Gravitational lensing for the Schwarzschild-de Sitter Metric

by

Wajeeha Pervaiz

A dissertation submitted in partial fulfillment of the requirements

for the degree of Master of Philosophy in Mathematics

Supervised by

Prof. Asghar Qadir

School of Natural Sciences

National University of Sciences and Technology

Islamabad, Pakistan

FORM TH-4

National University of Sciences & Technology

M.Phil THESIS WORK

We hereby recommend that the dissertation prepared under our supervision by: Wajeeha Pervaiz, Regn No. NUST201361936MSNS78013F Titled: Gravitational Lensing for Schwarzschild-de Sitter Metric be accepted in partial fulfillment of the requirements for the award of **M.Phil** degree.

Examination Committee Members

1. Name: Dr. Mubasher Jamil Signature: ______
2. Name: Dr. Adnan Aslam Signature: ______
3. Name: ______ Signature: ______
4. Name: Dr. Abid H. Mujtaba Signature: ______

Supervisor's Name: Prof. Asghar Qadir Signature: ______

______
Head of Department

05-01-2017
Date

COUNTERSINGED

Date: 05/01/17

______
Dean/Principal

Abstract

Gravitational lensing is considered one of the most important applications of General Relativity. The bending of the path of light by the Sun's gravitational field was one of the famous tests of General Relativity proposed by Einstein. While he calculated it for the weak field, currently it is being computed for the strong field as well. In this thesis we discuss the method given by V. Bozza to calculate gravitational lensing in the strong field region. We study strong gravitational lensing due to black holes and extend this work to the Schwarzschild-de Sitter space-time which describes the gravitational field of a spherically symmetric mass in a universe with a cosmological constant Λ. We calculate the effect of the cosmological constant on the deflection of light, effectively comparing the Schwarzschild and Schwarzschild-de Sitter space-times. We show that with the currently accepted value $\Lambda = 10^{-52} m^{-2}$ the change in deflection angle is negligible, however, larger values of Λ (for example $\Lambda = 10^{-10} m^{-2}$) result in a significant change in the deflection angle.

Acknowledgements

I express my heart-felt gratitude to my M.Phil supervisor, Prof. Asghar Qadir for his guidance and constant support during the research phase. He groomed me in several ways: to write thesis, be consistent with research work and how to be committed.

Next I would like to thank Dr. Mubasher Jamil who helped me whenever I encountered problems in mathematics. He gave me his precious time, and always encourged to ask questions. Also I would like to thank Dr. Adnan Aslam for his help and comments on an earlier draft of this thesis.

I would like to pay my gratitute to my family members especially my mother for her prayers, my friends played an important role by encourging me to work in a good way.

Wajeeha Pervaiz

Dedicated to
My Family
especially
My Father

Contents

Chapter 1

Introduction

Gravitational lensing (GL) is a term used for the deflection of light (or gravitational waves) by massive bodies. In astronomical situations, the light originates from a distant source and it undergoes deflection by the gravitational field of intervening matter distribution before arriving on the Earth. GL is widely used for finding extrasolar planets, dark matter and dark energy. There are three GL regimes [1], namely:

(a) Strong lensing (SL) is due to a sufficiently massive lens, which could be a galaxy or cluster of galaxies that produces multiple images of a background source or distorts the source into an arc or ring-like shape;

(b) Weak lensing (WL) is when the lens is not strong enough to require higher order corrections beyond the first general relativistic light bending effect, but it can weakly distort the images of extended sources;

(c) Microlensing (ML) is when the lens focuses to a point unresolvable by the measuring instrument used.

This division depends upon the location of the source from the lens, the observer and the resolutions/accuracy of measurement.

It had been proposed by Newton that masses should deflect light, but he did not estimate the deflection. The earliest calculations on the deflection of light from a distant source is due to Cavendish around 1784 [2]. Unfortunately, he did the calculations on an isolated scrap of paper. The next written account about the deflection of light by gravity appeared in the article entitled *"The Deflection Of Light Ray From Its Straight Motion Due To The Attraction Of A World Body Which It Passes Closely"* by Soldner in 1801 [3]. He predicted that a light ray passing close to the solar limb would be deflected by an angle of $\tilde{\alpha} = 0.84''$.

The GL effect inspired theorists as the potential it would have for astrophysics and cosmology. GL in the Strong Deflection (SD) regime is also a promising tool for black hole investigation. The reason for such an interest in GL in strong fields is that by the properties of the relativistic images it may be possible to investigate the regions immediately outside of the event horizon. High resolution imaging of black holes by VLBI (very large base line interferometry) could be able to detect the relativistic images and retrieve information about strong fields [4].

In 1959 [5], Darwin calculated the deflection angle due to a strong gravitational field using the Schwarzschild metric. Virbhadra and Ellis [6] obtained the lens equation and introduced a method to calculate the bending angle for strong gravitational lensing. Subsequently, Bozza provided a general method for the strong field to a static, spherically symmetric spacetime. He discovered that the parameters of the strong field expansion are directly connected to the observables [7].

The plan of the thesis is as follows. In the remaining part of Chapter 1, I will discuss some basic concepts of Special Relativity (SR) and General Relativity

(GR). I will discuss gravitational lensing, the general equation of motions for the propagation of light, the deflection angle and Bozza's method in chapter 2. I have reviewed the application of Bozza Method in chapter 3 and finally, I have applied this method to the Schwarzschild-de Sitter metric in chapter 4.

1.1 Brief Review of Special Relativity (SR)

In 1905, Einstein presented a simple resolution of the problem besetting Electrodynamics by only considering the kinematics of uniform linear motion, appealing to Galileo's principle of the relativity of such motion. On account of the restriction to constant velocities it was called the special (or restricted) theory of relativity, (henceforth called SR) [8]. SR deals with uniform motion. It connects space and time. SR grants channels that are crucial to understanding of the physical universe. Here I will define some basic concepts that will used in GR.

According to the principle of relativity, in all inertial frame of reference laws of physics are invariant. This principle also rejects the concept that there is existence of any universal frame of reference. Constancy of speed of light implies that ($c = 2.99792458 \times 10^8 m/s$) has absolute value in all inertial frame of reference in vaccum. Speed of light plays important role in theory.

1.1.1 Lorentz Transformation

Consider two observers O and O' with their rest frame R with coordinates (t, x, y, z) and for R' with coordinates (t', x', y', z') respectively. R' is moving with velocity v with respect to R along the $X - axis$. Consider that observer O fire a shot in the direction of negative $X - axis$ of velocity $-u$ of his rest frame R. As we know that by the Galilean transformation of coordinates; $x' = x - x_0$, here x_0 is distance of R' from R. Velocity of bullet according to O' (say u') is

$$u' = -u - v. \tag{1.1.1}$$

Now consider that the observer O sends a signal in the direction of the negative $X-$axis, then from equation (1.1.1) we can say that O' should find speed of the light signals greater then c. But this is inconsistent with the SR postulate that speed of light is same for all observers. We derive a new transformation law of coordinates to use in place of Galilean transformation that is Lorentz transformation. Thus the transformation from R to R' can be written as

$$x'^{\mu} = M^{\mu}_{\nu} x^{\nu}, \tag{1.1.2}$$

here M is a 4×4 matrix, which depends on v and c. since v has been taken in $X-$direction, therefore

$$y' = y, \;\; z' = z. \tag{1.1.3}$$

Let us consider that both the observers start their clocks ($t' = t = 0$) and emit a light signal when origins of R and R' coincide. According to R the light signals moves radially outward from the origin with speed c. The wavefront of light contains a sphere that satisfies the following conditions comprising of this sphere

$$s^2 = c^2 t^2 - x^2 - y^2 - z^2 = 0. \tag{1.1.4}$$

Also for R' must see the spherical wavefront as

$$s'^2 = c^2t'^2 - x'^2 - y'^2 - z'^2 = 0. \tag{1.1.5}$$

So under a transformation that connects two frames

$$s = 0 \iff s' = 0, \tag{1.1.6}$$

as the transformation is linear

$$s = ns'. \tag{1.1.7}$$

Now let us consider that speed is isotropic which allow us to reverse the direction of axis. The motion from O''s point of view the principle of relativity becomes

$$s' = ns. \tag{1.1.8}$$

Now from equation (1.1.4) and (1.1.7) implies that $n^2 = 1$. Thus it concludes that $n = 1$ and $s = s'$. Equation (1.1.3) gives

$$c^2t^2 - x^2 = c^2t'^2 - x'^2. \tag{1.1.9}$$

After introducing the imaginary coordinates and using these coordinates we have

$$x^2 + T'^2 = x'^2 + T'^2. \tag{1.1.10}$$

In the 2-dimensional space, the term $x^2 + T^2$ represents the distance of a point from the origin and it will remain invariant. Consider the rotation is in anti-clockwise direction and by an angle θ. The transformed coordinates look like

$$\begin{aligned} x' &= x\cos\theta + T\sin\theta, \\ T' &= -x\sin\theta + T\cos\theta. \end{aligned} \tag{1.1.11}$$

Since the origin of R' satisfies

$$x' = 0, x = vt. \tag{1.1.12}$$

After using the above equation and solving we get

$$\begin{aligned} x' &= \cos\theta(x + T\tan\theta), \\ T' &= \cos\theta(T - x\tan\theta). \end{aligned} \tag{1.1.13}$$

Using equation (1.1.13), we finally get

$$\begin{aligned} x' &= \gamma(x - vt), \\ t' &= \gamma\left(t - \frac{vx}{c^2}\right). \end{aligned} \tag{1.1.14}$$

Equations (1.1.3) and (1.1.14) with value of $\gamma = 1/\sqrt{1-\frac{v^2}{c^2}}$ is called Lorentz transformation.

Consider a time interval δt seen by the observer O. By this we mean there were two times t_1 and t_2. When in frame of O looked at his clock i.e, $\delta t = t_2 - t_1$, The position remains same for O so we can say that

$$x_1 = x_2. \tag{1.1.15}$$

The Lorentz transformation for the point (t_1, x_1), (t_2, x_2).

$$
\begin{aligned}
t_1' &= \gamma\left(t_1 - \frac{v}{c^2}x_1\right). \\
t_2' &= \gamma\left(t_2 - \frac{v}{c^2}x_2\right).
\end{aligned}
\tag{1.1.16}
$$

Then

$$
\delta t' = \gamma(t_2 - t_1) - \frac{v}{c^2}\gamma(x_2 - x_1). \tag{1.1.17}
$$

Hence we have, $\delta' = \gamma\delta t$. This is known as Time dilation formula. It states that the unit time measurement at O' is longer than that of O.

Another possibility is to measure the spatial interval. Now let us consider the spatial interval according to O is

$$
\delta x = x_2 - x_1. \tag{1.1.18}
$$

Observer O' see the two ends at the same time, for that $t_1' = t_2'$. Now using Lorentz transformation for (t_1, x_1) and (t_2, x_2). Also by doing algebra, we get

$$
\begin{aligned}
\delta x' &= \left(1 - \frac{v^2}{c^2}\right)\left(1/\sqrt{1 - \frac{v^2}{c^2}}\right)\delta x, \\
\delta x' &= \frac{\delta x}{\gamma}.
\end{aligned}
\tag{1.1.19}
$$

The above equation is called Lorentz-Fitzgerald length contraction.

Now consider two events simultaneous to O, i.e one event occur at x_1 and another occur at x_2 at the same time $(t_1 = t_2 = t)$.

