MHD Boundary Layer Flow and Heat Transfer Over a Stretching Sheet and a Moving Flat Plate

by

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Abstract

In the current thesis, a numerical solution for laminar boundary layer problems is developed for magnetohydrodynamics (MHD) fluids. In the first part of the thesis, numerical solutions are obtained for mixed convection flow by considering a nonlinear stretching sheet for incompressible electrically conducting fluid. A coupled nonlinear partial differential equations (PDEs) have been converted into system of nonlinear ordinary differential equations (ODEs) by means of similarity transformation. Numerical solutions have been attained for velocity and temperature gradient by using shooting technique and *bvp4c*, the built-in solver of MATLAB, in the presence of magnetic, suction/injection and velocity slip parameters. The effects of these parameters have also been presented and discussed for momentum and thermal boundary layer.

In the second part of the thesis, the constant and variable fluid properties have been analyzed in a parallel free stream over a moving flat plate. This work is extended for the case of MHD. The similarity transformation is introduced to reconstruct nonlinear partial differential equations (PDEs) into the system of nonlinear ordinary differential equations (ODEs). The shooting method is used to obtain a numerical solutions for constant fluid properties and temperature dependent viscosity in the presence of transverse magnetic field and results are compared with the built-in solver *bvp4c* of MATLAB.

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Dedicated to my parents

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Chapter 1

Preliminaries

The purpose of this chapter is to present some basic definitions and fundamental concepts related to the fluid mechanics. The governing equations of conservation of mass, conservation of linear momentum,and conservation of energy are derived. Numerical methods have also been presented at the end of this chapter.

1.1 Introduction

The present work is carried out for magnetohydrodynamic (MHD) laminar boundary layer flow and heat transfer over a stretching sheet as well as a moving flat plate. A uniform magnetic field is applied in the direction normal to the surface. Rossow [1] was perhaps first researcher who applied magnetic field to control the influence of the motion of electrically conducting fluid. He derived momentum and energy transport equations for incompressible boundary layer flow by taking constant magnetic field [1]. Moreover, applied magnetic field also plays a significant role in controlling momentum and heat transfer in the boundary layer flow of different fluids over a stretching sheet [2]. In recent times many researchers have shown interest in this field and solutions were obtained for variable surface temperature, the effect of slip, fluid injection and fluid suction and viscous dissipation over a linearly stretching sheet for MHD stagnation point flow.
However, work has also been done for nonlinear stretching sheet. In this regard, Shen *et al*[3] studied incompressible convection flow over a nonlinear stretching

sheet near a stagnation point in the presence of a magnetic field. He observed that heat flow rate rises with magnetic parameter when the free stream velocity exceeds the stretching velocity, although reverse trend is analyzed when stretching velocity exceeds the free stream velocity [3]. In the current work, Chapter 2 contains a review analysis of MHD convection flow over a nonlinear stretching sheet near a stagnation point [3].

The problem of laminar forced convection flow and heat transfer over a continuous flat surface were solved by Sakiadis [4]. Sakiadis obtained similarity solution for momentum and thermal boundary layer by considering ambient fluid. Thereafter, many solutions have been derived for different problems of this class of boundary layer flow. Most of the research is carried out for the ambient fluid by investigating the constant and variable fluid properties. In these cases a parabolic velocity profile is obtained for laminar boundary layer flow. Although, critical errors were predicted by Pop *et al* [5] for constant fluid properties. However, viscosity is considered as inverse linear function of temperature [6]. Andersson and Aarseth [7] considered variable fluid viscosity and revised the results for Sakiadis flow. Moreover, variable fluid properties for moving flat plate in a parallel free stream were analyzed by Bachok *et al* [8]. In Chapter 3, the effect of temperature dependent viscosity is taken into account to accurately determined flow [8]. However, in Chapter 4 an extended work is taken into consideration for constant and variable fluid viscosity in the presence of magnetic field over a moving flat plate in a parallel free stream.

1.2 Basic Definitions

In the following section, we present some definitions related to the current work.

1.2.1 Fluid Dynamics

A material that flows or deforms continuously when shear stress is applied known as *fluid*. Generally, it includes liquids and gases. If deformation increases continuously under applied shear stress then it is known as *flow*. The motion of fluids flow is studied under the subject of *fluid dynamics*. It also analyzes the effect of forces on the motion of fluids flow. The influence of motion of liquids and gases are studied under sub-discipline of fluid dynamics called *hydrodynamics* and *aerodynamics*, respectively. However, the problems associated to fluids flows and heat transfer are analysed numerically under the subject *computational fluid dynamics* (CFD). The technique is powerful and span a wide range of industrial and non-industrial application such as vehicles and aircraft (aerodynamics), ships (hydrodynamics), polymer molding (chemical engineering) etc [9].

1.2.2 Viscosity

It refers to physical property of fluids. The internal flow resistance or frictional force developed among the different layers of the particles of fluid is determined by its *absolute* or *dynamic* viscosity μ. Basically, it corresponds to the thickness of the fluid. It means those fluid which are thicker possess higher viscosity and their movement is also slow such as honey while thinner fluids are less viscous and also flows faster such as water. Moreover, the rate of momentum transferred between the particles of fluids is computed by *kinematic* viscosity. It is obtained by dividing the absolute viscosity to the density of a fluid. Mathematical expression of ν is

$$\nu = \frac{\mu}{\rho}, \tag{1.2.1}$$

It is also known as momentum diffusivity.
The temperature and pressure are one of the factors that affect viscosity of fluids. Both liquids and gases behave oppositely with the changes of temperature [10]. With the rise of temperature, viscosity of some liquids such as oil, grease etc reduces. Consequently, decreasing of temperature results in an increase in viscosity of fluids. This is due to the fact that strong intermolecular forces exist within the molecules of liquids which offers more internal resistance. Although, gases offer more viscosity with higher temperature. As the particles of gases are far apart, so increasing temperature particles move faster and collide with one another more vigorously, which causes an increase in internal friction which correspond to more viscosity. Unlike temperature, viscosity of fluids is moderately affected by pressure. With the

increase of pressure, viscosity of gases and some liquids also increases. An instrument used to measure viscosity of fluids, called viscometer. Under ordinary conditions the dynamic viscosity of air is 17.08×10^{-6} Pa.s and water is 1.793×10^{-3} Pa.s [11].

1.2.3 Newtonian Fluid

Those fluids for which shear stress is directly proportional to the rate of deformation are known as Newtonian fluids. Mathematically, it is written as

$$\tau = \mu \frac{du}{dy}, \tag{1.2.2}$$

where μ is a proportionality constant known as dynamic viscosity and $\frac{du}{dy}$ is the deformation rate which is also called velocity gradient or strain rate. The expression (1.2.2) is also known as Newton's law of viscosity as presented by *Isaac Newton* in 1687 [10]. Some fluids like water, air etc behave as a Newtonian fluid.

1.2.4 Compressible Flow

A fluid whose density varies sufficiently subject to high pressure gradient within a flow field are known as compressible fluid. Mostly, gases are considered more compressible (for air $1 \times 10^{-5} m^2/N$) than liquids (for water $4.6 \times 10^{-10} m^2/N$ at 25^oC) [12]. Compressibility can be analyzed and controlled more significantly by a dimensionless parameter called Mach number, which can be defined as:

$$Ma = \frac{v}{c}, \tag{1.2.3}$$

where v denotes speed of fluid flow and c is the speed of sound. Particularly, density variation occurs when Mach number exceeds 0.3. When the Mach number Ma approaches to unity and above, this effect becomes large.

1.2.5 Incompressible Flow

An incompressible flow is one in which the fluid density remains constant subject to pressure gradient. Generally, liquids are considered as incompressible. For instance,

$Ma < 0.3$ indicate this type of fluids.

1.2.6 Laminar Flow

A smooth orderly movement of viscous fluid in which fluid particles have a definite path in a parallel layer (laminae) with no unsteady macroscopic intersecting is known as laminar flow [12].
This flow occurs in the following cases:

1. Reynolds number is small.

2. Velocity of liquid is below at certain critical value.

3. Low shear stress.

1.2.7 Steady Flow

A steady flow is one in which fluid properties at any point do not vary with respect to time. Mathematically, it can be expressed as:

$$\frac{\partial P}{\partial t} = 0, \tag{1.2.4}$$

where P is any fluid property like density, temperature, pressure, velocity of flow etc. Thus,

$$P = P(x, y, z), \tag{1.2.5}$$

A water is flowing in a pipe with an increasing diameter is a steady flow because the velocity varying due to an increasing flow but at each point it does not change with time [14].

1.2.8 Reynolds Number

It is a dimensionless number that can be obtained by dividing the inertial forces by viscous forces.

Mathematically,

$$Re = \frac{\rho U L}{\mu} = \frac{UL}{\nu}, \tag{1.2.6}$$

where U, L and ν are known as the velocity of fluid, characteristics length, and coefficient of viscosity, respectively. It determined a standard of classifying the motion of fluid. For instance, in a pipe flow small Reynolds numbers correspond to large viscous forces. The flow becomes turbulent when large Reynolds number is observed. In this case inertial forces are important.

Magnetic Reynolds number (R_m) is another important dimensionless parameter used to analyze the effect of uniform magnetic field (magnetic advection) for an electrically conducting fluid. Mathematically, it can be expressed as:

$$R_m \equiv \frac{UL}{\eta}, \tag{1.2.7}$$

where η is the magnetic diffusivity [15].

1.2.9 Prandtl Number

It is a dimensionless number that can be obtained by dividing the momentum diffusivity (kinematic viscosity) to thermal diffusivity. Mathematically, it is represented as:

$$Pr = \frac{\nu}{\alpha} = \frac{\mu C_p}{\kappa}, \tag{1.2.8}$$

In Eq. (1.2.8), μ is the dynamic viscosity, C_p is the specific heat at constant pressure and κ is the thermal conductivity. So Pr is responsible for the growth of momentum and thermal boundary layer. It also influences the relative thickness of these two boundary layers. Moreover, when Prandtl number is large (Pr $>>$ 1), momentum diffusivity dominated, as a result momentum boundary layer becomes thicker while lower value of Prandtl number (Pr $<<$ 1) corresponds to thicker the thermal boundary layer as that thermal diffusivity is dominated in this case. This means that heat diffuses quickly in case of small Prandtl number. So Prandtl number Pr is the only dimensionless quantity that associates the thickness of both momentum and thermal boundary layer. Prandtl numbers for some of the fluid are given below:

$$\text{Pr} = 0.015 \text{ for mercury.}$$

$$\text{Pr} = 0.16 \text{ - } 0.7 \text{ for mixtures of noble gases.}$$

$$\text{Pr} = 0.7 \text{ - } 0.8 \text{ for air.}$$

$$\text{Pr} = 7 \text{ for water at } 20^0C.$$

1.2.10 Grashof Number

It is a dimensionless number that can be obtained by dividing buoyancy forces to viscous forces acting on a fluid. Mathematically, it can be expressed as

$$Gr = \frac{g\beta(T_s - T_\infty)L^3}{\nu^2} \tag{1.2.9}$$

where g is a gravitational acceleration, β is a coefficient of volume expansion, L is a characteristics length, T_s represents surface temperature, T_∞ is the temperature sufficiently far from the surface and ν is a kinematics viscosity. It determined the criteria whether a flow is laminar ($Gr < 10^9$) or turbulent ($Gr > 10^9$).

1.2.11 Thermal Conductivity

The ability of a substance to transmit heat is known as thermal conductivity. It is denoted by κ. Those substances which have low value of thermal conductivity such as air 0.024 $W/(m.K)$ at 0^oC, can conduct less amount of heat compared to material which have high thermal conductivity such as water 0.56 $W/(m.K)$ at 0^oC. The mathematical expression of κ is written as:

$$\kappa = \frac{\dot{Q}L}{A\Delta T}, \tag{1.2.10}$$

In Eq. (1.2.10), $\dot{Q}$ represent the amount of heat transmit through the substance per unit time, A is the area and ΔT is the temperature difference.

1.2.12 The Boundary Layer

In 1904, a German aerodynamicist *L. Prandtl* first introduced the concept of boundary layer flow. When the fluid flow on the surface, the adjacent fluid layer bears a maximum effect of that surface motion. This causes a maximum viscosity near the surface due to resistance. This influence of viscosity is continuously decreasing when it is moved away from the surface and this effect is practically neglected after a thin region in a normal to the surface. This thin region adjacent to a solid surface in which viscous effect is significant is known as *momentum boundary layer*.

The thickness of this region increases in the direction of flow from the surface to the fluid layer that attains a free stream velocity. Since the velocity profile merges smoothly into the free stream, thus boundary layer is difficult to measure. So this boundary layer flow is divided into two regions: a region near the surface in which viscous and inertial properties cannot be neglected and a region covering the rest of the fluid in which that properties can be negligible that is away from the surface of the body.

Different from momentum boundary layer, *thermal boundary layer* is the region which exists due to transfer of heat between the fluid and surface of the body as characterized by the temperature gradient. Like momentum boundary layer, its thickness also increases along the flow direction. These two boundary layers are similar only in the case when Pr=1.

1.2.13 Boundary Conditions

Boundary conditions are necessary to solve fluid flow governed by conservation equation of motion. These conditions specify the behavior of boundary of the region in which a set of differential conditions are to be solved. These conditions are generally provided on fluid-fluid interface and solid-fluid interface. When a fluid is flowing over a solid surface, the instant contact of fluid particles to the layer of solid surface is known as *no-slip boundary conditions*. Here, the velocity of fluid relative to boundary is zero and also there is no relative motion between the fluid particles and solid surface.

1.3 Governing Equations of Fluid Flow and Heat Transfer

In Computational Fluid Dynamics (CFD), the derivation of governing differential equations is based on fundamental laws of mechanics. These are continuity, momentum, and energy equations. These governing equations obey the following universal law.

1. Mass is conserved.

2. The rate of change of momentum is equal to the sum of forces acting on fluid particle. This is known as Newton's second law of motion [9].

3. According to first law of thermodynamics, the rate of change of internal energy is equivalent to the sum of the rate of the addition of heat and the rate of work done on a fluid particle. Hence energy is conserved [9].

So these governing equations are mathematical representation of conservation laws of mechanics.

1.3.1 Conservation of Mass

Mass of the fluid is strictly conserved. This is also known as equation of continuity. The principle of conservation of mass can be stated as [18]:

$$\dot{m}_{increment} = \dot{m}_{in} - \dot{m}_{out}, \tag{1.3.1}$$

where $\dot{m}_{increment}$ denotes increase in mass per unit time in the control volume, while $\dot{m}_{in}$ and $\dot{m}_{out}$ express inlet mass flow and outlet mass flow per unit time in a control volume, respectively.

The mass of fluid flow in a control volume is taken as $\rho\, dx\, dy\, dz$ and the mass increment in the control volume per unit time can be demonstrated as

$$\dot{m}_{increment} = \frac{\partial \rho}{\partial t}\, dx\, dy\, dz, \tag{1.3.2}$$

The mass of fluid flow (mass inlet) into the control volume per unit time in x-direction is given by

$$\dot{m}_{in} \;=\; \rho\, u\, dy\, dz,$$

Now, the mass of fluid flow out (mass outlet) of the control volume in a unit time in the x-direction is expressed as:

$$\dot{m}_{out} \;=\; \left[\rho u \;+\; \frac{\partial}{\partial x}\;(\rho u) dx\right]\; dydz,$$

Consequently, by using above equations of mass inflow and outflow, the mass increment per unit time in the control volume in the x-direction is stated as

$$\dot{m}_{increment} \;=\; \frac{\partial}{\partial x}\;(\rho u) dxdydz, \tag{1.3.3}$$

Similarly, the mass increments of fluid flow in the control volume per unit time in the y and z-directions per unit time are given by $\left(\frac{\partial}{\partial y}(\rho v)dxdydz\right)$ and $\left(\frac{\partial}{\partial z}(\rho w)dxdydz\right)$, respectively. Therefore, we obtained the following expression:

$$\dot{m}_{in} \;-\; \dot{m}_{out} \;=\; \left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right] dxdydz, \tag{1.3.4}$$

Comparing Eqs.(1.3.2) and (1.3.4), we attain the following form:

$$\frac{\partial \rho}{\partial t}dxdydz + \frac{\partial}{\partial x}(\rho u)dxdydz + \frac{\partial}{\partial y}(\rho v)dxdydz + \frac{\partial}{\partial z}(\rho w)dxdydz = 0,$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0, \tag{1.3.5}$$

where u, v and w are the velocity components in x, y and z directions, respectively. Eq.(1.3.5) is unsteady, three dimensional mass conservation equation for compressible fluid in a Cartesian coordinates. This equation is also known as continuity equation. The first term of continuity equation represents the rate of change in time of fluid density while second term the net flow of mass out of the element across its

boundaries [9].
In vector notation Eq.(1.3.5) becomes:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}.(\rho \overrightarrow{U}) = 0, \tag{1.3.6}$$

where $\overrightarrow{U} = ui + vj + wk$. For steady compressible flow

$$\boldsymbol{\nabla}.(\rho \overrightarrow{U}) = 0,$$

when density of the fluid ρ is constant for incompressible fluid then Eq.(1.3.6) can be written as:

$$\boldsymbol{\nabla}.\overrightarrow{\boldsymbol{U}} = 0,$$

In Cartesian coordinates for steady three dimensional for incompressible fluid, the Eq.(1.3.5) can be written as:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{1.3.7}$$

1.3.2 Conservation of Momentum

The conservation of momentum equation can be derived by applying Newton's second law on a moving fluid element. According to this law the rate of change of momentum is equal to the net force acting on the particle of fluid element [9]. Mathematically, this law can be written as:

$$\overrightarrow{F} = m\overrightarrow{a} \tag{1.3.8}$$

There are two types of forces acting on the fluid element.

