Depth and Stanley Depth of Powers of Edge Ideals Associated with Certain Trees

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Dedication

This thesis is dedicated to my respectable Supervisor, Teachers, Parents and Siblings for their endless support and encouragement.

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Zunaira Sajid

Abstract

Depth and Stanley depth are the algebraic and geometric invariants, respectively, which have been computed for various classes of graphs. Earlier, for the powers of edge ideal associated with trees and forests, the computed bounds were dependent upon diameter, power and number of connected components in the forest. The present dissertation is primarily concerned with the value of depth and Stanley depth of edge ideals associated with firecracker and three other classes of graphs. Also, the lower bounds have been given for any power of the edge ideals corresponding to the generalized banana tree and a class of the lobster tree, in terms of the total number of stars in the graph and power of the edge ideal, which is comparatively better than the existing bounds for trees.

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Introduction

At first sight, the utilization of graph theory for the study of edge ideals is advantageous in light of the fact that there are common objects of study. However, indeed some noteworthy results on certain graphs and power of their edge ideals have taken place in this dissertation. In 1982, Stanley [23] defined an invariant for finitely generated $\mathbb{Z}^n$-graded modules over the commutative ring, called Stanley depth. He also gave a conjecture relating the depth and Stanley depth of a module. It was later proved by Duval et al. [9] in 2015 that Stanley's conjecture does not hold generally for modules of the type S/I, where S is a polynomial ring in n variables and I is a monomial ideal. Nonetheless, discovering the classes that still satisfies the conjecture is yet an intriguing endeavor. In this thesis, some improved lower bounds are computed for the power of edge ideals of certain trees. Also, a detailed review of the work of Morey [14] and Pournaki et al. [18] on the power of edge ideals of the forest is given.

Chapter 1 summarizes the elementary details about commutative algebra. In particular, it covers the basics of ring and module theory. It also offers a short introduction to graph theory.

Chapter 2 presents the brief introduction of depth and Stanley depth of multigraded finitely generated modules. A method of computing Stanley depth for squarefree monomial ideals is also discussed. Furthermore, some known bounds of these two invariants are quoted along with Stanley's conjecture.

Chapter 3 is devoted to the study of the power of edge ideals. It comprises the bounds for depth and Stanley depth of kth power of edge ideals corresponding to trees and forests. These bounds are dependent upon number of components, diameter and powers of edge ideals.

In Chapter 4, firecracker, circular firecracker, paths attached to a path and cycle with a gradual decrease in length are introduced. Later their depth and Stanley depth are computed by using induction and Depth Lemma on short exact sequences.

Chapter 5 is about calculating the bound for depth and Stanley depth of powers of edge ideal corresponding to a class of lobster tree in terms of power k and the order of r-star in the graph. The bound is also computed for the generalized banana tree in terms of number of stars in the graph and power of its edge ideal.

Chapter 1

Preliminaries

Algebra is the abstract encapsulation of our intuition for the concept of two objects coming together to form a new one. This chapter comprises the fundamental concepts of abstract and commutative algebra; including rings, subrings, their quotients, ideals and modules along with examples. The study of rings is concerned with objects possessing two binary operations (called addition and multiplication) related by the distributive laws, whereas modules are the representation objects for rings, i.e., they are algebraic objects on which rings act. In addition, graph theory is also discussed here. Later, few elementary results are given in order to have a better understanding of the next chapters.

1.1 Ring theory

In algebra, the theory of rings [8] deals with the study of algebraic structures, called rings, which have defined operations of multiplication and addition.

Definition 1.1.1. A non-empty set R along with the two defined binary operations " $\times$ " and " $+$ " is called a ring if it satisfies the following axioms:

- R is a commutative group w.r.t " $+$ ".
- Associativity holds w.r.t " $\times$ " that is for all $s_1, s_2, s_3 \in R$

$$(s_1 \times s_2) \times s_3 = s_1 \times (s_2 \times s_3).$$

- The distributive laws hold in R, that is for all $s_1, s_2, s_3 \in R$

$$(s_1 + s_2) \times s_3 = (s_1 \times s_3) + (s_2 \times s_3),$$

$$s_1 \times (s_2 + s_3) = (s_1 \times s_2) + (s_1 \times s_3).$$

If ring R is commutative w.r.t multiplication then it is called a commutative ring. The ring R is said to have an identity if $\forall\ s \in R$, there exists an element $1 \in R$ such that

$$s \times 1 = 1 \times s = s.$$

Definition 1.1.2. Let R be a ring with unity. If every non-zero element of R has a multiplicative inverse then R is called a division ring. A division ring which is commutative w.r.t multiplication is called a field.

Example 1.1.3. Following are the examples of ring.

1. $\mathbb{Z}/n\mathbb{Z}$ with multiplicative identity 1 under multiplication and addition of residue classes, forms a commutative ring.

2. Let $R = \mathbb{R}^3$, then R is a non-commutative ring without unity, where the operation of addition to be the usual addition of vectors and multiplication is the cross product of vectors.

Proposition 1.1.4. *A non-empty subset I of a ring R is said to be an ideal if and only if $z_1 - z_2 \in I$, $zs \in I$ and $sz \in I$ for all $z_1, z_2, z \in I$ and $s \in R$.*

1.1.1 Ring of Polynomials

The polynomial ring is a particular type of ring which is formed by a set of polynomials. These polynomials are in one or more than one variable where the coefficients belong to a ring or maybe a field. Polynomial rings are used in several fields of mathematics and the investigation of their properties is among the primary inspirations for the advancement of commutative algebra and ring theory.

Definition 1.1.5. For a commutative ring R with unity, a polynomial in variable y while the coefficients belong to R has the form

$$s_0 + s_1 y + \cdots + s_{n-1} y^{n-1} + s_n y^n,$$

with $n \geq 0$ and each $s_i \in R$. If $s_n \neq 0$, then the polynomial is said to be of degree n, where $r_n y^n$ is called the leading term. The set of polynomials is denoted by $R[y]$. Thus

$$R[y] = \{s_0 + s_1 y + \cdots + s_{n-1} y^{n-1} + s_n y^n : n \geq 0, s_i \in R\}.$$

$R[y]$ is a commutative ring with unity under addition and multiplication of polynomials and the unity of $R[y]$ is the unity of R.

Definition 1.1.6. The polynomial ring in the variables $y_1, y_2, \ldots, y_n$ and coefficients belonging to R(commutative with identity) is represented by

$$R[y_1, y_2, \ldots, y_n] = R[y_1, y_2, \ldots, y_{n-1}][y_n].$$

1.1.2 Ring Homomorphism

In the field of ring theory, a ring homomorphism is a map from one ring to another that respects the same additive and multiplicative structures.

Definition 1.1.7. Consider two rings R_1 and R_2. A ring homomorphism is a map $H : R_1 \to R_2$ which satisfies the following axioms for all $s_1, s_2 \in R_1$

- $H(s_1 + s_2) = H(s_1) + H(s_2)$,
- $H(s_1 s_2) = H(s_1)H(s_2)$,
- $H(1) = H(1')$,

where 1 and $1'$ are multiplicative identities of R_1 and R_2, respectively. A ring homomorphism which is both injective and surjective is known as ring isomorphism.

Definition 1.1.8. For a proper ideal I, a quotient ring R/I can be formed, which consists of cosets $s + I$, where $s \in R$, and the product of cosets is defined as:

$$(s_1 + I)(s_2 + I) = s_1 s_2 + I.$$

Next there are the isomorphism theorems for rings.

Theorem 1.1.9. *(Isomorphism Theorems)*

1. *For a ring homomorphism $\gamma : R_1 \to R_2$, $\gamma(R_1)$ is isomorphic to $R_1/ker(\gamma)$, i.e.,*

$$R_1/ker(\gamma) \cong \gamma(R_1).$$

2. *For an ideal I and subring J of R_1,*

$$(J + I)/I \cong J/J \cap I.$$

3. *Consider the ideals I_1 and I_2 of ring R_1, with $I_1 \subseteq I_2$, then I_2/I_1 is an ideal of R_1/I_1. Also*

$$(R_1/I_1)/(I_2/I_1) \cong R_1/I_2.$$

For the ideals I and K of the ring R, the set of sums $a + b$ with $a \in I$ and $b \in K$ is not only a subring of R but is an ideal in R (the set is clearly closed under addition and $\alpha(a + b) = \alpha a + \alpha b \in I + K$ since $\alpha a \in I$ and $\alpha b \in K$). The product of ideals can also be defined in the following way.

Definition 1.1.10. Assume I_1 and I_2 be the ideals of ring R.

1. Product of two ideals, say I_1 and I_2, is a set consisting of all possible finite sums of the elements of the form $i_1 i_2$ where $i_1 \in I_1$ and $i_2 \in I_2$. It is denoted by $I_1 I_2$.

2. Similarly the t^{th} power of an ideal I, for $t \geq 1$, is a set consisting of all possible finite sums of the elements of the form $i_1 i_2 \dots i_t$ with $i_j \in I$ for all j. Equivalently, I^t is defined inductively by defining $I^1 = I$, and $I^t = II^{t-1}$ for $t = 2, 3, \dots$.

Example 1.1.11. Let $I = 9\mathbb{Z}$ and $J = 12\mathbb{Z}$ in $\mathbb{Z}$. Then $I + J$ consists of all integers of the form $9z_1 + 12z_2$ with $z_1, z_2 \in \mathbb{Z}$. Since every such integer is divisible by 3, so $9\mathbb{Z} + 12\mathbb{Z} \subseteq 3\mathbb{Z}$. On the other hand, $3 = 9(-1) + 12(1)$ shows that $3\mathbb{Z}$ is contained in $9\mathbb{Z} + 12\mathbb{Z}$, hence $9\mathbb{Z} + 12\mathbb{Z} = 3\mathbb{Z}$. In general, $q_1\mathbb{Z} + q_2\mathbb{Z} = d\mathbb{Z}$, whereas $d = (q_1, q_2)$.

1.1.3 Properties of Ideals

One important class of ideals are those which are not a subset of any other proper ideal of the ring.

Definition 1.1.12. For a ring R, a maximal ideal M is a proper ideal of the ring which is only contained in M itself and R.

Definition 1.1.13. For a commutative ring R, a proper ideal P is said to be a prime ideal if for $s_1, s_2 \in R$, $s_1 s_2 \in P$, then either $s_1 \in P$ or $s_2 \in P$.

Definition 1.1.14. For a ring R, consider two ideals I_1 and I_2. Then their ideal quotient is defined as

$$(I_1 : I_2) = \{s \in R \ : \ sI_2 \subseteq I_1\}.$$

Definition 1.1.15. The radical of an ideal I is defined as

$$\sqrt{I} = \{s \in R \ : \ s^k \in I \, , \text{ for } k > 0\}.$$

Definition 1.1.16. For a proper ideal N of ring R, N is called a primary ideal if $s_1 s_2 \in N$, for $s_1, s_2 \in R$, then either $s_1 \in N$ or $s_2^k \in N$ for some $k \geq 1$.

When N is a primary ideal, P is a prime ideal and also $P = \sqrt{N}$ then N is called P-primary.

1.1.4 Monomial Ideal

For a polynomial ring $S = K[x_1, \ldots, x_n]$ over the field K, monomials forms the natural K-basis. Let $\mathbb{R}_+^n$ be the set of vectors $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{R}^n$ where every $b_j \geq 0$ and $\mathbb{Z}_+^n = \mathbb{R}_+^n \cap \mathbb{Z}^n$. A monomial is any product of the form $x_1^{b_1} \ldots x_n^{b_n}$ with $b_j \in \mathbb{Z}_+$. If $w = x_1^{b_1} \ldots x_n^{b_n}$ is a monomial then $w = x^{\mathbf{b}}$ with $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{Z}_+^n$, and

$$x^{\mathbf{b_1}} x^{\mathbf{b_2}} = x^{\mathbf{b_1} + \mathbf{b_2}}.$$

An ideal whose generating set only consists of monomials is said to be a monomial ideal. Mon(S) denotes the set of all monomials in S and it forms the basis of S. For any polynomial $f \in S$ and for $b_w \in K$

$$f = \sum_{w \in Mon(S)} b_w w,$$

where support of f is defined as

$$\text{supp}(f) = \{w \in Mon(S) \, : \, b_w \neq 0\}.$$

Definition 1.1.17. Let $G(I) = \{u_1, \ldots, u_m\}$. The ideal I is called a complete intersection ideal if and only if, for all $i \neq j$,

$$\text{supp}(u_i) \cap \text{supp}(u_j) = \emptyset.$$

Proposition 1.1.18. *Consider two monomial ideals I_1 and I_2. Then*

1. *$I_1 \cap I_2$ is a monomial ideal, and $\{lcm(p, q) \, : \, p \in G(I_1), \, q \in G(I_2)\}$ is the generating set of $I_1 \cap I_2$.*

2. *$(I_1 : I_2)$ is a monomial ideal and $(I_1 : I_2) = \bigcap_{q \in G(I_2)} (I_1 : (q))$.*

A monomial $x^{\mathbf{b}}$ is said to be squarefree if $\mathbf{b}$ has components 0 and 1. An ideal with a generating set containing only squarefree monomials is known as squarefree monomial ideal.

Definition 1.1.19. For a prime ideal P, the height of P denoted by $\text{ht}(P)$, is defined as

$$\text{ht}(P) = \max\{m_i \, : \, P_0 \subset P_1 \subset \ldots \subset P_{m_i} = P\}.$$

For any proper ideal I

$$\text{ht}(I) = \min\{\text{ht}(P_i) \, : \, I \subset P_i \text{ where } P_i \text{ is a prime ideal}\}.$$

1.1.5 Primary Decomposition

For an ideal J, primary decomposition is a way of representing J as an intersection $J = \bigcap_{m=1}^{n} N_m$, whereas each N_m is a primary ideal. Let $\{P_m\} = \text{Ass}(N_m)$. If none of the N_m can be omitted in this intersection and $P_r \neq P_s$ for all $r \neq s$ then it is called irredundant primary decomposition.

Example 1.1.20. Let $I = (\, x_1^2 x_3\, ,\, x_4^3\, ,\, x_2^4 x_4^2\, ,\, x_1 x_2 x_3^3\,)$, then

$$\begin{aligned} I &= (x_1^2\, ,\, x_4^3\, ,\, x_2^4 x_4^2\, ,\, x_1 x_2 x_3^3\,) \cap (x_3\, ,\, x_4^3\, ,\, x_2^4 x_4^2\, ,\, x_1 x_2 x_3^3\,) \\ &= (x_1^2\, ,\, x_4^3\, ,\, x_2^4 x_4^2\, ,\, x_1 x_2 x_3^3\,) \cap (x_3\, ,\, x_4^3\, ,\, x_2^4 x_4^2\,) \\ &= (x_1^2\, ,\, x_4^3\, ,\, x_2^4\, ,\, x_1 x_2 x_3^3\,) \cap (x_1^2\, ,\, x_4^3\, ,\, x_4^2\, ,\, x_1 x_2 x_3^3\,) \cap (x_3\, ,\, x_4^3\, ,\, x_2^4\,) \cap (x_3\, ,\, x_4^3\, ,\, x_4^2\,) \\ &= (x_1^2\, ,\, x_4^3\, ,\, x_2^4\, ,\, x_1 x_2 x_3^3\,) \cap (x_1^2\, ,\, x_4^2\, ,\, x_1 x_2 x_3^3\,) \cap (x_3\, ,\, x_4^2\, ,\, x_2^4\,) \cap (x_3\, ,\, x_4^2\,). \end{aligned}$$

In the above example, the obtained primary decomposition is irredundant as $P_r \neq P_s$ for $1 \leq r, s \leq 4$. But generally it does not happen, as in the following example.

Example 1.1.21. Let $I = (\, x_2^4\, ,\, x_3^4\, ,\, x_2^3 x_4^3\, ,\, x_2 x_3 x_4^3\, ,\, x_3^3 x_4^3\,)$, then

$$\begin{aligned} I &= (\, x_2^4\, ,\, x_3^4\, ,\, x_2^3\, ,\, x_2 x_3 x_4^3\, ,\, x_3^3 x_4^3\,) \cap (\, x_2^4\, ,\, x_3^4\, ,\, x_4^3\, ,\, x_2 x_3 x_4^3\, ,\, x_3^3 x_4^3\,) \\ &= (\, x_2^3\, ,\, x_3^4\, ,\, x_2 x_3 x_4^3\, ,\, x_3^3 x_4^3\,) \cap (\, x_2^4\, ,\, x_3^4\, ,\, x_4^3\,) \\ &= (\, x_2^3\, ,\, x_3^4\, ,\, x_2\, ,\, x_3^3 x_4^3\,) \cap (\, x_2^3\, ,\, x_3^4\, ,\, x_3 x_4^3\, ,\, x_3^3 x_4^3\,) \cap (\, x_2^4\, ,\, x_3^4\, ,\, x_4^3\,) \\ &= (\, x_2\, ,\, x_3^4\, ,\, x_3^3 x_4^3\,) \cap (\, x_2^3\, ,\, x_3^4\, ,\, x_3 x_4^3\,) \cap (\, x_2^4\, ,\, x_3^3\, ,\, x_4^3\,) \\ &= (\, x_2\, ,\, x_3^4\, ,\, x_3^3\,) \cap (\, x_2\, ,\, x_3^4\, ,\, x_4^3\,) \cap (\, x_2^3\, ,\, x_3^4\, ,\, x_3\,) \cap (\, x_2^3\, ,\, x_3^4\, ,\, x_4^3\,) \cap (\, x_2^4\, ,\, x_3^3\, ,\, x_4^3\,) \\ &= (\, x_2\, ,\, x_3^3\,) \cap (\, x_2\, ,\, x_3^4\, ,\, x_4^3\,) \cap (\, x_2^3\, ,\, x_3\,) \cap (\, x_2^4\, ,\, x_3^4\, ,\, x_4^3\,) \\ &= (\, x_2\, ,\, x_3^3\,) \cap (\, x_2^3\, ,\, x_3\,) \cap (\, x_2^4\, ,\, x_3^4\, ,\, x_4^3\,). \end{aligned}$$

It is the primary decompostion of I but not irredundant. Here $\mathrm{Ass}(\, x_2\, ,\, x_3^3\,) = \mathrm{Ass}(\, x_2^3\, ,\, x_3\,) = \{(\, x_2\, ,\, x_3\,)\}$. Now for irredundant primary decomposition, take an intersection of $(\, x_2\, ,\, x_3^3\,)$ and $(\, x_2^3\, ,\, x_3\,)$, that is

$$(\, x_2\, ,\, x_3^3\,) \cap (\, x_2^3\, ,\, x_3\,) = (\, x_2^3\, ,\, x_2 x_3\, ,\, x_3^3\,).$$

Hence

$$I = (\, x_2^4\, ,\, x_3^4\, ,\, x_4^3\,) \cap (\, x_2^3\, ,\, x_2 x_3\, ,\, x_3^3\,).$$

1.2 Module Theory

For a ring R, the definition of R-module M [8] is quite similar to that of group action. Modules are the representation objects for rings, which implies they are the algebraic objects on which the rings act.

