Dual-Mode Soliton Solutions for Simplified Modified Camassa Holm and Gardner Equations

by

Sadia Sadiq

A thesis
submitted in partial fulfillment of the
requirements for the degree of
Master of Science
in
Mathematics

School of Natural Sciences,
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January, 2024

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Dedicated

to

My Beloved Parents

and

Honorable Teachers

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Islamabad **Sadia Sadiq**

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Abstract

The concept of dual-mode equations focuses on nonlinear models that describe the movement of waves in two directions at the same time, influenced by how their phase changes. Initially, Korsunsky introduced a dual-mode model that improved the Korteweg De Vries equation (KDVe) by turning it into a second-order formulation. In our research, our aim is to develop a new and simpler dual-mode model, which is based on the Simplified Modified Camassa-Holm (SMCH) model and the Gardner equation derived from an ideal fluid model. We will then systematically analyze its solutions and explore its geometric characteristics. To find analytical solutions for these bidirectional waves, we'll use various methods like the extended exponential function expansion, rational sine/cosine method, tanh/coth method, direct approach of tan/cot method and Kudryashov method. Furthermore, we will conduct a thorough examination of how the speed at which the phase changes affects the behavior of these dual waves, using both 2D and 3D graphs. The solutions we obtain in this study have significant implications for our understanding of how solitons propagate in the field of nonlinear optics. These models can be used in many different fields and the solutions we obtain will help make it clearer how various non-linear phenomena work. This includes areas like plasma physics, Bose-Einstein condensates, shallow water waves and more.

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Chapter 1

Introduction

Partial Differential Equations (PDEs) are important in our understanding of complex natural phenomena in the field of applied mathematics, physics and engineering. These equations are useful for understanding how different physical, biological and chemical processes occur across time and space. Their applications are extensive, encompassing fields such as physics, engineering, ecology, chemistry, finance, biology, telecommunications and ranging from heat transfer and wave propagation to population dynamics and chemical reactions. The study of nonlinear PDEs has acquired increasing attention, particularly in areas like engineering, physics, shallow water waves and earth sciences. In physics-related fields like nonlinear optics, plasma theory and fluid mechanics, researchers have discovered various solitary wave solutions for these equations [1–6].

Nonlinear evolution equations (NLEEs) are of central importance in mathematical physics and engineering, as they provide essential mathematical models for explaining physical processes in applied sciences. Exact solutions describing waves in these equations are critical for understanding these phenomena. Wave-like behavior is observed in a variety of systems such as structured materials, plasma, chemicals, elastic materials, optical cables, fluid flow and quantum mechanics. Several methods have been developed to tackle nonlinear evolution equations (NLEEs), making them key components for addressing problems in mathematical physics and engineering. These methods encompass various approaches including the G'/G-expansion method, tanh expansion approach, Kudryshov technique, extended F-expansion procedure, rational sin/cos strategy, inverse scattering transform, Hirota bilinear method, Darboux transformation, Painlevé analysis, Backlund Darboux transformation, tanh method, Exp-function method, improved F-expansion method, modified extended tanh-function method, $\exp(-\phi(\xi))$-expansion method, sine–cosine method, modified simple equation method, the new approach of the generalized G'/G-expansion method, the novel G'/G-expansion method, Jacobi elliptic function method, homogeneous balance method and the homotopy perturbation technique among others [7–14].

One specific PDE model that has found widespread use is the Simplified Modified Camassa-Holm (SMCH) equation, which is employed to study wave propagation and fluid dynamics phenomena. This equation is derived from the Camassa-Holm (CH) equation which in turn builds upon the Korteweg-de Vries (KdV) equation [15]. The KdV equation describes weakly nonlinear and dispersive waves, while the CH equation extends this to waves with both dispersion and another relevant characteristic, such as nonlinearity or wave steepening. The SMCH equation retains essential characteristics while making the model more manageable for certain applications. This equation finds application in various physical systems, including shallow water waves and plasma physics. This approach is commonly used to acquire insights into wave dynamics and its applications in geophysics, fluid mechanics and other sciences that study traveling waves. The choice of parameters and initial conditions can lead to different wave solutions, making it a flexible tool for representing complex wave phenomena.

In the context of understanding the behavior of waves, we can see the KdV equation which describes how small waves move in shallow water. This equation plays a crucial role in understanding the balance between linear dispersion and nonlinear steepening of waves. In situations where the size of waves differs from the depth of the water, the equation's behavior changes significantly, affecting wave motion. Higher-order KdV equations follow similar principles and the derivation of these equations can be associated with (2+1)-dimensional Gardner equations which is derived from ideal fluid model [16–19]. In the analysis of this systems non-dimensional parameters, $\alpha = a/H$ representing the amplitude parameter, $\beta = H^2/L^2$ representing the wavelength parameter, and $\tau = T/\rho g H^2$ representing the Bond number are of paramount importance in our research. These parameters are instrumental in gaining insights into the system's behavior that we are studying. In this context, g denotes the acceleration due to gravity, H signifies the average depth upstream, while a and L represent characteristic values for wave amplitude and wavelength. Additionally, T is the coefficient for surface tension and ρ denotes the density of water.

In summary, understanding and examining the exact solutions to basic equations that describe physical phenomena is vital. The exact solutions can serve as a model for testing the accuracy of different mathematical and computer-based methods, even when they may not have a clear physical explanation. Finding these exact solutions for nonlinear PDEs is an essential part of the field of nonlinear science, which has gained importance in recent decades due to advances in computer technology. This understanding is key to interpreting physical phenomena. The SMCH and Gardner equations have unique solutions as solitary wave, periodic wave and kink wave solutions. These solitons represent nonlinear wave processes in various fields like plasmas, fluid dynamics and shallow water waves. These solutions are often represented by specific mathematical functions like tanh, sin and sech. Many of these solutions are developed and calculations are done using computer software such as MAPLE, MATLAB, or similar programs.

1.1 Nonlinear Evolution Equations

Evolution equations are dynamic PDEs involving both temporal and spatial variables. A NLEEs is a mathematical equation that represents the time-dependent behavior of a physical system in which the connection between the variables involved is nonlinear. In such equations, the change in a field over time is impacted nonlinearly by the values of variables. NLEEs are widely used in physics, chemistry and biology to represent and comprehend complicated dynamic processes such as wave propagation, heat diffusion, chemical reactions and others. These equations frequently result in the creation of complex and nonlinear phenomena such as solitons and travelling waves, and they are solved using a range of mathematical and computer approaches.

Mathematical Framework

NLEEs serve as a foundational theme and offer insights into a wide range of physical phenomena. Their applications span diverse disciplines, including fluid mechanics, elasticity, solid-state physics, biomechanics, gas dynamics, relativity, chemical reactions, plasma physics, optical fiber systems, shallow water waves and various other nonlinear mathematical fields [20, 21]. Through investigating NLEEs, researchers can gain a deeper understanding of these complex physical occurrences and develop mathematical frameworks to describe and analyze them effectively.

For example:

$$\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0.$$

1.1.1 Dispersion

In the context of NLEEs, dispersion may have a significant influence on how wave patterns change over time. Dispersion is the phenomenon that occurs when distinct frequencies of a wave travel at different speeds. When dispersion exists, higher-frequency components of a wave often move faster than lower-frequency components. This causes wave packets to spread out or separate, which can affect the overall shape and behavior of the wave. The presence or lack of dispersion in a nonlinear evolution equation can impact whether waves keep their form or get distorted as they propagate. Media with this property is called dispersive media.

1.1.2 Nonlinearity

The behavior of waves in NLEEs is no longer simply dictated by linear superposition, because distinct components of the wave can impact each other. Nonlinearity describes how distinct components of a wave interact with one another. Solitons, travelling waves other complex wave phenomena can result from nonlinearity. In nonlinear evolution equations, the interaction between dispersion and nonlinearity frequently gives rise to complex and complicated wave dynamics.

1.2 Travelling wave solutions

The idea of mathematically representing this concept originates from watching water waves in oceans, rivers, or canals that keep their shape while moving. This concept is built on the notion of a "traveling wave," which is a wave that moves steadily in a specific direction. Through the study of these waves, efforts have been made to develop mathematical models that represent how waves behave and improve our knowledge of the principles behind these occurrences.

A **traveling wave** is a wave that travels across a medium without retaining its form or amplitude. When we solve mathematical models related to certain types of non-linear PDE, we get solutions that represent traveling waves. These traveling wave solutions can be described mathematically like this:

$$u(x,t) = \omega(\xi), \;\; \xi = x - \omega t.$$

This equation contains the symbols ω for wave speed, t for time, and x for space, which are used in travelling wave solutions. To simplify things, we introduce a new variable called (ξ) to represent the traveling wave. When the wave speed ω is zero, it means the wave isn't moving like a stationary wave. It's worth emphasizing that there are different types of traveling waves, but we will only focus on a specific types in this context. By applying the traveling wave transformation, we can change PDEs into ODEs. This transformation merges the two independent variables into a single variable, called the traveling wave variable ξ. Utilizing this transformation simplifies the PDEs and eases their analysis and solution.

1.2.1 Types of travelling wave solutions

There are several forms of travelling wave solutions that are of special interest in solitary wave theory, which is quickly growing in a variety of scientific domains ranging from shallow water waves to plasma physics. As previously said, there are many different forms of traveling waves, and only a few of them will be discussed:

- **Solitary Waves and Solitons:** Solitary waves are localized traveling waves that maintain a constant speed and shape, approaching zero as they extend to large distances. Solitons, a distinctive subtype of solitary waves, exhibit spatial localization with properties such as $u'(\xi),\ u''(\xi),\ u'''(\xi) \to 0$, and $\xi \to \pm\infty$, where $\xi = x - \omega t$. Notably, solitons possess a remarkable characteristic, preserving their identity even when interacting with other solitons.

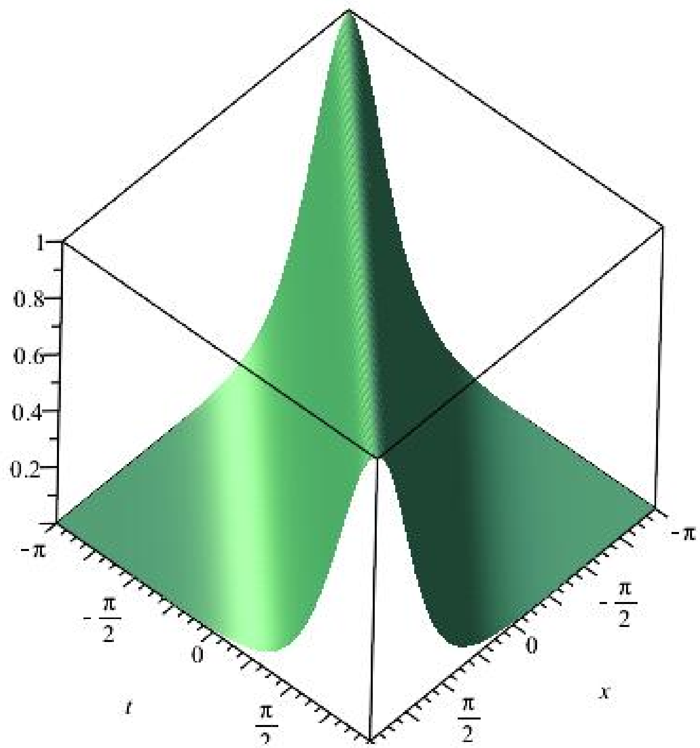


Figure 1.1: Graph of Soliton solution $u(x,t) = sech^2(x-t),\ -\pi \leq x,\ t \leq \pi$, that possesses unbounded support or endless tails.

The KdV equation serves as the fundamental model for analytical $sech^2$ solitary wave solutions, giving rise to bell-shaped profiles. The graph in above Fig. illustrates one of these solitary waves, featuring extended portions known as infinite wings or tails.

- **Kink Waves:** Kink waves are traveling waves that ascend or descend between two asymptotic states. The kink solution converges to a constant value as it extends towards infinity. The dissipative Burgers equation, a well-established mathematical expression, is renowned for yielding kink solutions.

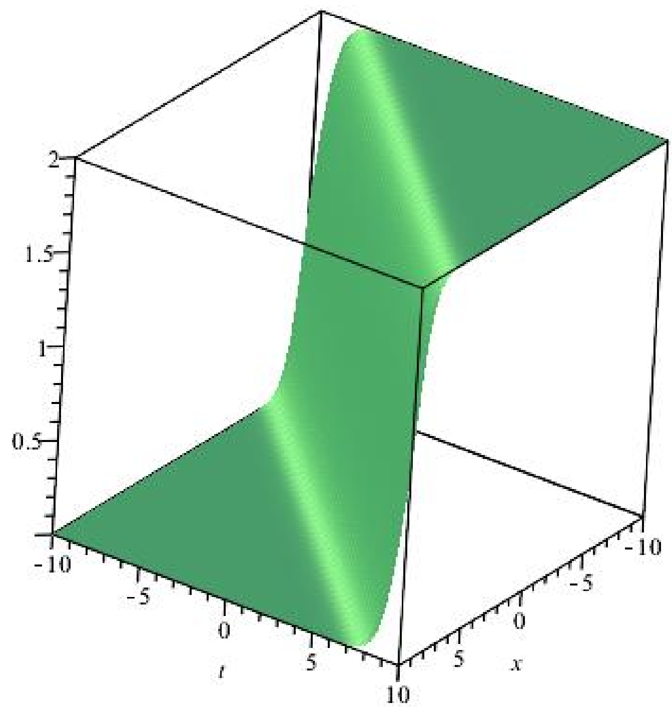


Figure 1.2: Graph of kink solution $u(x,t) = 1 - \tanh(x - t),\ -10 \leq x,\ t \leq 10$.

- **Peakons:** Peakons refer to solitary wave solutions characterized by sharp peaks. In this scenario, the traveling wave solutions exhibit smooth behavior, with the exception of a distinctive peak occurring at one corner of its crest. Peakons are identified as points where the spatial derivative undergoes a change in sign, resulting in a finite jump in the first derivative of the solution $u(x,t)$. This indicates that peakons possess discontinuities in the x-derivative, yet both one-sided derivatives exist and differ only in sign.

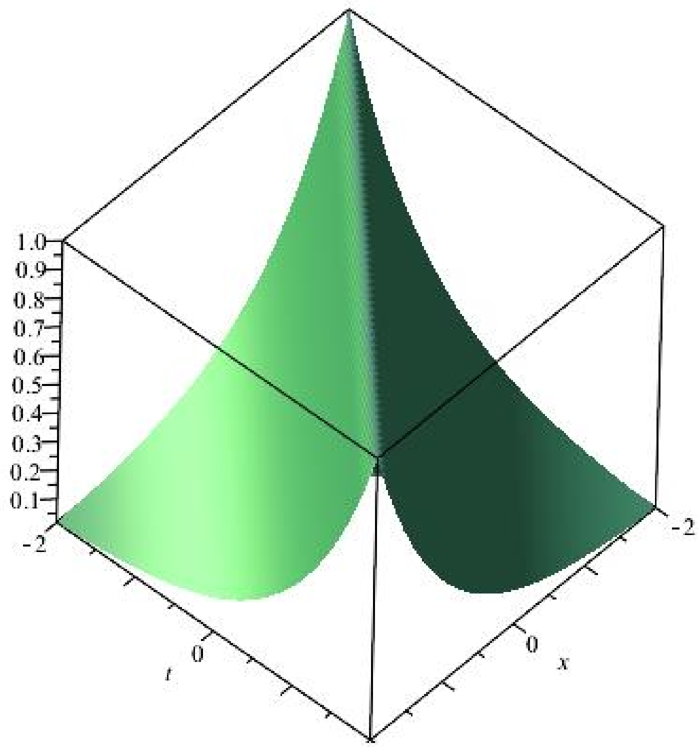


Figure 1.3: Graph of peakon solution $u(x, t) = \exp(-|x - t|),\ -2 \leq x,\ t \leq 2$.

