Numerical Analysis of Fractional Differential Equations

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MS THESIS WORK

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Dedicated

to

My Beloved Parents

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Abstract

It has been demonstrated that many physical systems can be modeled accurately by differential equation of arbitrary order rather than classical differential equation. The purpose of this thesis is to find solutions for differential equation of fractional order. We confer about some important functions and history related to fractional calculus. Some terminologies and important properties are also discussed for fractional calculus. Also we discuss some results from analysis. We set up results for uniqueness, stability and existence of solutions for initial value problem of ordinary differential equation of fractional order.

We proposed a quadrature rule to approximate solutions. The computational technique convert the given problem into system of algebraic equations which can easily be solvable by usual algebraic methods. To deal with non-linear fractional differential equations we convert the problem into recursive linear differential equation using quazi-linearization approach and then we utilize the proposed computational technique at each iteration to get solutions.

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Chapter 1

Introduction

1.1 Brief History

In endorsement of history J. Von Neuman's $(1903-1957)$ quoted that, "The calculus was the first achievement of modern mathematics and it is difficult to overestimate its importance. I think it defines more equivocally than anything else the inception of modern mathematics and the system of mathematical analysis, which is logical development, still constitute the greatest technical advance in exact thinking"[5, 6].
Gottfried Wihelm Leibniz $(1646-1716)$ introduced nth order derivative symbol $\frac{d^n}{dx^n}$ in 17^{th} century for the first time. L'Hospital asked Leibniz how this operator work if n is any fraction. Motivated from this question S. F. Lacroix formulated the formula for the fractional order derivative by use of Gamma function in his famous book published in 1819 [7]. The formula by Lacroix for α order derivative for x^m is $\frac{\Gamma(m+1)}{\Gamma(m-\alpha+1)}x^{m-\alpha}$. In [8], Joseph Liouville $(1809-1882)$ extended integer order derivative to fractional order α. This definition is considered to be the first formula for fractional derivative but that formula only applicable to the functions of the form $f(x)=\sum_{n=0}^{\infty}c_n e^{a_n x}$. Liouville formulated another formula to extend his first definition for fractional order derivative using the gamma function $D^\alpha x^{-m}=(-1)^\alpha\frac{\Gamma(\alpha+m)}{\Gamma(m)}x^{-\alpha-m}$ for $m>0$. This definition only applicable to rational functions. Most scientist prefer Liouville's definition, however Peacock support definition given by Lacroix. To avoid this conflict J. Fourier obtained the integral representation of function and its derivatives. Greer formulated fractional derivative for hyperbolic and trigonometric functions using first

definition of Liouville.

Various Mathematician including Hardy and Littlewood ,Davis, Pitcher and Sewell, Debnath and Grum published a lot of research on fractional integral and derivative in 20^{th} century [9, 6, 10].

1.2 Euler's Gamma function

Gamma function was presented by Leonhard Euler in inspiration to sum up the factorial to whole number esteem. We see that some well known scientific constants are happening in investigation of gamma function. It additionally shows up in different fields like number theory, geometric series and in definite integrals. The gamma function is defined as definite integral

$$\Gamma(t) = \int_0^\infty \xi^{t-1} \exp(-\xi) d\xi, \qquad t > 0. \tag{1.1}$$

It ought to be noted that the variable ξ is a dummy. We might set up two vital properties of the gamma function:

$$\Gamma(1) = 1,$$

$$\Gamma(t+1) = t\Gamma(t).$$

Eq. (1.1) characterizes the gamma function when $t > 0$. Using second property we can characterize gamma function for negative values of t as,

$$\Gamma(t) = \frac{\Gamma(1+t)}{t}. \tag{1.2}$$

Eq. (1.2) holds for $-1 < t < 0$. As $\Gamma(t+1) = \frac{\Gamma(t+2)}{t+1}$, thus Eq. (1.2) will become,

$$\Gamma(t) = \frac{\Gamma(t+2)}{t(t+1)}. \tag{1.3}$$

Eq. (1.3) valid for $-2 < t < 0, \quad t \neq 0, -1$. In same way for $m < t < 0, \quad t \neq -1, -2, \ldots, -m+1$, the gamma function is defined by:

$$\Gamma(t) = \frac{\Gamma(t+m)}{t(t+1)(t+2)\ldots(t+m-1)}. \tag{1.4}$$

A important relation of gamma function with beta function. For $t_1, t_2 > 0$, we have the relation

$$B(t_1, t_2) = \frac{\Gamma(t_1)\Gamma(t_2)}{\Gamma(t_1 + t_2)}. \tag{1.5}$$

The beta function $B(t_1, t_2)$ is the name utilized by Legendre and Whittaker and Watson in 1990. This also called the Eulerian integral of first kind. Where the beta function $B(t_1, t_2)$ defined by:

$$B(t_1, t_2) = \int_0^1 \xi^{t_1-1}(1-\xi)^{t_2-1} d\xi.$$

1.3 Mittag-Leffler Function

Mittag-Leffler function appears in problems of science and engineering. Mittag-Leffler functions directly involve in solutions for differential equations of fractional order and Volterra equations. This section deal with Mittag-Leffler function and its generalization.

Definition 1.3.1. *[11] For* $\alpha > 0$,

$$E_\alpha(t) = \sum_{m=0}^{\infty} \frac{t^m}{\Gamma(\alpha m + 1)}. \tag{1.6}$$

The function E_α is called one parameter Mittag-Leffler function defined in 1903 by G. M. Mittag. For $\alpha = 1$, series (1.6) gives exponential function. So, it is called generalization of exponential series.

Definition 1.3.2. *[11] For* $\alpha_1, \alpha_2 > 0$,

$$E_{\alpha_1,\alpha_2}(t) = \sum_{m=0}^{\infty} \frac{t^m}{\Gamma(\alpha_1 m + \alpha_2)}. \tag{1.7}$$

The function E_{α_1,α_2} is the generalized two parameter Mittag-Leffler function was defined by Wyman in 1905. For $\alpha_2 = 1$ in (1.7), two parameter Mittag Leffler function equal to one parameter Mittag Leffler function.

Theorem 1.3.1. *[11] Let $\alpha_1, \alpha_2 > 0$, and $t, \alpha_1, \alpha_2 \in \mathbb{C}$. Then*

$$E_{\alpha_1,\alpha_2}(t) = tE_{\alpha_1,\alpha_1+\alpha_2}(t) + \frac{1}{\Gamma(\alpha_2)}.$$

Proof. We start from left hand side. By the use of generalized Mittag-Leffler function we have

$$\begin{aligned} E_{\alpha_1,\alpha_2}(t) &= \sum_{m=0}^{\infty} \frac{t^m}{\Gamma(\alpha_1 m + \alpha_2)} \\ &= \frac{1}{\Gamma(\alpha_2)} + \sum_{m=1}^{\infty} \frac{t^m}{\Gamma(\alpha_1 m + \alpha_2)} \\ &= \frac{1}{\Gamma(\alpha_2)} + \sum_{m=1}^{\infty} \frac{t^m}{\Gamma(\alpha_1 m + \alpha_2)}. \end{aligned}$$

Replacing m by $m+1$, then we have,

$$\begin{aligned} E_{\alpha_1,\alpha_2}(t) &= \sum_{m=0}^{\infty} \frac{t^{m+1}}{\Gamma(\alpha_1(m+1) + \alpha_2)} + \frac{1}{\Gamma(\alpha_2)} \\ &= t\sum_{m=0}^{\infty} \frac{t^m}{\Gamma(\alpha_1 m + (\alpha_1 + \alpha_2))} + \frac{1}{\Gamma(\alpha_2)} \\ &= tE_{\alpha_1,\alpha_2}(t) + \frac{1}{\Gamma(\alpha_2)}. \end{aligned}$$

□

Theorem 1.3.2. *[11] Let $\alpha_1, \alpha_2 > 0$, Then we have following relation*

$$E_{\alpha_1,\alpha_2}(t) = \alpha_2 E_{\alpha_1,\alpha_2+1}(t) + \alpha_1 t \frac{d}{dt} E_{\alpha_1,\alpha_2+1}(t). \tag{1.8}$$

Proof. By definition we have

$$\begin{aligned} E_{\alpha_1,\alpha_2}(t) &= \sum_{m=0}^{\infty} \frac{t^m}{\Gamma(\alpha_1 m + \alpha_2)} \\ &= (\alpha_1 m + \alpha_2) \sum_{m=0}^{\infty} \frac{t^m}{(\alpha_1 m + \alpha_2)\Gamma(\alpha_1 m + \alpha_2)} \\ &= \alpha_1 m \sum_{m=0}^{\infty} \frac{t^m}{\Gamma(\alpha_1 m + \alpha_2 + 1)} + \alpha_2 \sum_{m=0}^{\infty} \frac{t^m}{\Gamma(\alpha_1 m + \alpha_2 + 1)} \\ &= \alpha_1 t \frac{d}{dt} \sum_{m=0}^{\infty} \frac{t^m}{\Gamma(\alpha_1 m + \alpha_2 + 1)} + \alpha_2 \sum_{m=0}^{\infty} \frac{t^m}{\Gamma(\alpha_1 m + \alpha_2 + 1)} \\ &= \alpha_2 E_{\alpha_1,\alpha_2+1}(t) + \alpha_1 t E_{\alpha_1,\alpha_2+1} t. \end{aligned}$$

□

Some particular values of α_1 and α_2 give some well known results.

Example 1.3.3. *Let* $\alpha_1 = 2, \alpha_2 = 2$, *then we have* $E_{2,2}(t) = \frac{e^t - 1}{t^{\frac{m+1}{2}}}$.

Example 1.3.4. *[11] For* $\alpha_1 = 2$ *and* $\alpha_2 = 1$, *we have* $E_{2,1}(-t^2) = E_2(-t^2) = \cos(t)$

Example 1.3.5. *[11] For* $\alpha_1 = 2$ *and* $\alpha_2 = 1$, *we have* $E_{2,1}(t^2) = E_2(t^2) = \cosh(t)$

Example 1.3.6. *[11] For* $\alpha_1 = \frac{1}{2}$ *and* $\alpha_2 = 1$, *we have* $E_{\frac{1}{2},1}(t^{\frac{1}{2}}) = (1 + erf(t))e^{t^2}$, *where*

$$erf(t) = \frac{2}{\sqrt{\pi}} \int_a^t e^{-\xi^2} d\xi,$$

is called error function.

1.4 Preliminaries of Fractional Calculus

This section discuss basic terminologies, lemmas and theorems of fractional integrals and derivatives.

1.4.1 Fractional Integral

Fractional integral and derivatives are defined by the use of Cauchy's integral formula.

$$I_a^k \phi(t) = \int_a^t frac(t - \xi)^{k-1} \phi(\xi)(k-1)! d\xi. \tag{1.9}$$

where $k \in \mathbb{N}$, and $\phi \in L_1[a, b]$, $a, b \in \mathbb{R}$.
If we replace factorial in (1.9) to gamma function then it leads to the definition of fractional integral.

Definition 1.4.1. *[11] Let the function* $\phi \in L_1[a, b]$ *and* $\alpha \in \mathbb{R}^+$. *Then the operator* I_c^α *is defined by*

$$I_c^\alpha \phi(t) = \int_c^t \frac{(t - \eta)^{\alpha - 1} \phi(\eta)}{\Gamma(\alpha)} d\eta.$$

The operator I_c^α *is called the* α *order Riemann Liouville integral operator.*

Integral $I_c^\alpha \phi(t)$ exist almost everywhere for $t \in [a, b]$ and $I_c^\alpha \phi(t)$ is also an element of $L_1[a, b]$. When $\alpha = 0$ then $I_c^0 = I$ is the identity operator.

Theorem 1.4.1. *[11] Let $\phi \in L_1[a, b]$ and $\alpha_1, \alpha_2 \in \mathbb{R}^+ U\{0\}$. Then,*

$$I_c^{\alpha_1} I_c^{\alpha_2} \phi(t) = I_c^{\alpha_1+\alpha_2} \phi(t) = I_c^{\alpha_2} I_c^{\alpha_1} \phi(t),$$

holds almost everywhere on $[a, b]$. Moreover if $\phi(t) \in C[a, b]$, then it is valid for all $t \in [a, b]$. Thus the operator satisfy semi-group property.

Thus we can say Riemann-Liouville fractional integral operator obeys a semi-group property.

Theorem 1.4.2. *[11] let $r > -1$ and $\alpha > 0$, and the function ϕ of the form $\phi(t) = (t + c)^r$. Then*

$$I_{-c}^\alpha \phi(t) = \frac{\Gamma(r + 1)}{\Gamma(\alpha + r + 1)} (t + c)^{\alpha + r}. \tag{1.10}$$

Definition 1.4.2. *Let $\alpha > 0$, and $\phi \in L_1[a, b]$. Then*

$$J^\alpha \phi(t) = \frac{1}{\Gamma(\alpha)} \int_t^\infty (\xi - t)^{\alpha-1} \phi(\xi) d\xi.$$

The operator J^α is called Wyel fractional integral of order α.

1.4.2 Riemann Liouville Fractional Derivative

Having built up these essential properties of Riemann-Liouville integral operator. We now go to the concept of relating fractional differential operators. The fundamental theorem of integer order calculus gives $D^m I^m \phi = \phi$, where $m \in \mathbb{N}$ denote the order of integral and differential operator. We can also write $D^{m_1-m_2} I^{m_1-m_2} \phi = \phi$, such that $m_1, m_2 \in \mathbb{N}$. By applying D^{m_2} on both side we have,

$$D^{m_2} \phi = D^{m_1} I^{m_1-m_2} \phi. \tag{1.11}$$

If m_2 is replaced by any $\alpha > 0$, relation (1.11) is valid for particular class of functions unless $m_1 - m_2 > 0$. This way Riemann Liouville introduce a fractional differential operator.

Definition 1.4.3. *[11] Let $\alpha > 0$ and $k = \lceil \alpha \rceil$, then,*

$$D_c^\alpha \phi = D^k I_c^{k-\alpha} \phi.$$

The operator D_c^α, is called Riemann Liouville differential operator of fractional order α.

For whole number estimation of α it turns into the customary derivative. For $\alpha = 0$, D_c^0 is the identity operator. So Riemann-Liouville fractional differential operator satisfies the zero property.

Theorem 1.4.3. *[11] Let $k(t) = (t-c)^p$ for some $p > -1$, then the $\alpha > 0$ order Riemann Liouville derivative of function k is,*

$$D_c^\alpha (t-c)^p = \frac{\Gamma(1+p)}{\Gamma(1+p-\alpha)}(t-c)^{p-\alpha}.$$

Example 1.4.4. *[12] Let $k(t) = e^{\lambda t}$, for some $\lambda > 0$ and $t > 0$, then the Riemann liouville fractional derivative of k is computed as follows.*
By definition we have

$$\begin{aligned} D_c^n e^{\lambda t} &= D^n \sum_{p=0}^{\infty} \frac{\lambda^p e^{\lambda c}(t-c)^p}{p!} \\ &= \sum_{p=0}^{\infty} \frac{\lambda^p e^{\lambda c}(t-c)^{p-n}\Gamma(1+p)}{p!\Gamma(1+p-n)} \\ &= \sum_{p=0}^{\infty} \frac{(\lambda(t-c))^p e^{\lambda c}\Gamma(1+p)(t-c)^{-n}}{p!\Gamma(1+p-n)}. \end{aligned}$$

Now as

$$\Gamma(1+p-n) = p!\left(1-\frac{n}{p}\right)\left(1-\frac{n}{p-1}\right)\ldots(1-n)\Gamma(1-n).$$

Thus, we have

$$\begin{aligned} D_c^n e^{\lambda t} &= \sum_{p=0}^{\infty} \frac{(\lambda(t-c))^p \Gamma(1+p) e^{\lambda c}(t-c)^{-n}}{p!p!\left(1-\frac{n}{p}\right)\left(1-\frac{n}{p-1}\right)\ldots(1-n)\Gamma(1-n)} \\ &= \frac{e^{\lambda a}(t-c)^{-n}}{\Gamma(1-n)} \sum_{p=0}^{\infty} \frac{(\lambda(t-c))^p}{= p!\left(1-\frac{n}{p}\right)\left(1-\frac{n}{p-1}\right)\ldots(1-n)\Gamma(1-n)} \\ &= \frac{e^{\lambda c}(t-c)^{-n}}{\Gamma(1-n)}\ {}_1K_1(1; 1-n; \lambda(t-c)), \end{aligned} \tag{1.12}$$

where ${}_1K_1(1; 1-n; \lambda(t-c))$ *is the Hypergeometric function of first kind.*

Example 1.4.5. *Let* $\phi(t) =$ *constant, then Riemann-Liouville derivative is,*

$$D_c^{\alpha}\phi(t) = \frac{(constant)(t-c)^{-\alpha}}{\Gamma(1-\alpha)} \neq 0.$$

It is to be observed that Riemann Liouville derivative of constant function is nonzero unless constant $\longrightarrow -\infty$. Some important properties of Riemann -Liouville fractional differential operator are:

Linearity: Let $\phi_1, \phi_2 \in L_1[a, b]$ and c_1, c_2 be any real constants. Moreover $D_c^{\alpha}\phi_1, D_c^{\alpha}\phi_2$ and $D_c^{\alpha}(c_1\phi_1 + c_2\phi_2)$ exist almost everywhere. Then we have,

$$D_c^{\alpha}(c_1\phi_1 + c_2\phi_2) = c_1 D_c^{\alpha}\phi_1 + c_2 D_c^{\alpha}\phi_2.$$

Semigroup: Let $\alpha_1, \alpha_2 \in \mathbb{R}^{+}\mathrm{U}\{0\}$, provided $\phi, \psi \in L_1[a, b]$ and $\phi = I_a^{\alpha_1+\alpha_2}\psi$. Then

$$D_c^{\alpha_1} D_c^{\alpha_2}\phi = D_c^{\alpha_1+\alpha_2}\phi. \tag{1.13}$$

Example 1.4.6. *[11] Let* $\phi(t) = t^{-\frac{1}{2}}$ *and* $\alpha_1 = \alpha_2 = \frac{1}{2}$, *then by Theorem 1.4.3,* $D_c^{\alpha_1}\phi(t) = D_c^{\alpha_2}\phi(t) = 0$ *and so* $D_c^{\alpha_1} D_c^{\alpha_2}\phi(t) = 0$. *Also* $D_c^{\alpha_2} D_c^{\alpha_1}\phi(t) = 0$. *On other hand* $D_c^{\alpha_1+\alpha_2}\phi(t) = D\phi(t) = -(2t^{\frac{3}{2}})^{-1}$.

Above example (1.4.6) shows that $D_c^{\alpha_1} D_c^{\alpha_2}\phi = D_c^{\alpha_2} D_c^{\alpha_2}\phi \neq D_c^{\alpha_1+\alpha_2}\phi$.

Example 1.4.7. *[11] Let* $\phi(t) = t^{\frac{1}{2}}$ *and* $\alpha_1 = \frac{1}{2}, \alpha_2 = \frac{3}{2}$. *Then* $D_c^{\alpha_1}\phi(t) = \frac{\sqrt{\pi}}{\sqrt{2}}$ *and* $D_c^{\alpha_2}\phi = 0$. *This implies* $D_c^{\alpha_1} D_c^{\alpha_2}\phi(t) = 0$ *while* $D_c^{\alpha_2} D_c^{\alpha_1}\phi(t) = \frac{-t^{\frac{-3}{4}}}{4}$.

The second example (1.4.7) shows that $D_c^{\alpha_1} D_c^{\alpha_2}\phi \neq D_c^{\alpha_2} D_c^{\alpha_1}\phi$.
Thus we can say that generally the semi-group property for Riemann-Liouville fractional differential operator not hold.

Theorem 1.4.4. *[11] Let* $\alpha \geq 0$. *Then, for every* $k \in L_1[a, b]$,

$$D_c^{\alpha} I_c^{\alpha} k = k,$$

almost everywhere.