Definition 1.2.1. For a commutative ring R, an R-module M is an commutative group w.r.t addition, along with a scalar multiplication map $\cdot : R \times M \to M$, defined as $\cdot((s,m)) = sm$, which holds the following axioms

1. $s(m_1 + m_2) = sm_1 + sm_2$,
2. $(s_1 + s_2)m = s_1 m + s_2 m$,
3. $(s_1 s_2)m = s_1(s_2 m)$,
4. $1m = m$, $\quad \forall\ s_1, s_2 \in \mathbb{R}$ and $m_1, m_2 \in M$.

Examples 1.2.2.

1. For a commutative group D, let $d \in D$ and $z \in \mathbb{Z}$, then define $\cdot : \mathbb{Z} \times D \to D$, such that

$$\cdot(z,d) = zd = \begin{cases} (-d) + \cdots + (-d)\,, & \text{if } \ z < 0; \\ d + d + \cdots + d\,, & \text{if } \ z > 0; \\ 0\,, & \text{if } \ z = 0. \end{cases}$$

   Then D is a $\mathbb{Z}$-module.

2. The ideals of the ring are also R-modules.

1.2.1 R-module Homomorphism

Definition 1.2.3. For a ring R, let K and S be R-modules. A function $F : K \to S$ is an R-module homomorphism if

- $F(k_1 + k_2) = F(k_1) + F(k_2)$, $\quad$ for all $k_1, k_2 \in K$.
- $F(rk) = rF(k)$, $\quad$ for all $r \in R\,,\ k \in K$.

If γ is injective and onto then it becomes an R-module isomorphism.

Examples 1.2.4.

1. For a ring R, consider R-module R. Then R-module homomorphism (even from R into itself) needs not to be a ring homomorphism. Consider $R = \mathbb{Z}$, the $\mathbb{Z}$-module homomorphism $x \mapsto 2x$ is not a ring homomorphism.

2. When $R = F[y]$, the ring homomorphism $\phi : h(y) \mapsto h(y^2)$ is not an $F[y]$-module homomorphism.

Definition 1.2.5. For a ring R, assume a submodule Q of R-module M. Then (additive abelian) quotient group M/Q becomes an R-module by defining the scalar multiplication, $\forall\, r \in R\,,\, m + Q \in M/Q$

$$r(m + Q) = rm + Q.$$

1.2.2 Generation of Modules

For any subset A of R-module M, let

$$RA = \{r_1a_1 + \cdots + r_na_n \,:\, r_1, \dots, r_n \in R\,,\, a_1, \dots, a_n \in A \text{ and } n \in \mathbb{Z}^+\}.$$

If A is the finite set, $\{a_1, \dots, a_n\}$, then $RA = Ra_1 + Ra_2 + \cdots + Ra_n$. Let $N = RA$ for some subset A of M and say N is a submodule of M. A is the generating set for N. A submodule N is said to be finitely generated if for $N = RA$, A is a finite subset of M. A R-submodule N may have many different generating sets (for instance the set N itself always generates N). Any generating set consisting of d number of elements (and for all $r < d$ it no longer remain a generating set) will be called a minimal generating set for N, whereas generally it is not unique and $d, r \in \mathbb{Z}^+ \cup \{0\}$.

Remark 1.2.6. *Submodules of finitely generated module needs not to be finitely generated. For a field F let $S = F[x_1, x_2, \dots]$. Consider S as a module over itself then R is a cyclic module generated by 1. Let $Q = (x_1, x_2, \dots)$ is a submodule of S, generated by $x_1, x_2, \dots$. Then Q cannot be generated by any finite set.*

Definition 1.2.7. Let F be an R-module then it is said to be free on the subset H of F if for $0 \neq f \in F$, there are unique non-zero elements $r_1, \dots, r_k$ of R and unique $h_1, \dots, h_k$ in H, such that

$$f = r_1h_1 + \cdots + r_kh_k.$$

Definition 1.2.8. For a commutative ring R, consider a chain of prime ideals in the ring, with length n_i

$$P_0 \subsetneq P_1 \subsetneq P_2 \subsetneq \cdots \subsetneq P_{n_i},$$

then dimension of ring R is defined as

$$\dim R = \sup\{n_i\}.$$

Let M be an R-module, then the Krull dimension of M is

$$\dim(M) = \dim(R/\text{Ann}(M)).$$

For the modules of the type R/I

$$\dim(R/I) = \max\{\dim(R/Q_i) \, : \, Q_i \in \text{Ass}(R/I)\}.$$

1.2.3 Exact Sequences

Definition 1.2.9. Consider a sequence of R-homomorphisms on R-modules

$$\ldots \longrightarrow \; M_{j-1} \xrightarrow{g_j} M_j \xrightarrow{g_{j+1}} M_{j+1} \xrightarrow{g_{j+2}} \ldots$$

it is exact at M_j if $Im(g_j) = ker(g_{j+1})$. The sequence is exact if it is exact at every M_j. Particularly, $0 \longrightarrow \; N' \xrightarrow{g} M$ is exact at N' if and only if g is one to one, and $M \xrightarrow{h} N'' \longrightarrow \; 0$ is exact at N'' if and only if h is onto.

Proposition 1.2.10. *The sequence*

$$0 \longrightarrow \; N' \xrightarrow{g} M \xrightarrow{h} N'' \longrightarrow \; 0$$

is an exact sequence if and only if g is one to one, h is onto and $Im(g) = ker(h)$.

Remark 1.2.11. *The sequence in Proposition 1.2.10 is called a short exact sequence.*

Examples 1.2.12. 1. Let D and E are R-modules, then

$$0 \longrightarrow \; D \xrightarrow{j} D \oplus E \xrightarrow{\pi} E \longrightarrow \; 0$$

is a short exact sequence, where $j(d) = (d, 0)$ and $\pi(d, e) = e$.

2. $0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$ is also an short exact sequence, here n denotes a map $x \mapsto nx$, given by multiplication by n.

Proposition 1.2.13. *Let Σ be a poset with respect to $\leq$. Then the following are equivalent.*

1. *Every increasing sequence $x_1 \leq x_2 \leq \ldots \leq x_n \leq \ldots$ in Σ is stationary, that is there exist $n \in \mathbb{N}$ for which $x_m = x_n$, for all $m \geq n$.*

2. *Every $\emptyset \neq A \subset \Sigma$ has a maximal element.*

Let Σ be the set of submodules of M which is ordered by the relation $\subseteq$ then *1* is called *ascending chain condition* and *2* is called the *maximal condition*.

Definition 1.2.14. For a ring R, an R-module M is said to be Noetherian if each ascending chain of R-submodules of M is stationary. A ring R is said to be Noetherian if R is Noetherian as an R-module.

Theorem 1.2.15. *(Hilbert Basis Theorem) Let R be a noetherian ring, then $R[y]$ is also a noetherian ring.*

Definition 1.2.16. Let M be finitely generated R-module where R is a Noetherian ring, an associated prime ideal of a module is a prime ideal P of the ring R such that $P = \text{Ann}(m)$, where $\text{Ann}(m) = \{\alpha \in R : \alpha m = 0\}$.

1.2.4 Graded Rings

Consider a commutative semigroup (w.r.t addition) U. An U-graded ring is a ring R along with a decomposition

$$R = \bigoplus_{u \in U} R_u \text{ (as a group)},$$

such that $R_u R_v \subset R_{u+v}$ for all $u, v \in U$.
Then for $r \in R$, we can write a unique expression

$$r = \sum_{u \in U} r_u,$$

where $r_u \in R_u$ and almost all $r_u = 0$. The element r_u is called the *uth* homogeneous component and if $r = r_u$, then r is homogeneous of degree u. $R[x]$ and $R[x, y]$ are $\mathbb{Z}$-graded rings as

- $R[x] = R \oplus Rx \oplus Rx^2 \oplus Rx^3 \oplus Rx^4 \oplus Rx^5 \oplus \cdots$.

- $R[x, y] = R \oplus (Rx + Ry) \oplus (Rx^2 + Rxy + Ry^2) \oplus (Rx^3 + Rx^2y + Rxy^2 + Ry^3) \oplus \cdots$.

For a U-graded ring R and R-module M

$$M = \bigoplus_{u \in U} M_u \text{ (as a group)},$$

with $R_u M_v \subset M_{u+v}$ for all $u, v \in U$, then M is said to be a U-graded module. A non zero element of M_u is called a homogeneous element of degree u.

For a polynomial ring S defined over the field K, suppose $\mathbf{b} \in \mathbb{Z}^n$, then $h \in S$ is said to be homogeneous of degree $\mathbf{b}$ when h has the form $\beta x^{\mathbf{b}}$, where $\beta \in K$. Also S is $\mathbb{Z}^n$-graded with graded components:

$$S_{\mathbf{b}} = \begin{cases} Kx^{\mathbf{b}}, & \text{if } \mathbf{b} \in \mathbb{Z}^n_+; \\ 0\,, & \text{otherwise.} \end{cases}$$

An S-module M is said to be $\mathbb{Z}^n$-graded if $M = \bigoplus_{\mathbf{b} \in \mathbb{Z}^n} M_{\mathbf{b}}$ and $S_{\mathbf{b_1}} M_{\mathbf{b_2}} \subset M_{\mathbf{b_1}+\mathbf{b_2}}$ for all $\mathbf{b_1}, \mathbf{b_2} \in \mathbb{Z}^n$.

1.3 Graph Theory

Graph theory comprises the study of graphs while the graphs are the mathematical structures that are used to model the relation between the objects. The present section gives an introduction to the primary fundamentals of graph theory [26].

Definition 1.3.1. A graph G is a triplet having a vertex set $V(G)$, an edge set $E(G)$ and a relation that associates two vertices with each edge of G, which are called the endpoints.

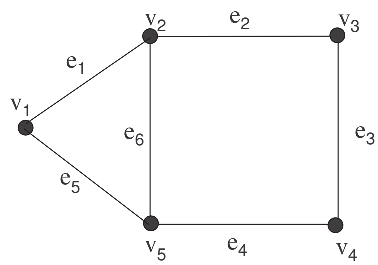

Figure 1.1: Simple graph

Definition 1.3.2. An edge with same end points is called a loop. The edges which share the same set of endpoints are called multiple edges. A simple graph is a graph with no multiple edges and loops.

Consider an edge with endpoints u_1, u_2. Then u_1, u_2 are said to be adjacent and they are neighbors of each other, written as $u_1 \leftrightarrow u_2$. In numerous significant applications, the attention is restricted to simple graphs only.

Definition 1.3.3. For a vertex w of a graph G, the total number of edges incident on it is known as the degree of w, which is usually denoted by $d_G(w)$ or $d(w)$. The set containing all the vertices adjacent to w forms the neighbourhood of w, denoted by $N_G(w)$.

Definition 1.3.4. The total number of vertices in $V(G)$ is known as the order of the graph G, denoted by $n(G)$. While the total number of edges in $E(G)$ determines the size of the graph, written as $e(G)$.

Proposition 1.3.5. *[26] For a graph G,*

$$\sum_{w \in V(G)} d(w) = 2e(G).$$

Definition 1.3.6. A graph G is said to be a path if $V(G)$ can be ordered in a way that whenever two vertices are consecutive in the list, there is an edge between them. A path P is said to maximal if it is not contained in any path longer than P.

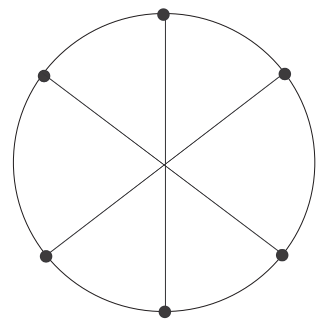

Figure 1.2: 3-regular graph

Definition 1.3.7. A graph whose vertex and edge sets have the same cardinality and vertices can be placed around the circle so that whenever two vertices appear consecutive along the circle, an edge lies between them, such a graph is known as a cycle. Deleting one edge from a cycle forms a path. A cycle and path on n vertices are represented by C_n and P_n, respectively.

Definition 1.3.8. Bipartite graph is a graph whose vertex set can be written as a union of two disjoint independent sets.

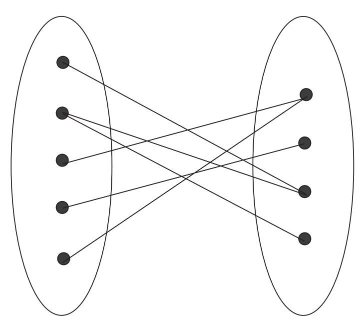

Figure 1.3: Bipartite graph

Definition 1.3.9. A simple graph in which there is an edge between every two vertices is known as a complete graph.

Definition 1.3.10. A subgraph J of a graph G, written as $J \subseteq G$, is a graph such that $V(J) \subseteq V(G)$ and $E(J) \subseteq E(G)$ and the endpoints of edges in J are the same as in G.

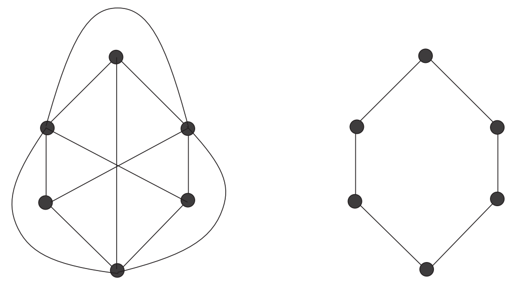

Figure 1.4: Graph and its subgraph

Definition 1.3.11. A graph is said to be connected if there is a path between every two vertices of the graph, otherwise graph is said to be disconnected.

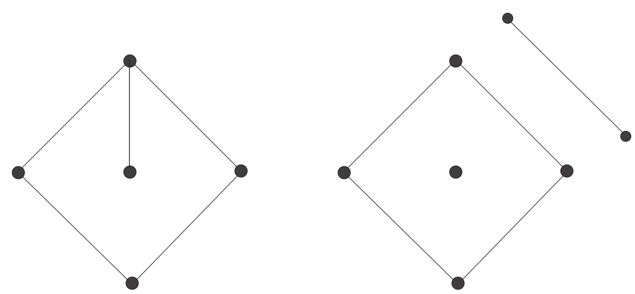

Figure 1.5: Connected and disconnected graphs

Definition 1.3.12. A subgraph of G which is connected and is not a part of any other connected subgraph of G is called the maximal connected subgraph. It is also known as the component of the graph.

Definition 1.3.13. An acyclic graph is a graph without cycles. A forest is an acyclic graph, while a connected graph without cycles is called a tree. A vertex w such that degree of w is 1, is called a leaf.

Every component of forest is a tree. Since trees and forests contain no cycles, therefore they are bipartite graphs.

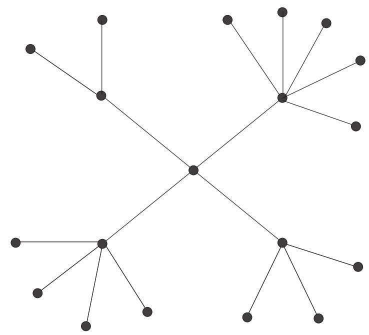

Figure 1.6: Tree

Definition 1.3.14. Consider a p,q-path in G. The distance from p to q is the minimum length of p,q-path, written as $d(p,q)$. The path with the maximum length in G determines the diameter i.e.,

$$diamG = \max_{p,q \in V(G)} d(p,q).$$

The eccentricity of a vertex p is the maximum distance from p to any other vertex of the graph.

Example 1.3.15. The cycle C_n has diameter $\lfloor \frac{n}{2} \rfloor$ and the path has the diameter $n-1$.

For an edge e with endpoints p and q, the contraction of e replaces p and q with a single vertex. The number of components in a graph can be increased by removing a vertex or an edge. Whenever an edge is deleted, the number of components are increased by 1, while the deletion of a vertex may cause the increment of more than one components. Removal of a vertex also removes the edges incident on it and the obtained graph is again a graph.

Chapter 2

Depth and Stanley Depth

The present chapter concerns the depth and Stanley depth (named after Richard Stanley in 1982) of $\mathbb{Z}^n$-graded modules over a commutative ring, including the Stanley's conjecture. It summarises the known values and bounds of depth and Stanley depth for monomial ideals of the polynomial rings and their quotients. Throughout this chapter, ring R has identity $1 \neq 0$.

2.1 Depth

Definition 2.1.1. Consider an R module M. A zero divisor of a module M is an element $0 \neq x \in R$ such that $xm = 0$, where $0 \neq m \in M$.

Definition 2.1.2. Let M be a module over a ring R. A non-zero element r of R is called M regular if for any $m \in M$, $rm = 0$ implies $m = 0$. In other words, the multiplication by r on M is an injective map.

Definition 2.1.3. A sequence $s = s_1, \ldots, s_n$ of elements of R is said to be M-regular if it satisfies the given axioms:

1. s_k is $M/(s_1, \ldots, s_{k-1})M$ regular for any k;

2. $M \neq (s)M$.

Definition 2.1.4. Let M be a finitely generated R-module and let m be the unique maximal ideal of local Noetherian ring R. Then depth of M is the common length of all maximal M-sequences in m, denoted by depth(M).