For O' events occur at

$$t'_2 = \gamma\left(t_2 - \frac{v}{c^2}x_2\right); \quad t'_1 = \gamma\left(t_1 - \frac{v}{c^2}x_1\right). \tag{1.1.20}$$

Subtracting the above equations and solving we get

$$t'_2 - t'_1 = \frac{v}{c^2}\gamma(x_1 - x_2). \tag{1.1.21}$$

So the simultaneity is relative.

Transformation of Velocities in SR. Contrary to Newtonian mechanics, velocities do not add up in SR, otherwise an observer moving towards a light source would measure the speed of light, larger than c which contradicts the second postulate. In order to obtain formula for the velocities. Consider a particle has velocity $u^{x'} \equiv \frac{dx'}{dt}$ relative to an inertial frame of reference $R' = \{t', x', y', z'\}$. We want to find the velocity u^x with respect to another interial frame of reference $R = \{t, x, y, z'\}$. Lorentz transformation can be written in differential form as

$$\begin{aligned} dt' &= \gamma(dt - \frac{v}{c^2}dx), \\ dx' &= \gamma(dx - vdt), \\ dy' &= dy, \\ dz' &= dz. \end{aligned} \tag{1.1.22}$$

Velocity components in R frame are

$$u^x = \frac{dx}{dt}, \quad u^y = \frac{dy}{dt}, \quad u^z = \frac{dz}{dt}. \tag{1.1.23}$$

In R' frame the components are

$$u^{x\prime} \equiv \frac{dx'}{dt'} = \frac{u^x - v}{1 - \frac{v}{c^2}u^x}. \tag{1.1.24}$$

Analogously, $dy' = dy, dz' = dz$. Similarly by following the above process we obtain the remaining components of velocities.

$$u^{y\prime} = u^y/\gamma\left(1 - \frac{v}{c^2}u^x\right). \tag{1.1.25}$$

$$u^{z\prime} = u^z/\gamma\left(1 - \frac{v}{c^2}u^x\right). \tag{1.1.26}$$

Minkowski Spacetime is the pair $(\mathbb{R}^4, \eta_{\mu\nu})$ consists of all spacetime events furnished with the Minkowski metric $\eta_{\mu\nu}$ [9]. It is flat spacetime. A spacetime consists of one spatial coordinate of time and three ordinary coordinates of space. In Minkowski spacetime, the measurement result for time as well as space are coordinate frame dependent means that time be treated on same footing as space coordinates. Let us consider $\{x^\nu\} = \{ct, x, y, z\}$ be coordinates in the Minkowski spacetime. The spacetime interval between points of coordinates $x^\nu = (ct, x, y, z)$ and for $x^\nu + dx^\nu = (ct + cdt, x + dx, y + dy, z + dz)$ is

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2. \tag{1.1.27}$$

The line element of Minkowski spacetime ds^2 can be obtained by taking the metric tensor. The Lorentzian signature $(-, +, +, +)$ shows that time is treated differently from space. For $c = 1$, the line element looks like

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2. \tag{1.1.28}$$

The above equation represents the line element for the Minkowski spacetime. The interval separating any two events is timelike if $ds^2 < 0$, spacelike if $ds^2 > 0$, and null if $ds^2 = 0$.

1.2 Brief Review of General Relativity (GR)

The General theory of Relativity was proposed fully by Einstein in 1915. This theory, irrespective of the special case deals with all kinds of motion and measurements made by accelerated observers. The conclusions were far more reaching than those of Special Relativity, it was sorted that the treatment of the effect of the gravitation can be included in the measurement of the accelerated system. Also these effects can be interpreted as if they comeforth from the situation of relative acceleration between an inertial observer Friedmann, de-Sitter and others. However GR reduces to the Newtonian theory under explicit circumstances when a particle is moving with slow velocity in a weak and static gravitational field.

The two assumptions on which GR is actually based are:

- *Principle of equivalence.*

  It states that there are no physical effects which can distinguish whether a system is accelerated with a constant acceleration or whether it is under the influence of a homogenous gravitational field.

- *Principle of general covariance.*

It states that all frame of reference are physically equivalent.

GR shows that the geometry of the spacetime is just the gravitational phenomena and determine by matter and energy.

Metric Tensor

GR describe the theory of curvature of spacetime. In order to discuss the Einstein Field Equations (EFEs), firstly we will discuss the metric tensor ($g^{\mu\nu}$). This tensor defines the coordinates of a spacetime in term of mass and gravity [10].

The equation (1.1.32), we have discussed in section (1.1.3), remains invariant under the Lorentz transformation. This equation can not be remain simplified when we move from an inertial frame to a non-inertial frame of reference. It appears as

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta, \tag{1.2.1}$$

here g_{ij} are the coefficients which defines the components of a tensor known as metric tensor, g_{ij} is symmetric.

In curved spacetime, in order to understand how basis vectors change from one point (x^0, x^1, x^2, x^3) to neighboring point $(x^0 + dx^0, x^1 + dx^1, x^2 + dx^2, x^3 + dx^3)$. Let e_α are basis vectors. The transformation law is

$$de_\alpha = \Gamma^\gamma{}_{\alpha\beta} e_\gamma dx^\beta, \tag{1.2.2}$$

here the coefficients $\Gamma^\gamma{}_{\alpha\beta}$ are called Christoffel symbols. Christoffel symbols can be defined in term of metric tensor as

$$\Gamma^\gamma{}_{\alpha\beta} = \frac{1}{2} g^{\gamma\sigma} \Big(g_{\sigma\alpha,\beta} + g_{\sigma\beta,\alpha} - g_{\alpha\beta,\sigma} \Big) \tag{1.2.3}$$

here, , represents partial derivative with respect to spatial coordinates.

Riemann curvature tensor

It describes the curvature of spacetime. It can be written as, [11].

$$R^{\sigma}{}_{\alpha\beta\gamma} = \Gamma^{\sigma}{}_{\alpha\gamma,\beta} - \Gamma^{\sigma}{}_{\alpha\beta,\gamma} + \Gamma^{\mu}{}_{\alpha\gamma}\Gamma^{\sigma}{}_{\mu\beta} - \Gamma^{\mu}{}_{\alpha\beta}\Gamma^{\sigma}{}_{\mu\gamma}. \tag{1.2.4}$$

Ricci curvature tensor

To contract the indices of curvature tensor, i.e, $R_{\alpha\beta} = R^{\sigma}_{\alpha\sigma\beta}$. We have

$$R_{\alpha\beta} = \Gamma^{\sigma}{}_{\beta\alpha,\sigma} - \Gamma^{\sigma}{}_{\sigma\alpha,\beta} + \Gamma^{\sigma}{}_{\sigma\rho}\Gamma^{\rho}{}_{\beta\alpha} - \Gamma^{\sigma}{}_{\beta\rho}\Gamma^{\rho}{}_{\sigma\alpha}, \Gamma^{\rho}{}_{\beta]\alpha}, \tag{1.2.5}$$

Ricci scalar

This is the resultant of the contraction of Ricci curvature tensor,

$$R = g^{\alpha\beta}R_{\alpha\beta}. \tag{1.2.6}$$

The Bianchi Identities

The Riemann curvature tensor satisfies some of the differential identities. This can be proved easily at a given point x^{ν} by using the geodesic coordinate system. In that coordinate system at a given point Christoffel symbols (but not there derivatives) vanish. So after differentiating $R^{\alpha}{}_{\beta\gamma\sigma}$ and substituting the Christoffel symbols equal to zero, we get

$$R^{\alpha}{}_{\beta\gamma\sigma;\rho} = \Gamma^{\alpha}{}_{\beta\sigma,\gamma\rho} - \Gamma^{\alpha}{}_{\beta\gamma,\sigma\rho}. \tag{1.2.7}$$

After permuting the indices γ, σ and ρ in equation (1.2.8) and adding the corresponding terms, we obtain

$$R^{\alpha}{}_{\beta\gamma\sigma,\rho} + R^{\alpha}{}_{\beta\sigma\rho;\gamma} + R^{\alpha}{}_{\beta\rho\gamma;\sigma} = 0. \tag{1.2.8}$$

As each term of equation above represents component of a tensor, this equation will hold in any coordinate system. The above equation is known as the Bianchi Identity. On contracting indices γ and α in above equation and using the symmetry property of the Riemann curvature tensor, we get

$$R_{\beta\sigma;\rho} - R\ {}_{\beta\rho;\sigma} + R^{\alpha}{}_{\beta\sigma\rho;\alpha} = 0. \tag{1.2.9}$$

After contracting indices β and σ, we have

$$\begin{aligned} R^{\sigma}{}_{\rho;\sigma} - \frac{1}{2}\delta^{\sigma}_{\rho} R_{;\sigma} = 0, \\ \mathcal{E}^{\sigma}_{\rho;\sigma} = 0. \end{aligned} \tag{1.2.10}$$

where

$$\mathcal{E}^{\sigma}_{\rho} = R^{\sigma}_{\rho} - \frac{1}{2}\delta^{\sigma}_{\rho} R, \tag{1.2.11}$$

or

$$\mathcal{E}_{\sigma\rho} = R_{\sigma\rho} - \frac{1}{2} R g_{\sigma\rho}, \tag{1.2.12}$$

is called the Einstein tensor.

Geodesics

A straight line is the shortest distance between two points. In 3-dimensional space, when we want to draw a triangle on curved sheet, the sum of all the interior angles will not equal to 180^0. The reason is that on a curved space it is not viable to draw actual straight lines. So for that we can only draw straight lines for shortest path between two points on the surface. By this we mean that a geodesic is basically a generalization of the straight line in Euclidean space to curved space.

To find the shortest path between two points P and R in arbitrary space, Euler-Lagrange Equation (ELE) is used. The distance between P and Q along γ is given by, [12].

$$\mathcal{L} = \sqrt{\pm g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}} \tag{1.2.13}$$

$$l = \int_{P}^{R} \sqrt{| \, g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} \, |} d\lambda, \tag{1.2.14}$$

$$l = \int_{P}^{R} \mathcal{L} d\lambda. \tag{1.2.15}$$

Here $\dot{x}^{\mu} = \frac{dx^{\mu}}{ds}$. The curve for which l is an extremium is determined by substituting $\mathcal{L}$ into the ELE.

$$\frac{d}{ds}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^{\gamma}}\right) - \frac{\partial \mathcal{L}}{\partial x^{\gamma}} = 0. \tag{1.2.16}$$

Using (1.2.14) and (1.2.15) in (1.2.16) and simplifying we get

$$\ddot{x}^{\sigma} + \Gamma^{\sigma}{}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} = 0. \tag{1.2.17}$$

The term $\Gamma^{\sigma}_{\mu\nu}$ is defined in equation (1.2.4).

