m-Convex Functions on Time Scale

By

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Dedication

This thesis is dedicated to my Parents, Supervisor and Teachers for their endless devotion, support and encouragement.

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Abstract

Convex functions play a key role in different field of mathematics such as in the theory of inequalities. m-convex function is the generalization of convex function. The calculus and applications of dynamic derivatives defined on time scales give unification as well as an extension of customary differential and difference equations. This dissertation deals with the notion of m-convex function on time scale. Some basic and well-known inequalities have been deduced such as Petrovic's, Jensen's, Hermite-Hadamard and Fejer's. Also Hermite-Hadamard and Petrovic's inequalities have been discussed in the two coordinated m-convex function on time scale.

List of Notations

Υ	Time Scale
$\mathfrak{R}$	Set of real numbers
$\mathfrak{R}^+$	Set of positive real numbers
$\mathbf{Z}$	Set of integers
$\mathbf{N}$	Set of natural numbers
$\mathbf{N}_0$	$\mathbf{N} \cup \{0\}$
$\mathbf{Q}$	Set of rational numbers
$\mathbf{Q}^{'}$	Set of irrational numbers
$\mathbf{C}$	Set of complex numbers
$L_1[a,b]$	Set of all integrable functions on $[a,b]$
$\mathbb{K}_m(a)$	set of all m-convex functions on $[0,a]$

Contents

Chapter 1

Introduction and Preliminaries

Convexity is a basic idea which can be followed back to Archimedes (around 250 B.C.), regarding his renowned gauge of the estimation of π. (utilizing engraved and restricted standard polygons). He observed the crucial actuality that the border of a convex figure is littler than the edge of some other convex figure, encompassing it.

Convex functions play a key role in different field of mathematics such as in the theory of inequalities. Convex functions are illustrated by appropriate properties when we are studying optimization problems. Such as, a local minimum of convex functions is a global minimum. The theory established by the concept of convex functions eagerly applied in real analysis as well as in economics.

In Chapter 1 we give some basic definitions and preliminaries such as convex function, m-convex function and time scale theory.

In Chapter 2 we give some basic results and well-known inequalities of m-convex function such as Jensen's inequality, Hermite-Hadamard inequality, Petrovic's inequality and Fejer's inequality.

In Chapter 3 we will apply the time scale theory on above mentioned inequalities.

In Chapter 4 we will generalize the definition of m-convex function on two coordinates. We then apply the time scale theory and derive Petrovic and Hermite-Hadamard type inequalities.

1.1 Convex Function

For the convexity of any set $\mathcal{C}$, it must contain the line segment between two distinct points, i.e. for $\tau_1, \tau_2 \in \mathcal{C}$ and $\alpha \in [0,1]$, we have

$$\alpha\tau_1 + (1-\alpha)\tau_2 \in \mathcal{C}.$$

Definition 1.1.1. *[19] A function $\varphi : I \to \Re$ is said to be convex if $\forall\ \tau_1, \tau_2 \in I$ and any arbitrary $0 \leq \alpha \leq 1$, the inequality*

$$\varphi(\alpha\tau_1 + (1-\alpha)\tau_2) \leq \alpha\varphi(\tau_1) + (1-\alpha)\varphi(\tau_2), \tag{1.1}$$

holds. For φ to be strictly convex on I

$$\varphi(\alpha\tau_1 + (1-\alpha)\tau_2) < \alpha\varphi(\tau_1) + (1-\alpha)\varphi(\tau_2), \tag{1.2}$$

holds.
The function φ is said to be concave (strictly concave) on I, if $-\varphi$ is convex (strictly convex).

1.2 m-Convex Function

In [21] G.H. Toader defined the m-convexity.

Definition 1.2.1. *If $0 \in I \subseteq \Re$ be an interval and any arbitrary number $0 \leq m \leq 1$, then a function $\varphi : I \to \Re$ is known as m-convex, if the inequality*

$$\varphi(\lambda\tau_1 + m(1-\lambda)\tau_2) \leq \lambda\varphi(\tau_1) + m(1-\lambda)\varphi(\tau_2), \tag{1.3}$$

is satisfied for arbitrary points $\tau_1, \tau_2 \in I$ and every coefficient $0 \leq \lambda \leq 1$.

Geometric presentation of the inequality of m-convexity in formula (1.3) shows that the line segment connecting the graph point $(\tau_1, \varphi(\tau_1))$ and the point $(m\tau_2, m\varphi(\tau_2))$ is above the graph of the restriction $\varphi/conv\{\tau_1, m\tau_2\}$. By the definition, a convex function is usually represented by a *1*-convex function.

1.3 Coordinated m-Convex Function

In [10], Farid et al. gave the definition of coordinated m-convex functions.

Definition 1.3.1. *Let $\Delta^2 = [0, p_2] \times [0, q_2] \subset [0, \infty)^2$, then a function $\varphi : \Delta^2 \to \mathfrak{R}$ is said to be coordinated m-convex if the partial mappings*

$$\varphi_y : [0, p_2] \to \mathfrak{R} \quad defined \ \ by \ \ \varphi_y(u) = \varphi(u, y),$$

and

$$\varphi_x : [0, q_2] \to \mathfrak{R} \quad defined \ \ by \ \ \varphi_x(v) = \varphi(x, v),$$

are m-convex on $[0, p_2]$ and $[0, q_2]$ respectively, $\forall\ y \in [0, q_2]$ and $x \in [0, p_2]$.

1.4 Time Scale

In 1988, Stefan Hilger (German mathematician) was the first man who proposed the theory of time scale in his PhD thesis [12]. The calculus and applications of dynamic derivatives defined on time scales give unification as well as an extension of customary differential and difference equations.

Any subset $\emptyset \neq \Upsilon$ of $\mathfrak{R}$ with induced topology of $\mathfrak{R}$ is said to be time scale if it is closed. For example $\mathfrak{R}$, $\mathbf{Z}$, $\mathbf{N}$, $\mathbf{N}_0$ and $[0, 1] \cup [2, 3] \cup \mathbf{N}$ are time scales. Whereas $\mathbf{Q}$, $\mathbf{Q}'$, $\mathbf{C}$ and $(0, 1)$ are not time scales. The intervals in time scale are denoted by I_Υ defined by $I_\Upsilon = I \cap \Upsilon$, where I is arbitrary interval of real numbers. For more details about fundamental rules of calculus related with the dynamic derivatives and integral operators, see [1], [4], [6], [12], and [20].

Here the idea of jump operators are required because the time scale Υ may or may not be connected.

Definition 1.4.1. *[3] For any $s \in \Upsilon$, the forward jump operator $\sigma : \Upsilon \to \Upsilon$ is defined as*

$$\sigma(s) = \inf\{\ell \in \Upsilon : \ell > s\}.$$

Definition 1.4.2. *[3] For any $s \in \Upsilon$, the backward jump operator $\rho : \Upsilon \to \Upsilon$ is defined by*

$$\rho(s) = \sup\{\ell \in \Upsilon : \ell < s\}.$$

We use the convention

$$\inf \emptyset = \sup \Upsilon \quad , \quad \sup \emptyset = \inf \Upsilon.$$

Definition 1.4.3. *[4] Any point $s \in \Upsilon$ is called right-scattered, if $\sigma(s) > s$ and it is called left-scattered if $\rho(s) < s$.*

Definition 1.4.4. *[4] The points are said to be isolated if they simultaneously act as right-scattered and left-scattered.*

Definition 1.4.5. *[4] If $\sigma(s) = s$ and $s < \sup \Upsilon$ then s is said to be right-dense and if $\rho(s) = s$ and if $s > \inf \Upsilon$, then s is said to be left-dense.*

Definition 1.4.6. *[4] If the points are simultaneously act as right-dense and left-dense then the points are said to be dense.*

Definition 1.4.7. *[3] The mappings $\mu, \nu : \Upsilon \to [0, \infty)$ defined by*

$$\mu(s) = \sigma(s) - s,$$

and

$$\nu(s) = s - \rho(s),$$

are known as the forward and backward graininess functions respectively.

We define $\Upsilon^k = \Upsilon \backslash M_1$, where $M_1 \in \Upsilon$ is a left-scattered maximum, otherwise $\Upsilon^k = \Upsilon$. In case of right-scattered minimum M_2, we define $\Upsilon_k = \Upsilon \backslash M_2$; otherwise $\Upsilon_k = \Upsilon$. Finally, we define $\Upsilon^* = \Upsilon^k \cap \Upsilon_k$.

Example 1.4.8. *Let $\Upsilon = \mathfrak{R}$ and any $s \in \mathfrak{R}$, we have*

$$\sigma(s) = \inf\{\ell \in \mathfrak{R} : \ell > s\} = \inf\{(s, \infty)\} = s,$$

$$\rho(s) = \sup\{\ell \in \mathfrak{R} : \ell < s\} = \sup\{(-\infty, s)\} = s.$$

Since, $\sigma(s) = s$ and $s < \sup \Upsilon = \sup \mathfrak{R} = \infty$, so s is right-dense. Also, $\rho(s) = s$ and $s > \inf \Upsilon = \inf \mathfrak{R} = -\infty$, so s is left-dense. Therefore, s is dense.

The Graininess function of $\Upsilon = \mathfrak{R}$ is

$$\begin{aligned} \mu(s) &= \sigma(s) - s \\ &= s - s \\ &= 0. \end{aligned}$$

Example 1.4.9. *Let $\Upsilon = \mathbf{Z}$ and any arbitrary $s \in \mathbf{Z}$*

$$\sigma(s) = \inf\{\ell \in \mathbf{Z} : \ell > s\} = \inf\{(s+1, s+2, s+3, ...)\} = s+1,$$

$$\rho(s) = \sup\{\ell \in \mathbf{Z} : \ell < s\} = \sup\{(..., s-3, s-2, s-1)\} = s-1.$$

Since, $\sigma(s) = s + 1 > s$, so s is right-scattered. Also, $\rho(s) = s - 1 < s$, so s is left-scattered. Hence, s is an isolated point.

The Graininess function of $\Upsilon = \mathbf{Z}$ is

$$\begin{aligned} \mu(s) &= \sigma(s) - s \\ &= s + 1 - s \\ &= 1. \end{aligned}$$

Definition 1.4.10. *[4] For a function* $\varphi : \Upsilon \to \Re$ *and for* $s \in \Upsilon^k$*, we define* $\varphi^\Delta(s)$ *in* s *(if exists) having a property that for any given* $\varepsilon > 0$, $\exists$ *a* U_Υ*(neighborhood) of* s *such that*

$$\left|\varphi(\sigma(s)) - \varphi(\ell) - \varphi^\Delta(s)[\sigma(s) - \ell]\right| \leq \varepsilon|\sigma(s) - \ell|, \quad \forall\ \ell \in U_\Upsilon\ ,\ \forall\ s \in \Upsilon^k.$$

Then we can say that φ *is delta differentiable on* Υ^k*.*

Definition 1.4.11. *[4] For a function* $\varphi : \Upsilon \to \Re$ *and for* $s \in \Upsilon_k$*, we define* $\varphi^\nabla(s)$ *in* s *(if exists) having a property that for any given* $\varepsilon > 0$, $\exists$ *a* U_Υ*(neighborhood) of* s *such that*

$$\left|\varphi(\rho(s)) - \varphi(\ell) - \varphi^\nabla(s)[\rho(s) - \ell]\right| \leq \varepsilon|\rho(s) - \ell|, \quad \forall\ \ell \in U_\Upsilon\ ,\ \forall\ s \in \Upsilon_k.$$

Then we can say that φ *is nabla differentiable on* Υ_k*.*

Theorem 1.4.12. *[4] Assume* $\varphi : \Upsilon \to \Re$ *and* $s \in \Upsilon^k$*. Then*

1. φ *is continuous at* s*, if* φ *is delta differentiable at* s*.*

2. *If* φ *is continuous at* s *where* s *is right-scattered, then* φ *is delta differentiable at* s *with*

$$\varphi^\Delta(s) = \frac{\varphi(\sigma(s)) - \varphi(s)}{\mu(s)}.$$

3. *If* s *is right-dense(rd) then* φ *is differentiable at* $s \Leftrightarrow$ *the limit*

$$\lim_{\tau \to s} \frac{\varphi(s) - \varphi(\tau)}{s - \tau},$$

exists as a finite number. In this case,

$$\varphi^\Delta(s) = \lim_{\tau \to s} \frac{\varphi(s) - \varphi(\tau)}{s - \tau}.$$

4. *If φ is delta differentiable at s, then*

$$\varphi(\sigma(s)) = \varphi(s) + \mu(s)\varphi^{\triangle}(s),$$

holds.

Theorem 1.4.13. *[3] Assume $\varphi : \Upsilon \to \mathfrak{R}$ and $s \in \Upsilon_k$. Then*

1. *φ is continuous at s, if φ is nabla differentiable at s.*

2. *If φ is continuous at s where s is left-scattered, then φ is nabla differentiable at s with*

$$\varphi^{\nabla}(s) = \frac{\varphi(s) - \varphi(\rho(s))}{\nu(s)}.$$

3. *If s is left-dense(ld), then φ is nabla differentiable at $s \Leftrightarrow$ the limit*

$$\lim_{\tau \to s} \frac{\varphi(s) - \varphi(\tau)}{s - \tau},$$

exists as a finite number. In this case,

$$\varphi^{\nabla}(s) = \lim_{\tau \to s} \frac{\varphi(s) - \varphi(\tau)}{s - \tau}.$$

4. *If φ is nabla differentiable at s, then*

$$\varphi(\rho(s)) = \varphi(s) + \nu(s)\varphi^{\nabla}(s),$$

holds.

Definition 1.4.14. *[3] Any function $\phi : \Upsilon \to \mathfrak{R}$ is said to be an rd-continuous, if continuity of ϕ holds at all right-dense points in Υ and at all left-dense points in Υ, its left-sided limits are finite. C_{rd} is the set containing all rd-continuous functions.*

Definition 1.4.15. *[3] Any function $\phi : \Upsilon \to \mathfrak{R}$ is said to be an ld-continuous, if continuity of ϕ holds at all left-dense points in Υ and at all right-dense points in Υ, its right-sided limits are finite. C_{ld} is the set of containing all ld-continuous functions.*

Both C_{rd} and C_{ld} are contained in set of continuous functions on Υ.

Definition 1.4.16. *[20] If $\varphi^{\Delta}(s) = \phi(s)$, $\forall\ \tau \in \Upsilon^k$, then $\varphi : \Upsilon \to \mathfrak{R}$ is said to be a Δ-antiderivative of $\phi : \Upsilon \to \mathfrak{R}$. We define Δ-integral by*

$$\int_{\alpha}^{s} \phi(\tau)\Delta\tau = \varphi(s) - \varphi(\alpha).$$

Theorem 1.4.17. *[5] Each rd-continuous(ld-continuous) function has a $\Delta(\nabla)$ antiderivative.*

Theorem 1.4.18. *[20] If $a, b, c \in \Upsilon, \alpha \in \mathfrak{R}$ and, $\varphi, \psi \in C_{rd}$, then*

1. $\int_a^b (\varphi(s) + \psi(s))\Delta s = \int_a^b \varphi(s)\Delta s + \int_a^b \psi(s)\Delta s.$
2. $\int_a^b \alpha\varphi(s)\Delta s = \alpha \int_a^b \varphi(s)\Delta s.$
3. $\int_a^b \varphi(s)\Delta s = -\int_b^a \varphi(s)\Delta s.$
4. $\int_a^a \varphi(s)\Delta s = 0.$
5. $\int_a^b \varphi(s)\Delta s = \int_a^c \varphi(s)\Delta s + \int_c^b \varphi(s)\Delta s.$
6. *If* $\forall\ s,\ \varphi(s) \geq 0 \Rightarrow \int_a^b \varphi(s)\Delta s \geq 0.$

Note that *Theorem 1.4.18* also holds for $\varphi, \psi \in C_{ld}$.

1.5 Convex Function on Time Scale

Definition 1.5.1. *[5] Any function $\varphi : I_\Upsilon \to \mathfrak{R}$ is said to be convex, if*

$$\varphi(\alpha\tau_1 + (1-\alpha)\tau_2) \leq \alpha\varphi(\tau_1) + (1-\alpha)\varphi(\tau_2), \tag{1.4}$$

$\forall\ \tau_1, \tau_2 \in I_\Upsilon$ and arbitrary $0 \leq \alpha \leq 1$, where $I_\Upsilon = I \cap \Upsilon$, I be an interval in $\mathfrak{R}$.

For φ to be strictly convex on I_Υ

$$\varphi(\alpha\tau_1 + (1-\alpha)\tau_2) < \alpha\varphi(\tau_1) + (1-\alpha)\varphi(\tau_2), \tag{1.5}$$

holds.
The function φ is said to be concave (strictly concave) on I_Υ, if $-\varphi$ is convex (strictly convex).

Chapter 2

Some Inequalities of m-Convex Function

In this chapter, some basic inequalities are considered related to m-convex functions. Then there are given some Jensen type inequalities, Hadamard type inequalities, Petrovic inequality and Fejer inequality.