Lemma 2.1.5. *(Depth Lemma) Let R be a noetherian ring where R_0 is local. Let*

$$0 \longrightarrow H_1 \longrightarrow H_2 \longrightarrow H_3 \longrightarrow 0$$

be a short exact sequence then [4, Proposition 1.2.9]
i) $\operatorname{depth}(H_1) \geq \min\{\operatorname{depth}(H_2)\,,\, 1+\operatorname{depth}(H_3)\}$,
ii) $\operatorname{depth}(H_2) \geq \min\{\operatorname{depth}(H_1)\,,\, \operatorname{depth}(H_3)\}$,
iii) $\operatorname{depth}(H_3) \geq \min\{\operatorname{depth}(H_1)-1\,,\, \operatorname{depth}(H_2)\}$.

Definition 2.1.6. Consider a polynomial ring S in n number of variables and let I be its ideal, then S/I is said to be Cohen-Macaulay if

$$\dim(S/I) = \operatorname{depth}(S/I).$$

2.2 Stanley Decomposition and Stanley Depth

Definition 2.2.1. Let $S = K[x_1, \ldots, x_n]$ be a polynomial ring, where K is a field and let M be a finitely generated $\mathbb{Z}^n$-graded S-module. Let $m \in M$ be a homogeneous element and consider a subset $N \subset \{x_1, \ldots, x_n\}$, then $mK[N]$ represents the K-subspace of M, whose generating set consists of homogeneous elements of the form mu, where u is a monomial in $K[N]$. The linear K-subspace $mK[N]$ is called a Stanley space of dimension $|N|$ if it is a free $K[N]$-module, whereas $|N|$ is the number of variables in N. A Stanley decomposition of M is a presentation of the K-vector space M as a finite direct sum of Stanley spaces.

$$\mathcal{D} \,:\, M = \bigoplus_{j=1}^{r} u_j K[N_j],$$

and the Stanley depth of a decomposition $\mathcal{D}$ is

$$\operatorname{sdepth} \mathcal{D} = \min\{\, |N_j|\,,\, j = 1, \ldots, r\}.$$

The Stanley depth of M is

$$\text{sdepth}_s(M) = \max\{\,\text{sdepth}\,\mathcal{D} \,:\, \mathcal{D}\,\text{is a Stanley decomposition of } M\}.$$

2.2.1 Stanley's Conjecture

In 1982, Stanley [23] gave a conjecture about an upper bound for the depth of a $\mathbb{Z}^n$-graded S-modules. This conjectured upper bound is called the Stanley depth of a module.

$$\text{depth}(M) \leq \text{sdepth}(M).$$

It has been immensely significant as it gave a comparison of two very different invariants of modules. For a polynomial ring S in n number of variables, let $I \subset S$ be a monomial ideal, then for $n \leq 3$, $n = 4$ and $n = 5$ the conjecture for S/I is proved by Apel [3], Anwar [2] and Popescu [17], respectively. Also, when I is an intersection of three monomial prime ideals, or three monomial primary ideals or four monomial prime ideals of S, the conjecture holds for I. But in 2016, Duval et al. [9] proved that Stanley's conjecture is generally false, by giving a counter example for the module of type S/I for which the conjecture does not hold.

2.2.2 Method of computing Stanley Depth for Squarefree Monomial Ideals

In 2009, Herzog et al. [13] gave a method of computing the lower bound for sdepth of squarefree monomial ideals in finite number of steps by using posets. Assume I be a squarefree monomial ideal and let $G(I) = (u_1, \ldots, u_m)$ is the minimal generating set of I. The characteristic poset of I w.r.t $g = (1, \ldots, 1)$, written as $\mathcal{P}_I^{(1,\ldots,1)}$ is defined as

$$\mathcal{P}_I^{(1,\ldots,1)} = \{\gamma \subset [n] \mid \gamma \text{ contains } \text{supp}(u_j) \ for\ some\ j\},$$

where $\text{supp}(u_j) = \{i \,:\, x_i|u_j\} \subseteq [n] := \{1, \ldots, n\}$. For each $\rho, \sigma \in \mathcal{P}_I^{(1,\ldots,1)}$ where $\rho \subseteq \sigma$, and

$$[\rho\,,\,\sigma] = \{\gamma \in \mathcal{P}_I^{(1,\ldots,1)} \,:\, \rho \subseteq \gamma \subseteq \sigma\}.$$

Let $\mathcal{P} : \mathcal{P}_I^{(1,\ldots,1)} = \cup_{j=1}^{k}[\gamma_j\,,\,\eta_j]$ be a partition of $\mathcal{P}_I^{(1,\ldots,1)}$, and for every j, suppose $s(j) \in \{0,1\}^n$ is the tuple with $\mathrm{supp}(x^{s(j)}) = \gamma_j$, then the Stanley decomposition $\mathcal{D}(\mathcal{P})$ of I is given by

$$\mathcal{D}(\mathcal{P}) \,:\, I = \bigoplus_{j=1}^{r} x^{s(j)} K[\{x_k \mid k \in \eta_j\}].$$

Clearly, $\mathrm{sdepth}\,\mathcal{D}(\mathcal{P}) = \min\{|\eta_1|, \ldots, |\eta_r|\}$ and

$$\mathrm{sdepth}(I) = \max\{\mathrm{sdepth}\,\mathcal{D}(\mathcal{P}) \mid \mathcal{P} \textit{ is a partition of } \mathcal{P}_I^{(1,\ldots,1)}\}.$$

Example 2.2.2. Consider $I = (y_1y_4, y_1y_2, y_2y_4, y_1y_3) \subset K[y_1, y_2, y_3, y_4]$ be a square-free monomial ideal and $J = 0$. Set $\sigma_1 = (1,0,0,1)$, $\sigma_2 = (1,1,0,0)$, $\sigma_3 = (0,1,0,1)$ and $\sigma_4 = (1,0,1,0)$. Thus I is generated by $y^{\sigma_1}, y^{\sigma_2}, y^{\sigma_3}, y^{\sigma_4}$ and choose $g = (1,1,1,1)$. The poset $P = P_{I/J}^{g}$ is given by

$$\begin{aligned} P = \{&(1,1,0,0),(1,0,1,0),(1,0,0,1),(0,1,0,1),(1,1,1,0),(1,1,0,1),(1,0,1,1),\\ &(0,1,1,1),(1,1,1,1)\}.\end{aligned}$$

Partitions of P are given by

$$\begin{aligned}\mathcal{P}_1 :\ &[(1,1,0,0),(1,1,0,0)] \bigcup [(1,0,1,0),(1,0,1,0)] \bigcup [(0,1,0,1),(0,1,0,1)] \bigcup\\ &[(1,0,0,1),(1,0,0,1)] \bigcup [(1,1,1,0),(1,1,1,0)] \bigcup [(1,1,0,1),(1,1,0,1)] \bigcup\\ &[(1,0,1,1),(1,0,1,1)] \bigcup [(0,1,1,1),(0,1,1,1)] \bigcup [(1,1,1,1),(1,1,1,1)].\end{aligned}$$

$$\begin{aligned}\mathcal{P}_2 :\ &[(1,1,0,0),(1,1,1,0)] \bigcup [(1,0,0,1),(1,1,0,1)] \bigcup [(1,0,1,0),(1,0,1,1)] \bigcup\\ &[(0,1,0,1),(0,1,1,1)] \bigcup [(1,1,1,1),(1,1,1,1)].\end{aligned}$$

and the corresponding Stanley decomposition is

$$\begin{aligned}\mathcal{D}(\mathcal{P}_1) \ :=\ & y_1y_2K[y_1,y_2] \oplus y_1y_3K[y_1,y_3] \oplus y_1y_4K[y_1,y_4] \oplus y_2y_4K[y_2,y_4] \oplus y_2y_3y_4K[y_2,y_3,y_4] \oplus\\ & y_1y_2y_4K[y_1,y_2,y_4] \oplus y_1y_3y_4K[y_1,y_3,y_4] \oplus y_1y_2y_3K[y_1,y_2,y_3] \oplus\\ & y_1y_2y_3y_4K[y_1,y_2,y_3,y_4].\end{aligned}$$

$$\begin{aligned}\mathcal{D}(\mathcal{P}_2) \ :=\ & y_1y_3K[y_1,y_3,y_4] \oplus y_1y_4K[y_1,y_2,y_4] \oplus y_1y_2K[y_1,y_2,y_3] \oplus y_2y_4K[y_2,y_3,y_4] \oplus\\ & y_1y_2y_3y_4K[y_1,y_2,y_3,y_4].\end{aligned}$$

Then

$$\begin{aligned}\text{sdepth}(I) &\geq \max\{\text{sdepth}(\mathcal{D}(\mathcal{P}_1))\,,\,\text{sdepth}(\mathcal{D}(\mathcal{P}_2))\}\\ &= \max\{2,3\}\\ &= 3.\end{aligned}$$

Since I is not principal, so $\text{sdepth}(I) = 3$.

Example 2.2.3. Consider $I = (y_1y_4, y_2y_5, y_3y_4y_5) \subset K[y_1, y_2, y_3, y_4, y_5]$ be a squarefree monomial ideal and $J = 0$. Set $\sigma_1 = (1,0,0,1,0)$, $\sigma_2 = (0,1,0,0,1)$ and $\sigma_3 = (0,0,1,1,1)$. Thus I is generated by $y^{\sigma_1}, y^{\sigma_2}, y^{\sigma_3}$ and choose $g = (1,1,1,1,1)$. The poset $P = P^g_{I/J}$ is given by

$$\begin{aligned}P = \{&(1,0,0,1,0),(0,1,0,0,1),(1,1,0,1,0),(1,1,0,0,1),(1,0,1,1,0),(1,0,0,1,1),\\ &(0,1,1,0,1),(0,1,0,1,1),(0,0,1,1,1),(1,1,1,1,0),(1,1,1,0,1),(1,1,0,1,1),\\ &(1,0,1,1,1),(0,1,1,1,1),(1,1,1,1,1)\}.\end{aligned}$$

Partitions of P are given by

$$\begin{aligned}\mathcal{P}_1:\ &[(1,0,0,1,0),(1,0,0,1,0)]\bigcup[(0,1,0,0,1),(0,1,0,0,1)]\bigcup\\ &[(1,1,0,1,0),(1,1,0,1,0)]\bigcup[(1,0,0,1,1),(1,0,0,1,1)]\bigcup\\ &[(1,1,0,0,1),(1,1,0,0,1)]\bigcup[(1,0,1,1,0),(1,0,1,1,0)]\bigcup\\ &[(0,1,1,0,1),(0,1,1,0,1)]\bigcup[(0,1,0,1,1),(0,1,0,1,1)]\bigcup\\ &[(0,0,1,1,1),(0,0,1,1,1)]\bigcup[(1,1,1,1,0),(1,1,1,1,0)]\bigcup\\ &[(1,1,1,0,1),(1,1,1,0,1)]\bigcup[(1,1,0,1,1),(1,1,0,1,1)]\bigcup\\ &[(1,0,1,1,1),(1,0,1,1,1)]\bigcup[(0,1,1,1,1),(0,1,1,1,1)]\bigcup\\ &[(1,1,1,1,1),(1,1,1,1,1)].\end{aligned}$$

$$\begin{aligned}\mathcal{P}_2:\ &[(1,0,0,1,0),(1,1,0,1,0)]\bigcup[(0,1,0,0,1),(1,1,0,0,1)]\bigcup\\ &[(1,0,1,1,0),(1,1,1,1,0)]\bigcup[(1,0,0,1,1),(1,1,0,1,1)]\bigcup\\ &[(0,1,1,0,1),(1,1,1,0,1)]\bigcup[(0,1,0,1,1),(0,1,1,1,1)]\bigcup\\ &[(0,0,1,1,1),(1,0,1,1,1)]\bigcup[(1,1,1,1,1),(1,1,1,1,1)].\end{aligned}$$

$$\begin{aligned}\mathcal{P}_3:\ & [(1,0,0,1,0),(1,1,1,1,0)]\bigcup[(0,1,0,0,1),(1,1,1,0,1)]\bigcup\\ & [(1,0,0,1,1),(1,1,0,1,1)]\bigcup[(0,1,0,1,1),(0,1,1,1,1)]\bigcup\\ & [(0,0,1,1,1),(1,0,1,1,1)]\bigcup[(1,1,1,1,1),(1,1,1,1,1)].\end{aligned}$$

and the corresponding Stanley decomposition is

$$\begin{aligned}\mathcal{D}(\mathcal{P}_1) &:= y_1y_4K[y_1,y_4]\oplus y_2y_5K[y_2,y_5]\oplus y_1y_2y_4K[y_1,y_2,y_4]\oplus y_1y_2y_5K[y_1,y_2,y_5]\oplus\\ & y_1y_3y_4K[y_1,y_3,y_4]\oplus y_1y_4y_5K[y_1y_4y_5]\oplus y_2y_3y_5K[y_2,y_3,y_5]\oplus\\ & y_2y_4y_5K[y_2,y_4,y_5]\oplus y_3y_4y_5K[y_3,y_4,y_5]\oplus y_1y_2y_3y_4K[y_1,y_2,y_3,y_4]\oplus\\ & y_1y_2y_3y_5K[y_1,y_2,y_3,y_5]\oplus y_1y_2y_4y_5K[y_1,y_2,y_4,y_5]\oplus y_1y_3y_4y_5K[y_1,y_3,y_4,y_5]\oplus\\ & y_2y_3y_4y_5K[y_2,y_3,y_4,y_5]\oplus y_1y_2y_3y_4y_5K[y_1,y_2,y_3,y_4,y_5].\end{aligned}$$

$$\begin{aligned}\mathcal{D}(\mathcal{P}_2) &:= y_1y_4K[y_1,y_2,y_4]\oplus y_2y_5K[y_1,y_2,y_5]\oplus y_1y_3y_4K[y_1,y_2,y_3,y_4]\oplus\\ & y_1y_4y_5K[y_1,y_2,y_4,y_5]\oplus y_2y_3y_5K[y_1,y_2,y_3,y_5]\oplus y_2y_4y_5K[y_2,y_3,y_4,y_5]\oplus\\ & y_3y_4y_5K[y_1,y_3,y_4,y_5]\oplus y_1y_2y_3y_4y_5K[y_1,y_2,y_3,y_4,y_5].\end{aligned}$$

$$\begin{aligned}\mathcal{D}(\mathcal{P}_3) &:= y_1y_4K[y_1,y_2,y_3,y_4]\oplus y_2y_5K[y_1,y_2,y_3,y_5]\oplus y_1y_4y_5K[y_1,y_2,y_4,y_5]\oplus\\ & y_2y_4y_5K[y_2,y_3,y_4,y_5]\oplus y_3y_4y_5K[y_1,y_3,y_4,y_5]\oplus y_1y_2y_3y_4y_5K[y_1,y_2,y_3,y_4,y_5].\end{aligned}$$

Then

$$\begin{aligned}\operatorname{sdepth}(I) &\geq \max\{\operatorname{sdepth}(\mathcal{D}(\mathcal{P}_1))\,,\operatorname{sdepth}(\mathcal{D}(\mathcal{P}_2))\,,\operatorname{sdepth}(\mathcal{D}(\mathcal{P}_3))\}\\ &= \max\{2,3,4\}\\ &= 4.\end{aligned}$$

Since I is not principal, so $\operatorname{sdepth}(I)=4$.

The next example illustrates the method of computing the Stanley depth of S/I.

Example 2.2.4. For $S=K[y_1,y_2,y_3,y_4,y_5]$, consider $I=(y_1y_5,y_2y_3y_4,y_1y_2,y_1y_4)$. Then choose $g=(1,1,1,1,1)$ and the poset $P=P^g_{S/I}$ is given by

$$\begin{aligned}P=\{&(0,0,0,0,0),(1,0,0,0,0),(0,1,0,0,0),(0,0,1,0,0),(0,0,0,1,0),(0,0,0,0,1),\\ &(1,0,1,0,0),(0,1,1,0,0),(0,1,0,1,0),(0,1,0,0,1),(0,0,1,1,0),(0,0,1,0,1),\\ &(0,0,0,1,1),(0,1,1,0,1),(0,1,0,1,1),(0,0,1,1,1)\}.\end{aligned}$$

Partitions of P are given by

$$\begin{aligned}\mathcal{P}_1: \ & [(0,0,0,0,0),(0,0,1,1,1)] \bigcup [(1,0,0,0,0),(1,0,0,0,0)] \bigcup \\ & [(0,1,0,0,0),(0,1,0,0,0)] \bigcup [(0,0,1,0,0),(0,0,1,0,0)] \bigcup \\ & [(0,0,0,1,0),(0,0,0,1,0)] \bigcup [(0,0,0,0,1),(0,0,0,0,1)] \bigcup \\ & [(1,0,1,0,0),(1,0,1,0,0)] \bigcup [(0,1,1,0,0),(0,1,1,0,0)] \bigcup \\ & [(0,1,0,1,0),(0,1,0,1,0)] \bigcup [(0,1,0,0,1),(0,1,0,0,1)] \bigcup \\ & [(0,0,1,1,0),(0,0,1,1,0)] \bigcup [(0,0,1,0,1),(0,0,1,0,1)] \bigcup \\ & [(0,0,0,1,1),(0,0,0,1,1)] \bigcup [(0,1,1,0,1),(0,1,1,0,1)] \bigcup \\ & [(0,1,0,1,1),(0,1,0,1,1)].\end{aligned}$$

$$\begin{aligned}\mathcal{P}_2: \ & [(0,0,0,0,0),(1,0,1,0,0)] \bigcup [(0,1,0,0,0),(0,1,1,0,0)] \bigcup \\ & [(0,0,0,1,0),(0,1,0,1,0)] \bigcup [(0,0,0,0,1),(0,1,0,0,1)] \bigcup \\ & [(0,0,0,1,1),(0,1,0,1,1)] \bigcup [(0,0,1,0,1),(0,1,1,0,1)] \bigcup \\ & [(0,0,1,1,0),(0,0,1,1,1)].\end{aligned}$$

and the corresponding Stanley decomposition is

$$\begin{aligned}\mathcal{D}(\mathcal{P}_1) \ := \ & K[y_3,y_4,y_5] \oplus y_1K[y_1] \oplus y_2K[y_2] \oplus y_3K[y_3] \oplus y_4K[y_4] \oplus y_5K[y_5] \oplus \\ & y_1y_3K[y_1,y_3] \oplus y_2y_3K[y_2,y_3] \oplus y_2y_4K[y_2,y_4] \oplus y_2y_5K[y_2,y_5] \oplus \\ & y_3y_4K[y_3,y_4] \oplus y_3y_5K[y_3,y_5] \oplus y_4y_5K[y_4,y_5] \oplus y_2y_3y_5K[y_2,y_3,y_5] \oplus \\ & y_2y_4y_5K[y_2,y_4,y_5].\end{aligned}$$

$$\begin{aligned}\mathcal{D}(\mathcal{P}_2) \ := \ & K[y_1,y_3] \oplus y_2K[y_2,y_3] \oplus y_4K[y_2,y_4] \oplus y_5K[y_2,y_5] \oplus y_4y_5K[y_2,y_4,y_5] \oplus \\ & y_3y_5K[y_2,y_3,y_5] \oplus y_3y_4K[y_3,y_4,y_5].\end{aligned}$$

Then

$$\begin{aligned}\operatorname{sdepth}(S/I) &\geq \max\{\operatorname{sdepth}(\mathcal{D}(\mathcal{P}_1)), \operatorname{sdepth}(\mathcal{D}(\mathcal{P}_2))\} \\ &= \max\{1,2\} \\ &= 2.\end{aligned}$$

2.2.3 Values and Bounds for Depth and Stanley Depth of Monomial Ideals and their Quotients

Some fundamental results on depth and Stanley depth of S-modules are given below.