Theorem 1.4.5. *[11] Let $\alpha > 0$ and k is assume to be such that $I_c^{\lceil\alpha\rceil-\alpha}k \in A^{\lceil\alpha\rceil}[a,b]$. Then,*

$$I_c^{\alpha} D_c^{\alpha} k(t) = k(t) - \sum_{i=0}^{\lfloor\alpha\rfloor} \frac{(t-c)^{\lceil\alpha-i-1\rceil}}{\Gamma(\alpha-i)} \lim_{z\to c} D_c^{\lceil\alpha\rceil-i-1} I_c^{\lceil\alpha\rceil-\alpha} k(z).$$

Particularly, for $0 < \alpha < 1$ we have

$$I_c^{\alpha} D_c^{\alpha} k(t) = k(t) - \frac{(t-c)^{\alpha-1}}{\Gamma(\alpha)} \lim_{z\to c} I_c^{1-\alpha} k(z).$$

1.4.3 The Caputo Differential Operator

The Rieman Liouville derivative assumed a noteworthy part in progression of hypothesis of fractional calculus. In theory of Riemann Liouville fractional derivative we have seen that their are certain impediments of utilizing the Riemann Liouville differential operator for demonstrating this present reality wonders. It is to be noted that the Riemann Liouville derivative of a constant is not zero. To deal with these circumstances, in 1967 Caputo presented another definition of fractional derivative. Here we talk about the Caputo derivative, its properties and it's connection with the Riemann Liouville operators for integral and derivative [11].

Definition 1.4.8. *[11] Assume $\alpha \in \mathbb{R}^{+} U \{0\}$, and $k = \lceil\alpha\rceil$ Then*

$$\begin{aligned} {}^{*}D_a^{\alpha}\phi(t) &= I_a^{k-\alpha} D^k \phi(t) \\ &= \int_a^t \frac{(t-\eta)^{k-\alpha-1} D^k \phi(\eta)}{\Gamma(k-\alpha)} d\eta, \qquad a < t < b, \end{aligned}$$

the operator ${}^{*}D_a^{\alpha}$ is said to be α order Caputo differential operator.

For whole number estimation of α the Caputo differential operator turns into customary derivative.

Theorem 1.4.6. *Assume $\alpha > 0$ and $\phi(t) = (t-a)^p$. Then for $p \in \mathbb{N}$ and $p \geq \lceil\alpha\rceil$*

$$ {}^{*}D_a^{\alpha}\phi(t) = \frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)} (t-a)^{p-\alpha}. \tag{1.14}$$

if $p \in \{0, 1, 2, \ldots, \lceil\alpha\rceil - 1\}$, then $D_*^{\alpha}\phi(t) = 0$.

Remark 1. *If we compare the differential operator of Riemann-Liouville and the Caputo differential operator, we notice that both operators have different kernels. In the Riemann Liouville differential operator one can utilize any $k \in \mathbb{N}$ with $k > \alpha$, while on other hand in definition of the Caputo derivative $k = \lceil \alpha \rceil$ is mandatory. Moreover the Caputo derivative of constant function is zero. On comparing we see that the Caputo approach for fractional derivative is better that Riemann-Liouville.*

Some properties of the Caputo derivative.

Linearity: Let ϕ_1 and ϕ_2 be two functions such that ${}^*D_a^\alpha \phi_1$ and ${}^*D_a^\alpha \phi_2$ exists almost everywhere on defined domain. Moreover ${}^*D_a^\alpha(c_1\phi_1 + c_2\phi_2)$ exist almost everywhere. Then

$$ {}^*D_a^\alpha(c_1\phi_1(t) + c_2\phi_2(t)) = c_1^* D_a^\alpha \phi_1(t) + c_2^* D_a^\alpha \phi_2(t). $$

Semi-group property: Assume $\phi \in C^n[a,b]$ and $n \in \mathbb{N}$. Moreover assume $\alpha, \epsilon > 0$ be such that there exists some $k \in \mathbb{N}$ with $k \leq n$ and $\alpha, \alpha+\epsilon \in [k-1,k]$. Then

$$ {}^*D_a^\alpha \, {}^*D_a^\epsilon \phi = {}^*D_a^{\alpha+\epsilon}\phi. $$

Theorem 1.4.7. *[11] Let $\alpha \geq 0$, and $k = \lceil \alpha \rceil$. Further assume that $\phi \in AC^k[a,b]$. Then*

$$ {}^*D_a^\alpha \phi = D_a^\alpha[\phi - T_{k-1}\phi(a)], $$

almost everywhere. Here D_a^α is Riemann-Liouville differential operator, and $T_{k-1}\phi(a)$ denote the $k-1$ degree Taylor polynomial for function ϕ centered at a.

Lemma 1.4.9. *[11] Assume $\alpha \geq 0$ and $k = \lceil \alpha \rceil$. Moreover, let $D_a^\alpha \phi$ and ${}^*D_a^\alpha \phi$ exist. Then*

$$ {}^*D_a^\alpha \phi(t) = D_a^\alpha \phi(t) - \sum_{j=0}^{k-1} \frac{D_a^j \phi(a)}{\Gamma(j-\alpha+1)}(t-a)^{j-\alpha} $$

holds almost everywhere.

As a consequence of Lemma 1.4.9, we have

Theorem 1.4.8. *[11] Assume $\alpha \geq 0$ and $k = \lceil \alpha \rceil$. Moreover, let $D_a^\alpha \phi$ and ${}^*D_a^\alpha \phi$ exist. Then*

$$ {}^*D_a^\alpha \phi = D_a^\alpha \phi $$

holds if and only if $D^m\phi$ is zero at centered point a for all $m \in [0, k]$.

Theorem 1.4.9. *[11] Assume $\alpha \geq 0$ and $\phi \in AC^k[a, b]$, then*

$$^*D_a^\alpha I_a^\alpha \phi = \phi.$$

Thus the composition of Riemann-Liouville integral and the Caputo differential operator we notice that the Caputo differential operator is left inverse of Riemann-Liouville integral operator.

Theorem 1.4.10. *[11] Assume $\alpha \geq 0$ and $k = \lceil \alpha \rceil$. Moreover $\phi \in AC^k[a, b]$. Then*

$$I_a^\alpha \ {}^*D_a^\alpha \phi(t) = \phi(t) - \sum_{j=0}^{k-1} \frac{D^j\phi(a)}{j!}(t-a)^j.$$

Thus from Theorem 1.4.10, we notice that the Caputo differential operator is not the right inverse of Rieman Liouville integral.

Corollary 1.4.10. *[11] Assume $\alpha \geq 0$ and $k = \lceil \alpha \rceil$. Moreover $\phi \in AC^k[a, b]$. Then the Taylor expansion of ϕ for the Caputo differential operator is*

$$\phi(t) = \sum_{j=0}^{\lceil \alpha \rceil - 1} \frac{D^j\phi(a)}{j!}(t-a)^j + I_a^\alpha \ {}^*D_a^\alpha \phi(t).$$

1.5 Applications of Fractional Calculus

Todays researcher can depicts and demonstrate numerous physical marvels with a customary differential equation. The instinct of scientist from integral and derivative depend on their physical and geometric interpretation. such as integral give the area under the curve. In traditional math 1^{st} and 2^{nd} derivative of distance give the speed and increasing speed respectively. Similarly 3^{rd} and 4^{th} derivative of displacement represent jerk and jounce.

Since the appearance of fractional integral and derivative have no satisfactory physical interpretation over multi year. The essential scientific thoughts of fractional calculus were grew long prior by mathematician Liouville, Riemann and other brought to consideration of the engineering field. Here we present some application of fractional calculus [13].

Shadow of Wall: In [14] G. L. Bullock gives a physical explanation of Riemann-Stieltjes integral. Using the concept of G. L. Bullock, I. Podlubny give a dynamical explanation of integral and derivative of fractional order. In [15] I. Podlubny proposed a new geometric interpretation of Stiltjes integral based on Riemann-Liouville integral in term of changing time scale. Consider Riemann-Liouville integral of order α.[15, 11]

$$\int^{\alpha} \phi(t)dt^{\alpha} = \frac{1}{\Gamma(\alpha)} \int_a^t \phi(\xi)(t-\xi)^{\alpha-1} d\xi. \tag{1.15}$$

Now if we assume $\Psi(\xi) = \frac{1}{\Gamma(\alpha+1)} (t^{\alpha} - (t-\xi)^{\alpha})$ then (1.15), becomes a stieltjes integral [15]

$$\int^{\alpha} \phi(t)dt^{\alpha} = \int_a^t \phi(\xi)d\Psi(\xi). \tag{1.16}$$

Now ξ, ϕ, Ψ considered to be three axis. In the plane (ξ, Ψ) we plot Ψ for $0 \le \xi \le t$. Along the acquired bend we manufacture a fence of the shifting stature $\phi(\xi)$, so the best edge of the fence for $0 \le \xi \le t$ is a three dimensional line $(\xi, \phi(\xi), \Psi(\xi))$.

Example 1.5.1. *Let* $f(t) = t + 0.5\sin(t)$ *and* $\alpha = 0.75$ *then the fractional order Riemann Liouville integral is [15]*

$$\begin{aligned} \int_c^{\alpha} f(t)dt &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-\xi)^{(\alpha-1)} f(\xi)d\xi \\ &= I_c^{\alpha} t + I_c^{\alpha} 0.5\sin(t) \\ &= \frac{\Gamma(2)}{\Gamma(1+1+0.75)} t^{1+o.75} + \ \sin\left(t - \frac{\phi\alpha}{2}\right). \end{aligned} \tag{1.17}$$

Figure (1.1) *shows that the fence projected on two plane* (τ, f) *and on the plane* (g, f)*. The projection on plane* (t, f) *is the region covered by the curve* $f(t)$ *correspond to the integral [15],* $\int_0^t f(\tau)d\tau$*. The projection of the area covered by the* $f(t)$ *on plane* (g, f) *relates the estimation of the area of integral* (1.16) *which is same to the integral in* (1.15)*.*

Thus we can say area covered by curve $f(t)$ *gives two shadows on two planes. One is the region covered by the given function on plane* (t, f) *which in general physical explanation* *of the integral* $\int_a^t f(\tau)d\tau$*. While the profile on second plane* (g, f) *is physical explanation of integral of fractional order in Eq.* (1.15).

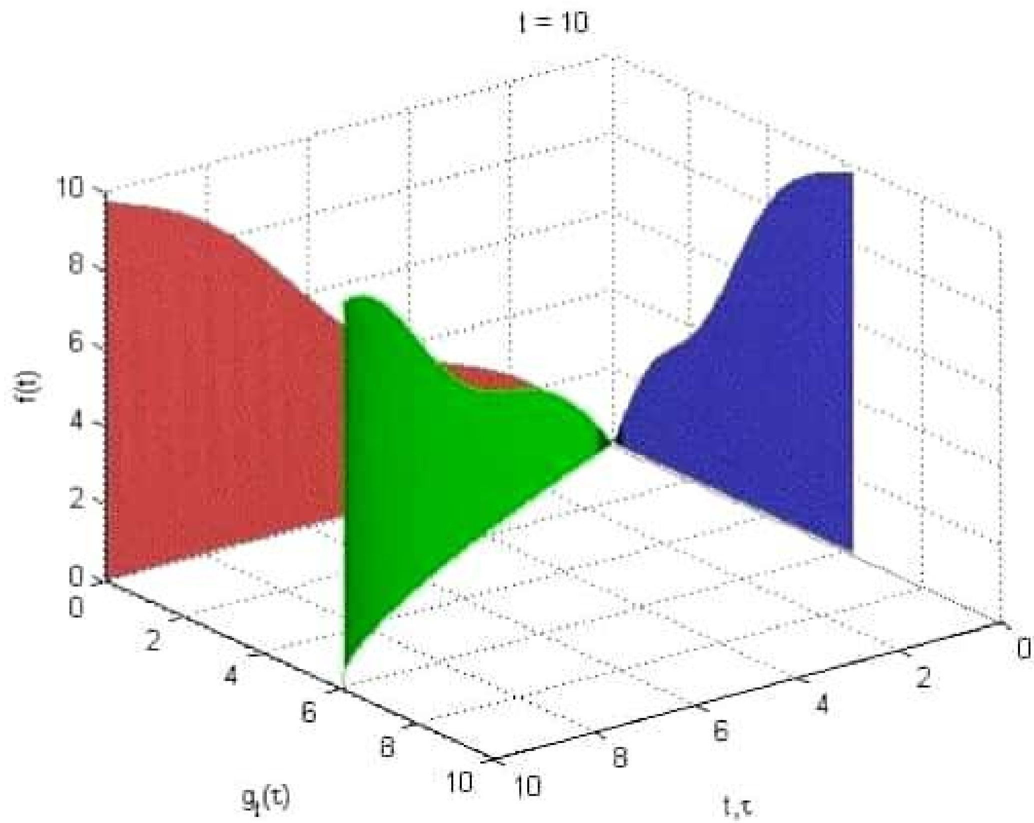


Figure 1.1: Shadow on wall

Electric transmission line:[14, 13, 16]
Amid the most recent many years of 19^{th} century, Heaviside effectively built up calculus which operate without thorough numerical contentions. In 1892 Heaviside presented possibility of fractional derivative in investigation of transmission of current in electric lines. In view of representative administrator shape arrangement of heat equation because of George. Letter ρ introduced by Heavside for differential operator $\frac{d}{dt}$ and proposed solution for diffusion equation [13].
Ultra Wave Propagation:[17, 13] Osteoporosis is an infection effected by hormonal changes and hormonal changes, influencing concordance between the declaration of

hard tissue and resorption. Early clinical location of this pomology by means of ultrasonic wave propagation would be a key intrigue. Since spongy bone (tissues of skeletal) is an in homogeneous permeable region, the connection among bone and ultrasound will exceedingly unpredictable. Ultrasonic wave engendering in human spongy bone is considered. Fractional calculus utilized to depict the collaborations amongst solid and liquid structure. Transmission and reflection diffusing administrators are inferred for section of spongy bone in the flexible casing utilizing Biot hypothesis. The Biot hypothesis is a set up method for foreseeing ultrasonic spread in an in homogeneous substance and initially connected to liquid soaked permeable rocks for geophysical testing. The Biot's hypothesis demonstrate both separate and doubled conduct of the casing and liquid [17].

Theory of Viscoelsticity:[18, 13]

The upside technique for fractional differentiation in principle of viscoelasticity is, it bears potential outcomes for acquiring fundamental conditions for flexible modulus of visco-elastic substance with some practical experiments decided variables. Likewise the fragmentary subsidiary technique had utilized as a part of investigations of the perplexing modulus and impedance for different viscoelastic materials models.

1.6 Basic Concepts From Analysis

This section explain important terminologies and results from analysis that will be used to set up results for stability, uniqueness and existence of solutions for differential equations of arbitrary order.

Definition 1.6.1. *[20, 21] Let A be the subset of $\mathbb{R}^n$ and φ is a complexed valued function. Then φ is continuous at $a_0 \in A$ iff for $\epsilon > 0$ there is a $\delta > 0$, that is $\|\varphi(a) - \varphi(a_0)\| < \epsilon, \quad whenever \quad \|a - a_0\| < \delta \ \ \forall a, a_0 \in A$.*

Definition 1.6.2. *[20, 21] The function φ is uniformly continuous if for each $\epsilon > 0$, there is a $\delta > 0$, thus whenever $\|a - a_0\| < \delta$, $\forall\ a, a_0 \in A$, we have $\|\varphi(a) - \varphi(a_0)\| < \epsilon$.*

Remark 2. *We notice that in definition of continuity δ depend on t_0 and ϵ while in* *uniform continuity δ depend only on ϵ. Therefore we can say every uniform continuous function must be continuous.*

Example 1.6.3. *Assume $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ and $\varphi(t) = 2t - 5$. Thus for arbitrary ϵ there is a $\delta = \frac{\epsilon}{2} > 0$. Moreover $|t - t_0| < \delta, \ \forall \ t, t_0 \in \mathbb{R}$. And,*

$$|\varphi(t) - \varphi t_0| = |2t - 5 - 2t_0 + 5| = 2|t - t_0| < 2\delta < \epsilon.$$

Thus φ is uniformly continuous.

Example 1.6.4. *Let $\varphi : B \longrightarrow \mathbb{R}$ and $B = \{t \in \mathbb{R} \ : \ 0 < t < \infty\}$. Assume $\varphi(t) = t^2$. Then for $\epsilon > 0$, there is $\delta > 0$ that is $|t - t_0| < \delta$ for $t, t_0 \in B$.*
$|\varphi(t) - \varphi(t_0)| = |t^2 - t_0^2|$. Now assume $s = t_0 + \delta_1$, $\delta_1 > 0$ and take $\delta = \min\left(\delta_1, \frac{\epsilon}{2s}\right)$. Thus $|t - t_0| < \delta_1 \implies t < t_0 + \delta_1 = s$. Now,

$$|\varphi(t) - \varphi(t_0)| < 2s|t - t_0| < 2s\delta_1 = 2s\left(\frac{\epsilon}{2s}\right) = \epsilon.$$

Since δ depend on s and s depend on t_0, so φ is continuous on B but not uniformly continuous. But if $B = [0, b]$ be any finite interval. Where b be any positive integer. Then for $\epsilon > 0$ there exist a $\delta = \frac{\epsilon}{2b}$ such that

$$|\varphi(t) - \varphi(t_0)| < 2b\delta < \epsilon,$$

whenever $|t - t_0| < \delta$.

Definition 1.6.5. *[20] Let $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, φ is said to be Lipschitz continuous if there exist $L > 0$ such that*

$$\|\varphi(a) - \varphi(b)\| \leq L\|a - b\|, \qquad \forall \ a, b \in domain(\varphi).$$

Example 1.6.6. *Let $\varphi : \mathbb{R} \longrightarrow \mathbb{R}^+$ and $\varphi(t) = |t|$. Thus for $\epsilon > 0 \ \exists \ \delta > 0$, that is $|t_1 - t_2| < \delta$. Now $|\varphi(t_1 - t_2)| = |t_1 - t_2| < \delta = \epsilon$.*
Thus $\varphi(t) = |t|$ is Lipschitz but clearly $\varphi(t) = |t|$ is not differentiable at 0.

Example 1.6.7. *The* sin *function is Lipschitz continuous because its derivative is bounded above by 1 in absolute value.*

Example 1.6.8. *Now let* $\varphi(t) = t^{\frac{3}{2}} \sin(\frac{1}{t})$ *where* $t \neq 0$. *It is differentiable on compact set* $[0,1]$ *but its derivative is not bounded so it is not Lipschitz continuous.*

Remark 3. *A differentiable function is Lipschitz continuous iff it has bounded first derivative.*

Definition 1.6.9. *[20] Let* A *be a finite set. Then the function* φ *is absolutely continuous on* A *if for each* $\epsilon > 0$, *there is a* $\delta > 0$, *then*

$$\sum_{i=0}^{n} |\varphi(b_i) - \varphi(a_i)| < \epsilon,$$

for any finite set $\{[a,b]\}$ *of disjoint intervals.*

Definition 1.6.10. *[20] Let* B *be subset of space of continuous function then* B *is equicontinuous if for each* $\epsilon > 0$ *there is a* $\delta > 0$, *then* $\|\varphi(a_1) - \varphi(a_2)\| < \epsilon$ *for all* $a_1, a_2 \in [a,b]$. *Whenever* $\|a_1 - a_2\| < \delta$, *for all* $\varphi \in B$.

Example 1.6.11. *[21] Let* B *be a closed subset of* $C[0,1]$ *and define* $\Phi(t) = \int_0^t \varphi(s)ds$ *for each* $\varphi \in B$. *Then for every* $\Phi \in B_1 \subset B$.

$$|\Phi(t_1)-\Phi(t_2)| \leq \left| \int_0^{t_1} \varphi(\tau)d\tau - \int_0^{t_2} \varphi(\tau)d\tau \right| \leq \left| \int_{t_2}^{t_1} \varphi(\tau)d\tau \right| \leq |t_1 - t_2| \, \|\varphi\| \leq |t_1 - t_2| \, .$$

Thus for each $\epsilon > 0$ *there is* $\delta > 0$. *Such that if we choose* $\delta = \epsilon$ *then for* $|t_1 - t_2|$ *we have* $|\Phi(t_1) - \Phi(t_2)| < \epsilon$. *Therefore* B *is equicontinuous and also absolutely continuous.*

Definition 1.6.12. *Consider an equation* $\varphi(t_1, t_2) = 0$, *then the functional equation is the equation in which the variables* t_1 *and* t_2 *are function.*

Example 1.6.13. *consider* $\varphi(t + 10) = \varphi(t) + 2$, *here the unknown* φ *is function of* t *so it is functional equation.*

Definition 1.6.14. *[22, 20] Let* $A, B \subset \mathbb{R}^n$, *consider a map* $\varphi : A \longrightarrow B$, *then* $a \in A$ *be the fixed point of operator* Ψ *if it satisfy the equation* $\varphi a = a$, *such that* a *must also belong to* B.