- **Cuspons:** Cuspons represent an alternative type of solitons characterized by the presence of cusps at their crests. In contrast to peakons, where the derivatives at the peak differ only by a sign, cuspons exhibit a distinct behavior. The derivatives at the cusp jump diverge. The following depicts a conceptual graph of a cuspon, which does not originate from a recognized model.

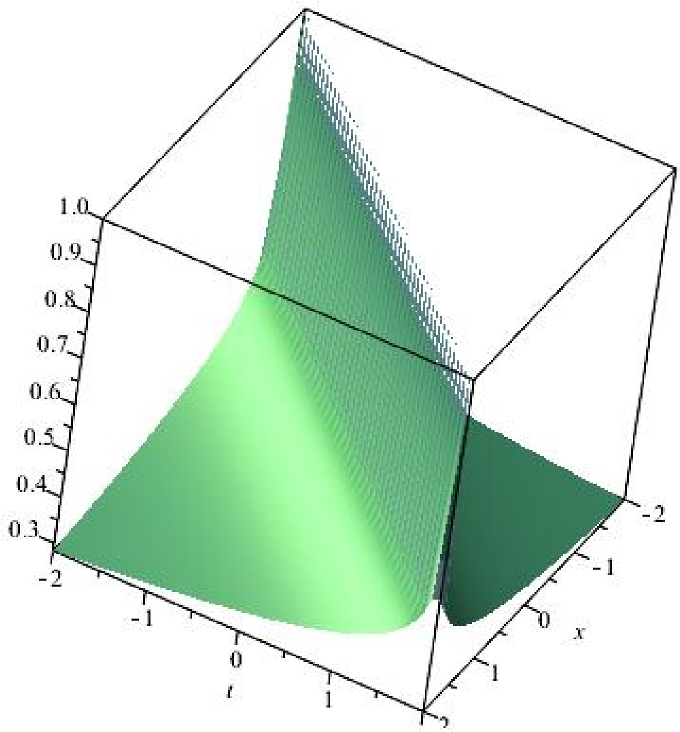


Figure 1.4: Graph of a cuspon $u(x,t) = \exp(-|x-t|^{1/6}),\ -2 \leq x,\ t \leq 2.$

- **Periodic waves:** A periodic wave exhibits a consistent and repeating pattern, defining its wavelength and frequency. The wave is characterized by its amplitude, which directly correlates with its energy and represents the highest and lowest points of the wave. The period is the time required to complete one cycle of the waveform, while frequency refers to the number of cycles per second.

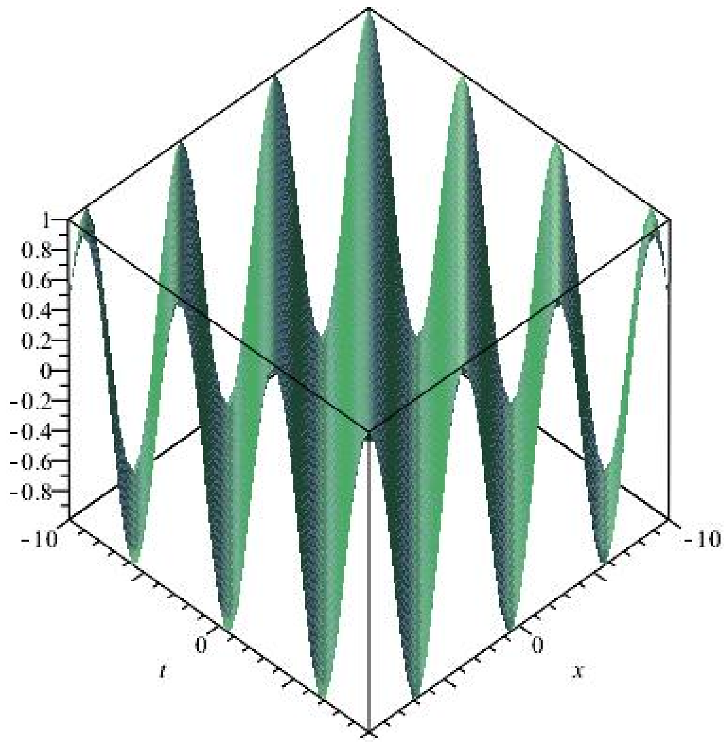


Figure 1.5: Graph of a periodic waves $u(x,t) = \cos(x - t),\ -10 \leq x,\ t \leq 10$.

1.2.2 Importance of traveling waves

Traveling waves are disturbances that transfer energy from one location to another within various mediums. These waves are essential for understanding and solving problems in fields such as fluid dynamics, chemistry, acoustics, elasticity, biology, and electromagnetic theory. They provide valuable insights into phenomena like ocean waves, combustion, the propagation of sound, seismic activity, disease transmission, optics, and electromagnetic wave transmission. By studying and mathematically modeling the propagation of energy through these waves, we can make significant advancements in these mathematical disciplines.

1.3 Solitons

The significant discovery of solitons within the Korteweg-de Vries (KdV) equation is credited to Martin Kruskal and Norman Zabusky [22]. Solitons, which represent solutions to complex mathematical equations known as nonlinear dispersive PDEs, serve as vital tools for explaining a wide array of physical phenomena.

A **soliton**, also recognized as a **solitary wave**, possesses an extraordinary property. It preserves its shape even in the presence of dispersion and nonlinear effects. When a soliton wave interacts with another wave of the same type, it retains its characteristic shape, unaffected by the presence of dispersion and nonlinearity. Solitary waves manifest in the context of light intensity within fiber optics and the elevation of water surfaces in channels with finite amplitudes. In the following sections, we will explore two distinct examples of solitary waves.

Figure 1.6: Ocean wave

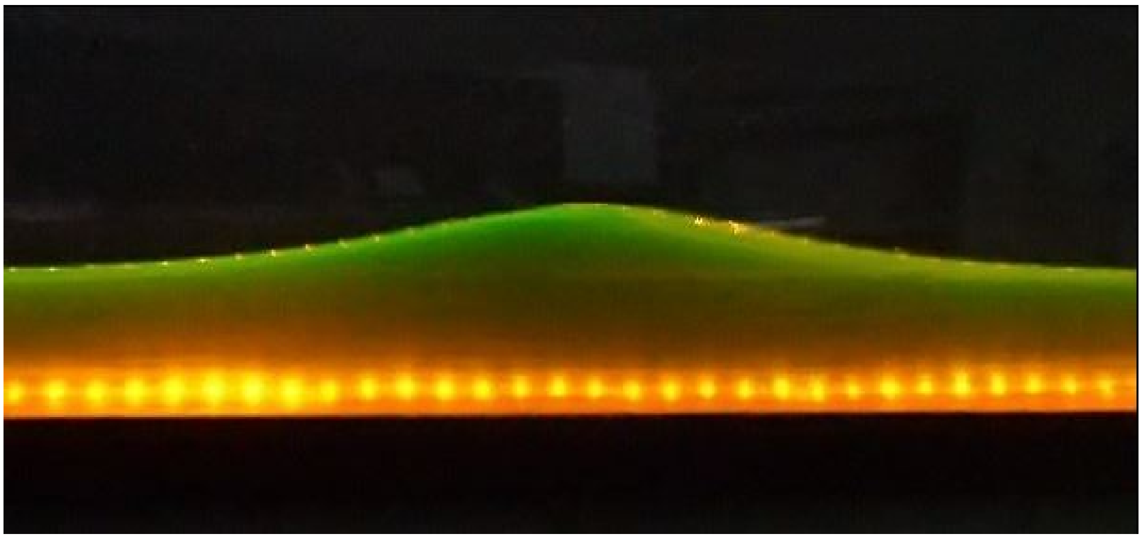
Figure 1.7: Heartbeat waves

1.3.1 Discovery of solitons

In 1834, John Scott Russell (1808-1882) a Scottish engineer made an interesting discovery of solitary waves in the Union Canal in Scotland. While engaged in canal work, he observed a substantial, steadily evolving water wave as it traversed the canal without altering its form. To understand this particular wave better, he recreated it in a small tank of water, and he called it the wave of translation. This was the first time that anyone had studied this unique wave phenomenon, now known as a soliton. Scott Russell personally expressed his observation as follows:

"I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped – not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually

diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation". In order to comprehend the idea of these waves, he conducted experiments at home in a water tank.



He identified a number of the most important features as follows:

- Solitary waves exhibit an exceptional ability to maintain their form and energy levels over considerable distances, even when subjected to external disturbances.
- Unlike most waves that disperse and expand as they travel, solitary waves are non-dispersive, keeping their shape and amplitude largely unchanged during propagation.
- Solitary waves are highly localized, existing as compact wave patterns that don't spread out spatially, which allows them to preserve their identity and energy over extended distances.
- When solitary waves interact with one another or with other waves, they adhere to the principle of retaining their original form and amplitude, known as the soliton property.

- The total energy they carry remains constant as they move through a medium, contributing to their exceptional stability.

Scott Russell's experiments challenged the Newton and Bernoulli wave theories of hydrodynamics. Airy and Stokes couldn't explain his observations. Henry Bazin reported more findings in 1862. It took until the 1870s for Boussinesq and Rayleigh to offer explanations. In 1895, Korteweg and de Vries introduced the KdV equation, which described solitary waves. Zabusky and Kruskal demonstrated soliton behavior in 1965. In 1967, Gardner, Greene, Kruskal and Miura found an inverse scattering transform for the KdV equation. Solitons keep their shape and speed when they collide with other waves. Solitary waves are similar but change slightly during interactions. Solitons are also studied in quantum mechanics as they could provide a new foundation for the field. An experiment in 2019 observed accelerating surface gravity water wave solitons and measured their phases using an external hydrodynamic linear potential.

1.3.2 Properties of Solitons [23]

Integrability:

Solitons are often solutions to integrable nonlinear (PDEs). Integrability implies the existence of an infinite number of conserved quantities, contributing to the stability and persistence of solitons. The inverse scattering transform is a powerful mathematical tool used to analyze and understand the integrability of soliton equations.

Particle-like Behavior:

Solitons exhibit a remarkable ability to maintain their shape and identity during collisions with other solitons. This particle-like behavior is a result of the balance between nonlinearity and dispersion in the governing equations. The interaction of solitons resembles the behavior of particles, making them distinct from conventional wave patterns.

Nonlinear Superposition:

Solitons display nonlinear superposition, meaning their interaction is not a simple linear addition of individual solutions. When solitons overlap, their interaction involves complex phenomena, such as phase shifts and changes in amplitude. It allows solitons to propagate through a medium without losing their individual characteristics.

Stability:

Solitons are inherently stable and can travel long distances without significant distortion. This stability arises from a delicate balance between nonlinear effects that tend to amplify disturbances and dispersive effects that tend to spread them out.

Conservation Laws:

Solitons are associated with a set of conserved quantities, such as energy and momentum. Conservation laws contribute to the robustness of solitons and their ability to persist over time.

Versatility of Existence:

Solitons are found in a wide range of physical systems, including water waves, optical fibers, plasmas, and biological systems. The ubiquity of solitons across different domains highlights their fundamental nature and applicability.

Mathematical Richness:

The study of solitons involves a rich mathematical framework, including techniques from nonlinear dynamics, differential geometry, and algebraic geometry. Solitons are often described using nonlinear integrable PDEs, leading to fascinating mathematical structures.

1.3.3 Applications of solitons

Solitons, which are unique wave phenomena seen in many disciplines of physics such as hydrodynamics, plasma physics, optics and others have a wide range of practical applications. They aid in river flow management, plasma stabilization in nuclear fusion, signal integrity in optical communication and particle behavior in materials. Solitons are also used in biophysics, low-temperature physics, astronomy and sectors as diverse as engineering and data transmission, making them important for understanding and progressing in a wide range of scientific and technical disciplines. The capacity of solitons to keep their shape and energy as they pass across different mediums distinguishes them as valuable instruments for studying and influencing complicated physical systems. Their consistency and predictability are critical for practical breakthroughs, driving continued study and development in these many disciplines.

In atmospheric science

Solitons play a significant role in atmospheric research, contributing to the understanding of abrupt stratospheric warming events that influence weather patterns and the polar vortex. Furthermore, they may be used to mimic internal ocean waves, offering insights into ocean circulation and its consequences on marine life. Solitons are extremely important in anticipating and managing catastrophic weather phenomena like hurricanes and tsunamis. Their uses include research into air turbulence, which benefits aviation and weather forecasting. Solitons are used by researchers to explore atmospheric gravity waves, revealing information on their origins and impact on climate. Furthermore, solitons are useful in the study of oceanic internal tides, which play an important role in nutrient transfer and ecosystem dynamics.

In Chemistry

Solitons provide a unique perspective on chemical processes in chemistry, enhancing reaction efficiency. They are essential for understanding reaction-diffusion systems and assisting in the creation of novel materials. Solitons can also anticipate and evaluate the kinetics of chemical reactions in combustion processes. In conclusion, solitons help us comprehend complicated processes and chemical structure development.

In Geophysics

Solitons are extremely useful in geophysics, where they play an important role in interpreting stress waves during earthquakes. This aids in the forecast of earthquakes and the assessment of associated risks. Solitons may also be used to represent how seismic energy moves and how fault lines shift during earthquakes. They are used to analyze volcanic eruptions and their associated seismic activity, which aids in volcano monitoring and risk assessment. These wave patterns are especially important for studying tsunamis and how they behave in various places of the ocean. Furthermore, solitons help us comprehend magma circulation and how it impacts the Earth's crust, which adds to our understanding of volcanoes and tectonic plate movements.

In Acoustics

Solitons are extremely valuable in the field of acoustics, particularly when investigating how waves behave nonlinearly. They're like useful tools for generating better sound barriers and making sounds quieter. Solitons in the undersea realm help us understand how sounds travel and how to lessen noise in the ocean. This enhances submarine navigation and makes underwater communication more efficient. Furthermore, solitons are extremely useful in the realm of medical ultrasonography. They enable us to photograph within the body without intrusive treatments.

In biology

Solitons have been discovered in proteins [24] and DNA [25], where they move slowly and synchronously. When chemical reactions occur, these solitons also carry energy in molecules. A recent hypothesis in neuroscience proposes that signals in neurons are similar to solitons. They manifest as waves of information in the brain. This novel hypothesis suggests that solitons are essential for signal movement and communication in the brain's networks. It opens up fresh perspectives on how the brain functions and processes information.

In fluid dynamics

Solitons are required in fluid dynamics to describe wave behavior in fluid flows, particularly in shallow water and open-channel flows. They are critical for flood prediction, understanding river dynamics, and maximizing water resource management. Solitons play an important role in the design of efficient water distribution systems and irrigation technologies, as well as in improving water conservation and agricultural practices. Furthermore, solitons play an important part in the creation of more efficient hydroelectric power generation as well as in optimizing the design of marine boats to reduce their environmental effect and improve maritime safety.

In field of plasma physics

Solitons are important in plasma physics [26] because they aid in the characterization of plasma waves in fusion experiments for clean energy production. They contribute to energy containment and stability, boosting nuclear fusion research. Solitons are also important in researching space weather, such as solar flares, and their influence on Earth. They describe particle and wave behavior in magnetospheres in plasma physics, which improves our knowledge of space environments and their consequences on spacecraft. Solitons are also employed in solar corona research, as well as the development of magnetic confinement devices and sophisticated space propulsion systems, all of which promise revolutionary energy solutions and space exploration possibilities.