Theorem 2.2.5. *[7, Theorem 1.3] Let $c_1, \dots, c_n$ be some positive integers, then*

$$\operatorname{sdepth}((x_1^{c_1}, \dots, x_n^{c_n})) = \operatorname{sdepth}((x_1, \dots, x_n)) = \lceil \frac{n}{2} \rceil.$$

In particular, for any $1 \leq m \leq n$

$$\operatorname{sdepth}((x_1^{c_1}, \dots, x_m^{c_m})) = n - m + \lceil \frac{m}{2} \rceil.$$

Proposition 2.2.6. *[5, Proposition 2.7] For $I \subset S$ and for every monomial $u \notin I$,*

1. $\operatorname{sdepth}_S(I : u) \geq \operatorname{sdepth}_S(I)$, *[17, Proposition 1.3]*
2. $\operatorname{depth}_S(S/(I : u)) \geq \operatorname{depth}_S(S/I)$, *[19, Corollary 1.3]*
3. $\operatorname{sdepth}_S(S/(I : u)) \geq \operatorname{sdepth}_S(S/I)$.

Theorem 2.2.7. *[20, Theorem 1.1] Let $J \subset S$ be a monomial ideal and $w \in S$ be a monomial regular on S/J, then*

$$\operatorname{sdepth}(S/(J, w)) = \operatorname{sdepth}(S/J) - 1.$$

Lemma 2.2.8. *[13, Lemma 3.6] For I and J be two monomial ideals with $J \subset I$, suppose $S' = S[x_{n+1}]$, then*

$$\operatorname{depth}(IS'/JS') = \operatorname{depth}(IS/JS) + 1.$$

$$\operatorname{sdepth}(IS'/JS') = \operatorname{sdepth}(IS/JS) + 1.$$

Lemma 2.2.9. *[5, Proposition 1.1] Let $I \subset S' = K[x_1, \dots, x_r], J \subset S'' = K[x_{r+1}, \dots, x_n]$ be monomial ideals, with $1 \leq r \leq n$, then*

$$\operatorname{depth}_S(S/(IS + JS)) = \operatorname{depth}_{S'}(S'/I) + \operatorname{depth}_{S''}(S''/J).$$

Theorem 2.2.10. *[19, Theorem 3.1] Let* $I \subset S' = K[x_1, \dots, x_r]$, $J \subset S'' = K[x_{r+1}, \dots, x_n]$ *be monomial ideals, with* $1 \le r \le n$, *then*

$$\operatorname{sdepth}_S(S/(IS + JS)) \ge \operatorname{sdepth}_{S'}(S'/I) + \operatorname{sdepth}_{S''}(S''/J).$$

Lemma 2.2.11. *[15, Lemma 2.4] Let* M *be a* $\mathbb{Z}^n$*-graded* S*-module. Suppose that* H_1 *and* H_2 *are two submodules of* M *and assume* $0 \to H_1 \to M \to H_2 \to 0$ *be a short exact sequence. Then*

$$\operatorname{sdepth}(M) \ge \min\{\operatorname{sdepth}(H_1), \operatorname{sdepth}(H_2)\}.$$

Chapter 3

Depth and Stanley depth of Power of the Edge Ideals of a Forest

This chapter is devoted to the detailed review of [14] and [18], which deals with the lower bounds for the depth and Stanley depth of powers of edge ideal corresponding to trees and forests. The Depth Lemma will mainly be used for the particular form of short exact sequences given below.

Lemma 3.0.12. *Let H be an ideal of S and let $s \in S$, then*

$$0 \longrightarrow S/(H:s) \xrightarrow{s} S/H \longrightarrow S/(H,s) \longrightarrow 0$$

is a short exact sequence.

Definition 3.0.13. The edge ideal $I(G)$ corresponding to the graph G is the squarefree ideal of polynomial ring S and the generating set contains monomials of the type $w_i w_j$, where w_i is adjacent to w_j in G.

Lemma 3.0.14. *Consider a polynomial ring S and squarefree monomial ideal I. Suppose $\mathcal{B} \in S$ be a monomial and z is an indeterminate in S so that z does not divide $\mathcal{B}$. Assume $\mathcal{G}'$ be minor formed by letting $z = 0$ in I and suppose E is the extension of $\mathcal{G}'$ in S, then for any $k \geq 1$*

$$((I^k : \mathcal{B}), z) = ((E^k : \mathcal{B}), z).$$

Lemma 3.0.15. *For a bipartite graph G and $k \geq 1$*

$$\operatorname{depth}(S/I^k(G)) \geq 1.$$

3.1 Powers of Paths

For small value of γ, the depth is computed for edge ideal of P_γ. Then it will be utilized as the base while proving bounds for other graphs.

Example 3.1.1. Let $I = I(P_\gamma)$. For $\gamma \leq 3$ and $k \geq 1$

$$\operatorname{depth}(S/I^k) = 1.$$

Proof. **Case 1:** When $\gamma = 1$
$I = I(P_1) = (0)$ since I is an edge ideal and P_1 contains no edge. Then for all k

$$\operatorname{depth}(S/I^k) = \operatorname{depth} K[y] = 1.$$

Case 2: When $\gamma = 2$
Then $I = I(P_2)$ becomes a complete intersection monomial ideal and therefore S/I^k becomes Cohen-Macaulay for all values of k. Thus

$$\operatorname{depth}(S/I^k) = dim(K[y,z]/(yz)^k) = 1.$$

Case 3: When $\gamma = 3$
Then $ht(I(P_3)) = 1$. As $\operatorname{Min}(S/I) = \operatorname{Min}(S/I^k)$ for all powers of any monomial ideal. It implies that, for all $k \geq 1$, S/I^k is mixed and hence it is not a Cohen-Macaulay. Now for $k \geq 1$

$$\operatorname{depth}(S/I^k) \leq \gamma - ht(I) - 1 = 3 - 1 - 1 = 1.$$

Notice that P_3 is a bipartite graph, Lemma 3.0.15 gives $\operatorname{depth}(S/I^k) \geq 1$. This yields $\operatorname{depth}(S/I^k) = 1$ for $\gamma = 3$ and $k \geq 1$. □

Let graph G be a tree or a forest. Initially, a bound for $\operatorname{depth}(S/I)$ is required, for computation of depth for power of edge ideals associated to G. Firstly, the depth of edge ideal of path will be computed. Notice that there exist a correspondence in between the graphs and two degree square-free monomial ideals which is basically the correspondence between the generators of an ideal and edges of a graph.

Lemma 3.1.2. *For $S = K[y_1, \dots, y_\gamma]$ and $I = I(P_\gamma)$*

$$\operatorname{depth}(S/I) = \lceil \frac{\gamma}{3} \rceil.$$

Proof. The proof is done by induction on γ. For $\gamma \leq 3$, the result is true from Example 3.1.1. Assume $\gamma \geq 4$, consider a short exact sequence

$$0 \longrightarrow S/(I : y_{\gamma-1}) \xrightarrow{y_{\gamma-1}} S/I \longrightarrow S/(I, y_{\gamma-1}) \longrightarrow 0$$

Then

$$(I : y_{\gamma-1}) = (P_{\gamma-3}, y_\gamma, y_{\gamma-2}).$$

Thus by using induction on γ and Lemma 2.2.8

$$\begin{aligned}
\operatorname{depth}(S/(I : y_{\gamma-1})) &= \operatorname{depth}(\Omega'[y_{\gamma-1}]/I(P_{\gamma-3})) \\
&= 1 + \operatorname{depth}(\Omega'/I(P_{\gamma-3})) \\
&= 1 + \lceil \frac{\gamma-3}{3} \rceil \\
&= \lceil \frac{\gamma}{3} \rceil,
\end{aligned}$$

where $\Omega' = S \backslash \{y_{\gamma-1}, y_{\gamma-2}, y_{\gamma-3}\}$. And

$$(I, y_{\gamma-1}) = (P_{\gamma-2}, y_{\gamma-1}).$$

Induction on γ and Lemma 2.2.8 implies

$$\begin{aligned}
\operatorname{depth}(S/(I, y_{\gamma-1})) &= \operatorname{depth}(\Omega''[y_\gamma]/I(P_{\gamma-2})) \\
&= 1 + \operatorname{depth}(\Omega''/I(P_{\gamma-2})) \\
&= 1 + \lceil \frac{\gamma-2}{3} \rceil,
\end{aligned}$$

where $\Omega'' = S \backslash \{y_\gamma, y_{\gamma-1}\}$. Now by Depth Lemma

$$\operatorname{depth}(S/I) \geq \lceil \frac{\gamma}{3} \rceil,$$

and as

$$\operatorname{depth}(S/I) \leq \operatorname{depth}(S/(I : y_{\gamma-1})) = \lceil \frac{\gamma}{3} \rceil,$$

therefore,

$$\operatorname{depth}(S/I) = \lceil \frac{\gamma}{3} \rceil.$$

□

The above proved value of depth of path can be useful in obtaining the lower bound of depth of trees. The path which determines the diameter of tree always join its two leaves.

Proposition 3.1.3. *For a tree G with diameter δ,*

$$\operatorname{depth}(S/I(G)) \geq \lceil \frac{\delta+1}{3} \rceil.$$

Proof. Let p and q be the nodes of graph, where δ is the distance between p and q. Assume $P_{\delta+1}$ be a path joining p and q which determines the diameter of G. Consider a short exact sequence

$$0 \longrightarrow S/(I:z) \longrightarrow S/I \longrightarrow S/(I,z) \longrightarrow 0$$

When $\delta \leq 2$ then

$$\lceil \frac{\delta+1}{3} \rceil = 1,$$

and Lemma 3.0.15 implies the result. So the result is true for $n \leq 3$. Now let $\delta \geq 3$. For a leaf p let z be its unique neighbor. As

$$(I,z) = (\lambda, z)$$

where λ is the edge ideal of deletion minor Δ' which is obtained by letting $z = 0$. Note that Δ' has at least $\delta - 2$ diameter and p is the isolated vertex in Δ'. If Ω' is formed by removing p from S, then by using induction on δ and Lemma 2.2.8

$$\begin{aligned} \operatorname{depth}(S/(I,z)) &= \operatorname{depth}(\Omega'[p]/(\lambda,z)) \\ &= 1 + \operatorname{depth}(\Omega'/(\lambda,z)) \\ &\geq 1 + \lceil \frac{(\delta-2)+1}{3} \rceil \\ &= \lceil \frac{\delta+2}{3} \rceil \\ &\geq \lceil \frac{\delta+1}{3} \rceil. \end{aligned}$$

Furthermore,

$$(I:z) = (\mathcal{K}, N(z))$$

here the edge ideal $\mathcal{K}$ is associated to the contraction minor Δ'' obtained by eliminating neighbors of z. Suppose Ω'' is a polynomial ring which is obtained by removing $\bar{N}(z) = z \cup N(z)$. So Δ'' has at least $\delta - 3$ diameter, also z is a vertex of degree zero. Therefore, induction and Lemma 2.2.8 implies

$$\begin{aligned} \operatorname{depth}(S/(I:z)) &= \operatorname{depth}(\Omega''[z]/\mathcal{K}) \\ &\geq \lceil \frac{(\delta-3)+1}{3} \rceil + 1. \end{aligned}$$

Then Depth Lemma implies

$$\operatorname{depth}(S/I) \geq \lceil \frac{\delta+1}{3} \rceil.$$

□

Lemma 3.1.4. *Let $I = I(G)$, for a graph G. Consider a leaf p in G and let q be the only vertex adjacent to it, then for any $k \geq 2$*

$$(I^k : pq) = I^{k-1}.$$

Proof. As $\{p, q\}$ is an edge in G and pq is in the generating set of I so, $I^{k-1} \subseteq (I^k : pq)$. For the other inclusion, let $b \in (I^k : pq)$. Then $b(pq) = g_1 g_2 \ldots g_k f$, for a monomial f and some two degree monomials g_i associated to the edges of G. Now supposing $b \notin I^{k-1}$, then p divides g_j, also q divides g_t, where $j \neq t$. Assume $j = k$. But as p is a leaf of G, so $g_k = pq$ and hence

$$b = g_1 \ldots g_{k-1} f \in I^{k-1}.$$

This completes the proof.

□

Corollary 3.1.5. *For $\gamma \geq 2$ and $k \geq 2$*

$$(P_\gamma^k : z_{\gamma-1} z_\gamma) = P_\gamma^{k-1}.$$

Proof. Since z_γ is leaf of P_γ so Lemma 3.2.3 implies the result. □

3.2 Power of Edge Ideals of Trees and Forests

The aim of this section is to give bound for the depth of power of edge ideals associated to tree and forest by using the graph invariants. As the Depth Lemma is repeatedly used to several short exact sequences, firstly the following Lemma is proved in order to make easy the proof of main result.

Lemma 3.2.1. *Let $I = I(G)$, for a graph G. Assume w_1, w_2 are the nodes of G and for some $r \geq 0$,* $\operatorname{depth}(S/(I^k : w_1w_2)) \geq r$, $\operatorname{depth}(S/(I^k, w_1)) \geq r$ *and* $\operatorname{depth}(S/((I^k : w_1), w_2)) \geq r$*, then*

$$\operatorname{depth}(S/I^k) \geq r.$$

Proof. Consider a short exact sequence

$$0 \longrightarrow S/(I^k : w_1) \xrightarrow{.w_1} S/I^k \longrightarrow S/(I^k, w_1) \longrightarrow 0$$

Then by given statement, $\operatorname{depth}(S/(I^k, w_1)) \geq r$ and for depth of $S/(I^k : w_1)$, consider another exact sequence

$$0 \longrightarrow S/(I^k : w_1w_2) \xrightarrow{.w_2} S/(I^k : w_1) \longrightarrow S/((I^k : w_1), w_2) \longrightarrow 0$$

Then by Depth Lemma

$$\operatorname{depth}(S/(I^k : w_1)) \geq \min\{\operatorname{depth}(S/(I^k : w_1w_2)), \operatorname{depth}(S/((I^k : w_1), w_2))\}.$$

Thus

$$\operatorname{depth}(S/(I^k : w_1)) \geq r,$$

and by again applying Depth Lemma on first exact sequence

$$\operatorname{depth}(S/I^k) \geq r.$$

□

Before moving towards computing the depth of power of edge ideals associated with trees and forest, a bound for the depth of powers of edge ideals corresponding to path is determined in the below proposition.