Remark 2.0.1. *Using* $m = 1$ *in* (1.3)*; we regain the definition of convex functions and by taking* $m = 0$*, we attain the star shaped functions. Remember that,* $\varphi : [0, a] \to \Re$ *is star shaped if*

$$\varphi(\lambda\tau) \leq \lambda\varphi(\tau),$$

holds, $\forall\ \lambda \in [0, 1]$ *and* $\tau \in [0, a]$.

Lemma 2.0.2. *[21] If* $\varphi \in \mathbb{K}_m(a)$*, then it is star shaped.*

Lemma 2.0.3. *[22] If* $\varphi \in \mathbb{K}_m(a)$ *and* $0 \leq n < m \leq 1$*, then* $\varphi \in \mathbb{K}_n(a)$.

2.1 Basic Inequalities

Some algebraic and topological properties of m-convex functions are discussed in this section.

Proposition 2.1.1. *[15] Suppose* $\varphi : [0, a] \to \Re$*, an* m_1*-convex function and* $\psi : [0, a] \to \Re$*, an* m_2*-convex function where* $m_1 \leq m_2$*, then* $\varphi + \psi$ *and* $\alpha\varphi$ *for* $\alpha \geq 0$*, are*

m_1-convex.

Proposition 2.1.2. *[15] Let $\varphi : [0, a] \to \mathfrak{R}$, $\psi : [0, b] \to \mathfrak{R}$ with range$(\varphi) \subseteq$ domain(ψ). If ψ is increasing and both φ and ψ are m-convex, then $\psi o \varphi$ is also m-convex on $[0, a]$.*

Proposition 2.1.3. *[15] If the two non-negative functions $\varphi, \psi : [0, b] \to \mathfrak{R}$ are m-convex as well as increasing, then $\varphi\psi$ is m-convex.*

Proposition 2.1.4. *[15] Consider an m-convex function $\varphi : [0, +\infty] \to \mathfrak{R}$, which is finite on $\left[a, \frac{b}{m}\right] \subset [0, +\infty)$, where $0 < m \leq 1$. Then in any arbitrary closed interval $[a, b]$, φ is bounded.*

Proposition 2.1.5. *[14] If $\varphi_1, \varphi_2 : [0, a] \to \mathfrak{R}$ are m-convex functions, then the function $\varphi(u)$ defined by*

$$\varphi(u) = \max_{u \in [0,a]} \{\varphi_1(u), \varphi_2(u)\},$$

is also m-convex.

Proposition 2.1.6. *[14] If the sequence of m-convex functions, $\varphi_n : [0, b] \to \mathfrak{R}$ converges pointwise to a function φ on $[0, b]$, then φ must be m-convex.*

Theorem 2.1.7. *[17] Suppose $0 < m \leq 1$ be an arbitrary number and a function $\varphi : I \to \mathfrak{R}$ defined on an interval $0 \in I$. Then the statements given below are equivalent:*

1. *For each pair of points $u, w \in I$, where $u \neq w$ and each coefficient $0 < \alpha \leq 1$, the function φ satisfies the inequality*

$$\varphi((1-\alpha)u + \alpha m w) \leq (1-\alpha)\varphi(u) + \alpha m \varphi(w). \tag{2.1}$$

2. *For every triple of points $u, v, w \in I$, where $u \neq mw$ and $v \in conv\{u, mw\}$, the function φ satisfies the inequality*

$$\varphi(v) \leq \frac{mw - v}{mw - u}\varphi(u) + \frac{v - u}{mw - u} m\varphi(w). \tag{2.2}$$

3. *For every triple of points $u, v, w \in I$, where $u < mw$ and $u \leq v \leq mw$, the function satisfies the inequality*

$$\det \begin{pmatrix} u & \varphi(u) & 1 \\ v & \varphi(v) & 1 \\ mw & m\varphi(w) & 1 \end{pmatrix} \geq 0. \tag{2.3}$$

If we replaced the first and the third row in (2.3)*, the above inequality also holds for $u > mw$ and $mw \leq v \leq u$.*

Corollary 2.1.8. *[17] Suppose I be an interval which contains zero and $0 < m \leq 1$ be an arbitrary number. Then for every triple of points $u, v, w \in I$, such that $u < v < mw$, every function $\varphi : I \to \mathfrak{R}$ which is m-convex must satisfies the inequality given below:*

$$\frac{\varphi(u) - \varphi(v)}{u - v} \leq \frac{\varphi(v) - m\varphi(w)}{v - mw}.$$

If $mw < v < u$, then the reverse inequality holds.

Corollary 2.1.9. *[17] If $0 < m \leq 1$ be an arbitrary number and I be an interval which contains zero. Then every function $\varphi : I \to \mathfrak{R}$ which is m-convex:*

If $p \le q < 0$ or $0 < p \le q$, then

$$\frac{\varphi(p)}{p} \le \frac{\varphi(q)}{q}. \tag{2.4}$$

If $p < 0 < q$,then

$$\frac{\varphi(p)}{p} - \frac{\varphi(q)}{q} \le \left(\frac{1}{p} - \frac{1}{q}\right) \frac{(m+1)\varphi(0)}{2m}. \tag{2.5}$$

Lemma 2.1.10. *[17] Let $0 < m \le 1$ be an arbitrary number and $[p, q]$ is an interval of $\mathfrak{R}$, where $p < 0 < q$. Then every function $\varphi : [p, q] \to \mathfrak{R}$ which is m-convex, is bounded by the affine functions as follows:*

If $p \le \tau \le 0$, then

$$\frac{\varphi(q) - \varphi(0)}{q}\tau + \frac{\varphi(0)}{m} \le \varphi(\tau) \le \frac{\varphi(p)}{p}\tau. \tag{2.6}$$

If $0 \le \tau \le q$, then

$$\frac{\varphi(p) - \varphi(0)}{p}\tau + \frac{\varphi(0)}{m} \le \varphi(\tau) \le \frac{\varphi(q)}{q}\tau. \tag{2.7}$$

Corollary 2.1.11. *[17] If $m \in (0, 1]$ be an arbitrary number and $[a, b]$ is an interval of $\mathfrak{R}$, where $a < 0 < b$. Then every m-convex function $\varphi : [a, b] \to \mathfrak{R}$, where $\varphi(0) = 0$ must satisfies the inequality*

$$\frac{b^3\varphi(a) - a^3\varphi(b)}{2ab} \le \int_a^b \varphi(\tau)d\tau \le \frac{b\varphi(b) - a\varphi(a)}{2}.$$

2.2 Petrovic's Inequality

In [2], M. Bakula et al. gave the Petrovic's inequality for m-convex functions as follows.

Theorem 2.2.1. *Let* $\varphi : [0,\infty) \to \mathfrak{R}$ *be an* m*-convex function and* $0 < m \leq 1$ *be any arbitrary number. Let* $(x_1, ..., x_n)$ *be non-negative* n*-tuples and* $(p_1, ..., p_n)$ *be positive* n*-tuples such that*

$$P_n := \sum_{k=1}^{n} p_k, \qquad 0 \neq \tilde{x_n} = \sum_{k=1}^{n} p_k x_k \geq x_i \qquad (i = 1, 2, ...),$$

then

$$\sum_{k=1}^{n} p_k \varphi(x_k) \leq \min \left\{ m\varphi\left(\frac{\tilde{x_n}}{m}\right) + (P_n - 1)\varphi(0), \varphi(\tilde{x_n}) + m(P_n - 1)\varphi(0) \right\}.$$

2.3 Jensen's Type Inequalities

Now we will discuss different forms of Jensen's inequality.

Theorem 2.3.1. *[17](**Jensen's Inequality in Discrete form**) Suppose* $0 \in I$ *be an interval and* $0 < m \leq 1$ *be an arbitrary number. If* $\sum_{i=1}^{n} \lambda_i \tau_i$ *be the convex combination of points* $\tau_i \in I$ *having coefficients* $\lambda_i \in [0,1]$*. Then, every function* $\varphi : I \to \mathfrak{R}$*, which is* m*-convex must satisfy the following inequality:*

$$\varphi\left(m \sum_{i=1}^{n} \lambda_i \tau_i\right) \leq m \sum_{i=1}^{n} \lambda_i \varphi(\tau_i). \tag{2.8}$$

Corollary 2.3.2. *Consider an interval,* $0 \in I$ *and an arbitrary number* $0 < m \leq 1$*. If* $\sum_{i=1}^{n} \lambda_i \tau_i$ *be the convex combination of points* $\tau_i \in I$ *having coefficients* $\lambda_i \in [0,1]$*. Then every function* $\varphi : I \to \mathfrak{R}$*, which is* $m-$*convex satisfy the following inequality:*

$$\varphi\left(\lambda_1 \tau_1 + m \sum_{i=2}^{n} \lambda_i \tau_i\right) \leq \lambda_1 \varphi(\tau_1) + m \sum_{i=2}^{n} \lambda_i \varphi(\tau_i).$$

Corollary 2.3.3. *[17] Suppose $m \in (0,1]$, an arbitrary number and $[c,d] \subset \mathfrak{R}$ be an interval which contains zero. If an integrable function $\psi : [c,d] \to \mathfrak{R}$ such that image$(\psi) \subseteq [c,d]$ then each continuous function $\varphi : [c,d] \to \mathfrak{R}$ which is also m-convex, satisfies the following inequality:*

$$\varphi\left(\frac{m}{d-c}\int_c^d \psi(\tau)d\tau\right) \leq \frac{m}{d-c}\int_c^d \varphi(\psi(\tau))d\tau.$$

Corollary 2.3.4. *[17] Suppose $0 < m \leq 1$ and $0 \in [c,d]$ be an interval. Consider an integrable functions $\psi : [c,d] \to \mathfrak{R}$ such that image$(\psi) \subseteq [c,d]$ and $\mho : [c,d] \to \mathfrak{R}$ which satisfies $\int_c^d \mid \mho(\tau) \mid d\tau \geq 0$. Then every continuous m-convex function $\varphi : [c,d] \to \mathfrak{R}$ satisfies the inequality:*

$$\varphi\left(m\frac{\int_c^d \psi(\tau)\mho(\tau)d\tau}{\int_c^d \mho(\tau)d\tau}\right) \leq m\frac{\int_c^d \varphi(\psi(\tau))\mho(\tau)d\tau}{\int_c^d \mho(\tau)d\tau}. \tag{2.9}$$

2.4 Hermite-Hadamard Type Inequalities

By [17], we have found an upper bound of Riemann integral for m-convex function by taking $mc \in [a,b]$.

Theorem 2.4.1. *Suppose $0 < m \leq 1$ an arbitrary number and $0 \in I \subseteq \mathfrak{R}$ be an interval. If $u,v,w \in I$ be any triplet of points with $u \leq mw \leq v$, then every m-convex function $\varphi : I \to \mathfrak{R}$ satisfies the following inequality:*

$$\int_u^v \varphi(\tau)d\tau \leq \frac{mw-u}{2}\varphi(u) + \frac{v-mw}{2}\varphi(v) + \frac{v-u}{2}m\varphi(w). \tag{2.10}$$

Proof. Let $u \leq \tau \leq mw$, with $u < mw$. Consider the convex combination

$$\tau = \frac{mw-\tau}{mw-u}u + \frac{\tau-u}{mw-u}mw. \tag{2.11}$$

By using the m-convexity of the function φ to (2.11), we attain

$$\varphi(\tau) \leq \frac{mw-\tau}{mw-u}\varphi(u) + \frac{\tau-u}{mw-u}m\varphi(w). \tag{2.12}$$

Now, integrate the inequality (2.12) over the interval $[u, mw]$, we have

$$\int_u^{mw} \varphi(\tau)d\tau \leq \frac{mw-u}{2}(\varphi(u) + m\varphi(w)). \tag{2.13}$$

Again integrating the inequality (2.12) over the interval $[mw, v]$, we obtain

$$\int_{mw}^{v} \varphi(\tau)d\tau \leq \frac{v-mw}{2}(\varphi(v) + m\varphi(w)), \tag{2.14}$$

we can write $\int_u^v \varphi(\tau)d\tau$ as

$$\int_u^v \varphi(\tau)d\tau = \int_u^{mw} \varphi(\tau)d\tau + \int_{mw}^{v} \varphi(\tau)d\tau. \tag{2.15}$$

By using the inequalities (2.13) and (2.14) in (2.15), we get the required result.

$\square$

If we use $w = u$ or $w = v$, in *Theorem 2.4.1*, we get the following appropriate inequality.

Corollary 2.4.2. *If* $0 < m \leq 1$ *and* $0 \in [c, d] \subset \mathfrak{R}$ *be an interval. Then every m-convex function* $\varphi : [c, d] \to \mathfrak{R}$ *satisfies the inequality*

$$\int_c^d \varphi(\tau)d\tau \leq \frac{md-c}{2}\varphi(c) + \frac{d-mc}{2}\varphi(d).$$

Theorem 2.4.3. *Suppose $m \in (0,1]$ be an arbitrary number and $\varphi : [0,\infty) \to \mathfrak{R}$, an m-convex function. If $0 \le c < d < \infty$ and $\varphi \in L_1[c,d]$, then inequality*

$$\frac{1}{d-c}\int_c^d \varphi(\tau)d\tau \le \min\left\{\frac{\varphi(c)+m\varphi(\frac{d}{m})}{2}, \frac{\varphi(d)+m\varphi(\frac{c}{m})}{2}\right\},$$

holds.

Theorem 2.4.4. *[8] Suppose $m \in (0,1]$ be an arbitrary number and $\varphi : [0,\infty) \to \mathfrak{R}$, an m-convex function. If $0 \le c < d < \infty$ and $\varphi \in L_1[c,d]$, then we have the following inequality:*

$$\begin{aligned}\varphi\left(\frac{c+d}{2}\right) &\le \frac{1}{d-c}\int_c^d \frac{\varphi(\tau)+m\varphi\left(\frac{\tau}{m}\right)}{2}d\tau \\ &\le \frac{m+1}{4}\left[\frac{\varphi(c)+\varphi(d)}{2} + m.\frac{\varphi\left(\frac{c}{m}\right)+\varphi\left(\frac{d}{m}\right)}{2}\right].\end{aligned} \tag{2.16}$$

Lemma 2.4.5. *[17] If $\hbar : [c,d] \to \mathfrak{R}$ is an integrable function, which is symmetric with respect to the midpoint of $[c,d]$, then every affine function $u : \mathfrak{R} \to \mathfrak{R}$ satisfies the following inequality:*

$$\int_c^d u(\tau)\hbar(\tau)d\tau \le u\left(\frac{c+d}{2}\right)\int_c^d \hbar(\tau)d\tau.$$

Lemma 2.4.6. *[17] Suppose $[u,w] \subset \mathfrak{R}$ be an interval which contains zero and $0 < m \le 1$, an arbitrary number. Consider any arbitrary point $v \in [a,b]$ and any integrable function $\hbar : [u,w] \to \mathfrak{R}$ which is positive and also symmetric with respect to the mid-points of $[u,mv]$ and $[mv,w]$. Then every m-convex function $\varphi : [u,w] \to \mathfrak{R}$*

must satisfies the following inequality:

$$\int_u^w \varphi(\tau)\hbar(\tau)d\tau \le \frac{\varphi(u)+m\varphi(v)}{2}\int_u^{mv}\hbar(\tau)d\tau + \frac{m\varphi(v)+\varphi(w)}{2}\int_{mv}^{w}\hbar(\tau)d\tau.$$

Theorem 2.4.7. *[8] Let $0 < m \le 1$ be an arbitrary number and an m-convex function $\varphi : [0,\infty) \to \Re$. If $\varphi \in L_1[cm,d]$, where $0 \le c < d$, then the inequality*

$$\frac{1}{m+1}\left[\int_c^{md}\varphi(\tau)d\tau + \frac{md-c}{d-mc}\int_{mc}^{d}\varphi(\tau)d\tau\right] \le (md-c)\frac{\varphi(c)+\varphi(d)}{2},$$

holds.

Theorem 2.4.8. *Suppose $\varphi,\psi : [0,\infty) \to [0,\infty)$ be two mappings such that $\varphi\psi \in L^1([c,d])$ where $0 \le c < d < \infty$. If φ is m_1-convex for some fixed $m_1 \in (0,1]$ and ψ is m_2-convex on $[c,d]$ for any fixed $m_2 \in (0,1]$ then*

$$\frac{1}{d-c}\int_c^d \varphi(\tau)\psi(\tau)d\tau \le \min\{M_1, M_2\},$$

where

$$M_1 = \frac{1}{3}\left[\varphi(c)\psi(c) + m_1m_2\varphi\left(\frac{d}{m_1}\right)\psi\left(\frac{d}{m_2}\right)\right]+\frac{1}{6}\left[m_2\varphi(c)\psi\left(\frac{d}{m_2}\right)+m_1\varphi\left(\frac{d}{m_1}\right)\psi(c)\right],$$

$$M_2 = \frac{1}{3}\left[\phi(d)\psi(d) + m_1m_2\varphi\left(\frac{c}{m_1}\right)\psi\left(\frac{c}{m_2}\right)\right]+\frac{1}{6}\left[m_2\varphi(d)\psi\left(\frac{c}{m_2}\right)+m_1\varphi\left(\frac{c}{m_1}\right)\psi(d)\right].$$

Theorem 2.4.9. *[17](**Fejer's Inequality**) Let $[u,w] \subset \Re$ be an interval which contains zero and $0 < m \le 1$ be an arbitrary number. Consider any arbitrary point*

$v \in [u, w]$ and an integrable function $\hbar : [u, w] \to \mathfrak{R}$, which is positive and also symmetric with respect to the mid-points of $[u, mv]$ and $[mv, w]$. Then every m-convex function $\varphi : [u, w] \to \mathfrak{R}$ must satisfies the following inequality:

$$\frac{1}{m}\varphi\left(m\frac{(u+mv)\int_u^{mv}\hbar(\tau)d\tau + (mv+w)\int_{mv}^{w}\hbar(\tau)d\tau}{2\int_u^w \hbar(\tau)d\tau}\right)$$

$$\leq \frac{\int_u^w \varphi(\tau)\hbar(\tau)d\tau}{\int_u^w \hbar(\tau)d\tau}$$

$$\leq \frac{(\varphi(u)+m\varphi(v))\int_u^{mv}\hbar(\tau)d\tau + (m\varphi(v)+\varphi(w))\int_{mv}^{w}\hbar(\tau)d\tau}{2\int_u^w \hbar(\tau)d\tau}.$$

2.5 Inequalities Related to m-Convex Functions on Two Coordinates

In [7] Dragomir gave the coordinated Hadamard inequality for convex functions.