In shallow water waves

Within the domain of shallow water waves, a adapted Camassa-Holm equation emerges as a crucial asset [27]. This mathematical model simplifies the intricate behaviors of waves along coastal zones, facilitating the anticipation of wave actions near shorelines. Consequently, this equation assists coastal and marine engineers in devising enhanced protective solutions for coastlines. It also aids in comprehending the influence of waves on coastal erosion and devising plans to oversee and protect susceptible coastal regions. This practical application offers valuable perspectives into the dynamic interplay of shallow water waves and their effects on coastal settings.

1.4 Literature review

In 1994, the dual-mode equations were introduced as mathematical models to describe the behavior of two waves that travel together and interact in a special way due to certain factors. These factors are nonlinearity, linearity and phase-velocity. One of the earliest forms of these equations, called the temporal-second-order Korteweg-de Vries (KdV) equation, was created by a scientist named Korsunsky in a published work [28]. Korsunsky studied how two long waves, each with their unique characteristics, interacted using the Hirota-Satsuma model. To understand this interaction and their simultaneous movement better, Korsunsky developed an improved version of the KdV equation known as the dual-mode KdV equation.

$$p_{tt} + (c_1 + c_2)p_{xt} + c_1c_2p_{xx} + ((\alpha_1 + \alpha_2)\frac{\partial}{\partial t} + (c_2\alpha_1 + c_1\alpha_2)\frac{\partial}{\partial x})pp_x + (\beta_1 + \beta_2)\frac{\partial}{\partial t} + (c_2\beta_1 + c_1\beta_2)\frac{\partial}{\partial x})p_{xxx} = 0. \tag{1.1}$$

In this context, the variables x and t are used to represent scaled space and time coordinates in the analysis. The function $p(x,t)$ describes the height of the water surface above a flat bottom. The two waves involved have their own phase velocities, labeled as c_1 and c_2, and their nonlinearity is indicated by α_1 and α_2. The dispersion of each wave is determined by the parameters β_1 and β_2 [28]. When there are no other influences, each wave mode is governed by its own KDV equation.

$$p_t + c_ip_x + \alpha_ipp_x + \beta_ip_{xxx} = 0, \tag{1.2}$$

this leads to a solution that can be expressed as:

$$p = A\text{sech}^2([x - (c_i + \frac{1}{3}\alpha_iA)t]/L_i),$$
$$12L_i^2 = A\alpha_i/\beta_i. \tag{1.3}$$

Now we find the solution of equation (1.2). Let we assume solution

$$
\begin{aligned}
&p(x,t) = q(v), \quad \text{where} \;\; v = x - ct,\\
&p_t = -cq'(v),\\
&p_x = q'(v),\\
&p_{xxx} = q'''(v),
\end{aligned}
\tag{1.4}
$$

by substituting these values into equation (1.2), we obtain the following result

$$
-cq'(v) + c_i q'(v) + \alpha_i q(v)q'(v) + \beta_i q'''(v) = 0, \tag{1.5}
$$

through the process of integrating the above equation, we achieve the following result

$$
-cq(v) + c_i q(v) + \frac{1}{2}\alpha_i (q(v))^2 q'(v) + \beta_i q''(v) = a, \tag{1.6}
$$

here a as a constant of integration, multiply the above equation by $q'(s)$ we get

$$
-cq(v)q'(v) + c_i q(v)q'(v) + \frac{1}{2}\alpha_i (q(v))^2 (q'(v))^2 + \beta_i q''(v)q'(v) = aq'(v), \tag{1.7}
$$

again integration

$$
-c\frac{1}{2}(q(v))^2 + \frac{1}{2}c_i (q(v))^2 + \frac{\alpha_i}{6}(q(v))^3 + \frac{\beta_i}{2}(q'(v))^2 = aq(v) + b, \tag{1.8}
$$

where b is constant of integration

$$
q, \; q', \; q'' \to 0, \quad as \;\; v \to \pm\infty, \tag{1.9}
$$

using equations (1.6) and (1.8), we can deduce that both $a = 0$ and $b = 0$. Therefore, based on equation (1.8)

$$
-c\frac{1}{2}(q(v))^2 + \frac{1}{2}c_i (q(v))^2 + \frac{\alpha_i}{6}(q(v))^3 + \frac{\beta_i}{2}(q'(v))^2 = 0, \tag{1.10}
$$

$$
\frac{\beta_i}{2}(q'(v))^2 = c\frac{1}{2}(q(v))^2 - \frac{1}{2}c_i (q(v))^2 - \frac{\alpha_i}{6}(q(v))^3, \tag{1.11}
$$

$$
\implies \frac{\beta_i}{2}(q'(v))^2 = (q(v))^2[c\frac{1}{2} - \frac{1}{2}c_i - \frac{\alpha_i}{6}(q(v))], \tag{1.12}
$$

$$\implies (q'(v))^2 = \frac{2}{\beta_i}(q(v))^2[c\frac{1}{2} - \frac{1}{2}c_i - \frac{\alpha_i}{6}(q(v))], \tag{1.13}$$

or

$$(q'(v))^2 = \beta_i(q(v))^2[c - c_i - \frac{\alpha_i}{3}(q(v))], \tag{1.14}$$

by taking square root on both side we get

$$q'(v) = q(v)\sqrt{\beta_i[c - c_i - \frac{\alpha_i}{3}(q(v))]}, \tag{1.15}$$

or

$$\frac{dq}{dv} = q(v)\sqrt{\beta_i[c - c_i - \frac{\alpha_i}{3}(q(v))]}, \tag{1.16}$$

$$\int dv = -\int \frac{\sqrt{q}}{q(v)\sqrt{\beta_i[c - c_i - \frac{\alpha_i}{3}(q(v))]}}, \tag{1.17}$$

$$v = -\int \frac{q}{q(v)\sqrt{\beta_i[c - c_i - \frac{\alpha_i}{3}(q(v))]}} + A, \tag{1.18}$$

assuming

$$q = \frac{c}{2}\text{sech}^2\theta, \tag{1.19}$$

$$dq = c\ \text{sech}^2\theta \tanh\theta d\theta, \tag{1.20}$$

by putting these values in (1.18) we get

$$v = -\int \frac{c\ \text{sech}^2\theta \tanh\theta d\theta}{\frac{c}{2}\text{sech}^2\theta\sqrt{\beta_i[c - c_i - \frac{\alpha_i}{3}(\frac{c}{2}\text{sech}^2\theta)]}} + A, \tag{1.21}$$

$$v = -\int \frac{\tanh\theta d\theta}{c\sqrt{\beta_i[c - c_i - \frac{\alpha_i}{3}(\frac{c}{2}\text{sech}^2\theta)]}} + A, \tag{1.22}$$

through a specific transformation

$$\xi = (\beta_1 + \beta_2)^{-\frac{1}{2}}(x - ct),\ T = (\beta_1 + \beta_2)^{-\frac{1}{2}}t,$$
$$c_0 = \frac{1}{2}(c_1 + c_2), \quad u(\xi, T) = (\alpha_1 + \alpha_2)p(x, t). \tag{1.23}$$

The equation (1.1) has been changed, giving rise to a new version presented in the following manner:

$$u_{TT} - s^2 u_{\xi\xi} + (\frac{\partial}{\partial T} - \alpha s \frac{\partial}{\partial \xi})(uu_\xi) + (\frac{\partial}{\partial T} - \beta s \frac{\partial}{\partial \xi})(u_{\xi\xi\xi}) = 0, \tag{1.24}$$

where

$$s = \frac{1}{2}(c_1 - c_2), \quad \alpha = \frac{\alpha_1 - \alpha_2}{\alpha_1 + \alpha_2} \leq 1, \quad \beta = \frac{\beta_1 - \beta_2}{\beta_1 + \beta_2} \leq 1, \quad c_2 \leq c_1. \tag{1.25}$$

In this equation, the variable $u = u(\xi, T)$ represents an unknown field function. The parameters β and α correspond to the dispersion and nonlinearity parameters, both are bounded by 1. The parameter s represents the phase velocity, which signifies the interaction of two moving waves. It's essential to note that when $s = 0$ there is no wave interaction, and integrating the equation once with respect to time t results in the well-known KdV equation, which describes the motion of a single wave. This modified form of the TMKdV equation has been verified and examined under the specified conditions. Furthermore, this equation possesses at least two conserved quantities.

$$\frac{\partial u}{\partial T} + \frac{\partial X}{\partial \xi} = 0, \tag{1.26}$$

here, when we have conserved flux denoted as X and conserved density represented as u, without any involvement of time derivatives w.r.t t, it's referred to as a consumption law. This means that X and u can vary based on ξ, T, u and u_ξ but they do not depend on u_T. In the many cases, the pairs of density and flux in these equations take the form of polynomials in u and its derivative with respect to ξ. For polynomial-type T and ξ, the integration of equation (1.26) is as follows:

$$Q = \int_{-\infty}^{\infty} u d\xi = \text{constant}, \tag{1.27}$$

provided that ξ vanishes at infinity.

$$p_t - 6pp_x + p_{xxx} = 0, \tag{1.28}$$

integrating

$$u = p, \quad \xi = p_{xx} - 3p^2,$$
$$\int pdx = \text{constant},$$

multiply (1.28) by p we get

$$\frac{\partial}{\partial t}(\frac{1}{2}p^2) + \frac{\partial}{\partial x}(pp_{xx} - \frac{1}{2}(p_x)^2 - 2p^3) = 0, \tag{1.29}$$

that gives second law of conservation

$$\int_{-\infty}^{\infty} p^2 dx = \text{constant}, \tag{1.30}$$

multiplying (1.28) by $3p^2$

$$3p^2(p_t - 6pp_x + p_{xxx}) = 0, \tag{1.31}$$

multiplying partial derivative of (1.28) w.r.t x by p_x

$$p_x(p_{xt} - 6(p_x)^2 + -6pp_{xx} + p_{xxxx} = 0, \tag{1.32}$$

adding last two equations, we get

$$\frac{\partial}{\partial t}(p^3 - \frac{1}{2}(p_x)^2) + \frac{\partial}{\partial x}(\frac{-9}{2}p^4 + 3p^2 p_{xx} - 6p(p_x)^2 + p_x p_{xxx} - \frac{1}{2}(p_{xx})^2 = 0, \tag{1.33}$$

this results in the third conservation law of the KDV equation

$$\int_{-\infty}^{\infty} (p^3 - \frac{1}{2}(p_x)^2) dx = \text{constant}, \tag{1.34}$$

the presence of these conservation laws has been regarded as an indicator of the integrability of the KDV equation. Upon substituting

$$u(\xi, T) = u(z), \quad z = \xi - \alpha T, \tag{1.35}$$

in (1.24) we get

$$(-\alpha s - \alpha)(u')^2 + (\alpha^2 - s^2)u'' + (-\alpha s - \alpha)uu'' + (-\beta s - \alpha)u^4 = 0, \tag{1.36}$$

$$\frac{1}{2}(\alpha s-\alpha)u^2+(\beta s-\alpha)u''+(\alpha^2-s^2)u+B=0, \tag{1.37}$$

$$\implies (\beta s-\alpha)u''+\frac{1}{2}(\alpha s-\alpha)u^2+(\alpha^2-s^2)u+B=0, \tag{1.38}$$

$$s(\alpha-\beta)u''+\frac{1}{2}(\alpha s^2-\alpha s)u^2+(\alpha^2 s^2-s^2)u+B=0, \tag{1.39}$$

$$\frac{1}{s}(s(\alpha-\beta)u''+\frac{1}{2}(\alpha s^2-\alpha s)u^2+(\alpha^2 s^2-s^2)u+B)=0, \tag{1.40}$$

$$(\beta-\alpha)u''-\frac{\alpha}{2}(s-\alpha)u^2+s(1-\alpha^2)u+B^*=0. \tag{1.41}$$

By solving this we get

$$u=\exp(\frac{\sqrt{s(\sqrt{\alpha^2-1})z}}{(-\beta+\alpha)})+\exp(\frac{\sqrt{s}(\sqrt{\alpha^2-1})z}{\sqrt{(-\beta+\alpha)}})+\frac{B(-\beta+\alpha)}{s(\alpha^2-1)}. \tag{1.42}$$

In 2017, Abdul Majid Wazwaz introduced a novel version of dual-mode Burgers equation [29]. In this study, the focus was on discovering multiple kink solutions, but this was only possible for specific values of nonlinearity and dispersion parameters. The research also explored different types of solutions, such as singular, periodic and rational solutions, each linked to specific parameter values. To do this, various mathematical methods were employed, including Hirota's method with the Cole-Hopf scheme, as well as analytical techniques like the tanh-coth method and tan-cot method. The research also explored rational solutions.

In the same year, another study led by M. Syam et al. [30] concentrated on investigating a system of dual-mode coupled modified Korteweg-de Vries (mKdV) equations. They used a simplified bilinear method to analyze this system and identify the necessary conditions for finding solutions. The study explored the properties of multiple singular soliton and multiple soliton solutions within the system while highlighting the key requirements for obtaining N soliton solutions. To validate their approach, the researchers also used a trigonometric-function-based approach, which yielded 27 distinct solutions for the system. Importantly, these solutions were consistent with those obtained using the simplified bilinear method, confirming the effectiveness of their proposed methodology. This research introduced new findings and suggested that the same approach could be applied to analyze other coupled systems.

In 2018, Imad Jaradat et al. [31] introduced a new category of nonlinear physical models known as dual-mode nonlinear equations. This study involved the establishment and examination of dual-mode Kuramoto–Sivashinsky equation. The solutions for solitary and singular solitary aspects of this equation were successfully obtained. They did this by using a new method called the Cole-Hopf transformation in the modified simplified bilinear method. Lastly, they figured out a periodic solution for TMKS using the tanh-coth expansion method.

In 2019, Marwan Alquran et al. [32] utilized three different ansatz methods, namely the rational sine-cosine method, the tanh-expansion method and the Kudryashov-expansion method, to explore solitary wave solutions for a recently developed nonlinear equation. This study focused on examining a novel generalized Hirota-Satsuma KdV model with second-order time derivatives. This model featured parameters associated with dispersion, phase-velocity and nonlinearity. It was categorized as a dual-mode system where each field function exhibited the propagation of dual-mode waves instead of single-mode waves. The researchers provided graphical explanations to illustrate how dispersion, nonlinearity and phase-velocity factors influenced the spacing of these dual-waves. Through the application of the three ansatz methods and a careful analysis of the system's parameters, the study contributed to a deeper understanding of the behavior and characteristics of the generalized Hirota-Satsuma coupled KdV system.

In 2020, Marwan Alquran and his team [33] introduced a novel dual-mode Kawahara equation, which introduced new parameters associated with dissipative effects, nonlinearity, and interaction phase velocity. The researchers investigated the solutions of this model using three different methods: the Kudryashov-scheme, the sine-cosine function methods, and the tanh-scheme. To gain a better understanding of the dynamics and characteristics of these solutions, the researchers used 2D and 3D graphical plots to visualize their shapes. Additionally, they analyzed how the interaction between the dual-waves within the model was affected by changes in the phase-velocity parameter. This study provided valuable insights into the behavior of the dual-mode Kawahara equation and its solutions, shedding light on how various parameters influenced the dynamics and interaction of the dual-waves within the system.