Proposition 3.2.2. *For $\gamma \geq 2$, suppose $I(P_\gamma)$ be the edge ideal of path, then*

$$\operatorname{depth}(S/I^k(P_\gamma)) \geq \max\{\lceil \frac{\gamma - k + 1}{3} \rceil, 1\}.$$

Proof. As paths are bipartite graphs so, [22] for all k

$$\operatorname{depth}(S/I^k(P_\gamma)) \geq 1.$$

Now it is only need to be proved that $\operatorname{depth}(S/I^k(P_\gamma)) \geq \lceil \frac{\gamma-k+1}{3} \rceil$. Note that for $\gamma \leq 3$ and for all k, the result holds by Example 3.1.1. For $k = 1$ and $\gamma \geq 1$, Lemma 3.1.2 implies the result. Let $\gamma \geq 4$ and $k \geq 2$. Consider a short exact sequence

$$0 \longrightarrow S/(I^k(P_\gamma) : y_{\gamma-1}) \longrightarrow S/I^k(P_\gamma) \longrightarrow S/(I^k(P_\gamma), y_{\gamma-1}) \longrightarrow 0$$

Since $y_{\gamma-1}$ is the unique neighbor of y_γ, so

$$(I^k(P_\gamma), y_{\gamma-1}) = (I^k(P_{\gamma-2}), y_{\gamma-1}).$$

By using induction on γ

$$\operatorname{depth}(\Omega''/I^k(P_{\gamma-2}) \geq \lceil \frac{(\gamma-2) - k + 1}{3} \rceil,$$

where $\Omega'' = K[y_1, \ldots, y_{\gamma-2}]$. Consequently,

$$\begin{aligned}
\operatorname{depth}(S/(I^k(P_\gamma), y_{\gamma-1})) &= \operatorname{depth}(\Omega''[y_{\gamma-1}, y_\gamma]/(I^k(P_{\gamma-2}), y_{\gamma-1})) \\
&= \operatorname{depth}(\Omega''/I^k(P_{\gamma-2})) + 1 \\
&\geq \lceil \frac{(\gamma-2) - k + 1}{3} \rceil + 1 \\
&= \lceil \frac{\gamma - k + 2}{3} \rceil.
\end{aligned}$$

Now for the depth of $S/(I^k(P_\gamma) : y_{\gamma-1})$, consider another short exact sequence

$$0 \longrightarrow S/(I^k(P_\gamma) : y_{\gamma-1}y_\gamma) \longrightarrow S/(I^k(P_\gamma) : y_{\gamma-1}) \longrightarrow S/((I^k(P_\gamma) : y_{\gamma-1}), y_\gamma) \longrightarrow 0$$

Then by Corollary 3.1.5 and induction on k

$$\begin{aligned}
\operatorname{depth}(S/(I^k(P_\gamma) : y_{\gamma-1}y_\gamma)) &= \operatorname{depth}(S/I^{k-1}(P_\gamma)) \\
&\geq \lceil \frac{\gamma - (k-1) + 1}{3} \rceil \\
&= \lceil \frac{\gamma - k + 2}{3} \rceil.
\end{aligned}$$

To compute the depth of $S/((I^k(P_\gamma) : y_{\gamma-1}), y_\gamma)$, notice that since y_γ does not divide $y_{\gamma-1}$, so by Lemma 3.0.14

$$((I^k(P_\gamma) : y_{\gamma-1}), y_\gamma) = ((I^k(P_{\gamma-1}) : y_{\gamma-1}), y_\gamma)$$

and let $\Omega' = K[y_1, \ldots, y_{\gamma-1}]$. Notice that

$$\operatorname{depth}(S/((I^k(P_\gamma) : y_{\gamma-1}), y_\gamma)) = \operatorname{depth}(\Omega'/(I^k(P_{\gamma-1}) : y_{\gamma-1})).$$

Consider a short exact sequence

$$0 \longrightarrow \Omega'/(I^k(P_{\gamma-1}) : y_{\gamma-1}y_{\gamma-2}) \longrightarrow \Omega'/(I^k(P_{\gamma-1}) : y_{\gamma-1}) \longrightarrow \Omega'/((I^k(P_{\gamma-1}) : y_{\gamma-1}), y_{\gamma-2}) \longrightarrow 0$$

Corollary 3.1.5 implies

$$(I^k(P_{\gamma-1}) : y_{\gamma-1}y_{\gamma-2}) = I^{k-1}(P_{\gamma-1})$$

and by induction on k and γ

$$\begin{aligned}
\operatorname{depth}(\Omega'/(I^k(P_{\gamma-1}) : y_{\gamma-1}y_{\gamma-2})) &= \operatorname{depth}(\Omega'/I^{k-1}(P_{\gamma-1})) \\
&\geq \lceil \frac{(\gamma-1)-(k-1)+1}{3} \rceil \\
&= \lceil \frac{\gamma-k+1}{3} \rceil.
\end{aligned}$$

Also since $y_{\gamma-2}$ does not divide $y_{\gamma-1}$, so by Lemma 3.0.14

$$\begin{aligned}
((I^k(P_{\gamma-1}) : y_{\gamma-1}), y_{\gamma-2}) &= ((I^k(P_{\gamma-3}) : y_{\gamma-1}), y_{\gamma-2}) \\
&= (I^k(P_{\gamma-3}), y_{\gamma-2}).
\end{aligned}$$

Then by using induction on γ

$$\begin{aligned}
\operatorname{depth}(\Omega'/((I^k(P_{\gamma-1}) : y_{\gamma-1}), y_{\gamma-2})) &= \operatorname{depth}(\Omega'/(I^k(P_{\gamma-3}), y_{\gamma-2})) \\
&= \operatorname{depth}(K[y_1, \ldots, y_{\gamma-3}, y_{\gamma-1}]/I^k(P_{\gamma-3})) \\
&= \operatorname{depth}(K[y_1, \ldots, y_{\gamma-3}] + 1 \\
&\geq \lceil \frac{(\gamma-3)-k+1}{3} \rceil + 1 \\
&= \lceil \frac{\gamma-k+1}{3} \rceil.
\end{aligned}$$

Then the Depth Lemma implies

$$\operatorname{depth}(\Omega'/(I^k(P_{\gamma-1}):y_{\gamma-1}))\geq\lceil\frac{\gamma-k+1}{3}\rceil,$$

consequently,

$$\operatorname{depth}(S/((I^k(P_\gamma):y_{\gamma-1}),y_\gamma))\geq\lceil\frac{\gamma-k+1}{3}\rceil.$$

Hence Lemma 3.2.1 implies

$$\operatorname{depth}(S/I^k(P_\gamma))\geq\lceil\frac{\gamma-k+1}{3}\rceil.$$

□

The next lemma is a fundamental result related to tree graphs which will be required in proving the lower bound for powers of tree.

Lemma 3.2.3. *Let the graph G be a tree and a path $P_{\delta+1}=\{y_1y_2,y_2y_3,\ldots,y_\delta y_{\delta+1}\}$ determines the diameter of G then there is at most one non-leaf vertex in the neighbors of y_δ.*

Proof. As $P_{\delta+1}$ is path whose length is maximal in G since there exists a unique path connecting any two nodes of a tree. Suppose $y\in N(y_\delta)$. Now if $y\neq y_{\delta-1}$ and y is a non-leaf then there exist a node $w\in N(y)$ with $w\neq y_\delta$. Then there exist a path

$$P=\{y_1y_2,\ldots,y_{\delta-1}y_\delta,y_\delta y,yw\},$$

of length $\delta+1$ in G, which is a contradiction. Hence, there is at most one non-leaf in the neighbors of y_δ. □

Following result is the main theorem about the depth of forests. Here the connected components $\Delta_i's$ of G contain at least two vertices whereas the component with a single vertex will not be a connected component.

Theorem 3.2.4. *Let G be a forest having s number of connected components Δ_1, Δ_2 , ..., Δ_s. Let $I=I(G)$ and Δ_j has diameter δ_j also suppose $\delta=\max\{\delta_j\}$. For $k\geq 1$*

$$\operatorname{depth}(S/I^k)\geq\max\{\lceil\frac{\delta-k+2}{3}\rceil+s-1,s\}.$$

Proof. The induction on γ and k is used in the proof, here γ be the total number of non-isolated vertices in G. Suppose that $\delta = \delta_j$. Proposition 3.1.3 follows the result for $s = 1$ and $k = 1$. Now let $k = 1$ and $s \geq 2$ then [25, Lemma 6.2.7] and Proposition 3.1.3 implies the result. Therefore, result is true for $k = 1$ and $\gamma \geq 1$. Let $k \geq 2$. For $\gamma = 2$, graph becomes a path on two vertices with $\delta = 1$ and by Proposition 3.2.2 the result follows for $k \geq 1$.

Now suppose $\gamma \geq 3$. Consider a diameter realizing path P_{d+1} in G. Let y_1 be the endpoint of P_{d+1} and hence a leaf of G. Suppose z be the only vertex adjacent to y_1 and assume $N(z) = \{y_1, y_2, \ldots, y_t\}$ be the neighbors of z, where $t \geq 1$ and t is finite. Also, by Lemma 3.2.3, at most one y_j is a non-leaf. Suppose that y_j is a leaf for $1 \leq j < t$. Consider an edge ideal $\mathcal{I}_r$ corresponding to the minor of G formed when $y_1, y_2, \ldots, y_r$ are deleted and let $\Omega_r = K[y_{r+1}, \ldots, y_{n-1}, z]$ be the subring of S excluding $y_1, \ldots, y_r$ and assume $\Omega'_r = K[y_{r+1}, \ldots, y_{n-1}]$. Also for every r, $\mathcal{I}_r \subset \Omega_r$ is a edge ideal of a graph having less than γ number of vertices. Let

$$q = \max\{\lceil \frac{\delta - k + 2}{3} \rceil + s - 1, s\}.$$

As y_1 is a leaf so by Lemma 3.1.4, $(I^k : y_1 z) = I^{k-1}$ and so by induction on k,

$$\begin{aligned}
\operatorname{depth}(S/(I^k : y_1 z)) &= \operatorname{depth}(S/I^{k-1}) \\
&\geq \max\{\lceil \frac{\delta - (k-1) + 2}{3} \rceil + s - 1, s\} \\
&= \max\{\lceil \frac{\delta - k + 3}{3} \rceil + s - 1, s\} \\
&\geq q.
\end{aligned}$$

To compute the $\operatorname{depth}(S/(I^k, z))$, notice that $(I^k, z) = (\lambda^k, z)$, where λ is the edge ideal of Δ', which is obtained by eliminating z. Therefore Δ' again becomes a forest having less than γ number of vertices, with the connected components at least $s - 1$. Also the generating set of λ lie in Ω_1'. So induction implies $\text{depth}(\Omega_1'/\lambda^k) \geq s-1$ and

$$\begin{aligned}\text{depth}(S/(I^k,z)) &= \text{depth}(\Omega_1[y_1]/(\lambda^k,z)) \\ &= 1+\text{depth}(\Omega_1'/\lambda^k) \\ &\geq 1+s-1 \\ &= s.\end{aligned}$$

Now assume $\delta \leq 3$, then for all $k \leq 2$

$$\lceil\frac{\delta-k+2}{3}\rceil \leq 1,$$

thus

$$\begin{aligned}q &= \max\{\lceil\frac{\delta-k+2}{3}\rceil + s-1, s\} \\ &\leq \max\{1+s-1,s\} \\ &= s,\end{aligned}$$

and

$$\text{depth}(S/(I^k,z)) \geq q.$$

For $\delta > 3$, note that there are at least s number of connected components in λ since $\Delta_2, \Delta_3, \ldots, \Delta_s$ remains the same and $\delta-2 \geq 1$ edges of $P_{\delta+1}$ are in Δ'. Thus the maximal diameter δ of a component of Δ' is at least $\delta-2$. Therefore,

$$\begin{aligned}\text{depth}(S/(I^k,z)) &= 1+\text{depth}(\Omega_1'/\lambda^k) \\ &\geq 1+\max\{\lceil\frac{(\delta-2)-k+2}{3}\rceil + s-1, s\} \\ &\geq q.\end{aligned}$$

Thus, $\text{depth}(S/(I^k,z)) \geq q$, for all d. Consider $((I^k:z),y_1)$. Since by Lemma 3.0.14

$$((I^k:z),y_1) = ((\mathcal{I}_1^k:z),y_1)$$

where $\mathcal{I}_1$ is the extension in S of the minor obtained by deleting y_1, therefore

$$\text{depth}(S/((I^k:z),y_1)) = \text{depth}(\delta_1/(\mathcal{I}_1^k:z)).$$

Notice that when $t = 1$ then diameter becomes one and $q = s$. Further for $t = 1$

$$(\mathcal{I}_1^k : z) = \mathcal{I}_1^k$$

and $\mathcal{I}_1 \subset \Omega_1'$ be the edge ideal corresponding to a forest having less than γ number of vertices with $s-1$ components. Therefore, by induction

$$\begin{aligned}
\operatorname{depth}(S/((I^k : z), y_1)) &= \operatorname{depth}(S/\mathcal{I}_1^k) \\
&= \operatorname{depth}(\Omega_1'[z]/\mathcal{I}_1^k) \\
&\geq \max\{\lceil \frac{3-k}{3} \rceil + (s-1) - 1, s-1\} + 1 \\
&= s - 1 + 1 \\
&= s.
\end{aligned}$$

So for $t = 1$

$$\operatorname{depth}(S/((I^k : z), y_1)) \geq q.$$

Now let $t \geq 2$, then $\delta \geq 2$ as y_1, z, y_t lies in $P_{\delta+1}$. Next reverse induction on t will be used to compute depth of $\Omega_1/(\mathcal{I}_1^k : z)$. Since z is an isolated vertex in $\mathcal{I}_t$, so

$$(\mathcal{I}_t^k : z) = \mathcal{I}_t^k$$

Note that all the generators of $\mathcal{I}_t^k$ lies in Ω_t' therefore by using Lemma 2.2.8

$$\begin{aligned}
\operatorname{depth}(\Omega_t/\mathcal{I}_t^k) &= \operatorname{depth}(\Omega_t'[z]/\mathcal{I}_t^k) \\
&= 1 + \operatorname{depth}(\Omega_t'/\mathcal{I}_t^k).
\end{aligned}$$

When $\delta \leq 3$, then $\mathcal{I}_t \subset \Omega_t'$ is the edge ideal associated to a graph having $s-1$ components. Thus induction on γ implies

$$\begin{aligned}
\operatorname{depth}(\Omega_t'/\mathcal{I}_t^k) &\geq \max\{1 + (s-1) - 1, s-1\} \\
&= s - 1.
\end{aligned}$$

So

$$\operatorname{depth}(\Omega_t/\mathcal{I}_t^k) \geq s = q.$$

Now for $\delta \geq 4$, $\mathcal{I}_t$ is associated to a graph having s connected components and at least $\delta - 3$ diameter. Then by induction on γ

$$\operatorname{depth}(\Omega_t/\mathcal{I}_t^k) \geq \max\{\lceil \frac{(\delta-3)-k+2}{3} \rceil + s - 1, s\}.$$

Consequently,

$$\begin{aligned}
\operatorname{depth}(\Omega_t/(\mathcal{I}_t^k : z)) &= \operatorname{depth}(\Omega_t/\mathcal{I}_t^k) \\
&= \operatorname{depth}(\Omega_t'/\mathcal{I}_t^k) + 1 \\
&\geq q.
\end{aligned}$$

Next suppose that $\operatorname{depth}(\Omega_j/(\mathcal{I}_j^k : z)) \geq q$ where $2 \leq j \leq t$. For a short exact sequence

$$0 \longrightarrow \Omega_{j-1}/(\mathcal{I}_{j-1}^k : zy_j) \longrightarrow \Omega_{j-1}/(\mathcal{I}_{j-1}^k : z) \longrightarrow \Omega_{j-1}/((\mathcal{I}_{j-1}^k : z), y_j) \longrightarrow 0$$

By Lemma 3.0.14 $((\mathcal{I}_{j-1}^k : z), y_j) = ((\mathcal{I}_j^k : z), y_j)$, so

$$\begin{aligned}
\operatorname{depth}(\Omega_{j-1}/((\mathcal{I}_{j-1}^k : z), y_j)) &= \operatorname{depth}(\Omega_j/(\mathcal{I}_{j-1}^k : z)) \\
&\geq q,
\end{aligned}$$

and Lemma 3.1.4 implies

$$(\mathcal{I}_{j-1}^k : zy_j) = \mathcal{I}_{j-1}^{k-1}$$

since y_j is a leaf for $j < t$ and when $j = t$ then z is a leaf. $\mathcal{I}_{j-1}$ corresponds to a graph with at least $\delta - 1$ diameter then by induction

$$\begin{aligned}
\operatorname{depth}(\Omega_{j-1}/(\mathcal{I}_{j-1}^k : zy_j)) &= \operatorname{depth}(\Omega_{j-1}/\mathcal{I}_{j-1}^{k-1}) \\
&\geq \max\{\lceil \frac{(\delta-1)-(k-1)+2}{3} \rceil + s - 1, s\} = q,
\end{aligned}$$

and hence by Depth Lemma

$$\operatorname{depth}(\Omega_{j-1}/(\mathcal{I}_{j-1}^k : z)) \geq q.$$

Therefore, by reverse induction

$$\operatorname{depth}(\Omega_1/(\mathcal{I}_1^k : z) \geq q.$$

Since $\operatorname{depth}(S/((I^k : z), y_1)) = \operatorname{depth}(\Omega_1/(\mathcal{I}_1^k : z))$ then

$$\operatorname{depth}(S/((I^k : z), y_1)) \geq q.$$

Now applying Lemma 3.2.1 with $w_1 = z$ and $w_2 = y_1$

$$\operatorname{depth}(S/I^k) \geq q.$$

□

Corollary 3.2.5. *For a tree G with diameter δ, let $I = I(G)$. Then for all $k \geq 1$*

$$\operatorname{depth}(S/I^k) \geq \max\{\lceil \frac{\delta - k + 2}{3} \rceil, 1\}.$$

Proof. By setting $s = 1$, Theorem 3.2.4 implies the result . □

Since P_γ has $\gamma - 1$ diameter therefore in this case Proposition 3.2.2 and Corollary 3.2.5 implies the same result. The above proof strongly relies on the existence of a node having at most one non-leaf in its neighbor. For tree of diameter $\delta \geq 3$, there must exist at least two such type of vertices that are non-leaves. These non-leaves are the neighbors of the end points of path which determines diameter. A node of a graph is known as *near leaf* if it is a non-leaf and its neighbors contain at most one non-leaf. Suppose α denotes the total number of near leaves in a graph. Next the bound computed in Theorem 3.2.4 will be improved using the same proof.

Lemma 3.2.6. *For a tree G with α near leaves, let δ be the diameter. Then*

$$\operatorname{depth}(S/I) \geq \lceil \frac{\delta + \alpha - 1}{3} \rceil.$$

Proof. When n is small, $\alpha \leq 2$ and the result is true by Proposition 3.1.3. Let $\alpha \geq 3$. Notice that when two near leaves are adjacent in a connected graph then $\delta = 3$ and $\alpha = 2$, as all remaining vertices must be leaves. Therefore for $\alpha \geq 3$, two near leaves will never be adjacent. Consider a diameter realizing path $P_{\delta+1}$ with vertices $y_1, y_2, \ldots, y_{\delta+1}$. Notice that both y_2 and y_δ are near leaves and $\delta \geq 4$ as for $\delta = 3$ there is an edge between y_2 and y_δ and for $\delta \leq 2$ there is atmost one near leaf in the tree.