1.3 The Einstein Field Equations

Einstein's gravitational theory lies in geometrizing the gravitational force, by which we mean that to map the impact of the gravitational force on physical processes and also properties of the gravitational force onto that of a Riemannian space. In view of GR, the thing that can produce a gravitational field entitled as matter. Einstein suggested that in a given spacetime a particle will travel on the straight path available to it and is called as geodesic. The idea to derive the EFEs is bascially that to extend the straightness of path in context of the spacetime curvature, so the gravitation might also be indicated in terms of spacetime curvature. SR states no essential difference between energy and matter. So we can conclude that in relativistic terms, curvature of spacetime must be related to the presence of matter-energy. Let the relationship be expressed by

$$\mathcal{E}_{\mu\nu}[g_{\alpha\beta}, R^{\alpha}_{\beta\rho\sigma}] = \kappa T_{\mu\nu}. \tag{1.3.1}$$

Here $\mathcal{E}_{\mu\nu}$ is symmetric and must be a divergence-free tensor. $\kappa = 8\pi G/c^4$ is constant. Here $T_{\mu\nu}$ is the for non-trivial case be linear function of curvature. As we know that (1.2.9) and (1.2.13) we get the Einstein tensor, by substituting this in above equation we obtain the Einstein field equations

$$R_{\sigma\rho} - \frac{1}{2}Rg_{\sigma\rho} = \kappa T_{\sigma\rho}. \tag{1.3.2}$$

After GR applied to Cosmology, Einstein added a constant of integration Λ, called the cosmological constant to $\mathcal{E}$ so that the above equation becomes

$$R_{\sigma\rho} - \frac{1}{2}Rg_{\sigma\rho} - \Lambda g_{\sigma\rho} = \kappa T_{\sigma\rho}. \tag{1.3.3}$$

1.3.1 The Schwarzschild Solution

The EFEs are second order non-linear and a set of 10 partial differential equations. The Schwarzschild solution is one of the most important and simplest of all exact solutions to EFEs. The solution is time independent and spherically symmetric. The German physicist, Schwarzschild calculated the space around a spherically symmetric body static at the origin. The line element for the static, spherically symmetric metric can be written as

$$ds^2 = -e^{\alpha(r)}dt^2 + e^{\beta(r)}dr^2 + r^2 d\Omega^2, \tag{1.3.4}$$

here α and β are functions of r only and $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the solid angle substended at the origin. In order to find out the value for unknown functions $\alpha(r)$ and $\beta(r)$, firstly we will find the Christoffel symbols using Equation (1.2.4). The non vanishing components are

$$\begin{aligned}
\Gamma^{t}{}_{tr} = -\frac{\alpha'}{2},\quad \Gamma^{r}{}_{tt} = \frac{\alpha'}{2}e^{\alpha-\beta},\quad \Gamma^{r}{}_{rr} = -\frac{1}{2}\beta',\\
\Gamma^{r}{}_{\theta\theta} = -re^{-\beta},\quad \Gamma^{r}{}_{\phi\phi} = -r\sin^2\theta e^{-\beta},\quad \Gamma^{\theta}{}_{r\theta} = \frac{1}{r} = \Gamma^{\phi}{}_{1\phi},\\
\Gamma^{r}{}_{\phi\phi} = -\sin\theta\cos\theta,\quad \Gamma^{\phi}{}_{\theta\phi} = \cot\theta,
\end{aligned} \tag{1.3.5}$$

Here the symbol $'$ denotes the derivative with respect to r. By using christoffel symbols we can write the above equation as follows

$$R_{00} = \alpha'' + \frac{1}{2}\alpha'(\alpha' - \beta') + 2\frac{\alpha'}{r} = 0. \tag{1.3.6}$$

$$R_{11} = -\alpha'' - \frac{1}{2}\alpha'(\alpha' - \beta') + 2\frac{\beta'}{r} = 0. \tag{1.3.7}$$

$$R_{22} = 1 - e^{-\beta} + \frac{1}{2}r(\beta' - \alpha')e^{-\beta} = 0. \tag{1.3.8}$$

$$R_{33} = R_{22}\sin^2\theta = 0. \tag{1.3.9}$$

Now adding equations (1.3.4) and (1.3.5) we have

$$(\alpha + \beta)' = 0, \tag{1.3.10}$$

which implies

$$\alpha + \beta = constant, \tag{1.3.11}$$

we can consider the constant to be zero and say that we have merged the constant into the units of time. So

$$\alpha(r) = -\beta(r). \tag{1.3.12}$$

After using equation (1.3.10) in equations (1.3.5) and (1.3.6), we obtain

$$e^{\alpha(r)} = 1 + \frac{\sigma}{r}. \tag{1.3.13}$$

Also

$$e^{\alpha(r)} = e^{-\beta}. \tag{1.3.14}$$

Now putting equations (1.3.11) and (1.3.12) in equation (1.3.2) we have

$$ds^2 = -\left(1 - \frac{\gamma}{r}\right)c^2dt^2 + \frac{dr^2}{\left(1 - \frac{\gamma}{r}\right)} + r^2d\Omega^2. \tag{1.3.15}$$

This is called Schwarzschild solution. To get the value γ one must look for weak field limit. By using correspondence principle γ may be easily determined. We get after comparing this

$$\gamma = -2Gm/c^2. \tag{1.3.16}$$

The above value can be obtained by using the relativistic equations of motion.

The Relativistic Equation of Motion

The relativistic equations are obtained by geodesic equation for the Schwarzschild metric. Following are the geodesic equations obtained from equation (1.2.16)

$$c\frac{d^2t}{ds^2} + c\alpha'\left(\frac{dt}{ds}\right)\left(\frac{dr}{ds}\right) = 0. \tag{1.3.17}$$

$$\frac{d^2r}{ds^2}+\frac{1}{2}c^2\alpha' e^{\alpha-\beta}\left(\frac{dt}{ds}\right)^2+\frac{1}{2}\beta'\left(\frac{dr}{ds}\right)^2-re^{-\beta}\left(\frac{d\theta}{ds}\right)^2-r\sin^2\theta e^{-\beta}\left(\frac{d\phi}{ds}\right)^2 = 0. \tag{1.3.18}$$

$$\frac{d^2\theta}{ds^2} + \frac{2}{r}\left(\frac{dr}{ds}\right)\left(\frac{d\theta}{ds}\right) - \sin\theta\cos\theta\left(\frac{d\phi}{ds}\right)^2 = 0. \tag{1.3.19}$$

$$\frac{d^2\phi}{ds^2} + \frac{2}{r}\left(\frac{dr}{ds}\right)\left(\frac{d\phi}{ds}\right) + 2\cot\theta\left(\frac{d\phi}{ds}\right)\left(\frac{d\theta}{ds}\right) = 0. \tag{1.3.20}$$

Taking $\theta = \frac{\pi}{2}$ and hence $\dot{\theta} = 0$ equations $(1.3.15), (1.3.16)$ and $(1.3.18)$ becomes

$$\frac{d^2t}{ds^2} + \alpha'\left(\frac{dt}{ds}\right)\left(\frac{dr}{ds}\right) = 0. \tag{1.3.21}$$

$$\frac{d^2r}{ds^2} + \frac{1}{2}c^2\alpha' e^{\alpha-\beta}\left(\frac{dt}{ds}\right)^2 + \frac{1}{2}\beta'\left(\frac{dr}{ds}\right)^2 - re^{-\beta}\left(\frac{d\phi}{ds}\right)^2 = 0. \tag{1.3.22}$$

$$\frac{d^2\phi}{ds^2} + \frac{2}{r}\left(\frac{dr}{ds}\right)\left(\frac{d\phi}{ds}\right) = 0. \tag{1.3.23}$$

Let us consider

$$\frac{d\alpha}{ds} = \frac{d\alpha}{dr}\frac{dr}{ds} = \alpha'\frac{dr}{ds}. \tag{1.3.24}$$

From equation (1.3.19) we can write

$$\frac{d}{ds}\left(e^{\alpha}\frac{dt}{ds}\right) = 0. \tag{1.3.25}$$

$$\frac{dt}{ds} = \frac{k_1}{e^{\alpha}} = \frac{k_1}{\left(1+\frac{\gamma}{r}\right)}. \tag{1.3.26}$$

For asymptotically flat, the spacetime must be the Minkowskian space. Mathematically we can say that as ($r \longrightarrow \infty$, $t \longrightarrow \frac{1}{c}$). From above equation we have

$$c\frac{dt}{ds} = e^{-\alpha}. \tag{1.3.27}$$

After multiplying with r^2 and solving the equation (1.3.21) becomes

$$r^2\frac{d\phi}{ds} = k_2. \tag{1.3.28}$$

Classically we have

$$r^2\frac{d\phi}{ds} = \frac{h}{c}. \tag{1.3.29}$$

Here c be speed of light. Now after comparing (1.3.26) and (1.3.27), we get

$$k_2 = \frac{h}{c}. \tag{1.3.30}$$

To simplify this we have

$$1 = e^{\alpha}\left(c\frac{dt}{ds}\right)^2 - e^{-\alpha}\left(\frac{dr}{ds}\right)^2 - r^2\left(\frac{d\phi}{ds}\right)^2. \tag{1.3.31}$$

Using (1.3.25) and (1.3.26) into equation (1.3.29) we have

$$e^{\alpha}\left(\frac{dr}{ds}\right)^2 = e^{-\alpha} - \frac{h^2/c^2}{r^2} - 1. \tag{1.3.32}$$

Using equations (1.3.25) and (1.3.29) and $(e^{\alpha})' = \frac{-\gamma}{r^2}$ in equation (1.3.20), we have

$$\frac{d^2r}{ds^2} - \frac{h^2/c^2}{r^3} = \frac{\gamma/2}{r^2} + \frac{3}{2}\frac{\gamma h^2/c^2}{r^4}. \tag{1.3.33}$$

From equation (1.3.27)

$$d\phi = \frac{h/c}{r^2}ds. \tag{1.3.34}$$

Also

$$\frac{d}{ds} = \frac{h/c}{r^2}\frac{d}{d\phi}. \tag{1.3.35}$$

Now taking $r = \frac{1}{u}$ in above equation

$$\frac{d^2r}{ds^2} = -\frac{h^2}{c^2}u^2\frac{d^2u}{d\phi^2}. \tag{1.3.36}$$

Putting the above equation in (1.3.31), we have

$$\frac{d^2u}{d\phi^2} + u = \left(\frac{-c^2\gamma/2}{h^2}\right) - \left(\frac{3\gamma}{2}\right)u^2. \tag{1.3.37}$$

From the above equation after considering $\gamma = \frac{-2Gm}{c^2}$. We have

$$\frac{d^2u}{d\phi^2} + u = \frac{Gm}{h^2} + \frac{3Gm}{c^2}u^2. \tag{1.3.38}$$

As we know that classical equation of motion for a particle in an external gravitational field is

$$\frac{d^2u}{d\phi^2} + u = \frac{Gm}{h^2}, \tag{1.3.39}$$

As we know that for $r \longrightarrow \infty$ the term u^2 is so small we can neglect it. The (1.3.13) takes the form

$$ds^2 = -\left(1 - \frac{2Gm}{c^2 r}\right)dt^2 + \frac{dr^2}{\left(1 - \frac{2Gm}{c^2 r}\right)} + r^2 d\Omega^2. \tag{1.3.40}$$

which is called the Schwarzschild metric. For gravitational units i.e $c = G = 1$, equation (1.3.13) can be written as

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \frac{dr^2}{\left(1 - \frac{2m}{r}\right)} + r^2 d\Omega^2. \tag{1.3.41}$$

1.4 The Three tests of GR

Einstein suggested three tests of general relativity, called the classical tests of GR. The perihelion precession of Mercury's orbit, the deflection of light by the Sun and the gravitational redshift of light.