Theorem 2.5.1. *Let $\varphi : [p_1, p_2] \times [q_1, q_2] \to \mathfrak{R}$ be such that the partial mappings*

$$\varphi_y : [p_1, p_2] \to \mathfrak{R},\ \ \varphi_y(u) := \varphi(u, y),$$

and

$$\varphi_x : [q_1, q_2] \to \mathfrak{R},\ \ \varphi_x(v) := \varphi(x, v),$$

defined for all $v \in [q_1, q_2]$ and $u \in [p_1, p_2]$, are convex. Then we have

$$
\begin{aligned}
&\varphi\left(\frac{p_1+p_2}{2},\frac{q_1+q_2}{2}\right)\\
&\leq \frac{1}{2}\left[\frac{1}{p_2-p_1}\int_{p_1}^{p_2}\varphi\left(x,\frac{q_1+q_2}{2}\right)dx+\frac{1}{q_2-q_1}\int_{q_1}^{q_2}\varphi\left(\frac{p_1+p_2}{2},y\right)dy\right]\\
&\leq \frac{1}{(p_2-p_1)(q_2-q_1)}\int_{p_1}^{p_2}\int_{q_1}^{q_2}\varphi(x,y)dydx\\
&\leq \frac{1}{4(p_2-p_1)}\int_{p_1}^{p_2}\left[\varphi(x,q_1)+\varphi(x,q_2)\right]dx+\frac{1}{4(q_2-q_1)}\int_{q_1}^{q_2}\left[\varphi(p_1,y)+\varphi(p_2,y)\right]dy\\
&\leq \frac{\varphi(p_1,q_1)+\varphi(p_1,q_2)+\varphi(p_2,q_1)+\varphi(p_2,q_2)}{4}.
\end{aligned}
$$

Theorem 2.5.2. *Let $\Delta^2=[0,p_2]\times[0,q_2]\subset[0,\infty)^2$ where $p_2,q_2>0$ and $\varphi:\Delta^2\to\Re$ be a coordinated m-convex function in Δ^2 where $0<m\leq 1$ be an arbitrary number. If $\varphi_x\in L_1[0,q_2]$ and $\varphi_y\in L_1[0,p_2]$, then*

$$
\begin{aligned}
2\varphi\left(\frac{p_1+p_2}{2},\frac{q_1+q_2}{2}\right)\leq{}&\frac{1}{p_2-p_1}\int_{p_1}^{p_2}\left(\frac{\varphi\left(x,\frac{q_1+q_2}{2}\right)+m\varphi\left(\frac{x}{m},\frac{q_1+q_2}{2}\right)}{2}\right)dx\\
&+\frac{1}{q_2-q_1}\int_{q_1}^{q_2}\left(\frac{\varphi\left(\frac{p_1+p_2}{2},y\right)+m\varphi\left(\frac{p_1+p_2}{2},\frac{y}{m}\right)}{2}\right)dy.
\end{aligned}
$$

Theorem 2.5.3. *Let φ,φ_x, and φ_y be defined (in Theorem 2.5.2). Then the following inequality*

$$\frac{2}{(p_2-p_1)(q_2-q_1)}\int_{p_1}^{p_2}\int_{q_1}^{q_2}\varphi(x,y)dydx$$

$$\leq \min\left\{\frac{1}{p_2-p_1}\int_{p_1}^{p_2}\left(\frac{\varphi(x,q_1)+m\varphi(x,\frac{q_2}{m})}{2}\right)dx,\right.$$

$$\left.\frac{1}{p_2-p_1}\int_{p_1}^{p_2}\left(\frac{\varphi(x,q_2)+m\varphi(x,\frac{q_1}{m})}{2}\right)dx\right\}$$

$$+\min\left\{\frac{1}{q_2-q_1}\int_{q_1}^{q_2}\left(\frac{\varphi(p_1,y)+m\varphi(\frac{p_2}{m},y)}{2}\right)dy,\right.$$

$$\left.\frac{1}{q_2-q_1}\int_{q_1}^{q_2}\left(\frac{\varphi(p_2,y)+m\varphi(\frac{p_1}{m},y)}{2}\right)dy\right\},$$

holds.

Theorem 2.5.4. *[10] (**Hadamard Type Inequalities for Coordinated m-Convex Functions**) Let $\Delta^2=[0,p_2]\times[0,q_2]\subset[0,\infty)^2$ where $p_2,q_2>0$. Let $\varphi:\Delta^2\to\Re$ be a coordinated m-convex function in Δ^2 where $0<m\leq 1$ be an arbitrary number. If $\varphi_x\in L_1[0,q_2]$ and $\varphi_y\in L_1[0,p_2]$, such that $0\leq p_1<p_2,\quad 0\leq q_1<q_2$. Then we have*

$$\frac{1}{p_2-p_1}\int_{p_1}^{p_2}\varphi\left(x,\frac{q_1+q_2}{2}\right)dx+\int_{q_1}^{q_2}\varphi\left(\frac{p_1+p_2}{2},y\right)dy$$

$$\leq\frac{1}{2(p_2-p_1)(q_2-q_1)}\int_{p_1}^{p_2}\int_{q_1}^{q_2}\left(2\varphi(x,y)+m\left(\varphi\left(x,\frac{y}{m}\right)+\varphi\left(\frac{x}{m},y\right)\right)\right)dydx$$

$$\leq\frac{(m+1)^2}{16}\left[\varphi(p_1,q_1)+\varphi(p_2,q_1)+\varphi(p_1,q_2)+\varphi(p_2,q_2)+m\left(\varphi\left(\frac{p_1}{m},q_1\right)+\varphi\left(\frac{p_2}{m},q_1\right)\right.\right. \tag{2.17}$$

$$+\varphi\left(\frac{p_1}{m},q_2\right)+\varphi\left(\frac{p_2}{m},q_2\right)+\varphi\left(p_1,\frac{q_1}{m}\right)+\varphi\left(p_1,\frac{q_2}{m}\right)+\varphi\left(p_2,\frac{q_1}{m}\right)+\varphi\left(p_2,\frac{q_2}{m}\right)\Bigg)$$

$$+m^2\left(\varphi\left(\frac{p_1}{m},\frac{q_1}{m}\right)+\varphi\left(\frac{p_2}{m},\frac{q_1}{m}\right)+\varphi\left(\frac{p_1}{m},\frac{q_2}{m}\right)+\varphi\left(\frac{p_2}{m},\frac{q_2}{m}\right)\right)\Bigg].$$

Proof. As mapping $\varphi : \Delta^2 \to \mathfrak{R}$ is coordinated m-convex so, the functions φ_x and φ_y are m-convex on $[0, q_2]$ and $[0, p_2]$, respectively. For the function φ_y, we use the inequality (2.16)

$$\varphi_y\left(\frac{p_1+p_2}{2}\right) \le \frac{1}{p_2-p_1}\int_{p_1}^{p_2}\left(\frac{\varphi_y(x)+m\varphi_y\left(\frac{x}{m}\right)}{2}\right)dx$$

$$\le \frac{m+1}{4}\left[\frac{\varphi_y(p_1)+\varphi_y(p_2)}{2}+m\frac{\varphi_y\left(\frac{p_1}{m}\right)+\varphi_y\left(\frac{p_2}{m}\right)}{2}\right].$$

We have

$$\varphi\left(\frac{p_1+p_2}{2},y\right) \le \frac{1}{p_2-p_1}\int_{p_1}^{p_2}\left(\frac{\varphi(x,y)+m\varphi\left(\frac{x}{m},y\right)}{2}\right)dx$$

$$\le \frac{m+1}{4}\left[\frac{\varphi(p_1,y)+\varphi(p_2,y)}{2}+m\frac{\varphi\left(\frac{p_1}{m},y\right)+\varphi\left(\frac{p_2}{m},y\right)}{2}\right].$$

From this one has

$$
\begin{aligned}
&\frac{1}{q_2-q_1}\int_{q_1}^{q_2}\varphi\left(\frac{p_1+p_2}{2},y\right)dy \\
&\leq \frac{1}{(p_2-p_1)(q_2-q_1)}\int_{p_1}^{p_2}\int_{q_1}^{q_2}\left(\frac{\varphi(x,y)+m\varphi\left(\frac{x}{m},y\right)}{2}\right)dydx \\
&\leq \frac{m+1}{4(q_2-q_1)}\int_{q_1}^{q_2}\left[\frac{\varphi(p_1,y)+\varphi(p_2,y)}{2}\right. \\
&\quad\left.+m\frac{\varphi\left(\frac{p_1}{m},y\right)+\varphi\left(\frac{p_2}{m},y\right)}{2}\right]dy.
\end{aligned} \tag{2.18}
$$

Now for the function φ_x, we get

$$
\begin{aligned}
&\frac{1}{p_2-p_1}\int_{p_1}^{p_2}\varphi\left(x,\frac{q_1+q_2}{2}\right)dx \\
&\leq \frac{1}{(p_2-p_1)(q_2-q_1)}\int_{p_1}^{p_2}\int_{q_1}^{q_2}\left(\frac{\varphi(x,y)+m\varphi\left(x,\frac{y}{m}\right)}{2}\right)dydx \\
&\leq \frac{m+1}{4(p_2-p_1)}\int_{p_1}^{p_2}\left[\frac{\varphi(x,q_1)+\varphi(x,q_2)}{2}\right. \\
&\quad\left.+m\frac{\varphi\left(x,\frac{q_1}{m}\right)+\varphi\left(x,\frac{q_2}{m}\right)}{2}\right]dx.
\end{aligned} \tag{2.19}
$$

Addition of (2.18) and (2.19) gives

$$
\begin{aligned}
&\frac{1}{p_2-p_1}\int_{p_1}^{p_2}\varphi\left(x,\frac{q_1+q_2}{2}\right)dx+\frac{1}{q_2-q_1}\int_{q_1}^{q_2}\varphi\left(\frac{p_1+p_2}{2},y\right)dy\\
&\le\frac{1}{2(p_2-p_1)(q_2-q_1)}\int_{p_1}^{p_2}\int_{q_1}^{q_2}\left(2\varphi(x,y)+m\left(\varphi\left(x,\frac{y}{m}\right)+\varphi\left(\frac{x}{m},y\right)\right)\right)dydx\\
&\le\frac{m+1}{4}\Bigg[\frac{1}{p_2-p_1}\int_{p_1}^{p_2}\left(\frac{\varphi(x,q_1)+\varphi(x,q_2)}{2}+m\frac{\varphi\left(x,\frac{q_1}{m}\right)+\varphi\left(x,\frac{q_2}{m}\right)}{2}\right)dx\\
&\quad+\frac{1}{q_2-q_1}\int_{q_1}^{q_2}\left(\frac{\varphi(p_1,y)+\varphi(p_2,y)}{2}+m\frac{\varphi\left(\frac{p_1}{m},y\right)+\varphi\left(\frac{p_2}{m},y\right)}{2}\right)dy\Bigg].
\end{aligned}
\tag{2.20}
$$

For a fixed y, and by using the m-convexity of φ_y, we get

$$\varphi(x,y)\le\frac{\varphi(x,y)+m\varphi\left(\frac{x}{m},y\right)}{2}.$$

Taking the average integral over the interval $[p_1,p_2]$ and use the inequality (2.16), we have

$$
\begin{aligned}
\frac{1}{p_2-p_1}\int_{p_1}^{p_2}\varphi(x,y)dx&\le\frac{1}{p_2-p_1}\int_{p_1}^{p_2}\left(\frac{\varphi(x,y)+m\varphi\left(\frac{x}{m},y\right)}{2}\right)dx\\
&\le\frac{m+1}{4}\left[\frac{\varphi(p_1,y)+\varphi(p_2,y)}{2}+m\frac{\varphi\left(\frac{p_1}{m},y\right)+\varphi\left(\frac{p_2}{m},y\right)}{2}\right].
\end{aligned}
\tag{2.21}
$$

In similar way, for a fixed x and by using the m-convexity of φ_x, we deduce

$$\frac{1}{q_2-q_1}\int_{q_1}^{q_2}\varphi(x,y)dy \le \frac{1}{q_2-q_1}\int_{q_1}^{q_2}\left(\frac{\varphi(x,y)+m\varphi\left(x,\frac{y}{m}\right)}{2}\right)dy$$

$$\le \frac{m+1}{4}\left[\frac{\varphi(x,q_1)+\varphi(x,q_2)}{2}+m\frac{\varphi\left(x,\frac{q_1}{m}\right)+\varphi\left(x,\frac{q_2}{m}\right)}{2}\right]. \tag{2.22}$$

First consider the inequality (2.21) for $y=q_1,q_2$, (2.22) for $x=p_1,p_2$, then (2.21) for $y=\frac{q_1}{m},\frac{q_2}{m}$ (2.22) for $x=\frac{p_1}{m},\frac{p_2}{m}$ to multiply later with m. By summing all these inequalities, we acquire the following inequality

$$\begin{aligned}
&\frac{1}{p_2-p_1}\int_{p_1}^{p_2}\left(\frac{\varphi(x,q_1)+\varphi(x,q_2)}{2}+m\frac{\varphi\left(x,\frac{q_1}{m}\right)+\varphi\left(x,\frac{q_2}{m}\right)}{2}\right)dx\\
&+\frac{1}{q_2-q_1}\int_{q_1}^{q_2}\left(\frac{\varphi(p_1,y)+\varphi(p_2,y)}{2}+m\frac{\varphi\left(\frac{p_1}{m},y\right)+\varphi\left(\frac{p_2}{m},y\right)}{2}\right)dy\\
&\le\frac{1}{2}\left[\frac{1}{p_2-p_1}\int_{p_1}^{p_2}\left(\frac{\varphi(x,q_1)+m\varphi\left(\frac{x}{m},q_1\right)+\varphi(x,q_2)+m\varphi(\frac{x}{m},q_2)}{2}\right.\right.\\
&\left.+m\frac{\varphi(x,\frac{q_1}{m})+m\varphi(\frac{x}{m},\frac{q_1}{m})+\varphi(x,\frac{q_2}{m})+m\varphi(\frac{x}{m},\frac{q_2}{m})}{2}\right)dx\\
&+\frac{1}{q_2-q_1}\int_{q_1}^{q_2}\left(\frac{\varphi(p_1,y)+m\varphi\left(p_1,\frac{y}{m}\right)+\varphi(p_2,y)+m\varphi(p_2,\frac{y}{m})}{2}\right.\\
&\left.\left.+m\frac{\varphi(\frac{p_1}{m},y)+m\varphi(\frac{p_1}{m},\frac{y}{m})+\varphi(\frac{p_2}{m},y)+m\varphi(\frac{p_2}{m},\frac{y}{m})}{2}\right)dy\right]
\end{aligned}\tag{2.23}$$

$$\leq \frac{1}{4}\Bigg[\varphi(p_1,q_1)+\varphi(p_2,q_1)+\varphi(p_1,q_2)+\varphi(p_2,q_2)+m\Bigg(\varphi\left(\frac{p_1}{m},q_1\right)+\varphi\left(\frac{p_2}{m},q_1\right)$$

$$+\varphi\left(\frac{p_1}{m},q_2\right)+\varphi\left(\frac{p_2}{m},q_2\right)+\varphi\left(p_1,\frac{q_1}{m}\right)+\varphi\left(p_1,\frac{q_2}{m}\right)+\varphi\left(p_2,\frac{q_1}{m}\right)+\varphi\left(p_2,\frac{q_2}{m}\right)\Bigg)$$

$$+m^2\left(\varphi\left(\frac{p_1}{m},\frac{q_1}{m}\right)+\varphi\left(\frac{p_2}{m},\frac{q_1}{m}\right)+\varphi\left(\frac{p_1}{m},\frac{q_2}{m}\right)+\varphi\left(\frac{p_2}{m},\frac{q_2}{m}\right)\right)\Bigg].$$