Let $(I, y_2) = (\lambda, y_2)$ where λ be the edge ideal associated with deletion minor Δ' which is obtained by eliminating y_2. Suppose Ω' be the polynomial ring obtained by removing y_1 and y_2. Then Δ' has the diameter at least $\delta - 2$ and near leaves at least $\alpha - 1$. So, by using induction on δ and α

$$\operatorname{depth}(\Omega'/\lambda) \geq \lceil \frac{(\delta-2)+(\alpha-1)-1}{3} \rceil = \lceil \frac{\delta+\alpha-4}{3} \rceil.$$

Consequently, from Lemma 2.2.8

$$\begin{aligned} \operatorname{depth}(S/(I, y_2)) &= \operatorname{depth}(\Omega'[y_1]/\lambda) \geq \lceil \frac{\delta+\alpha-3-1}{3} \rceil + 1 \\ &\geq \lceil \frac{\delta+\alpha-1}{3} \rceil. \end{aligned}$$

Let $(I : y_2) = (\psi, N(y_2))$, where ψ is the edge ideal of contraction minor Δ'' obtained by removing the neighbors of y_2. Then Δ'' is a tree with diameter at least $\delta - 3$. Suppose b be the number of near leaves in the neighbors of y_3 which does not belongs to $P_{\delta+1}$. The path joining y_3 to a leaf and not containing y_j for $j \neq 3$ has at most 2 length otherwise there exist another path having length greater than δ in G, which is a contradiction.

Thus a near leaf belonging to such a path is a neighbor of y_3. Assume $\delta = 4$. As $\alpha \geq 3$ and no near leaves are adjacent to either y_2 or $y_\delta = y_4$ then $b = \alpha - 2 \geq 1$. Notice that Δ'' has $b + 1$ components corresponding to near leaves adjacent to y_3 and to $(y_4 y_5)$, which is the path with $\delta - 3$ length. Note that y_2 is a vertex of degree one in Δ''. Since $\alpha \geq 3$, [25, Lemma 6.2.7] implies

$$\begin{aligned} \operatorname{depth}(S/(I : y_2)) &\geq 1 + b + 1 = \alpha \\ &= \lceil \frac{3\alpha}{3} \rceil \geq \lceil \frac{3+\alpha}{3} \rceil \\ &= \lceil \frac{4+\alpha-1}{3} \rceil. \end{aligned}$$

Next assume $\delta \geq 5$. Then Δ'' has at least $\delta - 3$ diameter and at least $\alpha - b - 1$ near leaves in the connected component which contains $P_{\delta-2}$. Considering that y_2 is an isolated vertex and there exist b additional connected components, so by applying induction

$$\operatorname{depth}(S/(I : y_2)) \geq \lceil \frac{(\delta-3)+(\alpha-b-1)-1}{3} \rceil + b + 1.$$

If b is a non-zero positive integer

$$\lceil \frac{(\delta-3)+(\alpha-b-1)-1}{3} \rceil + b + 1 \geq \lceil \frac{\delta+\alpha-1}{3} \rceil,$$

and in the case when $b=0$ if there are α near leaves in Δ'' then

$$\begin{aligned} \operatorname{depth}(S/(I:y_2)) &\geq \lceil \frac{(\delta-3)+\alpha-1}{3} \rceil + 1 \\ &= \lceil \frac{\delta+\alpha-1}{3} \rceil. \end{aligned}$$

When Δ'' contains $\alpha-1$ near leaves, then y_5 cannot be an additional near leaf. For $\delta=5$; y_5 is already a near leaf. As $b=0$ with $\alpha \geq 3$ then the path adjacent to y_4 other than $P_{\delta-2}$ must contain a near leaf. Δ'' has at least $\delta-2$ diameter as y_4 is non-leaf in Δ''. When $\delta \geq 6$ and y_5 is not a near leaf in Δ'' than either y_4 is non-leaf or there is another non-leaf else y_6 in the neighbors of y_5. In both cases, Δ'' has diameter at least $\delta-2$. Hence

$$\begin{aligned} \operatorname{depth}(S/(I:y_2)) &\geq \lceil \frac{(\delta-2)+(\alpha-1)-1}{3} \rceil + 1 \\ &= \lceil \frac{\delta+\alpha-1}{3} \rceil. \end{aligned}$$

Then by applying Depth Lemma to the sequence

$$0 \longrightarrow S/(I:y_2) \longrightarrow S/I \longrightarrow S/(I,y_2) \longrightarrow 0$$

the result follows.

□

Corollary 3.2.7. *Consider a forest G having s connected components $\Delta_1, \Delta_2, \ldots, \Delta_s$. Let $I=I(G)$ and Δ_j has diameter δ_j and suppose $\delta=\max\{\delta_j\}$ and component with diameter δ has α near leaves. For $k \geq 1$*

$$\operatorname{depth}(S/I^k) \geq \max\{\lceil \frac{\delta-k+\alpha}{3} \rceil + s - 1, s\}.$$

Proof. Let $\delta=\delta_1$ and α be the number of near leaves in Δ_1. Only the near leaves in the minor of Δ_1 will be counted while doing this proof. Let

$$q = \max\{\lceil \frac{\delta-k+\alpha}{3} \rceil + s - 1, s\}.$$

When $\delta \leq 3$ then $\alpha \leq 2$ and Theorem 3.2.4 follows the result for smaller values of n. For $t = 1$, Lemma 3.2.6 implies the result. Let $\delta \geq 4$ and $\alpha \geq 3$, so

$$(I^k : y_1 z) = I^{k-1}$$

and by applying induction on k

$$\begin{aligned} \operatorname{depth}(S/(I^k : y_1 z)) &= \operatorname{depth}(S/I^{k-1}) \\ &\geq \max\{\lceil \frac{\delta - (k-1) + \alpha}{3} \rceil + s - 1, s\} \\ &\geq q. \end{aligned}$$

Further

$$(I^k, z) = (\lambda^k, z)$$

where λ is the edge ideal corresponding to the minor Δ' obtained by removing z, where Δ' has diameter at least $\delta - 2$ and near leaves at least $\alpha - 1$. y_1 is an isolated vertex in Δ'. So by using induction on γ

$$\begin{aligned} \operatorname{depth}(S/(I^k, z)) &\geq \max\{\lceil \frac{(\delta - 2) - k + (\alpha - 1)}{3} \rceil + s - 1, s\} + 1 \\ &\geq q. \end{aligned}$$

Since $\delta \geq 4$ so $t \geq 2$ and

$$(\mathcal{I}_t^k : z) = \mathcal{I}_t^k$$

Note that the graph associated to $\mathcal{I}_t$ has diameter at least $\delta - 3$. Since y_t lies on a path realizing the diameter and it is at two distance from a leaf on the same path, any other path which joins a leaf to y_t and does not have any other node in $P_{\delta+1}$ must have at most two length. Therefore any near leaves on such a path are neighbors of y_t.

Suppose there are b number of near leaves in the neighbors of y_t and they are not on $P_{\delta+1}$. Accordingly, the graph of $\mathcal{I}_t$ contains $s + b$ connected components. There are at least $\alpha - b - 1$ near leaves of $\mathcal{I}_t$ in the component which contains $P_{\delta-2}$, as z is not a near leaf of $\mathcal{I}_t$.

When $\delta = 4$, there are zero near leaves in graph of $\mathcal{I}_t$ and $b = \alpha - 2$ because every near leaf is the neighbor of y_t, including both that are on $P_{\delta+1}$. Therefore the minor corresponding to $\mathcal{I}_t$ contains P_2 along with b extra components and at least one vertex with degree zero. Hence

$$\begin{aligned}
\operatorname{depth}(\Omega_t/(\mathcal{I}_t^k : z)) &= \operatorname{depth}(\Omega_t/\mathcal{I}_t^k) \\
&\geq 1 + b + s - 1 + 1 \\
&\geq \alpha + s - 1 \\
&\geq \lceil \frac{\alpha + 3}{3} \rceil + s - 1 \\
&= \lceil \frac{\delta + \alpha - 1}{3} \rceil + s - 1.
\end{aligned}$$

Let $\delta = 5$, then either there are two near leaves in the connected component which contains $P_{\delta-2}$ and has at least $\delta - 2$ diameter or $b \geq 1$. In case 1

$$\begin{aligned}
\operatorname{depth}(\Omega_t/\mathcal{I}_t^k) &= \operatorname{depth}(\Omega_t'/\mathcal{I}_t^k) + 1 \\
&\geq \max\{\lceil \frac{(\delta - 2) - k + (\alpha - 1)}{3} \rceil + s - 1, s\} + 1 \\
&\geq q.
\end{aligned}$$

Else $b \geq 1$, so

$$\begin{aligned}
\operatorname{depth}(\Omega_t/\mathcal{I}_t^k) &= \operatorname{depth}(\Omega_t'/\mathcal{I}_t^k) + 1 \\
&\geq \max\{\lceil \frac{(\delta - 3) - k + (\alpha - b - 1)}{3} \rceil + s - 1, s\} + b + 1 \\
&\geq q.
\end{aligned}$$

Assume $\delta > 5$, then $\delta - 3 \geq 4$ and therefore either there are two near leaves in $P_{\delta-3}$, where one of them was not a near leaf of I, or there exist a path other than $P_{\delta-3}$ that realizes the diameter. Hence the diameter of I_t is at least $\delta - 2$. In case 1, there are $\alpha - b$ near leaves and at least $\delta - 3$ diameter. Thus by induction

$$\begin{aligned}
\operatorname{depth}(\Omega_t/\mathcal{I}_t^k) &= 1 + \operatorname{depth}(\Omega_t'/\mathcal{I}_t^k) \\
&\geq 1 + \max\{\lceil \frac{(\delta - 3) - k + (\alpha - b)}{3} \rceil + s - 1, s\} + b \\
&\geq q,
\end{aligned}$$

otherwise

$$\begin{aligned} \operatorname{depth}(\Omega_t/\mathcal{I}_t^k) &= 1+\operatorname{depth}(\Omega_t'/\mathcal{I}_t^k) \\ &\geq 1+\max\{\lceil\frac{(\delta-3)-k+(\alpha-b-1)}{3}\rceil+s-1,s\}+b \\ &\geq q. \end{aligned}$$

Hence by reverse induction on j

$$\begin{aligned} \operatorname{depth}(\Omega_1/(\mathcal{I}_1^k:z)) &= \operatorname{depth}(S/((I^t:z),y_1)) \\ &\geq q, \end{aligned}$$

and Lemma 3.2.1 implies the required result. □

Example 3.2.8. Consider a graph G having edge set

$$\{y_1y_2, y_2y_3, y_3y_4, y_4y_5, y_3y_6, y_6y_7, y_3y_8, y_8y_9, y_3y_{10}, y_{10}y_{11}\},$$

then $\delta = 4$ and there are 5 near leaves in G. So,

$$\operatorname{depth}(S/I) = \lceil\frac{\delta-1+\alpha}{3}\rceil = \lceil\frac{4-1+5}{3}\rceil = 3,$$

while $\operatorname{depth}(S/I) = 5$, by using Macaulay 2 [11]. Therefore the bound in Corollary 3.2.7 is not always sharp. Since

$$\operatorname{depth}(S/I^2) \geq \lceil\frac{\delta-2+\alpha}{3}\rceil = \lceil\frac{4-2+5}{3}\rceil = 3,$$

whereas the actual value of depth is 5. This situation is encountered when $\delta = 4$ and m is very large. Still for large value of k, this bound gives accurate value. If $k = 5$, Macaulay 2 and Corollary 3.2.7 gives the same value i.e., 2 and when $k = 6$ it becomes 1. When δ gets large value, an improvised bound gives comparatively accurate values. For example, for a graph having 9 vertices and the edges

$$\{y_1y_2, y_2y_3, y_3y_4, y_4y_5, y_5y_6, y_6y_7, y_4y_8, y_8y_9\},$$

diameter is 6 with three near leaves. The bound in Corollary 3.2.7 and actual value of $\operatorname{depth}(S/I^k)$ are same for all $k \neq 3$ and $k \leq 6$. When $k = 5, 6$ both the bound and actual depth are 2 and 1 respectively. predicts where the depth will become one.

3.3 Stanley depth of Powers of Forest

Consider an edge ideal I of forest containing s connected components. Morey [14] proved that for $k \geq 1$

$$\operatorname{depth}(S/I^k) \geq \max\{\lceil \frac{\delta - k + 2}{3} \rceil + s - 1, s\}.$$

Later on, Pournaki et al. [18] proved the same bound for the Stanley depth for any power of I. For a bipartite graph, a fundamental lower bound exists for depth of S/I^k.

Theorem 3.3.1. *[6, Theorem 1.4] For a finitely generated $\mathbb{Z}^n$-graded S module M, if* $\operatorname{sdepth}(M) = 0$ *then* $\operatorname{depth}(M) = 0$*. Conversely, if* $\operatorname{depth}(M) = 0$ *and* $\dim_K(M_\alpha) \leq 1$ *for any $\alpha \in \mathbb{Z}^n$, then* $\operatorname{sdepth}(M) = 0$*.*

Proposition 3.3.2. *Let $S = K[y_1, \ldots, y_n]$ and $I = I(P_\gamma)$, then*

$$\operatorname{sdepth}(S/I) \geq \lceil \frac{\gamma}{3} \rceil.$$

Proposition 3.3.3. *Let $I = I(G)$ where G is a tree with diameter δ, then*

$$\operatorname{sdepth}(S/I) \geq \lceil \frac{\delta + 1}{3} \rceil.$$

Lemma 3.3.4. *For a graph G, let $I = I(G)$. Suppose w_1, w_2 are the nodes of G. If* $\operatorname{sdepth}(S/(I^k : w_1w_2)) \geq r$, $\operatorname{sdepth}(S/(I^k, w_1)) \geq r$ *and* $\operatorname{sdepth}(S/((I^k : w_1), w_2)) \geq r$ *for $r \geq 0$, then*

$$\operatorname{sdepth}(S/I^k) \geq r.$$

Proof. The proof is similar to the proof of Lemma 3.2.1. Lemma 2.2.11 is used instead of Depth lemma. □

Proposition 3.3.5. *Let $I(P_\gamma)$ be the edge ideal of path with $n \geq 2$, then*

$$\operatorname{sdepth}(S/I^k(P_\gamma)) \geq \max\ \{\lceil \frac{\gamma - k + 1}{3} \rceil, 1\}.$$

Theorem 3.3.6. *Let G be a forest having s number of connected components Δ_1, Δ_2, $\ldots$, Δ_s. Let $I = I(G)$ and Δ_j has diameter δ_j and suppose $\delta = \max\{\delta_j\}$. Then for $k \geq 1$*

$$\operatorname{sdepth}(S/I^k) \geq \max\{\lceil \frac{\delta - k + 2}{3} \rceil + s - 1, s\}.$$

Proof. The proof is done on the similar lines as the proof of Theorem 3.2.4, by replacing depth with Stanley depth and using Lemma 2.2.11, Lemma 3.3.4 and Proposition 3.3.5. □

Lemma 3.3.7. *For a tree G with diameter δ and α near leaves, then*

$$\operatorname{sdepth}(S/I) \geq \lceil \frac{\delta + \alpha - 1}{3} \rceil.$$

Corollary 3.3.8. *Consider a forest G having s connected components $\Delta_1, \Delta_2, \ldots, \Delta_s$. Let $I = I(G)$ and Δ_j has diameter δ_j and suppose $\delta = \max\{\delta_j\}$. Also the component with diameter δ has α near leaves. For $k \geq 1$*

$$\operatorname{sdepth}(S/I^k) \geq \max \{\lceil \frac{\delta - k + \alpha}{3} \rceil + s - 1, s\}.$$

Chapter 4

Depth and Stanley depth of some Graphs

4.1 Firecracker Graph

Definition 4.1.1. An (γ, z)-firecracker is a graph formed by the concatenation of γ number of z-stars by linking one leaf from each star. It is denoted by $F(\gamma, z)$.

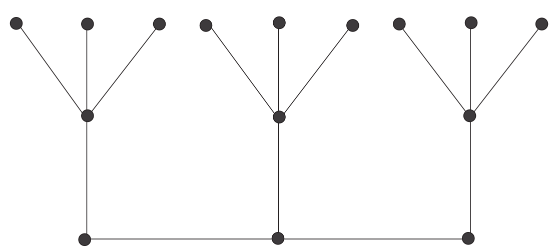

Figure 4.1: Firecracker $F(3, 5)$

Proposition 4.1.2. *[1] For a star graph $\mathcal{S}_k$, let $I = I(\mathcal{S}_k)$, then*

$$\operatorname{depth}(S/I) = \operatorname{sdepth}(S/I) = 1,$$

and

$$\operatorname{depth}(S/I^t), \operatorname{sdepth}(S/I^t) \geq 1.$$

Theorem 4.1.3. *For a firecracker graph $F(\gamma, z)$ let $I = I(F(\gamma, z))$, then for $\gamma \geq 2$ and $z \geq 3$*

$$\operatorname{depth}(S/I) = \gamma.$$

Proof. The proof is done by the induction on γ. Let e be the non-leaf of the last star in $F(\gamma, z)$. For short exact sequence

$$0 \longrightarrow S/(I:e) \longrightarrow S/I \longrightarrow S/(I,e) \longrightarrow 0$$

When $\gamma = 2$, $(I : e) = (I(\mathcal{S}_k), N(e))$, here $N(e)$ be the neighbors of e in $F(2, z)$. Then by Proposition 4.1.2 and Lemma 2.2.8, $\operatorname{depth}(S/(I : e)) = \operatorname{depth}(S'[e]/I(\mathcal{S}_z)) = \operatorname{depth}(S'/I(\mathcal{S}_z)) + 1 = 2$ and $S' = S\backslash N(e) \cup \{e\}$. Now let $W := (I, e)$. Let f be the leaf of second star in $F(2, z)$ which is linked with the leaf of previous star. Then for depth $S/(I, e)$ consider another short exact sequence

$$0 \longrightarrow S/(W:f) \longrightarrow S/W \longrightarrow S/(W,f) \longrightarrow 0$$

