5, Lemma 2.5.14 and theorem 4.0.6, we can get the required bound. □

Corollary 4.0.16. *For $n \geq 2$, $k \geq 3$, $p \geq 1$ and $l = k$,*

$$\operatorname{depth}(U/L^p) \geq \begin{cases} \max\{1, \big(\frac{n-p+1}{2}\big)k - 1\}, & \textit{when } n \textit{ and } p \textit{ have opposite parity;} \\ \max\{1, \big(\frac{n-p}{2}\big)k + 1\}, & \textit{when } n \textit{ and } p \textit{ have same parity and } 2 < l \leq k. \end{cases}$$

On the same lines for Stanley depth we have,

Corollary 4.0.17. *For $n \geq 2$, $k \geq 3$, $p \geq 1$ and $l = k$,*

$$\operatorname{sdepth}(U/L^p) \geq \begin{cases} \max\{1, \left(\frac{n-p+1}{2}\right)k - 1\}, & \textit{when } n \textit{ and } p \textit{ have opposite parity;} \\ \max\{1, \left(\frac{n-p}{2}\right)k + 1\}, & \textit{when } n \textit{ and } p \textit{ have same parity and } 2 < l \leq k. \end{cases}$$

Proposition 4.0.18. *Let $L = L(S_{n,k})$, for $n \geq 2$ and $k \geq 3$, then*

1. *when n is even,*
$$\left(\frac{n-2}{2}\right)k + 1 \leq \operatorname{depth}(U/L^2) \leq \left(\frac{n}{2}\right)k,$$

2. *when n is odd,*
$$\left(\frac{n-1}{2}\right)k - 1 \leq \operatorname{depth}(U/L^2) \leq \left(\frac{n+1}{2}\right)k - 1.$$

Proof. Let u, v be the end points of the path realizing the diameter and x, y be their unique neighbors, respectively then by Lemma 2.5.12, $(L^2 : uv) = (L, xy)$. Hence the result follows from Proposition 2.4.1, Corollary 4.0.16 and Theorem 4.0.12. $\square$

Bibliography

1. Alipour, A.and Tehranian, A. (2017). Depth and Stanley Depth of Edge Ideals of Star Graphs. *International Journal of applied mathematics and statistics*, 56(4), 63-69.

2. Atiyah, M. (2018). *Introduction to commutative algebra*. CRC Press.

3. Chen, J., Morey, S., and Sung, A. (2002). The stable set of associated primes of the ideal of a graph. *The Rocky Mountain journal of mathematics*, 71-89.

4. Cimpoeas, M. (2012). Several inequalities regarding Stanley depth. *Romanian Journal of Math. and Computer Science*, 2(1), 28-40.

5. Cimpoeas, M. (2008). Some remarks on the Stanley's depth for multigraded modules. *arXiv preprint arXiv:0808.2657*.

6. Cimpoeas, M. (2014). On the Stanley depth of edge ideals of line and cyclic graphs. *arXiv preprint arXiv:1411.0624*.

7. Fouli, L., and Morey, S. (2015). A lower bound for depths of powers of edge ideals. *Journal of Algebraic Combinatorics, 42*(3), 829-848.

8. Herzog, J., and Hibi, T. (2011). Monomial ideals. In *Monomial Ideals* (pp. 3-22). Springer, London.

9. Herzog, J., Vladoiu, M., and Zheng, X. (2009). How to compute the Stanley depth of a monomial ideal. *Journal of Algebra, 322*(9), 3151-3169.

10. Herzog, J. (2013). A survey on Stanley depth. In *Monomial ideals, computations and applications* (pp. 3-45). Springer, Berlin, Heidelberg.

11. Morey, S. (2010). Depths of powers of the edge ideal of a tree. *Communications in Algebra, 38*(11), 4042-4055.

12. Ortiz, C., and Villanueva, M. (2012). Maximal independent sets in caterpillar graphs. *Discrete Applied Mathematics, 160*(3), 259-266.

13. Popescu, D. (2009). An inequality between depth and Stanley depth. *Bulletin mathématique de la Société des Sciences Mathématiques de Roumanie*, 377-382.

14. Pournaki, M., Seyed Fakhari, S. A., and Yassemi, S. (2013). Stanley depth of powers of the edge ideal of a forest. *Proceedings of the American Mathematical Society, 141*(10), 3327-3336.

15. Rauf, A. (2010). Depth and Stanley depth of multigraded modules. *Communications in Algebra, 38*(2), 773-784.

16. Rinaldo, G. (2008). An algorithm to compute the Stanley depth of monomial ideals. *Le Matematiche, 63*(2), 243-256.

17. Stanley, R. P. (1982). Linear Diophantine equations and local cohomology. *Inventiones mathematicae, 68*(2), 175-193.

18. Ştefan, A. (2014). Stanley depth of powers of the path ideal. *arXiv preprint arXiv:1409.6072*.

19. Villarreal, R. H. Monomial Algebras (Monographs and Textbooks in Pure and Applied Mathematics 238, Marcel Dekker, New York, 2001). *Carlos RENTERÍA MÁRQUEZ, Departamento de Matemáticas, Escuela Superior de Fısica y Matemáticas, Instituto Politécnico Nacional, 7300.*