1. Body Forces: They act at a distance on entire body of the mass of the fluid element. These forces include electromagnetic forces, forces due to gravity, centrifugal forces etc.

2. Surface Forces: These forces act directly on the surface of the fluid elements. It is due to pressure, normal and shear stresses, surface tension etc [17].

Consider an enclosed surface S that contain a control volume. Let $\overrightarrow{F}_m$ be a force per unit mass of fluid and the total mass force in a control volume is represented by $\overrightarrow{F}_{m,v}$. Furthermore, $\overrightarrow{\tau}_n$ be a surface force per unit mass of fluid and $\overrightarrow{\tau}_{n,S}$ is a surface force containing in a control volume. Now, increase in momentum of the fluid in a control volume per unit time is taken as $G_{increment}$.

According to the conservation of momentum law, the increase in momentum of the fluid flow per unit time in the control volume is equal to the sum of total mass force and surface force in the same volume such that [18]:

$$\overrightarrow{G}_{increment} = \overrightarrow{F}_{m,v} + \overrightarrow{\tau}_{n,S}, \tag{1.3.9}$$

Total mass force per unit mass acting on fluid flow element within volume V is expressed as:

$$\overrightarrow{F}_{m,v} = \int_v \rho \overrightarrow{F}_m \, dV,$$

Total surface force per unit mass acting on fluid element within volume V is denoted by:

$$\overrightarrow{\tau}_{n,S} = \int_S \overrightarrow{\tau}_n \, dS,$$

Also the rate of increase of momentum of the mass contained in volume V is:

$$\overrightarrow{G}_{increment} = \frac{D}{Dt}\int_v \rho \overrightarrow{U} \, dV,$$

By using Gauss divergence theorem, the surface integral may be transform to volume integral. This follows to

$$\int_S \overrightarrow{\tau}_n dS = \int_v \nabla.\,[\tau] dV,$$

where $\nabla.[\tau]$ is known as surface shear tensor. Thus, the equation of motion (1.3.9) for the fluid can be rewritten as by using above relations:

$$\frac{D}{Dt}\int_v \rho\overrightarrow{U}dV \;=\; \int_v \rho\overrightarrow{F}_m dV \;+\; \int_v \nabla.[\tau]dV, \tag{1.3.10}$$

$$\int_v \left(\frac{D(\rho\overrightarrow{U})}{Dt} \;-\; \rho\overrightarrow{F}_m \;-\; \nabla.[\tau]\right) dV \;=\; 0,$$

It can be supposed that V is arbitrary and integrand is smooth.

$$\frac{D(\rho\overrightarrow{U})}{Dt} \;-\; \rho\overrightarrow{F}_m - \nabla.[\tau] \;=0,$$

$$\frac{D(\rho\overrightarrow{U})}{Dt} \;=\; \rho\overrightarrow{F}_m \;+\; \nabla.[\tau], \tag{1.3.11}$$

where $\overrightarrow{U} \;= ui+\; vj \;+\; wk$ is called fluid velocity. The Eq. (1.3.11) is known as momentum equation of the fluid flow.

For steady state, the momentum equation can be written as:

x-momentum equation:

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right) \;+\; u\left(u\frac{\partial \rho}{\partial x}+v\frac{\partial \rho}{\partial y}+w\frac{\partial \rho}{\partial z}\right) \;=\; -\;\frac{\partial p}{\partial x}$$

$$+2\;\frac{\partial}{\partial x}\;\left(\mu\frac{\partial u}{\partial x}\right) \;+\; \frac{\partial}{\partial y}\;\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right] \;+\; \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right]$$

$$-\;\frac{\partial}{\partial x}\;\left[\frac{2}{3}\mu\;\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\right] \;+\; \rho g_x, \tag{1.3.12}$$

y-momentum equation:

$$\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right) \;+\; v\left(u\frac{\partial \rho}{\partial x}+v\frac{\partial \rho}{\partial y}+w\frac{\partial \rho}{\partial z}\right) \;=\; -\;\frac{\partial p}{\partial y}$$

$$+\;\frac{\partial}{\partial x}\;\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right] \;+\; 2\frac{\partial}{\partial y}\left(\mu\frac{\partial v}{\partial y}\right) \;+\; \frac{\partial}{\partial z}\;\left[\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right]$$

$$-\;\frac{\partial}{\partial y}\;\left[\frac{2}{3}\mu\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\right] \;+\; \rho g_y, \tag{1.3.13}$$

z-momentum equation:

$$\rho\left(u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right) \;+\; w\left(u\frac{\partial \rho}{\partial x}+v\frac{\partial \rho}{\partial y}+w\frac{\partial \rho}{\partial z}\right) \;=\; -\;\frac{\partial p}{\partial z}$$

$$+\;\frac{\partial}{\partial x}\;\left[\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right] \;+\; \frac{\partial}{\partial y}\;\left[\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right] \;+\; 2\frac{\partial}{\partial z}\;\left(\mu\frac{\partial w}{\partial z}\right)$$

$$-\;\frac{\partial}{\partial z}\;\left[\frac{2}{3}\mu\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\right] \;+\; \rho g_z, \tag{1.3.14}$$

where g_x, g_y and g_z are gravitational accelerations in x, y, and z directions, respectively.

1.3.3 Conservation of Energy

The principle of conservation of energy is based on first law of thermodynamics. This law states that energy can be neither created nor destroyed during a mechanical process but it can only change from one form to another form. Therefore, every part of energy must be considered during a process. Thermal energy naturally moves in the direction of decreasing temperature and the transfer of thermal energy from one system to another as a result of a temperature difference.

Another way this law can be expressed that the rate of the energy of the particles of a fluid is equal to sum of the net rate of heat increment to the fluid particles and the net rate of work done on the particle [17].
Consider an arbitrary mass of fluid enclosed in volume. According to first law of thermodynamics:

$$\Delta \dot{E} = \dot{Q} + \dot{W}_{out}, \tag{1.3.15}$$

The total energy per unit time contained in volume V in the system is given as:

$$\Delta \dot{E} = \frac{D}{Dt} \int_v \rho \left(e + \frac{U^2}{2} \right) dV, \tag{1.3.16}$$

where t represents time, e is the internal energy per unit mass, U is the velocity of the fluid, and $\frac{U^2}{2}$ is the kinetic energy per unit mass of the fluid.
The work done on a fluid by the two external forces (surface and body forces) is the product of force vectors and fluid velocity. So the total work due to presence of these forces on a unit area will be:

$$\int_S \overrightarrow{\tau}_n . \overrightarrow{U}\, dS,$$

where $\overrightarrow{\tau}_n$ is the magnitude of surface force (stress) per unit area and S is the surface (boundary) enclosing volume V.
Similarly, the magnitude of body force per unit mass is vector $\overrightarrow{F}$. Then the total work done on the fluid particle due to mass force is

$$\int_v \rho \overrightarrow{F} . \overrightarrow{U}\, dV,$$

The total work done by the mass force and surface force per unit time on the system is represented as:

$$\dot{W}_{out} = \int_v \rho \overrightarrow{F} . \overrightarrow{U} dV + \int_S \overrightarrow{\tau}_n . \overrightarrow{U} dS, \tag{1.3.17}$$

The addition of heat is carried out by thermal conduction. So conduction is governed by Fouriers law[18]. Thus the net amount of heat entering the fluid per unit time is given as:

$$\dot{Q} \;=\; \int_S \kappa \, \frac{\partial T}{\partial n} \, dS, \tag{1.3.18}$$

where $\hat{n}$ is normal line of the surface, while κ represent the heat conduction coefficient.

By substituting value of Eq. (1.3.16)-(1.3.18), in Eq. (1.3.15) we obtained the following expression:

$$\frac{D}{Dt}\int_v \rho\left(e+\frac{U^2}{2}\right)dV \;= \int_v \rho\,\overrightarrow{F}.\overrightarrow{U}\;dV + \int_S \overrightarrow{\tau}_n.\overrightarrow{U}\,dS \;+ \int_S \kappa\,\frac{\partial T}{\partial n}dS, \tag{1.3.19}$$

where,

$$\frac{D}{Dt}\int_v \rho\;\left(e\;+\frac{U^2}{2}\right)\;dV = \int_v \frac{D}{Dt}\;\left[\rho\left(e+\frac{U^2}{2}\right)\right]\;dV,$$

The surface forces can be rearranged as:

$$\int_S \overrightarrow{\tau}_n.\overrightarrow{U}\,dS = \int_S \overrightarrow{n}[\tau].\overrightarrow{U}\,dS = \int_S \overrightarrow{n}([\tau].\overrightarrow{U})dS = \int_v \nabla.([\tau].\overrightarrow{U})dV$$

$$\int_S \kappa\,\frac{\partial T}{\partial n}\,dS \;\;=\;\; \int_v \nabla.\,(\kappa\nabla T)\;\,dV$$

By using above relations, the Eq. (1.3.19) can be rearranged as:

$$\int_v \frac{D}{Dt}\left[\rho\left(e+\frac{U^2}{2}\right)\right]dV = \int_v \rho\overrightarrow{F}.\overrightarrow{U}dV + \int_v \nabla.([\tau].\overrightarrow{U})dV + \int_v \nabla.(\kappa\nabla T)dV,$$

$$\frac{D}{Dt}\left[\rho\left(e+\frac{U^2}{2}\right)\right] = \rho\overrightarrow{F}.\overrightarrow{U} + \nabla.([\tau].\overrightarrow{U}) + \nabla.(\kappa\nabla T),$$

where $[\tau]$ is known as tensor of shear forces. The Eq. (1.3.20) is called the energy equation.

1.4 Numerical Methods

Initial value problems (IVPs) are much easier to solve rather than boundary value problems (BVPs). Any solver might be failed to solve BVPs. Moreover, the IVPs always have a unique solution and solution always exists but BVP may not have a solution or may or may not have finite number of solution. Following two numerical techniques have been applied in this work.

1.4.1 Shooting Method

The conceptual strategy of boundary value problem (BVP) depends on the solution of initial value problem (IVP) for ODEs along with the solution of nonlinear algebraic equation. Since there are useful programmes for both the problem to combine them in a program for the solution of BVPs. The strategy is known as shooting method. The name is derived from analogy with target shooting - take a shot and examine where it hits the target, then corrects the position and shoots again. Therefore, shooting method is most successful numerical technique to solve linear as well as nonlinear equations. Moreover, the fundamental algorithm of this numerical method is the supposition of trial value. The solution executes at one end of the boundary value problem (BVP) and shoots to the other end just like a cannon-ball (reaching its target under the impact of gravity) by using initial value solver until the boundary condition at the other end converges to its correct value. The superiority of the shooting method is the speed and adaptivity of methods for initial value problems. Consider a third order non-linear boundary value problem [19].

$$\begin{aligned} y''' &= f(x, y, y', y''), \qquad \forall a \leq x \leq b, \\ y(a) = \alpha, \qquad y'(a) &= \gamma, \qquad y'(b) = \beta \end{aligned} \tag{1.4.1}$$

where α, β and γ are constant parameters.

Choose an initial guess u_0 such that it to examine the solution of the derivatives $y'''(0)$. So

$$y'''(x,u) = f(x, y(x,u), y'(x,u), y''(x,u)), \qquad \forall a \leq x \leq b,$$
$$y(a) = \alpha, \qquad y'(a) = \gamma, \qquad y'(b) = \beta, \qquad y''(a,u) = u_0. \tag{1.4.2}$$

Now by differentiating Eq. (1.4.2) with respect to u, we obtained

$$\frac{\partial y'''}{\partial u}(x,u) = \frac{\partial f}{\partial x}(x, y(x,u), y'(x,u), y''(x,u))\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}(x, y(x,u), y'(x,u), y''(x,u))\frac{\partial y}{\partial u} +$$
$$\frac{\partial f}{\partial y'}(x, y(x,u), y'(x,u), y''(x,u))\frac{\partial y'}{\partial u} + \frac{\partial f}{\partial y''}(x, y(x,u), y'(x,u), y''(x,u))\frac{\partial y''}{\partial u},$$

As u and x are independent, so $\frac{\partial x}{\partial u} = 0$,

$$\frac{\partial y'''}{\partial u}(x,u) = \frac{\partial f}{\partial y}(x, y(x,u), y'(x,u), y''(x,u))\frac{\partial y}{\partial u} + \frac{\partial f}{\partial y'}(x, y(x,u), y'(x,u), y''(x,u))\frac{\partial y'}{\partial u} +$$
$$\frac{\partial f}{\partial y''}(x, y(x,u), y'(x,u), y''(x,u))\frac{\partial y''}{\partial u},$$

Now consider,

$$z(x,u) = \frac{\partial y}{\partial u}(x,u),$$

$$z'' = \frac{\partial f}{\partial y}(x,y,z,z')z + \frac{\partial f}{\partial y'}(x,y,z,z')z' + \frac{\partial f}{\partial y''}(x,y,z,z')z'',$$
$$z(a) = 0, \qquad z'(a) = 0, \qquad z'' = 1. \tag{1.4.3}$$

To select the value of u_0 such that,

$$y(b,u) - \beta = 0,$$

$$u_o = y'(a) = \frac{y(b) - y(a)}{b - a},$$

$$u_o = \frac{\beta - \alpha}{b - a},$$

Newton Raphson method is required to estimate the solution of $y(b, u) - \beta = 0$ and identify a next guess u_{t+1}.

$$u_{t+1} = u_t - \frac{y'(b, u_t) - \beta}{z(b, u_t)}, \tag{1.4.4}$$

Eq. (1.4.4) is transform into a first order ordinary differential equation. Then a first order ordinary differential equation can be solved by Runge-Kutta method. The process will stop until the error is $|\beta - y'(b, u_t)| \leqslant$ Tolerance value.

1.4.2 *bvp*4*c*

MATLAB programming requires a guess to solve boundary value problem (BVP). Finding a good guess is often the difficult part of solving a BVP. *bvp*4*c* is one such useful solver for solving BVP. It takes an exceptional approach to the control of error that assists to deal with poor guesses. It is a collocation method and starts solution with initial guess provided at initial mesh points. *bvp*4*c* is based on algorithm that are plausible even when the initial mesh is very poor, yet furnish the correct results. A step-size is also changed to acquire the specified accuracy. Contradictory to the shooting method, the solution approach over the whole interval and boundary conditions are taken all the time [20].

Chapter 2

Numerical Solution of MHD Mixed Convection Slip Flow near a Stagnation Point on a Nonlinearly Vertical Stretching Sheet

2.1 Introduction

This chapter is the review work of paper by Shen *et al* [3]. In recent years, the study of magnetohydrodynamics (MHD) has created much interest for many researchers and analysts due to extensive practical application especially in fluid dynamics. MHD is related to the interaction of electromagnetic fields and electrically conducting fluids. It has significantly played a vital role in industry and engineering such as the magnetic behavior of plasma in fusion reactors, liquid metal cooling of nuclear reactor, drawing of plastic sheet, metallurgy, electromagnetic casting, polymer extrusion etc. It also reveals specific characteristics in thermal conductivity. The study of MHD flow on stretching sheet has been investigated effects of slip condition [21-22], viscous dissipation[23] and unsteady flow and heat transfer [24]. The influence of viscosity on boundary layer flow and heat transfer is also analyzed for electrically conducting fluid over a linearly stretching sheet. For this purpose, Mukhopadhyay *et al* and Subhas *et al* [25-26] considered viscosity which was varying as a linear function of temperature.

Some work has also been done on the effect of mixed convection stagnation point flow on a stretching vertical sheet due to presence buoyancy forces. In this regard, the behaviour of MHD convection flow and heat transfer has been investigated by some researchers [33-35]. A substantial work is carried out for incompressible viscous nanofluid by considering MHD on convective flow. For this purpose, Ibrahim *et al* [30], Hamad [31], Ali *et al* [32] explored the effects of magnetic field on stagnation point flow and heat transfer of nanofluid over a stretching sheet and obtained a numerical results for velocity and temperature concentration.

In the recent time, some work has been carried out for fluid flow and heat transfer due to non-linearly stretching sheet. Rana and Bhargava [33], Ashraf *et al* [34] and Dhanai [35] evaluated results for the boundary layer fluid flow due to nonlinear stretching of a flat surface placed in nanofluid. With changing the type of nanofluid particles, fluid flow shows changes in behavior [36].