Now $(W : f) = (I(\mathcal{S}_{z-1}), N(f))$, here the neighbors of f in G' are denoted by $N(f)$. So by Lemma 2.2.8 and Proposition 4.1.2

$$\begin{aligned} \operatorname{depth}(S/(W:f)) &= \operatorname{depth}(S''[N(e)]/(I(\mathcal{S}_{z-1})) = \operatorname{depth}(S''/(I(\mathcal{S}_{z-1})) + |N(e)| \\ &= 1 + (z-1) = z, \end{aligned}$$

where $S'' = S'\backslash N(f)$. Next $(W, f) = (I(\mathcal{S}_z), f)$, and

$$\begin{aligned} \operatorname{depth}(S/(W,f)) &= \operatorname{depth}(S'[L]/I(\mathcal{S}_z)) = \operatorname{depth}(S'/I(\mathcal{S}_z)) + |L| \\ &= 1 + (z-2) = z-1, \end{aligned}$$

where L is a set of variables obtained by removing f from neighbors of e. Therefore by Depth Lemma, $\operatorname{depth}(S/W) \geq z - 1$. And again by applying Depth Lemma, $\operatorname{depth}(S/I) \geq 2$. Also, since $2 = \operatorname{depth}(S/(I : e)) \geq \operatorname{depth}(S/I)$, hence for $\gamma = 2$, $\operatorname{depth}(S/I) = 2$. Next for $\gamma = 3$, $(I : e) = (I(F(2, z)), N(e))$, where $N(e)$ be set of neighbors of e in $F(3, z)$. Then by Proposition 4.1.2 and Lemma 2.2.8, $\operatorname{depth}(S/(I : e)) = \operatorname{depth}(S'[e]/I(F(2, z))) = 1 + \operatorname{depth}(S'/I(F(2, z))) = 1 + 2 = 3$, where $S' =$

$S\backslash N(e) \cup \{e\}$. Now let $W := (I, e)$. Let f be the leaf of third star in $F(3, z)$ which is linked with the leaf of previous star. Then for depth $S/(I, e)$ consider another short exact sequence

$$0 \longrightarrow S/(W : f) \longrightarrow S/W \longrightarrow S/(W, f) \longrightarrow 0$$

Now $(W : f) = (I(\mathcal{S}_z), I(\mathcal{S}_{z-1}), N(f))$, here the neighbors of f in G' are $N(f)$. So by Lemma 2.2.8 and Proposition 4.1.2

$$\begin{aligned} \operatorname{depth}(S/(W : f)) &= \operatorname{depth}(S''[N(e)]/(I(\mathcal{S}_z), I(\mathcal{S}_{z-1})) \\ &= \operatorname{depth}(S''/(I(\mathcal{S}_z), I(\mathcal{S}_{z-1})) + |N(e)| \\ &= 1 + 1 + (z - 1) = z + 1, \end{aligned}$$

where $S'' = S'\backslash N(f)$. Next $(W, f) = (F(2, z)), f)$, then

$$\begin{aligned} \operatorname{depth}(S/(W, f)) &= \operatorname{depth}(S'[L]/I(F(2, z))) = \operatorname{depth}(S'/I(F(2, z))) + |L| \\ &= 2 + (z - 2) = z, \end{aligned}$$

where $L = N(e)\backslash\{f\}$. Then by Depth Lemma, $\operatorname{depth}(S/W) \geq z$. And again by applying Depth Lemma, $\operatorname{depth}(S/I) \geq 3$. Also, since $3 = \operatorname{depth}(S/(I : e)) \geq \operatorname{depth}(S/I)$, hence for $\gamma = 3$, $\operatorname{depth}(S/I) = 3$. Now assume $\gamma \geq 4$. Then

$$\begin{aligned} (I : e) &= (\Delta, N(e)) \\ &= (I(F(\gamma - 1, z)), N(e)) \end{aligned}$$

where the set of neighbors of e in $F(\gamma, z)$ is denoted by $N(e)$ and Δ be the edge ideal of contraction minor of $F(\gamma, z)$ obtained by removing $N(e)$. So, by induction and Lemma 2.2.8

$$\begin{aligned} \operatorname{depth}(S/(I : e)) &= \operatorname{depth}(S'[e]/I(F(\gamma - 1, z))) \\ &= 1 + \operatorname{depth}(S'/I(F(\gamma - 1, z))) \\ &= 1 + (\gamma - 1) \\ &= \gamma, \end{aligned}$$

where $S' = S\backslash N(e) \cup \{e\}$. Now let

$$W := (I, e) = (I', e)$$

and I' be the ideal associated to deletion minor G' of $F(\gamma, z)$ obtained by eliminating e. Let f be the leaf of last star in $F(\gamma, z)$ which is linked with the leaf of previous star. Then for depth $S/(I, e)$ consider another short exact sequence

$$0 \longrightarrow\ S/(W : f) \longrightarrow\ S/W \longrightarrow\ S/(W, f) \longrightarrow\ 0$$

By using Depth Lemma,

$$\operatorname{depth}(S/W) \geq \min\{\operatorname{depth}(S/(W : f)), \operatorname{depth}(S/(W, f)\}.$$

Now

$$\begin{aligned} (W : f) &= (W', N(f)) \\ &= (I(F(\gamma - 2, z)), I(\mathcal{S}_{z-1}), N(f)) \end{aligned}$$

where $N(f)$ be the neighbors of f in G' and W' is the edge ideal corresponding to contraction minor of G', obtained by eliminating $N(f)$. Then by induction on γ, Lemma 2.2.8 and Proposition 4.1.2

$$\begin{aligned} \operatorname{depth}(S/(W : f)) &= \operatorname{depth}(S''[N(e)]/(I(F(\gamma - 2, z)), I(\mathcal{S}_{z-1})) \\ &= \operatorname{depth}(S''/(I(F(\gamma - 2, z)), I(\mathcal{S}_{z-1})) + |N(e)| \\ &= (\gamma - 2) + 1 + (z - 1) \\ &= \gamma + z - 2, \end{aligned}$$

where $S'' = S'\backslash N(f)$. Note that, $(W, f) = (W'', f) = (I(F(\gamma - 1, z)), f)$. Here W'' is associated to the deletion minor of G' obtained by removing f. So by using induction and Lemma 2.2.8

$$\begin{aligned} \operatorname{depth}(S/(W, f)) &= \operatorname{depth}(S'[N']/I(F(\gamma - 1, z)) \\ &= \operatorname{depth}(S'/I(F(\gamma - 1, z)) + |N'| \\ &= (z - 2) + (\gamma - 1) \\ &= \gamma + z - 3, \end{aligned}$$

where $N' = N(e)\backslash\{f\}$. Then by Depth Lemma

$$\text{depth}(S/W) \geq \gamma + z - 3.$$

And again by applying Depth Lemma, $\text{depth}(S/I) \geq \gamma$. Since $\gamma = \text{depth}(S/(I : e)) \geq \text{depth}(S/I)$, hence

$$\text{depth}(S/I) = \gamma.$$

This completes the proof. □

On the lines of above proof, by using Proposition 2.2.6, Lemma 2.2.8, Proposition 4.1.2 and Lemma 2.2.11 the next result follows.

Proposition 4.1.4. *For a firecracker graph $F(\gamma, z)$ let $I = I(F(\gamma, z))$, then for $\gamma \geq 2$ and $z \geq 3$*

$$\text{sdepth}(S/I) = \gamma.$$

4.2 Circular Firecracker

Definition 4.2.1. The graph obtained by linking a leaf of first star to the leaf of last star in a firecracker graph is termed as a *circular firecracker*, denoted by $CF(\gamma, z)$.

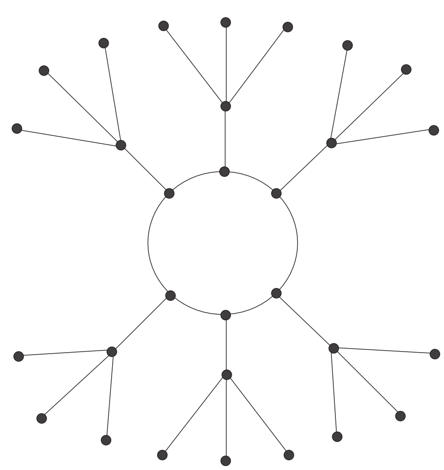

Figure 4.2: Circular firecracker $CF(6, 5)$

Theorem 4.2.2. *Let $I = I(CF(\gamma, z))$, then for any $\gamma \geq 3$ and $z \geq 3$*

$$\operatorname{depth}(S/I) = \gamma.$$

Proof. Let e be the central vertex of a star $\mathcal{S}_z$ in $CF(\gamma, z)$. For the short exact sequence

$$0 \longrightarrow S/(I : e) \longrightarrow S/I \longrightarrow S/(I, e) \longrightarrow 0$$

When $\gamma = 3$, $(I : e) = (I(F(2, z)), N(e))$, here the neighbors of e in $CF(3, z)$ forms the set $N(e)$. So, Lemma 2.2.8 and Theorem 4.1.3

$$\begin{aligned}\operatorname{depth}(S/(I : e)) &= \operatorname{depth}(S'[e]/I(F(2, z))) = 1 + \operatorname{depth}(S'/I(F(2, z))) \\ &= 1 + 2 = 3,\end{aligned}$$

where $S' = S \backslash N(e) \cup \{e\}$. Now let $W := (I, e) = (I', e)$ and let f be neighbor of e on cycle C_3. To compute $\operatorname{depth}(S/(I, e))$, consider another short exact sequence

$$0 \longrightarrow S/(W : f) \longrightarrow S/W \longrightarrow S/(W, f) \longrightarrow 0.$$

Then $(W : f) = (2(I(\mathcal{S}_{z-1})), N(f))$, here the neighbors of f in G' forms the set $N(f)$. Then by Lemma 2.2.8 and Proposition 4.1.2

$$\begin{aligned}\operatorname{depth}(S/(W : f)) &= \operatorname{depth}(S''[N(e)]/(2(I(\mathcal{S}_{z-1}))) = \operatorname{depth}(S''/(2(I(\mathcal{S}_{z-1}))) + |N(e)| \\ &= 2 + (z - 1) = z + 1,\end{aligned}$$

where $S'' = S' \backslash N(f)$. Furthermore, $(W, f) = (I(F(2, z)), f)$. Then by using Lemma 2.2.8

$$\begin{aligned}\operatorname{depth}(S/(W, f)) &= \operatorname{depth}(S'[L]/I(F(2, z)) = \operatorname{depth}(S'/I(F(2, z)) + |L| \\ &= 2 + (z - 2) = z,\end{aligned}$$

where $L = N(e) \backslash \{f\}$. Then by Depth Lemma $\operatorname{depth}(S/W) \geq z$. And by again applying Depth Lemma $\operatorname{depth}(S/I) \geq 3$. Since, $3 = \operatorname{depth}(S/(I : e)) \geq \operatorname{depth}(S/I)$, hence for $\gamma = 3$, $\operatorname{depth}(S/I) = 3$. Now for $\gamma = 4$, $(I : e) = (I(F(3, z)), N(e))$, where $N(e)$ be the neighbors of e in $CF(4, z)$. So, Lemma 2.2.8 and Theorem 4.1.3,

$\operatorname{depth}(S/(I:e)) = \operatorname{depth}(S'[e]/I(F(3,z))) = 1 + \operatorname{depth}(S'/I(F(3,z))) = 1 + 3 = 4$, where $S' = S \backslash N(e) \cup \{e\}$. Now let $W := (I,e) = (I',e)$ and let f be neighbor of e on cycle C_4. To compute $\operatorname{depth}(S/(I,e))$, consider another short exact sequence

$$0 \longrightarrow S/(W:f) \longrightarrow S/W \longrightarrow S/(W,f) \longrightarrow 0.$$

Then $(W:f) = (2(I(\mathcal{S}_{z-1})), I(\mathcal{S}_z), N(f))$, where the neighbors of f in G' forms the set $N(f)$. Then by Lemma 2.2.8 and Proposition 4.1.2

$$\begin{aligned}
\operatorname{depth}(S/(W:f)) &= \operatorname{depth}(S''[N(e)]/(2(I(\mathcal{S}_{z-1})), I(\mathcal{S}_z)) \\
&= \operatorname{depth}(S''/(2(I(\mathcal{S}_{z-1})), I(\mathcal{S}_z)) + |N(e)| \\
&= 2 + 1 + (z-1) = z + 2,
\end{aligned}$$

where $S'' = S' \backslash N(f)$. Furthermore, $(W,f) = (I(F(3,z)), f)$. Then by using Lemma 2.2.8

$$\begin{aligned}
\operatorname{depth}(S/(W,f)) &= \operatorname{depth}(S'[L]/I(F(3,z)) = \operatorname{depth}(S'/I(F(3,z)) + |L| \\
&= 3 + (z-2) = z + 1,
\end{aligned}$$

where $L = N(e) \backslash \{f\}$. Then by Depth Lemma $\operatorname{depth}(S/W) \geq z+1$. And by again applying Depth Lemma $\operatorname{depth}(S/I) \geq 4$. Since, $4 = \operatorname{depth}(S/(I:e)) \geq \operatorname{depth}(S/I)$, hence for $\gamma = 4$, $\operatorname{depth}(S/I) = 4$. Now assume $\gamma \geq 5$. Then

$$\begin{aligned}
(I:e) &= (\Delta, N(e)) \\
&= (I(F(\gamma-1,z)), N(e))
\end{aligned}$$

where $N(e)$ are the neighbors of e in $CF(\gamma,z)$ and Δ be the edge ideal of the contraction minor of $CF(\gamma,z)$ obtained by removing $N(e)$. So, Lemma 2.2.8 and Theorem 4.1.3

$$\begin{aligned}
\operatorname{depth}(S/(I:e)) &= \operatorname{depth}(S'[e]/I(F(\gamma-1,z))) \\
&= 1 + \operatorname{depth}(S'/I(F(\gamma-1,z))) \\
&= \gamma,
\end{aligned}$$

where $S' = S \backslash N(e) \cup \{e\}$. Now let

$$W := (I,e) = (I',e)$$

here I' is the ideal associated to deletion minor G' of $CF(\gamma, z)$ obtained by eliminating e. Let f be neighbor of e on C_γ. To compute depth$(S/(I,e))$, assume a short exact sequence

$$0 \longrightarrow S/(W:f) \longrightarrow S/W \longrightarrow S/(W,f) \longrightarrow 0$$

By using Depth Lemma

$$\text{depth}(S/W) \geq \min\{\text{depth}(S/(W:y)), \text{depth}(S/(W,y)\}.$$

Now

$$\begin{aligned}(W:f) &= (W', N(f))\\ &= (I(F(\gamma-3,z)), 2(I(\mathcal{S}_{z-1})), N(f))\end{aligned}$$

where the neighbors of y in G' forms $N(f)$ and W' is the ideal of contraction minor, which is obtained by removing $N(f)$. Then by Lemma 2.2.8 and Proposition 4.1.2

$$\begin{aligned}\text{depth}(S/(W:f)) &= \text{depth}(S''[N(e)]/(I(F(\gamma-3,z)), 2(I(\mathcal{S}_{z-1})))\\ &= \text{depth}(S''/(I(F(\gamma-3,z)), 2(I(\mathcal{S}_{z-1}))) + |N(e)|\\ &= (\gamma-3)+2+1+(z-2)\\ &= \gamma+z-2,\end{aligned}$$

where $S'' = S' \backslash N(f)$, and

$$\begin{aligned}(W,f) &= (W'', f)\\ &= (I(F(\gamma-1,z)), f)\end{aligned}$$

where W'' is the ideal of deletion minor of G' obtained after removing f. Then by using Lemma 2.2.8

$$\begin{aligned}\text{depth}(S/(W,f)) &= \text{depth}(S'[L]/I(F(\gamma-1,z))\\ &= \text{depth}(S'/I(F(\gamma-1,z)) + |L|\\ &= (\gamma-1)+(z-2)\\ &= \gamma+z-3,\end{aligned}$$

where $L = N(e)\backslash\{f\}$. Then by Depth Lemma

$$\text{depth}(S/W) \geq \gamma + z - 3.$$

And again by applying Depth Lemma, $\text{depth}(S/I) \geq \gamma$. Since $\gamma = \text{depth}(S/(I:e)) \geq \text{depth}(S/I)$, hence

$$\text{depth}(S/I) = \gamma.$$

This completes the proof. □

Similarly, by using Proposition 2.2.6, Proposition 4.1.4, Lemma 2.2.8, Proposition 4.1.2 and Lemma 2.2.11, the next result holds.

Proposition 4.2.3. *Let $I = I(CF(\gamma, z))$, then for any $\gamma \geq 3$ and $z \geq 3$*

$$\text{sdepth}(S/I) = \gamma.$$

4.3 Paths Attached to a Path with a Gradual Decrease in Length

Definition 4.3.1. Let $P_p(\gamma; m_1, \dots, m_\gamma)$ be a graph in which γ number of paths are attached in a way that the end vertex of each path P_{m_i} (with order $m_i = m_\gamma + \gamma - i$) is adjacent to the end vertex of path $P_{m_{i+1}}$ (with order $m_{i+1} = m_\gamma + \gamma - i - 1$), where $1 \leq i \leq \gamma - 1$ and m_γ is the order of P_{m_γ}.