By combining the inequalities (2.20) and (2.23), we get the inequality (2.17). □

Theorem 2.5.5. *[18](**Petrovic's Inequality for Coordinated m-Convex Functions**) Let $(x_1,...,x_n)$, $(y_1,...,y_n)$ be non-negative n-tuples and $(p_1,...,p_n)$, $(q_1,...,q_n)$ be positive n-tuples such that $\sum_{k=1}^{n} p_k \geq 1$, where $x_i, y_i, p_i, q_i \in [0,\infty)$.*

$$P_n := \sum_{k=1}^{n} p_k, \qquad 0 \neq \tilde{x}_n = \sum_{k=1}^{n} p_k x_k \geq x_i \quad for \quad each \quad i = 1,...,n,$$

and

$$Q_n := \sum_{j=1}^{n} q_j, \qquad 0 \neq \tilde{y}_n = \sum_{j=1}^{n} q_j y_j \geq y_j \quad for \quad each \quad j = 1,...,n.$$

If $\varphi : [0,\infty)^2 \to \mathfrak{R}$ be a coordinated m-convex function with $m \in (0,1]$, then

$$\sum_{k=1}^{n}\sum_{j=1}^{n} p_k q_j \varphi(x_k,y_j) \leq \min\{m\min\{G_{m,1}(\tilde{x}_n/m), G_{1,m}(\tilde{x}_n/m)\} + (P_n-1)$$
$$\times \min\{G_{m,1}(0), G_{1,m}(0)\}, \min\{G_{m,1}(\tilde{x}_n), G_{1,m}(\tilde{x}_n)\} \tag{2.24}$$
$$+ m(P_n-1)\min\{G_{m,1}(0), G_{1,m}(0)\}\},$$

where

$$G_{m,\widetilde{m}}(t) = m\varphi\left(t,\frac{\tilde{y}_n}{m}\right) + \widetilde{m}(Q_n-1)\varphi(t,0). \tag{2.25}$$

Corollary 2.5.6. *[18] If $\varphi : [0,\infty)^2 \to \mathfrak{R}$ be a coordinated m-convex function where $m \in (0,1]$ be an arbitrary number. Let $(x_1,...,x_n),(y_1,...,y_n)$ be non-negative n-tuples and $(p_1,...,p_n)$,$(q_1,...,q_n)$ be positive n-tuples such that $\sum_{j=1}^{n} p_j \geq 1$, where*

$$P_n := \sum_{j=1}^{n} p_j, \qquad 0 \neq \tilde{x}_n = \sum_{j=1}^{n} p_j x_j \geq x_i \quad for \quad each \quad i = 1,...,n,$$

then the following inequality holds.

$$\sum_{j=1}^{n} np_j\varphi(x_j) \leq \min\Bigg\{ m\min\{(m+n-1)\varphi(\tilde{x}_n/m),(mn-m+1)\varphi(\tilde{x}_n/m)\}$$

$$+ (P_n - 1)\min\{(m+n-1)\varphi(0),(mn-m+1)\varphi(0)\} \quad ,$$

$$\min\{(m+n-1)\varphi(\tilde{x}_n),(mn-m+1)\varphi(\tilde{x}_n)\}$$

$$+ m(P_n - 1)\min\{(m+n-1),(mn-m+1)\varphi(0)\}\Bigg\}.$$

Chapter 3

Time Scale Version of Some Inequalities of m-Convex Functions

In this chapter, we will extend the definition of m-convex function on time scale. Also, we will present time scale version of some basic and well-known inequalities namely, Hermite-Hadamard, Jensen's and Fejer's for m-convex functions.

3.1 m-Convex Function on Time Scale

Definition 3.1.1. *A function $\varphi : [a,b]_\Upsilon \to \mathfrak{R}$ is called m-convex on $[a,b]_\Upsilon$ if*

$$\varphi(\lambda\tau_1 + m(1-\lambda\tau_2)) \leq \lambda\varphi(\tau_1) + m(1-\lambda)\varphi(\tau_2),$$

$\forall\ \tau_1, \tau_2 \in [a,b]_\Upsilon$, and $\lambda, m \in [0,1]$ such that $\lambda\tau_1 + m(1-\lambda)\tau_2 \in [a,b]_\Upsilon$ where $[a,b]_\Upsilon$ is any interval containing zero.

3.2 Basic Inequalities

In this section, we present some basic inequalities of m-convex function on time scale.

Theorem 3.2.1. *Let $0 < m \leq 1$ be an arbitrary number and $\varphi : I_\Upsilon \to \mathfrak{R}$ be a function, where I_Υ be an interval which contains zero. Then the statements given* *below are equivalent:*

1. *For every pair of points* $u, w \in I_\Upsilon$*, where* $u \neq w$ *and each coefficient* $\lambda \in (0, 1]$*, the function* φ *satisfies the inequality*

$$\varphi((1-\lambda)u + \lambda m w) \leq (1-\lambda)\varphi(u) + \lambda m \varphi(w). \tag{3.1}$$

2. *For every triple of points* $u, v, w \in I_\Upsilon$*, where* $u \neq mw$ *and* $v \in conv\{u, mw\}$*, the function* φ *satisfies the inequality*

$$\varphi(v) \leq \frac{mw - v}{mw - u}\varphi(u) + \frac{v - u}{mw - u} m\varphi(w). \tag{3.2}$$

3. *For every triple of points* $u, v, w \in I_\Upsilon$*, where* $u < mw$ *and* $u \leq v \leq mw$*, the function satisfies the inequality*

$$\det \begin{pmatrix} u & \varphi(u) & 1 \\ v & \varphi(v) & 1 \\ mw & m\varphi(w) & 1 \end{pmatrix} \geq 0. \tag{3.3}$$

By replacing the first and the third row in (3.3)*, the above inequality also holds for* $u > mw$ *and* $mw \leq v \leq u$*.*

Each of the above statements (in *Theorem 3.2.1*) can also be used as a definition of m-convex function.

Corollary 3.2.2. *Suppose* $0 < m \leq 1$ *be an arbitrary number and* $0 \in I_\Upsilon$ *be an interval. Then for every triple of points* $p, q, r \in I_\Upsilon$*, such that* $p < q < mr$*, every m-convex function* $\varphi : I_\Upsilon \to \mathfrak{R}$ *must satisfies the inequality*

$$\frac{\varphi(p) - \varphi(q)}{p - q} \leq \frac{\varphi(q) - m\varphi(r)}{q - mr}.$$

If $mr < q < p$*, then the reverse inequality holds.*

Corollary 3.2.3. *Let $0 < m \leq 1$ be an arbitrary number and I_Υ be an interval which contains zero. Then every function $\varphi : I_\Upsilon \to \mathfrak{R}$ which is m-convex must satisfies the inequalities given below.*

If $p \leq q < 0$ or $0 < p \leq q$, then

$$\frac{\varphi(p)}{p} \leq \frac{\varphi(q)}{q}. \tag{3.4}$$

If $p < 0 < q$, then

$$\frac{\varphi(p)}{p} - \frac{\varphi(q)}{q} \leq \left(\frac{1}{p} - \frac{1}{q}\right)\frac{(m+1)\varphi(0)}{2m}. \tag{3.5}$$

Proof. We obtained inequality (3.4) by arranging (3.2) in the orders of $p \leq q < mq$ and $mp < p \leq q$. We obtained inequality (3.5) by adding the inequalities arising from (3.2) in the orders of $p < 0 < mq$ and $mp < 0 < q$. □

Lemma 3.2.4. *If $0 < m \leq 1$, an arbitrary number and $[c, d]_\Upsilon$ be an interval where $c < 0 < d$, then every function $\varphi : [c, d]_\Upsilon \to \mathfrak{R}$ which is m-convex, is bounded by the affine functions as follows:*

If $c \leq \tau \leq 0$, then

$$\frac{\varphi(d) - \varphi(0)}{d}\tau + \frac{\varphi(0)}{m} \leq \varphi(\tau) \leq \frac{\varphi(c)}{c}\tau. \tag{3.6}$$

If $0 \leq \tau \leq d$, *then*

$$\frac{\varphi(c)-\varphi(0)}{c}\tau + \frac{\varphi(0)}{m} \leq \varphi(\tau) \leq \frac{\varphi(d)}{d}\tau. \tag{3.7}$$

Proof. We obtained inequality (3.6) by using the m-convexity of φ to the ordered triplets $m\tau \leq 0 < d$ and $c \leq \tau \leq 0$. Similarly, we obtained inequality (3.7) by using the m-convexity of φ to the ordered triplets $c < 0 \leq m\tau$ and $0 \leq \tau \leq d$.

□

Now we estimate the $\Delta -$ *integral* of an m-convex function $\varphi : [c,d]_\Upsilon \to \mathfrak{R}$ for which $\varphi(0) = 0$. By adding (3.6) and (3.7) after integrating, we get the bound as in the following corollary.

Corollary 3.2.5. *If $0 < m \leq 1$ be an arbitrary number and $[c,d]_\Upsilon$ be any interval, where $c < 0 < d$. Then every m-convex function $\varphi : [c,d]_\Upsilon \to \mathfrak{R}$, where $\varphi(0) = 0$ satisfies the inequality*

$$\frac{d^2\varphi(c)\tau_d - c^2\varphi(d)\tau_c}{cd} \leq \int_c^d \varphi(\tau)\Delta\tau \leq \varphi(d)\tau_d - \varphi(c)\tau_c,$$

where

$$\tau_c = \frac{1}{0-c}\int_c^0 \tau\Delta\tau.$$

$$\tau_d = \frac{1}{d-0}\int_0^d \tau\Delta\tau.$$

Proposition 3.2.6. *Suppose $\varphi : [0,b]_\Upsilon \to \mathfrak{R}$, an m_1-convex and $\psi : [0,b]_\Upsilon \to \mathfrak{R}$, an m_2-convex where $m_1 \leq m_2$, then $\varphi + \psi$ and $\alpha\varphi$, $\alpha \geq 0$ an arbitrary constant, are m_1-convex.*

Proof. As ψ is m_2-convex with $m_1 \leq m_2$, so ψ is m_1-convex. So for any arbitrary $\tau_1, \tau_2 \in [0, b]_{\mathbb{T}}$ and $0 \leq \lambda \leq 1$, we have

$$(\phi + \psi)(\lambda\tau_1 + m_1(1-\lambda)\tau_2) = \varphi(\lambda\tau_1 + m_1(1-\lambda)\tau_2) + \psi(\lambda\tau_1 + m_1(1-\lambda)\tau_2)$$

$$\leq \lambda\varphi(\tau_1) + m_1(1-\lambda)\varphi(\tau_2) + \lambda\psi(\tau_1) + m_1(1-\lambda)\psi(\tau_2)$$

$$= \lambda(\varphi + \psi)(\tau_1) + m_1(1-\lambda)(\varphi + \psi)(\tau_2).$$

Also,

$$(\alpha\varphi)(\lambda\tau_1 + m_1(1-\lambda)\tau_2) \leq \alpha(\lambda\varphi(\tau_1) + m_1(1-\lambda)\varphi(\tau_2))$$

$$= \lambda(\alpha\varphi)(\tau_1) + m_1(1-\lambda)(\alpha\varphi)(\tau_2).$$

□

Proposition 3.2.7. *Let $\phi : [0, a]_{\Upsilon} \to \mathfrak{R}$, $\psi : [0, b]_{\Upsilon} \to \mathfrak{R}$ with range$(\phi) \subseteq$ domain(ψ), if ψ is increasing and both ϕ and ψ are m-convex and then $\psi o \phi$ is also m-convex on $[0, a]_{\Upsilon}$.*

Proof. For any $\tau_1, \tau_2 \in [0, a]_{\Upsilon}$ and $0 \leq \lambda \leq 1$, we have

$$(\psi o \phi)(\lambda\tau_1 + m(1-\lambda)\tau_2) = \psi(\phi(\lambda\tau_1 + m(1-\lambda)\tau_2))$$

$$\leq \lambda\psi(\phi(\tau_1)) + m(1-\lambda)\psi(\phi(\tau_2))$$

$$= \lambda(\psi o \phi)(\tau_1) + m(1-\lambda)(\psi o \phi)(\tau_2).$$

□

Proposition 3.2.8. *If $\varphi_1, \varphi_2 : [0, b]_\Upsilon \to \mathfrak{R}$ are m-convex functions then the function defined by*

$$\varphi(\tau) = \max_{\tau \in [0,b]_\Upsilon} \{\varphi_1(\tau), \varphi_2(\tau)\},$$

is also m-convex.

Proof. If $\tau_1, \tau_2 \in [0, b]_\Upsilon$ and $0 < \alpha \leq 1$, we have

$$\varphi_1(\alpha\tau_1 + m(1-\alpha)\tau_2) \leq \alpha\varphi_1(\tau_1) + m(1-\alpha)\varphi_1(\tau_2)$$

$$\leq \alpha\varphi(\tau_1) + m(1-\alpha)\varphi(\tau_2),$$

and

$$\varphi_2(\alpha\tau_1 + m(1-\alpha)\tau_2) \leq \alpha\varphi_2(\tau_1) + m(1-\alpha)\varphi_2(\tau_2)$$

$$\leq \alpha\varphi(\tau_1) + m(1-\alpha)\varphi(\tau_2),$$

where

$$\varphi(\alpha\tau_1 + m(1-\alpha)\tau_2) = \max\{\varphi_1(\alpha\tau_1 + m(1-\alpha)\tau_2), \varphi_2(\alpha\tau_1 + m(1-\alpha)\tau_2\}$$

$$\leq \alpha\varphi(\tau_1) + m(1-\alpha)\varphi(\tau_2).$$

□

Proposition 3.2.9. *If the sequence of m-convex functions $\varphi_n : [0, b]_\Upsilon \to \mathfrak{R}$, converges pointwise to a function φ on $[0, b]_\Upsilon$ then φ must be m-convex.*

Proof. For any $\tau_1, \tau_2 \in [0, b]_\Upsilon$ and $0 < \lambda \leq 1$, we have

$$
\begin{aligned}
\varphi(\lambda\tau_1 + m(1-\lambda)\tau_2) &= \lim_{n\to\infty} \varphi_n(\lambda\tau_1 + m(1-\lambda)\tau_2) \\
&\le \lim_{n\to\infty} (\lambda\varphi_n(\tau_1) + m(1-\lambda)\varphi_n(\tau_2)) \\
&= \lambda\varphi(\tau_1) + m(1-\lambda)\varphi(\tau_2).
\end{aligned}
$$

□

Proposition 3.2.10. *If $\varphi, \psi : [0,b]_\Upsilon \to \mathfrak{R}$ are both non-negative, m-convex functions as well as increasing, then $\varphi\psi$ is m-convex.*

Proof. If $(\tau_1 \le \tau_2)$ (for $(\tau_2 \le \tau_1)$ runs in the same manner) then $\varphi(\tau_1) - \varphi(\tau_2) \le 0$ and $\psi(\tau_2) - \psi(\tau_1) \ge 0$ which implies

$$
\varphi(\tau_1)\psi(\tau_2) + \varphi(\tau_2)\psi(\tau_1) \le \varphi(\tau_1)\psi(\tau_1) + \varphi(\tau_2)\psi(\tau_2). \tag{3.8}
$$

For any arbitrary $\tau_1, \tau_2 \in [0,b]_\Upsilon$ and $0 \le \lambda \le 1$, we have

$$
\begin{aligned}
(\varphi\psi)(\lambda\tau_1 + m(1-\lambda)\tau_2) &= \varphi(\lambda\tau_1 + m(1-\lambda)\tau_2)\psi(\lambda\tau_1 + m(1-\lambda)\tau_2) \\
&\le (\lambda\varphi(\tau_1) + m(1-\lambda)\varphi(\tau_2))(\lambda\psi(\tau_1) + m(1-\lambda)\psi(\tau_2)) \\
&= \lambda^2\varphi(\tau_1)\psi(\tau_1) + m^2(1-\lambda)^2\varphi(\tau_2)\psi(\tau_2) + m\lambda(1-\lambda)[\varphi(\tau_1)\psi(\tau_2) \\
&\quad + \varphi(\tau_2)\psi(\tau_1)].
\end{aligned}
$$

Using (3.8), we get

$$(\varphi\psi)(\lambda\tau_1 + m(1-\lambda)\tau_2) \leq \lambda^2\varphi(\tau_1)\psi(\tau_1) + m^2(1-\lambda)^2\varphi(\tau_2)\psi(\tau_2) + m\lambda(1-\lambda)[\varphi(\tau_1)\psi(\tau_2)$$

$$+ \varphi(\tau_2)\psi(\tau_1)]$$

$$= \lambda(\lambda + m(1-\lambda))\varphi(\tau_1)\psi(\tau_1) + m(1-\lambda)(\lambda + m(1-\lambda))\varphi(\tau_2)\psi(\tau_2).$$

But $\lambda + m(1-\lambda) \leq 1$, Therefore

$$(\varphi\psi)(\lambda\tau_1 + m(1-\lambda)\tau_2) \leq \lambda\varphi(\tau_1)\psi(\tau_1) + m(1-\lambda)\varphi(\tau_2)\psi(\tau_2)$$

$$= \lambda(\varphi\psi)(\tau_1) + m(1-\lambda)(\varphi\psi)(\tau_2).$$

□

Proposition 3.2.11. *If* $\varphi : [0, b_1]_\Upsilon \to \mathfrak{R}$ *be any finite function on* $\left[a, \frac{b}{m}\right)_\Upsilon \subset [0, b_1]_\Upsilon$ *with* $b < mb_1$, *m-convex where* $m \in (0, 1]$ *be any number. Then* φ *must be bounded on any arbitrary closed interval* $[a, b]_\Upsilon$.