Shen *et al* [3] analyzed the impact of the nonlinearity parameter, mixed convection parameter, magnetic field, suction and injection, on a MHD mixed convection incompressible flow near a stagnation point over a nonlinearly stretching sheet.

2.2 Mathematical Formulation

Consider a two dimensional incompressible laminar flow over a non-linearly stretching sheet that is emerged at the origin of Cartesian coordinate system (x, y). The flow is restricted to $y \geq 0$. The sheet moves with velocity $u_w(x) = cx^m$ in the positive x-direction, whereas y-axis is perpendicular to the stretching surface. The external velocity is taken as $u_e(x) = ax^m$, where a and c are considered as positive constants. The constant m is the non-linearity parameter. It should be mentioned that $m = 1$ represents a linear case while $m \neq 1$ corresponds a non-linearly stretching case. A uniform magnetic field of strength B is applied normally. The viscous dissipation effect is neglected. The basic boundary layer equations such that continuity, momentum and energy for incompressible flow are given by:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2.2.1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = u_e\frac{du_e}{dx} + \nu\frac{\partial^2 u}{\partial y^2} + \frac{\sigma B^2(x)}{\rho}(u_e - u) + g\beta(T - T_\infty), \tag{2.2.2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}, \tag{2.2.3}$$

where u is the velocity component along x-direction while v denotes a velocity component in y-direction, T is a temperature of fluid, ρ is a density of fluid, ν is a kinematics viscosity and σ represents electrical conductivity. The magnetic field, gravitational acceleration, thermal diffusivity, and coefficient of thermal expansion are denoted by $B(x)$, g, α and β, respectively. For subject problem, the related boundary condition are given by:

$$\begin{gathered} u = u_w(x) + \frac{2-\sigma_v}{\sigma_v}\lambda_0\frac{\partial u}{\partial y}, \quad v = v_w(x), \quad \frac{\partial T}{\partial y} = -\frac{q_w(x)}{k}, \ \ at\ y = 0 \\ u \to u_e(x) \qquad T \to T_\infty \qquad at\ \ y \to \infty \end{gathered} \tag{2.2.4}$$

In Eq. (2.2.4), κ, σ_v, λ_0, $v_w(x)$, and $q_w(x)$ corresponds to the thermal conductivity, coefficient of the tangential momentum accommodation, the mean free path, the suction (injection) velocity, and surface heat flux, respectively. So $B(x)$, $q_w(x)$, $v_w(x)$ are given by:

$$\begin{aligned} B(x) &= B_0\, x^{(m-1)/2}, \qquad q_w(x) = q_0\, x^{(5m-3)/2}, \\ v_w &= -\frac{\sqrt{a\nu}(m+1)}{2} x^{(m-1)/2} S, \end{aligned} \tag{2.2.5}$$

where B_0, S, and q_0 are constants. It is emphasized that $S > 0$ indicates suction while injection implies for $S < 0$. Now, we introduced the following similarity

transformations to transform Eqs. (2.2.1)-(2.2.3) into nonlinear ODEs along with relevant boundary conditions.
The following similarity variables are introduced:

$$\eta = \sqrt{\frac{a}{\nu}}\, y\, x^{(m-1)/2}, \tag{2.2.6}$$

$$\psi = \sqrt{a\nu}\, x^{(m+1)/2} f(\eta), \tag{2.2.7}$$

$$\theta = \sqrt{\frac{a}{\nu}}\, \frac{\kappa\,(T - T_\infty)}{q_0\, x^{2m-1}}, \tag{2.2.8}$$

In expression (2.2.7), ψ is the stream function. We define a ψ in such a manner that it satisfies mass conservation Eq. (2.2.1).

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}, \tag{2.2.9}$$

Moreover, the velocity component u and v are taken as:

$$u = ax^m f'(\eta) \quad and \quad v = -\sqrt{a\nu}x^{(m-1)/2}\left[\frac{m+1}{2}f(\eta) + \frac{m-1}{2}\eta f'(\eta)\right], \tag{2.2.10}$$

where prime denotes differentiation with respect to η.
Using Eq. (2.2.10), we evaluate $\frac{\partial u}{\partial x}$ as:

$$\begin{aligned}
\frac{\partial u}{\partial x} &= max^{m-1}f'(\eta) + ax^m f''(\eta)\frac{\partial \eta}{\partial x},\\
&= max^{m-1}f'(\eta) + ax^m f''(\eta)(y\sqrt{a/\nu}\frac{m-1}{2}x^{m-3/2}),\\
&= max^{m-1}f'(\eta) + a^{3/2}y\nu^{-1/2}x^{(3m-3)/2}\frac{m-1}{2}f''(\eta).
\end{aligned} \tag{2.2.11}$$

Now, differentiate Eq. (2.2.10) w.r.t y

$$\frac{\partial u}{\partial y} = ax^m f''(\eta)\frac{\partial \eta}{\partial y},$$

$$= a^{3/2}\nu^{-1/2}x^{3m-1/2}f''(\eta). \tag{2.2.12}$$

Again differentiate Eq. (2.2.12) w.r.t y

$$\frac{\partial^2 u}{\partial y^2} = a^{3/2}\nu^{-1/2}x^{3m-1/2}f'''(\eta)\frac{\partial \eta}{\partial y},$$

$$= a^2/\nu.x^{2m-1}f'''(\eta). \tag{2.2.13}$$

Now, taking square of $B(x)$ from Eq. (2.2.5), we get:

$$B^2(x) = B_0^2 x^{m-1},$$

by subtracting,

$$u_e - u = ax^m(1 - f'(\eta)),$$

by simplifying Eq. (2.2.8), we obtain the following relation:

$$T - T_\infty = \sqrt{\nu/a}\theta(q_0/\kappa)x^{2m-1},$$

by substituting Eqs. (2.2.11)-(2.2.13) into Eq. (2.2.2), we obtained

$$(ax^m f'(\eta))\left[(ax^{m-1}f'(\eta) + a^{3/2}\nu^{-1/2}y^{m-1/2}x^{3m-3/2}f''(\eta)\right] - \sqrt{a\nu}x^{(m-1)/2}$$

$$\left[\tfrac{m+1}{2}f(\eta) + \tfrac{m-1}{2}\eta f'(\eta)\right]\left[a^{3/2}\nu^{-1/2}x^{3m-1/2}f''(\eta)\right] = (ax^m)(amx^{m-1}) + \nu(a^2/\nu x^{2m-1}$$

$$f'''(\eta)) + \sigma B_0^2 x^{m-1}/\rho a x^m(1 - f') + g\beta(\sqrt{\nu/a}\theta(q_0/\kappa)x^{2m-1}),$$

$$a^2mx^{2m-1}f'^2+a^{5/2}\nu^{1/2}yx^{5m-3/2}f'f''-(a^2x^{2m-1}f''\left[\tfrac{m+1}{2}f+\tfrac{m-1}{2}f'\sqrt{a/\nu}yx^{(m-1/2)}\right]=$$

$$ma^2x^{2m-1}+a^2x^{2m-1}f'''(\eta)+(\sigma B_0^2a)/\rho x2m-1(1-f')+g\beta(\nu^{1/2}/a^{5/2})(q_0/\kappa)a^2x^{2m-1}\theta,$$

$$a^2mx^{2m-1}f'^2+a^{5/2}\nu^{1/2}yx^{5m-3/2}f'f''-(a^2x^{2m-1}f''\left[\tfrac{m+1}{2}f+\tfrac{m-1}{2}f'\sqrt{a/\nu}yx^{(m-1/2)}\right]=$$

$$ma^2x^{2m-1}+a^2x^{2m-1}f'''(\eta)+(\sigma B_0^2a)/\rho x^{2m-1}(1-f')+g\beta(\nu^{1/2}/a^{5/2})(q_0/\kappa)a^2x^{2m-1}\theta,$$

After applying transformation on momentum equation and simplifying it, the following nonlinear ordinary differential equation is obtained

$$f'''(\eta)+\frac{m+1}{2}f(\eta)f''(\eta)+m(1-f'^2(\eta))+M(1-f'(\eta))+\lambda\theta(\eta)=0, \quad (2.2.14)$$

where M denotes a magnetic parameter and λ is the mixed convection parameter. These constant parameters are given by

$$M=\frac{\sigma B_0^2}{a\rho}, \qquad \lambda=\frac{g\beta q_0\sqrt{\nu}}{\kappa a^{5/2}}=\frac{Gr_x}{Re_x^{5/2}},$$

where $Gr_x = gq_wx^4\beta/\nu^2\kappa$ is known as local Grashof number and $Re_x = u_ex/\nu$ is known as Reynolds number. It should be emphasized that $\lambda = 0$ correspond to pure forced convection flow, while $\lambda < 0$ represent that flow is opposing and $\lambda > 0$ represents accelerating flow. Now for transforming energy equation, we evaluate $\frac{\partial T}{\partial x}$ by using Eq. (2.2.8):

$$\frac{\partial T}{\partial x}=(q_0/\kappa)x^{2m-1}\sqrt{\nu/a}\theta'(\eta)\frac{\partial\eta}{\partial x}+\sqrt{\nu/a}(q_0/\kappa)(2m-1)x^{2m-2}\theta(\eta),$$

$$=(q_0/\kappa)x^{2m-1}\sqrt{\nu/a}\left((m-1/2)\sqrt{a/\nu}yx^{m-3/2}\right)\theta'(\eta),$$

$$+\sqrt{\nu/a}(q_0/\kappa)(2m-1)x^{2m-2}\theta(\eta). \quad (2.2.15)$$

Using Eq. (2.2.8), we evaluate $\frac{\partial T}{\partial y}$ as:

$$\frac{\partial T}{\partial y} = (q_0/\kappa)x^{2m-1}\sqrt{\nu/a}\frac{\partial \eta}{\partial y}\theta'(\eta),$$

$$= (q_0/\kappa)x^{5m-3/2}\theta'(\eta). \tag{2.2.16}$$

Again differentiate w.r.t y

$$\frac{\partial^2 T}{\partial y^2} = q_0/\kappa x^{3m-2}\theta''(\eta), \tag{2.2.17}$$

by substituting Eq. (2.2.15)-(2.2.17) in Eq. (2.2.3),

$(ax^m f'(\eta))\theta'(\eta)(q_0/\kappa)x^{2m-1}\sqrt{\nu/a}\left((m-1/2)\sqrt{a/\nu}yx^{m-3/2}\right) + \sqrt{\nu/a}(q_0/\kappa)$

$\theta(2m-1)x^{2m-2} - \sqrt{a\nu}x^{(m-1)/2}\left[\frac{m+1}{2}f(\eta) + \frac{m-1}{2}\eta f'(\eta)\right]\left[\theta'(\eta)(q_0/\kappa)x^{5m-3/2}\right]$

$= \alpha(q_0/\kappa)x^{3m-2}\theta''$,

$\sqrt{a\nu}(2m-1)f'\theta - \sqrt{a\nu}(m+1/2)\theta' f = \alpha\sqrt{a/\nu}\theta''$,

$\theta''(\eta) + \nu/\alpha(m+1/2)\theta'(\eta)f(\eta) - \nu/\alpha(2m-1)f'(\eta)\theta(\eta) = 0$,

Finally, the following ordinary differential equation is obtained by simplifying above expression:

$$\theta''(\eta) + \frac{Pr(m+1)}{2}f(\eta)\theta'(\eta) - Pr(2m-1)f'(\eta)\theta(\eta) = 0, \tag{2.2.18}$$

where Pr=$\frac{\nu}{\alpha}$ is the Prandtl number.

The boundary conditions (2.2.4) become

$$f(0) = S, \qquad f'(0) = \epsilon + \delta f''(0), \qquad \theta'(0) = -1,$$
$$f'(\infty) = 1, \qquad \theta(\infty) = 0, \tag{2.2.19}$$

where $\epsilon = \frac{c}{a}$ is known as velocity ratio parameter and $\delta = \frac{2-\sigma_v}{\sigma_v} Kn_x Re_x^{1/2}$ denotes velocity slip parameter with the local Knudsen number $Kn_x = \lambda_0/\sqrt{\epsilon}x$.
Furthermore, the skin friction coefficient C_f and the local Nusselt number Nu_x can be defined as

$$C_f = \frac{\tau_w(x)}{\rho u_e^2}, \qquad Nu_x = \frac{xq_w(x)}{\kappa(T_w - T_\infty)},$$

where $\tau_w(x) = \mu(\frac{\partial u}{\partial y})_{y=0}$ is the surface shear stress. Using similarity variables (2.2.6)-(2.2.8), we obtained

$$Re_x^{1/2} C_f = f''(0), \qquad Re_x^{-1/2} Nu_x = 1/\theta(0).$$

2.3 Numerical Methods

The system of coupled nonlinear ODEs (2.1.14) and (2.2.18) subject to the boundary conditions (2.2.19) has been solved numerically by using shooting method and *bvp4c*. Using shooting method boundary value problems (BVP) are transformed into an initial value problems (IVP). For this purpose an efficient initial guess is selected until the convergence is obtained. Shooting and *bvp4c* has already been explained in Chapter 1.
To convert **BVP** into **IVP**, we define a new variable as:

$$y = f = y_1, \qquad f' = y_1' = y_2, \qquad f'' = y_2' = y_3,$$
$$\theta = y_4, \qquad \theta' = y_4' = y_5. \tag{2.3.1}$$

By using shooting method the system of ODEs (2.2.14) and (2.2.18) are transformed to a system of five simultaneous equations with five unknowns. These transform ODEs are third order in f and second order in θ, and can be written as

$$y_3' = f''' = -\frac{m+1}{2} y_1 y_3 - m(1 - y_2^2) - M(1 - y_2) - \lambda y_4, \tag{2.3.2}$$

$$y_5' = \theta'' = -\frac{Pr(m+1)}{2} y_1 y_5 + Pr(2m+1) y_2 y_4, \tag{2.3.3}$$

The Eqs. (2.3.2) and (2.3.3) are transformed momentum and energy equations, respectively.

2.4 Results and Discussions

In this section the effect of different parameters on the skin friction coefficient, the local Nussetl number, the velocity and the temperature profile are analyzed numerically as well as graphically. In order to validate the accuracy of numerical procedure, results of $f''(0)$ and $1/\theta(0)$ [3] are compared with the results of Yaqob and Ishak [37] for Pr = 0.7, 1 at $m = 1$, $\lambda = 1$, $M = 0$, $\epsilon = 0$, $\delta = 0$, $S = 0$ in Table 2.1.
In Table 2.2 we present and compare the numerical results with the results in Table 2.1. We find a good agreement in Tables 2.1 and 2.2.

	Yaqob and Ishak [37]		Shen *et al*[3]		Shooting Results	
Pr	$f''(0)$	$1/\theta(0)$	$f''(0)$	$1/\theta(0)$	$f''(0)$	$1/\theta(0)$
0.7	1.8339	0.7776	1.8337	0.7771	1.8337	0.7776
1	1.7338	0.8780	1.7337	0.8780	1.7338	0.8780

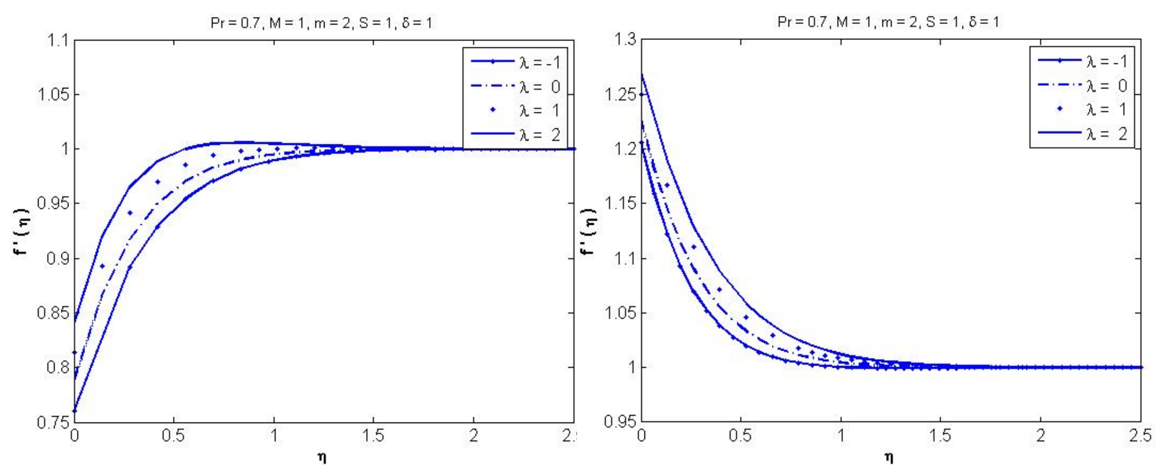
Table 2.1: Results of $f''(0)$ and $1/\theta(0)$.