In 2021, I. A. Jaradat and M. Alquran [34] introduced the two-mode Fisher-Burgers equation, relevant in biophysics for characterizing bidirectional wave motion. They used extended exponential-expansion and the extended tanh–coth methods to derive innovative solutions and provided insights into the equation's real-world applications. The study highlighted that the solutions to this equation exist when the nonlinearity and dispersion parameters are equal. Future research should explore scenarios where these parameters are not equal, broadening the equation's practical utility. The authors identified two types of solutions: kink and singular-kink and suggested further exploration to uncover additional solution types using alternative methods. Additionally, they proposed the investigation of Lie group analysis and conservation laws to gain deeper insights into the equation's behavior.

In 2022, Marwan Alquran et al. [35] introduced a new perspective on the foam drainage equation by showing the two-mode foam drainage equation. This novel model captures the movement of symmetric double-waves and is shaped by two key parameters: phase-velocity and nonlinearity. The researchers explored solutions using two recently developed methods, uncovering various physical structures, including periodic shapes and kink-type formations. They carefully examined the influence of both phase-velocity and nonlinearity through graphical analysis, providing a comprehensive grasp of their impact on the system's behavior. To ensure the model accuracy, they validated the obtained solutions through direct substitution, affirming their reliability. This study brings a unique angle to the foam drainage equation, emphasizing the interplay of nonlinearity and phase-velocity, and offers a detailed analysis of various physical structures through graphical techniques and rigorous validation. In 2022, Ismail et al. [36], the VietaLucas method's applicability in solving fractional order differential equations is explored. The research emphasizes the method's effectiveness in handling both linear and nonlinear fractional differential equations. It provides solutions in the form of convergent series with easily computable terms without the need for linearization or restrictive assumptions. The outcomes of this approach closely align with results obtained through other established techniques like the variation cycle method and the Adomian decomposition method. In general, this method can be used for a wide range of math and physics problems, whether they're linear or nonlinear.

In the same year, M. Alquran and R. Alhami [37] introduced a novel dual-mode generalization of the Cahn–Allen equation and explored its mathematical properties. This new model was thoroughly examined by three mathematical techniques Kudryashov-expansion, rational sin/cos method and rational sinh/cosh methods, leading to the discovery of various exact traveling solutions. Notably, these solutions were found to represent symmetric bidirectional waves, exhibiting unique characteristics, including convex-periodic, kink-periodic, peakon-soliton and kink-soliton patterns. This work demonstrates the versatility of two-mode equations, suggesting their potential role as information transmitters across different locations while preserving their fundamental physical attributes. The two-mode waves in these equations has the potential to be useful in many different areas. It can connect mathematical ideas with the needs of the real world, especially when it comes to transmitting information.

In 2023, Mohammed Ali et al. [38] introduced the dual-mode version of the generalized KdV equation (gTMKdV). This newly introduced model can be employed in scenarios involving weakly dispersive and nonlinear wave mediums, effectively depicting the behavior of asymmetric or symmetric binary waves depending on their individual amplitudes. The interaction and behavior of these binary waves are primarily governed by dispersion parameters and nonlinearity. To derive explicit solutions for various combinations of nonlinearity and dispersion factors, the researchers proposed two effective solution schemes. These schemes facilitated the extraction of specific solutions within the gTMKdV framework.

To further advance the understanding and application of the dual-mode model, this study will explore its relevance in the context of the SMCH and Gardner equations(derived from the ideal fluid model). The SMCH, a basic one-dimensional model described in [39], and the Gardner equation, which is derived from the ideal fluid model as outlined in [40], both represent different mathematical frameworks. However, it is noteworthy that both of these models have not yet been solved for the dual-mode model.

To address this gap in the existing literature and build upon the introduced TMKdV model, this research aims to extend the study of the SMCH and Gardner equations by solving them for dual-mode model. By doing so, the study will contribute to a more comprehensive understanding of how these models behave in the presence of weak dispersion and nonlinearity, showing how they behave in the dual-mode situation. To achieve this, we will propose and employ effective solution schemes, allowing for the extraction of specific solutions within the TMKdV framework, as well as extending the scope to encompass the SMCH and Gardner equations. This work will pave the way for a deeper exploration of the behavior and applications of these models, ultimately enhancing our understanding of wave dynamics in various physical systems.

This thesis is structured in the following manner:

- In chapter 2, we will illustrate the exponential expansion method, tanh/coth method, Kudryashov method, direct tan/cot approach and the rational sine/cosine method.
- In Chapters 3 and 4, we will apply the exponential expansion method, tanh/coth method, Kudryashov method, direct tan/cot approach and the rational sine/cosine method to address the dual mode SMCH equation and the Gardner equation (derived from the ideal fluid model).
- Chapter 5 provides a summary and draws conclusions.
- Chapter 6 presents an overview of potential future work.

Chapter 2

Methodologies

This section encompasses the explanation of the exponential expansion method, tanh/coth approach, rational sine/cosine methods, tan/cot technique and the contributions of Kudryashov approach.

2.1 Overview of tanh/coth expansion method

This section provides a brief overview of the tanh-coth method. E.J. Parkes, B.R. Duffy [41] introduced the tanh/coth method.

Now, let's examine the NLEE:

$$S(u, u_t, u_x, u_{xx}, u_{tt}, u_{tx}) = 0. \tag{2.1}$$

Here, S is a function involving the variables $u, u_t, u_x, u_{tt}, u_{xx}$, and $u_{tx} = 0$. The subscripts indicate partial derivatives with respect to t and x for the function $u(x,t)$. Assuming $u(x,t) = U(\xi)$, where $\xi = x \pm \omega t$ and ω is a constant representing the velocity of the traveling wave, Eq. (2.1) is simplified into a nonlinear ODE for $u = U(\xi)$.

$$R(U, U', U'', U''', ...) = 0. \tag{2.2}$$

In this context, R is a function that depends on U, U', U'', U''' and so forth. The Eq. (2.2) is integrated until all terms involve derivatives, with the integration constants being neglected. The tanh/coth method [42] follows the use of finite expansion. Consider a solution in the format of a traveling wave after integrated form of Eq. (2.2) as follows:

$$U(\xi) = \sum_{i=0}^{M} a_i Y^i, \tag{2.3}$$

$$Y = \tanh(\xi), \quad \text{or} \quad \coth(\xi).$$

Results in the modification of derivatives.

$$\begin{aligned} Y' &= \lambda(1 - Y^2), \\ Y'' &= -2\lambda^2 Y(1 - Y^2), \\ Y''' &= -2\lambda^3(1 - Y^2)(1 - 3Y^2). \end{aligned} \tag{2.4}$$

In the tanh/coth method, it is assumed that M is a positive integer, determined through the homogeneous balance approach. To ascertain the value of M, the highest order nonlinear terms and highest order linear terms are commonly set equal in the final equation. To address the case when parameter M is not an integer, a transformation technique is employed. By substituting (2.3) along with (2.4) into ODE (2.2), an algebraic system of equations in powers of $Y(\xi)$ is derived. This system allows the computation of the values of a_i (where $i = 0, ..., M$) and Y. The exact solution of $u(x, t)$ is then obtained in the form of tanh or coth. Additionally, various types of solutions, such as solitons, compactons, peakons, cuspons, kinks, periodic and traveling wave solutions, can be identified within the solutions obtained.

2.2 Overview of Kudryashov technique

This section provides a brief overview of the Kudryashov method, with a specific emphasis on its applicability to nonlinear PDEs. These are the Kudryashov method's primary steps:

Step 1: Assume that the solution to Eq. (2.2) can be expressed in the following manner:

$$U(\xi) = \sum_{i=0}^{M} a_i h^i(\xi). \tag{2.5}$$

Assuming that the constants a_i (where $i = 0, 1, ..., M$) are determined algebraically, with the condition $a_M \neq 0$, the function $h = h(\xi)$ fulfills the following Ricatti differential equation:

$$\frac{dh}{d\xi} = h^2(\xi) - h(\xi). \tag{2.6}$$

It is clear that the solution of Eq. (2.6) is

$$h(\xi) = \frac{1}{1 + d\exp(\xi)}, \quad d \neq 0. \tag{2.7}$$

Where d is Kudryashov constant.

Step 2: The determination of the positive integer M in Eq. (2.5) involves applying the homogeneous balance method to the highest power nonlinear terms and the highest order derivatives present in Eq. (2.2).

Step 3: By substituting the given $U(\xi)$ from Eq. (2.5), along with the necessary derivatives U, U'', U''', ... into Eq. (2.2), we derive a polynomial equation.

$$U(h(\xi)) = 0. \tag{2.8}$$

Step 4: We identify all terms with identical algebraic powers of h in the polynomial equation given by Eq. (2.8). By setting these terms equal to zero, we derive a system of algebraic equations with the unknowns $a_i(i = 0, 1, ..., M), k, \omega$. Utilizing calculation software like Maple, we can solve this system, taking into account the constraints of the model and ensuring that $a_M \neq 0$.

Step 5: By substituting the determined values and Eqs. (2.7) and (2.6) into Eq. (2.5), we generate all the solutions for Eq. (2.2) and consequently, for Eq. (2.1).

2.3 Overview of rational sine/cosine method

In this section, a brief explanation of the rational sine/cosine method is presented.

Step 1: Based on the findings outlined in reference [43], the solution of Eq. (2.2) can be expressed in the following manner:

$$U(\xi) = \frac{a_0}{1 + a_1 \sin(k\xi)}, \quad \sin(k\xi) \neq -1/a_1. \tag{2.9}$$

or

$$U(\xi) = \frac{a_0}{1 + a_1 \cos(k\xi)}, \quad \cos(k\xi) \neq -1/a_1. \tag{2.10}$$

Where $a_0,\ a_1$ and k are parameters that will be determined.

Step 2: Inserting either Eq. (2.9) or Eq. (2.10) into the equation derived in Eq. (2.2), equate the terms involving sine functions from Eq. (2.9) or cosine functions when Eq. (2.10) is applied. Utilize Maple to solve the resulting system of algebraic equations and determine all viable parameter values for $a_0,\ a_1,\ \omega$ and k.

2.4 Overview of exponential expansion method

In this section, concise summary of the novel exponential expansion approach is provided. Let's assume that the traveling wave solution for Eq. (2.2) takes the form:

$$U(\xi) = \sum_{i=0}^{M} a_i (\exp(-\phi(\xi)))^i, \quad a_M \neq 0. \tag{2.11}$$

Here, the coefficients a_i (where $0 \leq i \leq M$) are constants, and $\phi = \phi(\xi)$ satisfies the first-order nonlinear ODE:

$$\phi'(\xi) = \exp(-\phi(\xi)) + h_1 \exp(\phi(\xi)) + h_2. \tag{2.12}$$

Where h_1 and h_2 are real constants. By solving Eq. (2.12) we get:

- **Case 1.** When $h_2^2 - 4h_1 > 0$ and $h_1 \neq 0$;

$$\phi(\xi) = \ln\left(-\frac{1}{2}\frac{\sqrt{h_2^2 - 4h_1}\tanh\left(\frac{1}{2}\sqrt{h_2^2 - 4h_1}(\xi + A)\right)}{h_1} - h_1\right). \tag{2.13}$$

- **Case 2.** When $h_2^2 - 4h_1 < 0$ and $h_1 \neq 0$;

$$\phi(\xi) = \ln\left(\frac{-h_2 + \sqrt{-h_2^2 + 4h_1}\tan\left(\frac{1}{2}\sqrt{-h_2^2 + 4h_1}(\xi + A)\right)}{2h_1}\right). \tag{2.14}$$

- **Case 3.** When $h_2 \neq 0$ and $h_1 = 0$;

$$\phi(\xi) = -\ln\left(\frac{h_2}{\exp(h_2(\xi + A)) - 1}\right). \tag{2.15}$$

- **Case 4.** When $h_2^2 - 4h_1 = 0$ and $h_1 \neq 0$;

$$\phi(\xi) = \ln\left(\frac{2h_2(\xi + A) + 2}{h_2^2(\xi + A)}\right). \tag{2.16}$$

- **Case 5.** When $h_2 = 0$ and $h_1 = 0$;

$$\phi(\xi) = \ln(\xi + A). \tag{2.17}$$

By equating the highest-order derivatives with the highest-order nonlinear terms in Eq. (2.2), it becomes possible to determine the value of the positive integer M.

Upon substituting (2.11) into (2.2) and utilizing (2.12) when necessary, a system of algebraic equations for a_i (where $0 \leq i \leq M$), h_1 and h_2 is obtained. Through symbolic computation tools such as Maple, these equations can be solved to find the values of the unknowns.

2.5 Overview of direct tanh/coth and tan/cot method

This section provides an overview of the direct tanh/coth and tan/cot methods, previously investigated by Wazwaz in [44]. The initial approach, as previously described, exhibits minimal distinctions from the upcoming discussion. In the earlier method, the approach was employed on ODEs, leading to the derivation of a system of algebraic equations. In contrast, the present discussion directly applies this approach to PDEs. This application involves the extraction of coefficients associated with powers of tanh / coth or tan / cot, resulting in a system of algebraic equations.

The tanh/coth technique:

The tanh method permits using the assumption to rewrite it in a more advantageous manner.

$$u(x, t) = a_0 + a_1 \tanh^M(k\xi). \tag{2.18}$$

Parameters a_0, a_1 and c are to be determined. An analogous equation for the coth method can be formulated as follows:

$$u(x,t) = a_0 + a_1 \coth^M(k\xi). \tag{2.19}$$

The tan/cot approach

The tan method permits the utilization of the stated assumption;

$$u(x,t) = a_0 + a_1 \tan^M(k\xi). \tag{2.20}$$

Also, representing the cot method in a similar manner.

$$u(x,t) = a_0 + a_1 \cot^M(k\xi). \tag{2.21}$$

Chapter 3

Novel Solitary Wave Solutions of Dual-Mode Simplified Modified Camassa-Holm Equation in Shallow Water Waves

Within this chapter, we employ various methods, including tanh/coth, Kudryashov, rational sine/cosine, and Exp-expansion, to address the Simplified Modified Camassa-Holm (SMCH) Equation in Shallow Water Waves. This equation is prevalent in mathematical physics, and our aim is to provide a sophisticated and practical resolution to it. The primary objective of this research is to explore the behavior of two-waves in the context of SMCH and analyze their interactions under adjustable parameters. The model under consideration is outlined in previous works [45–48].

$$u_t + 2\mu u_x - u_{xxt} + \eta u^2 u_x = 0. \tag{3.1}$$

In this equation u stands for a field function, and constants η and μ are non-zero constants. Additionally, the investigation of the SMCH equation through Eq. (3.1) has provided insights into various significant and practical scenarios. It's worth noting that in theories related to shallow water waves, there is occasional interchangeability between the roles of the variables $x$and t [45, 49].

3.1 Dual-mode SMCH

In this study, our initial step involves expanding Eq. (3.1) into a dual-mode structure. This extension is achieved through the utilization of the formulations presented in Eqs. (1.24) and (1.25), expressed in a new form as follows:

$$u_{tt} - s^2 u_{xx} + \left(\frac{\partial}{\partial t} - \alpha s \frac{\partial}{\partial x}\right) \eta u^2 u_x + \left(\frac{\partial}{\partial t} - \beta s \frac{\partial}{\partial x}\right) (2\mu u_x - u_{xxt}) = 0. \tag{3.2}$$

The expression u_t has been modified to $u_{tt} - s^2 u_{xx}$, where s represents the phase velocity. In the second and third mappings, the manipulation involves the operator $(\frac{\partial}{\partial t} - \alpha s \frac{\partial}{\partial x})$ acting on non-linear terms and the operator $(\frac{\partial}{\partial t} - \beta s \frac{\partial}{\partial x})$ acting on linear terms. Here, β serves as the dispersion parameter, and α denotes the non-linearity.

