On Super Edge-Magic Total Labeling of Graphs

by

Muhammad Yasir Hayat Malik

Centre for Advanced Mathematics and Physics

National University of Sciences and Technology

Islamabad, Pakistan

July 2012

On Super Edge-Magic Total Labeling of Graphs

by

Muhammad Yasir Hayat Malik

A dissertation submitted in partial fulfillment of the requirements

for the degree of Master of Philosophy in Mathematics

Supervised by

Dr. Muhammad Imran

Centre for Advanced Mathematics and Physics

National University of Sciences and Technology

Islamabad, Pakistan

July 2012

Dedication

To my loving parents, preeminent friends

and

caring wife.

Abstract

A graph is said to be *labeled* if we assign different labels (usually non-negative integers) to the vertices or the edges (or both) of the graph, otherwise it is said to be *unlabeled*. These labels are used to identify the vertices and edges of a graph. The process of assigning labels to the vertices or edges of the graph is called *graph labeling*. A *super edge-magic total labeling* (SEMT labeling) of a graph G is a bijection $f : V(G) \cup E(G) \to \{1, 2, 3, \ldots, p+q\}$, such that in addition of being an edge-magic total labeling of G, it satisfies another property that $f : (V(G)) \to \{1, 2, 3, \ldots, p\}$. Also the egde-weights are calculated as: $\{f(u) + f(v) + f(uv) : xy \in E(G)\}$ such that all the edge-weights are same.

We constructed super edge magic total labeling of some families of acyclic graphs. The super edge-magic total labeling of *reflexive w-graphs, extended reflexive w-graphs, generalized reflexive w-graphs, generalized w-tree* and *generalized comb* was carried out and it was found that these graphs admit all the properties of super edge-magic total labeling.

Acknowledgement

I humbly bow my head before ALMIGHTY ALLAH, Who bestowed on me His blessings and gave me courage to present this piece of work. I invoke peace for Holy Prophet HAZRAT MUHAMMAD (PBUH) who is forever a torch of guidance for humanity as whole.

Being a student, I owe my thanks to Professor Dr. Azad Akhtar Siddiqui, Principal CAMP for giving me the opportunity to complete this project.

I am thankful to my supervisor Dr. Muhammad Imran for holding my hand and helping me to the way of light. His guidance to accomplish this work is note worthy. He was always there whenever I was in need.

I am also thankful to all the faculty members whose motivation and guidance worth a lot for me. I also owe my immense gratitude from the depth of my heart to my parents, my wife and other family members and friends, who provide me a full opportunity of devoting myself for M.Phil studies and whose prayers is the main cause of my success.

Muhammad Yasir Hayat Malik

Contents

Chapter 1

Introduction and Basic Concepts

In this chapter we give a brief introduction to the concept of graphs in discrete mathematics. We also include different graph theoretic terminologies and their explanation with examples. We also discuss about some graph classes and some examples related to these classes.

1.1 Graphs

How we can reduce the cost of cable to make telephone affordable to everyone? How we can create a fastest route from national to state capitals? How we can maximize people utilization by providing each of them with the required jobs? How we can assess the maximum flow/time from source to sink in a complex network of pipelines? How we can identify different layers; a computer chip needs so as to prevent the crossing over of wires from the same layers? How we can reschedule a sports league in order to reduce its time duration? What should be the visit schedule of a traveling salesman so that the traveling time can be minimized? How we can assign various colors to different regions of the map by using only four colors so that the adjacent sections received different colors? How we can reduce our diet in order to reduce our cholesterol level? How electricity can be made available to everyone by estimating our line losses? How we can raise the level of our school system by eliminating their drawbacks? How we can reduce the manufacturing cost of medicines in order to ensure their ease of availability? All of the above mentioned and many such closely related problems frequently involve graph-theory. Now we discuss some basic definitions correlated to graphs while some common families of graphs will be mentioned in detail in the upcoming chapters.

A *graph* G is a triple that comprises of a *vertex set* $V(G)$, an *edge set* $E(G)$, and a relationship which associates with each edge to vertices (not necessarily distinct) called its *endpoints*. A graph is usually drawn by placing each vertex at a point and expressing each edge by a curve linking the two vertices. Vertices are just points and edges are lines which join two distinct (or possibly same) points. We represent a graph by its vertex set and an edge set, with the edge set indicating a set of unordered pairs of vertices and writing $e = uv$ for an edge e with endpoints u and v.

Now we describe some basic definitions and terminologies related to graph theory.

Definition 1.1.1. The number of vertices in a graph G is called *order of G* and it is denoted by $n(G)$ while the number of edges in a graph G is called *size of G* and it is denoted by $e(G)$.

If the vertex v is the endpoint of an edge e, then e is said to be *incident* on v. The number of incident edges on the graph indicates the *degree* of a vertex v in a graph G; denoted by $d(v)$. A loop augments two in the degree of a vertex. The maximum degree of any vertex in a graph G is denoted as $\Delta(G)$, and the minimum degree in G is $\delta(G)$. A vertex of degree 0 is generally referred as an *isolated vertex* and a vertex of degree 1 is regarded as a *leaf*.

Definition 1.1.2. A graph having exactly one vertex represents a *trivial graph*. As it is known that the vertex set of every graph is non-empty and the order of every graph is at least 1, so the order of a *non-trivial graph* should be at least 2.

Definition 1.1.3. A graph H is recognized as a *subgraph* of a graph G, written $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

If the vertex set of a subgraph of a graph G and a graph G are the same, then it is called *spanning subgraph* of G.

A subgraph H of a graph G is called an *induced subgraph* of G if v and w are vertices of H and vw is an edge of G, then vw is an edge of H as well.

Definition 1.1.4. Two vertices are said to be *adjacent* if they are connected with an edge, otherwise they are called *non-adjacent vertices*. The *neighbourhood* of a vertex v in G, denoted as $N_G(v)$, is the set of all vertices adjacent to v in G.

Definition 1.1.5. A *loop* is an edge whose endpoints are similar. *Multiple edges* are the edges having the same pair of endpoints. A *simple graph* G generally has no loop and multiple edges while a graph with empty vertex and edge sets is said to be a *null graph*.

Definition 1.1.6. A graph G is said to be *finite* if its vertex set and edge set are finite, otherwise G will be an *infinite graph*.

Example 1.1.1. Consider a finite graph G, with the vertex set $V(G) = \{u, v, w, x\}$ and the edge set $E(G) = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$. The endpoints are associated to the edges as shown in the following diagram.

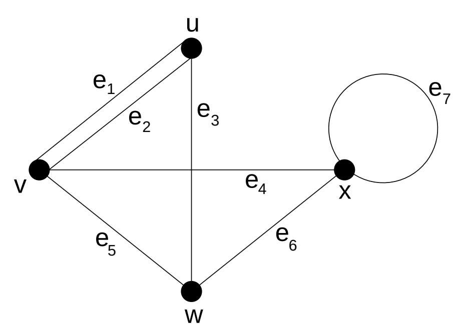

Figure 1.1: Graph

Here the edges e_1 and e_2 have the similar endpoints, so these are referred as multiple edges. The edges e_1,e_2,e_4 and e_5 are incident upon the vertex v, so the neighborhood of v in the graph G, denoted as $N_G(v)$ is $\{e_1, e_2, e_4, e_5\}$. The edge e_7 begins and terminates at the same vertex so it is a loop. Vertices u and x are not directly combined or connected with an edge so these are non-adjacent vertices. Clearly this graph does not appear to be simple.

Definition 1.1.7. A *clique* in a graph indicates a set of pairwise adjacent vertices whereas an *independent set* in a graph designates a set of pairwise non-adjacent vertices.

Definition 1.1.8. The *degree sequence* of a graph is a sequence formed by all the vertex degrees of the graph in ascending or descending order.

Example 1.1.2. Consider a finite graph G, with the vertex set $V(G) = \{u, w, x, y, z\}$ and the edge set $E(G) = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$. The endpoints are linked with the edges as shown in the diagram 1.2.

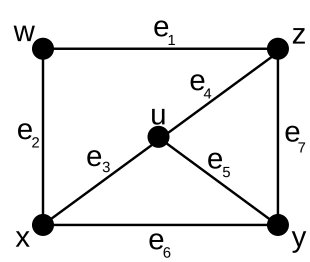

Figure 1.2: Graph

Here, the order and size of the graph G is 5 and 7 respectively. The independent set in the graph is $\{w, u\}$, as these vertices have been pairwise non-adjacent. The set of pairwise adjacent vertices or clique includes $\{x, y, u\}$.

Definition 1.1.9. A graph G is said to be *bipartite* if its vertex set $V(G)$ is the union of two disjoint collectively exhaustive independent sets, also known as *partite* sets of G.

A *bipartition* of G is a specification of two disjoint independent sets in G whose union is $V(G)$. A graph G is *k-partite* if $V(G)$ can be expressed as the union of k (possibly empty) independent sets. A disconnected bipartite graph has been associated with more than one bipartitions while a connected bipartite graph contains only one bipartition, except by interchanging the two partite sets.

Definition 1.1.10. A *cycle* refers to a closed path with an equal number of vertices and edges. Its vertices can be positioned around a circle so that two vertices are adjacent if and only if they appear consecutively along the circle. A cycle of order n is denoted as C_n. A cycle with an even number of vertices is called an *even cycle* and a cycle with an odd number of vertices is called an *odd cycle.*

Theorem 1.1.1 ([30])**.** *A graph G is bipartite if and only if it has no odd cycle.*

Example 1.1.3. Some bipartite graphs are shown in the following figure.

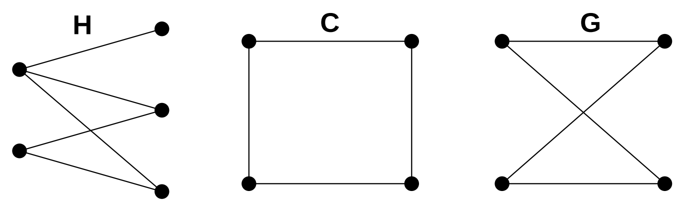

Figure 1.3: Some bipartite graphs

Definition 1.1.11. A *path* is a simple graph whose vertices can be ordered so that two vertices are adjacent if and only if they occur consecutively in the list. A path of order n is usually denoted as P_n. A path which begins and terminates at the same vertex is called a *closed path.* The *length* of a path is the number of edges in the path.

Definition 1.1.12. A graph G is said to be *connected* if each vertex in G is within reach from any other vertex of G, otherwise, G is *disconnected.* A vertex $u \in G$ is accessible from a vertex $v \in G$ if there is a sequence of vertices and edges from u to v.

Definition 1.1.13. A *component* of a graph G is the maximal connected subgraph of G. A component of a graph is trivial if it contains no edges; otherwise it is said to be nontrivial.

A *bridge* serves as an edge in such a way that its removal from a connected graph leaves a disconnected graph.

Remark 1.1.1. The components of a graph G are pairwise disjoint; no two share any vertex or edge. If we add an edge in the graph G with endpoints in different components, then both components will be merged into a single component. Similarly we can increase 0 or 1 to the number of components by omitting an edge from the graph.

Proposition 1.1.2 ([15]). *Every graph consisting of n vertices and k edges has at least $n-k$ components.*

Proof: Clearly a graph with n vertices and no edges comprises of n components. Since each added edge minimizes the number of components by at most 1, so by adding k edges, $n-k$ components will be left.

Next we present a theorem known as *handshaking lemma*, which provides a relation between the degree sum and size of a graph.

Proposition 1.1.3 ([30]). *If G is a graph, then*

$$\sum_{v\in V(G)} d(v) = 2e(G)$$

Proof: If the degrees of vertices of the graph are sum up then each edge will be counted twice. As each edge has two endpoints so it contributes to the degrees at each endpoint. It has been also termed as the degree sum formula.

1.2 Well-known Graph classes

In this section we present definitions of some well known and commonly used graph classes like grids, star, comb and ladder graphs, etc.

Definition 1.2.1. The *girth* of a graph with a cycle is the length of its shortest cycle.
A graph with no cycle indicates infinite girth.

Example 1.2.1. The following graph G is of order 7 and of size 11. Let $\{tu\}$ denote the edge between the vertices t and u.