Proof. Suppose $M = \max\{\varphi(a), \varphi\left(\frac{b}{m}\right)\}$ so by taking any arbitrary $z = \lambda a + m(1-\lambda)b \in [a, b]_\Upsilon$, we have

$$\varphi(z) = \varphi(\lambda a + (1-\lambda)b) = \varphi\left(\lambda a + m(1-\lambda)\frac{b}{m}\right)$$

$$\leq \lambda\varphi(a) + m(1-\lambda)\varphi\left(\frac{b}{m}\right)$$

$$\leq M.$$

Then φ is bounded above in $[a, b]_\Upsilon$.

Note that for $z \in [a,b]_\Upsilon$ can be written as $\frac{a+b}{2} + \lambda$ for $\mid \lambda \mid \leq \frac{b-a}{2}$. Hence,

$$\varphi\left(\frac{a+b}{2}\right) = \varphi\left(\frac{1}{2}\left(\frac{a+b}{2}+\lambda\right)+\frac{1}{2}\left(\frac{a+b}{2}-\lambda\right)\right) \leq \frac{1}{2}\varphi\left(\frac{a+b}{2}+\lambda\right)+\frac{m}{2}\varphi\left(\frac{\frac{a+b}{2}-\lambda}{m}\right).$$

In other words,

$$\varphi\left(\frac{a+b}{2}+\lambda\right) \geq 2\varphi\left(\frac{a+b}{2}\right) - m\varphi\left(\frac{\frac{a+b}{2}-\lambda}{m}\right)$$

$$\geq 2\varphi\left(\frac{a+b}{2}\right) - M,$$

and since $\frac{a+b}{2}+\lambda \in [a,b]_\Upsilon$ is an arbitrary element. So, φ is bounded below in $[a,b]_\Upsilon$. □

3.3 Petrovic's Inequality

Time scale version of Petrovic's inequality is derive in this section.

Corollary 3.3.1. *Suppose $\varphi : [0,\infty)_\Upsilon \to \mathfrak{R}$ be an m-convex function, where $0 < m \leq 1$ be an arbitrary number. Let x_i and p_i $(i = 1,2,...)$ be any non-negative numbers in $[0,\infty)_\Upsilon$. If*

$$0 \neq \tilde{x} = \sum_{k=1}^{n} p_k x_k \geq x_i \qquad (i = 1,2,...),$$

then

$$\sum_{k=1}^{n} p_k \varphi(x_k) \leq \min\left\{m\varphi\left(\frac{\tilde{x}}{m}\right) + (P_n - 1)\varphi(0), \varphi(\tilde{x}) + m(P_n - 1)\varphi(0)\right\},$$

where

$$P_n = \sum_{i=1}^{n} p_i \neq 0.$$

3.4 Jensen's Type Inequalities

Time scale version of Jensen's type inequalities of m-convex functions are discussed in this section.

Theorem 3.4.1. *Let $m \in (0, 1]$ be an arbitrary number and I_Υ be an interval which contains zero. Let $\sum_{i=1}^{n} \lambda_i \tau_i$ be the convex combination of points $\tau_i \in I_\Upsilon$ having coefficients $\lambda_i \in [0, 1]$. Then every m-convex function $\varphi : I_\Upsilon \to \mathfrak{R}$ satisfies*

$$\varphi\left(m \sum_{i=1}^{n} \lambda_i \tau_i\right) \leq m \sum_{i=1}^{n} \lambda_i \varphi(\tau_i). \tag{3.9}$$

Proof. Mathematical induction is being used on the number of points τ_i to prove this result. The basic step for $n = 1$ holds by the definition of m-convexity

$$\varphi(m\tau_1) \leq m\varphi(\tau_1).$$

For proving the induction step for $n \geq 2$, we take an assumption that the inequality in (3.9) holds for all those convex combinations, which contains $\leq n - 1$ members. Assume that $\lambda_1 < 1$, and using induction hypothesis to the point

$$s_1 = m \sum_{i=2}^{n} \frac{\lambda_i}{1 - \lambda_1} \tau_i,$$

where the sum $\sum_{i=2}^{n} \frac{\lambda_i}{1-\lambda_1} \tau_i$ is a convex combination in I_Υ,we get

$$\varphi(s_1) \leq m \sum_{i=2}^{n} \frac{\lambda_i}{1 - \lambda_1} \varphi(\tau_i).$$

$$
\begin{aligned}
\varphi\left(m\sum_{i=1}^{n}\lambda_i\tau_i\right) &= \varphi((1-\lambda_1)s_1 + m\lambda_1\tau_1) \\
&\leq (1-\lambda_1)\varphi(s_1) + m\lambda_1\varphi(\tau_1) \\
&\leq (1-\lambda_1)m\sum_{i=2}^{n}\frac{\lambda_i}{1-\lambda_1}\varphi(\tau_i) + m\lambda_1\varphi(\tau_1) \\
&= m\sum_{i=1}^{n}\lambda_i\varphi(\tau_i).
\end{aligned}
$$

$\square$

Corollary 3.4.2. *Let $0 < m \leq 1$ and $0 \in I_\Upsilon$ be an interval. Let $\sum_{i=1}^{n}\lambda_i\tau_i$ be the convex combination of points $\tau_i \in I_\Upsilon$ having coefficients $\lambda_i \in [0,1]$. Then for every m-convex function $\varphi : I_\Upsilon \to \Re$, we have*

$$
\varphi\left(\lambda_1\tau_1 + m\sum_{i=2}^{n}\lambda_i\tau_i\right) \leq \lambda_1\varphi(\tau_1) + m\sum_{i=2}^{n}\lambda_i\varphi(\tau_i).
$$

Corollary 3.4.3. *Suppose $0 < m \leq 1$ be an arbitrary number and $[u,v]_\Upsilon$ be an interval which contains zero. If $\psi : [u,v]_\Upsilon \to \Re$ be any Delta integrable function, such that image$(\psi) \subseteq [u,v]_\Upsilon$, then every continuous m-convex function $\varphi : [u,v]_\Upsilon \to \Re$ satisfies the inequality*

$$
\varphi\left(\frac{m}{v-u}\int_u^v \psi(\tau)\Delta\tau\right) \leq \frac{m}{v-u}\int_u^v \varphi(\psi(\tau))\Delta\tau.
$$

Proof. Let n be an arbitrary positive integer and $I = [u,v]_\Upsilon$. Suppose I_{n_j} be the disjoint subinterval such that $I = \bigcup_{j=1}^{n}\hat{I}_{n_j}$ i.e I_{n_j} be the partitioning of I, so that each of them contracts to the point as $n \to \infty$. By choosing a point τ_{n_j} from every subinterval I_{n_j} and form the convex combination

$$
\sum_{j=1}^{n}\frac{|\hat{I}_{n_j}|}{|I|}\psi(\tau_{n_j}),
$$

of points $s_{n_j} = \psi(\tau_{n_j})$ with coefficients $\lambda_{n_j} = \frac{|\hat{I}_{n_j}|}{|I|}$ where, $||$ denotes the length. Using (3.9) to the above mentioned convex combination, we get

$$\varphi\left(\frac{m}{|I|}\sum_{j=1}^{n}|\hat{I}_{n_j}|\psi(\tau_{n_j})\right) \leq \frac{m}{|I|}\sum_{j=1}^{n}|\hat{I}_{n_j}|\varphi(\psi(\tau_{n_j})),$$

and sending $n \to \infty$, we get the desired result. To providing the left side limit, we assumed that φ is continuous. Since the composition $\varphi(\psi)$ is bounded and almost everywhere continuous, so integrable. $\square$

Theorem 3.4.4. *Suppose* $0 < m \leq 1$ *be an arbitrary number and* $[u, v]_\Upsilon$ *be an interval which contains zero. Let* $\psi : [u, v]_\Upsilon \to \mathfrak{R}$ *be a* $\Delta -$ *integrable function, such that image*$(\psi) \subseteq [u, v]_\Upsilon$ *and let* $\mho : [u, v]_\Upsilon \to \mathfrak{R}$ *be a positive* $\Delta -$ *integrable function i.e* $\mho \in C_{rd}([u, v]_\Upsilon, \mathbb{R})$ *which satisfies* $\int_u^v \mid \mho(\tau) \mid \Delta\tau \geq 0$. *Then every m-convex function* $\varphi : [u, v]_\Upsilon \to \mathfrak{R}$,*which is also continuous must satisfies the inequality*

$$\varphi\left(m\frac{\int_u^v \psi(\tau)\mho(\tau)\Delta\tau}{\int_u^v \mho(\tau)\Delta\tau}\right) \leq m\frac{\int_u^v \varphi(\psi(\tau))\mho(\tau)\Delta\tau}{\int_u^v \mho(\tau)\Delta\tau}. \tag{3.10}$$

Proof. Let n be an arbitrary positive integer and I $= [u, v]_\Upsilon$. Suppose I_{n_j} be the disjoint subinterval such that $I = \bigcup_{j=1}^{n} \hat{I}_{n_j}$ i.e I_{n_j} be the partitioning of I, so that each of them contracts to the point as $n \to \infty$. By choosing a point τ_{n_j} from every subinterval I_{n_j} and form the convex combination

$$\sum_{j=1}^{n}\frac{(|I_{n_j}|\mho(\tau_{n_j}))}{\sum_{j=1}^{n}|I_{n_j}|\mho(\tau_{n_j})}\psi(\tau_{n_j}) = \sum_{j=1}^{n}\frac{(|I_{n_j}|\psi(\tau_{n_j}))}{\sum_{j=1}^{n}|I_{n_j}|\mho(\tau_{n_j})}\mho(\tau_{n_j}),$$

of points $s_{n_j} = \psi(\tau_{n_j})$ with coefficients $\lambda_{n_j} = \frac{|\hat{I}_{n_j}|}{|I|}$ where $||$ denotes the length. Applying the inequality (3.9) to the above mentioned convex combination, we get

$$\varphi\left(m\sum_{j=1}^{n}\frac{(|I_{n_j}|\mho(\tau_{n_j}))}{\sum_{j=1}^{n}|I_{n_j}|\mho(\tau_{n_j})}\psi(\tau_{n_j})\right) \leq m\sum_{j=1}^{n}\frac{(|I_{n_j}|\mho(\tau_{n_j}))}{\sum_{j=1}^{n}|I_{n_j}|\mho(\tau_{n_j})}\varphi(\psi(\tau_{n_j})).$$

As $n \to \infty$, we get (3.10). $\square$

3.5 Hermite-Hadamard Type Inequalities

In this section, we define Hermite-Hadamard type inequalities of m-convex function on time scale.

Theorem 3.5.1. *If $m \in (0, 1]$ be an arbitrary number and $[c, d]_\Upsilon$ be an interval, which contains zero, then every continuous m-convex function $\varphi : [c, d]_\Upsilon \to \mathfrak{R}$ satisfies the double inequality*

$$\frac{1}{m}\varphi\left(m\tau_\Delta\right) \leq \frac{1}{d-c}\int_c^d \varphi(\tau)\Delta\tau \leq \left(\frac{md-\tau_c}{d-c}\right)\varphi(c) + \left(\frac{m\tau_c - mc + \tau_d - m\tau_d}{d-c}\right)\varphi(d),$$

where

$$\tau_\Delta = \frac{1}{d-c}\int_c^d \tau\Delta\tau.$$

$$\tau_c = \frac{1}{md-c}\int_c^{md} \tau\Delta\tau.$$

$$\tau_d = \frac{1}{d-md}\int_{md}^d \tau\Delta\tau.$$

Proof. For every triple of points $c, \tau, d \in [c, d]_\Upsilon$, where $c \neq md$ and $\tau \in conv\{c, md\}$. The function φ satisfies the inequality

$$\varphi(\tau) \leq \frac{md-\tau}{md-c}\varphi(c) + \frac{\tau-c}{md-c}m\varphi(d).$$

By taking the Δ-integral over $[c, md]_\Upsilon$, we have

$$\int_c^{md} \varphi(\tau)\Delta\tau \leq md\varphi(c) - \tau_c\varphi(c) + m\tau_c\varphi(d) - dm\varphi(d). \tag{3.11}$$

Similarly the inequality for triplets $md, \tau, d \in [c, d]_\Upsilon$, where $md \neq d$ and $\tau \in conv\{md, d\}$

$$\varphi(\tau) \leq \frac{d-\tau}{d-md} m\varphi(d) + \frac{\tau - md}{d-md}\varphi(d).$$

By taking the Δ-integral over $[md, d]_\Upsilon$, we get

$$\int_{md}^{d} \varphi(\tau)\Delta\tau \leq \tau_d\varphi(d) - m\tau_d\varphi(d). \tag{3.12}$$

By adding (3.11) and (3.12) we get the required second inequality of *Theorem* (3.5.1).

For proving the first inequality of *Theorem 3.5.1*, we use *Corollary* 3.4.3 , by taking $g : \Upsilon \to \Upsilon$, $\psi(s) = s, \quad \forall s \in \mathbb{T}$. We get

$$\varphi\left(\frac{m\int_c^d s\Delta s}{d-c}\right) \leq \frac{m\int_c^d \varphi(s)\Delta s}{d-c}.$$

Hence

$$\varphi(m\tau_\Delta) \leq \frac{m}{d-c}\int_c^d \varphi(\tau)\Delta\tau.$$

□

Theorem 3.5.2. *Suppose* $m \in (0, 1]$ *be an arbitrary number and* $\varphi : [0, \infty)_\Upsilon \to \mathfrak{R}$ *be m-convex function as well as the* Δ*-integrable on* $[c, d]_\Upsilon$. *If* $0 \leq c < d < \infty$, *where* $c, d \in \Upsilon$ *then we have the following inequality.*

$$\frac{1}{d-c}\int_c^d \varphi(x)\Delta x \leq \min\left\{\frac{\varphi(c) + m\varphi(\frac{d}{m})}{2}, \frac{\varphi(d) + m\varphi(\frac{c}{m})}{2}\right\}.$$

Proof. As φ is m-convex, so

$$\varphi(tu+m(1-t)v) \leq t\varphi(u)+m(1-t)\varphi(v) \;\forall\; u,v \geq 0 \;\; and \;\; \forall\; 0 \leq t \leq 1,$$

we get

$$\varphi(tc+m(1-t)d) \leq t\varphi(c)+m(1-t)\varphi\left(\frac{d}{m}\right),$$

and

$$\varphi(td+m(1-t)c) \leq t\varphi(d)+m(1-t)\varphi\left(\frac{c}{m}\right).$$

Integrating on [0,1] we get

$$\int_0^1 \varphi(tc+(1-t)d)\Delta t \leq \left[\frac{\varphi(c)+m\varphi\left(\frac{d}{m}\right)}{2}\right],$$

and

$$\int_0^1 \varphi(td+(1-t)c)\Delta t \leq \left[\frac{\varphi(d)+m\varphi\left(\frac{c}{m}\right)}{2}\right].$$

However,

$$\int_0^1 \varphi(tc+(1-t)d)\Delta t = \int_0^1 \varphi(td+(1-t)c)\Delta t = \frac{1}{d-c}\int_c^d \varphi(x)\Delta x.$$

Hence

$$\frac{1}{d-c}\int_c^d \varphi(x)\Delta x \leq \min\left\{\frac{\varphi(c)+m\varphi(\frac{d}{m})}{2}, \frac{\varphi(d)+m\varphi(\frac{c}{m})}{2}\right\}.$$

$\square$

Theorem 3.5.3. *Suppose $m \in [0,1]$ ba an arbitrary number and $\varphi : [0,\infty)_\Upsilon \to \mathfrak{R}$ be m-convex function. Let $c, d \in \Upsilon$ with $0 \leq c < d < \infty$, φ is integrable function on $[c,d]_\Upsilon$ satisfies the inequality*

$$\varphi(\tau_\Delta) \leq \frac{1}{d-c}\int_c^d \left(\varphi(\tau) + m\varphi\left(\frac{\tau}{m}\right)\right)\Delta\tau$$

$$\leq \frac{m+1}{4}\Bigg[\left(\frac{d-\tau_\Delta}{d-c}\right)\varphi(c) + \left(\frac{\tau_\Delta - c}{d-c}\right)\varphi(d) \tag{3.13}$$

$$+ m\left(\left(\frac{\frac{d}{m}-\tau_\Delta}{\frac{d}{m}-\frac{c}{m}}\right)\varphi\left(\frac{c}{m}\right) + \left(\frac{\tau_\Delta - \frac{c}{m}}{\frac{d}{m}-\frac{c}{m}}\right)\varphi\left(\frac{d}{m}\right)\right)\Bigg].$$

Theorem 3.5.4. *Let $m \in (0,1]$ be an arbitrary number and I_Υ be an interval which contains zero. If $u, v, w \in I_\Upsilon$ be any triple of points, where $u \leq mw \leq v$, then every m-convex function $\varphi : I_\Upsilon \to \mathfrak{R}$ satisfy the following inequality*

$$\int_u^v \varphi(\tau)\Delta\tau \leq (mw - \tau_u)\varphi(u) + (\tau_v - mw)\varphi(v) + (\tau_u - \tau_v + v - u)m\varphi(w), \tag{3.14}$$

where

$$\tau_\Delta = \frac{1}{v-u}\int_u^v \tau\Delta\tau.$$

$$\tau_u = \frac{1}{mw-u}\int_u^{mw} \tau\Delta\tau.$$

$$\tau_v = \frac{1}{v-mw}\int_{mw}^v \tau\Delta\tau.$$

Proof. Consider the convex combination

$$\tau = \frac{mw-\tau}{mw-u}u + \frac{\tau-u}{mw-u}mw,$$

where $\tau \in conv\{u, mw\}$, where $u < mw$. Now, apply the m-convexity of the function φ to the above mentioned convex combination, we get

$$\varphi(\tau) \leq \frac{mw-\tau}{mw-u}\varphi(u) + \frac{\tau-u}{mw-u}m\varphi(w).$$

Taking the Δ-Integral over $[u, mw]_\Upsilon$, we deduce the following integral estimation,

$$\int_u^{mw} \varphi(\tau)\Delta\tau \leq (mw-\tau_u)\varphi(u) + (\tau_u - u)m\varphi(w). \tag{3.15}$$

In the same way, using the interval $[mw, v]_\Upsilon$, we deduce the following estimation

$$\int_{mw}^{v} \varphi(\tau)\Delta\tau \leq (v-\tau_v)m\varphi(w) + (\tau_v - mw)\varphi(v), \tag{3.16}$$

we can write $\int_u^v \varphi(\tau)\Delta\tau$ as

$$\int_u^v \varphi(\tau)\Delta\tau = \int_u^{mw} \varphi(\tau)\Delta\tau + \int_{mw}^{v} \varphi(\tau)\Delta\tau. \tag{3.17}$$

By using the inequalities (3.15) and (3.16) in (3.17), we get the required result.