Example 1.6.15. *let* $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ *and* $\varphi(t) = t^2$. *The the fixed points are* 0 *and* 1.

Definition 1.6.16. *[20] Let* $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^m$*,* φ *is said to be contraction if there exist* $0 < L < 1$*, such that*

$$\|\varphi(a) - \varphi(b)\| \le L\|a - b\|, \qquad \forall\ a, b \in domain(\varphi).$$

Theorem 1.6.1. *[20] Assume* B *be a subset of a banach space and* $\Psi : B \longrightarrow B$ *is contraction mapping then* B *has unique fixed point.*

Definition 1.6.17. *Let* $S = S_j|j \in I$ *is a cover for subset* B *of Banach space* X *and if each* S_i *is open in* X *then* S *is an open cover.*

Definition 1.6.18. *[20] Let* $B \subseteq X$ *where* X *be a Banach space. Then the set* B *is compact if every open cover for* B *has finite sub-cover.*

Example 1.6.19. *Assume a subset* $B = (0, 1)$ *of* $\mathbb{R}$*.*
Consider a family $S = \left\{(\frac{1}{n}, 1) : \ n \in \mathbb{N} \setminus \{1\}\right\}$*.* S *is cover for* B*. Let* $t \in B$*. For sufficiently large* n *we see that* $t > \frac{1}{n}$*. Thus* $t \in \left(\frac{1}{n}, 1\right)$*.*
This mean B *is contained in* S*. This shows that* S *is a cover for* $B = (0, 1)$*.*
Now if we take any finite subfamily ,

$$\left\{\left(\frac{1}{n}, 1\right), \left(\frac{1}{n^2}, 1\right), \left(\frac{1}{n^3}, 1\right), \ldots, \left(\frac{1}{n^7}, 1\right)\right\},$$

and $N = max\,\{n, n^2, n^3, \ldots, n^7\}$*. Then this subfamily cover only* $\left(\frac{1}{N}, 1\right)$ *not all* $(0, 1)$*. So* $(0, 1)$ *is not compact.*

Example 1.6.20. *Every finite set of a space* X *is compact.*
Consider a set $B = \{b_1, b_2, b_3, \ldots, b_n\}$ *of a space* X*. Let* $S = \{S_j\ :\ j \in I\}$ *is any open cover for* B*. where* I *is indexing set. For each* $i \in \{1, 2, 3, \ldots, n\}$ *we have* $b_i \in S_{ji}$ *for some* $ji \in I$*. Thus* $\{S_{i1}, S_{i2}, S_{i3}, \ldots, S_{in}\}$ *is a subfamily of* S *and is cover for* B*. Hence* B *is compact.*

Definition 1.6.21. *Assume* P *be subset of Banach space. Then* B *is relatively compact if its closure is compact.*

Theorem 1.6.2 (Arzela-Ascoli Theorem)**.** *[20] Assume* B *is subset of a Banach space and assume* $S \subset C(B)$*. Then* S *is relatively compact if and only if it is bounded and equi-continuous.*

Definition 1.6.22. *[20] Let A and D be Banach spaces and let $\phi : A \to D$, operator ϕ is compact if the image $\phi(S)$ of every bounded set S in A is relatively compact in D.*

Theorem 1.6.3. *[20] Assume Ψ be an operator mapping from a convex subset B to a Banach space X, i.e $\Psi : B \longrightarrow X$. Then the operator Ψ has fixed point if Ψ is compact and self map.*

Definition 1.6.23. *A set B is convex set if any line segment between arbitrary two points $t_1, t_2 \in B$ lies in B. That is for all $t_1, t_2 \in B$ and for $0 \leq \lambda \leq 1$,*

$$\lambda t_1 + (1 - \lambda)t_2 \in B.$$

Lemma 1.6.24. *[23] Assume F is bounded linear operator, F^{-1} exist iff there exist a bounded linear operator G in Banach space such that G^{-1} exist and $\|G - F\| \leq \frac{1}{\|G-1\|}$. If F^{-1} exist, then*

$$F^{-1} = \sum_{p=0}^{\infty} \left(I_G^{-1} F\right)^n G^{-1}, \quad \text{and} \quad \|F^{-1}\| \leq \frac{\|G^{-1}\|}{1 - \|G^{-1}\|\|G - F\|}. \tag{1.18}$$

Theorem 1.6.4 (Kantorovich Theorem)**.** *[20] Assume A, C be Banach spaces and $\Psi : A \longrightarrow C$ be a map. Assume $y \in C$ such that derivative F'_y exists and is invertible. Define $a_0 = \|F'^{-1}_y F_y\|$, $b_0 = \|F'^{-1}_y\|$, $S = \{x \in B : \|x - y\| \leq 2a_0\}$ and $K = Sup\left\{\frac{\|F'_x - F'_z\|}{\|x-z\|} : x \neq z\right\}$. If F is differentiable in S and if $a_0 b_0 k \leq \frac{1}{2}$, then F has zero in S and Newton iteration started at y converges quadratically to zero of F.*

Chapter 2

Existence and Uniqueness

In recent times, a large number of scientists put their contribution in field of fractional calculus just because of numerous applications of fractional differential equation in numerous fields like SIR model [24], models of viscoelasticity [25], control and electromagnetic theory [26] etc. The examination of the hypothesis of arbitrary order differential equations has picked up a great deal of intrigue on account difficulties it offers contrasted with investigation of the natural number order of differential equations. Several authors have set up results for uniqueness and existnece of solutions for differential equation of arbitrary order[11, 28]. It is to be noted that the solution structure of initial and the terminal value problem are non-identical. Here we analyse the question of uniqueness, stability and presence of solutions for differential equation. We will be focused on equations with the Caputo fractional differential operator. Consider the non-linear fractional differential problem:

$$\begin{aligned} {}^*D_a^{\alpha}\nu(t) &= k(t, \nu(t), {}^*D_a^{\beta}\nu(t)), \qquad 0 \leq \beta < \alpha, \\ \nu^{(i)}(0) &= \nu_i, \qquad i = 0, 1, 2, \ldots, \end{aligned} \tag{2.1}$$

where the function k is assumed to be continuous on $[a, b] \times \mathbb{R} \times \mathbb{R}$. Let $X = \{f : f \in C^{\beta}[a, b]\}$, then X is banach space with $\|f\| = \max_{t\in[a,b]} |f| + \max_{t\in[a,b]} |{}^*D_a^{\beta} f|$.

2.1 Existence of Solutions

Here in this portion we will investigate the presence of solutions for problem (2.1). Recently, various classical tools and fixed point theorems were exert to demonstrate the uniqueness and presence of solutions for differential equation of arbitrary order. We use Schauder theorem to verify the presence of solutions for arbitrary order differential equations.

Lemma 2.1.1. *[11] Assume k on $[a,b]\times\mathbb{R}\times\mathbb{R}$. $\nu\in C^{\lceil\alpha\rceil}[a,b]$ is solution of initial value problem* (2.1) *iff $\nu\in C^{\lceil\alpha\rceil}[a,b]$ is solution of* (2.2).

$$\nu(t)=g(t)+\int_a^t\frac{(t-\eta)^{\alpha-1}k(\eta,\nu(\eta),\widetilde{\nu}(\eta))}{\Gamma(\alpha)}d\eta, \tag{2.2}$$

where, $g(t)=\sum_{i=0}^{\lceil\alpha-1\rceil}(t-a)^i\nu_i$ and $\widetilde{\nu}={}^*D_a^\beta\nu$.

Proof. Assume $\nu(t)$ is solution Eq. (2.2). Now applying the differential operator ${}^*D_a^\alpha$ on Eq. (2.2), we get

$$^*D_a^\alpha\nu(t)={}^*D_a^\alpha g(t)+{}^*D_a^\alpha I_a^\alpha k(t,\nu(t),\widetilde{\nu}(t)). \tag{2.3}$$

As ${}^*D_a^\alpha g(t)={}^*D_a^\alpha\sum_{i=0}^{\lceil\alpha-1\rceil}(t-a)^i\nu_i=0$. Thus, by Lemma 1.4.3 and Theorem 1.4.4 Eq. (2.3) become,

$$^*D_a^\alpha\nu(t)=k(t,\nu(t),\widetilde{\nu}(t)).$$

Conversely, let $\nu(t)$ be the solution of problem 2.1. Applying Riemann-Liouville fractional integral operator on (2.1), we get

$$I_a^\alpha\,{}^*D_a^\alpha\nu(t)=I_a^\alpha k(t,\nu(t),\widetilde{\nu}(t)).$$

Again by Theorem (1.4.4) expand $I_a^\alpha\,{}^*D_a^\alpha\nu(t)$, the following expression obtained

$$\nu(t)-(\nu_0+\nu_1(t-a)+\nu_2(t-a)^2+\cdots+\nu_{\lceil\alpha-1\rceil}(t-a)^{\lceil\alpha-1\rceil})=\int_a^t\frac{(t-\eta)^{\alpha-1}k(\eta,\nu(\eta),\widetilde{\nu}(\eta))}{\Gamma(\alpha)}d\eta,$$

$$\nu(t)-g(t)=\int_a^t\frac{(t-\eta)^{\alpha-1}k(\eta,\nu(\eta),\widetilde{\nu}(\eta))}{\Gamma(\alpha)}d\eta.$$

□

Thus the solution of given problem is nothing but the solution of volterra intgeral equation.

Theorem 2.1.1. *Assume k defined on $[a,b]\times\mathbb{R}\times\mathbb{R}$ and continuous $|k(t,\nu(t),\widetilde{v}(s))|\leq G$, where G is +ve constant, then there is atleast one solution of problem* (2.1).

Proof. By Lemma 2.1.1 the integral representation of problem (2.1) is

$$\nu(t)=g(t)+\int_a^t\frac{(t-\eta)^{\alpha-1}k(\eta,\nu(\eta),\widetilde{\nu}(\eta))}{\Gamma(\alpha)}d\eta.$$

Let $P=\Big\{v\in C[a,b]:|\nu-g(t)|<\Upsilon_1,|^*D_a^\beta\nu-^*D_a^\beta g|<\Upsilon_2\Big\}$.
Where $\Upsilon_1=\frac{(b-a)^\alpha}{\Gamma(\alpha+1)}$ and $\Upsilon_2=\frac{(b-a)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}$. The set P is closed subset of given Banach space. Let $\Psi:P\longrightarrow P$ defined by

$$(\Psi\nu)(t)=g(t)+\int_a^t\frac{(t-\eta)^{\alpha-1}k(s,\nu(\eta),\widetilde{\nu}(\eta)}{\Gamma(\alpha)}d\eta. \tag{2.4}$$

Thus we can write $v=\Psi v$. Thus fixed point of the operator Ψ is solution of (2.1). Now prove operator Ψ is compact and self map. To show compactness just verify image set $\Psi(P)$ under the operation Ψ is relatively compact. Now let $\nu_1(t),\nu_2(t)\in P$, then for $t\in[a,b]$,

$$\begin{aligned}|\Psi\nu_1(t)-\Psi\nu_2(t)|&=|g(t)+\int_a^t\frac{(t-\eta)^{\alpha-1}k(\eta,\nu_1(\eta),\widetilde{\nu_1}(\eta))}{\Gamma(\alpha)}d\eta\\&\quad-g(t)+\int_a^t\frac{(t-\eta)^{\alpha-1}k(\eta,\nu_2(s),\widetilde{\nu_2}(\eta)}{\Gamma(\alpha)}d\eta|\\&\leq\int_a^t\frac{(t-\eta)^{\alpha-1}}{\Gamma(\alpha)}|k(\eta,\nu_1(\eta),\widetilde{\nu_1}(\eta)-k(\eta,\nu_2(\eta),\widetilde{\nu_2}(\eta)|d\eta.\end{aligned}$$

Since the function k is continuous. Thus for given ϵ, there exist $\delta>0$ such that

$$|k(t,\nu_1(t),\widetilde{\nu_1}(t)-k(t,\nu_2(t),\widetilde{\nu_2}(t)|<\epsilon,$$

whenever $\|\nu_1-\nu_2\|<\delta(\epsilon)$, $\forall t\in[a,b]$. Thus

$$\begin{aligned}|\Psi\nu_1(t)-\Psi\nu_2(t)|&\leq\frac{\epsilon}{\Gamma(\alpha)}\int_a^t(t-\eta)^{\alpha-1}d\eta\\&\leq\frac{(b-a)^\alpha\epsilon}{\Gamma(\alpha+1)}=\epsilon_1/2.\end{aligned} \tag{2.5}$$

Also for $t \in [a,b]$

$$\begin{aligned}
\left|D^{\beta}\Psi\nu_1(t) - D^{\beta}\Psi\nu_2(t)\right| &= \left|\int_a^t \frac{(t-\eta)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\left(k(\eta,\nu_1(\eta),\widetilde{\nu_1}(\eta)) - k(s,\nu_2(\eta),\widetilde{\nu_2}(\eta))\right)d\eta\right| \\
&\leq \int_a^t \frac{(t-\eta)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}(\eta-a)^0\epsilon\, d\eta \\
&= \frac{\epsilon(t-a)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \\
&\leq \frac{\epsilon_1}{2}.
\end{aligned} \tag{2.6}$$

Thus $\|\Psi(\nu_1) - \Psi(\nu_2)\| < \frac{\epsilon_1}{2} + \frac{\epsilon_1}{2} = \epsilon_1$, whenever $\|\nu_1 - \nu_2\| < \delta(\epsilon)$.

Thus, Eq.(2.5) and Eq.(2.6) reveal the continuity of operator Ψ. Next we have to show that ΨP is relatively compact set. Now for any $\phi \in \Psi P$ and $t \in [a,b]$[29, 12],

$$\begin{aligned}
|\phi(t)| &= |\Psi\nu(t)| \\
&\leq |g(t)| + \left|\int_a^t \frac{(t-\eta)^{\alpha-1}}{\Gamma(\alpha)}k(\eta,\nu(\eta),\widetilde{\nu}(\eta))d\eta\right| \\
&\leq \sum_{i=0}^{\alpha-1}\frac{(t-a)^i}{i!}\nu_i + \frac{\lambda(t-a)^{\alpha}}{\Gamma(\alpha+1)} \\
&\leq \sum_{i=0}^{\alpha-1}\frac{(b-a)^i}{i!}\nu_i + \Upsilon_1 = c_1.
\end{aligned}$$

Also

$$\begin{aligned}
\left|{}^*D_a^{\beta}\phi(t)\right| &= \left|{}^*D_a^{\beta}\Psi\nu(t)\right| \\
&\leq \left|D^{\beta}g(t)\right| + \left|\int_a^t \frac{(t-\eta)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}k(\eta,\nu(\eta),\widetilde{\nu}(\eta))d\eta\right| \\
&\leq \sum_{i=1}^{\alpha-1} i\frac{(t-a)^i}{i!}\nu_i + \frac{\lambda(t-a)^{-\beta+\alpha}}{\Gamma(1-\beta+\alpha)} \\
&\leq \sum_{i=1}^{\alpha-1} i\frac{(b-a)^i}{i!}\nu_i + \Upsilon_2 = c_2.
\end{aligned}$$

Now,

$$\begin{aligned}
\|\phi(t)\| &= \max_{t\in[a,b]}|\phi(t)| + \max_{t\in[a,b]}\left|{}^*D_a^{\beta}\phi(t)\right| \\
&\leq c_1 + c_2.
\end{aligned} \tag{2.7}$$

Eq. (2.7) show the boundedness of ΨP. As a further matter, for $a \leq t_1 \leq t_2 \leq b$

$$\begin{aligned}|\Psi\nu(t_1) - \Psi\nu(t_2)| &= \frac{1}{\Gamma(\alpha)}\left|\int_a^{t_1}(t_1-\eta)^{\alpha-1}k(\eta,\nu^\eta,\widetilde{\nu}(\eta))d\eta - \int_a^{t_2}(t_2-s)^{\alpha-1}k(\eta,\nu^\eta,\widetilde{\nu}(\eta))d\eta\right| \\ &\leq \frac{1}{\Gamma(\alpha)}\left|\int_a^{t_1}(t_1-\eta)^{\alpha-1}-(t_2-\eta)^{\alpha-1}k(\eta,\nu^\eta,\widetilde{\nu}(\eta))d\eta\right| \\ &\quad + \left|\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-\eta)^{\alpha-1}k(\eta,\nu(\eta),\widetilde{\nu}(\eta)d\eta\right|.\end{aligned}$$

Now the following cases.

Case 1:

If $t_1 = a$, and whenever $|t_1 - t_2| < \delta_1(\epsilon)$, we have

$$|\Psi\nu(t_1) - \Psi\nu(t_2)| \leq \frac{\lambda}{\Gamma(\alpha+1)} < \epsilon_1. \tag{2.8}$$

Case 2:

When $t_1 \neq a$, then

$$|\Psi\nu(t_1) - \Psi\nu(t_2)| \leq \lambda\left(\frac{2(b-a)^\alpha - (t_2-t_1)}{\Gamma(\alpha+1)}\right) + \frac{\lambda(t_2-t_1)}{\Gamma(\alpha+1)}.$$

Thus,

$$|\Psi\nu(t_1) - \Psi\nu(t_2)| \leq \frac{2\lambda(b-a)^\alpha}{\Gamma(\alpha)} = \epsilon_2. \tag{2.9}$$

Similarly for $a \leq t_1 \leq t_2 \leq b$

$$\begin{aligned}\left|{}^*D_a^\beta\Psi\nu(t_1) -{}^* D_a^\beta\Psi\nu(t_2)\right| &\leq \frac{1}{\Gamma(\alpha-\beta)}\left|\int_a^{t_1}(t_1-\eta)^{\alpha-\beta-1}-(t_2-\eta)^{\alpha-\beta-1}k(\eta,\nu^\eta,\widetilde{\nu}(\eta))d\eta\right| \\ &\quad + \left|\frac{1}{\Gamma(\alpha-\beta)}\int_{t_1}^{t_2}(t_2-\eta)^{\alpha-\beta-1}k(\eta,\nu(\eta),\widetilde{\nu}(\eta)d\eta\right| \\ &\leq \lambda\left(\frac{(t_1-a)^{\alpha-\beta}-(t_2-t_1)^{\alpha-\beta}+(t_2-a)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right) \\ &\quad + \frac{\lambda}{\Gamma(\alpha-\beta)}\int_{t_1}^{t_2}(t_2-\eta)^{\alpha-\beta-1}d\eta.\end{aligned}$$

Again we have two cases,

Case 1 :

If $t_1 = a$, then,

$$\begin{aligned}\left|{}^{*}D_a^{\beta}\Psi\nu(t_1) -{}^{*} D_a^{\beta}\Psi\nu(t_2)\right| &\leq \frac{-\lambda}{\Gamma(\alpha-\beta)}\left[\frac{(t_2-\eta)^{\alpha-\beta}}{\Gamma(\alpha-\beta)}\right]_{t_1}^{t_2} \\ &= \frac{\lambda}{\Gamma(\alpha+1)} < \epsilon_3\end{aligned} \tag{2.10}$$

Provided that $|t_2 - t_1| < \delta_3 = \left(\frac{\Gamma(1-\beta+\alpha)\epsilon_3}{\lambda}\right)^{\frac{1}{\alpha-\beta}}$.