The influence of mixed convection parameter λ on the velocity and temperature profile is depicted in Figs. 2.1-2.2 respectively for both cases of $\epsilon < 1$ and $\epsilon > 1$. Velocity profile shows a subsequent variation in boundary layer structure rather than thermal boundary layer. In Fig. 2.1(a), it is observed that thickness of velocity boundary layer decreases with an increasing value of λ when velocity ratio parameter is taken as $\epsilon = 0.1 < 1$, although flow shows a boundary layer structure in this case. On the other hand, for $\epsilon = 2 > 1$ the velocity boundary layer thickness increases

	Shooting Method		*bvp4c*	
Pr	$f''(0)$	$1/\theta(0)$	$f''(0)$	$1/\theta(0)$
0.7	1.8337	0.7776	1.8337	0.7776
1	1.7338	0.8780	1.7338	0.8780

Table 2.2: Results of $f''(0)$ and $1/\theta(0)$.

with λ because stretching surface velocity $u_w(x)$ exceeds the external velocity $u_e(x)$, as shown in Fig. 2.1(b).
Unlike velocity profile, a slight variation is observed in thermal boundary layer structure. Fig. 2.2(a-b) shows that temperature decreases for increasing values of mixed convection parameter λ. It is also observed that thermal boundary layer thickness decreases for both cases when $\epsilon < 1$ and $\epsilon > 1$. It is also analyzed that surface temperature $\theta(0)$ decreases with increasing values of λ, which indicates that heat transfer rate $1/\theta(0)$ at the surface increases.

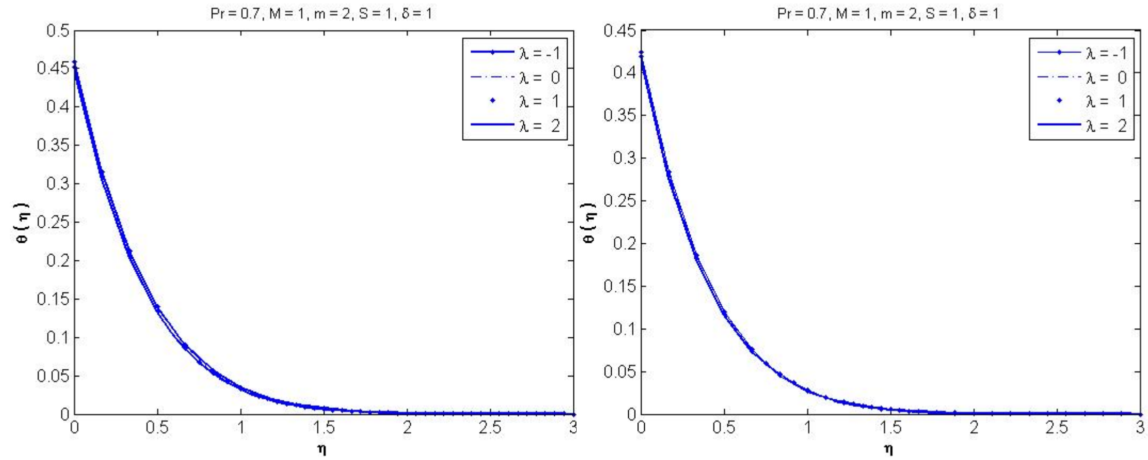


(a) Velocity profile for $\epsilon = 0.1$.

(b) Velocity profile for $\epsilon = 2$.

Figure 2.1: Velocity profile for different value of λ.

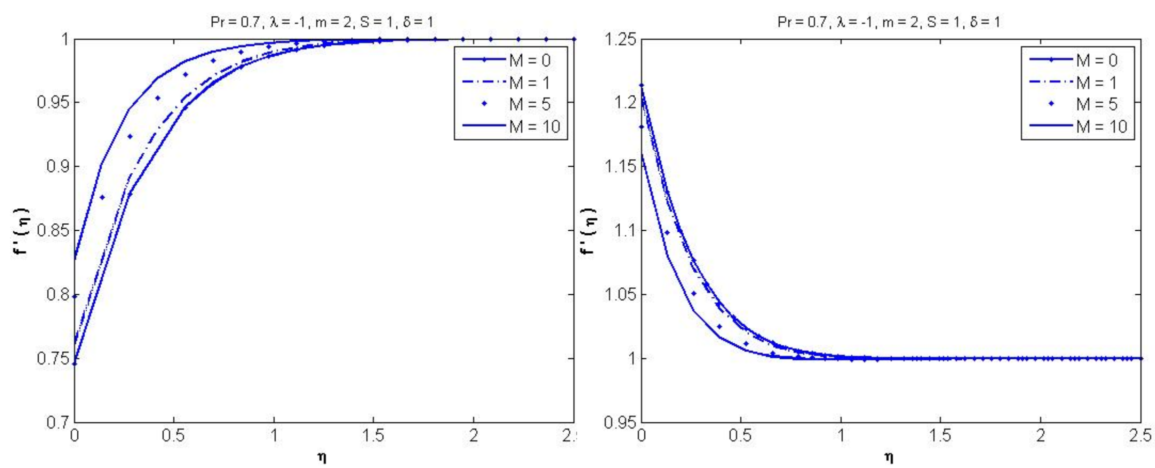
The effect of magnetic parameter M, injection (suction) parameter S and velocity slip parameter δ on velocity profile are illustrated in Figs. 2.3-2.5 for both cases of $\epsilon < 1$ and $\epsilon > 1$. Almost a similar impact is observed for these parameters



(a) Temperature profile for $\epsilon = 0.1$. (b) Temperature profile for $\epsilon = 2$.

Figure 2.2: Temperature profile for different value of λ.

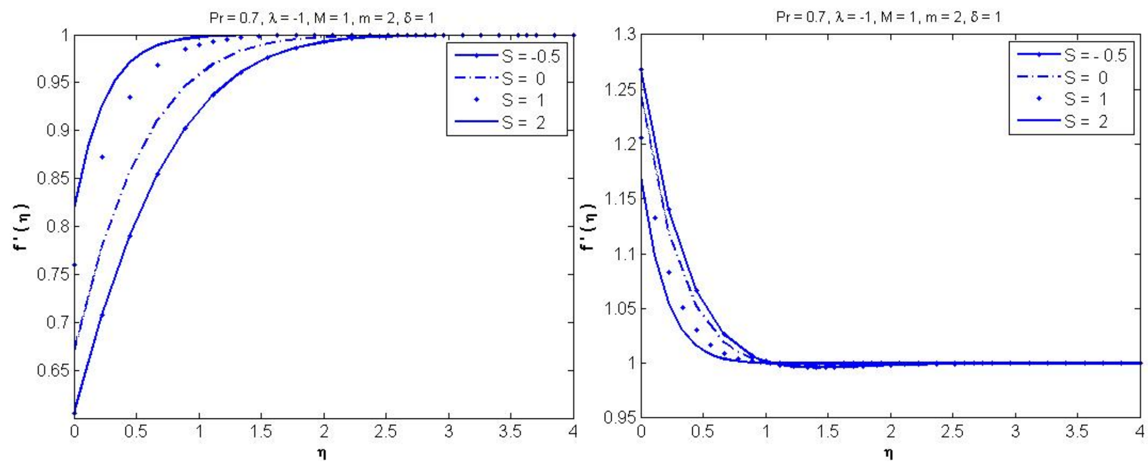
on momentum boundary layer structure. From these figures it is analyzed that increasing values of parameters M, S and δ leads to decrease the velocity boundary layer thickness for both cases. So that velocity gradient at the surface also increases. As a result the magnitude of coefficient of skin friction $f''(0)$ also increases.



(a) Velocity profile for $\epsilon = 0.1$. (b) Velocity profile for $\epsilon = 2$.

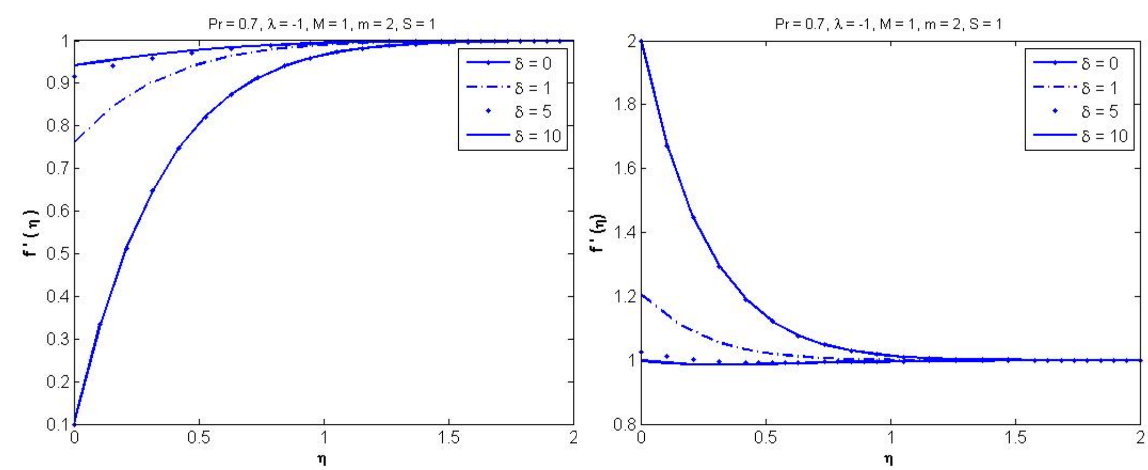
Figure 2.3: Velocity profile for different value of M.

Physically, it can be interpreted that the thickness of the momentum boundary layer decreases consequently as the strength of magnetic parameter M increases. This is due to the fact that the Lorentz force related with the magnetic field retards the motion of fluid which makes the boundary layer thinner so that the velocity profile $f'(\eta)$ along the surface decreases significantly.



(a) Velocity profile for $\epsilon = 0.1$. (b) Velocity profile for $\epsilon = 2$.

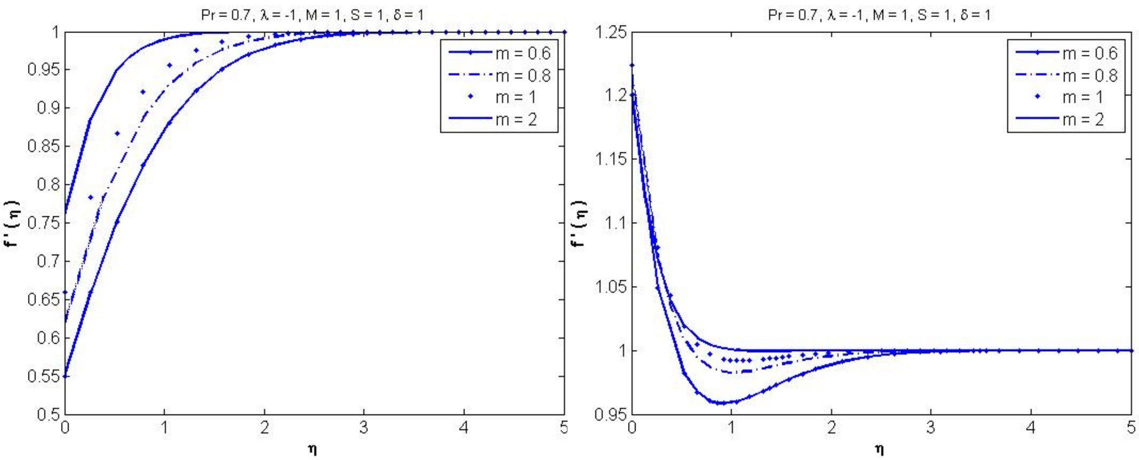
Figure 2.4: Velocity profile for different value of S.



(a) Velocity profile for $\epsilon = 0.1$. (b) Velocity profile for $\epsilon = 2$.

Figure 2.5: Velocity profile for different value of δ.

The effect of nonlinearity parameter m on velocity profile are illustrated in Figs. 2.6 for both cases of $\epsilon < 1$ and $\epsilon > 1$. In Fig. 2.6(a), it is observed that thickness of velocity boundary layer decreases with an increasing value of m when velocity ratio parameter is taken as $\epsilon = 0.1 < 1$. On the other hand, for $\epsilon = 2 > 1$ the velocity boundary layer thickness increases with an increasing value of m as shown in Fig. 2.6(b).

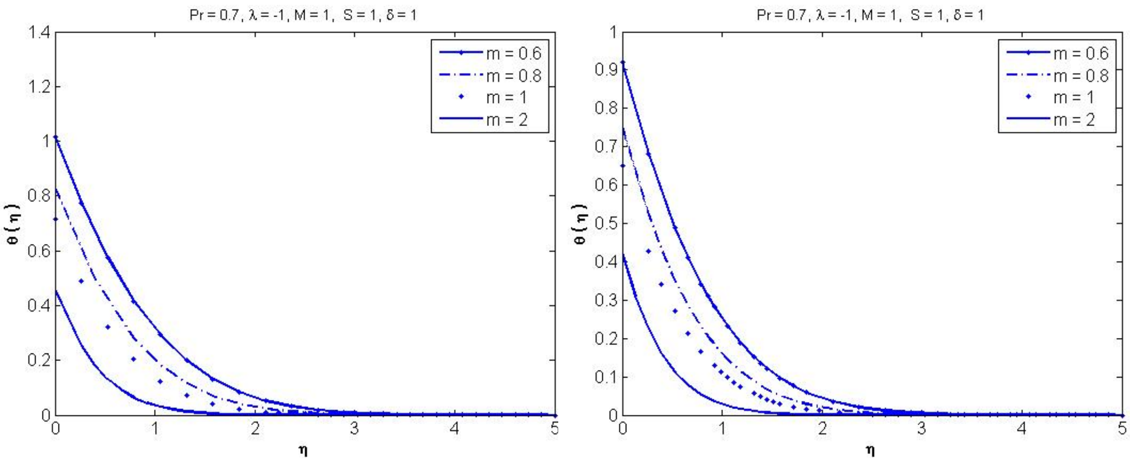


(a) Velocity profile for $\epsilon = 0.1$. (b) Velocity profile for $\epsilon = 2$.

Figure 2.6: Velocity profile for different value of m.

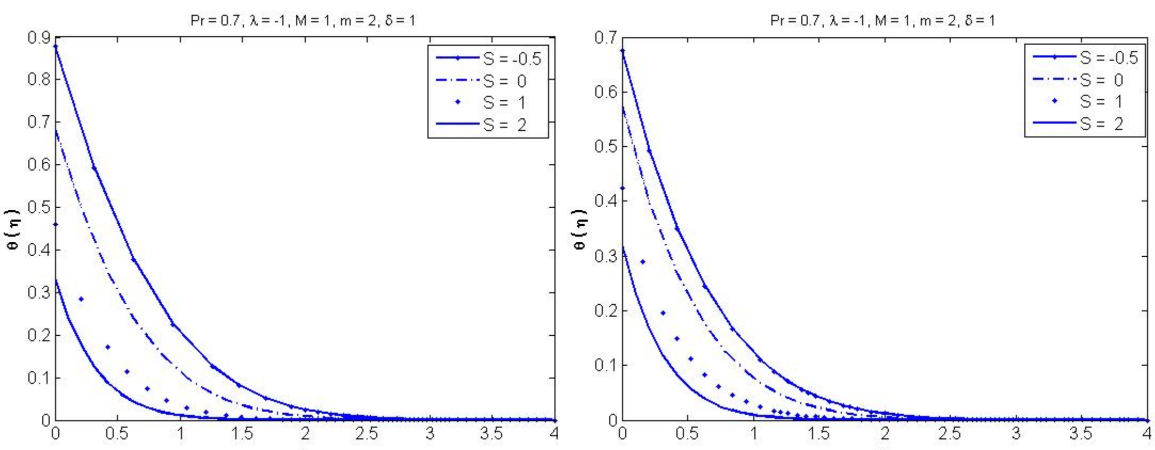
Figs. 2.7-2.8, depicts the effect of non-linearity parameter m and injection(suction) parameter S variations on temperature profile for both cases $\epsilon < 1$ and $\epsilon > 1$. It is observed that with an increasing value these two parameters the thermal boundary layer thickness decreases significantly for both cases. Moreover, it is also observed that for $\epsilon < 1$ the temperature and thermal boundary layer thickness is higher as compared to $\epsilon > 1$ by keeping other parameters are constant.

The temperature profile is less affected by velocity slip parameter δ and magnetic parameter M as shown in Fig. 2.9-2.10. These parameters M and δ show a reverse trend for both cases $\epsilon < 1$ and $\epsilon > 1$. Figs. 2.9 (a)-2.10 (a) a reduction in thermal boundary layer thickness is observed with an increasing values of M and δ for $\epsilon < 1$. This follows an increase in heat transfer rate at the surface. However, an opposite trend in characteristics is investigated for $\epsilon > 1$ for these two parameters as depicted in Figs. 2.9 (b)-2.10 (b). In this case thermal boundary layer thickness increases with an increasing values of M and δ which results in a reduction of heat transfer rate.