□

Since $I = [u, v]_\Upsilon$ is an arbitrary interval which contains zero, so we can take any point $w \in [u, v]_\Upsilon$ in (3.14) as $mw \in [u, v]_\Upsilon$. Taking $w = u$ or $w = v$, we deduce the following inequality.

Corollary 3.5.5. *If $0 < m \leq 1$ be an arbitrary number and $[u, v]_\Upsilon$ be an interval which contains zero, then every m-convex function $\varphi : [u, v]_\Upsilon \to \mathfrak{R}$ satisfies the inequality*

$$\int_u^v \varphi(\tau)\Delta\tau \leq (mv - \tau_u)\varphi(u) + (m\tau_u - mu + \tau_v - m\tau_v)\varphi(v).$$

Theorem 3.5.6. *Suppose $m \in (0,1]$ be an arbitrary number and let $\varphi : [0,\infty)_\Upsilon \to \Re$ be a m-convex function. If $\varphi \in L_1[cm,d]_\Upsilon$ where $0 \leq c \leq d$, then the following inequality holds.*

$$\left[\int_c^{md} \varphi(\tau)\Delta\tau + \frac{md-c}{d-mc}\int_{mc}^{d} \varphi(\tau)\Delta\tau\right]$$

$$\leq \frac{1}{d-mc}\Bigg[(md-\tau_a)(d-mc) + (m(d-\tau_{b'}))(md-c)\varphi(c) \tag{3.18}$$

$$+ (m(\tau_c - c))(d-mc) + (\tau_{b'} - mc)(md-c)\varphi(d)\Bigg],$$

where

$$\tau_c = \frac{1}{md-c}\int_c^{md} \tau\Delta\tau.$$

$$\tau_{d'} = \frac{1}{d-mc}\int_{mc}^{d} \tau\Delta\tau.$$

Proof. The function φ satisfies the inequality

$$\varphi(\tau) \leq \frac{md-\tau}{md-c}\varphi(c) + \frac{\tau-c}{md-c}m\varphi(d),$$

for every triple of points $a, \tau, mb \in [c,d]_\Upsilon$, where $c \neq md$ and $\tau \in conv\{c, md\}$. By taking the Δ-integral over $[c, md]_\Upsilon$, we get

$$\frac{1}{md-c}\int_c^{md}\varphi(\tau)\Delta\tau \le \frac{(md-\tau_c)}{md-c}\varphi(c)+\frac{(\tau_c-c)}{md-c}m\varphi(d). \tag{3.19}$$

Similarly by taking Δ-integral over $[md, d]_\Upsilon$, we get

$$\frac{1}{d-mc}\int_{mc}^{d}\varphi(\tau)\Delta\tau \le \frac{(d-\tau_{d'})}{d-mc}m\varphi(c)+\frac{(\tau_{d'}-mc)}{d-mc}\varphi(d). \tag{3.20}$$

Addition of (3.19) and (3.20), gives (3.18). $\square$

Theorem 3.5.7. *If $\varphi : [0,\infty)_\Upsilon \to [0,\infty)$ be an m_1-convex function and $\psi : [0,\infty)_\Upsilon \to [0,\infty)$ be an m_2-convex on $[c,d]_\Upsilon$, for fixed $m_1, m_2 \in (0,1]$, where $0 \le c < d < \infty$ and $\varphi\psi$ is in $L^1([c,d]_\Upsilon)$, then*

$$\frac{1}{d-c}\int_c^d \varphi(x)\psi(x)\Delta x \le \min\{M_1, M_2\},$$

where

$$M_1 = \frac{1}{3}\left[\varphi(c)\psi(c)+m_1m_2\varphi\left(\frac{d}{m_1}\right)\psi\left(\frac{d}{m_2}\right)\right]+\frac{1}{6}\left[m_2\varphi(c)\psi\left(\frac{d}{m_2}\right)+m_1\varphi\left(\frac{d}{m_1}\right)\psi(c)\right],$$

$$M_2 = \frac{1}{3}\left[\varphi(d)\psi(d)+m_1m_2\varphi\left(\frac{c}{m_1}\right)\psi\left(\frac{c}{m_2}\right)\right]+\frac{1}{6}\left[m_2\varphi(d)\psi\left(\frac{c}{m_2}\right)+m_1\varphi\left(\frac{c}{m_1}\right)\psi(d)\right].$$

Proof. We have

$$\varphi\left(tc+m_1(1-t)\frac{d}{m_1}\right) \le t\varphi(c)+m_1(1-t)\varphi\left(\frac{d}{m_1}\right).$$

$$\psi\left(tc+m_2(1-t)\frac{d}{m_2}\right) \le t\psi(c)+m_2(1-t)\psi\left(\frac{d}{m_2}\right),$$

$\forall\ t \in [0,1]$. Since φ and ψ are non-negative, so

$$\varphi\left(tc + m_1(1-t)\frac{d}{m_1}\right)\psi\left(tc + m_2(1-t)\frac{d}{m_2}\right)$$

$$\leq t^2\varphi(c)\psi(c) + m_2t(1-t)\varphi(c)\psi\left(\frac{d}{m_2}\right) + m_1t(1-t)\varphi\left(\frac{d}{m_1}\right)\psi(c)$$

$$+ m_1m_2(1-t)^2\varphi\left(\frac{d}{m_1}\right)\psi\left(\frac{d}{m_2}\right).$$

By taking the Δ-integral over $t \in [0,1]$, we get

$$\int_0^1 \varphi(tc + (1-t)d)\psi(tc + (1-t)d)\Delta t = \frac{1}{d-c}\int_c^d \varphi(x)\psi(x)\Delta x$$

$$\leq \frac{1}{3}\left[\varphi(c)\psi(c) + m_1m_2\varphi\left(\frac{d}{m_1}\right)\psi\left(\frac{d}{m_2}\right)\right] + \frac{1}{6}\left[m_2\varphi(c)\psi\left(\frac{d}{m_2}\right) + m_1\varphi\left(\frac{d}{m_1}\right)\psi(c)\right].$$

Similarly, we have

$$\frac{1}{d-c}\int_c^d \varphi(x)\psi(x)\Delta x \leq \frac{1}{3}\left[\varphi(d)\psi(d) + m_1m_2\varphi\left(\frac{c}{m_1}\right)\psi\left(\frac{d}{m_2}\right)\right]$$

$$+ \frac{1}{6}\left[m_2\varphi(d)\psi\left(\frac{c}{m_2}\right) + m_1\varphi\left(\frac{c}{m_1}\right)\psi(d)\right].$$

Hence,

$$\frac{1}{d-c}\int_c^d \varphi(x)\psi(x)\Delta x \leq \min\{M_1, M_2\}.$$

□

Lemma 3.5.8. *let $\hbar : [c,d]_\Upsilon \to \mathfrak{R}$ be a Δ-integrable function, which is symmetric with respect to the midpoint of $[c,d]_\Upsilon$. Then every affine function $u : \mathfrak{R} \to \mathfrak{R}$, satisfies the inequality*

$$\int_c^d u(\tau)\hbar(\tau)\Delta\tau = u(\tau_\Delta)\int_c^d \hbar(\tau)\Delta\tau, \tag{3.21}$$

where

$$\tau_\Delta = \frac{1}{d-c}\int_c^d \tau\Delta\tau. \tag{3.22}$$

Proof. Since the equation $\hbar(2\tau_\Delta - \tau) = \hbar(\tau)$ is satisfied by the function $\hbar$. Take the function $\overline{\hbar} : [c,d]_\Upsilon \to \mathfrak{R}$, which is defined by

$$\overline{\hbar}(\tau) = (\tau - \tau_\Delta)\hbar(\tau),$$

satisfies $\overline{\hbar}(2\tau_\Delta - \tau) = -\overline{\hbar}(\tau)$. In fact, this implies that $\hbar$ is antisymmetric with respect to the midpoint t_Δ, so we deduce

$$\int_c^d \overline{\hbar}(\tau)\Delta\tau = \int_c^d x\hbar(\tau)\Delta\tau - \tau_\Delta\int_c^d \hbar(\tau)\Delta\tau = 0,$$

subsequently,

$$\int_c^d \tau\hbar(\tau)\Delta\tau = \tau_\Delta\int_c^d \hbar(\tau)\Delta\tau. \tag{3.23}$$

The equality (3.22) can be obtained by using (3.23) and taking affine equation of u as $u(\tau) = m\tau + n$, where m and n are some real constants. □

Lemma 3.5.9. *Suppose* $0 \in [a,b]_\Upsilon$ *be an interval and* $0 < m \leq 1$ *an arbitrary number. Consider any arbitrary point* $c \in [a,b]_\Upsilon$ *and a positive* Δ*-integrable function which is symmetric with respect to the mid-points of* $[a,mc]_\Upsilon$ *and* $[mc,b]_\Upsilon$*. Then for every* m*-convex function* $\varphi : [a,b]_\Upsilon \to \mathfrak{R}$ *we have the following inequality*

$$\int_a^b \varphi(\tau)\hbar(\tau)\Delta\tau \leq \left(\frac{(mc-\tau_a)\varphi(a)+(\tau_a-a)m\varphi(c)}{mc-a}\right)\int_a^{mc}\hbar(\tau)\Delta\tau$$
$$+\left(\frac{(b-\tau_b)m\varphi(c)+(\tau_b-mc)\varphi(b)}{b-mc}\right)\int_{mc}^{b}\hbar(\tau)\Delta\tau, \qquad (3.24)$$

where

$$\tau_a = \frac{1}{mc-a}\int_a^{mc}\tau\Delta\tau.$$
$$\tau_b = \frac{1}{b-mc}\int_{mc}^{b}\tau\Delta\tau. \qquad (3.25)$$

Proof. The function φ satisfies the inequality

$$\varphi(\tau) \leq \frac{mc-\tau}{mc-a}\varphi(a) + \frac{\tau-a}{mc-a}m\varphi(c) \quad for \quad \tau \in [a, mc]_{\Upsilon},$$

and

$$\varphi(\tau) \leq \frac{m\varphi(c)-\varphi(a)}{mc-a}\tau + m\frac{c\varphi(a)-a\varphi(c)}{mc-a} \;\; for \; \tau \in [a, mc]_{\Upsilon}. \qquad (3.26)$$

Consequently, for the interval $[mc, b]_{\Upsilon}$ we have

$$\varphi(\tau) \leq \frac{\varphi(b)-m\varphi(c)}{b-mc}\tau + m\frac{b\varphi(c)-c\varphi(b)}{b-mc} \;\; for \; \tau \in [mc, b]_{\Upsilon}. \qquad (3.27)$$

Multiplying the inequalities (3.26),(3.27) with $\hbar(\tau)$ we get

$$\varphi(\tau)\hbar(\tau) \leq \frac{m\varphi(c)-\varphi(a)}{mc-a}\tau\hbar(\tau) + m\frac{c\varphi(a)-a\varphi(c)}{mc-a}\hbar(\tau) \;\; for \; \tau \in [a, mc]_{\Upsilon}. \qquad (3.28)$$

$$\varphi(\tau)\hbar(\tau) \leq \frac{\varphi(b) - m\varphi(c)}{b - mc}\tau\hbar(\tau) + m\frac{b\varphi(c) - c\varphi(b)}{b - mc}\hbar(\tau) \ for \ \tau \in [mc, b]_{\Upsilon}, \tag{3.29}$$

Δ-integrating of (3.28) over $[a, mc]_{\Upsilon}$ and (3.29) over the interval $[mc, b]_{\Upsilon}$ by using (3.23) gives

$$\int_{a}^{mc} \varphi(\tau)\hbar(\tau)\Delta\tau \leq \tau_a \left(\frac{m\varphi(c) - \varphi(a)}{mc - a}\right) \int_{a}^{mc} \hbar(\tau)\Delta\tau + \left(\frac{mc\varphi(a) - am\varphi(c)}{mc - a}\right) \int_{a}^{mc} \hbar(\tau)\Delta\tau. \tag{3.30}$$

$$\int_{mc}^{b} \varphi(\tau)\hbar(\tau)\Delta\tau \leq \tau_b \left(\frac{\varphi(b) - m\varphi(c)}{b - mc}\right) \int_{mc}^{b} \hbar(\tau)\Delta\tau + \left(\frac{mb\varphi(c) - mc\varphi(b)}{b - mc}\right) \int_{mc}^{b} \hbar(\tau)\Delta\tau. \tag{3.31}$$

Addition of (3.30) and (3.31), gives (3.24).

□

Theorem 3.5.10. ***(Fejer's Inequality)*** *Suppose $[u, w]_{\Upsilon}$ be an interval which contains zero and $0 < m \leq 1$ be an arbitrary number. Consider any arbitrary point $v \in [u, w]_{\Upsilon}$ and a positive Δ-integrable function, which is symmetric with respect to the mid-points of $[u, mv]_{\Upsilon}$ and $[mv, w]_{\Upsilon}$. Then every m-convex function $\varphi : [u, w]_{\Upsilon} \to \mathfrak{R}$ must satisfies the following inequality*

$$\begin{aligned} &\frac{1}{m}\varphi\left(m\frac{\tau_u \int_u^{mv} \hbar(\tau)\Delta\tau + \tau_w \int_{mv}^{w} \hbar(\tau)\Delta\tau}{\int_u^w \hbar(\tau)\Delta\tau}\right) \\ &\leq \frac{\int_u^w \varphi(\tau)\hbar(\tau)\Delta\tau}{\int_u^w \hbar(\tau)\Delta\tau} \qquad (3.32) \\ &\leq \frac{1}{\int_u^w \hbar(\tau)\Delta\tau}\left(\frac{(mv - \tau_u)\varphi(u) + (\tau_u - u)m\varphi(v)}{mv - u}\right) \int_u^{mv} \hbar(\tau)\Delta\tau \end{aligned}$$

$$+\left(\frac{(w-\tau_w)m\varphi(v)+(\tau_w-mv)\varphi(w)}{w-mv}\right)\int_{mv}^{w}\hbar(\tau)\Delta\tau,$$

where

$$\tau_u=\frac{1}{mv-u}\int_u^{mv}\tau\Delta\tau.$$

$$\tau_w=\frac{1}{w-mv}\int_{mv}^{w}\tau\Delta\tau.$$

Proof. For proving the above inequality we use (3.10) with identity function $\psi(\tau)=\tau$.

$$\varphi\left(m\frac{\int_u^w\tau\hbar(\tau)\Delta\tau}{\int_u^w\hbar(\tau)\Delta\tau}\right)\leq m\frac{\int_u^w\varphi(\tau)\hbar(\tau)\Delta\tau}{\int_u^w\hbar(\tau)\Delta\tau}. \tag{3.33}$$

By using *Lemma (3.5.9)* and (3.33), we get

$$\frac{\int_u^w\varphi(\tau)\hbar(\tau)\Delta\tau}{\int_u^w\hbar(\tau)\Delta\tau}\leq\frac{1}{\int_u^w\hbar(\tau)\Delta\tau}\left(\frac{(mv-\tau_u)\varphi(u)+(\tau_u-u)m\varphi(v)}{mv-u}\right)\int_u^{mv}\hbar(\tau)\Delta\tau$$

$$+\left(\frac{(w-\tau_w)m\varphi(v)+(\tau_w-mv)\varphi(w)}{w-mv}\right)\int_{mv}^{w}\hbar(\tau)\Delta\tau. \tag{3.34}$$

By using *Lemma* (*3.5.8*) and (3.33), we get

$$\frac{1}{m}\varphi\left(m\frac{\tau_u\int_u^{mv}\hbar(\tau)\Delta\tau+\tau_w\int_{mv}^{w}\hbar(\tau)\Delta\tau}{\int_u^w\hbar(\tau)\Delta\tau}\right)\leq\frac{\int_u^w\varphi(\tau)\hbar(\tau)\Delta\tau}{\int_u^w\hbar(\tau)\Delta\tau}. \tag{3.35}$$

By combining (3.34) and (3.35) we get the desired inequality. □

If we choose $\hbar(x)=1$, $v=u$ or $v=w$ in (3.32), we attain the Hermite-Hadamard inequality, which is defined in (*Theorem 3.5.1*).