Case 2:

If $t_1 \neq a$, then

$$\left|{}^{*}D_a^{\beta}\Psi\nu(t_1) -{}^{*} D_a^{\beta}\Psi\nu(t_2)\right| \leq \lambda\left(\frac{2(b-a)^{\alpha-\beta} - (t_2-t_1)}{\Gamma(\alpha+1-\beta)}\right) + \frac{\lambda(t_2-t_1)}{\Gamma(\alpha+1-\beta)}.$$

Thus,

$$\left|{}^{*}D_a^{\beta}\Psi\nu(t_1) -{}^{*} D_a^{\beta}\Psi\nu(t_2)\right| \leq \frac{2\lambda(b-a)^{\alpha-\beta-\beta}}{\Gamma(\alpha-\beta)} = \epsilon_4 \tag{2.11}$$

Thus the set ΨP is equicontinuous. Thus ΨP is relatively compact by Arzela Ascoli theorem. Moreover for any $\nu(t) \in P$ and $t \in [a,b]$, we find

$$\begin{aligned}|\Psi\nu(t) - g(t)| &= |\frac{1}{\Gamma(\alpha)}\int_a^t (t-\eta)^{\alpha-1} k(\eta, \nu(\eta), \widetilde{\nu}(\eta))d\eta| \\ &\leq \frac{\lambda}{\Gamma(\alpha)}\int_a^t (t-\eta)^{\alpha-1}(\eta-a)^0 d\eta \\ &= \frac{\lambda(t-a)^{\alpha}}{\Gamma(\alpha+1)} \leq \Upsilon_1.\end{aligned} \tag{2.12}$$

Also,

$$\begin{aligned}|{}^{*}D_a^{\beta}\Psi\nu(t) -{}^{*} D_a^{\beta}g(t)| &= |\int_a^t \frac{(t-\eta)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} k(\eta, \nu(\eta), \widetilde{\nu}(\eta))d\eta| \\ &\leq \int_a^t \frac{\lambda(t-\eta)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}(\eta-a)^0 d\eta \\ &= \frac{\lambda(t-a)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \leq \Upsilon_2.\end{aligned} \tag{2.13}$$

Thus, from Eq (2.12) and (2.13), we have shown that $\Psi : P \to P$. Therefore the Schauder theorem asserts that there is at-least one continuous solution $\nu(t)$ defined on $[a,b]$ of (2.1). □

2.2 Uniqueness of Solution

This section, we set up results for the uniqueness of solutions for fractional differential equation, particularly problem (2.1).

Lemma 2.2.1. *[29, 30] Assume a closed non-empty subset P of Banach space X, assume for every n there is sequence $\gamma_n \geq 0$ with condition $\sum_{n=0}^{\infty} \gamma_n$ must be convergent. Furthermore, let $\Psi : P \to P$ a self map satisfying following relation for all $n \in \mathbb{N}$ and any $u, v \in P$.*

$$\|\Psi^n u - \Psi^n v\| \leq \gamma_n \|u - v\|$$

Then, Ψ has a uniquely defined fixed point u^*. Moreover,sequence $(\Psi^n u_0)_{n=1}^{\infty}$ converges for any $u_0 \in P$ to this fixed point u^*.

Theorem 2.2.1. *Assume $0 \leq \beta < 1$, $1 \leq \alpha \leq 2$. Moreover k is continuous on $[a, b] \times \mathbb{R} \times \mathbb{R}$. Assume further*

$$|k(t, \nu_1(t), \widetilde{\nu_1}(t)) - k(t, \nu_2(t), \widetilde{\nu_2}(t))| \leq L \left(|\nu_1(t) - \nu_2(t)| + |\widetilde{\nu_1}(t) - \widetilde{\nu_2}(t)|\right), \qquad (2.14)$$

where L is any +ive constant independent of $\nu_1, \nu_2, \widetilde{\nu_1}, \widetilde{\nu_2}$. Then Eq. (2.1) has a unique continuous solution on $[a, b]$.

Proof. Let $P = \left\{v \in C[a, b] : |\nu - g(t)| < \Upsilon_1, |{}^*D_a^{\beta}\nu -{}^* D_a^{\beta} g| < \Upsilon_2\right\}$.
The set P is convex subset of Banach space X. Where $\Upsilon_1 = \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}$ and $\Upsilon_2 = \frac{(b-a)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}$.
Let $\Psi : P \longrightarrow P$ defined as

$$(\Psi\nu)(t) = g(t) + \frac{1}{\alpha} \int_a^t (t - s)^{\alpha-1} k(s, \nu(s), \widetilde{\nu}(s)ds.$$

The operator Ψ is well defined and completely continuous. It is enough to prove that Ψ^n is contraction mapping for large n. That is for $\nu_1, \nu_2 \in P$,

$$\|\Psi^n \nu_1 - \Psi^n \nu_2\| \leq (\Upsilon_1 - \Upsilon_2)^n L^n \|\nu_1 - \nu_2\|. \qquad (2.15)$$

We will prove Eq. (2.15) by induction. Let $\nu_1, \nu_2 \in P$, $t \in [a,b]$. Then, for $n = 1$, we have to show Ψ is a contraction mapping.

$$\begin{aligned}
\sup_{t\in[a,b]} |\Psi\nu_1 - \Psi\nu_2| &= \sup_{t\in[a,b]} \left| \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left(k(s,\nu_1(s),\widetilde{\nu_1}(s)) - k(s,\nu_2(s),\widetilde{\nu_2}(s))\right) ds \right| \\
&\leq \sup_{t\in[a,b]} \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left|(k(s,\nu_1(s),\widetilde{\nu_1}(s)) - k(s,\nu_2(s),\widetilde{\nu_2}(s)))\right| ds \\
&\leq \sup_{t\in[a,b]} \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} L\left(|\nu_1 - \nu_2| + |\widetilde{\nu_1} - \widetilde{\nu_2}|\right) ds \\
&= \frac{(t-a)^{\alpha} L \|\nu_1 - \nu_2\|}{\Gamma(\alpha+1)} \\
&\leq \Upsilon_1 L \|\nu_1 - \nu_2\|.
\end{aligned} \tag{2.16}$$

Also for $t \in [a,b]$

$$\begin{aligned}
\sup_{t\in[a,b]} |{}^{*}D_a^{\beta}\Psi\nu_1(t) -{}^{*} D_a^{\beta}\Psi\nu_2(t)| &\leq \frac{\sup_{t\in[a,b]}}{\Gamma(\alpha-\beta)} \int_a^t (t-s)^{\alpha-\beta-1} |k(s,\nu_1(s),\widetilde{\nu_1}(s)) \\
&\qquad - k(s,\nu_2(s),\widetilde{\nu_2}(s))| ds \\
&\leq \frac{\sup_{t\in[a,b]}}{\Gamma(\alpha-\beta)} \int_a^t (t-s)^{\alpha-\beta-1} L\left(|\nu_1 - \nu_2| + |\widetilde{\nu_1} - \widetilde{\nu_2}|\right) ds \\
&= \frac{(t-a)^{\alpha-\beta} L \|\nu_1 - \nu_2\|}{\Gamma(\alpha-\beta+1)} \\
&\leq \Upsilon_2 L \|\nu_1 - \nu_2\|.
\end{aligned} \tag{2.17}$$

Therefore, from Eq. (2.16) and (2.17) we have,

$$\begin{aligned}
\|\Psi\nu_1 - \Psi\nu_2\| &= \sup_{t\in[a,b]} |\Psi\nu_1 - \Psi\nu_2| + \sup_{t\in[a,b]} |{}^{*}D_a^{\beta}\Psi\nu_1(t) -{}^{*} D_a^{\beta}\Psi\nu_2(t)| \\
&\leq (\Upsilon_1 + \Upsilon_2) L \|\nu_1 - \nu_2\|.
\end{aligned}$$

Thus, Ψ is contraction mapping and relation (2.15) holds for $n = 1$. Now we have to assume that relation is true for (2.15) for $n = k$ such that

$$\|\Psi^k\nu_1 - \Psi^k\nu_2\| \leq (\Upsilon_1 - \Upsilon_2)^k L^k \|\nu_1 - \nu_2\|. \tag{2.18}$$

Now we have to prove (2.15) is true for $n = k+1$

$$\begin{aligned}
\|\Psi^{k+1}\nu_1 - \Psi^{k+1}\nu_2\| &= \sup_{t\in[a,b]} |\Psi^{k+1}\nu_1 - \Psi^{k+1}\nu_2| + \sup_{t\in[a,b]} |{}^*D_a^\beta \Psi^{k+1}\nu_1 -{}^* D_a^\beta \Psi^{k+1}\nu_2| \\
&\le \sup_{t\in[a,b]} |\Psi o \Psi^k \nu_1 - \Psi o \Psi^k \nu_2| + \sup_{t\in[a,b]} |{}^*D^\beta(\Psi o\Psi^k\nu_1) -{}^* D_a^\beta(\Psi o \Psi^k \nu_2)| \\
&\le \frac{\sup_{t\in[a,b]}}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} |k(s, \Psi^k\nu_1(s),{}^* D_a^\beta \Psi^k \nu_1(s)) \\
&\quad - k(s, \Psi^k\nu_2(s),{}^* D_a^\beta \Psi^k\nu_2(s))|ds \\
&\quad + \frac{\sup_{t\in[a,b]}}{\Gamma(\alpha-\beta)} \int_a^t (t-s)^{\alpha-\beta-1} |k(s, \Psi^k\nu_1(s),{}^* D_a^\beta \Psi^k\nu_1(s)) \\
&\quad - k(s, \Psi^k\nu_2(s),{}^* D_a^\beta\Psi^k\nu_2(s))|ds.
\end{aligned}$$

Using (2.14), we have

$$\begin{aligned}
\|\Psi^{k+1}\nu_1 - \Psi^{k+1}\nu_2\| &\le \frac{\sup_{t\in[a,b]}}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} L \left(|\Psi^k\nu_1 - \Psi^k\nu_2| + |{}^*D_a^\beta\Psi^k\nu_1 -{}^* D_a^\beta \Psi^k\nu_2| \right) ds \\
&\quad + \frac{\sup_{t\in[a,b]}}{\Gamma(\alpha-\beta)} \int_a^t (t-s)^{\alpha-\beta-1} L \left(|\Psi^k\nu_1 - \Psi^k\nu_2| + |{}^*D_a^\beta\Psi^k\nu_1 - D^\beta\Psi^k\nu_2| \right) \\
&= \frac{L}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \|\Psi^k\nu_1 - \Psi^k\nu_2\| ds \\
&\quad + \frac{L}{\Gamma(\alpha-\beta)} \int_a^t (t-s)^{\alpha-\beta-1} \|\Psi^k\nu_1 - \Psi^k\nu_2\| ds \\
&= \|\Psi^k\nu_1 - \Psi^k\nu_2\| \left(\frac{L}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} + \frac{L}{\Gamma(\alpha-\beta)} \int_a^t (t-s)^{\alpha-\beta-1} ds \right).
\end{aligned}$$

Thus, using (2.18), we have

$$\begin{aligned}
\|\Psi^{k+1}\nu_1 - \Psi^{k+1}\nu_2\| &\le (\Upsilon_1 + \Upsilon_2)^k L^k \|\nu_1 - \nu_2\| \left(\frac{L(t-a)^\alpha}{\Gamma(\alpha+1)} + \frac{L(t-a)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \right) \\
&\le (\Upsilon_1 + \Upsilon_2)^{k+1} L^{k+1} \|\nu_1 - \nu_2\|.
\end{aligned}$$

Therefore by induction, we can say Ψ^n is a contraction mapping for large n. Moreover we see the series (2.19)

$$\sum_{n=0}^{\infty} \left(\frac{(b-a)^\alpha}{\Gamma(\alpha+1)} + \frac{(b-a)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \right)^n L^n \tag{2.19}$$

converges to zero as $\left(\frac{(b-a)^\alpha}{\Gamma(\alpha+1)} + \frac{(b-a)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \right) < 1$. Thus by Lemma 2.2.1, Ψ has a fixed point which is the the unique solution. □

Example 2.2.2. *[32] Consider the initial value problem*

$$ {}^*D_0^{1.5}\varphi(t) = \frac{\sin\sqrt{t^3}}{5}\left(\frac{\varphi}{1+|\varphi|} + \frac{\varphi'}{1+|\varphi'|}\right), \qquad t \in [0,1]. \tag{2.20} $$

Here, $k(t,\varphi,\varphi') = \frac{\sin\sqrt{t^3}}{5}\left(\frac{\varphi}{1+|\varphi|} + \frac{\varphi'}{1+|\varphi'|}\right)$*, clearly it is continuous in* $[a,b]\times\mathbb{R}\times\mathbb{R}$*. Now*

$$ \begin{aligned} |k(t,\varphi_1,\varphi_1') - k(t,\varphi_2,\varphi_2')| &= \left|\frac{\sin\sqrt{t^3}}{5}\left(\frac{\varphi_1}{1+|\varphi_1|} + \frac{\varphi_1'}{1+|\varphi_1'|}\right) - \frac{\sin\sqrt{t^3}}{5}\left(\frac{\varphi_2}{1+|\varphi_2|} + \frac{\varphi_2'}{1+|\varphi_2'|}\right)\right| \\ &\le \left|\frac{sin\sqrt{t^3}}{5}\right|\left(\left|\frac{\varphi_1}{1+|\varphi_1|} - \frac{\varphi_2}{1+|\varphi_2|}\right| + \left|\frac{\varphi_1'}{1+|\varphi_1'|} - \frac{\varphi_2'}{1+|\varphi_2'|}\right|\right) \\ &\le \frac{1}{5}\left(\frac{|\varphi_1-\varphi_2|}{(1+|\varphi_1|)(1+|\varphi_2|)} + \frac{|\varphi_1'|-\varphi_2'|}{(1+|\varphi_1'|)(1+|\varphi_2'|)}\right). \end{aligned} $$

As $\frac{1}{(1+|\varphi_1|)(1+|\varphi_2|)} < 1$ *and* $\frac{1}{\left(1+|\varphi_1'|\right)\left(1+|\varphi_2'|\right)} < 1$*. So we have*

$$ |k(t,\varphi_1,\varphi_1') - k(t,\varphi_2,\varphi_2')| < \frac{1}{5}|\varphi_1 - v\varphi_2| + |\varphi_1' - \varphi_2'|. \tag{2.21} $$

Thus by Theorem 2.2.1, there exist a unique solution of problem 2.20.

Theorem 2.2.2. *[29, 31] let* $0 \le \beta < 1$, $1 \le \alpha < \alpha+\epsilon \le 2$ *and* k *is continuous and bounded on* $[a,b]\times\mathbb{R}\times\mathbb{R}$*. Further assume that*

$$ |k(t,\nu_1(t),\widetilde{\nu_1}(t)) - k(t,\nu_2(t),\widetilde{\nu_2}(t))| < L(|\nu_1-\nu_2| + |\widetilde{\nu_1}-\widetilde{\nu_2}|), \tag{2.22} $$

for some +ive constant L *independent of* $\nu_1,\nu_2,\widetilde{\nu_1},\widetilde{\nu_2} \in P$.
Where $P = \left\{\nu(t) \in C : |\nu(t) - g(t)| < \Upsilon_1, |{}^*D_a^{\beta}\nu(t) - {}^*D_a^{\beta}g(t)| < \Upsilon_2\right\}$*. Assume that* $\nu_1(t)$ *and* $\nu_2(t)$ *are unique solutions of initial value problems*

$$ \begin{aligned} {}^*D_a^{\alpha+\epsilon}\nu_1(t) &= k(t,\nu_1(t),\widetilde{\nu_1}(t)), \qquad \nu_1(a)=\nu_{10}, \qquad \nu_1'(a)=\nu_{11}. \\ {}^*D_a^{\alpha}\nu_2(t) &= k(t,\nu_2(t),\widetilde{\nu_2}(t)), \qquad \nu_2(a)=\nu_{10}, \qquad \nu_2'(a)=\nu_{11}. \end{aligned} \tag{2.23} $$

respectively then we have the relation.

$$ \|\nu_1-\nu_2\|_\infty = O(\epsilon). \tag{2.24} $$

Proof. It is given that $\nu_1(t)$ and $\nu_2(t)$ be the unique solutions of initial value problems (2.23). Applying the integral operator $I_a^{\alpha+\epsilon}$ and I_a^{α}, the corresponding integral equations are of the form as in [31]

$$\nu_1(t) = \nu_{10} + \nu_{11}(t-a) + \frac{1}{\Gamma(\alpha+\epsilon)} \int_a^t (t-s)^{\alpha+\epsilon-1} k(s, \nu_1(s), \widetilde{\nu_1}(s))ds.$$

$$y(t) = \nu_{10} + \nu_{11}(t-a) + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} k(s, \nu_2(s), \widetilde{\nu_2}(s))ds.$$

Now

$$\begin{aligned}
\sup_{t\in[a,b]} |\nu_1(t) - \nu_2(t)| &= | \sup_{t\in[a,b]} \frac{1}{\Gamma(\alpha+\epsilon)} \int_a^t (t-s)^{\alpha+\epsilon} k(s, \nu_1(s), \widetilde{\nu_1}(s))ds \\
&\qquad - \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha} k(s, \nu_2(s), \widetilde{\nu_2}(s))ds| \\
&\leq \sup_{t\in[a,b]} \left| \frac{1}{\Gamma(\alpha+\epsilon)} \int_a^t (t-s)^{\alpha+\epsilon-1} k(s, \nu_1(s), \widetilde{\nu_1}(s)) - k(s, \nu_2(s), \widetilde{\nu_2}(s))ds \right| \\
&\quad + \sup_{t\in[a,b]} \left| \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} k(s, \nu_2(s), \widetilde{\nu_2}(s)) \left(1 - \frac{\Gamma(\alpha)(t-s)^{\epsilon}}{\Gamma(\alpha+\epsilon)}\right) ds \right| \\
&\leq \frac{1}{\Gamma(\alpha+\epsilon)} \int_a^t (t-s)^{\alpha+\epsilon-1} \sup_{t\in[a,b]} |k(s, \nu_1(s), \widetilde{\nu_1}(s)) - k(s, \nu_2(s), \widetilde{\nu_2}(s))| \, ds \\
&\quad + \sup_{t\in[a,b]} \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} |k(s, \nu_2(s), \widetilde{\nu_2}(s))| \left(1 - \frac{\Gamma(\alpha)(t-s)^{\epsilon}}{\Gamma(\alpha+\epsilon)}\right) ds \\
&\leq \frac{l}{\Gamma(\alpha+\epsilon)} \int_a^t (t-s)^{\alpha+\epsilon-1} \left(\sup_{t\in[a,b]} |\nu_1(s) - \nu_2(s)| + \sup_{t\in[a,b]} |\widetilde{\nu_1}(s) - \widetilde{\nu_2}((s)| \right) ds \\
&\quad + \sup_{t\in[a,b]} \frac{\lambda}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \left(1 - \frac{\Gamma(\alpha)(t-s)^{\epsilon}}{\Gamma(\alpha+\epsilon)}\right) ds \\
&\leq L\|x-y\| I_a^{\alpha+\epsilon}(t-a)^0 + \sup_{t\in[a,b]} \frac{\lambda}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \left(1 - \frac{\Gamma(\alpha)(t-s)^{\epsilon}}{\Gamma(\alpha+\epsilon)}\right) ds.
\end{aligned} \tag{2.25}$$

As

$$\lim_{\epsilon\to 0} \frac{1 - \frac{\Gamma(\alpha)(t-s)^{\epsilon}}{\Gamma(\alpha+\epsilon)}}{\epsilon} = -1.$$

Thus, we can obtain

$$1 - \frac{\Gamma(\alpha)(t-s)^{\epsilon}}{\Gamma(\alpha+\epsilon)} = O(\epsilon).$$

Then,

$$\begin{aligned}\sup_{t\in[a,b]}|\nu_1(t)-\nu_2(t)| &\leq L\|\nu_1-\nu_2\|\frac{1}{\Gamma(\alpha+\epsilon+1)}+\frac{\lambda}{\Gamma()\alpha}\int_a^t(t-s)^{\alpha-1}O(\epsilon)ds \\ &\leq L\|\nu_1-\nu_2\|\frac{1}{\Gamma(\alpha+\epsilon+1)}+\frac{\lambda O(\epsilon)}{\Gamma(\alpha+1)}.\end{aligned} \tag{2.26}$$