1.4.1 The Perihelion Shift of Mercury

Mercury's orbit is shifting it's position slowly over time, due to the curvature of spacetime around the sun. There is an extra 43 seconds of arc per century in this precession are observed to occur. Mercury's orbit orientation is found to precess in space over time, this is commonly called the precession of the perihelion, because of it position of the perihelion is moving. In Newton's theory, only this part can be accounted for perturbations. Although this effect is extremely small, but the measurements are precise and can detect such effects very well.

1.4.2 The Gravitational Red-Shift

Einstein predicted the gravitational red-shift of light, but it is very difficult to measure astrophysically. Although it was measured by Walter in 1925, it was only conclusively tested when the Pound-Rebka experiment in 1959 measured the relative red-shift of two sources situated at the top and bottom of Harvard University's Jefferson tower using an extremely sensitive phenomenon called the Mossbauer effect. The result was in excellent agreement with general relativity. This was one of the first precision experiments testing general relativity. Light coming from a strong gravitational field have its wavelength shifted to larger values, and it's magnitude is correctly given by Einstein's theory, known as red-shift. Now let us consider a light signal of n waves sent from point a, which is at a distance r_a from the gravitational source to a point b and which is at distance r_b. Now we have to consider frequency of emission be v_a and reception be v_b. The proper time can be taken by considering $dr, d\theta, d\phi = 0$. From equation (1.3.14) we have

$$d\tau_a = \sqrt{1 - 2Gm/c^2 r_a dt}, \tag{1.4.1}$$

also for $d\tau_b$ we have

$$d\tau_b = \sqrt{1 - 2Gm/c^2 r_b dt}. \tag{1.4.2}$$

As the change in frequency is given by

$$\frac{v_b}{v_a} = \sqrt{\frac{1 - 2Gm/c^2 r_a}{1 - 2Gm/c^2 r_b}}. \tag{1.4.3}$$

Now if consider the surface of star with radius R so the frequency for very far away is given by $v + \delta v$. So for $r_a = R$ also $r_b = \infty$.

$$\frac{\delta v}{v} = \sqrt{1 - \frac{2Gm}{c^2 R}} - 1 \approx -\frac{Gm}{c^2 R}. \tag{1.4.4}$$

The above equation gives the red-shift of light. For case of the Earth light entering into the gravitational potential would suffer blue-shift and is given by

$$\frac{\delta v}{v} \approx -\frac{Gm_E}{c^2 R_E}. \tag{1.4.5}$$

1.4.3 The Gravitational Deflection of Light

For light $d\tau = 0$. Thus from equation (1.3.34) as $h \to \infty$ for light and hence the first term from equation (1.3.38) vanishes. Thus for light the equation of motion becomes

$$\frac{d^2 u}{d\phi^2} + u = \frac{3Gm}{c^2} u^2. \tag{1.4.6}$$

As we know that the above equation is non-linear differential equation and we can not solve it exactly. We will first assume that the first term on right is negligible as compared to u.

We consider that the right hand side of above equation equal to zero, we will denote the solution of this equation by u_0. Then the equation looks like

$$u_0 = \frac{1}{r_0} = \frac{\cos(\phi - \phi_0)}{R}. \tag{1.4.7}$$

In the above equation ϕ_0 and R and constants of integration of second order linear differential equation. In order to obtain a better approximation, we have to solve the right hand side of above equation by replacing u_0 with u

$$\frac{d^2 u_1}{d\phi^2} + u_1 = \frac{3Gm}{c^2 R^2} \cos^2(\phi - \phi_0). \tag{1.4.8}$$

To solve the above equation and by first order approximation to the solution of equation (1.4.6)

$$u_1(\phi) = \frac{1}{r_1} = \frac{1}{R} \cos(\phi - \phi_0) + \frac{Gm}{c^2 R^2}[\cos^2(\phi - \phi_0) + 2\sin^2(\phi - \phi_0)]. \tag{1.4.9}$$

We can obtain the second, third and so on nth order approximation for u_n.

In order to evaluate u_0 so that we get the classical path of light, R, being the "impact parameter"and also the ϕ_0 be the choice of that phase. To observe this, consider a light coming from a distant stargrazing by Sun and observed at the Earth. The first order approximation given by u_1 gives the "corrected" path of the light from here we get the gravitational deflection of the light. Now substituting $x = r_1 \cos(\phi - \phi_0)$ and $y = r_1 \sin(\phi - \phi_0)$ in equation (1.4.9). Thus we get $r_1 = \sqrt{x^2 + y^2}$. Multiplying the equation (1.4.9) by Rr_1 we get

$$x = R - \frac{2Gm}{c^2 R} \frac{x^2 + 2y^2}{\sqrt{x^2 + y^2}}. \tag{1.4.10}$$

The deflection of the light is given by

$$\theta \approx \tan\theta = \frac{x - R}{r_1} \approx -\frac{2Gm}{c^2 R}. \tag{1.4.11}$$

Chapter 2

Black Holes and Cosmology

2.1 Black Hole

A black hole is a region in spacetime where the gravitational field is strong enough that even light cannot escape from it. The thermonuclear reactions in a star generate pressure due to which a star supports itself against gravity. Since these are not forever continuing reactions, so when the nuclear fuel is exhausted the pressure reduces which break down it's balance with gravity. As a consequence the star starts contracting. In case the mass of the star is enough such that all outward acting forces get overcome by the inward acting force of gravity then the collapse continues. As a result density increases when the volume of star decreases and escape velocity tends to exceed the speed of light at some place. The star is then said to be a *black hole* (a name given by J.A. Wheeler) [13].

Their most interesting physical property is that they are described by only three parameters: mass M, electric charge Q and angular momentum J. Due to its structural simplicity, John Wheeler coined a famous phrase: *A black hole has no* *hair*, which was called the 'no hair theorem'. As a consequence, there are now four solutions of Einstein's field equations that represent black holes: The most general spherically symmetric, static solution of the EFE's was obtained by K. Schwarzschild [14] and is known as the Schwarzschild metric. The most general static, spherically symmetric solution of the Einstein-Maxwell equations is the Reissner-Nordström (RN) metric [15, 16]. The most general axi-symmetric and stationary solution of the vacuum Einstein equation is Kerr metric, was discovered by Roy Kerr. Kerr also found another solution to the Einstein-Maxwell equations that describes a charged, rotating black hole, the generalization of the (RN) metric. It is called the charged-Kerr or Kerr-Newmann, solution [17, 18].

A spacetime is said to be *stationary* if and only if there exists a timelike coordinate t such that the metric is independent of t. The Schwarzschild spacetime possesses time translation symmetry as the metric remains unchanged under $t \longrightarrow t+t_0$. Where t_0 is an arbitrary constant. Clearly the equation (1.3.15) is independent of the timelike coordinate t and is, therefore, stationary. A stationary spacetime is also *static* if the metric does not change under a time reversal $t \longrightarrow -t$.

In GR the presence of a singularity must be explored very carefully. If the curvature becomes infinite then singularity is said to be essential. The Riemann tensor is helpful for determining whether a singularity is essential or coordinate. The problem is that $R^{\sigma}{}_{\alpha\beta\gamma}$ given by equation (1.2.4), is expressed in coordinate form. Thus the trouble of the coordinate system will effect the components of Riemann tensor. However, scalar quantities are invariant under coordinate transformations. By constructing the scalars quantities from the Riemann tensor we could easily check if they become infinite. There are only finitely many independent scalars. We can express all the others in term of these. We only need to construct the simplest scalars. These are

$$
\begin{aligned}
I_1 &= R = g^{ab}R_{ab}, \\
I_2 &= R^{ab}{}_{cd}R^{cd}{}_{ab}, \\
I_3 &= R^{ab}{}_{cd}R^{cd}{}_{ef}R^{ef}{}_{ab}.
\end{aligned} \tag{2.1.1}
$$

If the independent curvature invariants, discussed above are all finite the singularity will be coordinate. If any of them is infinite the singularity will be essential. From equation (1.3.15), it is clear that the Schwarzschild black hole metric coefficients become infinite at $r = 0$ and $r = 2m$. The question arises which type of singularity is essential or coordinate in the Schwarzschild metric. The curvature invariants at any point are given by

$$I_1 = 0, \tag{2.1.2}$$

$$I_2 = \frac{48m^2}{r^6}, \tag{2.1.3}$$

$$I_3 = \frac{64m^3}{r^6}. \tag{2.1.4}$$

It is clear from the above expression that the curvature invariants are regular at $r = 2m$, while I_2 and I_3 are infinite at $r = 0$. As we know that they are scalars, so their value will remain same in any coordinate system. So we can say that the spacetime curvature is well behaved at $r = 2m$ and is singular at $r = 0$. Thus we conclude that the singularity at $r = 0$ is an essential singularity, whereas at $r = 2m$ is a coordinate singularity and can be removed by a good choice of coordinates.

2.2 Cosmology

The word Cosmology is the combination of two Greek words "cosmos"($\kappa\omega\sigma\mu\omega\sigma$) and "logos"($\lambda\omega\gamma\omega\sigma$). The former means "the Universe" and the latter means "the study of". Hence it is the study of the Universe [19].

The *Cosmological Principle* states that on a sufficiently large scale, the properties of the Universe are the same for all observers. The first cosmological model was proposed by Einstein in 1917. He assumed not only homogeniety and isotropy in space but also in time. This assumption is called the *Strong Cosmological Principle.* It was later dropped due to the discovery of the expansion of the Universe by Edwin Hubble [12].

The stress-energy tensor for a perfect fluid is

$$T^{\mu\nu} = (\rho + p)\dot{x}^{\mu}\dot{x}^{\nu} - pg^{\mu\nu}. \tag{2.2.1}$$

For the rest frame $x^{\mu} = (x^0, x^1, x^2, x^3)$ so

$$\dot{x}^{\mu} = \left(dx^0/d\tau, 0, 0, 0\right), \tag{2.2.2}$$

since the fluid is at rest. For the time-like geodesics

$$\dot{x}^{\mu}\dot{x}^{\nu}g_{\mu\nu} = c^2, \tag{2.2.3}$$

where c is the speed of light. Hence we can write it as

$$\dot{x}^0 = (g_{00})^{-1/2}, \quad \dot{x}_0 = (g_{00})^{1/2}. \tag{2.2.4}$$

Thus we can write equation (2.2.1) in the form

$$T^{\mu\nu} = (\rho + p)(g_{00})^{-1}\delta_0^\mu \delta_0^\nu - pg^{\mu\nu}. \tag{2.2.5}$$

Using the law of conservation of energy-momentum ($T^{\mu\nu}_{;\nu} = 0$). The equation becomes

$$0 = ((\rho + p)g^{00}\delta_0^\mu \delta_0^\nu)_{;\nu} - (pg^{\mu\nu})_{;\nu}. \tag{2.2.6}$$

Since the covariant derivative of the metric tensor vanishes and p is constant, hence the second term of the above equation also disappears. The spherically symmetric metric is given by [12]

$$ds^2 = e^{\nu(r,t)}dt^2 - e^{\lambda(r,t)}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.2.7}$$

Now using equations (2.2.6) and (2.2.7) we have

$$(\rho + p)[(e^{-\nu(r)}\delta_0^\mu \delta_0^\nu)_{,\nu} + e^{-\nu(r)}\Gamma^\mu{}_{\nu 0}\delta_0^\nu + \Gamma^\nu{}_{\nu 0}\delta_0^\mu] = 0. \tag{2.2.8}$$