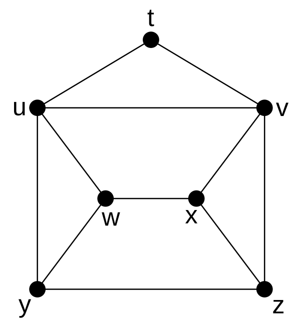

Figure 1.4: Graph

Let $\{tuy\}$ denote the sequence of vertices and edges such that the vertex t is connected with vertex u through an edge and u is connected with another vertex y. In the above figure, $\{tuyz\}$ and $\{tvxwyz\}$ are paths of orders 4 and 6 respectively. Also there are cycles $\{tuvt\}$, $\{tuyzvt\}$ and $\{wxyzw\}$ of orders 3, 5 and 4 in the above graph.

Definition 1.2.2. The *complete graph* is a simple graph in which every pair of vertices has been connected through an edge. A complete graph of order n is denoted as K_n. A complete bipartite graph, denoted as $K_{m,n}$ is a complete graph with m vertices in one partite set and n vertices in the other.

Definition 1.2.3. A graph all of whose vertex degrees are same is called a *regular graph.* All complete graph and complete bipartite graphs are regular.

Example 1.2.2. The complete graphs K_4 of order $n(K_4) = |V(K_4)| = 4$ and K_5 of order $|V(K_4)| = 5$ and complete bipartite graph $K_{3,3}$ are in Figure 1.5.

Definition 1.2.4. A graph having no cycle is called an *acyclic* graph.
A connected acyclic graph is referred as a *tree.* A tree of order n is denoted as T_n.
A *spanning tree* is a spanning subgraph which is tree.
The disjoint union of acyclic graphs is referred as a *forest.*

Lemma 1.2.1 ([7])**.** *Every tree with at least two vertices has at least two leaves. If we omit a leaf from an n-vertex tree then a tree with $n-1$ vertices is produced.*

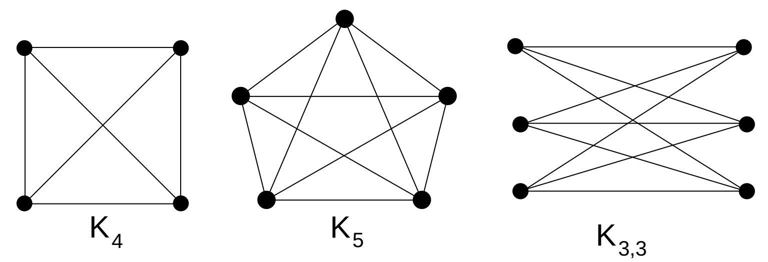

Figure 1.5: Complete graphs

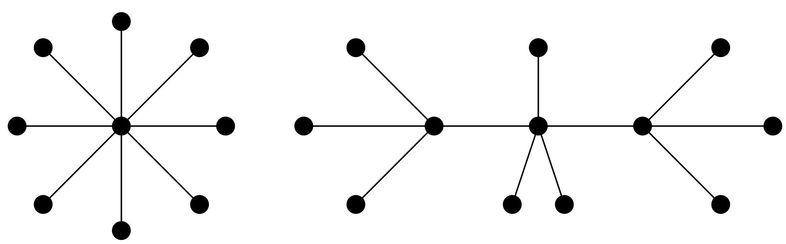

Figure 1.6: Trees

Theorem 1.2.2 ([15]). *A graph T is a tree if and only if it is connected and every edge in it is a* bridge.

Definition 1.2.5. If G contains a u, v-path, then the *distance* from u to v, written as $d_G(u, v)$, is the length of the minimal u, v-path. If G has no such path, then $d_G(u, v) = \infty$.

Definition 1.2.6. The *graph cartesian product* $G_1 \square G_2$, sometimes simply called the product of graphs G_1 and G_2 with vertex set $V(G_1 \square G_2) = V(G_1) \square V(G_2)$ and edge set $E(G_1 \square G_2) = E(G_1) \square E(G_2)$.

Definition 1.2.7. A *star* S_n is a complete bipartite graph $K_{1,n}$; a tree with one internal node and n leaves. A disjoint union of stars is called *galaxy*.

Definition 1.2.8. A *comb* Cb_n is a tree obtained by joining a path P_n with edges $\{x_{i+1}y_i : 1 \leq i \leq n-1\}$.

Definition 1.2.9. A *ladder* L_n is defined as $P_n \square P_2$, and a *circular ladder* (prism) D_n is defined as $C_n \square P_2$.

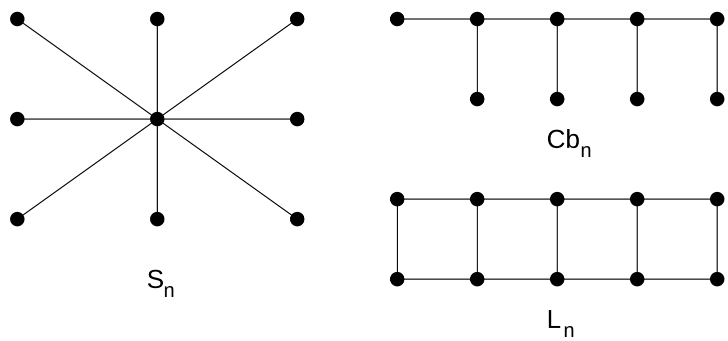

Figure 1.7: Star, Comb and Ladder graph

Definition 1.2.10. A *two-dimensional grid graph* $G_{m,n}$ is a graph cartesian product $P_m \square P_n$ of paths P_m and P_n.

Definition 1.2.11. The *n-book graph* B_n is defined as the graph Cartesian product $S_{n+1} \square P_2$, where S_{n+1} is a star graph and P_2 is the path graph.

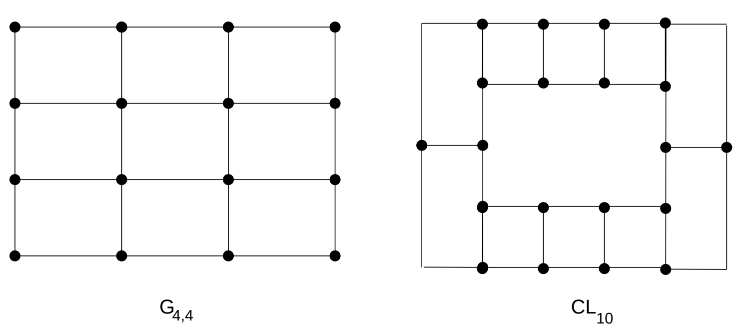

Figure 1.8: Grid and Circular ladder graph

Chapter 2

Graph Labeling

Assigning labels to vertices and edges of a graph has a long history in graph theory. Probably the most famous example is the Four Color Problem [8]. If a map is partitioned into regions in some manner, what is the minimum number of colors required if the neighboring regions have different colors? The conjecture that every map can be colored with at least four colors is known as the Four Color Conjecture.

The process of assigning labels to the vertices or edges of the graph is called *graph labeling*. In theory of graph labeling, the labels are always mathematical objects, for example integers, prime numbers or elements of a group, their mathematical properties are used via an evaluating function which assigns a value to a vertex or edge. The evaluating function produces partial sums called vertex-weights or edge-weights. A vertex-weight is the sum of labels of the vertex and its incident edges. Similarly an edge-weight is the sum of the labels of edges and its end vertices.

When all edge-weights or all vertex-weights are the same then the labeled graph is called edge-magic or vertex-magic, respectively. In another situation, when all edge-weights or all vertex-weights are different then the labeled graph is called edge antimagic or vertex antimagic, respectively.

Sedláček [26] defined a graph to be *magic* if it has an edge labeling with real numbers as its range, such that the sum of the labels of the edges incident to a vertex is constant, independent of the choice of the vertex. Stewart [28] called this labeling *super-magic* if the labels are consecutive integers starting from 1. In 1970, Kotzig et al. [21] defined a magic labeling of a graph $G(V, E)$ as a bijection $f : V(G) \to E(G)$ such that there exists an integer λ satisfying $f(u) + f(uv) + f(v) = \lambda$ for any edge $uv \in E(G)$. The integer λ is called the *magic constant*. To avoid ambiguity from the notation used by Stewart [28] we will call this labeling an edge-magic total labeling. Enomoto et al. [9] call the labeling f as a super edge-magic total labeling when the vertices are labeled with the smallest possible labels. Wallis [29] call this labeling strongly edge-magic. Enomoto et al. [9] conjectured that every tree admits a super edge-magic total labeling. To prove this conjecture, many authors have considered super edge-magic total labeling for some particular trees. Lee and Shah [22] have proved this conjecture for the trees with upto 17 vertices using computer. Earlier, Kotzig et al. [21] proved that every caterpillar is super edge-magic total.

The interest on this subject is due to the wide range of applications in other branches of science such as crystallography, radar theory, coding theory, x-ray, chemical compounds in organic chemistry, circuit design and communication networking.

A graph is said to be *labeled* if we assign different labels to the vertices or the edges (or both) of the graph, otherwise it is said to be *unlabeled*. These labels are used to identify the vertices and edges of a graph. We can use numerical digits or weights to label the vertices and edges. In this thesis we are interested in the study of labeled and simple graphs.

The main classes of labelings are:

- Edge labeling - When all the edges of the graph are labeled.
- Vertex labeling - When all the vertices of the graph are labeled.
- Total labeling - When all the vertices and edges of the graph are labeled.
- Super total labeling - When all the vertices and edges of the graph are labeled and the smallest possible labels are assigned to the vertices of the graph.

A labeling is any mapping that sends some set of graph elements to the set of non-negative integers. If the domain of this mapping is the vertex-set then the mapping is called *vertex labeling*. If the domain of this mapping is the edge-set then the mapping is called *edge labeling*. When the domain of this mapping is $\{V \cup E\}$, we label all the vertices and edges of the graph and the labeling is called *total labeling*. In a total labeling if the smallest possible labels are assigned to the vertices of the graph, then the labeling is called *super total labeling*.

Informally, by a graph labeling we mean an assignment of integers to the elements of a graph, such as the vertices, or edges or both, subject to some specified conditions. These conditions are usually expressed on the basis of the values (called weights) of some evaluating function. In our case, the evaluating function will be simply to produce the partial sums of the labeled elements of the graph. The partial sums will be either a (multi) set of vertex-weights, obtained for each vertex by adding all the labels of a vertex and its adjacent edges, or a (multi) set of edge-weights, obtained for each edge by adding the labels of an edge and its endpoints.

2.1 Graceful labeling

Rosa [25] in 1967 introduced a vertex labeling as a bijective function f from the set of vertices in a graph G to the set $\{0, 1, 2, 3, \ldots, q\}$, where q is the number of edges in G, so that each edge $xy \in E(G)$ is assigned the label $|f(x) - f(y)|$, where all the labels are distinct, and the absolute value of the difference of $f(x)$ and $f(y)$ is called the weight of the edge xy. Rosa [25] called this type of labeling β-valuation. Golomb [12] also searched out the same kind of labeling independently, and called this labeling *graceful labeling*. This labeling was developed to solve the problem of the decomposition of a complete graph into isomorphic subgraphs.

If the graph G admits a graceful labeling, it is said to be a *graceful graph*. Applications of the graph labeling has been found in x-ray crystallography, coding theory, radar, circuit design, astronomy and communication design. The graceful labelings of the graphs K_4, C_4 and C_5 are shown in the following figure. The numbers in italic shows the edge-weights which are the absolute differences of the labels of the adjacent vertices.

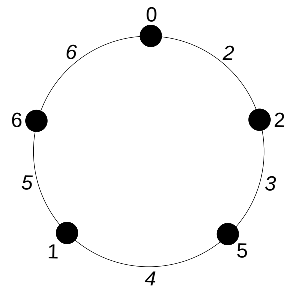

Figure 2.1: Graceful Graph

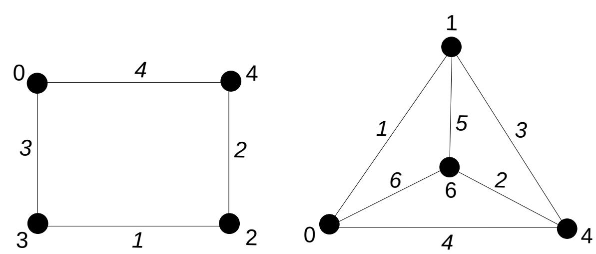

Figure 2.2: Graceful Graphs

2.2 Cordial labeling

Let G be a graph and f be a function from the set of vertices $V(G)$ to $\{0,1\}$ and for each edge $uv \in E(G)$ assign the label $|f(u) - f(v)|$. The graph G is said to be a *cordial* if

(i) the number of vertices labeled 0 and the number of vertices labeled 1 differ by at most 1.