Also,

$$\begin{aligned}\sup_{t\in[a,b]}|^*D_a^\beta\nu_1(t)-^*D_a^\beta\nu_2(t)| &= |\frac{\sup_{t\in[a,b]}}{\Gamma(\alpha+\epsilon-\beta)}\int_a^t(t-s)^{\alpha+\epsilon-\beta-1}k(s,\nu_1(s),\widetilde{\nu_1}(s))ds \\ &\quad-\frac{1}{\Gamma(\alpha-\beta)}\int_a^t(t-s)^{\alpha-\beta-1}k(s,\nu_2(s),\widetilde{\nu_2}(s))ds| \\ &\leq\left|\frac{\sup_{t\in[a,b]}}{\Gamma(\alpha-\beta+\epsilon)}\int_a^t(t-s)^{\alpha-\beta+\epsilon-1}k(s,\nu_1(s),\widetilde{\nu_2}(s))-k(s,\nu_2(s),\widetilde{\nu_2}(s))ds\right| \\ &\quad+\left|\frac{\sup_{t\in[a,b]}}{\Gamma(\alpha-\beta)}\int_a^t(t-s)^{\alpha-\beta-1}k(s,\nu_2(s),\widetilde{\nu_2}(s))\left(1-\frac{\Gamma(\alpha-\beta)(t-s)^\epsilon}{\Gamma(\alpha-\beta+\epsilon)}\right)ds\right| \\ &\leq\frac{1}{\Gamma(\alpha-\beta+\epsilon)}\int_a^t(t-s)^{\alpha-\beta+\epsilon-1}\sup_{t\in[a,b]}|k(s,\nu_1(s),\widetilde{\nu_1}(s))-k(s,\nu_2(s),\widetilde{\nu_2}(s))|\,ds \\ &\quad+\frac{\sup_{t\in[a,b]}}{\Gamma(\alpha-\beta)}\int_a^t(t-s)^{\alpha-\beta-1}\,|k(s,\nu_2(s),\widetilde{\nu_2}(s))|\left(1-\frac{\Gamma(\alpha-\beta)(t-s)^\epsilon}{\Gamma(\alpha-\beta+\epsilon)}\right)ds \\ &\leq\frac{L}{\Gamma(\alpha-\beta+\epsilon)}\int_a^t(t-s)^{\alpha+-\beta\epsilon-1}\sup_{t\in[a,b]}\left(|\nu_1(s)-\nu_2(s)|+/|\widetilde{\nu_1}(s)-\widetilde{\nu_2}((s)|\right)ds \\ &\quad+\sup_{t\in[a,b]}\frac{\lambda}{\Gamma(\alpha-\beta)}\int_a^t(t-s)^{\alpha-\beta-1}\left(1-\frac{\Gamma(\alpha-\beta)(t-s)^\epsilon}{\Gamma(\alpha-\beta+\epsilon)}\right)ds \\ &\leq L\|\nu_1-\nu_2\|I_a^{\alpha-\beta+\epsilon}(t-a)^0 \\ &\quad+\sup_{t\in[a,b]}\frac{\lambda}{\Gamma(\alpha-\beta)}\int_a^t(t-s)^{\alpha-\beta-1}\left(1-\frac{\Gamma(\alpha-\beta)(t-s)^\epsilon}{\Gamma(\alpha-\beta+\epsilon)}\right)ds \\ &\leq\frac{L\|\nu_1-\nu_2\|}{\Gamma(\alpha-\beta+\epsilon+1)}+\frac{\lambda O(\epsilon)}{\Gamma(\alpha-\beta+1)}.\end{aligned} \tag{2.27}$$

Now, from Eq. (2.26) and (2.27), we have

$$
\begin{aligned}
\|\nu_1(t)-\nu_2(t)\| &= \sup_{t\in[a,b]}|x(t)-y(t)| + \sup_{t\in[a,b]}|\widetilde{x}(t)-\widetilde{y}(t)| \\
&\leq \frac{L\|\nu_1-\nu_2\|}{\Gamma(\alpha+\epsilon+1)} + \frac{\lambda O(\epsilon)}{\Gamma(\alpha+1)} + \frac{L\|\nu_1-\nu_2\|}{\Gamma(\alpha-\beta+\epsilon+1)} + \frac{\lambda O(\epsilon)}{\Gamma(\alpha-\beta+1)} \\
&\leq \left(\frac{L}{\Gamma(\alpha+\epsilon+1)} + \frac{L}{\Gamma(\alpha-\beta+\epsilon+1)}\right)\|\nu_1-\nu_2\| \\
&\quad + \left(\frac{\lambda}{\Gamma(\alpha+1)} + \frac{\lambda}{\Gamma(\alpha-\beta+1)}\right) \\
&\leq \frac{L\left(\Gamma(\alpha+\epsilon+1)+\Gamma(\alpha-\beta+\epsilon+1)\right)}{\Gamma(\alpha+\epsilon+1)\Gamma(\alpha-\beta+\epsilon+1)}\|x-y\| \\
&\quad + \lambda\left(\frac{\Gamma(\alpha-\beta+1)+\Gamma(\alpha+1)}{\Gamma(\alpha-\beta+1)\Gamma(\alpha+1)}\right)O(\epsilon).
\end{aligned}
$$

$$
\begin{aligned}
\|\nu_1-\nu_2\|\left(\frac{\Gamma(\alpha+\epsilon+1)\Gamma(\alpha+-\beta+\epsilon+1) - L\left(\Gamma(\alpha+\epsilon+1)+\Gamma(\alpha-\beta+\epsilon+1)\right)}{\Gamma(\alpha+\epsilon+1)\Gamma(\alpha-\beta+\epsilon+1)}\right) \\
\leq \lambda\left(\frac{\Gamma(\alpha-\beta+1)+\Gamma(\alpha+1)}{\Gamma(\alpha-\beta+1)\Gamma(\alpha+1)}\right)O(\epsilon)
\end{aligned}
$$

$$
\|x-y\| \leq \frac{\lambda\Gamma(\alpha+\epsilon+1)\Gamma(\alpha-\beta+\epsilon+1)(\Gamma(\alpha-\beta+1)+\Gamma(\alpha+1))O(\epsilon)}{\Gamma(\alpha)\Gamma(\alpha-\beta+1)(\Gamma(\alpha+\epsilon+1)\Gamma(\alpha-\beta+\epsilon+1)-\Gamma(\alpha+\epsilon+1)\Gamma(\alpha-\beta+\epsilon+1))}.
$$

Thus over a compact set P, we have,

$$\|x-y\| \leq O(\epsilon).$$

□

Chapter 3

Quaziliearization

Initially quasilinearization lies in principles of kinematic problems. In modern science many engineering and physical problems are modeled with differential equation. Most of these differential equations are nonlinear. In this chapter we discuss how to solve non-linear problems.
Quasilinearization and invariant imbedding are two helpful systems for getting numerical answers for such kind of problems. These two methods show two orderly ways to deal with boundary value problems. It ought to be stressed that quasilinearization and invariant imbedding are totally different concepts. Quasilinearization is a numerical strategy where as invariant imbedding represents a completely different fomulation of problem. This chapter we focused on quasilinearization numerical technique.

3.1 Quazilinearization

The quasilinerarization was invented by R. E. Belman and R. E. Kalaba [33, 34]. In numerous angles the quasilinearization system is basically a generalized Newton's iterative method which solve non-linear differential equation either it is in system of non-linear equation or individual non-linear equation [3]. With quasilinearization approach, the nonlinear differential equation solved recursively. The quasilinearization approach linearizes the non-linear equation as well as gives an arrangement of functions which generally converges. It converges quaratically to the exact solution and it has monotone convergence. A large portion of this work concerned with numerical solution of non-linear boundary value problem. Motivated from the work of U. Saeed and M. Rehman [3], we employ this approach to fractional order differential equations. Consider the non-linear differential equation [33, 34, 3].

$$ {}^{*}D_{a}^{\alpha}\nu(t) = \phi(\nu(t), t), \qquad 1 < \alpha \leq 2, \quad \nu(0) = c_1, \qquad \nu'(0) = c_2, \qquad a < t < b. \tag{3.1} $$

The function ϕ is now a function of ν. Pick a sensible beginning guess of the function ν, call it $\nu_0(t)$. This initial guess can be acquired in many ways. For problems where convergence requires better initial approximations, this estimation can be acquired by some scientific gadgets, for example, the invariant imbedding system (the method of converting problem with boundary conditions to problem with initial conditions). For engineering problem, this estimation can be acquired from the physical circumstance and by practicing building judgment. Nonetheless, for various issues, a rough initial estimation is sufficient for the strategy to converge. The most self-evident one would be $\nu_0(t) = c_1$. The function ϕ can now be expanded around by the utilization of the Taylor series expansion, ignoring second and higher order term. The Eq.(3.1) becomes[33, 34, 3],

$$ {}^{*}D_{a}^{\alpha}\nu(t) = \phi(\nu_0(t), t) + (\nu(t), \nu_0(t)) \frac{\partial\phi(\nu_0(t), t)}{\partial\nu_0}. \tag{3.2} $$

Now Eq.(3.2) is linear equation with variable co-efficient. Since $\nu_0(t)$ is known. So now we can solve (3.2) for $\nu(t)$ by using any single step or multi-step method and get second approximation ν_1. Expand $\phi(\nu(t), t)$ around ν_1 then Eq.(3.1) becomes

$$ {}^{*}D_{a}^{\alpha}\nu(t) = \phi(\nu_1(t), t) + (\nu(t), \nu_1(t)) \frac{\partial\phi(\nu_1(t), t)}{\partial\nu_1}. \tag{3.3} $$

Solving Eq.(3.3) we get third approximation $\nu_2(t)$ continuing the procedure and recurrence relation can now be constructed in same way as for algebraic equation. The recurrence relation is of the form

$$ {}^{*}D_{a}^{\alpha}\nu_{r+1}(t) = \phi(\nu_r(t), t) + (\nu_{r+1}(t) - \nu_r(t))\phi_{\nu_r(t)}(\nu_r(t), t), \tag{3.4} $$

and the conditions are $\nu_{r+1}(0) = c_1$, $\nu'_{r+1}(0) = c_2$ for $a < t < b$.

Now if $\phi = \phi({}^{*}D_{a}^{\beta}\nu(t), \nu(t), t)$ and $0 < \beta \leq 1$, the function ϕ is function of two unknown

$\nu^{(\beta)}(t)$ and $\nu(t)$. So, we have to take two initial guesses call them $\nu_0^{(\beta)}(t)$ and $\nu_0(t)$. Expand ϕ around $\nu_0(t)$ and $\nu_0^{(\beta)}(t)$ by using Taylor expansion, neglecting higher order term we have

$$\phi({}^*D_a^\beta\nu(t),\nu(t),t) = \phi({}^*D_a^\beta\nu_0(t),\nu_0(t),t) + ({}^*D_a^\beta\nu(t) - {}^*D_a^\beta\nu_0(t))\frac{\partial\phi({}^*D_a^\beta\nu_0(t),\nu_0(t),t)}{\partial^*D_a^\beta\nu_0(t)} + (\nu(t)-\nu_0(t))\frac{\partial\phi({}^*D_a^\beta\nu_0(t),\nu_0(t),t)}{\partial v_0(t)}. \tag{3.5}$$

Thus the linearized form of Eq.(3.1) is

$$ {}^*D_a^\alpha\nu(t) = \phi({}^*D_a^\beta\nu_0(t),\nu_0(t),t) + ({}^*D_a^\beta\nu(t) - {}^*D_a^\beta\nu_0(t))\frac{\partial\phi({}^*D_a^\beta\nu_0(t),\nu_0(t),t)}{\partial^*D_a^\beta\nu_0(t)} + (\nu(t)-\nu_0(t))\frac{\partial\phi({}^*D_a^\beta\nu_0(t),\nu_0(t),t)}{\partial \nu_0(t)}. \tag{3.6}$$

Now Eq. (3.6) is linear differential equation. Solving Eq. (3.6) for $\nu(t)$ and get second approximation. Following the above procedure and the recurrence relation is of the form:

$$ {}^*D_a^\alpha\nu_{r+1}(t) = \phi({}^*D_a^\beta\nu_r(t),\nu_r(t),t) + ({}^*D_a^\beta\nu_{r+1}(t) - {}^*D_a^\beta\nu_r(t))\frac{\partial\phi({}^*D_a^\beta\nu_r(t),\nu_r(t),t)}{\partial^*D_a^\beta\nu_r(t)} + (\nu_{r+1}(t)-\nu_r(t))\frac{\partial\phi({}^*D_a^\beta\nu_r(t),\nu_r(t),t)}{\partial \nu_r(t)}. \tag{3.7}$$

Similarly for higher order non linear differential equations such that $0 < \alpha \leq n$, where n denote the nth order derivative and so nth order differential equation can solve recursively. The recurrence relation is of the form[3]:

$$ {}^*D_a^\alpha\nu_{r+1} = \phi(\nu_r,\nu_r',\therefore,\nu_r^{\lfloor\alpha\rfloor},t) + \sum_{i=0}^{\lfloor\alpha\rfloor}(\nu_{r+1}^{(i)} - \nu_r^{(i)})\phi_{\nu_r^{(i)}}(\nu_r,\nu_r',\therefore,\nu_r^{\lfloor\alpha\rfloor},t). \tag{3.8}$$

3.2 Applications of Quasilinearization

Consider the non-linear algebraic equation

$$t + t^m = c, \tag{3.9}$$

where $c > 0$ and $m > 1$. We see how qusilinearization procedure allow us to acquire a portrayal for a unique solution in term of maximum operation as in [33]. Let

$$t^m = \max_{s \geq 0}[mts^{m-1} - (m-1)s^m], \qquad s \geq 0. \tag{3.10}$$

Thus Eq. (3.9) will be

$$t + \max_{s \geq 0}[mts^{m-1} - (m-1)s^m] = c. \tag{3.11}$$

Eq. (3.11) is equivalent to,

$$t + mts^{m-1} - (m-1)s^m \leq c \tag{3.12}$$

$$t \leq \frac{c + (m-1)s^m}{1 + ms^{m-1}} \tag{3.13}$$

for all $s \geq 0$, as we have equality at $s = t$. So, we can write as

$$t = \min_{s \geq 0}[\frac{c + (m-1)s^m}{1 + ms^{m-1}}] \tag{3.14}$$

an explicit analytic representation for desired solution [33].

Example 3.2.1. *We now find solution of Riccati equation using quazilinearization in term of maximum operation. Consider the most common Riccati equation*

$$\nu'(t) = 1 + \nu^2(t), \tag{3.15}$$

with $\nu(0) = 0,\ 0 \leq t < \frac{\pi}{2}$. *Applying quazilinearization technique and non-linear term* ν^2 *is replaced by,*

$$\nu(t) = \min_{z}\ [-\nu(t)^2 + 2\nu(t)z(t)]. \tag{3.16}$$

Thus the equivalent form of Eq.(3.15)*,*

$$\begin{aligned} \nu'(t) &= 1 + \max_{z}[-z(t)^2 + 2\nu(t)z(t)] \\ &= -\min_{z}[-1 + z(t)^2 - 2z(t)\nu(t)]. \end{aligned} \tag{3.17}$$

The equation has a number of properties inherent to linear equation and in order to solve from Eq. (3.17)*, it is to be observed,*

$$\nu'(t) \leq 1 - z(t)^2 + 2z(t)\nu(t).$$

Thus for some $\psi(t) > 0$ *for* $0 \leq t \leq \frac{\pi}{2}$.

$$\nu(t)' = 1 + 2z(t)\nu(t) - z(t)^2 - \psi(t). \tag{3.18}$$

Eq. (3.18) *is a linearized differential equation which can be solveable analytically using integration factor (I.F). Multiplying I.F*$=e^{\int_0^t -2z(s)ds}$ *to the Eq.* (3.18) *we obtained,*

$$\nu(t) = e^{\int_0^t 2z(s)ds} \int_0^t \left(1 - z^2(s) - g(s)\right) e^{\int_0^s -2z(\xi)d\xi} ds. \tag{3.19}$$

Thus an analytic solution of Ricatti equation is obtained by use of Quasilinearization approach.

Many author used Quazilinearization for solving differential equation numerically. Kaur et al.[35] used Harr wavelets with quasilinearization to approximate nonlinear boundary value problem. Jiwari [36, 37] apply uniform Haar wavelet with quazilinearization to solve Lane Emden equations and Burgers equation.

Example 3.2.2. *[33] Consider another non-linear differential equation*

$$\begin{gathered} \nu''(t) = -1 - 0.49\nu'(t)^2, \\ \nu(0) = 0, \qquad \nu(1) = 0. \end{gathered} \tag{3.20}$$

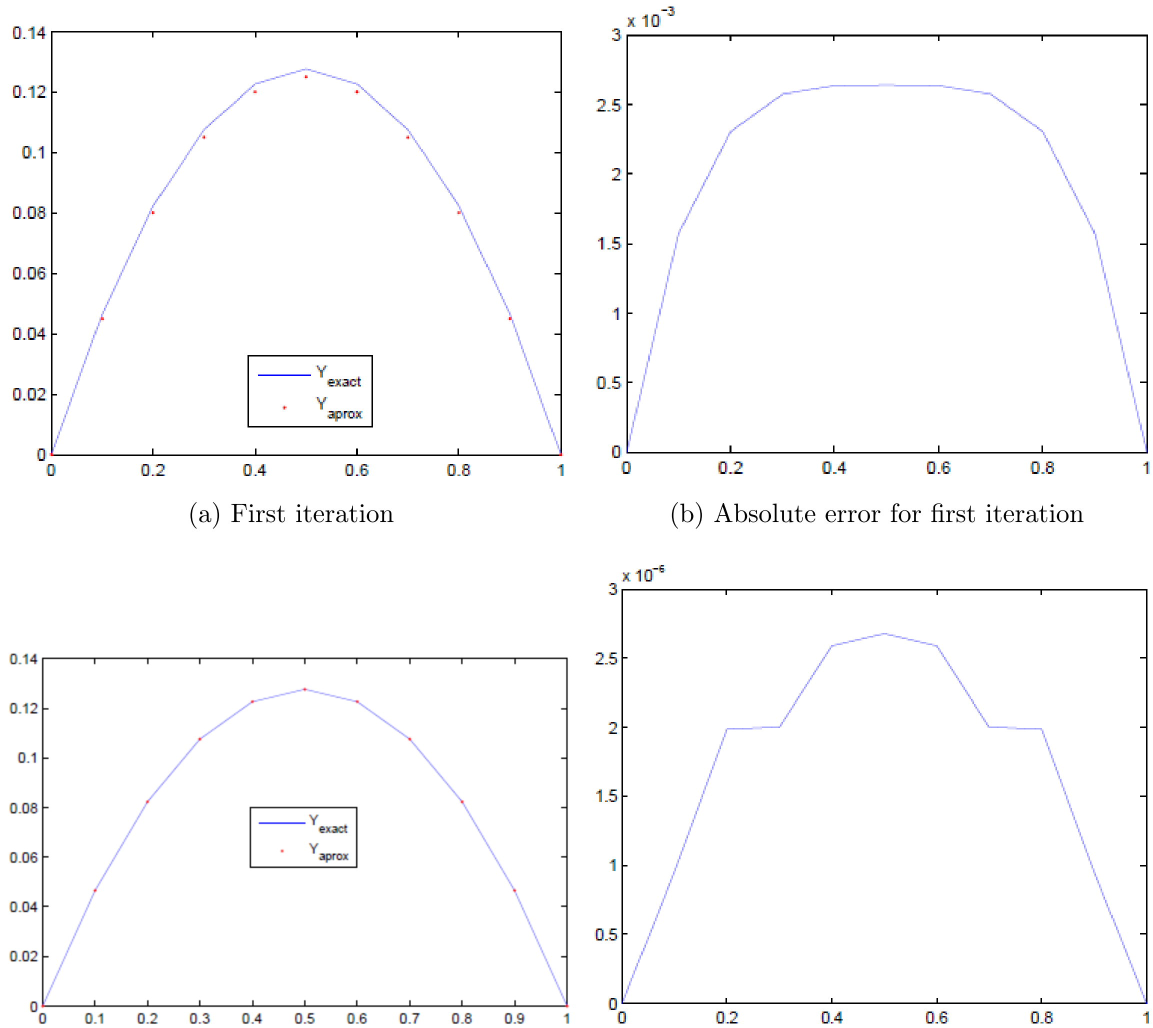
The exact solution of (3.20) *is [33]*

$$\nu(t) = 2.04log\left(\frac{\cos\left(\sqrt{0.49}t - 0.245\right)}{\cos 0.245}\right).$$

Applying quasi-linearization to problem (3.20), *we get*

$$\begin{gathered} \nu''_{r+1}(t) = -0.98\nu'_r(t)\nu'_{r+1}(t) - 1 + 0.49\nu'_r(t)^2, \\ \nu_{r+1}(0) = 0, \qquad v_{r+1}(1) = 0. \end{gathered} \tag{3.21}$$

Applying Rk-4 at each iteration to get approximate solution. Figure (3.1a)-(3.1d) *depicts the exact and approximate solutions graphs after first and second iterations. It shows that solution converges very fast to exact solution.*



(a) First iteration

(b) Absolute error for first iteration

(c) Second iteration

(d) Absolute error for second iteration

Figure 3.1: Comparison of exact and numerical solution for first three iterations.