There is no time dependence so the first term and the last term of the above equation vanishes. The above equation then becomes

$$\dot{\rho} = -\frac{(\rho + p)}{2}\dot{\lambda}, \tag{2.2.9}$$

using the Strong Cosmological Principle, we obtain

$$\frac{(\rho + p)}{2}\dot{\lambda} = 0. \tag{2.2.10}$$

From the above equation we have two cases:

$$\rho + p = 0, \tag{2.2.11}$$

$$\rho + p \neq 0, \;\; \dot{\lambda} = 0. \tag{2.2.12}$$

Equation (2.2.11) gives the de Sitter Universe, whereas the second one give the Einstein Universe. We will discuss here only the Einstein Static Universe model (as we will discuss the Schwarzschild de-Sitter lensing in the next chapter). Taking $\mu = 1$ in

$$\mathrm{T}^{1\nu}_{;\nu} = 0. \tag{2.2.13}$$

Solving the above equation and incorporating equation (2.2.7) and using the assumption of homogeneity and isotropy we have

$$\dot{\nu} = 0. \tag{2.2.14}$$

Now to solve the equation (1.3.3) the mixed tensor form as follows;

$$R^{\alpha}_{\rho} - \frac{1}{2} R \delta^{\alpha}_{\rho} = \kappa T^{\alpha}_{\rho} + \Lambda \delta^{\alpha}_{\rho}. \tag{2.2.15}$$

Here $\alpha, \rho = 0, 1, 2, 3$. From equation (2.2.7) after getting the Christoffel symbols, the Ricci tensors R^{α}_{ρ} are

$$\begin{aligned} R^0_0 &= \frac{e^{-\lambda}}{2}\left[\nu'' + \frac{\nu'}{2}(\nu' - \lambda') + \frac{2\nu'}{r}\right], \\ R^1_1 &= \frac{e^{-\lambda}}{2}\left[\nu'' + \frac{\nu'}{2}(\nu' - \lambda') - \frac{2\lambda'}{r}\right], \\ R^2_2 &= -\frac{1}{r^2}\left[1 - (re^{-\lambda}) - \left(\frac{\nu' + \lambda'}{2}\right)re^{-\lambda}\right] = R^3_3. \end{aligned} \tag{2.2.16}$$

The Ricci scalar, R is

$$\begin{aligned} R &= R^0_0 + R^1_1 + R^2_2 + R^3_3, \\ &= e^{-\lambda}\left[\nu'' + \frac{\nu'}{2}(\nu' - \lambda') + \frac{2\nu'}{r}\right] - \frac{2}{r^2}\left[1 - (re^{-\lambda})\right]. \end{aligned} \tag{2.2.17}$$

The stress-energy tensor is calculated as

$$\mathrm{T}^{\alpha}_{\beta} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \tag{2.2.18}$$

Now the zeroth component of equations $(2.2.15), (2.2.16)$ and $(2.3.17)$ yields

$$1 - (8\pi\rho + \Lambda)r^2 = (re^{-\lambda})'. \tag{2.2.19}$$

Integrating the above equation we have

$$re^{-\lambda} = r - \left(\frac{8\pi\rho + \Lambda}{3}\right)r^3 + \alpha_1. \tag{2.2.20}$$

As $r \longrightarrow 0, \alpha_1 = 0$.

Let

$$\left(\frac{8\pi\rho + \Lambda}{3}\right) = \frac{1}{R^2}. \tag{2.2.21}$$

The equation (2.2.20), looks like

$$e^{-\lambda} = 1 - \frac{r^2}{R^2}. \tag{2.2.22}$$

Also the equations (2.2.16) and (2.2.17) and (2.2.19) yields

$$\frac{1}{r^2}[1 - e^{-\lambda}(r\nu' + 1)] = -8\pi p + \Lambda. \tag{2.2.23}$$

After subtracting the above equation from (2.2.20) we have

$$e^{-\lambda}\left(\frac{\nu' + \lambda'}{r}\right) = 8\pi(p + \rho). \tag{2.2.24}$$

After that we use $\dot{\nu} = 0$ and equation (2.2.22), the above equation looks like

$$\frac{\nu' e^{\nu/2}}{2} + \frac{2r/R^2}{2(1 - r^2/R^2)} e^{\nu/2} = \frac{4\pi\kappa r}{2(1 - r^2/R^2)}. \tag{2.2.25}$$

Solving the above equation we have

$$\left[\frac{e^{\nu/2}}{(1 - r^2/R^2)^{1/2}}\right]' = \frac{4\pi\kappa r}{[1 - r^2/R^2]^{3/2}}. \tag{2.2.26}$$

After integrating above equation we have

$$\frac{e^{\nu/2}}{(1 - r^2/R^2)^{1/2}} = \frac{4\pi\kappa R}{(1 - r^2/R^2)^{1/2}} + C, \tag{2.2.27}$$

here C is the constant of integration, so we have

$$e^{\nu} = \left(A + C\sqrt{1 - \frac{r^2}{R^2}}\right)^2. \tag{2.2.28}$$

Finally using equations (2.2.22) and (2.2.28) in equation (2.2.7) we have

$$ds^2 = -\left(A + C\sqrt{1 - \frac{r^2}{R^2}}\right)^2 dt^2 + \frac{dr^2}{\left(1 - \frac{r^2}{R^2}\right)} + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.2.29}$$

From equation (2.2.15) it follows that $C = 0$. Taking $A = 1$ equation (2.2.29) will become

$$ds^2 = -dt^2 + \frac{dr^2}{\left(1 - \frac{r^2}{R^2}\right)} + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.2.30}$$

The de Sitter Universe model was suggested in 1917. The static Universe model was proposed in the same year by Einstein. Equation (2.2.11) is taken for the de Sitter Universe, when both the pressure and density taken to be zero.

$$\rho = p = 0. \tag{2.2.31}$$

Using the above equation, equation (2.2.24) becomes

$$\nu = -\lambda. \tag{2.2.32}$$

Now the equation (2.2.22) can be written as

$$e^{-\lambda} = 1 - \frac{r^2}{R^2} = e^{\nu}. \tag{2.2.33}$$

We obtained the de Sitter metric by incorporating above equation

$$ds^2 = -\left(1 - \frac{r^2}{R^2}\right)dt^2 + \frac{dr^2}{\left(1 - \frac{r^2}{R^2}\right)} + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.2.34}$$

As $\rho = 0$ therefore the equation (2.2.21) gives $R^2 = \frac{3}{\Lambda}$. So

$$ds^2 = -\left(1 - \frac{r^2\Lambda}{3}\right)dt^2 + \frac{dr^2}{\left(1 - \frac{r^2\Lambda}{3}\right)} + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.2.35}$$

Chapter 3

Review of Gravitational Lensing

3.1 General Equations of Motion

We now consider the motion of a freely falling material particle or photon in a static isotropic gravitational field. First we consider the most general such metric in the standard form [20]

$$d\tau^2 = -B(r)dt^2 + A(r)dr^2 + D(r)d\Omega^2, \tag{3.1.1}$$

where $d\Omega^2{=}d\theta^2 + \sin^2\theta d\phi^2$, $d\tau^2$ is the time interval and A, B, D are arbitrary functions of r. The equations of free fall are

$$\ddot{x}^\gamma + \Gamma^\gamma_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = 0. \tag{3.1.2}$$

The nonzero Christoffel's symbols for the above metric are

$$
\begin{gathered}
\Gamma^r_{\phi\phi} = -\frac{r\sin^2\theta}{A(r)}, \qquad \Gamma^r_{rr} = \frac{A'(r)}{2A(r)}, \qquad \Gamma^r_{\theta\theta} = -\frac{r}{A(r)}, \\
\Gamma^r_{tt} = \frac{B'(r)}{2A(r)}, \qquad \Gamma^\theta_{r\theta} = \frac{1}{r}, \qquad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta, \\
\Gamma^\phi_{\phi r} = \frac{1}{r}, \qquad \Gamma^\phi_{\theta\phi} = \cot\theta, \qquad \Gamma^t_{tr} = \frac{B'(r)}{2B(r)}.
\end{gathered} \tag{3.1.3}
$$

From (3.1.2), the equations for t, r, θ, ϕ are:

$$
\begin{aligned}
0 &= \frac{d^2x^t}{dp^2} + \Gamma^t_{\alpha\beta}\frac{dx^\alpha}{dp}\frac{dx^\beta}{dp} \\
&= \frac{d^2t}{dp^2} + \frac{B'(r)}{B(r)}\frac{dt}{dp}\frac{dr}{dp};
\end{aligned} \tag{3.1.4}
$$

$$
\begin{aligned}
0 &= \frac{d^2x^r}{dp^2} + \Gamma^r_{\alpha\beta}\frac{dx^\alpha}{dp}\frac{dx^\beta}{dp} \\
&= \frac{d^2r}{dp^2} + \frac{A'(r)}{2A(r)}\left(\frac{dr}{dp}\right)^2 - \frac{r}{A(r)}\left(\frac{d\theta}{dp}\right)^2 - \frac{r\sin^2\theta}{A(r)}\left(\frac{d\phi}{dp}\right)^2 + \frac{B'(r)}{2A(r)}\left(\frac{dt}{dp}\right)^2;
\end{aligned} \tag{3.1.5}
$$

$$
\begin{aligned}
0 &= \frac{d^2x^\theta}{dp^2} + \Gamma^\theta_{\alpha\beta}\frac{dx^\alpha}{dp}\frac{dx^\beta}{dp} \\
&= \frac{d^2\theta}{dp^2} + \frac{2}{r}\frac{d\theta}{dp}\frac{dr}{dp} - \sin\theta\cos\theta\left(\frac{d\phi}{dp}\right)^2;
\end{aligned} \tag{3.1.6}
$$

$$
\begin{aligned}
0 &= \frac{d^2x^\phi}{dp^2} + \Gamma^\phi_{\alpha\beta}\frac{dx^\alpha}{dp}\frac{dx^\beta}{dp} \\
&= \frac{d^2\phi}{dp^2} + \frac{2}{r}\frac{d\phi}{dp}\frac{dr}{dp} + 2\cot\theta\frac{d\phi}{dp}\frac{d\theta}{dp}.
\end{aligned} \tag{3.1.7}
$$

Since the field is isotropic, we may consider the orbit of our particle to be confined to the equatorial plane ($\theta = \frac{\pi}{2}$). In this case, (3.1.6) is satisfied. Finally (3.1.4), (3.1.5) and (3.1.7) take the form

$$\frac{d^2r}{dp^2} + \frac{A'(r)}{2A(r)}\left(\frac{dr}{dp}\right)^2 + \frac{B'(r)}{2A(r)}\left(\frac{dt}{dp}\right)^2 - \frac{r}{A(r)}\left(\frac{d\phi}{dp}\right)^2 = 0, \tag{3.1.8}$$

$$\frac{d^2\phi}{dp^2} + \frac{2}{r}\frac{d\phi}{dp}\frac{dr}{dp} = 0, \tag{3.1.9}$$

$$\frac{d^2t}{dp^2} + \frac{B'(r)}{B(r)}\frac{dt}{dp}\frac{dr}{dp} = 0. \tag{3.1.10}$$

Calling $d\phi/dp, v$ in (3.1.9) and $dt/dp, u$ in (3.1.10), we obtain

$$0 = \frac{d}{dp}[\ln(vr^2)], \tag{3.1.11}$$

$$0 = \frac{d}{dp}[\ln uB(r)], \tag{3.1.12}$$

$$J = r^2\frac{d\phi}{dp}, \tag{3.1.13}$$

where J is constant. From (3.1.12) we see that

$$\frac{dt}{dp} = \frac{1}{B(r)}. \tag{3.1.14}$$

Inserting (3.1.13) and (3.1.14) in (3.1.8) we get

$$0 = 2A(r)\frac{d^2r}{dp^2} + \frac{B'(r)}{B^2(r)} + A'(r)\left(\frac{dr}{dp}\right)^2 - 2r\frac{J^2}{r^4}. \tag{3.1.15}$$

Equation (3.1.15) can be written as

$$0 = \frac{d}{dr}\left[A(r)\left(\frac{dr}{dp}\right)^2 - \frac{1}{B(r)} + \frac{J^2}{r^2}\right], \tag{3.1.16}$$

which, on integration yields

$$A(r)\left(\frac{dr}{dp}\right)^2 - \frac{1}{B(r)} + \frac{J^2}{r^2} = -E. \tag{3.1.17}$$

From the metric (3.1.1), the above equations yield

$$E = \left(\frac{1}{B(r)}\right) - \left[\frac{1}{B(r)} - \frac{J^2}{r^2} - E\right] - \frac{J^2}{r^2}. \tag{3.1.18}$$

We see that $E > 0$ for matter and $E = 0$ for light.