(ii) the number of edges labeled 0 and the number of edges labeled 1 differ by at most 1.

The cordial labeling of a graph is shown in figure 2.3.

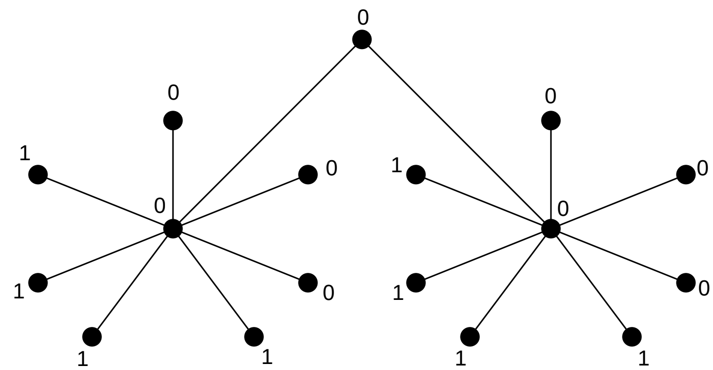

Figure 2.3: Cordial labeling of a graph

2.3 Equitable labeling

A labeling f of the vertices of a graph G is called k-*equitable* if each weight induced by f on edges of G appears exactly k times. A graph G is called *equitable* if for every proper divisor k (say) of its size, the graph G is k-equitable.

The 5-equitable labeling of a graph is shown in the figure 2.4.

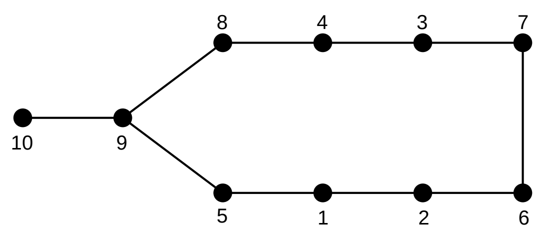

Figure 2.4: Equitable labeling of a graph

2.4 Harmonious labeling

This is another kind of vertex labeling first introduced by Graham and Sloan [13] in 1980 to study additive basis. A vertex labeling of a graph G is called *harmonious* if f is an injective mapping from the vertex-set $V(G)$ to the additive group Z_E, such that the mapping $f^{\cdot}$ from the edge-set $E(G)$ to Z_q defined by

$$f^{\cdot}(uv) = f(u) + f(v)$$

for every $uv \in E(G)$, and this mapping assigns different labels to the edges of G. If the graph G admits a harmonious labeling then it said to be a *harmonious graph*.

Recently, it is proved that no graph is neither graceful (Erdöes unpublished results) nor harmonious [13]. A graph labeling is an injective (one-one) function whose domain is the elements of a graph (vertices, edges or faces), to the set of labels (usually a subset of the set of natural numbers or non-negative integers). According to the different types of above mentioned labelings, there are five possibilities of the evaluations of a specific labeling of a graph; vertex evaluation, edge evaluation, face evaluation, total evaluation or super total evaluation.

2.5 Magic labeling

In a magic labeling, the weights of all the vertices (or edges) in the graph are same, so there are two possibilities for a magic labeling to be;

- Edge-magic labeling: when the weights of all the edges are same.
- Vertex-magic labeling: when the weights of all the vertices are same.

The study of face labeling is beyond the interest of this thesis.

2.5.1 Edge-magic labeling

Sedlàček [26] discussed a labeling in his paper, which was inspired by the *magic square* of the number theory. He called that labeling *magic*. In a magic square of order n, all the rows, columns and both diagonals of the magic square sum to the same constant. The constant is called the *magic constant*. According to his definition, a magic labeling is a function f from the set of edges of a graph G to the finite subset of the set of real numbers, such that the sums of the edge labels of the edges incident upon a vertex in G is the same, and is equal to a fix constant, for every vertex. That constant was named as the *index* of the labeling. We can observe that Sedlàček used the real labels in his labeling.

The magic labeling of $K_{3,3}$ and the corresponding magic square is shown in the following diagram.

	u_1	u_2	u_3
v_1	2	7	6
v_2	9	5	1
v_3	4	3	8

Magic square of order 3

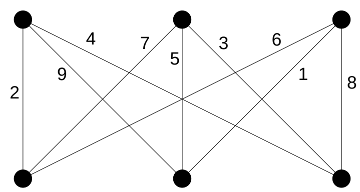

Figure 2.5: Edge labeling

The edge-magic labelings of some other graphs are shown in the following figure, with magic constants 83 and 28 respectively.

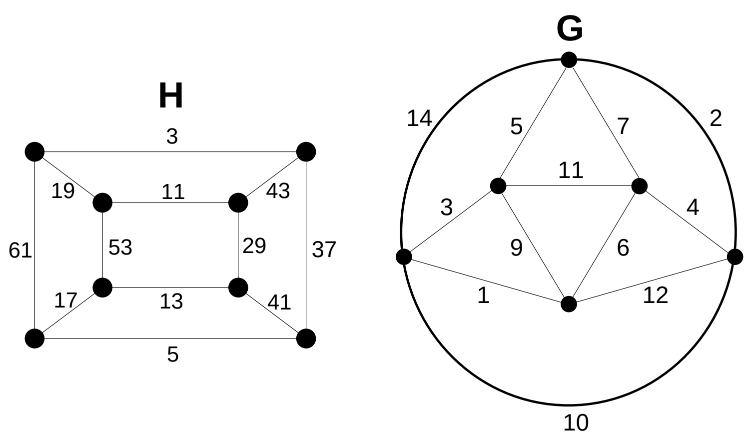

Figure 2.6: Magic Graphs

Stewart [28] proved that for complete graph K_n, for $n \geq 5$ and basket graph B_n for $n = 4$ and $n \geq 6$, are magic, while fan graphs F_n are magic only for n odd and for $n \geq 3$.

Stewart called the magic labeling, *super-magic*, if the set of edge labels consisted of consecutive integers.

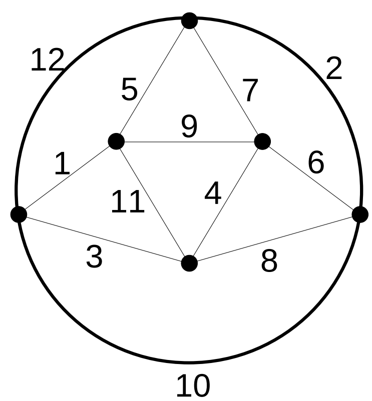

Figure 2.7: Super-Magic Graph

Stewart, in his paper about complete graphs, introduced a magic labeling where the labels of edges do not need to start from 1, but can start from any positive integer. He called this labeling *semi magic*. If distinct integers are used to label the edges, then he called the same labeling, *magic*. The *super-magic labeling* is a magic labeling such that the set of edge labels is a set of consecutive integers.

Theorem 2.5.1 ([3]). *If a bipartite graph G is decomposable into two hamiltonion cycles then G is super-magic.*

Theorem 2.5.2 ([2]). *The complete graph K_n is super-magic if and only if either $n \geq 6$ and $n \not\equiv 0\,(mod\ 4)$, or $n = 2$.*

Kong, Lee and Sun [20] defined the concept of magic labeling in another way. They called a labeling *magic* if the edges are labeled with non-negative integers such that for each vertex v, the sum of the labels of all the edges incident on v is the same for all vertices.

Edge-magic total labeling

The edge-magic total labeling was introduced by Kotzig and Rosa [21] in 1970. They called this labeling *magic*, but to distinguish the magic labeling defined by Stewart it has been agreed to call this labeling *edge-magic total.*

An edge-magic total labeling of a graph G is a bijection f from $V(G) \cup E(G)$ onto the set of integers $\{1, 2, 3, \ldots, p + q\}$ such that for all edges $uv \in E(G)$

$$f(u) + f(v) + f(uv) = k$$

for some constant k. The constant k is called a magic sum of G. A graph with an edge-magic total labeling is called *edge-magic total.* For the sake of ease, the edge-magic total labeling is abbreviated as EMT labeling.

In the Figure 2.8 , the graphs $K_4 - e$ and C_6 are labeled using edge-magic total labeling with magic constants 12 and 17 respectively, are shown.

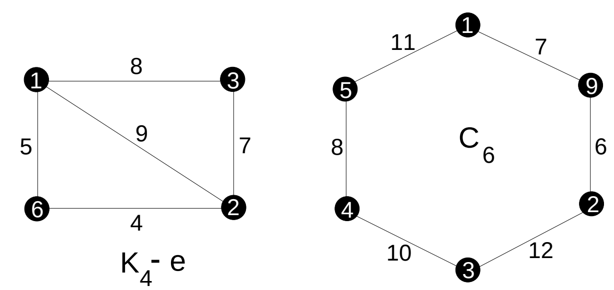

Figure 2.8: Edge-Magic Total Graphs

Kotzig and Rosa [21] proved that complete bipartite graph $K_{m,n}$ is edge-magic total, for any m and n, and also proved that the class C_n has an Edge-magic total labeling for all $n \geq 3$.

Baskoro et al. [4] proposed a secret sharing scheme construction using edge-magic labeling, which was based on Bloom and Golomb's results. Wallis [29] proposed edge-magic total labeling for assigning addresses of communication network and radar pulse codes.

Enomoto et al. [9] defined the concept of *super edge-magic total labeling* (SEMT labeling) of a graph G as a bijection $f : V(G) \cup E(G) \rightarrow \{1, 2, 3, \ldots, p+q\}$, such that in addition of being an edge-magic total labeling of G, it satisfies another property that $f : (V(G)) \to \{1, 2, 3, \ldots, p\}$. Wallis [29] calls this labeling *strongly edge-magic.*

The super edge-magic total labelings of C_3 and C_7 are shown as follows. The magic constant of C_3 is 9 and that of C_7 is 19.

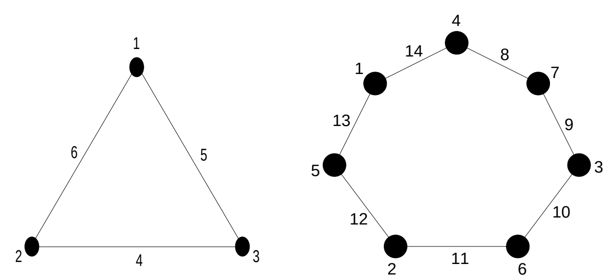

Figure 2.9: Super edge-magic total graphs

We have a lemma which provides a more easier necessary and sufficient condition for a graph to be super-edge-magic total.

Lemma 2.5.3 ([10])**.** *A (p, q) graph G is said to be super edge-magic total if and only if there exists a bijective function $f : V(G) \to \{1, 2, 3, \ldots, p\}$ such that the set*

$$S = \{f(u) + f(v) : uv \in E(G)\}$$

consists of q consecutive integers.
In such a case, f extends to a super edge-magic total labeling of G with constant $k = p + q + s$, where $s = min(S)$ and

$$S = \{k - (p + 1), k - (p + 2), k - (p + 3), \ldots, k - (p + q)\}.$$

Result: Enomoto et al. [9] proved that

- The cycle C_n is super edge-magic total if and only if n is odd.
- The complete bipartite graph $K_{m,n}$ is super edge-magic total if and only if $m = 1$ or $n = 1$.

- The complete graph K_n is super edge-magic total if and only if $n = 1, 2$ or 3.

Enomoto et al. proposed the following conjecture.

Conjecture 2.5.4 ([9]). *Every tree is super edge-magic total.*

Lee and Shah [22] have verified this conjecture for the trees with up to 17 vertices, by using a computer.

Kotzig and Rosa [21] proved that all caterpillars are super edge-magic total.

In the following, we present a very comprehensive list of graphs labeled with SEMT labeling and some graphs which are not SEMT. The source of this data is a survey on graph labelings conducted by J.A. Gallian [11].