Chapter 4

Time Scale Version of some Inequalities of m-Convex Functions on Two Coordinates

In this chapter, we will present the time scale version of m-convex functions on two coordinates. Also, we extend a time scale version of the well known inequality namely, Hermite-Hadamard, for coordinated m-convex functions.

4.1 Coordinated Convex Function

Definition 4.1.1. *Suppose $p_1, p_2, q_1, q_2 \in \Upsilon$ where $p_1 < p_2$ and $q_1 < q_2$. Define $\Delta^2_\Upsilon := [p_1, p_2]_\Upsilon \times [q_1, q_2]_\Upsilon$. Any function $\varphi : \Delta^2_\Upsilon \to \mathfrak{R}$ is said to be convex on the coordinates if the following partial mappings*

$$\varphi_y : [p_1, p_2]_\Upsilon \to \mathfrak{R}, \quad defined \ \ by \ \ \varphi_y(u) = \varphi(u, y),$$

and

$$\varphi_x : [q_1, q_2]_\Upsilon \to \mathfrak{R}, \quad defined \ \ by \ \ \varphi_x(v) = \varphi(x, v),$$

are convex, where these mappings are defined $\forall \ \ y \in [q_1, q_2]_\Upsilon$ and $x \in [p_1, p_2]_\Upsilon$.

In other words, a mapping $\varphi : \Delta^2_\Upsilon \to \mathfrak{R}$ is called convex in Δ^2_Υ if for every $(a, b), (c, d) \in \Delta^2_\Upsilon$ and $0 \leq \lambda \leq 1$ we have the following inequality:

$$\varphi(\lambda(a,b)+(1-\lambda)(c,d)) \leq \lambda\varphi(a,b)+(1-\lambda)\varphi(c,d).$$

4.2 Coordinated m-Convex Function

Definition 4.2.1. *Let $\Delta^2_\Upsilon = [0,p_2]_\Upsilon \times [0,q_2]_\Upsilon \subset [0,\infty)^2$ then a function $\varphi : \Delta^2_\Upsilon \to \Re$ is called two coordinated m-convex if the following mappings*

$$\varphi_y : [0,p_2]_\Upsilon \to \Re, \quad defined \;\; by \;\; \varphi_y(u) = \varphi(u,y),$$

and

$$\varphi_x : [0,q_2]_\Upsilon \to \Re, \quad defined \;\; by \;\; \varphi_x(v) = \varphi(x,v),$$

are m-convex on $[0,p_2]_\Upsilon$ and $[0,q_2]_\Upsilon$ respectively, where these mappings are defined $\forall$ $y \in [0,q_2]_\Upsilon$ and $x \in [0,p_2]_\Upsilon$.

4.3 Hermite-Hadamard Type Inequalities

In this section, we develop Hermite-Hadamard type inequalities of m-convex function on two coordinates on timescale.

Theorem 4.3.1. *If $0 < p_1 < p_2$, $\;0 < q_1 < q_2$, where $p_1, p_2, q_1, q_2, x, y \in \Upsilon$, and $\Delta^2_\Upsilon = [0,p_2]_\Upsilon \times [0,q_2]_\Upsilon \subset [0,\infty)^2$ with $p_2, q_2 > 0$ and $\varphi : \Delta^2_\Upsilon \to \Re$ be such that the partial mappings*

$$\varphi_y : [0,p_2]_\Upsilon \to \Re, \quad defined \;\; by \;\; \varphi_y(u) = \varphi(u,y),$$

and

$$\varphi_x : [0,q_2]_\Upsilon \to \Re, \quad defined \;\; by \;\; \varphi_x(v) = \varphi(x,v),$$

are continuous and m-convex, where mappings are defined, $\forall$ $y \in [0,q_2]_\Upsilon$ and $x \in [0,p_2]_\Upsilon$ then the following inequalities

$$\frac{1}{m}\left[\frac{1}{p_2-p_1}\int_{p_1}^{p_2}\varphi(x,ms_\Delta)\Delta x+\frac{1}{q_2-q_1}\int_{q_1}^{q_2}\varphi(mt_\Delta,y)\Delta y\right]$$

$$\leq \frac{2}{(p_2-p_1)(q_2-q_1)}\int_{p_1}^{p_2}\int_{q_1}^{q_2}\varphi(x,y)\Delta y\Delta x$$

$$\leq \frac{1}{(q_2-q_1)(p_2-p_1)}\int_{q_1}^{q_2}\left((mp_2-t_{p_1})\varphi(p_1,y)+(mt_{p_1}-mp_1+t_{p_2}-mt_{p_2})\varphi(p_2,y)\right)\Delta y$$

$$+\frac{1}{(p_2-p_1)(q_2-q_1)}\int_{p_1}^{p_2}\left((mq_2-s_{q_1})\varphi(x,q_1)+(ms_{q_1}-mq_1+s_{q_2}-ms_{q_2})\varphi(x,q_2)\right)\Delta x,$$

holds, where

$$t_\Delta=\frac{1}{p_2-p_1}\int_{p_1}^{p_2}x\Delta x.$$

$$s_\Delta=\frac{1}{q_2-q_1}\int_{q_1}^{q_2}y\Delta y.$$

$$t_{p_1}=\frac{1}{mp_2-p_1}\int_{p_1}^{mp_2}x\Delta x.$$

$$t_{p_2}=\frac{1}{p_2-mp_2}\int_{mp_2}^{p_2}x\Delta x.$$

$$s_{q_1}=\frac{1}{mq_2-q_1}\int_{q_1}^{mq_2}y\Delta y.$$

$$s_{q_2}=\frac{1}{q_2-mq_2}\int_{mq_2}^{q_2}y\Delta y.$$

Proof. Applying *Theorem (3.5.1)* for the function φ_y, we obtain

$$\frac{1}{m}\varphi_y(mt_\Delta) \leq \frac{1}{p_2-p_1}\int_{p_1}^{p_2}\varphi_y(x)\Delta x \leq \left(\frac{mp_2-t_{p_1}}{p_2-p_1}\right)\varphi_y(p_1)$$

$$+\left(\frac{mt_{p_1}-mp_1+t_{p_2}-mt_{p_2}}{p_2-p_1}\right)\varphi_y(p_2),$$

$\forall\ y \in [0, q_2]_\Upsilon$. i.e,

$$\frac{1}{m}\varphi(mt_\Delta, y) \leq \frac{1}{p_2-p_1}\int_{p_1}^{p_2}\varphi(x,y)\Delta x \leq \left(\frac{mp_2-t_{p_1}}{p_2-p_1}\right)\varphi(p_1,y)$$

$$+\left(\frac{mt_{p_1}-mp_1+t_{p_2}-mt_{p_2}}{p_2-p_1}\right)\varphi(p_2,y). \tag{4.1}$$

Taking Δ-integral of (4.1) over the interval $[q_1, q_2]_\Upsilon$, and then dividing the resulting inequality by $q_2 - q_1$, we obtain

$$\frac{1}{m(q_2-q_1)}\int_{q_1}^{q_2}\varphi(mt_\Delta, y)\Delta y$$

$$\leq \frac{1}{(p_2-p_1)(q_2-q_1)}\int_{q_1}^{q_2}\int_{p_1}^{p_2}\varphi(x,y)\Delta x\Delta y$$

$$\leq \left(\frac{mp_2-t_{p_1}}{(p_2-p_1)(q_2-q_1)}\right)\int_{q_1}^{q_2}\varphi(p_1,y)\Delta y + \left(\frac{mt_{p_1}-p_1m+t_{p_2}-mt_{p_2}}{(p_2-p_1)(q_2-q_1)}\right)\int_{q_1}^{q_2}\varphi(p_2,y)\Delta y. \tag{4.2}$$

Similarly by applying *Theorem (3.5.1)* for the function φ_x, we obtain

$$\frac{1}{m}\varphi_x(ms_\Delta) \leq \frac{1}{q_2-q_1}\int_{q_1}^{q_2}\varphi_x(y)\Delta y \leq \left(\frac{mq_2-s_{q_1}}{q_2-q_1}\right)\varphi_x(q_1) + \left(\frac{ms_{q_1}-mq_1+s_{q_2}-ms_{q_2}}{q_2-q_1}\right)\varphi_x(q_2),$$

$\forall\ x \in [0,p_2]_\Upsilon$. i.e,

$$\frac{1}{m}\varphi(x,ms_\Delta) \leq \frac{1}{q_2-q_1}\int_{q_1}^{q_2}\varphi(x,y)\Delta y \leq \left(\frac{mq_2-s_{q_1}}{q_2-q_1}\right)\varphi(x,q_1) + \left(\frac{ms_{q_1}-mq_1+s_{q_2}-ms_{q_2}}{q_2-q_1}\right)\varphi(x,q_2). \tag{4.3}$$

Taking Δ-integral of (4.3) over the interval $[p_1,p_2]_\Upsilon$, and then dividing the resulting inequality by p_2-p_1, we obtain

$$\begin{aligned}\frac{1}{m(p_2-p_1)}\int_{p_1}^{p_2}\varphi(x,ms_\Delta)\Delta x &\leq \frac{1}{(q_2-q_1)(p_2-p_1)}\int_{p_1}^{p_2}\int_{q_1}^{q_2}\varphi(x,y)\Delta y\Delta x \\ &\leq \left(\frac{mq_2-s_{q_1}}{(q_2-q_1)(p_2-p_1)}\right)\int_{p_1}^{p_2}\varphi(x,q_1)\Delta x \\ &\quad + \left(\frac{ms_{q_1}-mq_1+s_{q_2}-ms_{q_2}}{(q_2-q_1)(p_2-p_1)}\right)\int_{p_1}^{p_2}\varphi(x,q_2)\Delta x.\end{aligned} \tag{4.4}$$

Now by adding (4.2) and (4.4),

$$\frac{1}{m}\left[\frac{1}{p_2-p_1}\int_{p_1}^{p_2}\varphi(x,ms_\Delta)\Delta x+\frac{1}{q_2-q_1}\int_{q_1}^{q_2}\varphi(mt_\Delta,y)\Delta y\right]$$

$$\leq\frac{2}{(p_2-p_1)(q_2-q_1)}\int_{p_1}^{p_2}\int_{q_1}^{q_2}\varphi(x,y)\Delta y\Delta x$$

$$\leq\frac{1}{(q_2-q_1)(p_2-p_1)}\int_{q_1}^{q_2}\left((mp_2-t_{p_1})\varphi(p_1,y)+(mt_{p_1}-mp_1+t_{p_2}-mt_{p_2})\varphi(p_2,y)\right)\Delta y$$

$$+\frac{1}{(p_2-p_1)(q_2-q_1)}\int_{p_1}^{p_2}\left((mq_2-s_{q_1})\varphi(x,q_1)+(ms_{q_1}-mq_1+s_{q_2}-ms_{q_2})\varphi(x,q_2)\right)\Delta x,$$

we get the desired inequality. □

Theorem 4.3.2. *Under the assumption of Theorem (4.3.1), and suppose also the intervals contain the midpoints, then the inequality*

$$\frac{1}{m}\left[\varphi\left(\frac{p_1+p_2}{2},ms_\Delta\right)+\varphi\left(mt_\Delta,\frac{q_1+q_2}{2}\right)\right]\leq\frac{1}{p_2-p_1}\int_{p_1}^{p_2}\varphi\left(x,\frac{q_1+q_2}{2}\right)\Delta x$$
$$+\frac{1}{q_2-q_1}\int_{q_1}^{q_2}\varphi\left(\frac{p_1+p_2}{2},y\right)\Delta y, \tag{4.5}$$

holds.

Proof. Since the inequality (4.1) is true $\forall\, y\in[0,q_2]_\Upsilon$, and by postulation, for $y=\frac{q_1+q_2}{2}$, the following inequality holds:

$$\frac{1}{m}\varphi\left(mt_\Delta,\frac{q_1+q_2}{2}\right)\leq\frac{1}{p_2-p_1}\int_{p_1}^{p_2}\varphi\left(x,\frac{q_1+q_2}{2}\right)\Delta x, \tag{4.6}$$

similarly, from (4.3)

$$\frac{1}{m}\varphi\left(\frac{p_1+p_2}{2}, ms_\Delta\right) \leq \frac{1}{q_2-q_1}\int_{q_1}^{q_2} \varphi\left(\frac{p_1+p_2}{2}, y\right)\Delta y. \tag{4.7}$$

Add the inequalities (4.3) and (4.7), we get

$$\frac{1}{m}\left[\varphi\left(\frac{p_1+p_2}{2}, ms_\Delta\right) + \varphi\left(mt_\Delta, \frac{q_1+q_2}{2}\right)\right] \leq \frac{1}{p_2-p_1}\int_{p_1}^{p_2} \varphi\left(x, \frac{q_1+q_2}{2}\right)\Delta x$$

$$+ \frac{1}{q_2-q_1}\int_{q_1}^{q_2} \varphi\left(\frac{p_1+p_2}{2}, y\right)\Delta y,$$

the required result. □

Theorem 4.3.3. *By postulation of Theorem (4.3.1), the following inequality*

$$\frac{1}{p_2-p_1}\int_{p_1}^{p_2} (\varphi(x,q_1)+\varphi(x,q_2))\,\Delta x + \frac{1}{q_2-q_1}\int_{q_1}^{q_2} (\varphi(p_1,y)+\varphi(p_2,y))\,\Delta y \tag{4.8}$$

$$\leq A_1\varphi(p_1,q_1) + A_2\varphi(p_2,q_1) + A_3\varphi(p_1,q_2) + A_4\varphi(p_2,q_2),$$

holds, where

$$A_1 = \left(\frac{mp_2 - t_{p_1}}{p_2-p_1} + \frac{mq_2 - s_{q_1}}{q_2-q_1}\right).$$

$$A_2 = \left(\frac{mt_{p_1} - mp_1 + t_{p_2} - mt_{p_2}}{p_2-p_1} + \frac{mq_2 - s_{q_1}}{q_2-q_1}\right).$$

$$A_3 = \left(\frac{mp_2 - t_{p_1}}{p_2-p_1} + \frac{ms_{q_1} - mq_1 + s_{q_2} - ms_{q_2}}{q_2-q_1}\right).$$

$$A_4 = \left(\frac{mt_{p_1} - mp_1 + t_{p_2} - mt_{p_2}}{p_2-p_1} + \frac{ms_{q_1} - mq_1 + s_{q_2} - ms_{q_2}}{q_2-q_1}\right).$$

Proof. By choosing $y = q_1$ in (4.1), we have

$$\frac{1}{p_2 - p_1}\int_{p_1}^{p_2} \varphi(x, q_1)\Delta x \le \left(\frac{mp_2 - t_{p_1}}{p_2 - p_1}\right)\varphi(p_1, q_1) + \left(\frac{mt_{p_1} - mp_1 + t_{p_2} - mt_{p_2}}{p_2 - p_1}\right)\varphi(p_2, q_1), \tag{4.9}$$

for $y = q_2$, we have

$$\frac{1}{p_2 - p_1}\int_{p_1}^{p_2} \varphi(x, q_2)\Delta x \le \left(\frac{mp_2 - t_{p_1}}{p_2 - p_1}\right)\varphi(p_1, q_2) + \left(\frac{mt_{p_1} - mp_1 + t_{p_2} - mt_{p_2}}{p_2 - p_1}\right)\varphi(p_2, q_2). \tag{4.10}$$

By choosing $x = p_1$ in (4.3), we have

$$\frac{1}{q_2 - q_1}\int_{q_1}^{q_2} \varphi(p_1, y)\Delta y \le \left(\frac{mq_2 - s_{q_1}}{q_2 - q_1}\right)\varphi(p_1, q_1) + \left(\frac{ms_{q_1} - mq_1 + s_{q_2} - ms_{q_2}}{q_2 - q_1}\right)\varphi(p_1, q_2). \tag{4.11}$$

For $x = p_2$, we have

$$\frac{1}{q_2 - q_1}\int_{q_1}^{q_2} \varphi(p_2, y)\Delta y \le \left(\frac{mq_2 - s_{q_1}}{q_2 - q_1}\right)\varphi(p_2, q_1) + \left(\frac{ms_{q_1} - mq_1 + s_{q_2} - ms_{q_2}}{q_2 - q_1}\right)\varphi(p_2, q_2). \tag{4.12}$$