Deflection Angle

In GR one of the applications in which we are interested is the shape of the orbits, i.e r as a function of ϕ. From (3.1.13), (3.1.14) and (3.1.17) and simplifying we have

$$-E = \frac{A(r)}{B^2(r)}\left(\frac{dr}{dt}\right)^2 - \frac{1}{B(r)} + \frac{J^2}{r^2}. \tag{3.1.19}$$

Clearly

$$J = r^2 \frac{1}{B(r)}\frac{d\phi}{dt}. \tag{3.1.20}$$

From (3.1.19), we have

$$\frac{d\phi}{dr} = \pm\frac{1}{r^2}A^{1/2}(r)\Big[\frac{1}{J^2B(r)} - \frac{1}{r^2} - \frac{E}{J^2}\Big]^{-1/2}. \tag{3.1.21}$$

Integrating the above equation we get

$$\phi = \pm\int \frac{1}{r^2}A^{1/2}(r)\Big[\frac{1}{J^2B(r)} - \frac{1}{r^2} - \frac{E}{J^2}\Big]^{-1/2} dr + K. \tag{3.1.22}$$

here K is a constant of integration. Now, to find the deflection angle consider a particle or photon approaching from very great distances. As the metric is asymptotically flat we get constant velocity, $\underline{V}$. Leading to [20]

$$b \backsimeq r\sin(\phi-\phi_\infty) \backsimeq r(\phi-\phi_\infty)), \qquad -V \backsimeq \frac{dr}{dt}(r\cos(\phi-\phi_\infty)) \backsimeq \frac{dr}{dt}, \quad (3.1.23)$$

where b is the impact parameter and ϕ_∞ is the incident direction. Substituting in (3.1.19), the constant of motion is $E = 1 - V^2$ and since $V = 1$ for light, $E = 0$.

To express J in term of r_0 of the closest approach to the distribution of mass, rather than the impact parameter b. At r_0, $dr/d\phi$ vanishes and from (3.1.21), we have

$$J = r_0\Big[\frac{1}{B(r_0)} - 1 + V^2\Big]^{1/2}. \quad (3.1.24)$$

Using the above equation in (3.1.22) we have

$$\phi(r) = \phi_\infty + \int_r^\infty \frac{1}{r^2}A^{1/2}\Big[\frac{1}{J^2B(r_0)} - \frac{1}{r^2}\Big]^{-1/2}dr. \quad (3.1.25)$$

The total change in ϕ as r decreases gradually from infinity to its minimum value r_0 and then increases again to infinity is just twice its change from ∞ to r_0, that is, $2|\phi(r_0)-\phi_\infty|$. Since the particles move in straight lines, so this would just be π, i.e.

$$\Delta\phi = \hat{\alpha}(r_0) = 2|\phi(r_0)-\phi_\infty| - \pi. \quad (3.1.26)$$

For the positive direction, the angle ϕ changes by more than 180° that is, the trajectory is bent towards the mass distribution and for negative, the trajectory is bent away from the mass distribution.

We get the following expression by simplifying and inserting (3.1.26) in (3.1.27)

$$\hat{\alpha}(r_0) = 2\int_{r_0}^{\infty} \frac{1}{r}\sqrt{A(r)}\left\{\left(\frac{r}{r_0}\right)^2 \frac{B(r_0)}{B(r)} - 1\right\}^{-1/2} dr - \pi. \tag{3.1.27}$$

Photon Sphere

The photon sphere (which was earlier known as photo sphere in literature) is the region of a spacetime where gravity is strong enough that light is forced to travel in closed orbits. This mean that the Einstein bending angle of light ray with the closest approach r_0 becomes large as r_0 tends to r_{sp}. In order to obtain the equation for the photon sphere, set $r = constant$ in (3.1.8)

$$\frac{B'(r)}{A(r)}\left(\frac{dt}{dp}\right)^2 - \frac{2r}{A(r)}\left(\frac{d\phi}{dp}\right)^2 = 0. \tag{3.1.28}$$

Since it is photon sphere, so from equation (3.1.1) taking ($\theta = \frac{\pi}{2}$), $d\tau^2 = 0$ and simplifying we get

$$B(r)D'(r) = B'(r)D(r). \tag{3.1.29}$$

The above equation has at least one positive solution. The largest root will be the radius of the photon sphere (r_m).

Bozza's Method

Bozza's method is widely used to calculate the deflection angle in the strong field limit taking the photon sphere as the starting point. This is universal method and can be applied to any spacetime provided that the photons satisfy the standard geodesic equation. We are going to show that as we approach near the photon sphere the deflection angle diverges logarithmically. Bozza divided the deflection angle into the divergent and regular terms.

3.2 Divergent Term of the Deflection Angle

Define two new variables

$$
\begin{aligned}
y &= A(r), \\
z &= \frac{y - y_0}{1 - y_0},
\end{aligned}
\tag{3.2.1}
$$

z is the function of r only. The above new variables will help in calculating the divergent and regular term of the deflection. Here $y_0 = A(r_0)$, the equation (3.1.27) become

$$
\begin{aligned}
\hat{\alpha}(r_0) &= 2\int_{x_0}^{\infty} \left[\frac{B(r)}{D(r)}\right]^{1/2} \left[\frac{D(r)}{D_0}\frac{y_0}{y} - 1\right]^{-1/2} dr - \pi \\
&= 2\int_{r_0}^{\infty} \frac{B(r)y}{D(r)}\sqrt{D_0}\frac{dz}{\left[y_0 - [(1 - y_0)z + y_0]\frac{D_0}{D(r)}\right]^{1/2}} - \pi.
\end{aligned}
\tag{3.2.2}
$$

Using (3.2.1), the above equation takes the form

$$
\hat{\alpha}(r_0) = 2\int_0^1 \frac{\sqrt{B(r)y}}{D(r)A'(r)}(1 - y_0)\sqrt{D_0}\frac{dz}{\left[y_0 - [(1 - y_0)z + y_0]\frac{D_0}{D(r)}\right]^{1/2}} - \pi. \tag{3.2.3}
$$

Calling

$$
\begin{aligned}
2\frac{\sqrt{B(r)y}}{D(r)A'(r)}(1 - y_0)\sqrt{D_0} &= R(z, r_0), \\
\frac{1}{\left[y_0 - [(1 - y_0)z + y_0]\frac{D_0}{D(r)}\right]^{1/2}} &= h(z, r_0),
\end{aligned}
\tag{3.2.4}
$$

(3.2.3) can be expressed as

$$\hat{\alpha}(r_0) = \int_0^1 R(z, r_0) h(z, r_0) dz - \pi. \tag{3.2.5}$$

The function $R(z, r_0)$ is regular for all the values of z and r_0, whereas $h(z, r_0)$ diverges for $z \to 0$. In order to find the order of divergence of the integrand, the square root in $h(z, r_0)$ is be expanded upto the second order in z. For $z_0 = 0$ the expansion of $l(z, r_0) = y_0 - [(1 - y_0)z + y_0]D_0/D(r)$ is

$$\begin{aligned} l(z, r_0) &= \sum_{n=0}^{\infty} \frac{l^{(n)} z_0}{n!}(z - z_0)^n & (3.2.6) \\ &= l(0) + l'(0)z + \frac{l''(0)}{2} z^2 + ... & (3.2.7) \\ &= l(0) + pz + qz^2 + ... & (3.2.8) \end{aligned}$$

In order to find p and q, also from (3.2.1) we know that z is the function of x only for this reason we have

$$\begin{aligned} l'(z) &= \tfrac{d}{dr}\left\{y_0 - [(1 - y_0)z + y_0]\tfrac{D_0}{D(r)}\right\}\left[\tfrac{dz}{dr}\right]^{-1} & (3.2.9) \\ &= \tfrac{D_0[(1-y_0)z+y_0]D(r)'}{D(r)}\left[\tfrac{dz}{dr}\right]^{-1} - \tfrac{(1-y_0)D_0}{D(r)}. & (3.2.10) \end{aligned}$$

For $z = 0$, $r = r_0$, the above equation becomes

$$\begin{aligned} p = l'(z = 0) &= \frac{D_0 D_0'(1 - y_0)y_0}{D_0^2 A_0'} - \frac{(1 - y_0)D_0}{D_0} \\ &= \frac{(1 - y_0)}{D_0 A_0'}[D_0' y_0 - D_0 A_0']. \end{aligned} \tag{3.2.11}$$

Now to find the standard value for σ, we have

$$l''(z, r_0) = \frac{d}{dz}\left(\frac{dl}{dz}\right) = \frac{d}{dr}\left(\frac{dl}{dz}\right)\frac{dr}{dz} \tag{3.2.12}$$

$$\begin{aligned}\frac{d}{dr}\left(\frac{dl}{dz}\right) &= \frac{d}{dr}\left[\frac{(1-y_0)D_0}{D^2A'}[D'[(1-y_0)z+y_0]-DA']\right] \\ &= \frac{1-y_0D_0}{D^3A'^2}-(2D'A'+A''D)D'[(1-y_0)z+y_0] \\ &\quad + DA'((1-y_0(D''z+D'z')+y_0D'')+DD'A'^2.\end{aligned} \tag{3.2.13}$$

Since for $z=0$, that is $r=r_0$ (3.2.13) becomes

$$l''(0, r_0) = \frac{(1-y_0)^2}{D_0^2A_0'^3}\left[2D_0D_0'A_0'^2+(D_0D_0''-2D_0'^2)y_0A_0'-D_0y_0D_0'A_0''\right]. \tag{3.2.14}$$

Finally q looks like

$$q = \frac{1}{2}l''(0, r_0) = \frac{(1-y_0)^2}{2D_0^2A_0'^3}\left[2D_0D_0'A_0'^2+(D_0D_0''-2D_0'^2)y_0A_0'-D_0y_0D_0'A_0''\right]. \tag{3.2.15}$$