Graph	**Types**	**Notes**
C_n	SEM	iff n is odd
caterpillars	SEM	
$K_{m,n}$	SEM	iff $m = 1$ or $n = 1$
K_n	SEM	iff $n = 1, 2$ or 3
trees	SEM?	
nK_2	SEM	iff n odd
nG	SEM	if G is a bipartite or tripartite SEM graph and n odd
$K_{1,m} \cup K_{1,n}$	SEM	if m is a multiple of $n+1$
$K_{1,m} \cup K_{1,n}$	SEM?	iff m is a multiple of $n+1$
$K_{1,2} \cup K_{1,n}$	SEM	iff n is a multiple of 3
$K_{1,3} \cup K_{1,n}$	SEM	iff n is a multiple of 4
$P_m \cup K_{1,n}$	SEM	if $m \geq 4$ is even
$2P_n$	SEM	iff n is not 2 or 3
$2P_{4n}$	SEM	for all n
$K_{1,m} \cup 2nK_{1,2}$	SEM	for all m and n

Graph	**Types**	**Notes**
$C_3 \cup C_n$	SEM	iff $n \geq 6$ even
$C_4 \cup C_n$	SEM	iff $n \geq 5$ odd
$C_5 \cup C_n$	SEM	iff $n \geq 4$ even
$C_m \cup C_n$	SEM	if $m \geq 6$ even and n odd, $n \geq m/2 + 2$
$C_m \cup C_n$	SEM?	iff $m + n \geq 9$ and $m + n$ odd
$C_4 \cup P_n$	SEM	iff $n \in 3$
$C_5 \cup P_n$	SEM	if $n \in 4$
$C_m \cup P_n$	SEM	if $m \geq 6$ even and $n \geq m/2 + 2$
$P_m \cup P_n$	SEM	iff $(m, n) \in (2, 2)$ or $(3, 3)$
corona $C_n \odot \bar{K_m}$	SEM	$n \geq 3$
$St(m, n)$	SEM	$n \equiv 0 \, mod(m + 1)$
$St(1, k, n)$	SEM	$k = 1, 2$ or n
$St(2, k, n)$	SEM	$k = 2, 3$
$St(1, 1, k, n)$	SEM	$k = 2, 3$
$St(k, 2, 2, n)$	SEM	$k = 1, 2$
$St(a_1, ..., a_n)$	SEM?	for $n > 1$ odd
friendship graph of n triangles	SEM	iff $n = 3, 4, 5,$ or 7
generalized Petersen graph $P(n, 2)$	SEM	if $n \geq 3$ odd
nP_3	SEM	if $n \geq 4$ even
$P_3 \cup kP_2$	SEM	for all k
kP_n	SEM	if k is odd
$k(P_2 \cup P_n)$	SEM	if k is odd and $n = 3, 4$
fans F_n	SEM	iff $n \leq 6$
books B_n	SEM	if n even
books B_n	SEM?	if n even or $n \equiv 5 \, mod(8)$

Graph	Types	Notes
$G \odot \bar{K_n}$	SEM	if G is SEM 2-regular graph
if G is k-regular SEM graph		then $k \leq 3$
G is connected (p, q)-graph	SEM	G exists iff $p-1 \leq q \leq 2p-3$
G is connected 3-regular graph on p vertices	SEM	iff $p \equiv 2\,mod(4)$
$nK_2 + nK_2$	not SEM	

2.5.2 Vertex-magic labeling

MacDougall et al. [23] introduced the concept of *vertex-magic total labeling.*

A bijection from $V(G) \cup E(G)$ onto the set of integers $\{1, 2, 3, \ldots, p+q\}$ is called a *vertex-magic total labeling* (VMT labeling), if there is a constant so that, for every vertex $v \in V(G)$

$$f(v) + \sum_{u \in N_u(G)} f(uv) = k.$$

The constant k is called the magic constant for f. A graph that admits a vertex-magic total labeling is called a *vertex-magic total graph.*

So, for a graph to be a vertex-magic total graph, all of its vertex-weights must be same.

The vertex-magic total labeling of a tree T_6 and a cycle C_7 is shown below.

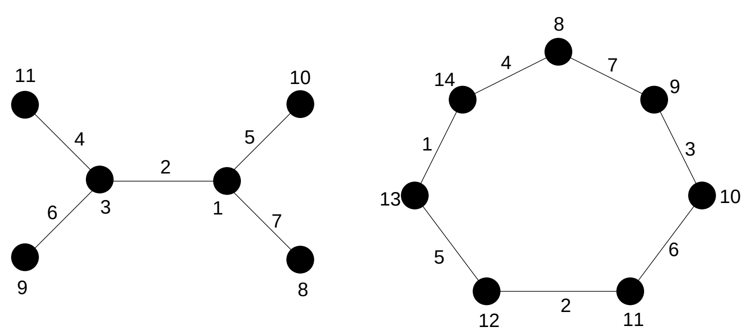

Figure 2.10: Vertex-Magic Total Graphs

Lin and Miller [2] proved that all complete graphs of order divisible by 4 are vertex-magic total. Afterwards it was also shown that all other complete graphs are also vertex-magic total.

Gray et al. [14] examined the existence of vertex-magic total labelings of trees and forests.

2.6 Antimagic labeling

An *antimagic square* of order n is an arrangement of numbers 1 to n^2 in the square, such that the sum of n rows, n columns and both diagonals form a sequence of $2n+2$ consecutive integers.
Two antimagic squares of order 4 are shown below.

4	13	12	1
11	6	2	14
5	15	10	8
16	3	7	9

1	13	3	12
15	9	4	10
7	2	16	8
14	6	11	5

In each of these two antimagic squares, the rows, columns, and the diagonals sum to ten different numbers in the range $29-38$. As the order increases, the construction becomes easier. The smallest antimagic square is of order 4.
When all the weights of the graph are not equal, then it is called *antimagic labeling*. For a particular labeling, if all the calculated weights of the elements of a graph form an arithmetic progression, starting from any constant a (>0) and having common difference d (≥ 0), the labeling is called an (a, d)-*antimagic labeling*. A graph having an antimagic labeling is called an *antimagic graph*. If $d = 0$ the labeling becomes a magic labeling. Hence the magic labeling is a spacial case of (a, d)-antimagic labeling when common difference of the arithmetic progression is zero.

The notion of an antimagic graph was first introduced by Hartsfield and Ringel [16] in 1989, afterwards Nicholas et al. Bodendiek and Walther [5] in 1996 were the first to introduce the concept of (a, d)-vertex antimagic edge labelings. They called that labeling (a, d)-*antimagic labeling*.

Hartsfield and Ringel [16] pointed out that all paths P_n for $n \geq 3$, cycles C_n, wheels W_n, and complete graphs K_n for $n \geq 3$, are antimagic.

Antimagic labelings can be subcategorized in the following two ways.

- Edge antimagic labeling - When edge-weights form an arithmetic progression.
- Vertex antimagic labeling - When vertex-weights form an arithmetic progression.

2.6.1 Edge antimagic labeling

In an *edge antimagic labeling*, the vertices or the edges and vertices both are labeled. In such a labeling, the set of edge-weights (say) $W = \{w(uv) : uv \in E(G)\}$ admits an arithmetic progression, starting from a constant a and having common difference d. Formal definition of this kind of labeling is given below.

Edge antimagic labeling can be further classified in the following two ways, according to the choice of labeling different elements of the graph.

- Edge antimagic vertex labeling (EAV labeling),
- Edge antimagic total labeling (EAT labeling).

First we discuss the edge antimagic vertex labelings of the graphs.

Edge antimagic vertex labeling

An (a, d)-*edge antimagic vertex labeling* of a graph $G(V, E)$ is an injection f from $f : V(G) \to \{1, 2, 3, \ldots, p\}$ such that the set of edge-weights of all the edges in G is $W = \{f(u) + f(v) : uv \in E(G)\} = \{a, a + d, a + 2d, \ldots, a + (q - 1)d\}$, where $a > 0$ and $d \geq 0$ are two fixed integers.

In short this labeling is called an (a, d)-*EAV labeling.*

Figueroa et al. [10] gave a very useful lemma which provides a relationship between the edge antimagic vertex labeling and the super edge magic total labeling.

Lemma 2.6.1 ([10])**.** *A graph G with p vertices and q edges is super edge-magic total if and only if there exists a bijective function $\lambda : V(G) \to \{1, 2, \cdots, p\}$ such that the set $S = \{\lambda(x)+\lambda(y)|xy \in E(G)\}$ consists of q consecutive integers. In such a case, λ extends to a super edge-magic total labeling of G.*

The Figure 2.11 is the $(5, 1)$-edge antimagic vertex labeling of a cycle C_7, where the set of edge-weights is $\{5, 6, 7, 8, 9, 10, 11\}$.

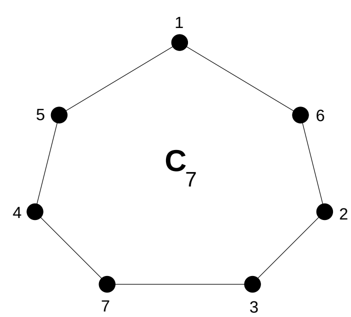

Figure 2.11: (5,1)-edge antimagic labeling of C_7

Lemma 2.6.2 ([3])**.** *Every path P_n has* (3,2)-EAV labeling*.*

Conjecture 2.6.3 ([16])**.** *Every connected graph except K_2 is antimagic.*

Edge antimagic total labeling

The notions (a, d)-*edge antimagic total labeling* and *super*(a, d)-*edge antimagic total labeling* are the natural extensions of the notions of *edge-magic labeling* which was introduced by Kotzig et al. [21], and *super-magic labeling* which was introduced by Enomoto et al [9]. Wallis et al. [29] use the term *strongly edge-magic labeling* in place of *super edge-magic total labeling.*

The definition of an (a,d)-*edge antimagic total labeling* was introduced by Simanjuntak et al. [27], and *super*(a,d)-*edge antimagic total labeling* was introduced by Enomoto et al. [9].

An (a,d)-*edge antimagic total labeling* of a graph $G(V,E)$ is an injection f from $V(G)\cup E(G)$ onto the set $\{1,2,3,\ldots,p+q\}$ such that the set of edge-weights of all the edges in G is

$$W=\{a,a+d,a+2d,\ldots,a+(q-1)d\}$$

where $a>0$ and $d\geq 0$ are fixed integers and the edge-weights under this labeling are calculated as

$$\{f(u)+f(v)+f(uv): uv\in E(G)\}.$$

This labeling is also referred as (a,d)-*EAT labeling*.

Here we can observe that for $d=0$, an $(a,0)$-EAT labeling is the same as the EMT labeling.

The edge antimagic total labeling of C_6 with a chord is shown in Figure 2.12. Here the set of edge-weights is $\{11,13,15,17,19,21,23\}$.

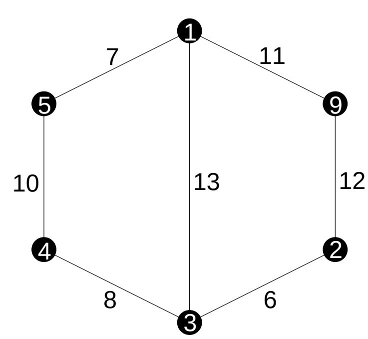

Figure 2.12: $(11,2)$-edge antimagic total labeling of C_6 with a chord

In an (a,d)-*edge antimagic total labeling*, if $f(V(G))=\{1,2,3,\ldots,p\}$ and the remaining labels are assigned to the edges in the graph, then this total labeling is called *super*(a, d)-*edge antimagic total labeling*. In simple words, an (a, d)-*edge antimagic total labeling* is called *super*, if the smallest possible labels in the labeling are assigned to the vertices of the graph.

The Super $(12, 2)$-edge antimagic total labeling of C_7 with a chord is shown in Figure 2.13 . The set of edge-weights in this labeling is $\{12, 14, 16, 18, 20, 22, 24, 26\}$.

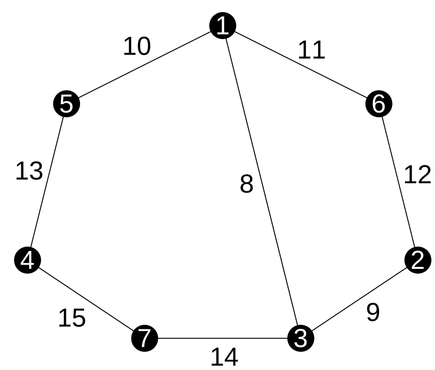

Figure 2.13: Super $(12, 2)$-edge antimagic total labeling of C_7 with a chord

2.6.2 Vertex antimagic labeling

Generally, in a vertex antimagic labeling, we label some elements of a graph and then calculate the weights of all the vertices of the graph, which form an arithmetic progression with initial term a and the common difference d. This labeling is called an *(a,d)-vertex antimagic labeling*. The formal definition is given below.

The vertex antimagic labelings can be further classified in two subclasses with respect to the choice under which the specific elements of the graph are labeled.

- Vertex antimagic edge labeling (VAE labeling).
- Vertex antimagic total labeling (VAT labeling).

First we discuss the vertex antimagic edge labeling of the graphs.