Chapter 4

A Newton-Cotes Method for Numerical Solution of Fractional Differntial Equation

It is not easy to evaluate exact solution of differential equation of arbitrary order, so we construct a computational scheme for solution of differential equation of arbitrary order. In this chapter, we have developed a quadrature rule based on Newton-Cotes method to solve class of fractional differential equations. We first convert given problem into integral equation and then we apply the proposed method. The proposed technique converts the problem into algebraic equations. And for non-linear problem we use first quasi-linearization approach and then use computational technique.
This chapter is further divided into following four subsections. In first section we develop a new quadrature rule and a method developed for solution of problem is discussed in second section. Convergence of proposed method is explained in third section and in section four we apply the presented method on some specific problems.

4.1 Integration Technique

Engineers and scientist often encounter with integrals that are not easy to solve analytically. For example error function, the Fresnel integral and many other such type of integrals that cannot be evaluated by usual methods of integration defined in calculus [38]. So we need some numerical methods to evaluate such integrals. The basic method involved in approximation such type of integrals is called quadrature rule. Quadrature rule approximate $\int_a^b \phi(t)dt$ by the sum $\sum_{k=0}^{m} \alpha_k \phi(t_i)$. There are generally two types of quadrature rule one is Gaussian quadrature rules and other one is Newton-Cotes quadrature rule.

The method in this section is based on Newton-Cotes quadrature rule. Assume $a = t_0 < t_1 < t_2 < \therefore < t_m = b$ be partition of $[a, b]$ and $h = \frac{b-a}{m}$ be the step size with equally spaced nodes $t_i = a + kh$, $k \in \mathbb{N}$. The assumed numerical technique is of the form.

$$\int_a^b \phi(t)dt = \int_{t_0}^{t_m} \sum_{k=0}^{m} L_{k,m}(t)\phi(t_k)dt + \int_{t_0}^{t_m} \prod_{k=0}^{m}(t-t_k)\frac{\phi^{m+1}\zeta(t)}{(m+1)!}dt. \tag{4.1}$$

where $L_{k,m}(t) = \prod_{k=1,j\neq k}^{m} \frac{(t-t_k)}{(t_j-t_k)}$ is called the Lagrange interpoltaing polynomial of degree m and $E(\phi) = \int_{t_0}^{t_m} \prod_{k=0}^{m}(t-t_k)\frac{\phi^{(m+1)\zeta(t)}}{(m+1)!}dt$ is the error. We obtain the new integration technique using fifth degree Lagrange integrating polynomial. Then Eq. (4.1) takes the form, thus the integration technique is:

$$\begin{aligned}\int_a^b \phi(t)dt =& \frac{95h}{288}\phi(t_0) + \frac{375h}{288}\phi(t_1) + \frac{250h}{288}\phi(t_2) + \frac{250h}{288}\phi(t_3) + \frac{375h}{288}\phi(t_4) + \frac{95h}{288}\phi(t_5) \\ &- \frac{275h^7}{12096}\phi^{(6)}(\zeta).\end{aligned} \tag{4.2}$$

And the error term $E(t) = -\frac{275h^7}{12096}\phi^{(6)}(\zeta)$. Now we can observe from error term $E(t) = \frac{275h^7}{12096}\phi^{(6)}(\zeta)$ that the proposed integration technique gives exact solution for functions of degree less than five or equal to five. We use this technique to develop solution for fractional order differential equation. Now suppose $[a, b]$ is subdivided into $5M$ subintervals $[t_k, t_{km}]$ of equal step size h using $t_k = a + kh$ for $k = 0, 1, 2, \ldots, 5M$. The composite rule for $5M$ subintervals can be expressed as

$$\begin{aligned}\int_a^b \phi(t)dt =& \frac{5h}{288}\left(\phi(a) + \phi(b)\right) + \frac{190h}{288}\sum_{k=1}^{M}\phi(t_{5k}) + \frac{375h}{288}\sum_{k=1}^{M}\phi(t_{5k-1}) + \frac{375h}{288}\phi(t_{5k-4}) \\ &+ \frac{250h}{288}\sum_{k=1}^{M}\phi(t_{5k-1}) + \frac{250h}{288}\sum_{k=1}^{M}\phi(t_{5k-3}).\end{aligned} \tag{4.3}$$

4.2 Development of Method

Consider the class of non-linear differential equation of arbitrary order,

$$^{*}D_0^{\alpha}\phi(t) = \psi(t, \phi(t), ^{*}D_0^{\beta}\phi(t)), \tag{4.4}$$

with $1 \leq \alpha \leq 2$ and $0 \leq \beta \leq 1$. The initial conditions are $\phi(0) = \phi_0$, $\phi'(0) = \phi_1$, where ϕ_0, ϕ_1 are any real numbers. Since the problem is non-liner, we apply quasilinearization technique to make it linear problem. Using quasilinearization technique Eq. (4.4)takes the form:

$$\begin{aligned} ^{*}D_0^{\alpha}\phi^{r+1}(t) =& \psi(t, \phi^r(t), \varphi^r(t)) + (\phi^{r+1}(t) - \phi^r(t))\frac{\partial\psi(t, \phi^r(t), \varphi^r(t))}{\partial\phi} \\ &+ (\varphi^{r+1}(t) - \varphi^r(t))\frac{\partial\psi(t, \phi^r(t), \varphi^r(t))}{\partial\varphi}. \end{aligned} \tag{4.5}$$

Now define $h^r(t) := \psi(t, \phi^r(t), \varphi^r(t)) - \phi^r(t)\frac{\partial\psi(t,\phi^r(t),\varphi^r(t))}{\partial\phi} - \varphi^r(t)\frac{\partial\psi(t,\phi^r(t),\varphi^r(t))}{\partial\varphi}$, where $\varphi(t) :=^{*} D_0^{\beta}\phi^r(t)$.
$\Phi^r(t) := \frac{\partial\psi(t,\phi^r(t),\varphi^r(t))}{\partial\phi}$, $\Psi^r(t) := \frac{\partial\psi(t,\phi^r(t),\varphi^r(t))}{\partial\varphi}$.
Thus Eq. (4.5) becomes

$$^{*}D_0^{\alpha}\phi^{r+1}(t) = h^r(t) + \Psi^r(t)^{*}D_0^{\beta}\phi^{r+1}(t) + \Phi^r(t)\phi^{r+1}(t), \tag{4.6}$$

with initial conditions $\phi^{r+1}(0) = \phi_0$ and $D\phi^{r+1}(0) = \phi_1$. Where D is ordinary differential operator. Now applying Riemann-Liouville integral operator I_0^{α} on both sides of Eq. (4.6).

$$\phi^{r+1}(t) - \phi_0 - t\phi_1 = H^r(t) + \int_0^t \left(\frac{(t-\tau)^{(\alpha-\beta-1)}\Psi^r(t)}{\Gamma(\alpha-\beta)} + \frac{\Phi^r(t)(t-\tau)^(\alpha-1)}{\Gamma(\alpha)}\right)\phi^{r+1}(\tau)d\tau.$$

$$\phi^{r+1}(t) = G^r(t) + \int_a^t \kappa^r(t,\tau)\phi^{r+1}(\tau)d\tau, \tag{4.7}$$

where $G^r(t) = H^r(t) + \phi_0 + \phi_1(t-0)$ and $H^r(t) = I_0^{\alpha}h^r(t)$.
And $\kappa^r(t,\tau) = \left(\frac{(t-\tau)^{(\alpha-\beta-1)}\Psi^r(t)}{\Gamma(\alpha-\beta)} + \frac{\Phi^r(t)(t-\tau)^{(\alpha-1)}}{\Gamma(\alpha)}\right)$ is the kernel function. Thus to approximate solution we combine trapezoidal, Simpson's $\frac{1}{3^{rd}}$, Simpson $\frac{3}{8^{th}}$, Boole's rule and modified Boole's rule.

Let $a = t_0 < t_1 < \cdots < t_n = b$ be the partion of interval $[a, b]$ and $h = \frac{b-a}{n}$ be the step size, where n is any positive integer. Let $t_k = t_0 + kh$ for $k = 0, 1, 2, 3, \ldots$ are equally spaced nodes. We write $\phi^r(t_i) = \phi_i^r$, $G^r(t_i) = G_i^r$, $\kappa^r(t_i, t_j) = \kappa_{ij}^r$ to make notation simple. We approximate solution at nodes $t = t_k$. Thus Eq. (4.7) takes the form

$$\phi^{r+1}(t_k) = G^r(t_k) + \int_a^{t_k} \kappa^r(t_k, \tau)\phi^{r+1}(\tau)d\tau. \tag{4.8}$$

At $k = 0$, $\phi^{r+1}(t_0) = G^r(t_0)$. When $k = 5j$, $j \in \mathbb{N}$, we use repeatedly modified Boole's rule. Thus Eq. (4.8) becomes

$$\begin{aligned}
\phi^{r+1}(t_{5j}) &= G^r(t_{5j}) + \int_a^{t_{5j}} \kappa^r(t_{5j}, \tau)\phi^{r+1}(\tau)d\tau. \\
&= G_{5j}^r + T_j + \int_{t_{5(j-1)}}^{t_{5j}} \kappa_{5j}^r(\tau)\phi^{r+1}(\tau)d\tau \\
&= G_{5j}^r + T_j + \frac{5h}{288}[19\kappa_{5j,5(j-1)}^r\phi_{5(j-1)}^{r+1} + 75\kappa_{5j,5j-4}^r\phi_{5j-4}^{r+1} + 50\kappa_{5j,5j-3}^r\phi_{5j-3}^{r+1} \\
&\quad + 50\kappa_{5j,5j-2}^r\phi_{5j-2}^{r+1} + 75\kappa_{5j,5j-1}^r\phi_{5j-1}^{r+1} + 19\kappa_{5j,5j}^r\phi_{5j}^{r+1}].
\end{aligned} \tag{4.9}$$

Where,

$$\begin{aligned}
T_j &= \sum_{i=1}^{j-1} \int_{5i-5}^{5i} \kappa^r(t_{5j}, \tau)\phi^{r+1}(\tau)d\tau \\
&= \sum_{i=1}^{j-1} \frac{5h}{288}[19\kappa_{5j,5(i-1)}^r\phi_{5(i-1)}^{r+1} + 75\kappa_{5j,5i-4}^r\phi_{5i-4}^{r+1} + 50\kappa_{5j,5i-3}^r\phi_{5i-3}^{r+1} \\
&\quad + 50\kappa_{5j,5i-2}^r\phi_{5i-2}^{r+1} + 75\kappa_{5j,5i-1}^r\phi_{5i-1}^{r+1} + 19\kappa_{5j,5i}^r\phi_{5i}^{r+1}],
\end{aligned} \tag{4.10}$$

for $j = 1$, $T_1 = 0$. Now when $k = 5j + 1$ for $j = 0, 1, 2, 3, \cdots$, we use combination of trapezoidal and modified Boole's rule. Thus Eq. (4.8) takes the form

$$\begin{aligned}
\phi^{r+1}(t_{5j+1}) &= G^r(t_{5j+1}) + \int_a^{t_{5j+1}} \kappa^r(t_{5j+1}, \tau)\phi^{r+1}(\tau)d\tau. \\
&= G_{5j+1}^r + T_{j+1} + \int_{t_{5j}}^{t_{5j+1}} \kappa_{5j+1}^r(\tau)\phi^{r+1}(\tau)d\tau \\
&= G_{5j+1}^r + T_{j+1} + \frac{h}{2}\kappa_{5j+1,5j}^r\phi_{5j}^{r+1} + \frac{h}{2}\kappa_{5j+1,5j+1}^r\phi_{5j+1}^{r+1},
\end{aligned} \tag{4.11}$$

where

$$T_{j+1} = \sum_{i=1}^{j} \int_{5i-5}^{5i} \kappa^r(t_{5j},\tau)\phi^{r+1}(\tau)d\tau$$
$$= \sum_{i=1}^{j} \frac{5h}{288}[19\kappa^r_{5j,5(i-1)}\phi^{r+1}_{5(i-1)} + 75\kappa^r_{5j,5i-4}\phi^{r+1}_{5i-4} + 50\kappa^r_{5j,5i-3}\phi^{r+1}_{5i-3}$$
$$+ 50\kappa^r_{5j,5i-2}\phi^{r+1}_{5i-2} + 75\kappa^r_{5j,5i-1}\phi^{r+1}_{5i-1} + 19\kappa^r_{5j,5i}\phi^{r+1}_{5i}], \tag{4.12}$$

with $T_1 = 0$. And for $k = 5j+2,\ j = 0,1,2,3,\cdots$ combination of Simpson $\frac{1}{3^{rd}}$ and modified Boole's rule is used and (4.8) takes the form

$$\phi^{r+1}(t_{5j+2}) = G^r(t_{5j+2}) + \int_a^{t_{5j+2}} \kappa^r(t_{5j+2},\tau)\phi^{r+1}(\tau)d\tau.$$
$$= G^r_{5j+2} + T_{j+1} + \int_{t_{5j}}^{t_{5j+2}} \kappa^r_{5j+2}(\tau)\phi^{r+1}(\tau)d\tau$$
$$= G^r_{5j+2} + T_{j+1} + \frac{h}{3}\kappa^r_{5j+2,5j}\phi^{r+1}_{5j} + \frac{4h}{3}\kappa^r_{5j+2,5j+1}\phi^{r+1}_{5j+1} + \frac{h}{3}\kappa^r_{5j+2,5j+2}\phi^{r+1}_{5j+2}. \tag{4.13}$$

When $k = 5j+3$ for $j = 0,1,2,3,\ldots,$ we use combination of Simpson $\frac{3}{8}$ and modified Boole's rule. Thus Eq. (4.8) takes the form

$$\phi^{r+1}(t_{5j+3}) = G^r(t_{5j+3}) + \int_a^{t_{5j+3}} \kappa^r(t_{5j+3},\tau)\phi^{r+1}(\tau)d\tau.$$
$$= G^r_{5j+3} + T_{j+1} + \int_{t_{5j}}^{t_{5j+3}} \kappa^r_{5j+3}(\tau)\phi^{r+1}(\tau)d\tau$$
$$= G^r_{5j+3} + T_{j+1} + \frac{3h}{8}\kappa^r_{5j+3,5j}\phi^{r+1}_{5j} + \frac{9h}{8}\kappa^r_{5j+3,5j+1}\phi^{r+1}_{5j+1} + \frac{9h}{8}\kappa^r_{5j+3,5j+2}\phi^{r+1}_{5j+2}$$
$$+ \frac{3h}{8}\kappa^r_{5j+3,5j+3}\phi^{r+1}_{5j+3}. \tag{4.14}$$

And if $k = 5j + 4$ for $j = 0, 1, 2, 3, \ldots$, combination of modified Boole's rule and Boole rule is used. Thus Eq. (4.8) takes the form

$$\begin{aligned}
\phi^{r+1}(t_{5j+4}) &= G^r(t_{5j+4}) + \int_a^{t_{5j+4}} \kappa^r(t_{5j+4}, \tau)\phi^{r+1}(\tau)d\tau. \\
&= G^r_{5j+4} + T_{j+1} + \int_{t_{5j}}^{t_{5j+4}} \kappa^r_{5j+4}(\tau)\phi^{r+1}(\tau)d\tau \\
&= G^r_{5j+4} + T_{j+1} + \frac{14h}{45}\kappa^r_{5j+4,5j}\phi^{r+1}_{5j} + \frac{64h}{45}\kappa^r_{5j+4,5j+1}\phi^{r+1}_{5j+1} + \frac{24h}{45}\kappa^r_{5j+4,5j+2}\phi^{r+1}_{5j+2} \\
&\quad + \frac{64h}{45}\kappa^r_{5j+4,5j+3}\phi^{r+1}_{5j+3} + \frac{14h}{45}\kappa^r_{5j+4,5j+4}\phi^{r+1}_{5j+4}.
\end{aligned} \tag{4.15}$$

Thus Eqs. (4.9)-(4.15) leads to system of algebraic equations at rth iteration. Algebraic equation written in matrix form as $N^r\Phi^r = \widetilde{G^r}$, where $N^r = I - K^r$, $\Phi^r = [\phi^{r+1}(t_0), \phi^{r+1}(t_1), \phi^{r+1}(t_2), \ldots, \phi^{r+1}(t_m)]^T$, $\widetilde{G^r} = [G^r(t_0), G^r(t_1), G^r(t_2), \ldots, G^r(t_m)]^T$. For $n = 5$ the matrix N^r can be written as,

$$N^r = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
\frac{-h}{2}\kappa^r_{10} & 1 - \frac{h}{2}\kappa^r_{11} & 0 & 0 & 0 & 0 \\
-\frac{h}{3}\kappa^r_{20} & -\frac{4h}{3}\kappa^r_{21} & 1 - \frac{h}{3}\kappa^r_{22} & 0 & 0 & 0 \\
-\frac{3h}{8}\kappa^r_{30} & -\frac{9h}{8}\kappa^r_{31} & -\frac{9h}{8}\kappa^r_{32} & 1 - \frac{3h}{8}\kappa^r_{33} & 0 & 0 \\
-\frac{14h}{45}\kappa^r_{40} & -\frac{32h}{45}\kappa^r_{41} & -\frac{12h}{45}\kappa^r_{42} & -\frac{32h}{45}\kappa^r_{43} & 1 - \frac{14h}{45}\kappa^r_{44} & 0 \\
-\frac{95h}{288}\kappa^r_{50} & -\frac{375h}{288}\kappa^r_{51} & -\frac{250h}{288}\kappa^r_{52} & -\frac{250h}{288}\kappa^r_{53} & -\frac{375h}{288}\kappa^r_{54} & 1 - \frac{95h}{h288}\kappa^r_{55}
\end{bmatrix}.$$

4.3 Convergence

Whenever we approximate the solution of our problem by using some numerical technique we must discuss about convergence, consistency and stability of numerical scheme. In this section we make convergence analysis of our method.

Definition 4.3.1. *A numerical scheme is convergent if*

$$|\phi_n - \phi| \longrightarrow 0, \qquad as \qquad n \longrightarrow 0,$$

where ϕ_n is sequence of approximate solutions and ϕ is exact solution. And the speed at which the approximate solution approach exact solution is called rate of convergence.

Definition 4.3.2. *Let ϕ_n be the sequence converges to ϕ. And there exist α with +ive constant λ. Such that*

$$\lim_{n\to\infty} \frac{|\phi_{n+1} - \phi|}{|\phi_n - \phi|^{\alpha}} = \lambda,$$

then the sequence ϕ_n converges to ϕ of order α with error constanat λ.
If $\alpha = 1$ then scheme is linearly convergent.
If $\alpha = 2$ then scheme is quadratically convergent.
If $\alpha = 3$ then scheme is cubic convergent.
If $0 < \alpha < 1$ then it is sub-linear convergent.
If $1 < \alpha < 2$ then the numerical scheme is super-linear convergent.

Example 4.3.3. *Bisection method is linear convergent. Newton's method is quadratic and secant method is superlinear convergent.*

Definition 4.3.4. *A numerical technique is consistant if by increasing number of nodes error reduces.*

Definition 4.3.5. *A numerical scheme is stable if error does not grow un-boundedly.*

Theorem 4.3.1. *Suppose $|\psi_{t_i}(t, t_1, t_2)| \leq M_i$, $M_i > 0$ for $i = 1, 2$. Furthermore, assume*

(i) $M\left(\frac{(b-a)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{(b\ -\ a)^{\alpha}}{\Gamma(1+\alpha)}\right) < 01,$

(ii) $\frac{\Gamma(\alpha-\beta+1)\Gamma(1+\alpha)\xi}{\Gamma(\alpha-\beta+1)\ \Gamma(1+\alpha) - M\left(\Gamma(1+\alpha)(b-a)^{\alpha-\beta} + \Gamma(1+\alpha-\beta)\ (b-a)^{\alpha}\right)} \leq \frac{1}{4},$

where $\xi \geq \|T\phi^r\|$.
Then the iterative scheme (4.6) converges quadratically to unique solution.