We can write $h(z, r_0)$ as

$$h(z, r_0) \sim h_0(z, r_0) = \frac{1}{\sqrt{pz+qz^2}}. \tag{3.2.16}$$

The integral diverges, when p vanishes. By examining the form p, we see that it vanishes at $r_0 = r_m$. To solve the integral we will split the integral into two pieces

$$I(r_0) = I_R(r_0) + I_D(r_0), \tag{3.2.17}$$

where

$$I_D(r_0) = \int_0^1 R(z, r_m) h_0(z, r_0) dz, \tag{3.2.18}$$

which contains the divergence and

$$I_R(r_0) = \int_0^1 [R(z, r_0) h(z, r_0) - R(0, r_m) h_0(z, r_0)] dz. \tag{3.2.19}$$

The divergent term is being substracted from the original term. To derive the expression $I_D(r_0)$ using the Taylor expansion for $h(z, r_0)$ we have

$$I_D(r_0) = R(0, r_m) \int_0^1 \frac{1}{\sqrt{pz + qz^2}} dz. \tag{3.2.20}$$

After solving, the integral becomes

$$I_D(r_0) = R(0, r_m) \frac{2}{\sqrt{q}} \ln\left(\frac{\sqrt{q} + \sqrt{q + p}}{\sqrt{q}}\right). \tag{3.2.21}$$

3.3 Regular Term Of The Deflection Angle

We have to expand $I_R(r_0)$ in powers of $(r_0 - r_m)$

$$I_R(r_0) = \sum_{n=0}^{\infty} \frac{1}{n!} (r_0 - r_m)^n \int_0^1 \frac{\partial^n g}{\partial r_0^n} \mid r_0 = r_m \; dz, \tag{3.3.1}$$

and evaluate the single coefficients. If we had not subtracted the singular part from $R(z, r_0) h(z, r_0)$, we would have an infinite coefficient for $n = 0$, while all other coefficients would be finite. However the function $g(z, r_0)$ is regular in $z = 0$. Since we are interested in terms upto $\mathcal{O}(r_0 - r_m)$, we will just retain the $n = 0$ term

$$I_R(r_0) = \int_0^1 g(z, r_m) dz + \mathcal{O}(r_0 - r_m), \tag{3.3.2}$$

and then

$$a_R = I_R(r_m), \tag{3.3.3}$$

is the term we need to add to b_D in order to get the regular coefficient.

Chapter 4

Applications of Bozza's Method

4.1 Schwarzschild BH Lensing

The simplest black hole solution is the Schwarzschild exterior metric

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{4.1.1}$$

where $f(r)$=$1 - 2m/r$.

4.1.1 Schwarzschild BH deflection angle

It is useful to write (4.1.1) in terms of Schwarzschild radius in terms of the dimensionles variable x =r/r_s, where $r_s = 2GM/c^2$. For the deflection angle, we must determine values for $R(z, x_0)$ and $h(z, x_0)$. Substituting values in (3.2.4) we obtain

$$R(z, x_0) = 2, \tag{4.1.2}$$

and

$$
\begin{aligned}
h(z, x_0) &= 1/\sqrt{(1-1/x_0) - [(1-(1-1/x_0))z + (1-1/x_0)]x_0^2/x^2} \\
&= 1/\sqrt{(2-3/x_0)z + (3/x_0-1)z^2 - z^3/x_0}.
\end{aligned}
\tag{4.1.3}
$$

The next step is to determine the photon sphere for the Schwarzschild metric using (4.1.2) in (3.2.11) and (3.2.15) to get

$$
p(x_0) = 2 - 3/x_0, \quad q(x_0) = 3/x_0 - 1. \tag{4.1.4}
$$

Using (4.1.4) in (3.2.16) we have

$$
h_0(z, x_0) = 1/\sqrt{(2-3/x_0)z + (3/x_0-1)z^2}. \tag{4.1.5}
$$

The regular term of the deflection angle can be calculated after substituting in (3.2.19)

$$
\begin{aligned}
I_R(x_0) &= \int_0^1 [R(z, x_m)h(z, x_m) - R(0, x_m)h_0(z, x_m)]dz \\
&= \int_0^1 [2/\sqrt{z^2 - 2/3z^3} - 2/z^2]dz.
\end{aligned}
\tag{4.1.6}
$$

The Schwarzschild deflection angle in the strong field becomes [7]

$$
\begin{aligned}
\alpha(\theta) &= -a/2\ln[(\theta D_{ol}/u_m - 1)] + \bar{b} - \pi \\
&= -\ln\left(\frac{2\theta D_{ol}}{3\sqrt{3}-1}\right) + \ln[216(7-4\sqrt{3})] - \pi,
\end{aligned}
\tag{4.1.7}
$$

where D_{ol} is the distance between the observer and the lens.

4.2 Reissner-Nordström Lensing

The Reissner-Nordström metric describes the gravitational field of a spherically symmetric massive object endowed with an electric charge q. The metric functions in standard coordinates are

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{4.2.1}$$

where $f(r) = 1 - 2m/r^2 + q^2/r^2$. For the deflection angle, we determine $R(z, x_0)$ and $h(z, x_0)$ respectively using values in (3.2.4)

$$\begin{aligned} R(z, x_0) &= 2\sqrt{(1 - 1/x + q^2/x^2)(1 - 1/x + q^2/x^2)^{-1}}/(1/x^2 - 2q^2/x^3), \\ h(z, x_0) &= 1/\sqrt{(2 - 3/x_0 + 3q^2/x_0)z + (3/x_0 - 1 - q^2/x_0^2)z^2 + (-1/x_0 + q^2/x_0)z^3}. \end{aligned} \tag{4.2.2}$$

From (3.2.11) and (3.2.15) we have

$$\begin{aligned} p(x_0) &= [2 - 3/x_0 + 4q^2/x_0^2](x_0 - q^2)/(x_0 - 2q^2), \\ q(x_0) &= [3/x_0 - 1 - 9q^2/x_0^2 + 8q^4/x_0^3]x_0(x_0 - q^2)^2/(x_0 - 2q^2)^3. \end{aligned} \tag{4.2.3}$$

To determine the photon sphere for the Reissner-Nordström metric, from (3.2.4) we have

$$0 = (2 - 3/x_0 + 4q^2/x_0^2)(x_0 - q^2)/(x_0 - 2q^2). \tag{4.2.4}$$

From the above equation we get

$$x_{m1} = 3 + \sqrt{9 - 32q^2}/4, \quad x_{m2} = 3 - \sqrt{9 - 32q^2}/4. \tag{4.2.5}$$

We chose x_{m1} as the photon sphere radius. To obtain $R(0, x_m)$, we set $z = 0$ and x_0=x_m

$$R(0, x_m) = 2\frac{(x_m - q^2)(x_m + \sqrt{x_m^2 - 4q^2x_m + 4q^2})}{x_m(x_m + \sqrt{x_m^2 - 4q^2x_m + 4q^2})) - 4q^2(x_m - q^2)}. \tag{4.2.6}$$

For the Reissner-Nordström metric, the regular term $I_R(x_0)$ cannot be calculated analytically. We can expand the integrand in (3.3.2) in powers of q

$$\begin{aligned} I_R(x_0) &= R(z, x_m)h(z, x_m) - R(0, x_m)h_0(z, x_m) \\ &= [R(z, x_m)h(z, x_m) - R(0, x_m)h_0(z, x_m)]|_{q=0} + q[\dot{R}(z, x_m)h(z, x_m) \\ &+ R(z, x_m)\dot{h}(z, x_m) - \dot{R}(0, x_m)h_0(z, x_m) - R(z, x_m)\dot{h_0}(z, x_m)]|_{q=0} \\ &+ \frac{q^2}{2}[\ddot{R}(z, x_m)h(z, x_m) + 2\dot{R}(z, x_m)\dot{h}(z, x_m) + R(z, x_m)\ddot{h}(z, x_m) \\ &- \ddot{R}(0, x_m)h_0(z, x_m) + 2\dot{R}(0, x_m)\dot{h_0}(z, x_m) - R(0, x_m)\ddot{h}_0(z, x_m)]|_{q=0} \\ &+ \mathcal{O}(q^4). \end{aligned} \tag{4.2.7}$$

Substituting (4.2.7) in (3.2.19) and simplifying we get

$$I_R(x_0) = 1/2\int_0^1 \left(8/9z - \frac{16/9 - 80/27z^3 + 24/27z^2}{(z\sqrt{1 - 2/3}z)^3}\right)dz. \tag{4.2.8}$$

Chapter 5

Deflection angle for the Schwarzschild-de Sitter Metric

The bending of the path of light by a gravitational field was one of the classical tests of general relativity proposed by Einstein [12]. While he calculated it for a very weak field, nowadays it is being computed for strong fields as well. This is relevant when light comes from a distant source and passes through a very massive galaxy, or galactic cluster or when light goes by a supermassive black hole and gets bent practically right round [20]. Many papers ignore cosmological effects (see for example [6, 21, 22]) but others incorporate them (see for example [23, 24]). Here the procedure and notation of Bozza [7] will be used to calculate the lensing in the Schwarzschild-de Sitter metric.

The Schwarzschild-de Sitter line element, which represents a black hole in an accelerating Universe, is [25]

$$ds^2 = -f(r)dt^2 + dr^2/f(r) - r^2(d\theta^2 + \sin^2\theta d\phi^2) \ ,$$
$$f(r) = (1 - 2m/r + \Lambda r^2/3) \ , \tag{5.0.1}$$

where m is the black hole mass and Λ is the cosmological constant. For light in the equatorial plane the geodesic equation yields the first integral

$$\dot{r} = L\sqrt{1/b^2 - 1/f(r)r^2} \ , \tag{5.0.2}$$

where L is a spin-like parameter and b is the impact parameter for the light. This leads to the equation

$$\dot{r}^2 + Lf(r)/r^2 = E \ , \tag{5.0.3}$$

where E is a constant corresponding to the total energy. It is useful to write the above metric in dimensionless variables: $\tau = s/2m$; $T = t/2m$; $X = r/2m$ and $\Lambda_1 = \Lambda \ 4m^2$; as

$$d\tau^2 = -f(x)dT^2 + f^{-1}(x)dX^2 + x^2(d\theta^2 + \sin^2\theta d\phi^2) \ . \tag{5.0.4}$$

The radius of the sphere of photons, x_p, is the largest root of the equation [9], $f'(x)/f(x) = 2/x$, which gives $x_p = 3/2$. The deflection angle for a photon coming from infinity with a distance of closest approach x_0 is

$$\alpha(x_0) = \int_{x_0}^{\infty} 2x_0 dx/[x\sqrt{f(x_0)x^2 - f(x)}] - \pi \ . \tag{5.0.5}$$

The deflection angle depends on the relation between x_0 and x_p which increases as

x_0 approaches to x_p.