Vertex antimagic edge labeling

Bodendiek et al. [6] in 1996, introduced the concept of (a,d)-vertex antimagic edge labeling. They called this labeling (a,d)-antimagic labeling.

It is an edge labeling, i.e., we just label the edges of the graph. In this labeling the vertex-weights form an arithmetic progression. The formal definition of (a,d)-vertex antimagic edge labeling is given as:

A bijectiion $f : E \rightarrow \{1,2,3,\ldots,q\}$ is an (a,d)-*vertex antimagic edge labeling* ((a,d)-*VAE labeling*), of the graph $G=(V,E)$, if the set of all vertex-weights of the vertices of G admits an arithmetic progression

$$\{a, a+d, a+2d, \ldots, a+(p-1)d\}$$

with starting term a and common difference d, where $a>0$ and $d \geq 0$ are fixed positive integers. The weight of a vertex under this labeling is calculated as $\omega(x) = \sum_{y \in N(x)} f(xy)\ \forall\ x \in V(G)$, and $N(x)$ is the neighbourhood of the vertex x, which is the set of all the vertices adjacent to vertex x.

The $(20,1)$-*VAE labeling* of peterson graph $P_{6,5}$ is shown in Figure 2.14. The numbers inside the vertices are showing the vertex-weights.

Vertex antimagic total labeling

In a vertex antimagic total labeling we label all the vertices and edges of the graph using $p+q$ distinct numbers, such that the vertex-weights form an arithmetic progression. Formally, a vertex antimagic total labeling is defined as follows.

A bijection $f : V \cup E \rightarrow \{1,2,3,\ldots,p+q\}$ is called an (a,d)-*vertex antimagic total labeling* ((a,d)-*VAT labeling*), of a graph $G=(V,E)$, if the set of vertex-weights of all the vertices in G is as

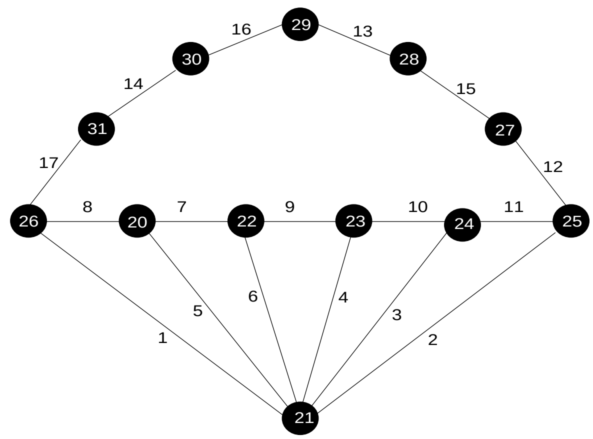

Figure 2.14: Vertex Antimagic Edge labeling

$W = \{\omega(x) : x \in V(G)\} = \{a, a+d, a+2d, \ldots, a+(p-1)d\}$, where $a > 0$ and $d \geq 0$ are fixed positive integers. Here the vertex-weights are calculated as $\omega(x) = f(x) + \sum\limits_{y \in N(x)} f(xy)$, $\forall\, x \in V(G)$, $f(x) \neq 0$.

The *VAT labeling* of the graph $K_4 - \{e\}$ is shown below.

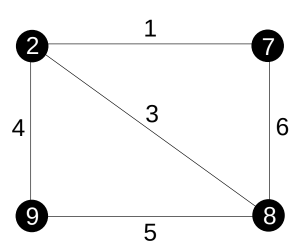

Figure 2.15: (10,4)-vertex antimagic total labeling of $(K_4 - e)$

Lemma 2.6.4 ([1])**.** *Every super magic graph G has an* (a,1)-VAT labeling.

An (a, d)-vertex antimagic total labeling f is called *super(a, d)-vertex antimagic total labeling* (*super(a, d)-VAT labeling*), if the vertex-set is assigned with the smallest possible labels, and the remaining labels are assigned to the edges of the graph G. That is, $f(V) = \{1, 2, 3, \ldots, p\}$ and $f(E) = \{p+1, p+2, p+3, \ldots, p+q\}$.

The *super* $(55, 1)$*-vertex antimagic total labeling* of the peterson graph $P_{5,2}$ is shown in the following diagram.

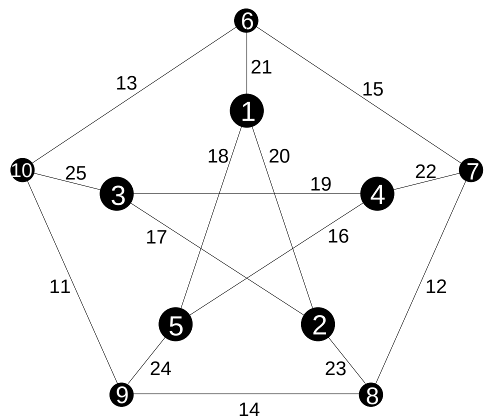

Figure 2.16: Super $(55, 1)$-vertex antimagic total labeling of $P_{2,5}$

Theorem 2.6.5 ([3])**.** *For every cycle with at least one tail and even number of vertices, there is no* super$(a, 1)$-vertex antimagic total labeling*.*

Theorem 2.6.6 ([3])**.** *Every tree with even number of vertices has no* super$(a, 1)$-VAT labeling*.*

MacDougall et al. [24] in 2002, introduced a special case of (a, d)-vertex antimagic total labeling. That is, for $d = 0$, (a, d)-vertex antimagic total labeling is called *vertex magic total labeling* (*VMT labeling*). This labeling is called *super* (*SVMT labeling*) if $f(V) = \{1, 2, 3, \ldots, p\}$ and $f(E) = \{p+1, p+2, p+3, \ldots, p+q\}$.

Chapter 3

SEMT labeling of some families of acyclic graphs

In this chapter, we construct a family of acyclic graphs referred as *w-graphs*, the *super edge-magic total labeling* of *w-graphs* and some of its generalizations. We also study the super edge-magic total labeling of *generalized combs* and *generalized w-trees*.

Hussain et al. [17] study the *super-edge magic total labeling* of *w-graphs* and *w-trees* (denoted by $W(n)$ and $WT(n,k)$ respectively). Javaid et al. [18] constructed super edge-magic total labeling of extended w-trees and disjoint union of extended w-trees. Hussain et al. [19] constructed super edge-magic total labeling of banana trees. They define a *w-graph* to be a graph G with vertex and edge sets.
$V(G) = \{c_1, c_2, b, w, d\} \cup \{x^1, x^2, \ldots, x^n\} \cup \{y^1, y^2, \ldots, y^n\}$ and
$E(G) = \{c_1x^i, c_2y^i,\ 1 \le i \le n\} \cup \{c_1b, c_1w, c_2w, c_2d\}$.

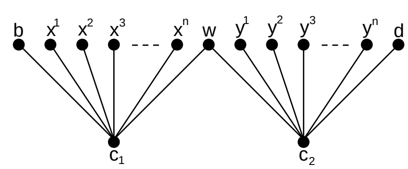

Figure 3.1: W(n)

Suppose that we have K isomorphic copies of w-graphs $W(n)$. A $w\!-\!tree\, WT(n,k)$ is a tree obtained by taking a new vertex a and joining it with $\{d_i : 1 \leq i \leq k\}$, where $n \geq 2$ and $k \geq 3$.

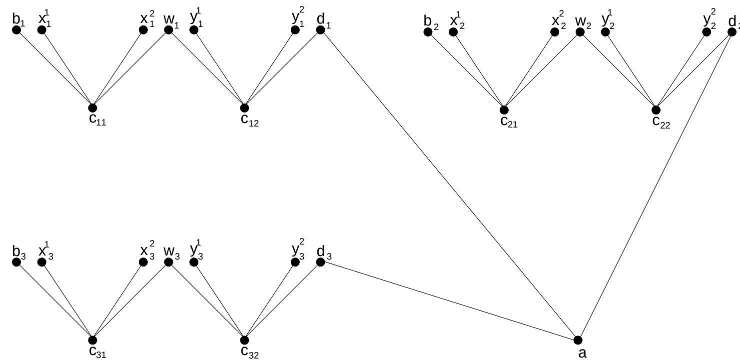

Figure 3.2: WT(2,3)

Let $K_{1,n_1}, K_{1,n_2}, K_{1,n_3}, ..., K_{1,n_k}$ be a family of disjoint stars with the vertex-sets $V(K_{1,n_i})$=$\{c_i, a_{i1}, ..., a_{in_i}\}$ and $deg(c_i)$=n_i, $1 \leq i \leq k$. A *banana tree* $BT(n_1, n_2, ..., n_k)$ is a tree obtained by adding a new vertex a and joining it to $a_{11}, a_{21}, ..., a_{k1}$.

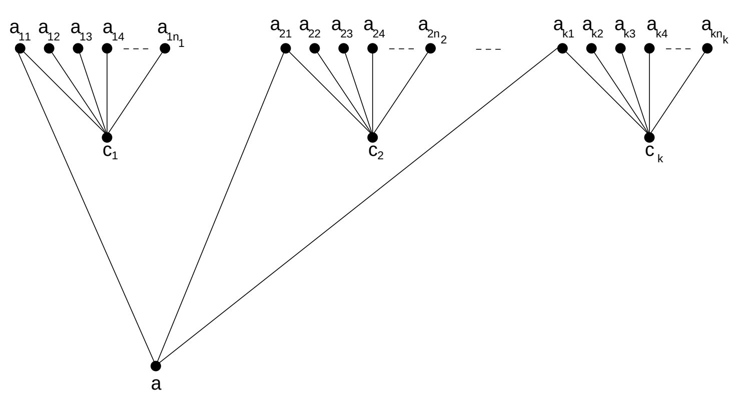

Figure 3.3: $BT(n_1, n_2, ..., n_k)$

3.1 SEMT labeling of reflexive w-graphs

We define a new family of acyclic graphs and refer them to be *reflexive w-graphs*. We also discussed some generalizations of reflexive w-graphs and their SEMT labelings.

Definition 3.1.1. The reflexive w-graph denoted by $RW(m, n)$ is obtained by connecting two copies of a w-graph by a path of order m in such a way that the path joins any two corresponding central vertices.

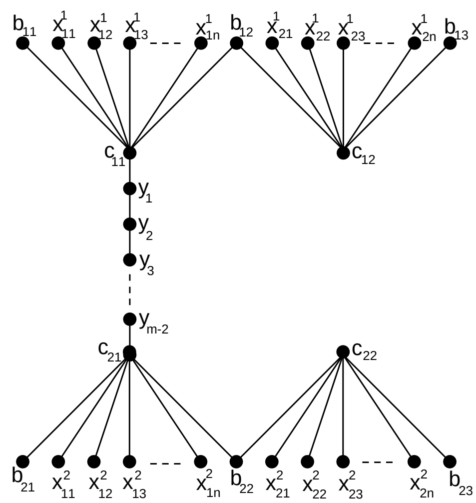

Figure 3.4: RW(m,n)

Theorem 3.1.1. *For any $m, n \in \mathbb{Z}^+$, the graph $G \cong RW(m, n)$ admits super edge-magic total labeling.*

Proof. The graph $G \cong RW(m, n)$ is of order $4n + m + 8$ and size $4n + m + 7$. The vertex and edge set of G are defined below as

$$\begin{aligned} V(G) &= \{b_{i,j} : 1 \leq i \leq 2,\ 1 \leq j \leq 3\} \cup \{c_{i,j} : 1 \leq i, j \leq 2\} \cup \\ &\quad \{x^l_{i,t} : 1 \leq i, l \leq 2,\ 1 \leq t \leq n\} \cup \{y_i : 1 \leq i \leq m-2\}. \end{aligned}$$

$$\begin{aligned} E(G) &= \{b_{i,j}c_{i,j}, b_{i,j+1}c_{i,j} : 1 \leq i, j \leq 2\} \cup \{y_1c_{11}, y_{m-2}c_{21}\} \cup \\ &\quad \{y_iy_{i+1} : 1 \leq i \leq m-3\} \cup \{x^l_{i,t}c_{l,i} : 1 \leq i, l \leq 2,\ 1 \leq t \leq n\}. \end{aligned}$$

We define the labeling of $RW(m,n)$ as

$$f: V(RW(m,n)) \rightarrow \{1,2,\ldots,4n+m+8\}$$

in two cases depending upon the order $m-2$ of the path y_i in $RW(m,n)$ as follows.
Case 1: When m is odd.
The outer leaves $b_{i,j}$ of the stars are labeled as

$$f(b_{i,j}) = 2n + \lceil\frac{m-2}{2}\rceil(i-1) + (-1)^i(nj+j-n) - i + 5,\ 1 \leq i \leq 2,\ 1 \leq j \leq 3.$$