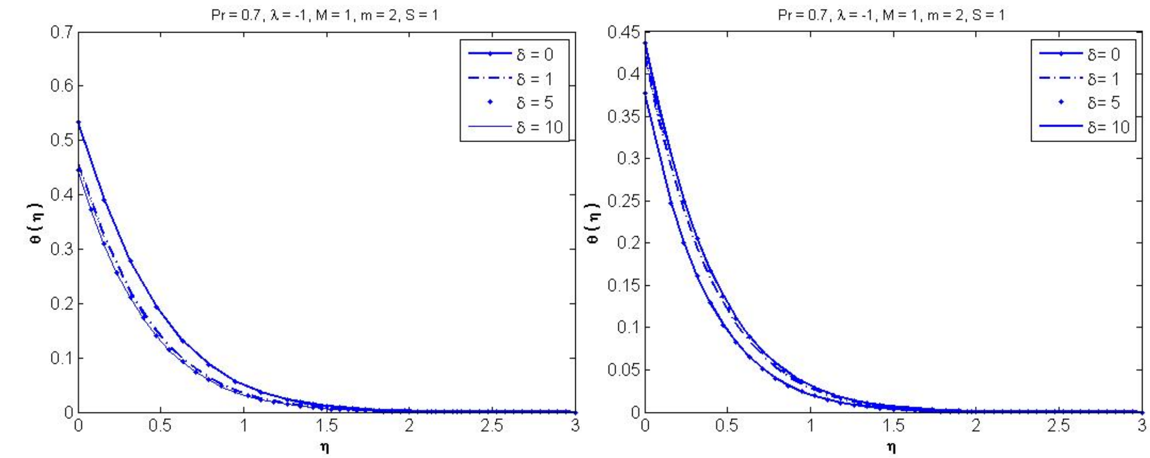
(a) Temperature profile for $\epsilon = 0.1$. (b) Temperature profile for $\epsilon = 2$.

Figure 2.7: Temperature profile for different value of m.



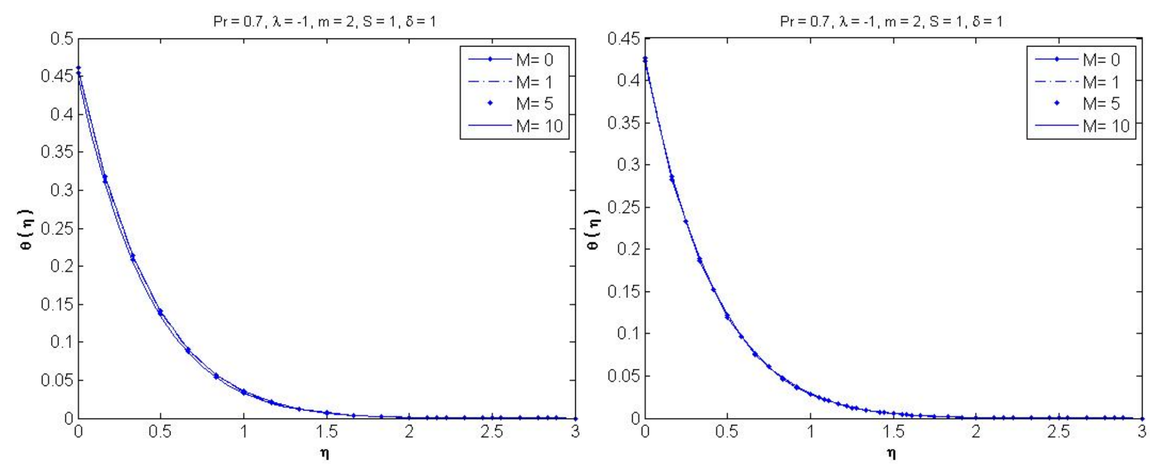
(a) Temperature profile for $\epsilon = 0.1$. (b) Temperature profile for $\epsilon = 2$.

Figure 2.8: Temperature profile for different value of S.



(a) Temperature profile for $\epsilon = 0.1$. (b) Temperature profile for $\epsilon = 2$.

Figure 2.9: Temperature profile for different value of δ.



(a) Temperature profile for $\epsilon = 0.1$. (b) Temperature profile for $\epsilon = 2$.

Figure 2.10: Temperature profile for different value of M.

Chapter 3

Numerical Solution for Boundary Layer Flow and Heat Transfer with Variable Fluid Properties

3.1 Introduction

This chapter contains a review work of Bachok *et al* [8]. The study of boundary layer on a continuous moving flat plate has significant role in Newtonian as well as non-Newtonian flows problems in fluid mechanics. Although, it has received considerable attention for many researchers due to its crucial practical applications in engineering and industrial process such as materials manufactured by extrusion processes, melt-spinning, wire drawing, polymer filaments and sheet, manufacture of plastic and rubber sheets, the hot rolling, the fabrication of sheet glass, electroplating of steel sheets and copper wire etc. It was the Sakiadis [4] who first studied the boundary layer flow with constant velocity induced by a solid surface in an otherwise ambient fluid. Tsou *et al* [38] extended the Sakiadis [4] work and analyzed a results for heat transfer in a thermal boundary layer by considering continuous moving surface and ambient fluid. Afterwards, a series of research is carried out on this class of boundary layer flow problems and many investigators [39-42] have studied different aspects of this problem such as the effect of mass and heat transfer, chemical effects, constant and variable surface temperature, the effect of magnetic field etc.

Later on, a substantial work is examined on temperature dependent fluid properties for Sakiadis flow [4]. Pop *et al* [5] studied variation in viscosity of fluid with temperature and derived a similarity solution. The influence of temperature dependent viscosity on heat transfer was also reconsidered by Bazid [6] and Pantokratoras [43] over a continuous surface. Although, Andersson and Aarseth [7] presented a detailed analysis about the influence of variable fluid properties for Sakiadis [4] flow and construct a new similarity transformation for boundary layer inspired by the Howarth-Dorodnitsyn [7] transformation. In spite of the fact that, Bachok *et al* [8] extended the work of Andersson and Aarseth [7] for the case when a flat plate moves in a parallel free stream.

3.2 Mathematical Formulation

Consider a steady, two-dimensional flow of a viscous fluid due to moving flat plate in a parallel free stream. The flow is assumed to be Newtonian and is placed at $y \geq 0$. The x-axis is taken along the direction of motion of moving plate while y-axis is normal to the plate. Let U_w be the uniform velocity of moving plate along a parallel free stream of constant velocity U_0 in the same or opposite direction. Let T_w and T_0 be constant temperature of moving flat plate and free stream temperature, respectively. So under these assumptions, the boundary layer equations governing the flow and heat transfers over a moving flat plate are given by:

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0, \tag{3.2.1}$$

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right), \tag{3.2.2}$$

$$\rho C_p\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = \frac{\partial}{\partial y}\left(\kappa\frac{\partial T}{\partial y}\right), \tag{3.2.3}$$

where u and v are the the velocity components along x-axis and y-axis, respectively. And T is the fluid temperature while κ, ρ, μ and C_p are the thermal conductivity, fluid density, dynamic viscosity and specific heat at constant pressure respectively. The boundary conditions for the subject problem are:

$$\begin{gathered} u = U_w, \quad v = 0, \quad T = T_w, \quad at\ \ y = 0, \\ u \to U_0 \quad T \to T_o \quad as\ \ y \to \infty, \end{gathered} \tag{3.2.4}$$

Now, similarity transformation is introduced to convert PDEs (3.2.1 - 3.2.3) into ODEs along with boundary conditions (3.2.4). For this purpose, it is necessary to define a stream function first $\psi(x, y)$ such that it satisfies the conservation equations. Hence, we define stream function as:

$$\rho u = \frac{\partial \psi}{\partial y}, \qquad -\rho v = \frac{\partial \psi}{\partial x}.$$

The similarity variable η is introduced to simplify the mathematical analysis of the problem along with the new dependent variable $f(\eta)$ and $\theta(\eta)$, such that

$$\eta = \sqrt{\frac{U}{a\nu_0 x}} \int \frac{\rho}{\rho_0} dy, \tag{3.2.5}$$

$$\psi(x, y) = \rho_0 \sqrt{a\nu_0 x U} f(\eta), \tag{3.2.6}$$

$$\theta(\eta) = \frac{T - T_0}{T_w - T_0}, \tag{3.2.7}$$

where U denotes the composite velocity, which is defined as $U = U_w + U_0$ and a is a dimensionless positive constant. It should be noticed that the transformation (3.2.7) exists only if $T_w \neq T_0$. Further, ρ_0, κ_0, ν_0 and C_{p0} are the values of fluid properties at temperature T_0 and ν_0 is corresponding kinematic viscosity i.e. $\nu_0 \equiv \mu_0/\rho_0$.

For applying transformation η can be represented as

$$\eta = \rho/\rho_0 \left(\frac{U}{a\nu_0 x}\right)^{1/2} y,$$

Differentiate η w.r.t x

$$\begin{aligned}\frac{\partial \eta}{\partial x} &= (\rho/2\rho_0)\left(\frac{U}{a\nu_0 x}\right)^{-1/2}\left(\frac{-U}{a\nu_0 x^2}\right) y,\\ &= -\rho/2\rho_0 \left(\frac{U}{a\nu_0}\right)^{1/2} x^{-3/2} y.\end{aligned}$$

Differentiate η w.r.t y

$$\frac{\partial \eta}{\partial y} = \rho/\rho_0 \left(\frac{U}{a\nu_0 x}\right)^{1/2}.$$

Differentiate Eq. (3.2.6) w.r.t x

$$\begin{aligned}\frac{\partial \psi}{\partial x} &= \rho_0/2(a\nu_0 xU)^{-1/2}(a\nu_0 U)f(\eta) + \rho_0(a\nu_0 xU)^{1/2} f'(\eta)\frac{\partial \eta}{\partial x},\\ &= \rho_0/2(a\nu_o xU)^{1/2} x^{-1/2} f(\eta) - \rho/2Ux^{-1} y f'(\eta).\end{aligned}$$

Differentiate Eq. (3.2.6) w.r.t y

$$\begin{aligned}\frac{\partial \psi}{\partial y} &= \rho_0(a\nu_o xU)^{1/2} f'(\eta)\frac{\partial \eta}{\partial y},\\ &= \rho U f'(\eta).\end{aligned}$$

Now, u can be expressed as:

$$\begin{aligned}u &= \frac{1}{\rho}\frac{\partial \psi}{\partial y},\\ &= U\ f'(\eta).\end{aligned} \tag{3.2.8}$$

Differentiate Eq. (3.2.8) w.r.t x

$$\begin{aligned}\frac{\partial u}{\partial x} &= U\, f''(\eta)\, \frac{\partial \eta}{\partial x}, \\ &= -\frac{\rho}{2\rho_0} \left(\frac{1}{a\nu_0}\right) U^{3/2}\, x^{-3/2} y\, f''(\eta).\end{aligned} \tag{3.2.9}$$

Differentiate Eq. (3.2.8) w.r.t y

$$\begin{aligned}\frac{\partial u}{\partial y} &= U\, f''(\eta) \frac{\partial \eta}{\partial y}, \\ &= \frac{\rho}{\rho_0}\, U\, f''(\eta) \left(\frac{U}{a\nu_0 x}\right)^{1/2}.\end{aligned} \tag{3.2.10}$$

Now v can be written in the form:

$$v = \frac{-1}{\rho}\, \frac{\partial \psi}{\partial x},$$

$$= \left(\frac{U}{2x}\right) y\, f'(\eta) - \rho_0/2\rho \left(\frac{a\nu_0 U}{x}\right)^{1/2} f(\eta). \tag{3.2.11}$$

By substituting Eqs. (3.2.8) - (3.2.11) into Eq. (3.2.2), we obtained the following expression:

$$U\, f'(\eta)\; - \frac{\rho}{2\rho_0 a\nu_0}\, (U/x)^{-3/2}\, y\, f''(\eta) + \rho/\rho_0 U f''(\eta)(U/a\nu_0 x)^{1/2}(U/2x)y$$

$$f'(\eta)\; -\; \rho_0/2\rho\, (a\nu_0\, U/x)^{1/2} f(\eta) \;=\; \frac{1}{\rho}\frac{\partial}{\partial y}\left[\mu\rho/\rho_0\, U f''(\eta) \left(\frac{U}{a\nu_0 x}\right)^{1/2}\right],$$

$$-\, (\rho\, U^2/2x)\, f(\eta)\, f''(\eta) \;=\; \mu\, (\rho^2/\rho_0^2)\, U^2(a\nu_0 x)^{-1} 1\, f'''(\eta),$$

$$\rho U^2/x\ \left(\frac{\rho}{\rho_0^2 a\nu_0}\ f'''(\eta)\ +\ \frac{1}{2}f(\eta)\ f''(\eta)\right) = 0,$$

$$\frac{\rho}{\rho_0^2 a\nu_0}\ f'''(\eta)\ +\ \frac{1}{2}f(\eta)f''(\eta) = 0,$$

$$\frac{2\rho\mu}{a\rho_o\mu_o}f'''(\eta) + \frac{1}{2}f(\eta)f''(\eta) = 0,$$

The momentum equation can be reduced into the following nonlinear ordinary differential equation:

$$\frac{2}{a}\left(\frac{\rho\mu}{\rho_o\mu_o}\ f''(\eta)\right)'\ +\ f(\eta)\ f''(\eta)\ =\ 0, \tag{3.2.12}$$

Now by simplifying Eq. (3.2.7), we obtain:

$$T = (T_w - T_0)\theta(\eta) + T_0,$$

Differentiate T w.r.t x

$$\begin{aligned}\frac{\partial T}{\partial x} &= (T_w - T_0)\theta'(\eta)\frac{\partial \eta}{\partial x},\\ &= -\frac{\rho(T_w - T_0)}{2\rho_0}\left(\frac{U}{a\nu_0}\right)^{1/2} x^{-3/2}y\theta'(\eta).\end{aligned} \tag{3.2.13}$$

Differentiate T w.r.t y

$$\begin{aligned}\frac{\partial T}{\partial y} &= (T_w - T_o)\theta'(\eta)\frac{\partial \eta}{\partial y},\\ &= \frac{\rho(T_w - T_0)}{\rho_0}\left(\frac{U}{a\nu_0 x}\right)^{1/2} \theta'(\eta).\end{aligned} \tag{3.2.14}$$

By substituting Eqs. (3.2.13) - (3.2.14) into Eq. (3.2.3), we obtained the following relation:

$$-Uf'(\eta)\tfrac{\rho(T_w-T_0)}{2\rho_0}\left(\tfrac{U}{a\nu_0}\right)^{1/2}x^{-3/2}y\theta'(\eta)+\left(\tfrac{U}{2x}\right)yf'(\eta)-\rho_0/2\rho\left(\tfrac{a\nu_0 U}{x}\right)^{1/2}f(\eta)\tfrac{\rho(T_w-T_0)}{\rho_0}$$

$$\left(\tfrac{U}{a\nu_0 x}\right)^{1/2}\theta'(\eta)=\tfrac{1}{\rho C_p}\tfrac{\partial}{\partial y}\left[\tfrac{\rho(T_w-T_0)}{\rho_0}\left(\tfrac{U}{a\nu_0 x}\right)^{1/2}\theta'(\eta)\right],$$

$$-Uf'(\eta)\tfrac{\rho(T_w-T_0)}{2\rho_0}\left(\tfrac{U}{a\nu_0}\right)^{1/2}x^{-3/2}y\theta'(\eta)+\left(\tfrac{U}{2x}\right)yf'(\eta)-\rho_0/2\rho\left(\tfrac{a\nu_0 U}{x}\right)^{1/2}f(\eta)\tfrac{\rho(T_w-T_0)}{\rho_0}$$

$$\left(\tfrac{U}{a\nu_0 x}\right)^{1/2}\theta'(\eta)=\tfrac{\kappa\rho^2(T_w-T_0)}{\rho_0^2}\left(\tfrac{U}{a\nu_0 x}\right)\theta''(\eta),$$

$$(-C_p/2)(a\nu_0)^{1/2}f(\eta)\theta'(\eta)=(\kappa\rho/\rho_0^2)(1/a\nu_0)^{1/2}\theta''(\eta),$$

$$\frac{\kappa\rho}{a\rho_0\mu_0}\theta''(\eta)=-\frac{C_p}{2}f(\eta)\theta'(\eta),$$

$$\frac{\kappa\rho}{\rho_0\ \kappa_0}\theta''(\eta)\ =\ -\ \frac{C_p\ \ a\mu_0\ C_{p0}}{2\ \kappa_0\ C_{p0}}\ f(\eta)\theta'(\eta),$$

$$(\rho\kappa/\rho_0\kappa_0)\theta''(\eta)+(aC_p/2C_{p0}Pr_0f(\eta)\theta'(\eta)=0,$$

Finally, the partial differential Eq. (3.2.3) is reduced to nonlinear ordinary differential equation:

$$\left(\frac{\rho\kappa}{\rho_0\kappa_0}\,\theta'(\eta)\right)'\ +\ \frac{aC_p}{2C_{p0}}\,Pr_0\,f(\eta)\,\theta'(\eta)\ =\ 0, \tag{3.2.15}$$

where $Pr_0\equiv\frac{\mu_0 C_{p0}}{\kappa_0}$ is Prandtl number in free stream at temperature T_0.