Proof. To show the quadratic convergence for iterative scheme (4.6), it is sufficient to show that the the iterative scheme given in (4.8) converges quadratically. Define a linear operator $T : C^{\lceil\alpha\rceil}[a, b] \longrightarrow C^{\lceil\alpha\rceil}[a, b]$ as

$$(T\phi^{r+1})(t) = \phi^{r+1}(t) - G^r(t) - \int_a^t \kappa^r(t, \tau)\phi^{r+1}(\tau)d\tau. \tag{4.16}$$

The Frechet derivative of T is given by

$$(T'_{\phi^{r+1}})g(t) = g(t) - \int_a^t \kappa^r(t,\tau)h(\tau)d\tau. \tag{4.17}$$

Now for $g \in C[a,b]$ with $\|g\| < 1$, we have

$$\begin{aligned}
|\left(I - T'_{\phi^{r+1}}\right) g(t)| &= |\int_a^t \kappa^r(t,\tau)g(\tau)d\tau| \leq |\int_a^t \kappa^r(t,\tau)g(\tau)|d\tau \\
&\leq \int_a^t |\frac{\Psi^r(t)\ (t-\tau)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} + \frac{\Phi^r(t)(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}|d\tau \\
&\leq \int_a^t \frac{M(t-\tau)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} + \frac{M(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}d\tau \\
&= M\left(\frac{(t-a)^{\alpha-\beta}}{(\alpha-\beta)\Gamma(\alpha-\beta)} + \frac{(t-a)^{\alpha}}{\alpha\Gamma(\alpha)}\right) \\
&\leq M\left(\frac{(b-a)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\right).
\end{aligned}$$

Thus, $\|I - T'_{\phi^{r+1}}\| \leq M\left(\frac{(b-a)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\right) < 1$, and by Lemma 1.6.24, $T'_{\phi^{r+1}}$ is invertible and $T'^{-1}_{\phi^{r+1}} = \sum_{n=0}^{\infty}(I - T'_{\phi^{r+1}})$. The norm $\|T'^{-1}_{\phi^{r+1}}\| \leq \frac{\Gamma(\alpha-\beta+1)\Gamma(1+\alpha)}{\Gamma(\alpha-\beta+1)\Gamma(1+\alpha) - M\left(\Gamma(\alpha+1)(b-a)^{\alpha-\beta} + \Gamma(\alpha-\beta+1)(b-a)^{\alpha}\right)}$. For $u_{n+1}, \tilde{u}_{n+1} \in C[a,b]$, we have

$$\begin{aligned}
|(T'_{u_{n+1}} - T'_{\tilde{u}_{n+1}})h(t)| &\leq M_1\frac{(b-a)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}|z_n - \tilde{z}_n| + M_2\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}|u_n - \tilde{u}_n| \\
&\leq \Upsilon\|u_n - \tilde{u}_n\|.
\end{aligned}$$

Hence $\|T'_{u_{n+1}} - T'_{\tilde{u}_{n+1}}\| \leq \Upsilon\|u_n - \tilde{u}_n\|$, where $\Upsilon := \min\left\{M_1\frac{(b-a)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}, M_2\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\right\}$. Also note that $\|T'^{-1}_{\phi^{r+1}}T(\phi^{r+1})\| \leq \xi\|T'^{-1}_{\phi^{r+1}}\|$. Moreover $T'_{\phi^{r+1}}(\phi^{r+2} - \phi^{r+1}) = -T(\phi^{r+1})$. This leads to

$$\begin{aligned}
\phi^{r+2}(t) - \phi^{r+1}(t) - &\int_a^t \kappa^r(t,\tau)\left(\phi^{r+2}(\tau) - \phi^{r+1}(\tau)\right)d\tau \\
&= -\phi^{r+1}(t) + G^r(t) + \int_a^t \kappa^r(t,\tau)\phi^{r+1}(\tau)d\tau
\end{aligned}$$

$$\phi_{r+2} = G^r(t) + \int_a^t \kappa^r(t,\tau)\phi^{r+1}(\tau)d\tau. \tag{4.18}$$

Thus, for the choice of $b_0 = \|T'^{-1}_{\phi^{r+1}}\|$, $k = 2\Upsilon$ and $a_0 = \|T'^{-1}_{\phi^{r+1}}T(\phi^{r+1})\|$ all the hypothesis of the Kantorovich Theorem 1.6.4 are satisfied. Thus the iterative scheme (4.6) converges quadratically to the unique solution. $\square$

For fixed n the quadrature rule for equation (4.8) is given by

$$\phi^{r+1,m}(t_i) = G^r(t_i) - \sum_{j=1}^{m} w_j \kappa^r(t_i, t_j)\phi^{r+1}(t_j). \tag{4.19}$$

where $m = 0, 1, 2, 3, \cdots$ and w_j, $j = 1, 2, 3, \cdots$ are the weights. Following theorem , ensures the convergence of quadrature scheme.

Theorem 4.3.2. *[39] Assume the weight function* $|w_j| \leq \frac{\Gamma(\alpha)}{(b-a)^{\alpha-1}}$*, then*

$$|\phi^{r+1,m}(t_i) - \phi^{r+1}(t_i)| \leq \frac{R_{mi}}{1 - |w_m|\kappa_{n,im}} \prod_{j=0}^{m-1} \left(1 + \frac{|w_j| K_{n,ij}}{1 - |w_m|\kappa_{n,im}}\right),$$

where

$$R_{mi} = \sum_{j=1}^{m} w_j \kappa^r(t_i, t_j)\phi^{r+1}(t_j) - \int_a^t \kappa^r(t_i, \tau)\phi^{r+1}(\tau) d\tau.$$

4.4 Numerical Examples

In this section we solve fractional order differential equation using our method. For non-linear problem we combine quadrature rule with quasilinearization. For efficiency of our method we compare our results with exact solution and results obtained from other methods.

Example 4.4.1. *Consider the problem*

$$\begin{aligned} D^\alpha \phi(t) + \phi(t) = t^\alpha + \Gamma(1+\alpha), \quad t > 0, \quad 0 \leq \alpha \leq 2, \\ \phi(0) = 0, \quad \phi'(0) = 0, \end{aligned}, \tag{4.20}$$

with exact solution t^α*. The corresponding integral form of Eq.* (4.20) *is*

$$\phi(t) = I_0 * \alpha(t^\alpha + \Gamma(1+\alpha) - \phi(y)). \tag{4.21}$$

Haar wavelet operational matrix method used by F.A. Shah et al. [1] to solve (4.20)*. Absolute error for different number of interval* n *and for different* α *are given in Table 4.1. Table 4.2 predict that our method give robust results. Figure 4.1 shows that graph of solution for non-integer order* α *equation lie inside the graphs of integer order* α*.*

Table 4.1: Absolute error at different value of α.

n	$\alpha = 1$	$\alpha = 1.25$	$\alpha = 1.75$	$\alpha = 2$
10	1.1102×10^{-16}	9.2680×10^{-3}	8.2613×10^{-4}	1.7447×10^{-4}
50	1.4408×10^{-16}	1.8643×10^{-3}	7.2198×10^{-5}	2.3318×10^{-6}
100	3.3306×10^{-16}	8.1507×10^{-4}	2.2231×10^{-5}	3.0558×10^{-7}
200	3.3306×10^{-16}	3.4923×10^{-4}	6.7221×10^{-6}	3.9073×10^{-8}
500	5.5511×10^{-16}	1.1234×10^{-4}	1.3659×10^{-6}	2.5342×10^{-9}

Table 4.2: Comparison of absolute error between presented method and Harr Wavelet method [1] for $\alpha = 1$.

n	Presented method	Haar Wavelet [1]
8	$1.43E-004$	$3.9E-004$
16	$1.91E-005$	$8.2E-005$
32	$2.46E-006$	$2.5E-005$
64	$3.12E-007$	$4.9E-006$

Example 4.4.2. *Consider fractional harmonic oscillator equation*

$$\begin{aligned} {}^{*}D_0^{\alpha}\phi(t) + \omega^{\alpha-\beta}\, {}^{*}D_0^{\beta}\phi(t) = 0 \qquad 1 \le \alpha \le 2, \qquad 0 \le \beta \le 1, \\ \phi(0) = \phi_0, \qquad \phi'(0) = \phi_1. \end{aligned} \tag{4.22}$$

Where ϕ_0 and ϕ_1 are the displacement and velocity respectively at $t = 0$, ω is angular frequency of harmonic oscillator. Harmonic oscillator equation arises in many fields like in classical mechanics, quantum mechanics and in electronic engineering. The corresponding equivalent integral equation for Eq.(4.22) is

$$\phi(t) = \phi_0 + \phi_1 t - \omega^{\alpha-\beta} \int_0^t \frac{(t-\tau)^{\alpha-\beta-1}\phi(\tau)}{\Gamma(\alpha-\beta)} d\tau. \tag{4.23}$$

Let $\phi_0 = 0$, $\phi_1 = 1$ then exact solution of (4.22) is $\phi(t) = \frac{1}{\omega}\sin(\omega t)$ for $\alpha = 2$, $\beta = 0$ and $\phi(t) = \frac{1}{\omega}(1 - e^{-\omega t})$ is the exact solution for $\alpha = 2$ and $\beta = 1$. It is observed from Figure 4.2 that graphs of exact and approximate solution overlap for integer values of α and β. For fractional α and β the approximate solutions approaches the exact solution as α and β close to integer values.

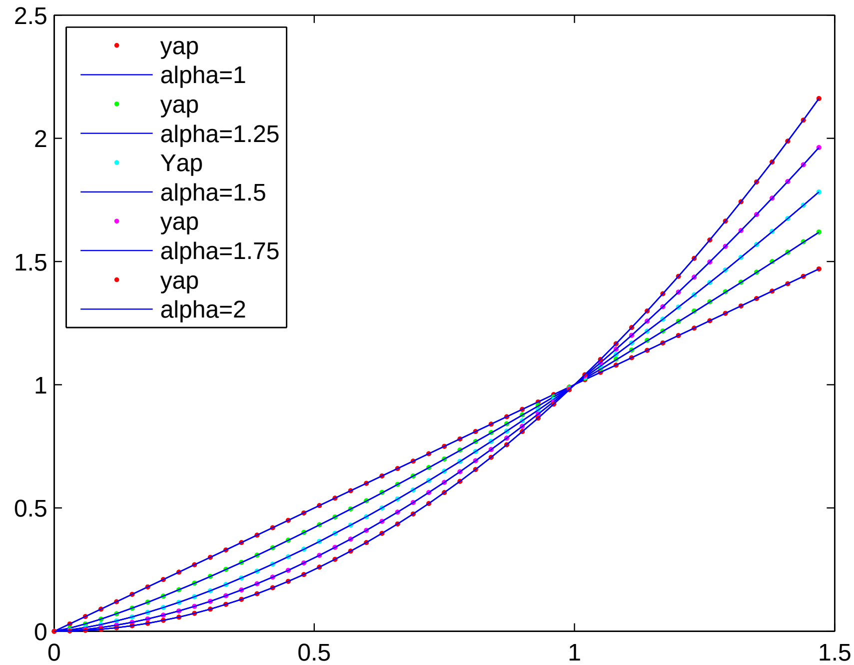


Figure 4.1: Numerical and exact solution for different for $\alpha = 1, 1.25, 1.75, 1.5, 2$.

Example 4.4.3. *Let us consider another problem of fractional order*

$$^{*}D_0^{\alpha}\phi(t) + \phi(t) = 0, \quad 0 < \alpha \leq 2, \quad \phi(0) = 1, \quad \phi'(0) = 0. \tag{4.24}$$

$\phi(t) = E_{\alpha}(-t^{\alpha})$ *is the exact solution of equation* (4.24)*. Applying Riemann liouville integral operator on both side of Eq.* (4.24) *and the integral form of* (4.24) *is*

$$\phi(t) = 1 + \phi'(0)t - \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\phi(\tau)d\tau \tag{4.25}$$

$\phi(t) = e^{-t}$ *is exact solution when* $\alpha = 1$*, when* $\alpha = 2$ *the exact solution is given as* $\phi(t) = \cos(t)$*. Absolute error are shown in Table 4.3 for different values of* α*. Table 4.3 shows the solutions obtained by presented method are close to exact solutions. Graph of numerical and exact solutions are shown in Figure 4.3 which shows that numerical solutions .*

Example 4.4.4. *Consider a multi-term ordinary differential equation of arbitrary or-*

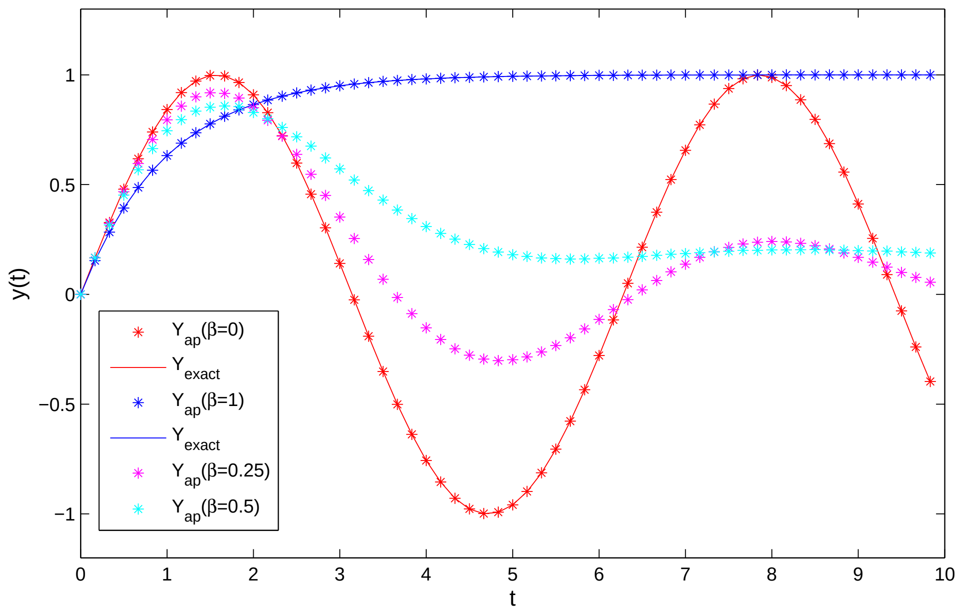


Figure 4.2: Numerical and exact solution for different values of α.

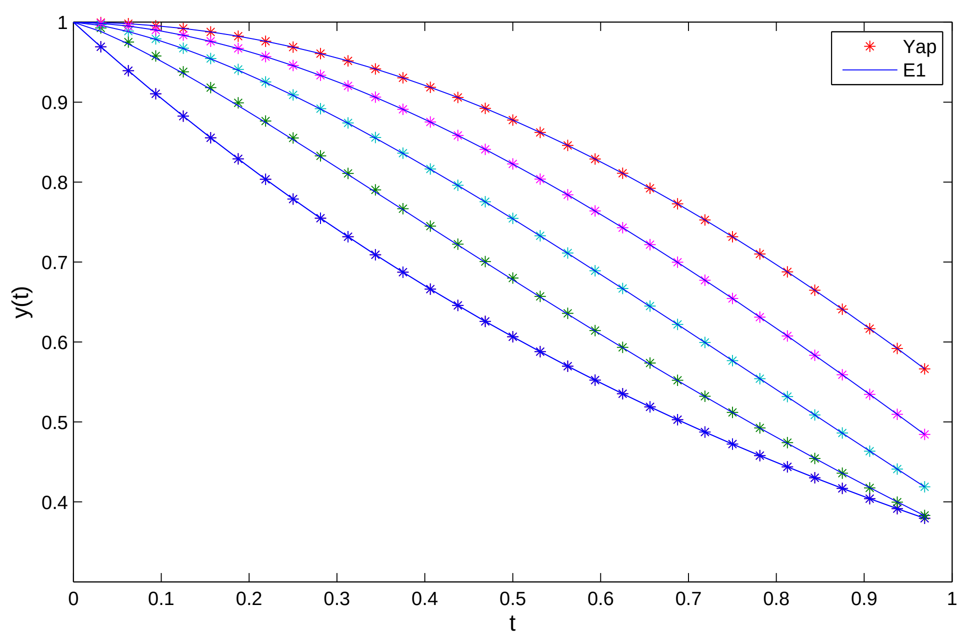


Figure 4.3: Numerical solutions for different values of $\alpha = 1, 1.25, 1.5, 1.75, 2$.

Table 4.3: Absolute error at different value of α.

n	$\alpha = 1$	$\alpha = 1.5$	$\alpha = 2$	$\alpha = 2.5$	$\alpha = 3$
10	7.5513×10^{-5}	5.7810×10^{-3}	8.3267×10^{-5}	2.3796×10^{-4}	8.3334×10^{-5}
50	6.5350×10^{-7}	5.3059×10^{-4}	1.0094×10^{-6}	4.2819×10^{-6}	6.6667×10^{-7}
100	8.2505×10^{-8}	1.8789×10^{-4}	1.3012×10^{-7}	7.5762×10^{-7}	8.3334×10^{-8}
200	1.0364×10^{-8}	6.6469×10^{-5}	1.6469×10^{-8}	1.3395×10^{-7}	1.0416×10^{-8}
500	6.6533×10^{-10}	1.6877×10^{-5}	1.0641×10^{-9}	1.3556×10^{-8}	6.6667×10^{-10}

der,

$$\begin{aligned} &{}^{*}D_0^{\alpha}\phi(t) + {}^{*}D_0^{\beta}\phi(t) + \phi(t) = 8, \qquad t > 0, \qquad 0 \leq \beta \leq 1 \qquad 1 \leq \alpha \leq 2, \\ &\phi(0) = 0, \qquad \phi'(0) = 0. \end{aligned} \tag{4.26}$$

For $\alpha = 2$ *and* $\beta = \frac{3}{2}$, (4.26) *is the most famous Bragley Torvik equation. After applying integral operator then the corresponding integral representation of* (4.26) *is*

$$\phi(t) = I_0^{\alpha}(8 - \phi(t)) - I_0^{\alpha-\beta}(\phi(t))$$

For $\alpha = 2$ *and* $\beta = 0.5$ *Y. Li et al. [40] used generalized Block-pulse operational matrix method and A. Arikoglu [41] apply decomposition method and transform method. A comparison with these method are shown in Table 4.4. It is observed that our results are in robust agreement with ADM, FDTM and generalized Block-pulse operational matrix method.*

Numerical solutions for non-integer values of β *are made in Figure 4.4. For* $\alpha = 2$ *and* $0 \leq \beta \leq 1$ *numerical solutions are depicted in Figure 4.5.*

For $\alpha = 2$ *and* $\beta = 1$, *the exact solution is* $\phi(t) = -8e^{\frac{-t}{2}} cos(\frac{\sqrt{3}t}{2}) + \frac{8}{\sqrt{3}} e^{\frac{-t}{2}} sin(\frac{\sqrt{3}t}{2}) + 8$. *It is observed for* $n = 20$, *the maximum absolute error is* 7.7256×10^{-5}. *For* $\alpha = 2$ *and* $\beta = 0$ *the exact solution is* $\phi(t) = 4 - 4cos(\sqrt{2}t)$ *and maximum absolute error for* $n = 20$ *is* 1.9728×10^{-4}.