The central vertices $c_{i,j}$ of the stars are labeled as

$$f(c_{i,j}) = 4n + \lfloor\frac{m-2}{2}\rfloor(i-1) + \lceil\frac{m-2}{2}\rceil + (-1)^i j - i + 10,\ 1 \leq i \leq 2,\ 1 \leq j \leq 2.$$

The labeling pattern of the inner leaves $x^l_{i,t}$ of the stars is

$$f(x^l_{i,t}) = 2n + \big(\lceil\frac{m-2}{2}\rceil + 1\big)(l-1) + (-1)^l(ni-n+i+t-1) + 3,\ 1 \leq i \leq 2,\ 1 \leq t \leq n,\ 1 \leq l \leq 2.$$

The vertices of the path y_i are labeled as

$$f(y_{2i-1}) = 2n + 3 + i, \quad 1 \leq i \leq \lceil\frac{m-2}{2}\rceil,$$
$$f(y_{2i}) = 4n + \lceil\frac{m-2}{2}\rceil + 8 + i, \quad 1 \leq i \leq \lfloor\frac{m-2}{2}\rfloor.$$

The edge weights under this labeling scheme constitute a sequence of $4n+m+7$ consecutive integers

$$\left\{4n + \lceil\frac{m-2}{2}\rceil + 8, 4n + \lceil\frac{m-2}{2}\rceil + 9, \ldots, 8n + m + \lceil\frac{m-2}{2}\rceil + 14\right\}$$

and hence by lemma 2.5.3, the labeling f can be extended to the *super edge magic total labeling* with magic constant $12n + 2m + \lceil\frac{m-2}{2}\rceil + 23$.
Case 2: When m is even.
The outer leaves $b_{i,j}$ of the stars are labeled as

$$f(b_{i,j}) = 2n + 3 + (m+3)(i-1) + (-1)^i(nj+j-n-1),\ 1 \leq i \leq 2\ \ 1 \leq j \leq 3.$$

The central vertices $c_{i,j}$ of the stars have labels

$$f(c_{i,j}) = 2n + \frac{m-2}{2} + (-1)^i j - 5i + 13, \quad 1 \leq i \leq 2, 1 \leq j \leq 2.$$

The labels of the inner leaves $x_{i,t}^l$ of the stars are

$$\begin{aligned} f(x_{i,t}^l) &= 2n + (m+3)(l-1) + (-1)^l (n(i-1) + t + i - 1) + 3, \\ & \qquad 1 \leq i \leq 2, 1 \leq t \leq n, 1 \leq l \leq 2. \end{aligned}$$

The vertices of the path y_i are labeled as

$$\begin{aligned} f(y_{2i-1}) &= 2n + 3 + i, \quad 1 \leq i \leq \frac{m-2}{2}, \\ f(y_{2i}) &= 2n + \frac{m-2}{2} + i + 7, \quad 1 \leq i \leq \frac{m-2}{2}. \end{aligned}$$

The edge weights under this labeling pattern form a sequence of $4n + m + 7$ consecutive integers

$$\left\{2n + \frac{m-2}{2} + 7, 2n + \frac{m-2}{2} + 8, \ldots, 6n + m + \frac{m-2}{2} + 13\right\}.$$

So by lemma 2.5.3, the graph $RW(m,n)$ is *super edge magic total* with magic constant $10n + 2m + \frac{m-2}{2} + 22$. □

In the next theorem, we show that the graph obtained by extending the number of stars in $RW(m,n)$ to any positive integer r also admits *super edge-magic total labeling.* This graph is referred as *Extended Reflexive w-graph*, denoted by $RW_E(m,n,r)$.

Theorem 3.1.2. *For any $m, n, r \in \mathbb{Z}^+$, the graph $G \cong RW_E(m,n,r)$ is super edge-magic total.*

Proof. The graph $G \cong RW_E(m,n,r)$ is of order $2nr+4r+m$ and size $2nr+4r+m-1$. The vertex and edge set of G are defined below as

$$V(G) = \{b_{i,j} : 1 \leq i \leq 2, 1 \leq j \leq r+1\} \cup \{c_{i,j} : 1 \leq i, j \leq r\} \cup$$

$$\{x_{i,t}^l : 1 \leq l \leq 2, 1 \leq i \leq r, 1 \leq t \leq n\} \cup \{y_i : 1 \leq i \leq m-2\}.$$

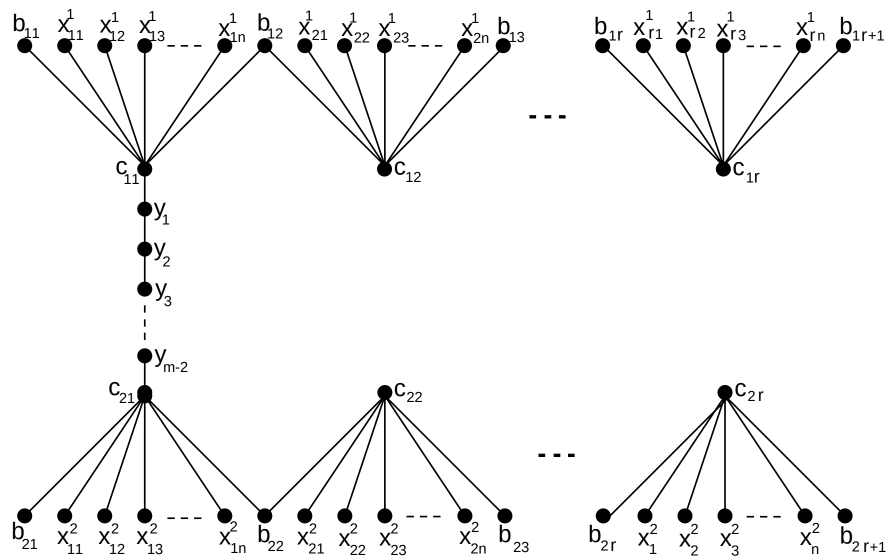

Figure 3.5: $RW_E(m,n,r)$

$$E(G) = \{b_{i,j}c_{i,j}, b_{i,j+1}c_{i,j} : 1 \leq i \leq 2, 1 \leq j \leq r\} \cup \{y_1c_{11}, y_{m-2}c_{21}\} \cup$$
$$\{x^l_{i,t}c_{l,i} : 1 \leq l \leq 2, 1 \leq i \leq r,\ 1 \leq t \leq n\} \cup \{y_iy_{i+1} : 1 \leq i \leq m-3\}.$$

We define the labeling of $RW_E(m,n,r)$ as

$$f : V(RW_E(m,n)) \to \{1, 2, \ldots, 2nr + 4r + m\}$$

in two cases depending upon the order $m-2$ of the path y_i in $RW_E(m,n,r)$ as follows.

Case 1: When m is odd.

The labels of the outer leaves $b_{i,j}$ of the stars are

$$\begin{aligned} f(b_{i,j}) &= (nr + r + 1) + \big(\lceil \frac{m-2}{2} \rceil + 1\big)(i-1) + (-1)^i(n+1)(j-1), \\ &\qquad\qquad\qquad\qquad 1 \leq i \leq 2,\ 1 \leq j \leq r+1. \end{aligned}$$

The labels of the central vertices $c_{i,j}$ of the stars are

$$\begin{aligned} f(c_{i,j}) &= (2nr + \lceil \frac{m-2}{2} \rceil + 3r + 4) + \lfloor \frac{m-2}{2} \rfloor (i-1) + i(2j-1) - 3j, \\ &\qquad\qquad\qquad\qquad 1 \leq i \leq 2,\ 1 \leq j \leq r. \end{aligned}$$

The inner leaves $x^l_{i,t}$ of the stars have labels

$$\begin{aligned} f(x^l_{i,t}) &= (nr+r+1) + \left(\lceil \frac{m-2}{2} \rceil + 1\right)(l-1) + (-1)^l(ni-n+i+t-1), \\ & \qquad 1 \le i \le r,\ 1 \le t \le n,\ 1 \le l \le 2. \end{aligned}$$

The vertices of the path y_i are labeled as

$$\begin{aligned} f(y_{2i-1}) &= nr+r+i+1, \quad 1 \le i \le \lceil \frac{m-2}{2} \rceil, \\ f(y_{2i}) &= 2nr+3r+\lceil \frac{m-2}{2} \rceil + i + 2, \quad 1 \le i \le \lfloor \frac{m-2}{2} \rfloor. \end{aligned}$$

The edge weights under this labeling scheme constitute a sequence of $2nr+4r+m-1$ consecutive integers

$$\left\{ 2(nr+r)+\lceil \frac{m-2}{2} \rceil+4, 2(nr+r)+\lceil \frac{m-2}{2} \rceil+5, \ldots, 4nr+6r+m+\lceil \frac{m-2}{2} \rceil+2 \right\}.$$

So by lemma 2.5.3, the graph $RW_E(m,n,r)$ admits *super edge magic total labeling* with magic constant $6nr+10r+2m+\lceil \frac{m-2}{2} \rceil + 3$.

Case 2: When m is even.

The labels of the outer leaves $b_{i,j}$ of the stars are

$$\begin{aligned} f(b_{ij}) &= (nr+r+1) + (m+2r-1)(i-1) + (-1)^i(n+1)(j-1), \\ & \qquad 1 \le i \le 2,\ \ 1 \le j \le r+1. \end{aligned}$$

The labels of the central vertices $c_{i,j}$ of the stars are

$$\begin{aligned} f(c_{ij}) &= (nr+r+1) + \frac{m-2}{2} + (2r+1)(2-i) + (-1)^i j, \\ & \qquad 1 \le i \le 2,\ 1 \le j \le r. \end{aligned}$$

The inner leaves $x^l_{i,t}$ of the stars have labels

$$\begin{aligned} f(x^l_{it}) &= (nr+r+1) + (m+2r-1)(l-1) + (-1)^l(n(i-1)+i+t-1), \\ & \qquad 1 \le i \le r,\ 1 \le t \le n,\ 1 \le l \le 2. \end{aligned}$$

The vertices of the path y_i are labeled as

$$\begin{aligned} f(y_{2i-1}) &= nr+r+i+1, \quad 1 \le i \le \frac{m-2}{2}, \\ f(y_{2i}) &= nr+3r+\frac{m-2}{2}+i+1, \quad 1 \le i \le \frac{m-2}{2}. \end{aligned}$$

The edge weights under this labeling pattern constitute a sequence of $2nr+4r+m-1$ consecutive integers

$$\left\{nr+2r+(\frac{m-2}{2})+3, nr+2r+(\frac{m-2}{2})+4, \ldots, 3nr+6r+m+(\frac{m-2}{2})+1\right\}.$$

So by lemma 2.5.3, the graph $RW_E(m,n,r)$ is *super edge magic total* with magic constant $5nr+10r+2m+\left(\frac{m-2}{2}\right)+2$. □

Now we generalize the graph $RW(m,n)$ in such a way that both copies of $W(n)$ have different number of stars and the number of vertices in every star is also arbitrary. The graph thus obtained is *Generalized Reflexive w-graph*, denoted by $RW_G(m;n_{i,j};r_i)$.

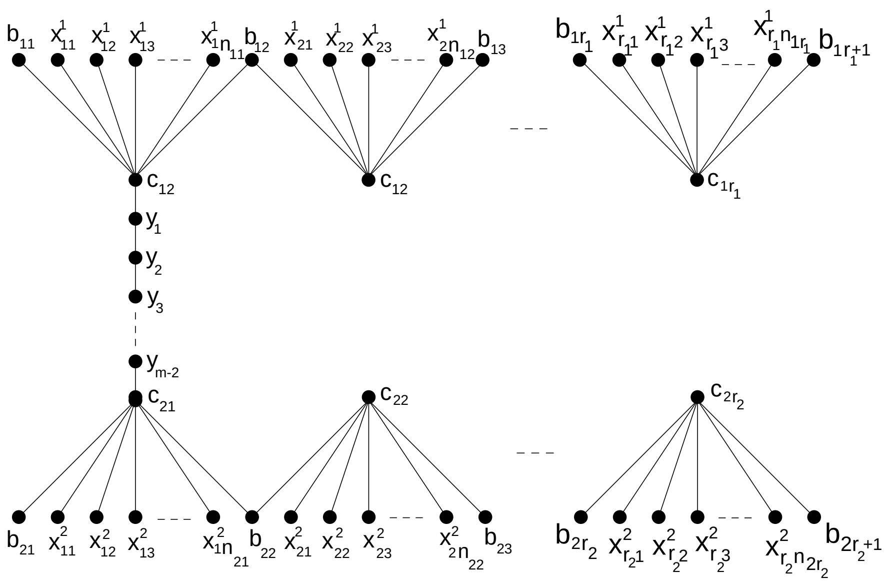

Figure 3.6: $RW_G(m;n_{i,j};r_i)$

From now on we will follow a convention that $\sum\limits_{1}^{0}$ will be considered as zero.