Thus transformed boundary conditions are given by:

$$f(0) = 0, \qquad f'(0) = 1 - \varepsilon, \qquad \theta(0) = 1,$$
$$f'(\eta) = \epsilon, \qquad \theta(\eta) = 0, \qquad as \quad \eta \to \infty \tag{3.2.16}$$

where, $\epsilon = \frac{U_0}{U} = \frac{U_0}{U_0+U_w}$ is a known free stream parameter. It represents the relative significance of free stream velocity.

It should be emphasized that when velocity of moving plate becomes equal to free stream velocity then $\epsilon = 1/2$, while $\epsilon = 0$ and $\epsilon = 1$ denotes the case of Sakiadis or quiescent fluid and classical Blasius fluid for moving flat plate, respectively. It should be noticed that $0 < \epsilon < 1$ is considered when both the free stream velocity and the plate velocity are in the same direction. Although, $\epsilon > 1$ corresponds to the free stream is directed towards the positive x-direction while the plate moves towards the negative x-direction and $\epsilon < 0$ denotes to the free stream is directed towards the negative x-direction while the plate moves towards the positive x-direction [47]. However, $\epsilon \geq 0$ is considered into account.i.e, when both free stream and plate moves towards the positive x-direction [8].

The surface shear stress τ_w and surface heat flux q_w can be written of the form

$$\tau_w = \mu_w \left(\frac{U^3}{a\nu_0 x} \right)^{1/2} f''(0), \qquad q_w = \frac{\mu_w C_{p0}}{Pr_0} \Delta T \left(\frac{U}{a\nu_0 x} \right)^{1/2} [-\theta'(0)]. \tag{3.2.17}$$

3.3 Special Cases

3.3.1 Case A: Constant Fluid Properties

In specific case for constant physical fluid properties, the similarity variable η reduces to the Blasius [51] variable:

$$\eta = \sqrt{\frac{U}{av_0 x}} y \tag{3.3.1}$$

Therefore Eqs. (3.2.12) and (3.2.15) can be written of the form

$$\frac{2}{a} f'''(\eta) + f(\eta) f''(\eta) = 0, \tag{3.3.2}$$

$$\theta''(\eta) + \frac{a}{2}Pr_0 f(\eta)\theta'(\eta) = 0, \tag{3.3.3}$$

Eqs. (3.3.2) and (3.3.3) are subject to the same boundary conditions as mentioned in Eq. (3.2.16).

3.3.2 Case B: Variable Viscosity

Viscosity is considered as temperature dependent $\mu(T)$ by Pop et al [5] by assuming other properties of fluid are constant. This assumption was also followed by Elbashbeshy and Bazid [6], Pantokratoras [43] and Anderson [7]. This form of variable viscosity $\mu(T)$ is recommended by Lai and Kulacki [53] and then allowed by Andersson and Aarseth [7], by assuming viscosity as an inverse linear function of temperature. So $\mu(T)$ is taken as

$$\mu(T) = \frac{\mu_{ref}}{1 + \gamma(T - T_{ref})}, \tag{3.3.4}$$

In Eq. (3.3.4), γ is known as fluid property. It depends on reference temperature (T_{ref}). For $\gamma > 0$, with the rise of temperature the viscosity of liquids decreases while for $\gamma < 0$ viscosity of gases increases with decreasing temperature. By using reference temperature $T_{ref} \approx T_o$ the relation (3.3.4) can be written as

$$\mu(T) = \frac{\mu_0}{1 - (T - T_0)/(T_w - T_0)\theta_{ref}}, \tag{3.3.5}$$

In Eq. (3.3.5), $T_w - T_0$ is an operating temperature difference while θ_{ref} is a dimensionless constant. It is defined as $\theta_{ref} \equiv -1/[\gamma(T_w - T_0)]$. Using this relation in above equation we obtained:

$$\frac{\mu}{\mu_0} = \frac{\theta_{ref}}{(\theta_{ref} - \theta)}, \tag{3.3.6}$$

It should be mentioned that θ_{ref} is taken positive for gases and negative for liquid [54]. By using approximation (3.3.5), the momentum boundary layer Eq. (3.2.12) can be written as

$$\frac{2}{a}\left(\frac{\mu}{\mu_0}f''\right)' + ff'' = 0, \tag{3.3.7}$$

From Eq. (3.3.6)

$$\frac{2}{a}\left(\frac{\theta_{ref}}{(\theta_{ref}-\theta)}f''\right)' + ff'' = 0, \tag{3.3.8}$$

The thermal boundary layer equation is considered same as in Eq. (3.3.3).

3.4 Numerical Methods

The system of coupled nonlinear ODEs. (3.2.12)- (3.2.15) subject to the boundary conditions (3.2.16) has been solved numerically by using shooting method and *bvp4c*. Using these methods boundary value problem (BVP) are converted into an initial value problems (IVP).

To convert **BVP** into an **IVP**, we define a new variable as:

$$\begin{aligned} y = f = y_1, \qquad f' = y_1' = y_2, \qquad & f'' = y_2' = y_3. \\ \theta = y_4, \qquad & \theta' = y_4' = y_5. \end{aligned} \tag{3.4.1}$$

The system of ODEs. (3.2.12)- (3.2.15) are transformed to a system of five simultaneous equations with five unknowns.

For constant fluid properties (Case A), momentum and energy equations can be written as:

$$y_3' = f''' = -\frac{a}{2}y_1y_3, \tag{3.4.2}$$

and

$$y_5' = \theta'' = -\frac{a}{2}Pr_o y_1 y_5, \tag{3.4.3}$$

For variable fluid viscosity (Case B), momentum equation an be written as

$$y_3' = f''' = -\frac{a}{2}(\frac{\theta_{ref}-y_4}{\theta_{ref}})y_1y_3 - (\frac{1}{\theta_{ref}-y_4})y_3y_5. \tag{3.4.4}$$

While energy equation remains same for Case B as Eq.(3.4.3).

3.5 Results and Discussions

Like Andersson and Aarseth[7], Bachok *et al* [8] found numerical results for considering temperature dependent viscosity $\mu(T)$ by keeping other fluid properties are constant. First, numerical values are calculated for Sakiadis fluid, $\epsilon = 0$ at Prandtl number 0.7 and 10 by considering constant fluid properties and variable viscosity. Then these results for characteristic surface gradient $f''(0)$ and $\theta'(0)$ are compared with Andersson and Aarseth [7] in Table 3.1. The purpose is to check the validity of Bachok *et al* [8] technique.

				Anderson [7]		Bachok *et al* [8]		Shooting Results	
Case	ϵ	Pr_0	a	$-f''(0)$	-$\theta'(0)$	$-f''(0)$	-$\theta'(0)$	$-f''(0)$	-$\theta'(0)$
A	0	0.7	1	0.44374	0.34923	0.4437	0.3492	0.44375	0.34929
-	0	1	1	-	-	0.4437	0.4437	0.44375	0.44375
-	0	10	1	0.44374	1.68029	0.4437	1.6803	0.44375	1.68031
B	0	1	1	-	-	1.0381	0.3181	1.03814	0.31812
-	0	10	1	1.30055	1.52915	1.3006	1.5292	1.30065	1.52920

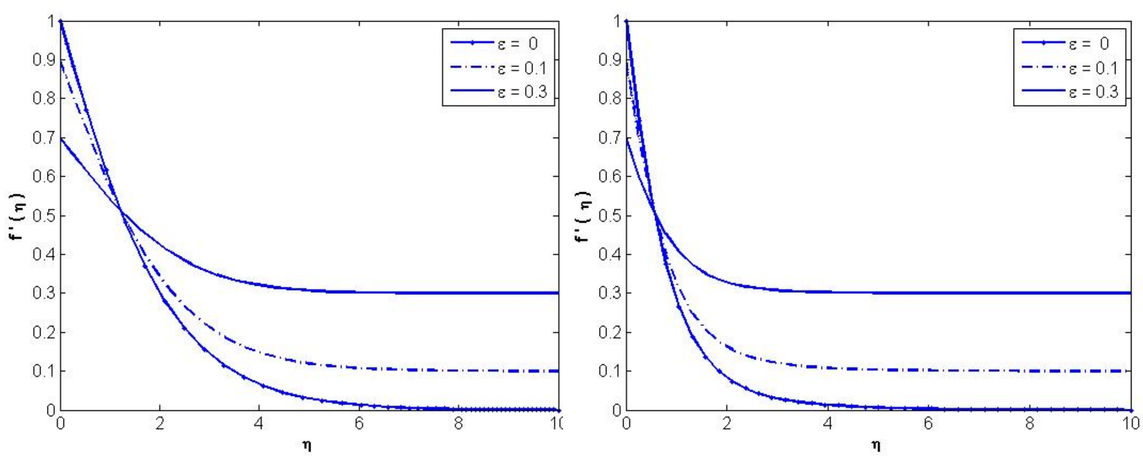
Table 3.1: Results of $f''(0)$ and $\theta'(0)$ for Cases A and B.

Two different cases have been considered to analyze the effect of viscosity that depends on temperature. For this purpose, numerical computation is obtained for water as a fluid at temperature $T_0 = 5^oC$ or $278K$. The temperature of a moving flat plate T_w is taken as 85^oCor $358K$. So the operating temperature difference ΔT is $80K$ i.e. $T_w - T_0 = 80K$. For variable fluid viscosity $\theta_{ref} = -0.25$ for water at $T_0 = 5^oC(278K)$ [7]. Results for skin friction coefficient $f''(0)$ and local Nusselt number $-\theta'(0)$ are compared for Case A (constant fluid properties) and Case B (variable fluid viscosity) for Sakaidis fluid with free stream parameter ϵ in Table 3.2. In these numerical results it is observed that the absolute value of $f''(0)$ is larger for Case B as compared to Case A. This increase in velocity gradient shows that a skin friction is developed when temperature dependent viscosity is taken into consideration. Although, a small reduction is noticed for temperature gradient $\theta'(0)$ in Case B as compared to Case A.

Case	Shooting Method $-f''(0)$	$-\theta'(0)$	*bvp4c* $-f''(0)$	$-\theta'(0)$
A	0.44375	1.68031	0.44374	1.68031
B	1.30065	1.52920	1.30060	1.52920

Table 3.2: Results of $f''(0)$ and $\theta'(0)$ for $Pr_0 = 10$ and $\epsilon = 0$.

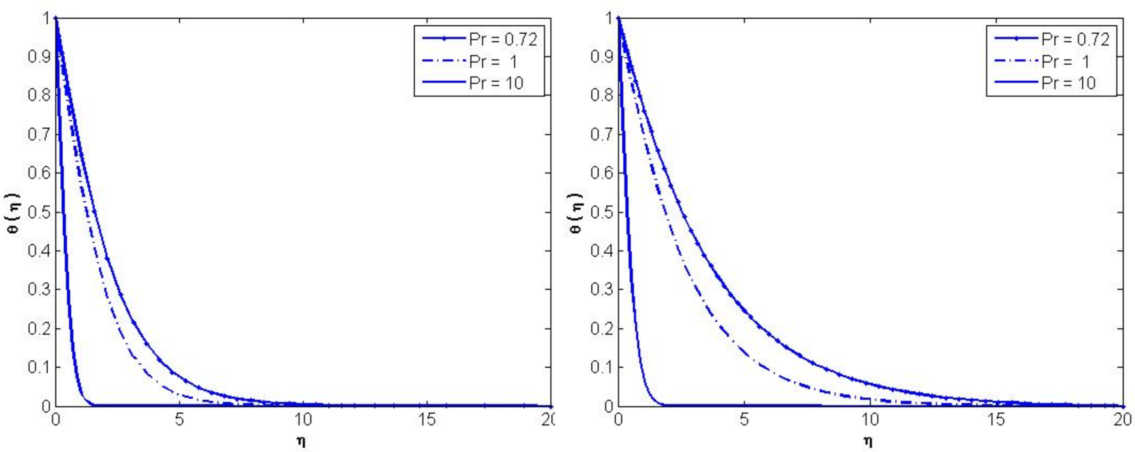
When velocity profile $f'(\eta)$ of constant fluid properties (Case A) are compared with those of variable viscosity (Case B), it is observed that $f'(\eta)$ is decreased near the moving surface for Case B as shown in Fig. 3.1. This is because adjacent fluid is heated due to moving surface as a result its viscosity is decreased. Inspite of this, it also shows a reduction in the viscous diffusion of stream-wise momentum from the surface in the inner part of momentum boundary layer. This reduction in $f'(\eta)$ is caused by the advection term $(1/2a)Pr_0 f(\eta)\theta'(\eta)$ in transformed energy Eq. (3.2.15) as a result viscosity is decreased and generates higher temperature close to the moving surface.



(a) Case A. (b) Case B.

Figure 3.1: Velocity profile for different value of ϵ.

In temperature profile $\theta(\eta)$, as viscosity is reduces so higher temperature is observed near the moving surface for Case B rather then Case A as shown in Fig. 3.2. However, thermal boundary layer thickness decreases for both cases with increasing value Pr_0. It is worth to mention that the result described in Figs. 3.1 and 3.2 were generated with $\eta_\infty = 30$. This integration length is enough longer to justify $f' \longrightarrow 0$ and $\theta' \longrightarrow 0$ which is a necessary condition indicated by Andersson and Aarseth [7].



(a) Case A.

(b) Case B.

Figure 3.2: Temperature profile for different value of Pr_0.

Chapter 4

MHD Boundary Layer Flow and Heat Transfer over a Moving Flat Plate in a Parallel Free Stream

4.1 Introduction

In a fluid dynamics, the study of magnetohydrodynamics (MHD) viscous boundary layer flow and heat transfer in a parallel free stream have numerous practical applications in industry and engineering. Thus many researchers have shown interest in the study of MHD in recent time. Rossow [1], Watanabe [45] and Das [46] studied the flow of electrically conducting fluid over a flat plate in presence of transverse magnetic field. Furthermore, thermal boundary layer flow in a parallel free stream for a moving flat plate were also analyzed by some researchers for different flow pattern and boundary condition. Afzal *et al* [47], Bianchi *et al* [48], Lin and Huang [49], Chen [50], Ishak *et al* [42] and Bachok *et al* [8] studied characteristics for momentum and heat transfer on a continuously moving flat plate in a parallel free stream.

However, in this chapter an extensive work is carried out to analyze the motion of fluid in the presence of magnetic field by considering variable viscosity of a fluid in a parallel free stream. Soundalgekar*et al* [44], Pop *et al*[5], Elbashbeshy and Bazid

[6], Pantokratoras [43], Andersson and Aareseth [7], Bachok *et al* [8] revised the Sakiadis flow [4] by taking variable fluid properties into account. Although, this extensive work deals both the constant fluid properties and temperature-dependent viscosity for a fluid flow over a moving flat plate with a parallel free stream in the presence of magnetic field. To our knowledge, the present analysis is not yet been studied before.

4.2 Mathematical Formulation

Let us consider steady two dimensional magneto-hydrodynamic laminar boundary layer flow on a fixed or continuously moving flat plate in a parallel free stream of an electrically conducting viscous fluid. Suppose that the surface is moving with constant velocity U_w in the same or opposite direction with respect to the constant free stream velocity U_0. A uniform magnetic field of strength B_0 is assumed to be applied in the positive y-direction. In this case the magnetic Reynold number of the flow is assumed to be small. The aim of taking low value of R_m is to make possible to neglect the induced magnetic field in the comparison to the applied magnetic field. Electric field is also supposed to be zero i.e. $E = 0$. Under the boundary layer assumptions the conservation equations for MHD flow for a moving flat plate in the absence of body forces are as follows.