Given that the function $u(x,t)$ in Eq. (3.2) is real-valued, we proceed with the following assumption:

$$u(x,t) = U(\xi), \quad \xi = x + \omega t. \tag{3.3}$$

Where the wave transformation is denoted as ξ. Substituting Eq. (3.3) into Eq. (3.2) results in the following ODE:

$$\begin{aligned} &(-2\alpha\eta s + 2\eta\omega)U(U')^2 + (-\alpha\eta s + \eta\omega)U^2 U'' \\ &+(-2\beta\mu s + 2\mu\omega + \omega^2 - s^2)U'' + (\beta\omega s - \omega^2)U^{(IV)} = 0. \end{aligned} \tag{3.4}$$

Now, we employ strategies such as tanh/coth, Kudryashov, rational sine/cosine, and exponential expansion methods [50–52] to explore solutions for the aforementioned system.

3.1.1 Application of tanh/coth method

The tanh/coth method, as described by [50], anticipates that the solution to (3.4) assumes the following expression:

$$U(\xi) = \sum_{i=0}^{M} a_i Y^i, \qquad a_M \neq 0. \tag{3.5}$$

To determine the appropriate index M in Eq. (3.5), we set the highest nonlinear term U^2U'' equal to the highest order derivative term $U^{(IV)}$ from Eq. (3.4). This comparison yields $M = 1$. Consequently, the suggested solution for Eq. (3.5) is:

$$U(\xi) = a_0 + a_1 Y(\xi). \tag{3.6}$$

For $Y(\xi)$ the auxiliary differential equation is given as:

$$\begin{aligned} Y' &= \lambda(1 - Y^2), \\ Y'' &= -2\lambda^2 Y(1 - Y^2), \\ Y''' &= -2\lambda^3(1 - Y^2)(1 - 3Y^2). \end{aligned} \tag{3.7}$$

The solutions can take the form of either $Y(\xi) = \tanh(k\xi)$ or $Y(\xi) = \coth(k\xi)$. Upon differentiating Eq. (3.6), we deduce that:

$$\begin{aligned} \frac{dU}{d\xi} &= a_1\lambda(1 - Y(\xi)^2), \\ \frac{d^2U}{d\xi^2} &= -2a_1\lambda^2 Y(\xi)(1 - Y(\xi)^2), \\ \frac{d^4U}{d\xi^4} &= 16a_1\lambda^4(1 - Y(\xi)^2)^2 Y(\xi) - 8a_1\lambda^4 Y(\xi)^3(1 - Y(\xi)^2). \end{aligned} \tag{3.8}$$

By inserting the values from Eqs. (3.6) and (3.8) into Eq. (3.4) and equating the coefficients of $Y(\xi)^i$ for i = 0, 1, 2, 3, 4, 5 to zero, we derive an algebraic system. Solving this system by Maple yields following values of unknowns:

$$\alpha = \pm 1,\ \beta = \pm 1,\ \omega = \beta s,\ a_0 = 0,\ a_1 = \mu,\ \lambda = \mp\frac{\sqrt{s\mu}}{s}. \tag{3.9}$$

Substituting these values in Eq. (3.2) gives the following exact solution:

$$u_1(x,t) = \mu \tanh\left(\frac{\sqrt{s\mu}(\beta s t + x)}{s}\right), \tag{3.10}$$

or,

$$u_2(x,t) = \mu \coth\left(\frac{\sqrt{s\mu}(\beta s t + x)}{s}\right). \tag{3.11}$$

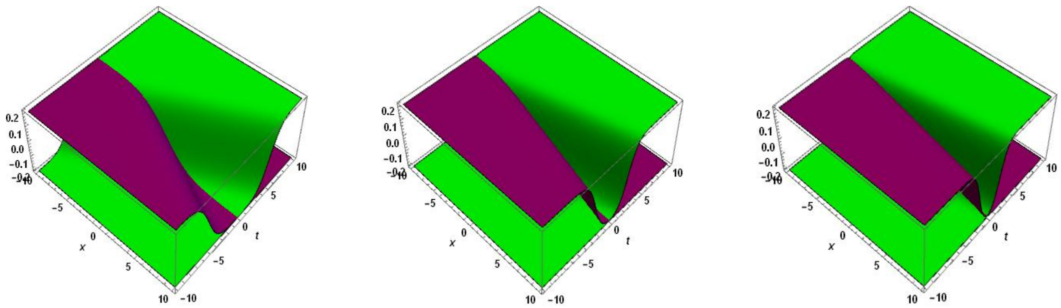


Figure 3.1: Kink wave solution of $u_1(x,t)$ for all parameters $\mu = 0.2,\ \beta = \pm 1,\ s = 2,\ 4,\ 6$ respectively.

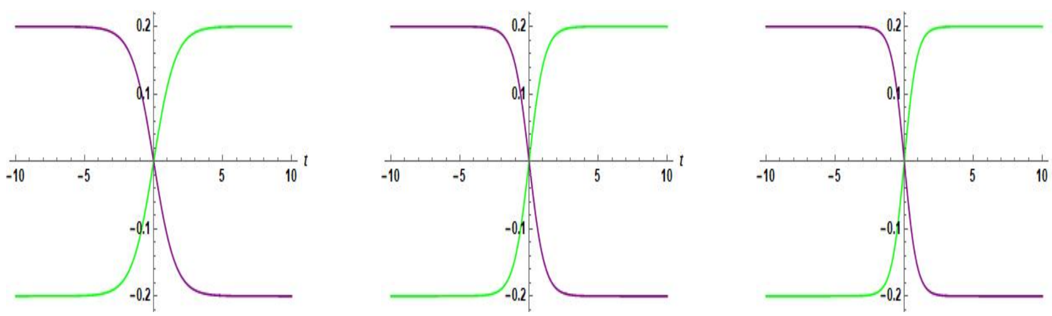


Figure 3.2: 2D profile of kink wave solutions $u_1(x,t)$ for parameters $\mu = 0.2,\ \beta = \pm 1,\ s = 2,\ 4,\ 6$ respectively.

3.1.2 Application of Exp-function Expansion method

Here, we use a modification of exponensial function expansion method [53] to solve Eq. (3.2). After balancing the highest degree nonlinear term with highest order dispersive term in Eq. (3.4) gives $M = 1$. Substitute the value of M in Eq. (2.11) and get:

$$U(\xi) = a_0 + a_1 \exp(-\phi(\xi)). \tag{3.12}$$

Where $\phi(\xi)$ satisfied the equation:

$$\phi'(\xi) = \exp\left(-\phi(\xi)\right) + h_1 \exp\left(\phi(\xi)\right) + h_2. \tag{3.13}$$

Where h_1 and h_2 are real constants. For Eq. (3.13), other solutions are discussed in [53], but we are specifically seeking only this solution,

$$\phi(\xi) = \ln\left(-\frac{1}{2}\frac{\sqrt{h_2^2-4h_1}\tanh\left(\frac{1}{2}\sqrt{h_2^2-4h_1}(\xi+A)\right)}{h_1} - h_1\right). \tag{3.14}$$

Under the conditions $h_2^2 - 4h_1 > 0$, $h_1 \neq 0$, $a_1 \neq 0$, and a_0 being a constant to be determined, we substitute Eq. (3.12) along with Eqs. (3.13) and (3.14) into Eq. (3.4). By equating different powers of $\exp(-\phi(\xi))$ to zero, a system of algebraic equations can be obtained. Solving this resulting system for the unknowns α, β, ω, h_1, h_2, a_0, and a_1 yields the following outcomes:

$$\alpha = \pm 1,\ \beta = \pm 1,\ \omega = \beta s,\ a_0 = a_0,\ a_1 = \eta,\ h_1 = -\frac{1}{2}\eta,\ h_2 = \mu. \tag{3.15}$$

The solution to the SMCH equation using the Exp-function method is expressed as:

$$u_3(x,t) = a_0 + \frac{\eta}{\frac{\sqrt{4\mu^2+2\eta}\tanh\left(1/2\sqrt{4\mu^2+2\eta}(\beta st+A+x)\right)}{\eta} + \frac{1}{2}\eta}. \tag{3.16}$$

With condition $\sqrt{4\mu^2+2\eta} > 0$ and $\eta \neq 0$.

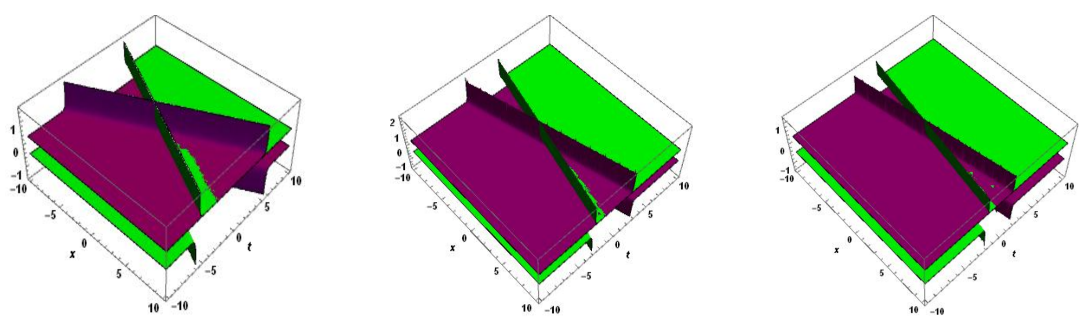


Figure 3.3: Singular solution of $u_3(x,t)$ for $a_0 = 0.1,\ \mu = 0.2,\ \eta = 0.3,\ A = 1,\ \beta = \pm 1,\ s = 2,\ 4,\ 6$ respectively.

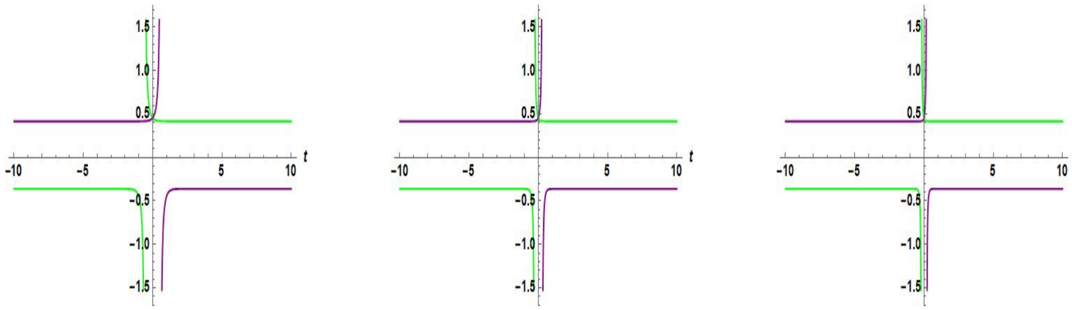


Figure 3.4: 2D profile of singular solution $u_3(x,t)$ with $a_0 = 0.1,\ \mu = 0.2,\ \eta = 0.3,\ A = 1,\ \beta = \pm 1,\ s = 2,\ 4,\ 6$ respectively.

3.1.3 Application of Kudryashov method

To find more possible solutions to Eq. (3.2), we plan to apply the Kudryashov technique [54].The method proposed for achieving this is the extended Kudryashov-expansion method [55]. This methodology exhibits parallels with the tanh / coth function approach, as it employs the identical proposed solution and leverages the inherent property of the balancing procedure. However, it differs in the type of variables used. Specifically, the proposed solution of Eq. (3.2) is employed in this extended Kudryashov-expansion method for obtaining the desired solitary wave solutions:

$$U(\xi) = a_0 + a_1 Y(\xi). \tag{3.17}$$

Here $Y(\xi)$ represents the solution of the auxiliary differential equation,

$$Y'(\xi) = \lambda Y(\xi)(Y(\xi) - 1). \tag{3.18}$$

The result for Eq. (3.18) is:

$$Y(\xi) = \frac{1}{1 + d\ \exp(\lambda\ (\xi))}. \tag{3.19}$$

Where d is Kudryashov constant, the first second and fourth derivatives of Eq. (3.17) are:

$$\begin{aligned}
\frac{dU}{d\xi} &= a_1\lambda Y(\xi)(Y(\xi)-1),\\
\frac{d^2U}{d\xi^2} &= a_1\lambda^2 Y(\xi)(Y(\xi)-1)^2 + a_1\lambda^2 Y(\xi)^2(Y(\xi)-1),\\
\frac{d^4U}{d\xi^4} &= a_1\lambda^4 Y(\xi)(Y(\xi)-1)^4 + 11a_1\lambda^4 Y(\xi)^2(Y(\xi)-1)^3 + 11a_1\lambda^4 Y(\xi)^3(Y(\xi)-1)^2 +\\
&\quad a_1\lambda^4 Y(\xi)^4(Y(\xi)-1).
\end{aligned} \tag{3.20}$$

Subsequently, the incorporation of Eqs. (3.17) and (3.20) along with Eq. (3.19) into Eq. (3.4) allows for the aggregation of coefficients corresponding to identical powers of $Y(\xi)$. By equating each coefficient to zero, an algebraic system emerges, involving the variables α, β, ω, λ, a_0, and a_1. The resolution of this recently established system yields the following result for unknown parameters:

$$\alpha = \pm 1,\ \beta = \pm 1,\ a_0 = -\frac{1}{2}\eta,\ a_1 = \eta,\ \omega = \beta s,\ \lambda = 2\sqrt{\frac{\mu}{s}}. \tag{3.21}$$

Through the application of the Kudryashov scheme, we have derived two-wave solutions for Eq. (3.2), taking the form of kink waves when $d > 0$, as illustrated in Figs. (3.1) and (3.2). Conversely, it yields a singular solution when $d < 0$, as depicted in Figs. (3.3) and (3.4).

$$u_4(x,t) = -\frac{1}{2}\eta + \frac{\eta}{1 + d\exp\left(2\sqrt{\frac{\mu}{s}}(\beta s t + x)\right)}. \tag{3.22}$$

3.1.4 Application of rational sine-cosine method

The application of the rational sin / cos technique allows for the utilization of the ansatz [56].

$$U(\xi) = \frac{a_0}{1 + a_1\cos(\lambda\xi)}, \quad a_1\cos(\lambda\xi) \neq -1, \tag{3.23}$$

and

$$U(\xi) = \frac{a_0}{1 + a_1\sin(\lambda\xi)}, \quad a_1\sin(\lambda\xi) \neq -1. \tag{3.24}$$

By incorporating either Eq. (3.23) or Eq. (3.24) into Eq. (3.4), a polynomial equation is formed, containing terms with cos or sin functions. The coefficients of these trigonometric functions, when combined and set to zero, give rise to a system of algebraic equations involving the variables a_0, a_1, λ, and ω. This conversion effectively shifts the problem from a differential domain to an algebraic one. Solving this system of algebraic equations enables the determination of the values for the unknown parameters as described below:

$$\lambda = \frac{\sqrt{2\mu s}}{s},\ \alpha = \pm 1,\ \beta = \pm 1,\ \omega = \beta s. \tag{3.25}$$

Where a_0 and a_1 are free parameters and μ is constant. After substituting these values in Eq. (3.2) we have these solutions:

$$u_5(x,t) = \frac{a_0}{1 + a_1 \cos((\beta st + x)\frac{\sqrt{2\mu s}}{s})}, \qquad a_1 \cos(\beta st + x) \neq -1,\ \sqrt{2\mu s} > 0, \tag{3.26}$$

or

$$u_6(x,t) = \frac{a_0}{1 + a_1 \sin((\beta st + x)\frac{\sqrt{2\mu s}}{s})}, \qquad a_1 \sin(\beta st + x) \neq -1,\ \sqrt{2\mu s} > 0. \tag{3.27}$$

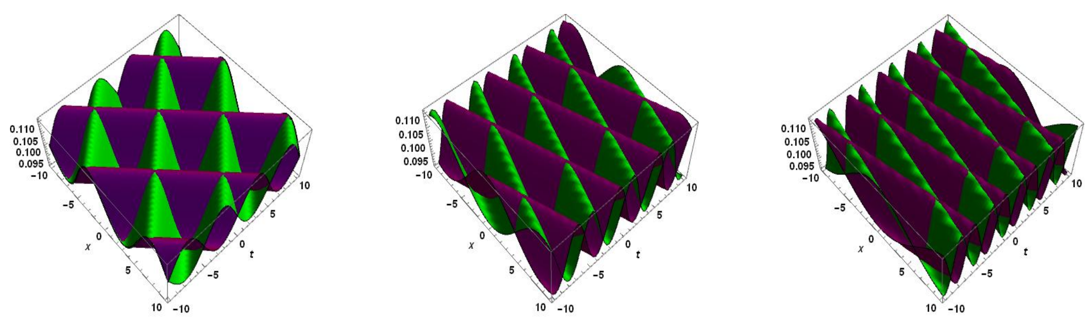


Figure 3.5: Periodic solution $u_6(x,t)$ for $a_0 = 0.1$, $a_1 = 0.1$, $\mu = 0.2$, $\beta = \pm 1$, $s = 2,\ 4,\ 6$ respectively.