Theorem 3.1.3. *For $1 \le i \le 2, 1 \le j \le r_i$ and any $m, n_{i,j}, r_i \in \mathbb{Z}^+$, the graph $G \cong RW_G(m;n_{i,j};r_i)$ admits super edge-magic total labeling.*

Proof. The graph $G \cong RW_G(m; n_{i,j}; r_i)$ is of order

$$\sum_{i=1}^{2}\sum_{j=1}^{r_i} n_{ij} + 2(r_1+r_2) + m$$

and size

$$\sum_{i=1}^{2}\sum_{j=1}^{r_i} n_{ij} + 2(r_1+r_2) + m - 1.$$

The vertex and edge set of $RW_G(m; n_{i,j}; r_i)$ is defined below as

$$V(G) = \{b_{i,j} : 1 \le i \le 2,\ 1 \le j \le r_i + 1\} \cup \{c_{i,j} : 1 \le i \le 2, 1 \le j \le r_i\} \cup$$

$$\{y_i : 1 \le i \le m-2\} \cup \{x^l_{i,t} : 1 \le l \le 2, 1 \le i \le r_l,\ 1 \le t \le n_{li}\}.$$

$$E(G) = \{y_1 c_{11}, y_{m-2} c_{21}\} \cup \{x^l_{i,t} c_{l,i} : 1 \le l \le 2, 1 \le i \le r_l,\ 1 \le t \le n_{li}\} \cup$$

$$\{b_{i,j} c_{i,j}, b_{i,j+1} c_{i,j} : 1 \le i \le 2, 1 \le j \le r_i\} \cup \{y_i y_{i+1} : 1 \le i \le m-3\}.$$

We define the labeling of $RW_G(m; n_{i,j}; r_i)$ as

$$f : V\big(RW_G(m; n_{i,j}; r_i)\big) \to \left\{1, 2, \ldots, \sum_{i=1}^{2}\sum_{j=1}^{r_i} n_{ij} + 2(r_1+r_2) + m\right\}$$

in two cases depending upon the order $m-2$ of the path y_i in $RW_G(m; n_{i,j}; r_i)$ as follows.

Case 1: When m is odd.

For $1 \le i \le 2$, $1 \le j \le r_i + 1$

$$f(b_{ij}) = \big(\sum_{k=1}^{r_1-j+1} n_{1,r_1-k+1}\big)(2-i) + \Big(\lceil \frac{m-2}{2} \rceil + \sum_{k=1}^{r_1} n_{1,k} + \sum_{k=1}^{j-1} n_{2,k} - 1\Big)(i-1)$$
$$+(-1)^i j + r_1 + 2,$$

For $1 \le i \le 2$, $1 \le j \le r_i$

$$f(c_{ij}) = \sum_{k=1}^{2}\sum_{s=1}^{r_k} n_{ks} + \lceil \frac{m-2}{2} \rceil + 2r_1 + r_2 + \Big(\lfloor \frac{m-2}{2} \rfloor - 1\Big)(i-1) + (-1)^i j + 3,$$

For $1 \le l \le 2, 1 \le i \le r_l, 1 \le t \le n_{li}$

$$f(x^l_{it}) = \big(\sum_{k=1}^{r_1} n_{1,k} + \sum_{k=1}^{i-1} n_{2,k} + \lceil \frac{m-2}{2} \rceil + r_1\big)(l-1) + \big(\sum_{k=1}^{r_1-i+1} n_{1,r_1-k+1} + r_1 + 2\big)(2-l)$$

$$+(-1)^l(t+i),$$

$$f(y_{2i-1}) = \sum_{k=1}^{r_1} n_{1,k} + r_1 + 1 + i, \quad 1 \le i \le \lceil \frac{m-2}{2} \rceil,$$

$$f(y_{2i}) = \sum_{k=1}^{2}\sum_{s=1}^{r_k} n_{ks} + \lceil \frac{m-2}{2} \rceil + 2r_1 + r_2 + i + 2, \quad 1 \le i \le \lfloor \frac{m-2}{2} \rfloor.$$

The edge weights under this labeling pattern forms a sequence of $\sum_{i=1}^{2}\sum_{j=1}^{r_i} n_{ij} + 2(r_1+r_2) + m - 1$ consecutive integers

$$\Big\{ \sum_{k=1}^{2}\sum_{s=1}^{r_k} n_{ks} + \lceil \frac{m-2}{2} \rceil + r_1 + r_2 + 4, \sum_{k=1}^{2}\sum_{s=1}^{r_k} n_{ks} + \lceil \frac{m-2}{2} \rceil + r_1 + r_2 + 5, \ldots$$

$$\ldots, 2\sum_{k=1}^{2}\sum_{s=1}^{r_k} n_{ks} + 3(r_1+r_2) + \lceil \frac{m-2}{2} \rceil + m + 2 \Big\}.$$

So by lemma 2.5.3, the labeling f admits *super edge magic total labeling* with magic constant

$$3\sum_{k=1}^{2}\sum_{s=1}^{r_k} n_{ks} + 5(r_1+r_2) + \lceil \frac{m-2}{2} \rceil + 2m + 3.$$

Case 2: When m is even.

For $1 \le i \le 2, 1 \le j \le r_i + 1$

$$f(b_{ij}) = \big(\sum_{k=1}^{r_1} n_{1,k} + \sum_{k=1}^{j-1} n_{2,k} + r_1 + r_2 + m - 3 \big)(i-1) + r_1 + 2 + (-1)^i j$$

$$+\big(\sum_{k=1}^{r_1-j+1} n_{1,r_1-k+1} \big)(2-i),$$

For $1 \le i \le 2, 1 \le j \le r_i$

$$f(c_{ij}) = \sum_{k=1}^{r_1} n_{1,k} + \big(\frac{m-2}{2}\big) + (r_1+1) + (-1)^i j + (r_1+r_2+1)(2-i),$$

For $1 \le l \le 2, 1 \le i \le r_l, 1 \le t \le n_{li}$

$$f(x_{it}^l) = \sum_{k=1}^{r_1-i+1} n_{1,r_1-k+1}(2-l) + \big(\sum_{k=1}^{r_1} n_{1,k} + \sum_{k=1}^{i-1} n_{2,k} + r_1 + r_2 + m - 3 \big)(l-1)$$

$$+(-1)^l(i+t)+r_1+2,$$

$$f(y_{2i-1})=\sum_{k=1}^{r_1} n_{1,k}+r_1+i+1, \quad 1\leq i\leq (\frac{m-2}{2}),$$

$$f(y_{2i})=\sum_{k=1}^{r_1} n_{1,k}+(\frac{m-2}{2})+2r_1+r_2+i+1, \quad 1\leq i\leq (\frac{m-2}{2}).$$

The edge weights under this labeling scheme constitute a sequence of $\sum\limits_{i=1}^{2}\sum\limits_{j=1}^{r_i} n_{ij}+2(r_1+r_2)+m-1$ consecutive integers

$$\Big\{\sum_{k=1}^{r_1} n_{1,k}+(\frac{m-2}{2})+r_1+r_2+3, \sum_{k=1}^{r_1} n_{1,k}+(\frac{m-2}{2})+r_1+r_2+3,$$

$$\dots, 2\sum_{k=1}^{r_1} n_{1,k}+\sum_{k=1}^{r_2} n_{2,k}+3(r_1+r_2)+(\frac{m-2}{2})+m+1\Big\}.$$

Hence by lemma 2.5.3, the labeling f can be extended to the *super edge-magic total labeling* with magic constant

$$3\sum_{k=1}^{r_1} n_{1,k}+2\sum_{k=1}^{r_2} n_{2,k}+5(r_1+r_2)+(\frac{m-2}{2})+2m+2.$$

$\square$

3.2 SEMT labeling of generalized comb

A *generalized comb* is a graph derived from the path $P_{n+1}: x_{1,0}, x_{1,1}, ..., x_{1,n}$, $n\geq 2$ by adding n new paths $x_{2,j}, x_{3,j}, ..., x_{l_i,j}$ of lengths l_i-2, where $l_i\geq 2$, $1\leq i\leq n$ and new edges $x_{1,j}x_{2,j}$ for $1\leq j\leq n$ and this is denoted by $Cb_n(l_1, l_2, ..., l_n)$. $Cb_n(2, 2, ..., 2)$ is simply called a comb and it is denoted by Cb_n.

Theorem 3.2.1. *For $l\geq 1$, $n\geq 2$, the graph $G\cong 2Cb_n(l, l, ..., l)+e$ is SEMT.*

Proof. The two isomorphic copies of generalized comb $Cb_n(l, l, ..., l)$ are super edge magic total if we add an edge to it. The veretx and edge set of this graph can be

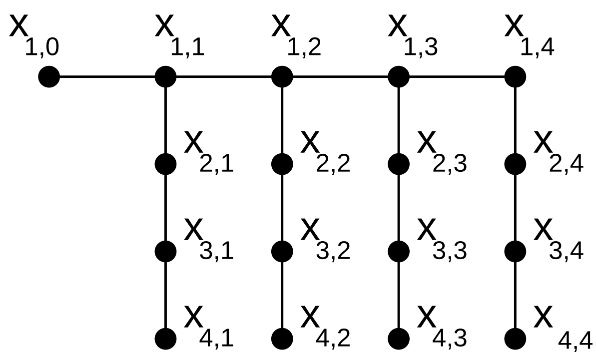

Figure 3.7: $Cb_n(4,4,4,4)$

represented as:

$$V(G) = \{x_{i,j}^k \,:\, 1 \le i \le l, 1 \le j \le n, 1 \le k \le 2\} \cup \{x_{1,0}^k\}$$

and

$$\begin{aligned} E(G) \;&=\; \{x_{i,j}^k\, x_{i+1,j}^k \,:\, 1 \le i \le l-1, 1 \le j \le n, 1 \le k \le 2\} \cup \\ &\quad\; \{x_{i,j}^k\, x_{i,j+1}^k \,:\, 0 \le j \le n-1, 1 \le k \le 2\} \cup \{e\}. \end{aligned}$$

where $p = 2(ln+1)$, $q = 2n(l-1) + 2n + 1 \;=\; 2ln + 1$,

$$e \;=\; \begin{cases} x_{1,n}^1\, x_{1,0}^2, & \text{l=2 , n=odd,} \\ x_{l,n}^1\, x_{1,0}^2, & \text{l=odd , n=odd,} \\ x_{l-1,n}^1\, x_{1,0}^2, & \text{l=even , n=odd,} \\ x_{2,n}^1\, x_{1,0}^2, & \text{n=even.} \end{cases}$$

Labeling is defined for $1 \le k \le 2$ as

$$f(x_{1,0}^K) \;=\; (k+1)\lceil \frac{nl}{2} \rceil - 1.$$

$$f(x_{i,j}^k) = \begin{cases} \lceil \frac{nl}{2} \rceil + \frac{jl+2-i}{2}, & 2 \leq i \leq l \, even, 2 \leq j \leq n \, even, \\ \lceil \frac{nl}{2} \rceil + \frac{(i+1)+l(j-1)}{2}, & 1 \leq i \leq l \, odd, 1 \leq j \leq n \, odd, \\ (k+1)\lceil \frac{nl}{2} \rceil + \frac{(i+1)+l(j-1)}{2}, & 2 \leq i \leq l \, even, 1 \leq j \leq n \, odd, \\ (k+1)\lceil \frac{nl}{2} \rceil + \frac{(3-i)+lj}{2}, & 1 \leq i \leq l \, odd, 2 \leq j \leq n \, even. \end{cases}$$

□

The labeling scheme of the above theorem is given as:

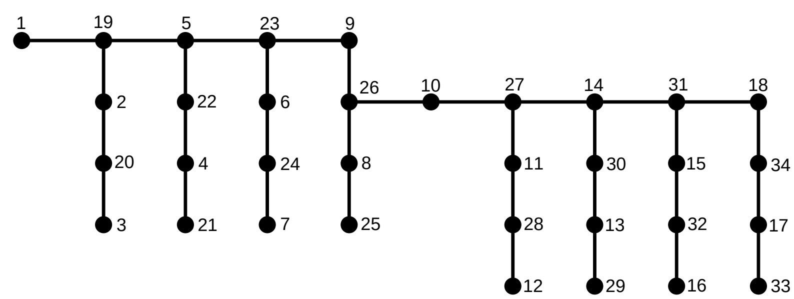

Figure 3.8: Generalized Comb

3.3 SEMT labeling of generalized w-tree

Suppose that we have K isomorphic copies of w-graphs $W(n)$. A *generalized w-tree* is a tree obtained by increasing the number of vertices to any arbitrary number.