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0, \tag{4.2.1}$$

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{dp}{dx} + \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) + (J_c \times B)_x, \tag{4.2.2}$$

$$\rho C_p\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = \frac{\partial}{\partial y}\left(\kappa\frac{\partial T}{\partial y}\right), \tag{4.2.3}$$

The relevant boundary condition for subject problem are given by [8]:

$$u = U_w, \quad v = 0, \quad T = T_w, \quad at \;\; y = 0,$$
$$u \to U_0 \quad T \to T_0 \quad as \;\; y \to \infty, \tag{4.2.4}$$

where x and y are Cartesian coordinates along the surface and normal to it, respectively. Further, u is the component of velocity in the x-direction and v is the component of velocity in the y-directions. Also ρ, μ, κ and C_p are the fluid density, the dynamic viscosity, thermal conductivity, strength of magnetic field and specific heat at constant pressure, respectively. Moreover, T_0 and T_w are temperature at free stream and moving plate temperature, respectively where $T_w > T_0$. The last term in Eq. (4.2.2) is known as Lorentz force where J_c represents conduction current and B_0 is the strength of magnetic field. For a given problem, conduction current can be defined as:

$$J_c \;=\; \sigma\;(\overrightarrow{q}\;\times\;\overrightarrow{B}\;), \qquad\qquad \overrightarrow{E} = 0, \tag{4.2.5}$$

where σ is constant electrical conductivity and $\overrightarrow{q} = (u, v) = u\hat{i} + v\hat{j}$ is a velocity field and $\overrightarrow{B} = (0, B_0, 0)$ is a magnetic field.
Taking vector product we obtained

$$\overrightarrow{q} \times \overrightarrow{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u & v & 0 \\ 0 & B_0 & 0 \end{vmatrix} = B_0 u\hat{k},$$

By using above expression, the Eq. (4.25) can be written of the form

$$J_c = \sigma B_0 u\hat{k},$$

So the Lorentz force for subject problem are given by:

$$\overrightarrow{J_c} \times \overrightarrow{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & \sigma B_0 u \\ 0 & B_0 & 0 \end{vmatrix} = -\sigma B_0^2 u\hat{i}, \tag{4.2.6}$$

Also,

$$\frac{dp}{dx} = -\sigma B_0^2 U_0, \tag{4.2.7}$$

Under above assumptions, the boundary layer Eqs. (4.2.1)- (4.2.3) becomes:

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0, \tag{4.2.8}$$

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) - \sigma_0 B_0^2(u - U_0), \tag{4.2.9}$$

$$\rho C_p\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = \frac{\partial}{\partial y}\left(\kappa\frac{\partial T}{\partial y}\right), \tag{4.2.10}$$

However, boundary condition (4.2.4) remains the same.
Now, to solve the system of Equation numerically it is useful to express it through transformed variable. Let us first choose a stream function $\psi(x, y)$ which defines the velocity component u and v as:

$$\rho u = \frac{\partial \psi}{\partial y}, \qquad -\rho v = \frac{\partial \psi}{\partial x}.$$

such that the mass conservation equation is satisfied.
Now, the following transformation is introduced to examine the flow [8]. Hence, the similarity variable η and the new dependent variable f and θ are defined as:

$$\eta = \sqrt{\frac{U}{a\nu_0 x}} \int \frac{\rho}{\rho_0} dy, \tag{4.2.11}$$

$$\psi(x, y) = \rho_0\sqrt{a\nu_0 x U}\; f(\eta), \tag{4.2.12}$$

$$\theta(\eta) = \frac{T - T_0}{T_w - T_0}, \tag{4.2.13}$$

where $U = U_w + U_0$ is composite velocity and a is a dimensionless positive constant. Further, ρ_0, μ_0, κ_0, ν_0 and C_{p0} are the values of fluid properties at temperature T_0.

The momentum and energy equation after applying transformation can be reduced to the following nonlinear ordinary differential equations:

$$\frac{2}{a}\left(\frac{\rho\mu}{\rho_0\mu_0}\, f''(\eta)\right)' + f(\eta)\, f''(\eta) - M\, f'(\eta) + M\,\epsilon = 0, \tag{4.2.14}$$

$$\left(\frac{\rho\kappa}{\rho_0\kappa_0}\,\theta'(\eta)\right)' + \frac{aC_p}{2C_{p0}}\, Pr_0\, f(\eta)\,\theta'(\eta) = 0, \tag{4.2.15}$$

where M is the dimensionless magnetic parameter Pr_0 is Prandtl number at free stream. These parameters are given by:

$$M = \frac{2\sigma_0 B_0^2 x}{\rho U}, \qquad Pr_0 = \frac{\mu_0 C_{p0}}{\kappa_0}$$

The Eqs. (4.2.14) and (4.2.15) with subject to the boundary conditions (4.2.4) which becomes

$$\begin{aligned} &f(0) = 0, \qquad f'(0) = 1-\epsilon, \qquad \theta(0) = 1, \\ &f'(\eta) = \epsilon, \qquad \theta(\eta) = 0, \qquad as \quad \eta \to \infty \end{aligned} \tag{4.2.16}$$

where $\epsilon = \frac{U_0}{U}$ is the free stream parameter of the fluid. For current analysis we consider the case when the plate is moved in the same direction as the free stream velocity i.e. $\epsilon \geq 0$ as taken by Bachok *et al* [8]. The surface shear stress τ_w and surface heat flux q_w can be written of the form

$$\tau_w = \mu_w \left(\frac{U^3}{a\nu_0 x}\right)^{1/2} f''(0), \qquad q_w = \mu_w \frac{C_{p0}}{Pr_0}\Delta T \left(\frac{U}{a\nu_0 x}\right)^{1/2} [-\theta'(0)]. \tag{4.2.17}$$

4.3 Special Cases

4.3.1 Case A: Constant Fluid Properties

By means of temperature-dependent density and viscosity, the ODE in Eq. (4.2.14) governing the flow in the momentum boundary layer is paired to the thermal boundary layer Eq. (4.2.15). In specific case for constant thermophysical fluid properties, the similarity variable η reduce to the Blasius [51] variable:

$$\eta = \sqrt{\frac{U}{av_0x}}y \tag{4.3.1}$$

Hence, Eqs. (4.2.14) and (4.2.15)can be written of the form

$$\frac{2}{a}f'''(\eta) + f(\eta)f''(\eta) - Mf'(\eta) + M\epsilon = 0, \tag{4.3.2}$$

$$\theta''(\eta) + \frac{a}{2}Pr_0 f(\eta)\theta'(\eta) = 0, \tag{4.3.3}$$

Still these equations are subjected to same boundary conditions as mentioned in Eq. (4.2.16). Fang [52] investigated the solution for $\epsilon = 1$, by taking same boundary condition (4.2.16).

4.3.2 Case B: Variable Viscosity

Generally, constant thermophysical properties of fluid are analyzed for considering problem of convection flow and heat transfer. As in these thermal properties variation can be observed especially in viscosity with temperature, so it is necessary to examine the change of viscosity to correctly predict the momentum and heat transfer flow rate.

Viscosity is considered as temperature dependent $\mu(T)$ by Pop et al [5] by assuming other physical properties of fluid are constant. This assumption was also followed by Elbashbeshy and Bazid [6], Pantokratoras [43], Andersson and Aarseth[7] and Bachok *et al* [8]. This form of temperature-dependent variable viscosity $\mu(T)$ is also recommended by Lai and Kulacki [53] and then allowed by Andersson and Aarseth [7] by assuming viscosity as a inverse linear function of temperature. So temperature dependent viscosity $\mu(T)$ is considered as

$$\mu(T) = \frac{\mu_{ref}}{1 + \gamma(T - T_0)}, \tag{4.3.4}$$

In expression (4.3.4), γ is a fluid property. It depends on reference temperature T_{ref}. The inversely linear temperature viscosity correlation mentioned in expression

(4.3.4) was also been used by Bachok *et al* [8]. By considering reference temperature $T_{ref} \approx T_0$, the expression (4.3.4) can be written of the form:

$$\mu(T) = \frac{\mu_0}{1 - (T - T_0)/(T_w - T_0)\theta_{ref}}, \tag{4.3.5}$$

where $T_w - T_0$ is a temperature difference of moving plate and free stream. Also $\theta_{ref} \equiv -1/[\gamma(T_w - T_0)]$ is a dimensionless constant. Using this relation in Eq. (4.3.5) we obtained the following ratio:

$$\frac{\mu}{\mu_0} = \frac{\theta_{ref}}{(\theta_{ref} - \theta)}, \tag{4.3.6}$$

By using above assumption, the momentum boundary layer Eq. (4.2.14) can be represented as:

$$\frac{2}{a}\left(\frac{\mu}{\mu_o} f''(\eta)\right)' + f(\eta)f''(\eta) - Mf'(\eta) + M\epsilon = 0, \tag{4.3.7}$$

Using relation (4.3.6) in above equation, we obtained

$$\frac{2}{a}\left(\frac{\theta_{ref}}{(\theta_{ref} - \theta)} f''(\eta)\right)' + f(\eta)f''(\eta) - Mf'(\eta) + M\epsilon = 0, \tag{4.3.8}$$

$$f''' = \frac{a}{2}\left(\frac{(\theta_{ref} - \theta)}{\theta_{ref}}\right)[f'' + Mf'(\eta) - M\epsilon] - \left(\frac{1}{(\theta_{ref} - \theta)}\right) f''\theta', \tag{4.3.9}$$

The thermal boundary layer equation takes the same form as of Eq. (4.3.3).

4.4 Numerical Methods

The shooting method by means of fifth order Runge-Kutta integration scheme and MATLAB built-in solver *bvp4c* are used to solve the system of coupled nonlinear ODEs (4.2.14)-(4.2.15) subject to the boundary conditions (4.2.16).

To convert **BVP** into **IVP**, we define a new variable as:

$$\begin{aligned} y = f = y_1, \quad f' = y_1' = y_2, \quad f'' = y_2' = y_3 \\ \theta = y_4 \quad \theta' = y_4' = y_5 \end{aligned} \tag{4.4.1}$$

The system of ODEs. (4.2.14)-(4.2.15) are transformed to a system of five simultaneous equations with five unknowns.
For constant physical fluid properties (Case A) the momentum and thermal equations in the form of new variable f and θ can be written of the form:

$$y_3' = f''' = \frac{a}{2}(-y_1 y_3 + M y_2 - M\epsilon), \tag{4.4.2}$$

$$y_5' = \theta'' = -\frac{a}{2} Pr_o y_1 y_5, \tag{4.4.3}$$

For variable fluid viscosity (Case B) the momentum boundary layer equation can be written as

$$y_3' = f''' = \frac{a}{2}(\frac{\theta_{ref} - y_4}{\theta_{ref}})(-y_1 y_3 + M y_2 - M\epsilon) - (\frac{1}{\theta_{ref} - y_4}) y_3 y_5, \tag{4.4.4}$$

While thermal energy equation remains same for Case B as Eq. (4.4.3).

4.5 Results and Discussions

A numerical study is considered to investigate the effect of magnetic parameter M and Prandtl number Pr_0 with free stream ϵ parameter upon the nature of the flow. Like Andersson and Aarseth[7] and Bachok *et al* [8], the extensive work is also focused on impact of temperature dependent viscosity by keeping other thermophysical properties are constant. Although, numerical computation has been done in the presence of magnetic field. First the numerical study is computed for Pr_0 at 0.71, 1 and 10 and then compared the numerical value of characteristics surface gradients $f''(0)$ and $\theta'(0)$ with the solution obtained by Bachok *et al* [8] of the quiescent fluid, for $\epsilon = 0$ in Table 4.1. The purpose is to check the accuracy of present solution technique in the absence of magnetic field i.e, $M = 0$.

Two different cases have been considered to investigate the effect of magnetic field on boundary layer structure for constant and variable fluid properties in a parallel

				Bachok *et al* [8]		Shooting Results ($M = 0$)	
Case	ϵ	Pr_0	a	$-f''(0)$	-$\theta'(0)$	$-f''(0)$	-$\theta'(0)$
A	0	0.71	1	0.4437	0.3492	0.4437	0.3492
-	0	1	1	0.4437	0.4437	0.4437	0.4437
-	0	10	1	0.4437	1.6803	0.4437	1.6803
B	0	1	1	1.0381	0.3181	1.0381	0.3181
-	0	10	1	1.3006	1.5292	1.3006	1.5292

Table 4.1: Results of $f''(0)$ and $\theta'(0)$ for Case A and B.

free stream. Case A constitutes constant fluid properties while Case B show variable fluid viscosity. Water is considered as a fluid for Prandtl 7 and 10 at temperature $278K$ along with surface temperature $358K$, so that $\Delta T \cong (T_w - T_0) \cong 80k$, whereas Prandtl 0.71 is taken for air at temperature $293K$. For variable fluid properties Lai and Kulacki [53] declared $\theta_{ref} = -0.37$ for water and $\theta_{ref} = 5.62$ for air but we have reviewed $\theta_{ref} = -0.25$ for water as reported by Andersson and Aarseth[7], Bachok *et al* [8] and Ling and Dybb [54].

The influence of magnetic parameter on the coefficient of reduced skin friction $f''(0)$ and the local Nusselt number $\theta'(0)$ is illustrated in Table 4.2 for constant and variable fluid properties respectively. Numerical values reveal that the skin friction coefficient $f''(0)$ increase consistently with an increasing values of magnetic field whereas local Nusselt number $\theta'(0)$ decreases with an increasing values of magnetic parameter. Actually, the effect of M on a viscous fluid is to suppress the velocity field due to the enhanced Lorentz force which in turn causes the increase in skin friction coefficient. Almost three fold increase in skin friction coefficient is observed when temperature dependent viscosity is considered into account in the presence of magnetic field. Similarly, for different value of Prandtl number it is noticed that skin friction coefficient $f''(0)$ shows a very small change in the solution for Case A as compared to Case B, but $\theta'(0)$ increases with an increasing Prandtl number. Also numerical value indicate that for $Pr_0 < 1$, thermal diffusion occurs at a faster

			Case A		Case B	
ϵ	Pr_o	M	$-f''(0)$	-$\theta'(0)$	$-f''(0)$	-$\theta'(0)$
0.01	10	0	0.4371	1.6719	1.2811	1.5215
0.01	10	0.1	0.4852	1.6616	1.4117	1.4953
0.01	10	0.2	0.5301	1.6520	1.5265	1.4719
0.01	10	0.3	0.5721	1.6430	1.6293	1.4508
0.01	10	0.4	0.6110	1.6341	1.7225	1.4316
0.01	0.7	0.1	0.4852	0.3395	1.1207	0.2284
0.01	1	0.1	0.4852	0.4323	1.1337	0.3011
0.01	3	0.1	0.4852	0.8515	1.2223	0.7043
0.01	7	0.1	0.4852	1.3699	1.3463	1.2106
0.02	1	0.1	0.4783	0.4300	1.1177	1.2995
0.03	1	0.1	0.4714	0.4278	1.1009	1.2972

Table 4.2: Results of $f''(0)$ and $\theta'(0)$ for Case A and B.

	Shooting Method		*bvp4c*	
Case	$-f''(0)$	-$\theta'(0)$	$-f''(0)$	-$\theta'(0)$
A	0.4852	1.6616	0.4852	1.6617
B	1.4117	1.4953	1.4115	1.4952

Table 4.3: Results of $f''(0)$ and $\theta'(0)$ for $Pr_0 = 10$, $M = 0.1$ and $\epsilon = 0.01$.

rate rather than momentum diffusion while for $Pr_0 > 1$, momentum diffuses at a faster rate. However, in the presence of magnetic field atleast 11 % reduction in temperature gradient at the surface is found for variable fluid viscosity as compared to constant fluid property. Also same effects are analyzed by ϵ on both $f''(0)$ and $\theta'(0)$. With incresing value of ϵ, both $f''(0)$ and $\theta'(0)$ decreases consistently.

The velocity $f'(\eta)$ profile is illustrated in Fig. 4.1 for both cases i.e. constant fluid properties (Case A) and variable fluid properties (Case B). The magnetic parameter M and Prandtl number Pr_0 are taken 0.5 and 10, respectively. In Fig. 4.1, a substantial variation is observed in velocity profile. It is noticed that the velocity $f'(\eta)$ profile is significantly reduced near the moving surface for case B when temperature dependent viscosity is taken into account. The moving surface heats the adjacent fluid as a result temperature increases so that it reduces the viscosity of fluid. Also momentum boundary layer thickness decreases for case B by comparing it with Case A.

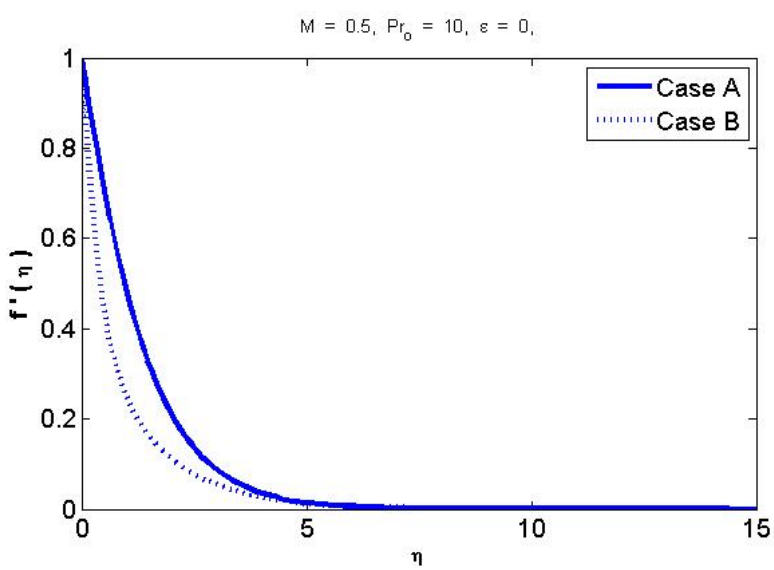


Figure 4.1: Velocity profile for Case A and Case B.

The temperature profile $\theta(\eta)$ is illustrated in Fig. 4.2 for Case A and Case B. The magnetic parameter M and Prandtl number Pr_0 are taken 0.5 and 10, respectively. A slight change is observed in temperature $\theta(\eta)$. Thermal boundary layer thickness decreases for Case A as compared to Case B. So a higher temperature is analyzed near a moving surface because of reduction in viscosity. This indirect effect come due to the advection term $1/2aPr_o\, f(\eta)\theta'(\eta)$ in the thermal boundary layer Eq. (4.3.3).