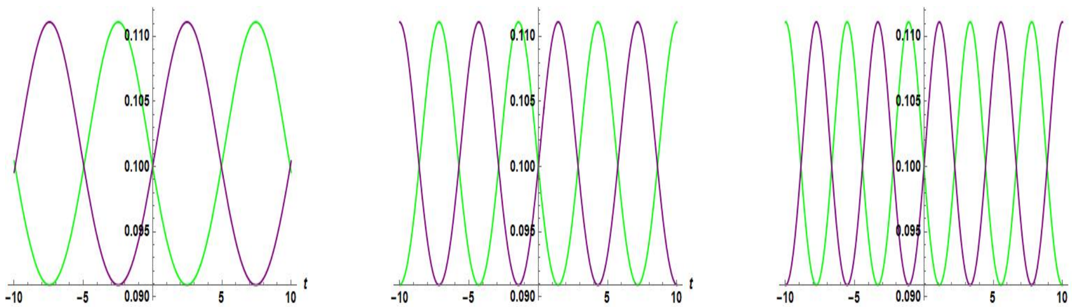


Figure 3.6: Periodic solution for $u_6(x,t)$ with parametersr $a_0 = 0.1,\ a_1 = 0.1,\ \mu = 0.2,\ \beta = \pm 1,\ s = 2,\ 4,\ 6$ respectively.

3.2 Discussion, graphical interpretation and importance of method

In this study, a novel dual-mode SMCH model is introduced. To extract exact solutions for this model, four solitary ansatz methods are employed. The results of this investigation are summarized, emphasizing the visual representation of solutions obtained through these methods. Specifically, tanh / coth method is utilized to derive the kink wave solution for $u_1(x,t)$, as illustrated in Figs. (3.1) and (3.2). Singular kink wave type solution obtained through Exp-function expansion technique as described 2D and 3D graphs in Figs. (3.3) and (3.4). By applying Kudryashov-expansion technique we get a solution $u_4(x,t)$, in this solution the graphical representation depends on d which is Kudryashov constant. When $d > 0$ kink wave solution obtained, when $d < 0$ singular kink wave type solution obtained. We didn't describe the graphical representation for $u_4(x,t)$ because they are similar as described in Figs. (3.1), (3.2), (3.3) and (3.4). A periodic solution is attained through the application of the rational sin / cos method as described in Figs. (3.5) and (3.6).

Ultimately, our study explores the influence of both the linearity-dispersive parameter β and the phase velocity s on the propagation characteristics of the dual waves in the proposed model. It is observed that as the phase velocity increases, there is a convergence of the left and right waves. This phenomenon of left and right waves converging as phase velocity increases holds significance in applications such as communication systems, turbulence management and shallow water wave patterns. However, these methods play a crucial role in solving nonlinear problems efficiently. The Kudryashov method uses a series representation, the tanh method employs hyperbolic tangent functions, Exp-function expansion utilizes exponential functions, and the rational sin/cos method applies sine and cosine functions. The choice depends on the specific equation and problem. These approaches are vital in fields like signal processing. They offer unique insights into solitons and wave behavior, with kink wave solutions providing valuable information. Singular solutions, revealing phenomena like shockwave propagation, demonstrate their effectiveness compared to traditional methods. These findings enhance our understanding of wave behavior and broaden their practical applications.

Limitations of methods: The Kudryashov method, under consideration, has two distinct limitations. Firstly, its efficacy in solving equations with multiple branches in the Laurent series expansion of the general solution is constrained. It's noteworthy that this limitation is not unique to the Kudryashov method; many other methods face similar challenges in such cases. Secondly, the technique is not well-suited for identifying two periodic solutions. It is crucial to emphasize that when applied to equations featuring non-zero real balancing numbers, both the tanh method and the Exp-function expansion method prove ineffective.

Chapter 4

Analytical Solutions of Dual-Mode Gardner Equation derived from Ideal Fluid Model

In this chapter, we apply direct techniques such as tanh / coth and tan / cot. Widely recognized in mathematical physics, Gardner equation derived from ideal fluid model is a focal point of our investigation, with the goal of providing an advanced and practical solution. The principal goal of this study is to investigate the dynamics of two-waves within the framework of GE, examining their interactions across adjustable parameters. The theoretical framework for this model was recently introduced and documented in a publication by [57].

$$u_t + u_x + \frac{3}{2}\,\alpha\, u u_x - \frac{3}{8}\,\alpha^2 u^2 u_x + \beta\,\left(\frac{1-3\tau}{6}\right) u_{xxx} = 0. \tag{4.1}$$

This equation is fundamental to the analysis and is employed to investigate the behavior of certain non-dimensional parameters: $\alpha = a/H$, wavelength parameter $\beta = H^2/L^2$ and the bond number $\tau = T/\rho g H^2$. These non-dimensional parameters are essential to our research because they help us to understand the behavior of the system under consideration. In this context, g represents the gravitational acceleration, H signifies the average upstream depth, a and L denote characteristic values for wave amplitude and wavelength. We also have T as the coefficient for surface tension and ρ representing the density of water.

4.1 Dual-mode Gardner equation

In this study, we first formulate Eq. (4.1) as a two-mode framework. Using the formulations described in Eqs. (1.24) and (1.25), which have been reformulated as follows, this expansion is achieved:

$$u_{tt} - s^2 u_{xx} + \left(\frac{\partial}{\partial t} - \eta s \frac{\partial}{\partial x}\right)\left(\frac{3}{2}\alpha u u_x - \frac{3}{8}\alpha^2 u^2 u_x\right) + \left(\frac{\partial}{\partial t} - \mu s \frac{\partial}{\partial x}\right) \left(\frac{\beta(1-3\tau)}{6} u_{xxx} + u_x\right) = 0. \tag{4.2}$$

The expression u_t has been modified to $u_{tt} - s^2 u_{xx}$, where s represents the phase velocity. In the second and third mappings, the manipulation involves the operator $(\frac{\partial}{\partial t} - \eta s \frac{\partial}{\partial x})$ acting on non-linear terms and the operator $(\frac{\partial}{\partial t} - \mu s \frac{\partial}{\partial x})$ acting on linear terms. Here, μ serves as the dispersion parameter, and η denotes the non-linearity.

In this section, the methods discussed earlier will be employed to derive novel exact solutions for the dual-mode Gardner equation, as presented in Eq. (4.2). A concise overview of these techniques was provided in the preceding section. Assuming that the function $u(x,t)$ in Eq. (4.2) is a real-valued function, the following assumption is made:

$$u(x,t) = f(\xi), \qquad \xi = k(x - ct). \tag{4.3}$$

The transformation is referred to as the travelling wave transformation. By substituting this expression into Eq. (4.2), the subsequent ODE is obtained:

$$\left(\frac{-3}{2}\eta\, s\alpha\, k^2 - \frac{3}{2}\alpha\, k^2 c\right)(f')^2 + \left(\frac{3}{4}\alpha^2 k^2 c + \frac{3}{4}\eta\, s\alpha^2 k^2\right) f (f')^2 + (k^2 c^2 - \mu\, s k^2) f'' \tag{4.4}$$
$$(-s^2 k^2 - k^2 c) f'' + \left(\frac{-3}{2}\eta\, s\alpha\, k^2 - \frac{3}{2}\alpha\, k^2 c\right) f f'' + (\frac{3}{8}\alpha^2 k^2 c + \frac{3}{8}\eta\, s\alpha^2 k^2) f^2 f''$$
$$+ \left(-\frac{1}{6}\beta\, k^4 c - \frac{1}{6}\mu\, s\beta\, k^4 + \frac{1}{2}\mu\, s\beta\, k^4 \tau + \frac{1}{2}\beta\, k^4 c\tau\right) f'''' = 0.$$

Upon performing integration on this differential equation and assuming that the constants of integration are set to zero, we obtain the following simplified form:

$$\frac{1}{8}\left(\alpha^2\eta\, s+\,\alpha^2 c\right) f^3+\frac{3}{4}(-\,\alpha\,\eta\, s-\,\alpha\, c)f^2+\left(c^2-\,\mu\, s-\,s^2-c\right) f \tag{4.5}$$
$$+\,\frac{1}{3}\,k^2\,(3\,\beta\,\mu\, s\tau+3\,\beta\, c\tau-\beta\,\mu\, s-4\beta\, c)\, f''=0.$$

Now, we employ direct methods such as tanh/coth and tan/cot techniques, as outlined in Section 2.6, to explore potential solutions.

4.1.1 Applications of direct tan/cot method

By equating the leading nonlinear term with the dispersive term in Eq. (4.5), it is determined that M equals 1. Substituting this value of M into Eq. (2.20) yields the following result:

$$u(x,t)=a_0+a_1\tan(k(x-ct)). \tag{4.6}$$

Upon substituting this assumption into Eq. (4.2) and isolating the coefficients corresponding to $\tan^i(k(x-ct))$ for $i=0,\ 2,\ 4,\ 6$, solving the resulting system provides the following values for unknown parameters:

Case 1: $\eta\neq\mu$

$$c=\pm s,\ a_0=\frac{2}{\alpha}, \tag{4.7}$$
$$a_1=\frac{2}{\alpha}(\sqrt{-\frac{9\beta\tau-3\beta}{3\beta k^2\tau-\beta k^2-3}}k),\qquad k^2(3\tau-1)-3>0.$$

As a result, this equation yields a periodic solution,

$$u_1(x,t)=\frac{2}{\alpha}(1+\sqrt{-\frac{9\beta\tau+3\beta}{3k^2\beta\tau-k^2\beta+3}}k\tan(k(st+x))). \tag{4.8}$$

Similarly for cot function we have following singular solution;

$$u_2(x,t)=\frac{2}{\alpha}(1+\sqrt{-\frac{9\beta\tau+3\beta}{3k^2\beta\tau-k^2\beta+3}}k\cot(k(st+x))). \tag{4.9}$$

Here is a graphical representation of $u_1(x,t)$, depicting the individual left and right waves, as well as a dual-modes graph.

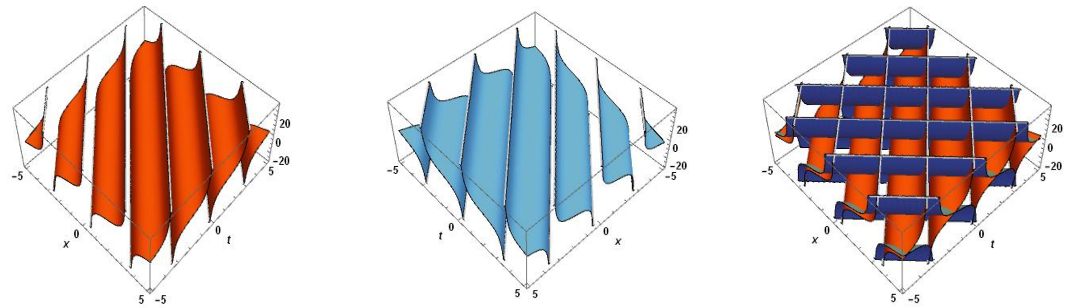


Figure 4.1: Left, right and dual waves for $u_1(x,t)$ when $s = 1,\ \tau = 1,\ k = 1,\ \alpha = 1$ and $\beta = 1$.

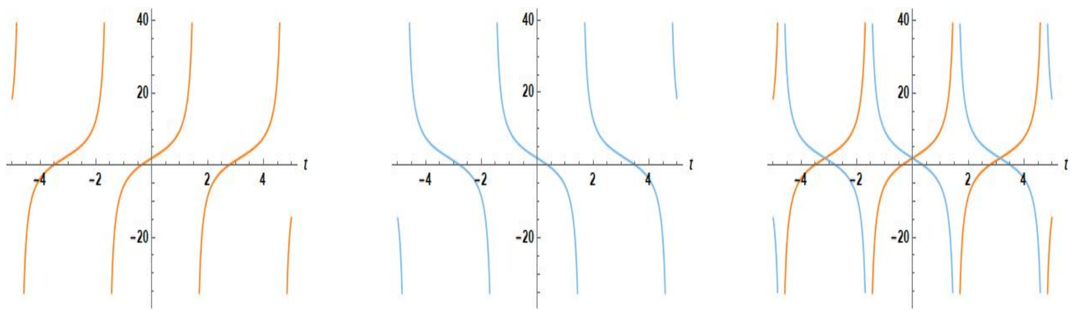


Figure 4.2: 2D graph of left, right and combined/dual waves for $u_1(x,t)$ when $s = 1,\ \tau = 1,\ k = 1,\ \alpha = 1$ and $\beta = 1$.

Next, 3D and 2D graphs depicting the variation in dual-mode behavior as the phase velocity is altered.

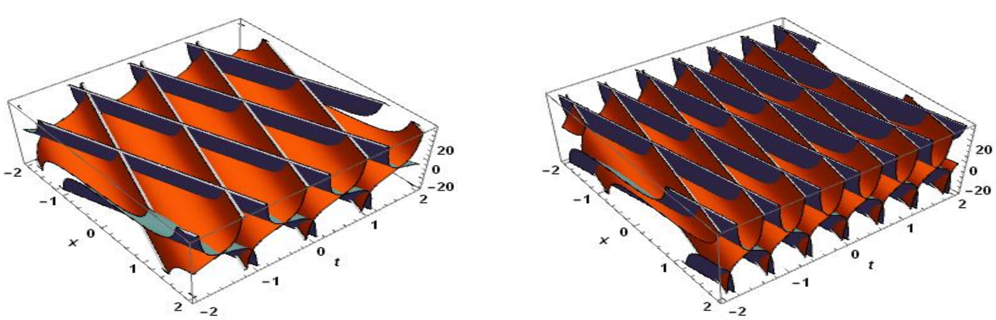


Figure 4.3: 3D graph of dual waves for $u_1(x,t)$ when $\beta = 1,\ \tau = 1,\ k = 1,\ \alpha = 1$ and $s = 3\ and\ 5$.