The super edge magic total labeling of *generalized w-tree* $WT(n_1, n_2, ..., n_{2k}; k)$ is discussed in the next theorem.

Theorem 3.3.1. *For* $1 \leq i \leq k-2$*,* $k \in \mathbb{N}$ *and* $n_{4i-2} \geq 2$*, the graph* $G \cong WT(n_1, n_2, ..., n_{2k}; k)$ *is SEMT.*

Proof. The veretx and edge set of this graph can be represented as:

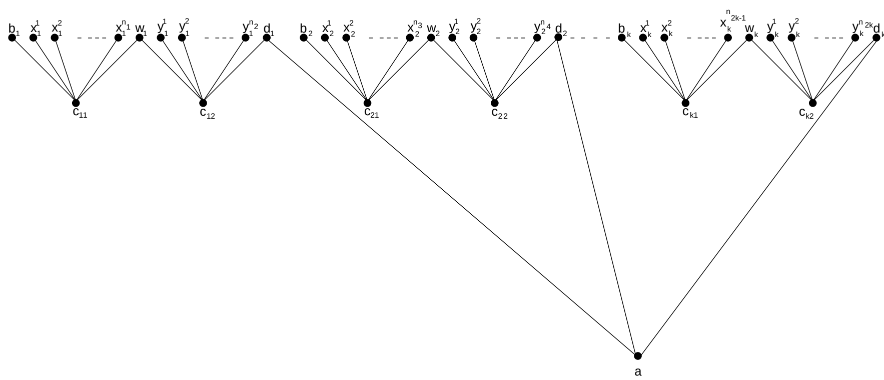

Figure 3.9: Generalized w-tree

$$
\begin{aligned}
V(G) &= \{a\} \cup \{b_i, w_i, d_i, c_{i1}, c_{i2};\, 1 \leq i \leq k\} \cup \\
&\quad \{x_{i1}^{l_{i1}},\, 1 \leq i \leq k,\, 1 \leq l_{i1} \leq n_{2i-1}\} \cup \\
&\quad \{y_{i2}^{l_{i2}},\, 1 \leq i \leq k,\, 1 \leq l_{i2} \leq n_{2i}\}.
\end{aligned}
$$

$$
\begin{aligned}
E(G) &= \{b_i\, c_{i1}, d_i\, c_{i2}, w_i\, c_{i1}, w_i\, c_{i2}, a\, d_i\,;\, 1 \leq i \leq k\} \cup \\
&\quad \{c_{i1}\, x_{i1}^{l_{i1}}\,;\, 1 \leq i \leq k,\, 1 \leq l_{i1} \leq n_{2i-1}\} \cup \\
&\quad \{c_{i2}\, y_{i2}^{l_{i2}}\,;\, 1 \leq i \leq k,\, 1 \leq l_{i2} \leq n_{2i}\}.
\end{aligned}
$$

where $\nu = |V(G)| = \sum_{i=1}^{2k} n_i + 5k + 1$ and $\epsilon = |E(G| = \sum_{i=1}^{2k} n_i + 5k$.

Consider $\lambda : V(G) \rightarrow 1, 2, ..., |V(G)|$.

Define $s = \lfloor \frac{k}{2} \rfloor$ and $\lambda(a) = \nu - 2\lceil \frac{k}{2} \rceil$.

$$
\lambda(c_{i1}) = \begin{cases} \nu - 2k + 2i - 2, & 1 \leq i \leq s, \\ \nu - 2k + 2i, & s + 1 \leq i \leq k. \end{cases}
$$

$$
\lambda(c_{i2}) = \nu - 2k = 2i - 1\,;\, 1 \leq i \leq k.
$$

for $s=1$

$$\lambda(b_i)=\begin{cases} 1, & \text{i=1 , s=1,}\\ \sum_{t=1}^{3i} n_t+3i, & s+1\le i\le k.\end{cases}$$

for $s\ge 2$

$$\lambda(b_i)=\begin{cases} 1, & \text{i=1,}\\ \sum_{t=1}^{2i-2} n_t+3(i-1)+1, & 2\le i\le s,\\ \sum_{t=1}^{2i} n_t+3i, & s+1\le i\le k.\end{cases}$$

$$\lambda(w_i)=\begin{cases} \sum_{t=1}^{2i-1} n_t+3i-1, & 1\le i\le s,\\ \sum_{t=1}^{2i} n_t-n_{2i-1}+3i-1, & s+1\le i\le k.\end{cases}$$

$$\lambda(d_i)=\begin{cases} \lambda(w_i)+2i+n_{2i}-2s+1, & 1\le i\le s,\\ \lambda(w_i)+2i-n_{2i}-2s-3, & s+1\le i\le k.\end{cases}$$

$x_{i1}^{l_{i1}}$ occur in l_{i1} consecutive integers.

$$\lambda x_{i1}^{l_{i1}}=\begin{cases} \{\lambda(b_i)+1,\lambda(b_i)+2,...,\lambda(b_i)+n_{2i-1}\}, & 1\le i\le s,\\ \{\lambda(b_i)-1,\lambda(b_i)2,...,\lambda(b_i)-n_{2i-1}\}, & s+1\le i\le k.\end{cases}$$

Similarly, $\{d_i\cup y_{i2}^{l_{i2}}\}$ occur in $l_{i2}+1$ consecutive integers.

$$\lambda(y_{i2}^{l_{i2}})=\begin{cases} \{\lambda(w_i)+1,\lambda(w_i)+2,...,\lambda(w_i)+n_{2i},\lambda(w_i)+n_{2i}+1\}\setminus\lambda d_i, & 1\le i\le s,\\ \{\lambda(w_i)-1,\lambda(w_i)-2,...,\lambda(w_i)-n_{2i},\lambda(w_i)-n_{2i}-1\}\setminus\lambda d_i\}, & s+1\le i\le k.\end{cases}$$

The edge weights are

$$\Big\{\sum_{i=1}^{2k} n_i+3k+2,\sum_{i=1}^{2k} n_i+3k+3,...,\sum_{i=1}^{2k} n_i+3k+\nu\Big\}.$$

So by lemma 2.5.3, the graph $WT(n_1, n_2, ..., n_{2k}; k)$ is *super edge magic total* with magic constant $2\sum_{i=1}^{2k} n_i + 8k + \nu + i$. $\square$

3.4 Concluding Remarks

The main objective of the work was to verify the *tree conjecture*; that all trees are super edge-magic total. But in this thesis we just constructed a couple of tree classes and their super edge-magic total labeling.

Within the thesis, we invite the readers to investigate the super edge-magic total labeling of a forest of *reflexive w-graphs, extended* and *generalized reflexive w-graphs* and also super edge-magic total labeling of a forest of *generalized w-tree* and *generalized comb.*

Bibliography

[1] M. Bača, F. Bertault, J.A. MacDougall, M. Miller, R.Simanjuntak and Slamin, *Vertex antimagic total labelings of graphs*, Discuss. Math. Graph Theory, **23** (2003), 67–83.

[2] M. Bača, Y. Lin, M. Miller, R. Simanjuntak, *New construction of magic and antimagic graph labelings*, Utilitas Math. **60** (2001), 229–239.

[3] M. Bača, M. Miller, *Super Edge Antimagic Graphs*, Brown Walker Press, USA, (2008).

[4] E.T. Baskoro, R. Simanjuntak, and M.T. Adithia, *Secret Sharing Scheme based on Magic Labeling*, Proc. of the 12th National Conference on Mathematics, Bali, 23-27 July (2004), 139-145.

[5] R. Bodendiek, G. Walther, *Aritmetisch antimagische Graphen*, in: K. Wagner and R. Bodendiek, ed., Graphentheorie III, BI-Wiss. Verl., Mannheim, (1993).

[6] R. Bodendiek, G. Walther, *On arithmetic antimagic edge labelings of graphs*, Mitt. Math. Ges. Hamburg **17** (1998), 85–99.

[7] J.A. Bondy, U.S.R. Murty, *Graph theory with applications*, Department of Combinatorics and Optimization, University of Waterloo, Ontario, North - Holland, New York, Amsterdam, London, 1976.

[8] Cayley, Arthur, *On the colourings of maps*, Proc. Royal Geographical Society (Blackwell Publishing) **1** (4), (1879), 259–261.

[9] H. Enomoto, A.S. Llado, T. Nakamigawa, G. Ringel, *Super edge magic graphs*, SUTJ. Math. **2** (1998), 105–109.

[10] R. Figueroa-Centeno, R. Ichishima, F.A. Muntaner-Batle, *The place of super edge magic labelings among other classes of labelings*, Discrete Math. **231** (2001), 153–168.

[11] J.A. Gallian, *A dynamic survey of graph lebeling*, The Electronic Journal of Combinatorics, #05C78 (Nov. 2011 version).

[12] S.W. Golomb, *How to number a graph*, in Graph Theory and Computing, R. C. Read, ed., Accademic Press, New York (1972), 23–37.

[13] R.L. Graham, N.J.A. Sloan, *On additive bases and harmonious graphs*, SIAM J. Alg. Discrete Math. **1** (1980), 382–404.

[14] I.D. Gray, J.A. MacDougall, W.D. Wallis, *A unified construction of vertex-magic total labelings for complete graphs*, Australasian Journal of Combinatorics **26** (2002), 49–63.

[15] J.L. Gross, J. Yellen, *Graph Theory and its Applications*, Chapman and Hall/CRC, USA, 2006.

[16] N. Hartsfield, G. Ringel, *Pearls in Graph Theory*, Academic Press, Boston - San Diego - New York - London, (1990).

[17] M. Hussain, K. Ali, M. Javaid, K.H. Dar, *Super edge-magic total labelings on w-trees*, Utilitus Math. **86** (2011), 183–191.

[18] M. Javaid, M. Hussain, K. Ali, K.H. Dar, *Super edge-magic total labeling on forest of extended w-trees*, Utilitus Math., in press.

[19] M. Hussain, E.T. Baskoro, Slamin, *On super edge magic total labeling of banana trees*, Utilitus Math. (2009), in press.

[20] M.C.Kong, S-M Lee, and Hugo Sun, *On magic strength of graphs*, Ars Combinatoria **45** (1997), 193–200.

[21] A. Kotzig, A. Rosa, *Magic valuations of finite graphs*, Canad. Math. Bull. **13** (1970), 451–461.

[22] S.M. Lee and Q.X. Shah, *All trees with at most 17 vertices are super edgemagic*, 16th MCCCC Conference, Carbondale, University Southern Illinois, Nov. (2002).

[23] J.A. MacDougall, M. Miller, Slamin, W.D. Wallis, *Vertex magic total labelings*, Utilitas Math. (to appear).

[24] J.A. MacDougall, M. Miller, W.D. Wallis, *Vertex magic total labelings of wheels and related graphs*, Utilitas Math. **62** (2002), 175–183.

[25] A. Rosa, *On certain valuations of the vertices of a graphs*, Theory of Graphs (International Symposium, Rome, July 1966), Gordon and Breach, N. Y. and Dunod Paris (1967), 349–355.

[26] J. Sedláček, Problem 27, *Theory of graphs and its applications.* (Smolenice, 1963), 163–164, Publ. House Czechoslovak Acad. Sci., Prague (1964).

[27] R. Simanjuntak, F. Bertault, M. Miller, *Two new (a, d)-antimagic graph labelings*, Proc. of Eleventh Australian Workshop of Combinatorial Algorithms (2000), 179–189.

[28] B.M. Stewart, *Magic Graphs*, Canad. J. Math. **18** (1966), 1031–1059.

[29] W.D. Wallis, *Magic Graphs*, Brikhäuser, Boston - Basel - Berlin, 2001.

[30] D.B. West, *An Introduction to Graph Theory*, Prentice - Hall, 1996.