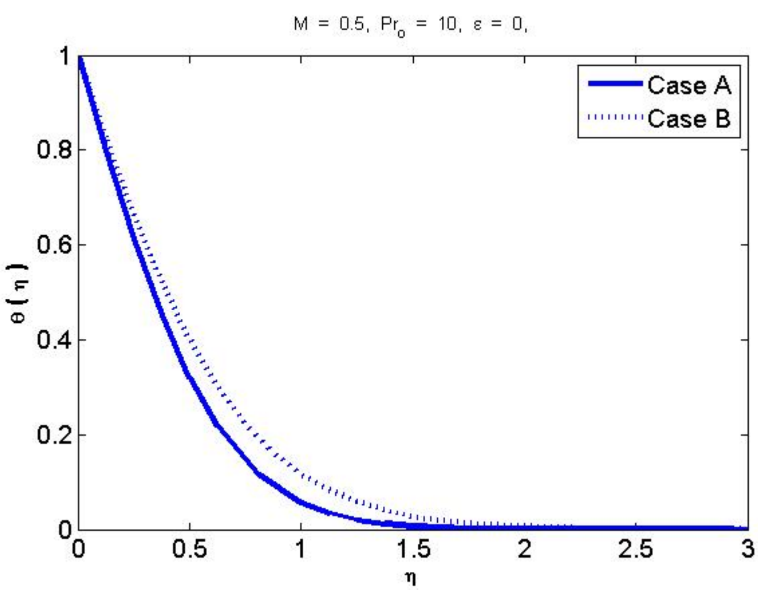
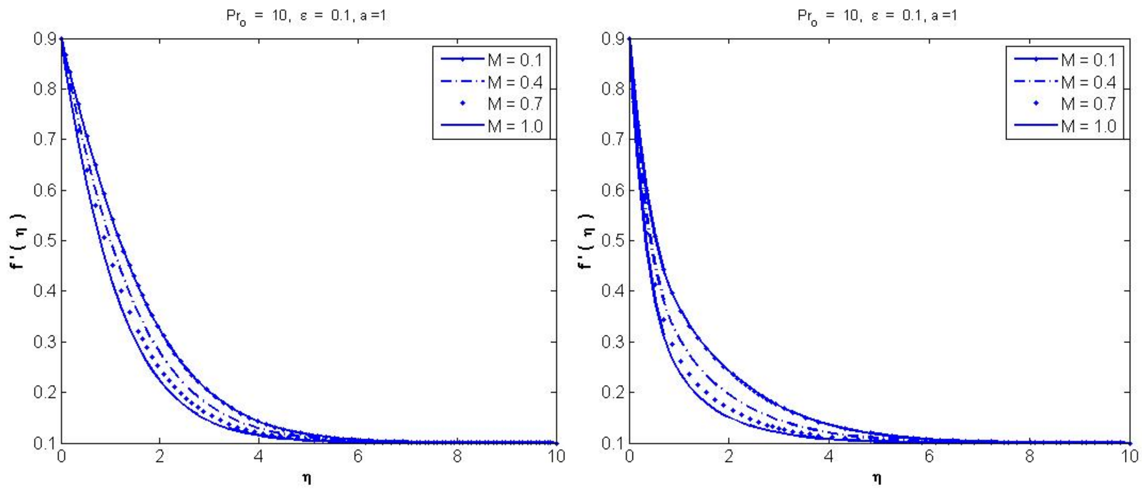


Figure 4.2: Temperature profile for Case A and Case B.

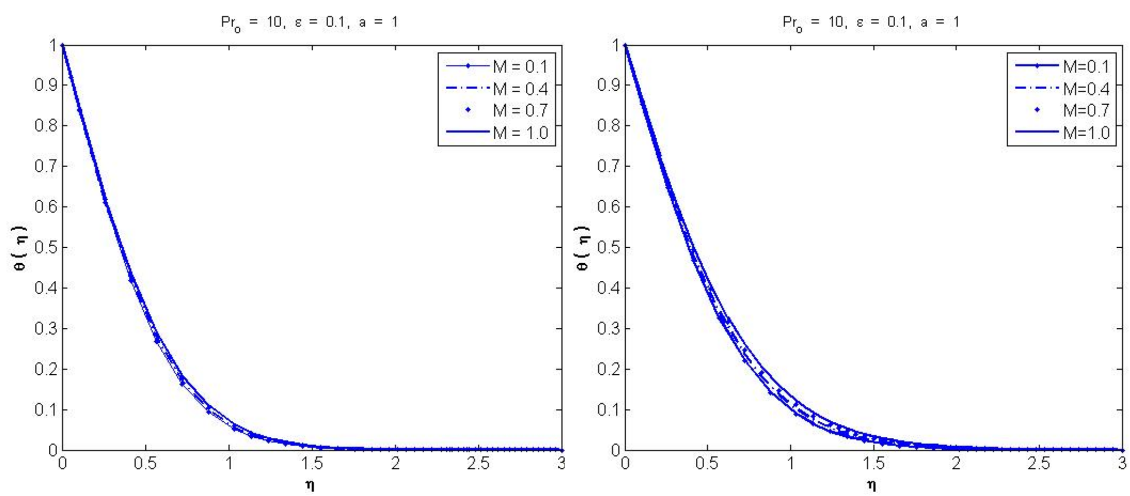
In Fig. 4.3(a-b) the variation in the velocity profile is demonstrated for different value of magnetic parameter M for constant and variable fluid properties over a moving flat plate in a parallel free stream. As the magnetic parameter M increases consequently the thickness of the momentum boundary layer decreases. Actually, rate of transport decreases with increasing magnetic parameter M. This is caused due to the fact that the Lorentz force retards the fluid motion which makes the boundary layer thinner.

The effect of magnetic field is also analyzed for thermal boundary layer in Fig 4.4(a-b) for both constant and variable fluid properties respectively. The thermal boundary layer thickness slightly increases for both cases as the Lorentz force increased with increasing value of M. A small variation is observed in $\theta(\eta)$ for Case B when it is compared with Case A.



(a) Case A. (b) Case B.

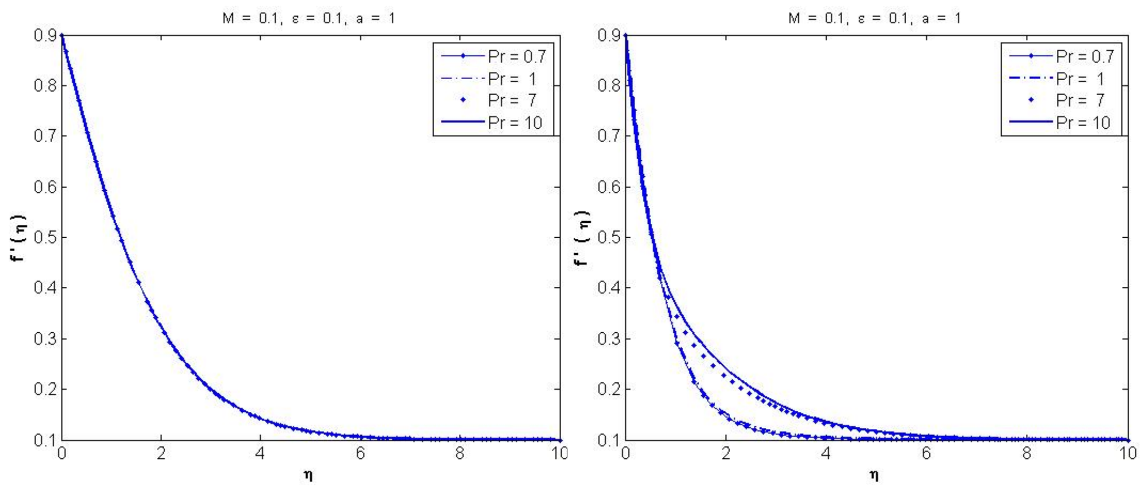
Figure 4.3: Velocity profile for different values of M.



(a) Case A. (b) Case B.

Figure 4.4: Temperature profile for different values of M.

In Fig. 4.5(a-b) the influence of Prandtl number on momentum boundary layer is studied in the presence of magnetic field for constant and variable fluid properties. For velocity profile constant fluid properties shows no remarkable effect with an increasing values of Prandtl number in the presence of magnetic field but for Case B due to presence temperature dependent viscosity a velocity boundary layer thickness slightly increases with an increasing values of Prandtl number.

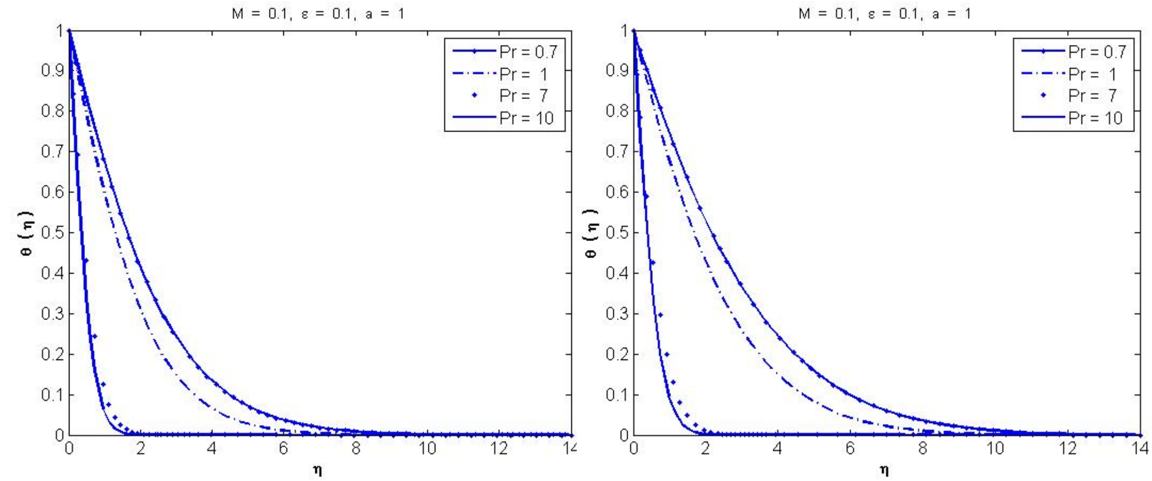


(a) Case A. (b) Case B.

Figure 4.5: Velocity profile for different values of Pr_0.

Fig. 4.6(a-b) shows the impact of Prandtl number on thermal boundary layer structure in the presence of magnetic field for constant and variable fluid properties. It is observed that thermal boundary layer thickness decreases with an increasing values of Prandtl number. With the increasing value of Prandtl number leads to increase the heat transfer rate at the surface which indicates a higher temperature near the moving surface.

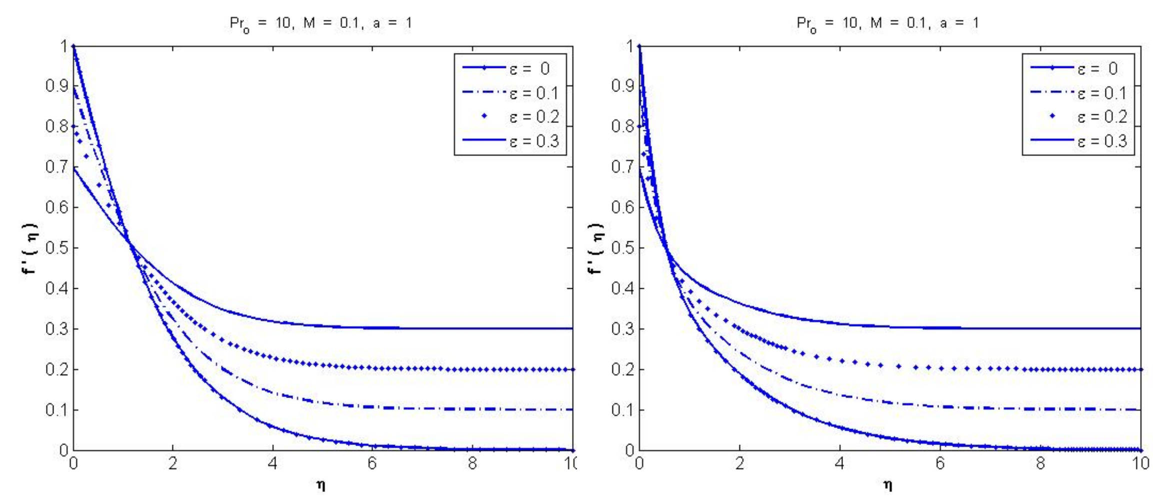
The effect of the parameter ϵ on the velocity and temperature profile is depicted in Figs. (4.7)-(4.8) for the case when both the fluid and plate move in same direction in the presence of magnetic field. With an increasing values of free stream parameter momentum boundary layer thickness increases significantly for constant as well as



(a) Case A. (b) Case B.

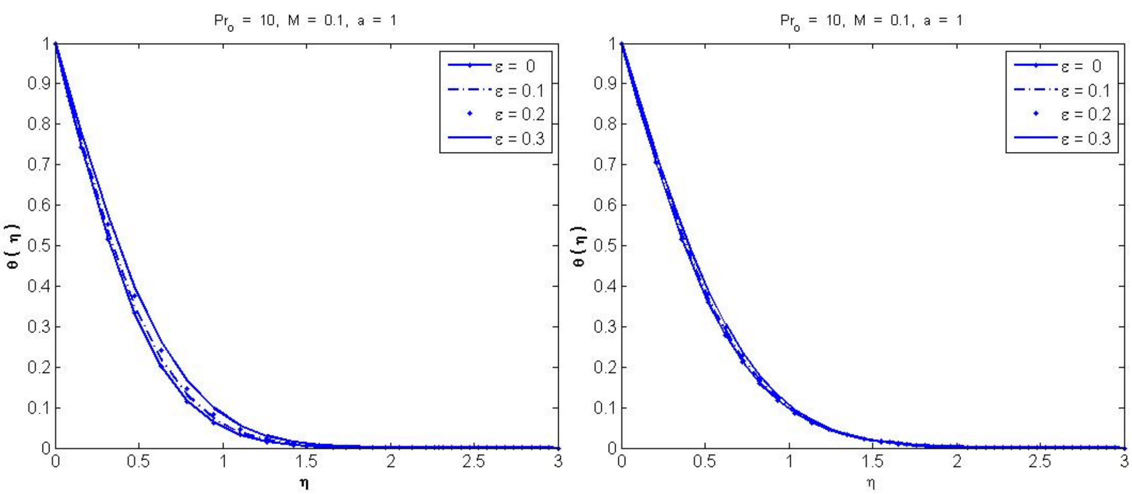
Figure 4.6: Temperature profile for different values of Pr_0.

variable fluid viscosity. However, insignificant effect of ϵ is measured for temperature profile for both cases.



(a) Case A. (b) Case B.

Figure 4.7: Velocity profile for different values of ϵ.



(a) Case A. (b) Case B.

Figure 4.8: Temperature profile for different values of ϵ.

Chapter 5

Conclusions and Outlook

Numerical solution of incompressible and compressible fluid were studied for non-linearly stretching sheet as well as for moving flat plate, respectively.

In the first part of current thesis, we studied the impact for the MHD stagnation point flow with mixed convection. A laminar boundary layer solutions are obtained for an incomprassible electrically conducting fluid over a non-linearly stretching sheet specify the change that will brought about by a magnetic field applied normally. Non-linear partial differential equations (PDEs) are converted into non-linear ordinary differential equations (ODEs) with the help of similarity transformation. Numerical solutions for momentum and thermal boundary layer are obtained by means of *bvp4c* and shooting method. Results are plotted for key embedding physical parameters such as suction (injection) parameter, magnetic parameter, velocity slip parameter and non-linearity parameter. The temperature decreases inside a thermal boundary layer with increasing these parameters. However, increasing mixed convection parameter, results an increased both skin friction coefficient and heat transfer rate. It is also observed that when the free stream velocity dominates the stretching velocity and heat transfer rate increases with velocity slip and magnetic field and they suppress the heat transfer rate when the stretching velocity dominates the free stream velocity.

The influence of temperature dependent viscosity for Sakiadis problem is studied in the second part of thesis. A momentum and thermal boundary layer flow behaviour has been explored on a continuous moving flat plate in a parallel free stream . A uniform magnetic field is applied normal to the plate. By using similarity transformation the governing PDEs are reduced into the system of coupled ODEs. Numerical methods namely shooting method and *bvp4c* are applied to solve resultant ODEs. Results are investigated for constant fluid properties (Case A) and variable fluid viscosity (Case B). It is observed that:

- The skin friction coefficient $f''(0)$ increases consistently with an increasing values of magnetic field whereas local Nusselt number $\theta'(0)$ decreases with an increasing values of magnetic parameter.
- A three fold increase in skin friction coefficient is calculated for Case B as compared to Case A.
- In the presence of magnetic field atleast 11 % reduction in temperature gradient at the surface is found for variable fluid viscosity as compared to constant fluid property.
- As the magnetic parameter M increases consequently the thickness of the momentum boundary layer decreases while thermal boundary layer thickness slightly increases for constant as well as variable fluid properties.

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