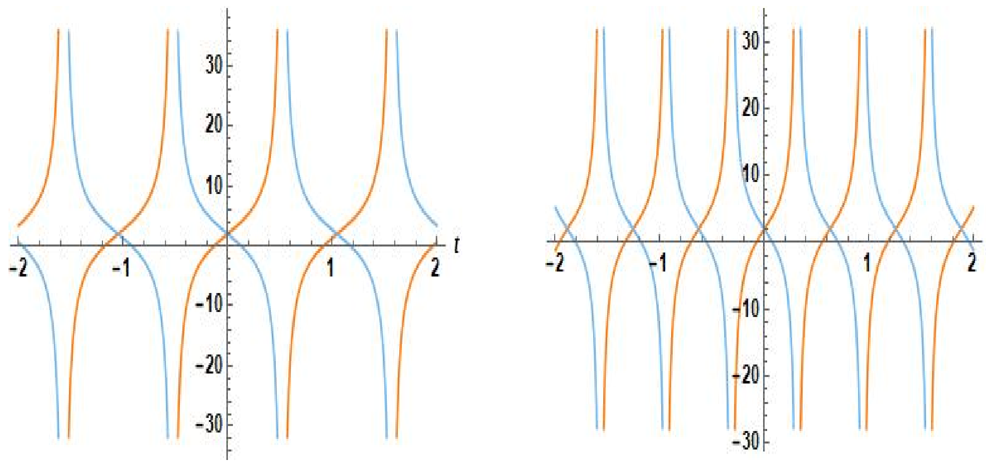


Figure 4.4: 2D graph of dual waves for $u_1(x,t)$ when $\beta = 1,\ \tau = 1,\ k = 1,\ \alpha = 1$ and $s = 3$ and 5.

Case 2: $\mu = \eta$

After substituting this assumption in Eq. (4.2) with $\mu = \eta$ and collecting the coefficients associated with the terms involving $\tan^i(k(x-ct))$ for values of i including 0, 2, 4 and 6. Subsequently, by solving the resulting system, which yields the following set of values for unknown parameters:

$$a_0 = \frac{2}{\alpha},\ a_1 = \frac{2\sqrt{-2\beta\tau + \frac{2}{3}\beta}k}{\alpha}, \quad \alpha \neq 0, \quad -2\beta\tau + \frac{2}{3}\beta > 0, \tag{4.10}$$

$$c = \pm\frac{1}{12}\sqrt{36(\tau - \frac{1}{3})^2k^4\beta^2 - 144(\tau - \frac{1}{3})k^2(\eta s + \frac{5}{4})\beta + 144s^2 + 360\eta s + 225} + \frac{5}{4} + \frac{1}{12}(-6\tau + 2)k^2\beta.$$

Where

$$(36(\tau - \frac{1}{3})^2k^4\beta^2 - 144(\tau - \frac{1}{3})k^2(\eta s + \frac{5}{4})\beta + 144s^2 + 360\eta s + 225) > 0.$$

As a result, this leads to the periodic solution.

$$
\begin{aligned}
u_3(x,t) &= \frac{2}{\alpha} + \frac{2\sqrt{-2\beta\tau + \frac{2}{3}\beta k}}{\alpha} \\
&\times \tan(k(\frac{1}{12}\sqrt{36(\tau-\frac{1}{3})^2k^4\beta^2 - 144(\tau-\frac{1}{3})k^2(\eta s+\frac{5}{4})\beta + 144s^2 + 360\eta s + 225} \\
&\quad + \frac{5}{4} + \frac{1}{12}(-6\tau+2)k^2\beta)t + x).
\end{aligned} \tag{4.11}
$$

Likewise, when considering the cot function, the following solution emerges:

$$
\begin{aligned}
u_4(x,t) &= \frac{2}{\alpha} + \frac{2\sqrt{-2\beta\tau + \frac{2}{3}\beta k}}{\alpha} \\
&\times \cot(k(\frac{1}{12}\sqrt{36(\tau-\frac{1}{3})^2k^4\beta^2 - 144(\tau-\frac{1}{3})k^2(\eta s+\frac{5}{4})\beta + 144s^2 + 360\eta s + 225} \\
&\quad + \frac{5}{4} + \frac{1}{12}(-6\tau+2)k^2\beta)t + x).
\end{aligned} \tag{4.12}
$$

Now, showing 2D and 3D graphical representations of the function $u_4(x,t)$.

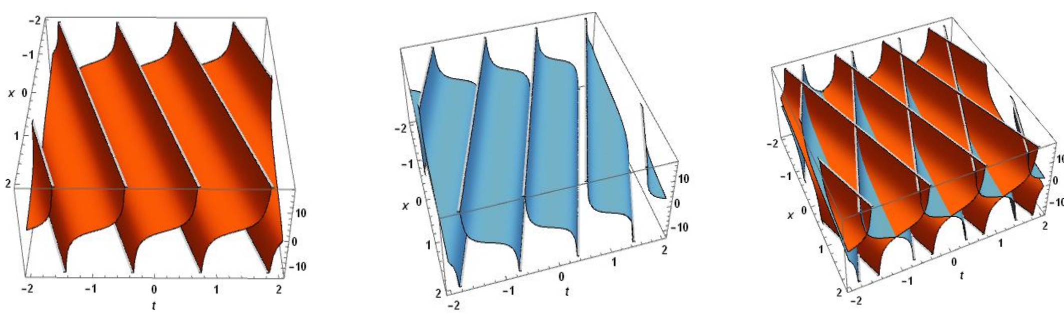


Figure 4.5: Left, right and dual waves for $u_4(x,t)$ when $s=1,\ \tau=-1,\ k=1,\ \alpha=1,\ \eta=0.1$ and $\beta=0.5$.

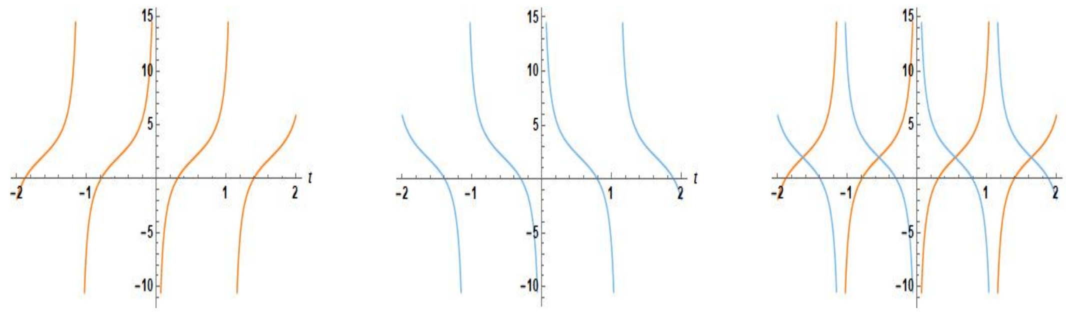


Figure 4.6: 2D graph of left, right and combined/dual waves for $u_4(x,t)$ when $s = 1,\ \tau = -1,\ k = 1,\ \alpha = 1,\ \eta = 0.1$ and $\beta = 0.5$.

Following this, 3D and 2D plots are presented to illustrate the changes in dual-mode behavior in $u_3(x,t)$ as the phase velocity is modified.

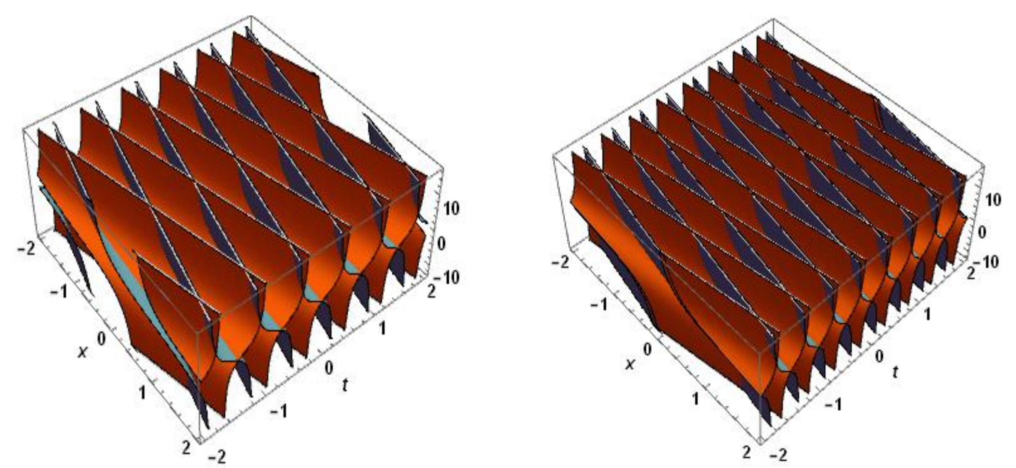


Figure 4.7: 3D graph of dual waves for $u_4(x,t)$ when $\beta = 0.5,\ \tau = -1,\ k = 1,\ \alpha = 1,\ \eta = 0.1$ and $s = 3$ and 5.

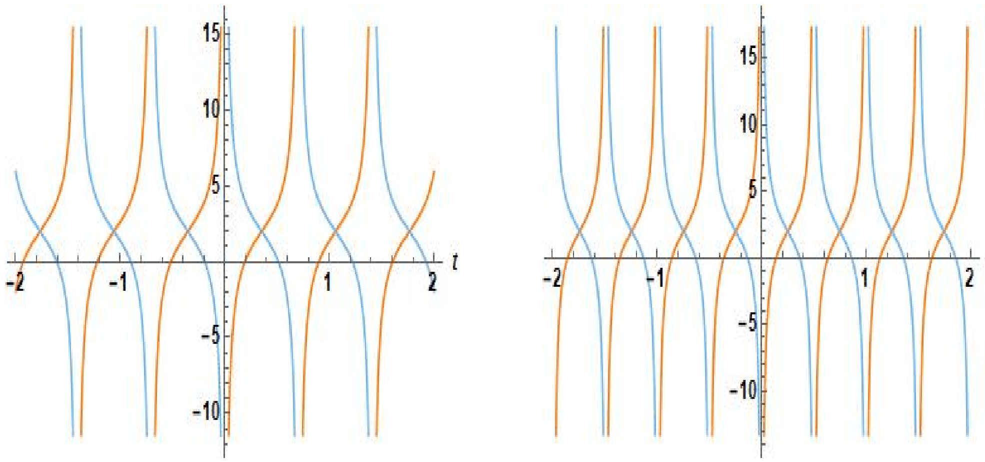


Figure 4.8: 2D graph of dual waves for $u_4(x,t)$ when $\beta = 0.5,\ \tau = -1,\ k = 1,\ \alpha = 1,\ \eta = 0.1$ and $s = 3$ and 5.

4.1.2 Applications of direct tanh/coth method

In the analysis, the tanh / coth approach is used to address the equation. By setting the higher-order nonlinear term equal to the dispersive term in Eq. (4.5), the value of $M = 1$ is determined. This value is then substituted into Eq. (2.18) to obtain the resulting outcome.

$$u(x,t) = a_0 + a_1 \tanh(k(x-ct)). \tag{4.13}$$

After substituting this assumption into Eq. (4.2) and collecting the coefficients related to $\tanh^i(k(x-ct))$ for $i = 0,\ 2,\ 4$ and 6 the resulting system is solved, providing the following values for unknown parameters:

Case 1: $\eta \neq \mu$

$$c = \pm s,\ a_0 = \frac{2}{\alpha}, \tag{4.14}$$
$$a_1 = \frac{2}{\alpha}\sqrt{-\frac{-9\beta\tau + 3\beta}{3\beta k^2\tau - \beta k^2 + 3}}k,$$
$$\text{where} \quad (36(\tau - \frac{1}{3})^2 k^4 \beta^2 - 144(\tau - \frac{1}{3})k^2(\eta s + \frac{5}{4})\beta + 144s^2 + 360\eta s + 225) > 0.$$

As a result, the obtained values for the parameters, when substituted into Eq. (4.13), gives a kink wave solution.

$$u_5(x,t) = \frac{2}{\alpha}(1 + \sqrt{-\frac{-9\beta\tau + 3\beta}{3\beta k^2\tau - \beta k^2 + 3}}k\tanh(k(st + x))). \tag{4.15}$$

Similarly, for $\eta \neq \mu$, we obtain the singular solution using the coth method as outlined earlier in Eq. (2.19).

$$u_6(x,t) = \frac{2}{\alpha}(1 + \sqrt{-\frac{-9\beta\tau + 3\beta}{3\beta k^2\tau - \beta k^2 + 3}}k\coth(k(st + x))). \tag{4.16}$$

Displaying 2D and 3D plots representing the left, right and dual-mode graphs of $u_6(x,t)$.

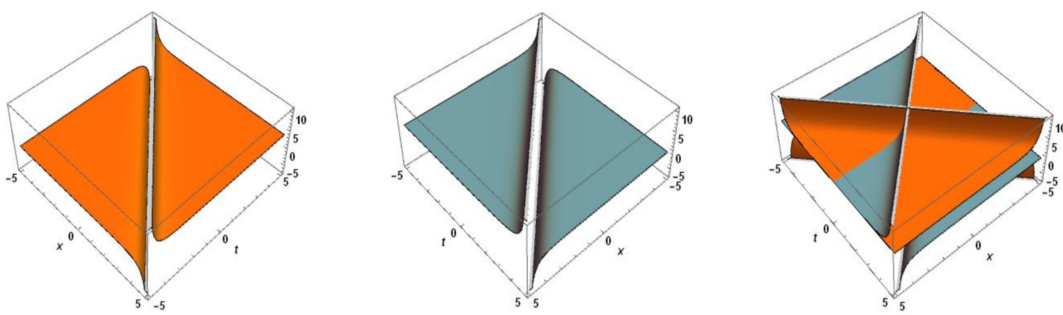


Figure 4.9: Left, right and dual waves for $u_6(x,t)$ when $s = 1,\ \tau = 1,\ k = 1,\ \alpha = 1$ and $\beta = 1$.

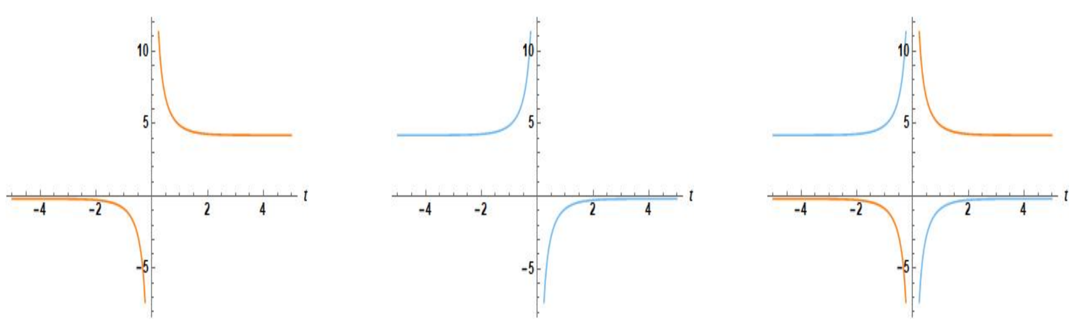


Figure 4.10: 2D graph of left, right and combined/dual waves for $u_6(x,t)$ when $s = 1,\ \tau = 1,\ k = 1,\ \alpha = 1$ and $\beta = 1$.

Next, presenting 3D and 2D graphs depicting the variation in dual mode behavior as the value of phase velocity is altered.