By adding the inequalities (4.9), (4.10), (4.11) and (4.12), we get

$$\frac{1}{p_2-p_1}\int_{p_1}^{p_2}\left(\varphi(x,q_1)+\varphi(x,q_2)\right)\Delta x+\frac{1}{q_2-q_1}\int_{q_1}^{q_2}\left(\varphi(p_1,y)+\varphi(p_2,y)\right)\Delta y$$

$$\leq\left(\frac{mp_2-t_{p_1}}{p_2-p_1}+\frac{mq_2-s_{q_1}}{q_2-q_1}\right)\varphi(p_1,q_1)+\left(\frac{mt_{p_1}-mp_1+t_{p_2}-mt_{p_2}}{p_2-p_1}+\frac{mq_2-s_{q_1}}{q_2-q_1}\right)\varphi(p_2,q_1)$$

$$+\left(\frac{mp_2-t_{p_1}}{p_2-p_1}+\frac{ms_{q_1}-mq_1+s_{q_2}-ms_{q_2}}{q_2-q_1}\right)\varphi(p_1,q_2)$$

$$+\left(\frac{mt_{p_1}-mp_1+t_{p_2}-mt_{p_2}}{p_2-p_1}+\frac{ms_{q_1}-mq_1+s_{q_2}-ms_{q_2}}{q_2-q_1}\right)\varphi(p_2,q_2).$$

□

Theorem 4.3.4. *Let φ,φ_x and φ_y be defined in the same way as in Theorem (4.3.1), Then the the following inequality*

$$\frac{2}{(p_2-p_1)(q_2-q_1)}\int_{p_1}^{p_2}\int_{q_1}^{q_2}\varphi(x,y)\Delta y\Delta x$$

$$\leq\min\left\{\frac{1}{p_2-p_1}\int_{p_1}^{p_2}\left(\frac{\varphi(x,q_1)+m\varphi\left(x,\frac{q_2}{m}\right)}{2}\right)\Delta x\ ,\right. \tag{4.13}$$

$$\left.\frac{1}{p_2-p_1}\int_{p_1}^{p_2}\left(\frac{\varphi(x,q_2)+m\varphi\left(x,\frac{q_1}{m}\right)}{2}\right)\Delta x\right\}$$

$$+\min\left\{\frac{1}{q_2-q_1}\int_{q_1}^{q_2}\left(\frac{\varphi(p_1,y)+m\varphi\left(\frac{p_2}{m},y\right)}{2}\right)\Delta y\ ,\right.$$

$$\left.\frac{1}{q_2-q_1}\int_{q_1}^{q_2}\left(\frac{\varphi(p_2,y)+m\varphi\left(\frac{p_1}{m},y\right)}{2}\right)\Delta y\right\},$$

holds.

Proof. As mapping $\varphi : \Delta^2_\Upsilon \to \mathfrak{R}$ is two coordinated m-convex, then the function φ_x and φ_y are m-convex on $[0, q_2]_\Upsilon$ and $[0, p_2]_\Upsilon$, respectively. Hence, we have

$$\frac{1}{p_2 - p_1}\int_{p_1}^{p_2} \varphi_y(x)\Delta x \leq \min\left\{\frac{\varphi_y(x_a) + \varphi_y\left(\frac{p_2}{m}\right)}{2}, \frac{\varphi_y(p_2) + \varphi_y\left(\frac{p_1}{m}\right)}{2}\right\},$$

that is

$$\frac{1}{p_2 - p_1}\int_{p_1}^{p_2} \varphi(x, y)\Delta x \leq \min\left\{\frac{\varphi(p_1, y) + \varphi\left(\frac{p_2}{m}, y\right)}{2}, \frac{\varphi(p_2, y) + \varphi\left(\frac{p_1}{m}, y\right)}{2}\right\}.$$

By taking the average $\Delta - integration$ over the interval $[q_1, q_2]_\Upsilon$, we have

$$\frac{1}{(p_2 - p_1)(q_2 - q_1)}\int_{p_1}^{p_2}\int_{q_1}^{q_2} \varphi(x, y)\Delta y\Delta x$$

$$\leq \min\left\{\frac{1}{q_2 - q_1}\int_{q_1}^{q_2}\left(\frac{\varphi(p_1, y) + \varphi\left(\frac{p_2}{m}, y\right)}{2}\right)\Delta y\ ,\right. \tag{4.14}$$

$$\left.\frac{1}{q_2 - q_1}\int_{q_1}^{q_2}\left(\frac{\varphi(p_2, y) + \varphi\left(\frac{p_1}{m}, y\right)}{2}\right)\Delta y\right\}.$$

Similarly for φ_x one has

$$\frac{1}{(p_2 - p_1)(q_2 - q_1)}\int_{p_1}^{p_2}\int_{q_1}^{q_2} \varphi(x, y)\Delta y\Delta x$$

$$\leq \min\left\{\frac{1}{p_2 - p_1}\int_{p_1}^{p_2}\left(\frac{\varphi(x, q_1) + \varphi\left(x, \frac{q_2}{m}\right)}{2}\right)\Delta x\ ,\right. \tag{4.15}$$

$$\left.\frac{1}{p_2 - p_1}\int_{p_1}^{p_2}\left(\frac{\varphi(x, q_2) + \varphi\left(x, \frac{q_1}{m}\right)}{2}\right)\Delta x\right\}.$$

By adding the inequalities (4.14) and (4.15), we get

$$\frac{2}{(p_2-p_1)(q_2-q_1)}\int_{p_1}^{p_2}\int_{q_1}^{q_2}\varphi(x,y)\Delta y\Delta x$$

$$\leq \min\left\{\frac{1}{p_2-p_1}\int_{p_1}^{p_2}\left(\frac{\varphi(x,q_1)+m\varphi\left(x,\frac{q_2}{m}\right)}{2}\right)\Delta x\ ,\right.$$

$$\left.\frac{1}{p_2-p_1}\int_{p_1}^{p_2}\left(\frac{\varphi(x,q_2)+m\varphi\left(x,\frac{q_1}{m}\right)}{2}\right)\Delta x\right\}$$

$$+\min\left\{\frac{1}{q_2-q_1}\int_{q_1}^{q_2}\left(\frac{\varphi(p_1,y)+m\varphi\left(\frac{p_2}{m},y\right)}{2}\right)\Delta y\ ,\right.$$

$$\left.\frac{1}{q_2-q_1}\int_{q_1}^{q_2}\left(\frac{\varphi(p_2,y)+m\varphi\left(\frac{p_1}{m},y\right)}{2}\right)\Delta y\right\},$$

which is the required result. □

Theorem 4.3.5. *Let $\Delta_\Upsilon^2=[0,p_2]_\Upsilon\times[0,q_2]_\Upsilon\subset[0,\infty)^2$ with $0<p_1<p_2, 0<q_1<q_2$ and $\varphi:\Delta_\Upsilon^2\to\Re$ be m-convex on two coordinate in Δ_Υ^2 where $0<m\leq 1$. If $\varphi_x\in L_1[0,q_2]_\Upsilon$ and $\varphi_y\in L_1[0,p_2]_\Upsilon$, then*

$$2\varphi\left(t_\Delta,s_\Delta\right)\leq\frac{1}{p_2-p_1}\int_{p_1}^{p_2}\left(\frac{\varphi(x,s_\Delta)+m\varphi\left(\frac{x}{m},s_\Delta\right)}{2}\right)\Delta x$$
$$+\frac{1}{q_2-q_1}\int_{q_1}^{q_2}\left(\frac{\varphi(t_\Delta,y)+m\varphi\left(t_\Delta,\frac{y}{m}\right)}{2}\right)\Delta y. \tag{4.16}$$

Proof. Applying the first inequality of (3.13) for the function φ_y, we get

$$\varphi_y(t_\Delta) \leq \frac{1}{p_2 - p_1} \int_{p_1}^{p_2} \left(\varphi_y(x) + m\varphi_y\left(\frac{x}{m}\right)\right) \Delta x,$$

that is,

$$\varphi(t_\Delta, y) \leq \frac{1}{p_2 - p_1} \int_{p_1}^{p_2} \left(\varphi(x, y) + mf\left(\frac{x}{m}, y\right)\right) \Delta x,$$

for $y = s_\Delta$

$$\varphi\left(t_\Delta, s_\Delta\right) \leq \frac{1}{p_2 - p_1} \int_{p_1}^{p_2} \left(\frac{\varphi(x, s_\Delta) + m\varphi(\frac{x}{m}, s_\Delta)}{2}\right) \Delta x. \tag{4.17}$$

Similarly for φ_x, we obtain

$$\varphi\left(x, s_\Delta\right) \leq \frac{1}{c - d} \int_{c}^{d} \left(\frac{\varphi(x, y) + m\varphi(x, \frac{y}{m})}{2}\right) \Delta y,$$

put $x = t_\Delta$, we get

$$\varphi\left(t_\Delta, s_\Delta\right) \leq \frac{1}{q_1 - q_2} \int_{q_1}^{q_2} \left(\frac{\varphi(t_\Delta, y) + m\varphi(t_\Delta, \frac{y}{m})}{2}\right) \Delta y. \tag{4.18}$$

Addition of (4.17) and (4.18) gives

$$\begin{aligned} 2\varphi\left(t_\Delta, s_\Delta\right) \leq & \frac{1}{p_2 - p_1} \int_{p_1}^{p_2} \left(\frac{\varphi(x, s_\Delta) + m\varphi\left(\frac{x}{m}, s_\Delta\right)}{2}\right) \Delta x \\ & + \frac{1}{q_2 - q_1} \int_{q_1}^{q_2} \left(\frac{\varphi(t_\Delta, y) + m\varphi\left(t_\Delta, \frac{y}{m}\right)}{2}\right) \Delta y, \end{aligned} \tag{4.19}$$

which is the required result. □

4.4 Petrovic's Inequality

Time scale version of Petrovic's inequality for coordinated m-Convex function is defined in this section.

Theorem 4.4.1. *If $\varphi : [0,\infty)^2_\Upsilon \to \mathfrak{R}$, an m-convex function on coordinates where $0 < m \leq 1$ be an arbitrary number. Let $(x_1,...,x_n),(y_1,...,y_n)$ be non-negative n-tuples and $(p_1,...,p_n)$,$(q_1,...,q_n)$ be positive n-tuples such that $\sum_{j=1}^n p_j \geq 1$, where $x_i, y_i, p_i, q_i \in [0,\infty)_\Upsilon$.*

$$P_n := \sum_{j=1}^n p_j, \qquad 0 \neq \tilde{x}_n = \sum_{j=1}^n p_j x_j \geq x_i \quad for \quad every \quad i = 1,...,n,$$

and

$$Q_n := \sum_{k=1}^n q_k, \qquad 0 \neq \tilde{y}_n = \sum_{k=1}^n q_k y_k \geq y_i \quad for \quad every \quad i = 1,...,n,$$

then

$$\begin{aligned} \sum_{j=1}^n \sum_{k=1}^n p_j q_k \varphi(x_j, y_k) \leq \min\{& m \min\{G_{m,1}(\tilde{x}_n/m), G_{1,m}(\tilde{x}_n/m)\} + (P_n - 1) \\ & \times \min\{G_{m,1}(0), G_{1,m}(0)\}, \min\{G_{m,1}(\tilde{x}_n), G_{1,m}(\tilde{x}_n)\} \\ & + m(P_n - 1)\min\{G_{m,1}(0), G_{1,m}(0)\}\}, \end{aligned} \tag{4.20}$$

where

$$G_{m,\widetilde{m}}(t) = m\varphi\left(t, \frac{\tilde{y}_n}{m}\right) + \widetilde{m}(Q_n - 1)\varphi(t, 0). \tag{4.21}$$

Proof. Let $\varphi_x : [0,\infty)_\Upsilon \to \mathfrak{R}$ and $\varphi_y : [0,\infty)_\Upsilon \to \mathfrak{R}$ be mappings such that $\varphi_x(v) = \varphi(x, v)$ and $\varphi_y(u) = \varphi(u, y)$. As φ is coordinated m-convex on $[0,\infty)^2_\Upsilon$, so φ_y is m-convex on $[0,\infty)_\Upsilon$. By using *Corollary 3.3.1* for φ_y, we have

$$\sum_{j=1}^{n} p_j \varphi_y(x_j) \le \min\{m\varphi_y(\tilde{x}_n/m) + (P_n - 1)\varphi_y(0) \quad , \quad \varphi_y(\tilde{x}_n) + m(P_n - 1)\varphi_y(0)\}.$$

This is equivalent to

$$\sum_{j=1}^{n} p_j \varphi(x_j, y) \le \min\{m\varphi(\tilde{x}_n/m, y) + (P_n - 1)\varphi(0, y) \quad , \quad \varphi(\tilde{x}_n, y) + m(P_n - 1)\varphi(0, y)\}.$$

By setting $y = y_k$, we have

$$\sum_{j=1}^{n} p_j \varphi(x_j, y_k) \le \min\{m\varphi(\tilde{x}_n/m, y_k) + (P_n - 1)\varphi(0, y_k) \; , \; \varphi(\tilde{x}_n, y_k) + m(P_n - 1)\varphi(0, y_k)\},$$

That gives

$$\sum_{j=1}^{n}\sum_{k=1}^{n} p_j q_k \varphi(x_j, y_k) \le \min\left\{ m\sum_{k=1}^{n} q_k \varphi(\tilde{x}_n/m, y_k) + (P_n - 1)\sum_{k=1}^{n} q_k \varphi(0, y_k) \quad , \right.$$
$$\left. \sum_{k=1}^{n} q_k \varphi(\tilde{x}_n, y_k) + m(P_n - 1)\sum_{k=1}^{n} q_j \varphi(0, y_k) \right\}. \tag{4.22}$$

Similarly, one has

$$\sum_{k=1}^{n} q_k \varphi(\tilde{x}_n/m, y_k) \le \min\{m\varphi(\tilde{x}_n/m, \tilde{y}_n/m) + (Q_n - 1)\varphi(\tilde{x}_n/m, 0) \quad ,$$
$$\varphi(\tilde{x}_n/m, \tilde{y}_n) + m(Q_n - 1)\varphi(\tilde{x}_n/m, 0)\}.$$

$$\sum_{k=1}^{n} q_k \varphi(0, y_k) \le \min\{m\varphi(0, \tilde{y}_n/m) + (Q_n - 1)\varphi(0, 0) \quad ,$$
$$\varphi(0, \tilde{y}_n) + m(Q_n - 1)\varphi(0, 0)\},$$

and

$$\sum_{k=1}^{n} q_k\varphi(\tilde{x}_n, y_k) \leq \min\{m\varphi(\tilde{x}_n, \tilde{y}_n/m) + (Q_n - 1)\varphi(\tilde{x}_n, 0) \quad ,$$

$$\varphi(\tilde{x}_n, \tilde{y}_n) + m(Q_n - 1)\varphi(\tilde{x}_n, 0)\}.$$

Putting all these values in inequality (4.22), and using the notation in (4.21), one has the required result. □

Corollary 4.4.2. *If $\varphi : [0,\infty)^2_\Upsilon \to \mathfrak{R}$, an m-convex function on two coordinates where $m \in (0,1]$ be an arbitrary number. Let $(x_1, ..., x_n), (y_1, ..., y_n)$ be non-negative n-tuples and $(p_1, ..., p_n)$,$(q_1, ..., q_n)$ be positive n-tuples such that $\sum_{j=1}^{n} p_j \geq 1$, where $x_i, y_i, p_i, q_i \in [0,\infty)_\Upsilon$.*

$$P_n := \sum_{j=1}^{n} p_j, \qquad 0 \neq \tilde{x}_n = \sum_{j=1}^{n} p_j x_j \geq x_i \quad for \quad each \quad i = 1, ..., n,$$

then the following inequality holds:

$$\sum_{j=1}^{n} np_j\varphi(x_j) \leq \min\Bigg\{m\min\{(m+n-1)\varphi(\tilde{x}_n/m), (mn-m+1)\varphi(\tilde{x}_n/m)\}$$

$$+ (P_n - 1)\min\{(m+n-1)\varphi(0), (mn-m+1)\varphi(0)\} \quad ,$$

$$\min\{(m+n-1)\varphi(\tilde{x}_n), (mn-m+1)\varphi(\tilde{x}_n)\}$$

$$+ m(P_n - 1)\min\{(m+n-1), (mn-m+1)\varphi(0)\}\Bigg\}.$$

Proof. If we put $y_k = 0$ and $q_k = 1$, $k = 1, ..., n$ where $\varphi(x, 0) \mapsto \varphi(x)$ in inequality (4.20), we get the desired result. $\square$

Conclusion

In this thesis, we discussed the concept of m-convex function on time scale. In chapter 3, we deduce some basic and well-known inequalities such as Petrovic's, Jensen's, Hermite-Hadamerd and Fejer's of m-convex function on timescale. In chapter 4, we converted the definition of m-convex function in two coordinates of real numbers into timescale. Using this definition we deduce some inequalities such as Hermite-Hadamard, Petrovic's and other results.

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