The impact parameter b obtained by the conservation of angular momentum is $b = x_0/\sqrt{f(x_0)}$. The deflection angle for photons near the photon sphere

$$\alpha(x_0) = -a \ln(x/x_0 - 1) + a_1 + O(x - x_p) \ , \tag{5.0.6}$$

which has a logarithmic divergence. Here

$$a = \frac{(1 + \Lambda_1 x_p^3/3)/(1 - 2\Lambda_1 x_p/3)}{\sqrt{(1 + x^3\Lambda_1/3)^2/(1 - 2\Lambda_1 x_p^3/3)^3[3/x_p - 1 - 5\Lambda_1 x_p^2 + 8\Lambda_1 x_p^3/3]}} \ , \tag{5.0.7}$$

and

$$a_1 = -\pi + a_R + a \ln 2\eta \ , \tag{5.0.8}$$

where

$$\eta = \frac{(1 + x_p^3\Lambda_1/3)^2}{(1 - 2\Lambda_1 x_p^3/3)^3(3/x_p - 1 - 5\Lambda_1 x_p^2 + 8\Lambda_1 x_p^3)(1 - 1/x_p - \Lambda_1 x_p^2/3)} \ . \tag{5.0.9}$$

The value of the impact parameter is $b_p = x_p/\sqrt{3(x_p - 1) + \Lambda_1 x_p^3}$ at $x_0 = x_p$. The term a_R is evaluated up to second order in Λ_1, i.e.

$$a_R = a_{R,0} + a_{R,2}\Lambda_1^2 + O(\Lambda_1^4) \ , \tag{5.0.10}$$

where $a_{R,0}$ is the value for the unchanged black hole (Schwarzschild). The lens equation shown in Figure 1 (as modified from [26]) relates the positions and the magnification of the relativistic images and the deflection angle for the strong field

$$\beta = \theta - D_{LS}/D_S \Delta\alpha_n. \tag{5.0.11}$$

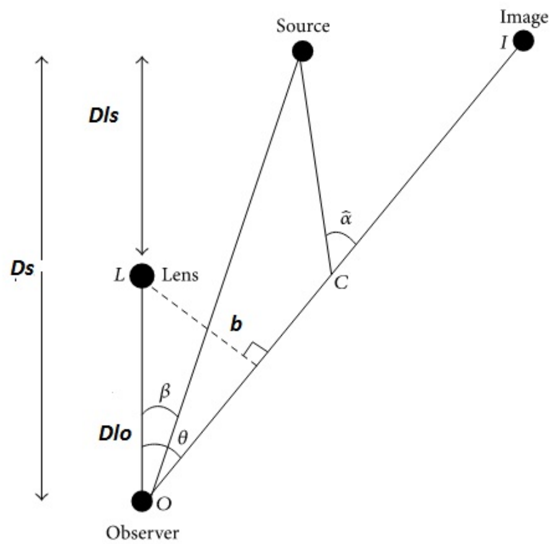

Figure 5.1: Basic lens formalism: Position of image, lens, source and observer are represented by I, L, S and O. The distances between the lens-observer, lens-source and source-observer are denoted by D_{LO}, D_{LS} and D_S. The line connecting O and L is called the optical axis of the lens geometry, The angle between the line joining the source and observer to the optical axis is β and of the image and observer is θ. The impact parameter is given by b.

Now according to the lens geometry, (5.0.6) can be written as

$$\alpha(\theta) = -a \ln \left(\frac{\theta D_{LO}}{b_p} - 1 \right) + a_1 \ , \tag{5.0.12}$$

where we have $b = D_{LO} \sin\theta \approx D_{LO}\theta$. D_{LO} is the distance between the lens and the observer. The n^{th} image position can be obtained by using the Taylor series.

$$\theta_n = \theta_n^0 + \frac{b_p e^{(a_1 - 2n\pi/a)}(\beta - \theta_n^0) D_S}{a_1 D_{LS} D_{OL}}. \tag{5.0.13}$$

Here $\theta_n^0 = b_p/D_{OL}[1+e^{(a_1-2n\pi/a)}]$ and $\xi_n = b_p e^{(a_1-2n\pi/a)}/a_1 D_{OL}$. The above formula is valid both for the images on the same side of the source and images on the opposite side. An infinite sequence of Einstein rings is obtained when $\beta = 0$ in (5.0.13)

$$\theta_n^E = \left(1 - \frac{\xi_n D_{OL}}{D_{LS}}\right)\theta_n^0 \ . \tag{5.0.14}$$

Another important quantity is the magnification of images. The magnification of the n^{th} relativistic image is given by

$$\mu_n = \frac{\sin\theta}{\sin\beta}\frac{\partial\beta}{\partial\theta}. \tag{5.0.15}$$

Letting the angle be small and using equation (5.0.13) and equation (5.0.15), the n^{th} relativistic image is

$$\mu_n = \frac{1}{\beta}\left(\theta_n^0 + \frac{\xi_n D_S}{D_{LS}}(\beta - \theta_n^0)\right)\frac{\xi_n D_S}{D_{LS}}. \tag{5.0.16}$$

Finally the magnification is

$$\mu_n = e^{(a_1-2n\pi)/a}\frac{b_p^2(1+e^{(a_1-2n\pi)/a})D_S}{a\beta D_{LO}^2 D_{LS}} \ , \tag{5.0.17}$$

which decreases very quickly as n increases.

The minimum impact parameter is now, $b_p = D_{LO}\theta_\infty$ (shown in Figure 5.2).

Here θ_∞ represents the asymptotic position of the set of the relativistic images obtained in the limit $n \to \infty$. For a single image we get the angular separation between the first and the limiting value, $s = \theta_1 - \theta_\infty$. The ratio between the flux of

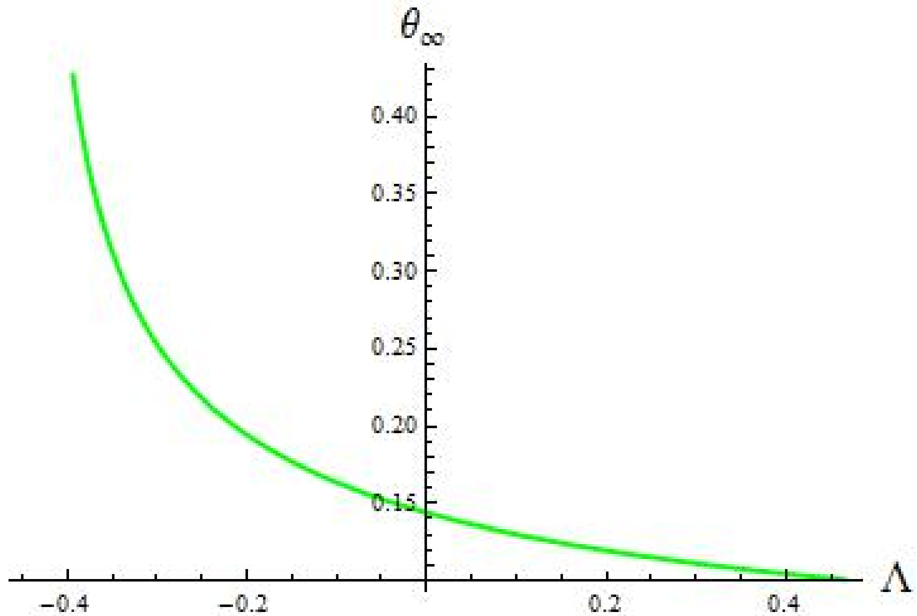

Figure 5.2: The asymptotic position of the set of images plotted against the dimensionless cosmological constant Λ.

the first image and the total flux from all images is

$$r = \mu_1 / \sum_{n=2}^{+\infty} \mu_n. \tag{5.0.18}$$

Chapter 6

Conclusion

In this dissertation, we have reviewed a general method to compute the coefficient of leading order divergent term and the regular term. We have studied the gravitational lensing in the strong field and extended this work to the Schwarzschild-de Sitter black hole.

We first reviewed the simplest case for the Schwarzschild black hole and calculated its bending angle. Here we used the procedure and notation of Bozza.

Furthermore we have calculated the bending angle for the Reissner-Nordström black hole. Finally we studied the bending angle for the Schwarzschild-de Sitter black hole. We plotted graphs to show the deflection angle.

Bibliography

[1] A. O. Peters, H. Levine and J. Wambsganss, *Singularity Theory and Gravitational Lensing*, Springer Science and Business Media, (2012).

[2] C. M. Will, "Henry Cavendish, John Soldner and the deflection of light", *Am. J. Phys.,* **56** (1988) 413-415.

[3] J. Soldner, "On the deflection of a light ray from its rectilinear motion", *Berliner Astronomisches Jahrbuch,* **27** (1801) 161-172.

[4] V. Bozza, "Extreme gravitational lensing by supermassive blackholes", *Nuovo Cimento B.* **122** (2007) 547-556.

[5] C. Darwin,"The gravity field of a particle", *Proc. R. Soc. Lond. A.* **249** (1959) 180-194.

[6] K. S. Virbhadra and G. F. R. Ellis, "Schwarzschild blackhole lensing", *Phys. Rev. D.* **62** (2000) (084003-084008).

[7] V. Bozza, "Gravitational lensing in the strong field limit", *Phys. Rev. D.* **66** (2002) (103001-103009 pages).

[8] A. Qadir, *Relativity: An Introduction to the Special Theory*, World Scientific Publishing Co. Ltd, (1989).

[9] V. Faraoni, *Special Relativity*, Springer International Publishing Switzerland, (2013).

[10] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields*, Butterworth-Heinemann, (1980).

[11] S. M. Carroll, *Spacetime and Geometry: An Introduction to General Relativity*, Addison-Wesley San Francisco, (2004).

[12] A. Qadir, *Einstein's General Theory of Relativity*, (unpublished).

[13] R. Ruffini and J. A. Wheeler, "Introducing the black hole", *Phys. Today.* **24** (1971) 30-34.

[14] K. Schwarzschild, B. Sitzungsbesichte, "On the gravitational field of mass point according to Einstein's theory", *Phys. Math Klasse.* **92** (1916) 189-196.

[15] H. Reissner, "Uber die Eigengravitation des elektrischen Feldes nach der Einsteinschen Theorie", *Ann. Phys.* **50** (1916) 106-120.

[16] G. Nordström, "On the energy of the gravitational field in Einstein's theory", *Proc. Kon. Ned. Akad. Wet.* **20** (1918) 1238-1245.

[17] R. Kerr and A. Schild, "A new class of vacuum solutions of the Einstein field equations", in Atti del convegno sulla relativita generale; problemi dellenergia e ondi gravitationali., *Barbra, Firenze.* **26** (1965) p222.

[18] E. T. Newmann and R. Penrose, "10 exact gravitationally conserved quantities", *Phys. Rev. Lett.* **15** (1965) 231-233.

[19] E. V. Linder, *First Principles of Cosmology*, Longman Singapore Publishers Ltd, (1997).

[20] S. Weinberg, *Gravitational and Cosmology: Principles and Applications of the General Theory of Relativity*, John Wiley and Sons, (1972).

[21] A. Younas, S. Hussain, M. Jamil and S. Bahamonde, "Strong gravitational lensing by Kiselev black hole", *Phys. Rev. D.* **92** (2015) (084042-084047).

[22] G. N. Gyulchev and S. S. Yazadjiev, "Gravitational lensing by Kerr-Sen dilaton-axion black hole in the weak deflection limit", *AIP Conf. Proc.* **1301** (2010) p327.

[23] S. Chakraborty and A. K. Sen, "Light deflection due to a charged rotating body", *Class. Quantum Grav.* **32** (2015) (115011-115017).

[24] E. F. Eiroa, G. E. Romero and D. F. Torres, "Reissner-Nordström black hole lensing", *Phys. Rev. D.* **66** (2002) (024010-024015).

[25] V. Kagrmanova, J. Kunz and C. Lammerzahl, "Solar system effects in the Schwarzschild-de Sitter spacetime", *Phys. Lett. B.* **634** (2006) (229-236).

[26] S. Sahu, M. Patil, D. Narasimha and P. S. Joshi, "Can strong gravitational lensing distinguish naked singularities from black hole?", *Phys. Rev. D.* **86** (2012) (475-480).