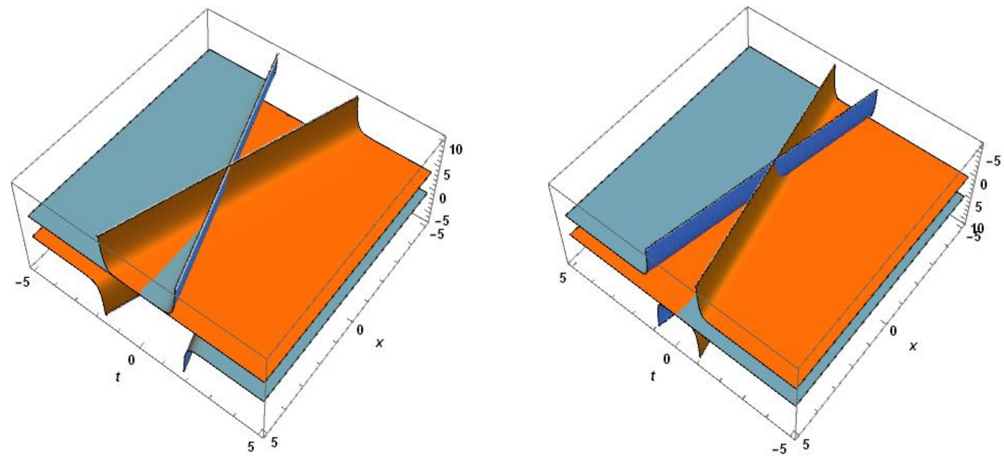


Figure 4.11: 3D graph of dual waves for $u_6(x,t)$ when $\beta = 1,\ \tau = 1,\ k = 1,\ \alpha = 1$ and $s = 3,\ 5$.

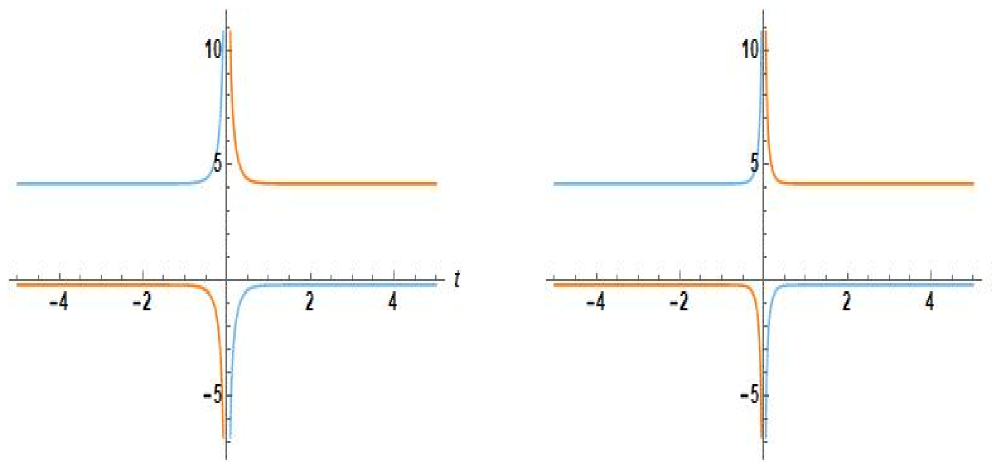


Figure 4.12: 2D graph of dual waves for $u_6(x,t)$ when $\beta = 1,\ \tau = 1,\ k = 1,\ \alpha = 1$ and $s = 3,\ 5$.

Case 2: $\mu = \eta$:

Upon substituting this assumption in (4.2) with $\mu = \eta$ and then collecting the coefficients of the terms containing $\tanh^i(k(x - ct))$ for $i = 0,\ 2,\ 4$ and 6, proceeding to solve the resulting system, leading to the subsequent set of solutions for the unknown parameters:

$$a_0 = \frac{2}{\alpha},\ a_1 = \frac{2}{3}\frac{k}{\alpha}\sqrt{-18\beta\tau + 6\beta}, \quad \alpha \neq 0, \quad (-18\beta\tau + 6\beta) > 0, \tag{4.17}$$

$$c = \pm\frac{1}{12}\sqrt{36(\tau - \frac{1}{3})^2 k^4\beta^2 + 144(\eta s + \frac{5}{4})(\tau - \frac{1}{3})k^2\beta + 144s^2 + 360\eta s + 225}$$
$$+ \frac{5}{4} + \frac{1}{12}(6\tau - 2)k^2\beta.$$

Where

$$(36(\tau - \frac{1}{3})^2 k^4\beta^2 + 144(\eta s + \frac{5}{4})(\tau - \frac{1}{3})k^2\beta + 144s^2 + 360\eta s + 225) > 0.$$

Consequently, this leads to the formation of kink wave solution,

$$u_7(x,t) = \frac{2}{\alpha} + \frac{2}{3}\frac{k}{\alpha}\sqrt{-18\beta\tau + 6\beta}$$
$$\times \tanh(k(\frac{1}{12}\sqrt{36(\tau - \frac{1}{3})^2 k^4\beta^2 + 144(\eta s + \frac{5}{4})(\tau - \frac{1}{3})k^2\beta + 144s^2 + 360\eta s + 225} \tag{4.18}$$
$$+ \frac{5}{4} + \frac{1}{12}(6\tau - 2)k^2\beta)t + x).$$

However, employing the coth method as outlined previously in Eq. (8) and substituting $\mu = \eta$ leads to a singular solution.

$$u_8(x,t) = \frac{2}{\alpha} + \frac{2}{3}\frac{k}{\alpha}\sqrt{-18\beta\tau + 6\beta}$$
$$\times \coth(k(\frac{1}{12}\sqrt{36(\tau - \frac{1}{3})^2 k^4\beta^2 + 144(\eta s + \frac{5}{4})(\tau - \frac{1}{3})k^2\beta + 144s^2 + 360\eta s + 225} \tag{4.19}$$
$$+ \frac{5}{4} + \frac{1}{12}(6\tau - 2)k^2\beta)t + x).$$

Next, presenting 2D and 3D plots illustrating the left, right, and dual-mode representation of $u_7(x,t)$.

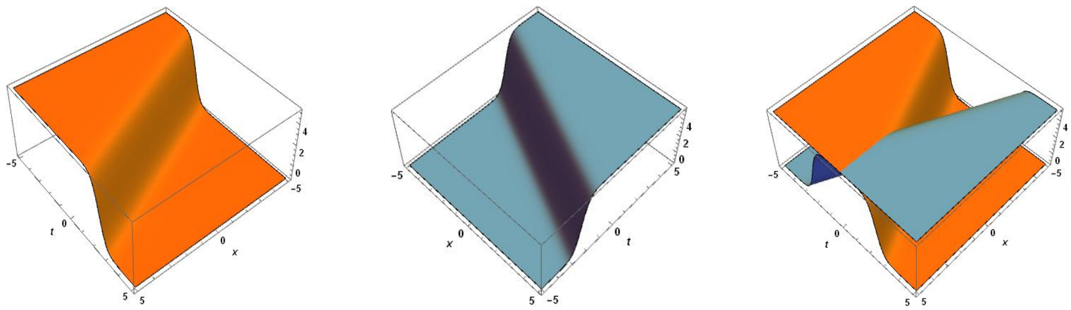


Figure 4.13: Left, right and dual waves for $u_7(x,t)$ when $s = 1,\ \tau = -1,\ k = 1,\ \alpha = 1,\ \eta = 1$ and $\beta = 1$.

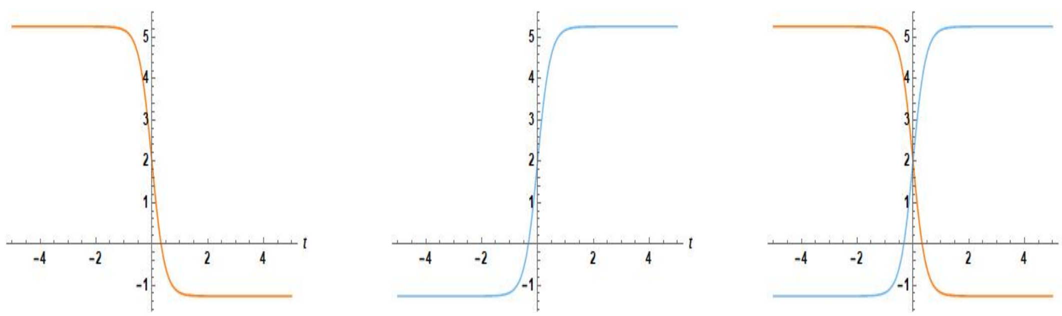


Figure 4.14: 2D graph of left, right and combined/dual waves for $u_7(x,t)$ when $s = 1,\ \tau = -1,\ k = 1,\ \alpha = 1,\ \eta = 1$ and $\beta = 1$.

In the following, 3D and 2D graphical representations are shown to capture how the dual mode behavior evolves with changes in the phase velocity.

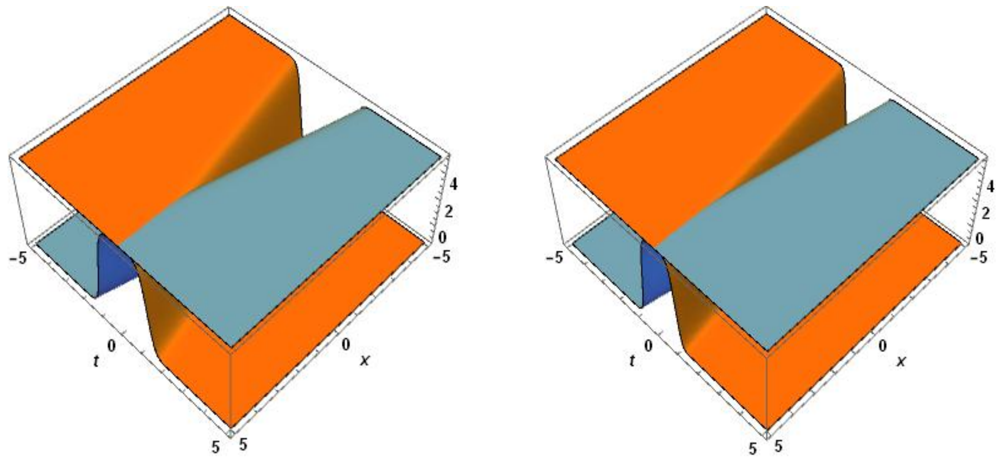


Figure 4.15: 3D graph of dual waves for $u_7(x,t)$ when $\beta = 1,\ \tau = -1,\ k = 1,\ \alpha = 1,\ \eta = 1$ and $s = 3$ and 5.

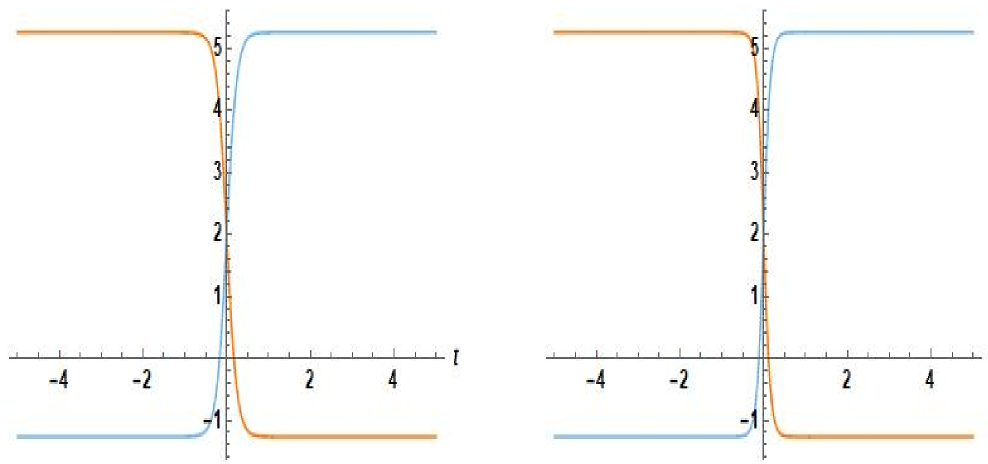


Figure 4.16: 2D graph of dual waves for $u_7(x,t)$ when $\beta = 1,\ \tau = -1,\ k = 1,\ \alpha = 1,\ \eta = 1$ and $s = 3$ and 5.

4.2 Results and graphical analysis

It is important to understand that the symbol c assumes two distinct values within this context, signifying that nonlinear equation under examination displays a unique dual-mode motion involving dual waves. Specifically, left-wave and right-wave exhibit simultaneous propagation. In this study, we focused on two distinct methods on dual mode Gardner equation derived from ideal fluid model: tan / cot method and tanh / coth method.

In Figs. (4.1) and (4.2), we depict the left, right and dual-mode graphs of $u_1(x,t)$. Figs. (4.3) and (4.4) illustrate the effect of the phase velocity parameter s. Notably, as we increase the value of the phase velocity parameter, the left and right waves converge, moving closer to each other. In the second case, when the dispersive parameter equals the nonlinearity parameter while employing the tan / cot method, we obtain two types of solutions, $u_3(x,t)$ and $u_4(x,t)$. We have generated 3D and 2D graphs of $u_4(x,t)$ to visualize the impact of the phase velocity in Figs. (4.5) and (4.6). As we increase the value of s, the waves again draw closer to each other.

Subsequently, when applying the tanh / coth method, we explore a scenario where the nonlinearity and dispersive parameters are not equal. We have plotted 2D and 3D graphs of $u_6(x,t)$ in Figs. (4.9) and (4.10) which reveal a singular solution. Once more, the effect of the phase velocity is evident as we manipulate its value in Figs. (4.11) and (4.12). In the final case, we discover two solutions: singular and kink waves. We have constructed graphs of $u_7(x,t)$ to visualize these solutions through Figs. (4.13) and (4.14). Throughout these investigations, we observe the consistent impact of the phase velocity by varying the values of the associated constants. The employed methods, tan / cot and tanh / coth, offer valuable insights into nonlinear wave behavior. Their results provide a deeper understanding of how parameters affect wave propagation. In our study, we've discovered a range of wave behaviors, including kink waves, singular solutions and periodic patterns. Each of these findings carries its unique significance. For instance, kink waves play a role in fluid dynamics, singular solutions can apply to the study of shock waves and periodic patterns are adaptable in fields like signal processing and communication theory.

Chapter 5

Conclusions

In this study, we have introduced an innovative two-mode Simplified Modified Camassa Holm (SMCH) equation and Gardner Equation (which is derived from an ideal fluid model). Our investigation primarily focused on identifying exact traveling solutions and examining their mathematical characteristics. To pursue this goal, we employed four widely recognized and dependable techniques: tanh / coth, Kudryashov, rational sine/cosine method, Exp-function expansion methods, and the direct approach of tanh / coth or tan / cot. A geometric analysis was conducted to assess the impact of the phase velocity parameter, s, on dual-waves. We observed that an increase in the phase velocity parameter approximately doubles the number of two-waves obtained, while the coordinate region remains constant. This analysis revealed that the solutions propagate as symmetric bi-directional waves. The results were presented through 3D plots of the obtained solutions, and we also identified a singular dual-mode solution using a negative value for Kudryashov's parameter, d. Additionally, 2D plots were provided for selected solutions. The implications of our findings extend to the advancement of data transmission in integrated telecommunication systems. Specifically, the phenomenon of dual-wave doubling has the potential to serve as a carrier wave for transmitting certain types of data in multiple directions.

Chapter 6

Future work

An overview of possible research topics that present chances for further investigation is given in this section. These fields have room for more research and development and cover a broad spectrum of disciplines.

- While the stability of the dual-mode SMCH and Gardner equations was not specifically examined in this study, it stands out as a noteworthy area for subsequent exploration. The investigation and comprehension of the stability properties inherent in the dual-mode NLSE will be a central point of focus for future research endeavors.

- Additionally, there are plans to broaden the application of the dual-mode concept, extending it to (n+1) dimensions and exploring its adaptation to coupled systems. These extensions aim to unveil new insights and complexities within the studied models.

- Moreover, the physical implications of the traveling wave solutions produced in this study will be a primary focus of the significance assessment. This thorough analysis aims to clarify the usefulness and possible uses of the detected wave patterns in actual situations.

- In upcoming research, a potential avenue for exploration involves examining a fractional version of dual-mode equations and undertaking a similar analytical approach.

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