Example 4.4.5. *Consider a non-linear ordinary differential equation of fractional order. The Eq.* (4.27) *explains the cooling effect of semi-infinite body by radiation.*

$$\begin{aligned} &{}^{*}D_0^{\alpha}\phi(t) = (1 - \phi(t))^4, \qquad t > 0, \qquad 1 \leq \alpha < 2, \\ &\phi(0) = 0, \qquad \phi'(0) = 0. \end{aligned} \tag{4.27}$$

Table 4.4: Comparison of our method with adomian decomposition (ADM) [41] and block pulse operational matrix method (BPOM) [40] for $\alpha = 2$ and $\beta = 0.5$.

t	$\phi_{ADM}, m = 30$	$\phi_{BPOM}, m = 1000$	$\phi_{App}, m = 20$	ϕ_{Exact} [41]
0.1	0.03987	0.03975	0.03979	0.03975
0.2	0.15851	0.15704	0.15718	0.15703
0.3	0.34841	0.34737	0.34825	0.34737
0.4	0.62208	0.60469	0.60543	0.60469
0.5	0.96004	0.92176	0.92260	0.92176
0.6	1.36309	1.29045	1.29160	1.29045
0.7	1.82625	1.70200	1.70321	1.70200
0.8	2.34422	2.14728	2.15169	2.14728
0.9	3.52146	2.61699	2.61954	2.61700

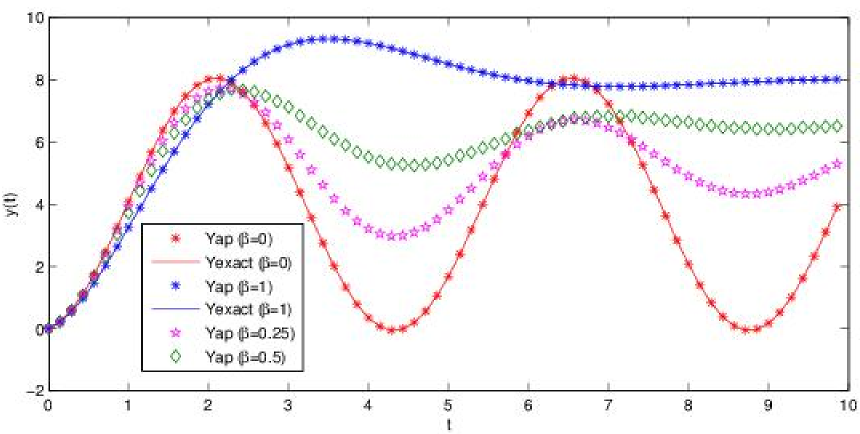


Figure 4.4: Approximate solution for different values of β.

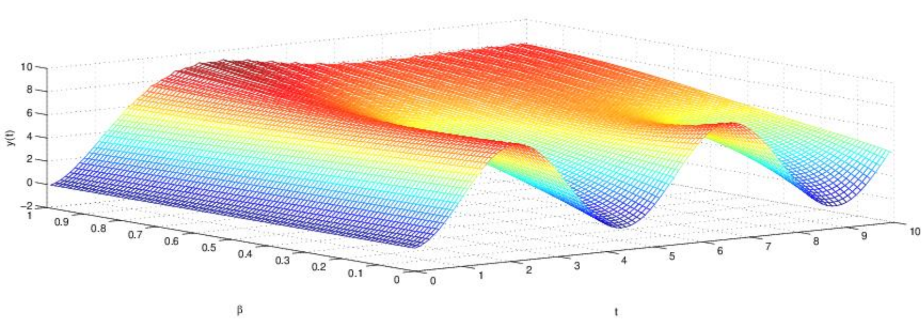


Figure 4.5: Approximate solutions for $0 \leq \beta \leq 1$ and for $\alpha = 2$.

A. Al-Rabtah et al. [2] implemented Spline function to solve (4.27). *Adomian decomposition and fractional difference method used by S. Momani et al. [42] to solve* (4.27).
We use quasilinerization technique to Eq. (4.27) *to make it linear. Thus Eq.* (4.27) *becomes*

$$\begin{aligned} {}^{*}D_0^{\alpha}\phi^{r+1}(t) =& (\phi^r)^4(t) + 4(\phi^r)^3(t)\left(\phi^{r+1}(t) - \phi^r(t)\right) - 4(\phi^r)^3(t) - 12(\phi^r)^2(t)\left(\phi^{r+1}(t) - \phi^r(t)\right) \\ &+ 6(\phi^r)^2 + 12(\phi^r)\left(\phi^{r+1}(t) - \phi^r(t)\right) - 4\phi^{r+1}(t) + 1, \end{aligned} \tag{4.28}$$

with initial condition $\phi^{r+1}(0) = 0$, *and the corresponding integral representation of* (4.27) *is*

$$\begin{aligned} \phi^{r+1}(t) =& \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \int_0^t -6(\phi^r)^2(\tau) + 8(\phi^r)^3(\tau) - 3(\phi^r)^4(\tau)d\tau \\ &+ \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}(-4 + 12\phi^r(\tau) - 12(\phi^r)^2(\tau) + 4(\phi^r)^3(\tau))\phi^{r+1}(\tau)d\tau \end{aligned} \tag{4.29}$$

For $\alpha = 1$ *the exact solution is* $\phi(t) = \frac{1+3t-(1+6t+9t^2)^{(\frac{1}{3})}}{1+3t}$. *The exact solution and approximate solutions are presented in Figure 4.6 . It is to be noticed that solutions for non integer order* α *equation* (4.27) *are approaching the graph of integer order differential equations* (4.27). *By increasing number of iteration Table 4.5 shows solutions obtained by given method are in robust agreement with exact soutions. At* $\alpha = 1$ *Table 4.5 shows that presented method works better than the method used by A. Al-Rabtah [2].*

Table 4.5: Approximate solution at third iteration and absolute error compared with result obtained by A. Al-Rabtah using Spline function [2].

t	$itr = 1$	$itr = 2$	$itr = 3$	ϕ_{Spline}
0.1	8.1546×10^{-4}	1.1157×10^{-5}	1.1629×10^{-6}	9.3660×10^{-5}
0.3	1.5893×10^{-2}	5.5682×10^{-5}	8.5812×10^{-6}	1.1162×10^{-4}
0.5	4.4381×10^{-2}	6.4055×10^{-4}	3.2067×10^{-6}	9.3223×10^{-5}
0.7	7.9378×10^{-2}	2.6330×10^{-3}	1.1129×10^{-6}	7.5850×10^{-5}
0.9	1.0781×10^{-1}	6.5844×10^{-3}	9.0917×10^{-6}	6.2520×10^{-5}

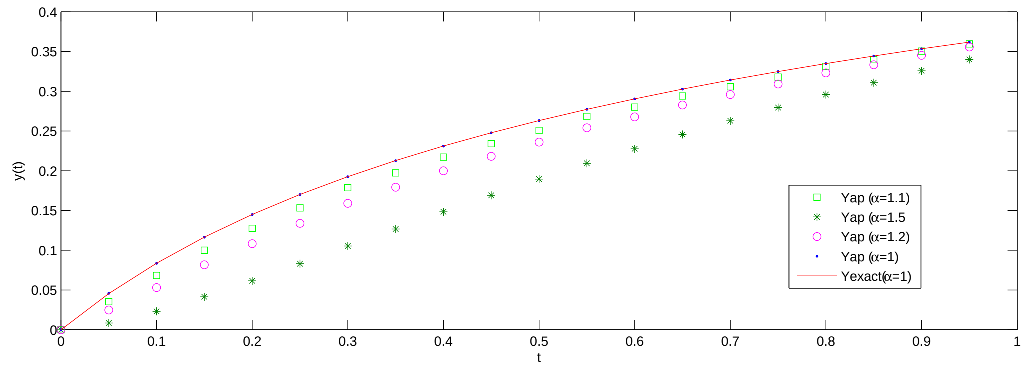


Figure 4.6: Approxmate solution at third iteration for different α.

Example 4.4.6. *Consider the famous Riccati Equation of fractional order [2, 3]*

$$\begin{aligned} {}^*D_0^\alpha \phi(t) + \phi^2(t) = 1, \qquad t > 0, \qquad 0 < \alpha \leq 1, \\ \phi(0) = 0. \end{aligned} \tag{4.30}$$

For $\alpha = 1$ the exact solution is $\phi(t) = \frac{e^{2t}-1}{e^{2t}+1}$.
Riccati equation played a very important role in optimal control development and Kalman filtering. It arises in various classes of Nash, Stackelbegr differential and dynamic games [43]. A brief history of Riccati equation was explained by M. Jungers [44]. Applying quasilinearization approach, the problem (4.30) *is*

$$\begin{aligned} {}^*D_0^\alpha \phi^{r+1}(t) + (\phi^r)^2(t) + (\phi^{r+1}(t) - \phi^r(t))2\phi^r(t) = 1, \quad t > 0, \qquad 0 \leq \alpha \leq 1, \\ \phi_{r+1}(0) = 0. \end{aligned} \tag{4.31}$$

Corresponding integral equation of Eq. (4.31) *is,*

$$\phi^{r+1}(t) = J_0^\alpha((\phi^r)^2(t) - 2\phi^r(t)\phi^{r+1}(t) + 1).$$

Graphs in Figure (4.7a-4.7f) narrate by increasing numbers of iterations absolute error reduced for fix value $\alpha = 1$. Table 4.6 gives a comparison of the proposed method with Quasi-Haarwavelet [3] and Homotopy Perturbation method [4]. It shows that solutions are in robust agreement up to four digit with exact solution and proposed method work efficiently.

Table 4.6: Comparison of solution at fourth iteration with quasi-Haar wavelet[3] and Homotopy perturbation method (HPM) [4].

t	$\phi_{App}, m = 30$	$\phi_{Haar}, m = 70$	ϕ_{HPM}	ϕ_{Exact}
0.1	0.099668	0.099668	0.099668	0.099668
0.2	0.197375	0.197375	0.197375	0.197375
0.3	0.291316	0.291310	0.291312	0.291313
0.4	0.379950	0.379948	0.379944	0.379949
0.5	0.462118	0.462116	0.462078	0.462117
0.6	0.537052	0.537049	0.536857	0.537050
0.7	0.604369	0.604367	0.603631	0.604368
0.8	0.664038	0.664036	0.661706	0.664037
0.9	0.716299	0.716297	0.716298	0.716298

Example 4.4.7. *Consider another type of fractional Riccati Equation*

$$\begin{aligned} {}^{*}D_0^{\alpha}\phi(t) =& 1 + \phi(t) - \phi^2(t), \qquad t > 0, \qquad 0 < \alpha \leq 1, \\ \phi(0) =& 0. \end{aligned} \tag{4.32}$$

For $\alpha = 1$ the exact solution as in [3]

$$\phi(t) = 1 + \sqrt{2}t + 0.5 \tanh\left(1.13t + 0.5 \log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right).$$

Applying quasilinearization to problem (4.32), we get

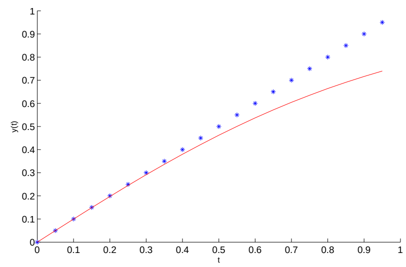
$$\begin{aligned} {}^{*}D_0^{\alpha}\phi^{r+1}(t) &= 1 + (\phi^r)^2(t) + (2 - 2\phi^r(t))\phi^{r+1}(t), \\ \phi^{r+1}(0) &= 0. \end{aligned}$$

The integral form of (4.32) is

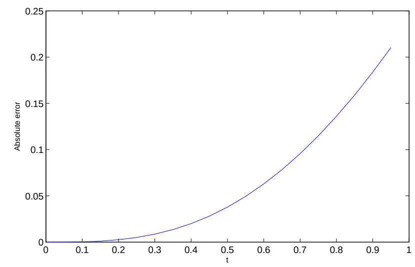
$$\pi^{r+1} = I_0^{\alpha}(1 + (\phi^r)^2(t) + (2 - 2\phi^r(t))\phi^{r+1}(t)).$$

For fix order of differential equation Eq. (4.32), $\alpha = 1$, the solution at three iterations are shown in Figure 4.8. It is observed that by increasing number of iteration solutions converges the exact solution rapidly.

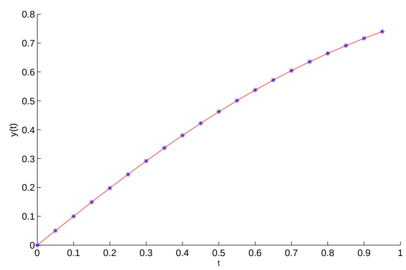
Table 4.7 shows the approximate solutions at fourth iteration for fixed $\alpha = 1$. Solution obtain from proposed method compared with Homotopy perturbation method [4], quasi-harr [3] and exact solution. It is noticed that solutions at given nodes points are equal



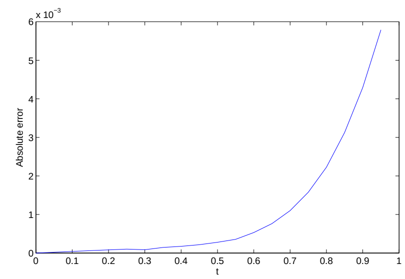
(a) First iteration



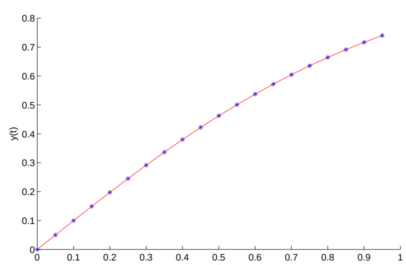
(b) Absolute error for first iteration



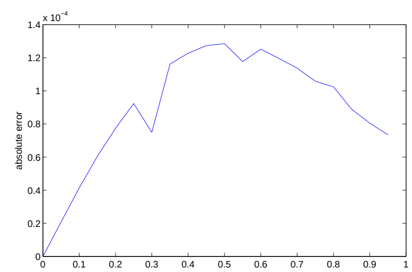
(c) Second iteration



(d) Absolute error for second iteration



(e) Third iteration



(f) Absolute error for third iteration

Figure 4.7: Comparison of exact and numerical solution for first three iterations.

to exact solution up to four digits and developed method gives almost same results as in [3, 4].

Table 4.7: Comparison of Proposed method with quasi-harrwavelet [3] and [4], $\alpha = 1$.

t	$\phi_{Aproximate}$	$\phi_{HARR}[3]$	$\phi_{HPM}[4]$	ϕ_{exact}
0.1	0.110295	0.110296	0.110294	0.110295
0.2	0.241977	0.241977	0.241965	0.241977
0.3	0.395106	0.395105	0.395106	0.395105
0.4	0.567814	0.567813	0.56815	0.567812
0.5	0.756017	0.756015	0.757564	0.756014
0.6	0.953569	0.953567	0.958259	0.953566
0.7	1.152951	1.152949	1.163459	1.152949
0.8	1.346366	1.346364	1.365240	1.346364
0.9	1.526911	1.526911	1.554960	1.526911

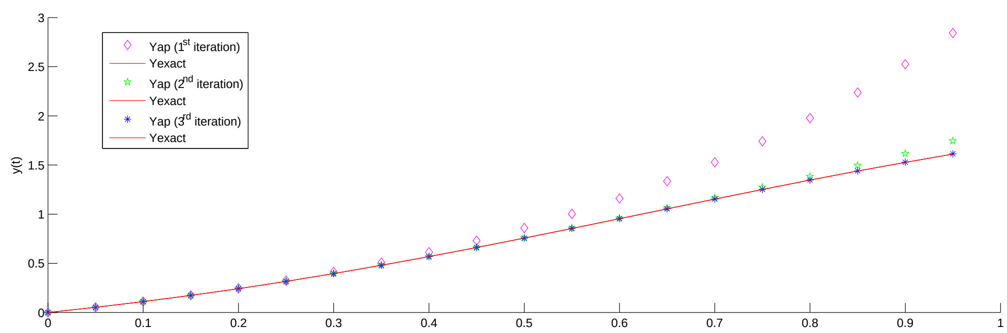


Figure 4.8: Approximate solutions at first three iteration, $\alpha = 1$.

Chapter 5

Summary

We have made a feasible and an efficient computational technique for different types of differential equations of fractional order. Results for existence of solutions, stability of solution and uniqueness of solution are also established. We developed an integration technique for definite integrals. And developed a quadrature rule for solving fractional equations. We combine quadrature method with quasi-linearization technique to solve non-linear differential equations. The proposed technique convert the given problem into system linear algebraic equation. Convergence analysis of presented method is also discussed. The proposed technique is tested on many bench mark problems from the literature. We have checked the supremacy of presented method by comparing results with results obtained using Adomian decomposition method [41], differential transform method [41] , operational matrix method [40], Homotopy method , quasi haar wavelet [3] and spline function approximation [2]. Comparison shows that proposed method work slightly better than haar wavelet method and adomian decomposition method. Comparison shows that approximate solutions obtained by proposed methods are equal to exact solution at nodes points upto some significant digit and work efficiently. Proposed technique can be applied to many problems of same kind.

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Appendix

Matlab Code for proposed method in chapter 4.

```
function VIEK(f,k,b)
formate long
a=c_{1};b=c_{2};h=(b-a)/n;
g=@(x);k=@(x,t);
for i=1:n;
x=a+(i-1)*h;
% vector F
if i<2;
G(i)=feval(g,a);
elseif i>=2;
G(i)=feval(g,x);
end
for j=1:n;
t=a+(j-1)*h;
% matrix K
if i>=j
K(i,j)=feval(k,x,t);
elseif j>i
K(i,j)=0;
end
end
end
for k=1:n./5;
for j=1:n;
for i=5*k+2:n;
if j==5*k-3;
M(i,j)=(375/288)*h*K(i,j);
elseif j==5*k-2;
M(i,j)=(250/288)*h*K(i,j);
elseif j==5*k-1;
```

```
M(i,j)=(250/288)*h*K(i,j);
elseif j==5*k
M(i,j)=(375/288)*h*K(i,j);
end
end
end
end
for i=7:n
for j=1:n
if j==1
M(i,j)=(95*h/288)*K(i,j);
end
end
end
for k=1:n./5;
for i=7:n;
for j=1:n;
if i==5*k+2;
if j==5*k+1;
M(i,j)=(95/288 +1/2)*h*K(i,j);
elseif j==5*k+2;
M(i,j)=(h./2).*K(i,j);
end
end
end
end
end
for k=1:n./5;
for i=7:n;
for j=1:n;
if i==5*k+3;
if j==5*k+1;
M(i,j)=(95/288 +1/3)*h*K(i,j);
elseif j==5*k+2
M(i,j)=(4*h./3).*K(i,j);
elseif j==5*k+3
M(i,j)=(h./3).*K(i,j);
end
end
end
end
```

```
end
for k=1:n./5;
for i=7:n;
for j=1:n;
if i==5*k+4
if j==5*k+1
M(i,j)=(95/288 +3/8)*h*K(i,j);
elseif j==5*k+2
M(i,j)=(9*h./8).*K(i,j);
elseif j==5*k+3
M(i,j)=(9*h./8).*K(i,j);
elseif j==5*k+4
M(i,j)=(3*h./8).*K(i,j);
end
end
end
end
end
for k=1:n./5;
for i=7:n;
for j=1:n;
if i==5*k+5
if j==5*k+1
M(i,j)=(95/288 +14/45)*h*K(i,j);
elseif j==5*k+2
M(i,j)=(64*h./45).*K(i,j);
elseif j==5*k+3
M(i,j)=(24*h./45).*K(i,j);
elseif j==5*k+4
M(i,j)=(64*h./45).*K(i,j);
elseif j==5*k+5
M(i,j)=(14*h./45).*K(i,j);
end
end
end
end
end
for k=1:n./5;
for i=7:n;
for j=1:n;
if i==5*k+6
```

```
if j==5*k+1
M(i,j)=(95/288 +95/288)*h*K(i,j);
elseif j==5*k+2
M(i,j)=(375*h./288).*K(i,j);
elseif j==5*k+3
M(i,j)=(250*h./288).*K(i,j);
elseif j==5*k+4
M(i,j)=(250*h./288).*K(i,j);
elseif j==5*k+5
M(i,j)=(375*h./288).*K(i,j);
elseif j==5*k+6
M(i,j)=(95*h./288).*K(i,j);
end
end
end
end
end
for k=1:n./5;
for j=1:n;
if j==5*k+1
for i=(5*k+7):n;
M(i,j)=(190/288)*h*K(i,j);
end
end
end
end
I=eye(n);

A=I-M;
Yap=inv(A)*G
```