Theorem 4.3.2. *Let $I = I(P_p(\gamma; m_1, \dots, m_\gamma))$, then for $\gamma \geq 3$ and $m_\gamma \geq 3$*

$$\text{depth}(S/I) \geq 2\gamma + m_\gamma - 5.$$

Proof. Let $I = I(P_p(\gamma; m_1, \dots, m_\gamma))$ and suppose e be the end vertex of P_{m_γ} then for $\gamma = 3$ consider a short exact sequence

$$0 \longrightarrow S/(I:e) \longrightarrow S/I \longrightarrow S/(I,e) \longrightarrow 0$$

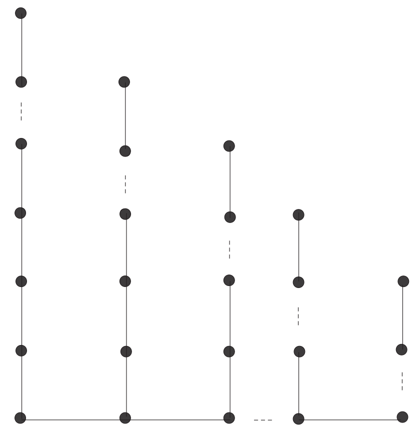

Figure 4.3: $P_p(\gamma\,; m_\gamma+\gamma-1, m_\gamma+\gamma-2, \ldots, m_\gamma+1, m_\gamma)$

Now $(I : e) = (\Delta, N(e))$, here Δ be the edge ideal associated to contraction minor obtained by removing $N(e)$. So, $(I : e) = (I(P_{m_\gamma+2}), I(P_{m_\gamma}), I(P_{m_\gamma-2}), N(e))$. So by Lemma 2.2.8

$$\operatorname{depth}(S/(I:e)) = \operatorname{depth}(S'/(I(P_{m_\gamma+2}), I(P_{m_\gamma}), I(P_{m_\gamma-2}))) + 1,$$

where $S' = S\backslash(N(e)\cup\{e\})$. Then by Lemma 3.1.2

$$\begin{aligned}\operatorname{depth}(S/(I:e)) &= \lceil\frac{m_\gamma+2}{3}\rceil + \lceil\frac{m_\gamma}{3}\rceil + \lceil\frac{m_\gamma-2}{3}\rceil + 1\\ &\geq m_\gamma+1.\end{aligned}$$

Furthermore,

$$\begin{aligned}(I,e) &= (\Delta', e)\\ &= (I(P_{2m_\gamma+3}), I(P_{m_\gamma-1}), e),\end{aligned}$$

and Δ' be the edge ideal corresponding to the deletion minor obtained by removing e and all the edges incident on it. Now by Lemma 3.1.2

$$\begin{aligned}\operatorname{depth}(S/(I,e)) &= \operatorname{depth}(S''/(I(P_{2m_\gamma+3}), I(P_{m_\gamma-1}))\\ &= \lceil\frac{2m_\gamma+3}{3}\rceil + \lceil\frac{m_\gamma-1}{3}\rceil\\ &\geq m_\gamma+1,\end{aligned}$$

where $S'' = S\backslash\{e\}$. Hence Depth lemma implies, $\operatorname{depth}(S/I) \geq m_\gamma + 1$ for $\gamma = 3$. Assume $\gamma \geq 4$. Then

$$(I : e) = (I(P_p(\gamma - 2; m_\gamma + \gamma - 1, \dots, m_\gamma + 2)), I(P_{m_\gamma}), I(P_{m_\gamma - 2}), N(e))$$

and from Lemma 2.2.8

$$\operatorname{depth}(S/(I : e)) = \operatorname{depth}(S'/(I(P_p(\gamma-2; m_\gamma+\gamma-1, \dots, m_\gamma+2)), I(P_{m_\gamma}), I(P_{m_\gamma-2})))+1,$$

where $S' = S\backslash(N(e) \cup \{e\})$. Then by induction on n and Lemma 3.1.2

$$\begin{aligned} \operatorname{depth}(S/(I : e)) &= 2(\gamma - 2) + (m_\gamma + 2) - 5 + \lceil\frac{m_\gamma}{3}\rceil + \lceil\frac{m_\gamma - 2}{3}\rceil + 1 \\ &= 2\gamma + m_\gamma - 6 + \lceil\frac{m_\gamma}{3}\rceil + \lceil\frac{m_\gamma - 2}{3}\rceil \\ &\geq 2\gamma + m_\gamma - 5. \end{aligned}$$

Also

$$\begin{aligned} (I, e) &= (\Delta', e) \\ &= (I(P_p(\gamma - 1; m_\gamma + \gamma - 1, \dots, m_\gamma + 1)), I(P_{m_\gamma - 1}), y) \end{aligned}$$

where Δ' be the edge ideal corresponding to the deletion minor obtained by removing e and all the edges incident on it. Now by using induction and Lemma 3.1.2

$$\begin{aligned} \operatorname{depth}(S/(I, e)) &= \operatorname{depth}(S''/(I(P_p(\gamma - 1; m_\gamma + \gamma - 1, \dots, m_\gamma + 1)), I(P_{m_\gamma - 1}))) \\ &= 2(\gamma - 1) + (m_\gamma + 1) - 5 + \lceil\frac{m_\gamma - 1}{3}\rceil \\ &= 2\gamma + m_\gamma - 6 + (m_\gamma + 1) - 5 + \lceil\frac{m_\gamma - 1}{3}\rceil \\ &\geq 2\gamma + m_\gamma - 5, \end{aligned}$$

where $S'' = S\backslash\{e\}$. Hence by Depth Lemma

$$\operatorname{depth}(S/I) \geq 2\gamma + m_\gamma - 5.$$

□

Moreover, on the same steps of above proof by using Proposition 2.2.6, Lemma 2.2.8, Lemma 2.2.11 and Lemma 3.1.2 the following result holds.

Proposition 4.3.3. *Let $I = I(P_p(\gamma; m_1, \ldots, m_\gamma))$, then for $\gamma \geq 3$ and $m_\gamma \geq 3$*

$$\text{sdepth}(S/I) \geq 2\gamma + m_\gamma - 5.$$

4.4 Paths Attached to a Cycle with a Gradual Decrease in Length

Definition 4.4.1. Consider a graph $C_p(\gamma; m_1, \ldots, m_\gamma)$ in which the end vertex of each path P_{m_i} (having order $m_i = m_\gamma + \gamma - i$) is attached through an edge to the end vertex of path $P_{m_{i+1}}$ (having order $m_{i+1} = m_\gamma + \gamma - i - 1$), where $1 \leq i \leq \gamma - 1$ and m_γ is the order of P_{m_γ}. Also there is an edge between the end vertex of P_{m_1} and P_{m_γ}.

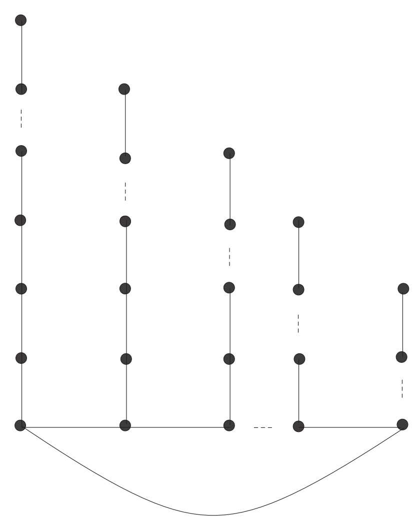

Figure 4.4: $C_p(\ \gamma\,; m_\gamma + \gamma - 1, \ldots, m_\gamma + 1, m_\gamma)$

Example 4.4.2. Let $I = I(C_p(\gamma; m_1, \ldots, m_\gamma))$. Then

$$\text{depth}(S/I) \geq \begin{cases} m_\gamma + 1, & \text{when } \gamma = 3; \\ \lceil \frac{4m_\gamma + 5}{3} \rceil, & \text{when } \gamma = 4; \\ \lceil \frac{4m_\gamma + 11}{3} \rceil, & \text{when } \gamma = 5. \end{cases}$$

Proof. Let e be the end vertex of path P_{m_γ} which is adjacent to the end vertex of $P_{m_{\gamma-1}}$. Consider a short exact sequence

$$0 \longrightarrow S/(I : e) \longrightarrow S/I \longrightarrow S/(I, e) \longrightarrow 0$$

Then consider the following cases.

Case 1: When $\gamma = 3$.

Then $(I : e) = (I(P_{m_\gamma-2}), I(P_{m_\gamma}), I(P_{m_\gamma+1}), N(e))$. Therefore, by Lemma 2.2.8 and Lemma 3.1.2

$$\begin{aligned}\operatorname{depth}(S/(I:e)) &= \operatorname{depth}(S'/(I(P_{m_\gamma-2})+1, I(P_{m_\gamma}), I(P_{m_\gamma+1})))\\ &= 1+\lceil\frac{m_\gamma-2}{3}\rceil+\lceil\frac{m_\gamma}{3}\rceil+\lceil\frac{m_\gamma+1}{3}\rceil\\ &\geq m_\gamma+1,\end{aligned}$$

where $S' = S\backslash N(e)\cup\{e\}$. Next $(I, e) = (e, I(P_{m_\gamma-1}), I(P_{2m_\gamma+3}))$. So, by Lemma 3.1.2

$$\begin{aligned}\operatorname{depth}(S/(I,e)) &= \operatorname{depth}(S^*/(I(P_{m_\gamma-1}), I(P_{2m_\gamma+3})))\\ &= \lceil\frac{m_\gamma-1}{3}\rceil+\lceil\frac{2m_\gamma+3}{3}\rceil\\ &\geq m_\gamma+1,\end{aligned}$$

where $S^* = S\backslash\{e\}$. Hence, by Depth Lemma, $\operatorname{depth}(S/I) \geq m_\gamma + 1$ for $\gamma = 3$.

Case 2: When $\gamma = 4$.

Now $(I : e) = (I(P_{m_\gamma-2}), I(P_{m_\gamma}), I(P_{m_\gamma+2}), N(e))$. Therefore, by Lemma 2.2.8 and Lemma 3.1.2

$$\begin{aligned}\operatorname{depth}(S/(I:e)) &= \operatorname{depth}(S'/(I(P_{m_\gamma-2}), I(P_{m_\gamma}), I(P_{m_\gamma+2})))+1\\ &= 1+\lceil\frac{m_\gamma-2}{3}\rceil+\lceil\frac{m_\gamma}{3}\rceil+\lceil\frac{m_\gamma+2}{3}\rceil\\ &\geq \lceil\frac{4m_\gamma+5}{3}\rceil,\end{aligned}$$

where $S' = S\backslash N(e)\cup\{e\}$. Next $(I, e) = (e, I(P_{m_\gamma-1}), I(P_p(3; m_\gamma+3, m_\gamma+2, m_\gamma+1)))$. So, by Lemma 3.1.2 and Theorem 4.3.2

$$\begin{aligned}\operatorname{depth}(S/(I,e)) &= \operatorname{depth}(S^*/(I(P_{m_\gamma-1}), I(P_p(3; m_\gamma+3, m_\gamma+2, m_\gamma+1))))\\ &= \lceil\frac{m_\gamma-1}{3}\rceil+2(3)+(m_\gamma+1)-5\\ &= \lceil\frac{m_\gamma-1}{3}\rceil+m_\gamma+2\\ &\geq \lceil\frac{4m_\gamma+5}{3}\rceil,\end{aligned}$$

where $S^* = S\backslash\{e\}$. Hence by Depth Lemma, $\text{depth}(S/I) \geq \lceil \frac{4m_\gamma+5}{3} \rceil$ for $\gamma = 4$.

Case 3: When $\gamma \geq 5$.

Then $(I : e) = (I(P_{m_\gamma - 2}), I(P_{m_\gamma}), I(P_{m_\gamma+3}), I(P_{2m_\gamma+5}), N(e))$. Therefore, by Lemma 2.2.8 and Lemma 3.1.2

$$\begin{aligned}
\text{depth}(S/(I:e)) &= \text{depth}(S'/(I(P_{m_\gamma-2}), I(P_{m_\gamma}), I(P_{m_\gamma+3}), I(P_{2m_\gamma+5}))) + 1 \\
&= 1 + \lceil \frac{m_\gamma - 2}{3} \rceil + \lceil \frac{m_\gamma}{3} \rceil + \lceil \frac{m_\gamma + 3}{3} \rceil + \lceil \frac{2m_\gamma + 5}{3} \rceil \\
&\geq \lceil \frac{4m_\gamma + 11}{3} \rceil,
\end{aligned}$$

where $S' = S\backslash N(e) \cup \{e\}$. Next $(I, e) = (e, I(P_{m_\gamma-1}), I(P_p(4; m_\gamma + 4, \ldots, m_\gamma + 1)))$. So, by Lemma 3.1.2 and Theorem 4.3.2

$$\begin{aligned}
\text{depth}(S/(I,e)) &= \text{depth}(S^*/(I(P_{m_\gamma-1}), I(P_p(4; m_\gamma + 4, \ldots, m_\gamma + 1)))) \\
&= \lceil \frac{m_\gamma - 1}{3} \rceil + 2(4) + (m_\gamma + 1) - 5 \\
&= \lceil \frac{m_\gamma - 1}{3} \rceil + m_\gamma + 4 \\
&\geq \lceil \frac{4m_\gamma + 11}{3} \rceil,
\end{aligned}$$

where $S^* = S\backslash\{e\}$. Hence by Depth Lemma, $\text{depth}(S/I) \geq \lceil \frac{4m_\gamma+11}{3} \rceil$ for $\gamma = 5$. □

Theorem 4.4.3. *Let $I = I(G)$ for $G = C_p(\gamma; m_1, \ldots, m_\gamma)$, then for $m_\gamma \geq 3$*

$$\text{depth}(S/I) \geq \begin{cases} m_\gamma + 1, & \textit{when } \gamma = 3; \\ \lceil \frac{4m_\gamma + 6\gamma - 19}{3} \rceil, & \textit{when } \gamma \geq 3. \end{cases}$$

Proof. For $\gamma \leq 5$, the result follows from Example 4.4.2. Assume $\gamma \geq 6$. Let e be the end vertex of path P_{m_γ} which is adjacent to the end vertex of $P_{m_{\gamma-1}}$. Consider a short exact sequence

$$0 \longrightarrow S/(I:e) \longrightarrow S/I \longrightarrow S/(I,e) \longrightarrow 0$$

Note that

$$(I : e) = (N(e), W)$$

where $W = (I(P_{m_\gamma-2}), I(P_{m_\gamma}), I(P_{m_\gamma+\gamma-2}), I(P_p(\gamma - 3; m_\gamma + \gamma - 4, m_\gamma + \gamma - 3, \ldots, m_\gamma + 2)))$ and the set of neighbors of e in G be $N(e)$. Then by Lemma 2.2.8, Lemma 3.1.2

and Theorem 4.3.2

$$\begin{aligned}\text{depth}(S/(I:e)) &= \text{depth}(S^*[e]/W)\\ &= 1+\text{depth}(S^*/W)\\ &\geq 1+\lceil\frac{m_\gamma-2}{3}\rceil+\lceil\frac{m_\gamma}{3}\rceil+\lceil\frac{m_\gamma+\gamma-2}{3}\rceil+2(\gamma-3)+(m_\gamma+2)-5\\ &= \lceil\frac{m_\gamma-2}{3}\rceil+\lceil\frac{m_\gamma}{3}\rceil+\lceil\frac{m_\gamma+\gamma-2}{3}\rceil+2\gamma+m_\gamma-8\\ &\geq \lceil\frac{4m_\gamma+6\gamma-19}{3}\rceil,\end{aligned}$$

where $S^*=S\backslash N(e)\cup\{e\}$. Further

$$(I,e)=(e,H)$$

where $H=(I(P_{m_\gamma-1}),I(P_p(n-1;m_\gamma+\gamma-1,m_\gamma+\gamma-2,\dots,m_\gamma+1)))$. Then

$$\begin{aligned}\text{depth}(S/(I,e)) &= \text{depth}(S'/H)\\ &\geq \lceil\frac{m_\gamma-1}{3}\rceil+2(\gamma-1)+(m_\gamma+1)-5\\ &= \lceil\frac{m_\gamma-1}{3}\rceil+2\gamma+m_\gamma-6\\ &\geq \lceil\frac{4m_\gamma+6\gamma-19}{3}\rceil,\end{aligned}$$

where $S'=S\backslash\{e\}$. Hence, the Depth Lemma implies the result. □

Proposition 4.4.4. *Let $G=C_p(\gamma;m_1,\dots,m_\gamma)$ and let $I=I(G)$. Then for $m_\gamma\geq 3$*

$$\text{sdepth}(S/I)\geq\begin{cases} m_\gamma+1, & \text{when } \gamma=3;\\ \lceil\frac{4m_\gamma+6\gamma-19}{3}\rceil, & \text{when } \gamma\geq 3.\end{cases}$$

Proof. The proof is same as that of Theorem 4.4.3. Proposition 2.2.6, Lemma 2.2.11, Lemma 3.1.2 and Proposition 4.3.3 implies the result. □

Chapter 5

Depth and Stanley depth of Power of Certain Trees

A caterpillar tree is a graph in which all the vertices are within distance 1 of a central path. A lobster tree is a tree having the property that the removal of leaf nodes leaves a caterpillar graph. This chapter covers the lower bounds for depth and Stanley depth of a class of the lobster trees.

5.1 A class of Lobster Trees

Definition 5.1.1. Consider a star graph $\mathcal{S}_r$ and let $l_i - 1$ number of vertices are adjacent to the ith leaf of $\mathcal{S}_r$ with at most one $l_i = 1$, where $1 \leq i \leq r-1$, such a graph is denoted by $S(r; l_1, \ldots, l_{r-1})$.

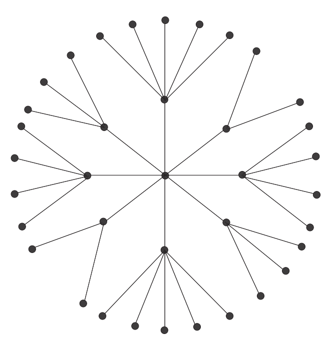

Figure 5.1: $S(5; 3, 6, 4, 5)$

Definition 5.1.2. Let $\mathcal{S}_k$ be a star. Then $\mathcal{L}(\mathcal{S}_k)$ denotes the set consisting of leaves of $\mathcal{S}_k$.

Lemma 5.1.3. *[10, Lemma 4.2] Consider a polynomial ring S over the field and an ideal I of S. Suppose $\sigma \in S$ be a monomial and let $\{y_1, y_2, \ldots, y_s\}$ be variables such that $\forall$ j, y_j does not divide σ. Assume $z_1, z_2 \in \mathbb{Z}^+\backslash\{0\}$. If* $\operatorname{depth} S_{j-1}/(I^k_{j-1} : \sigma y_j) \geq z_1$ $\forall$ $j \geq 1$ *and* $\operatorname{depth} S_s/(I^k_s : \sigma) \geq z_2$*, then* $\operatorname{depth} S_j/(I^k_j : \sigma) \geq \min\{z_1\,, z_2\}$ *for each* $i \geq 0$*. In particular,* $\operatorname{depth} S/(I^k : \sigma) \geq \min\{z_1\,, z_2\}$.

Proposition 5.1.4. *Let $I = I(S(3; l_1, l_2))$, then for $k \geq 2$*

$$\operatorname{depth}(S/I^k) \geq 1.$$

Proof. Let e be the central vertex of the graph and let w be the neighbor of e with $l-1$ leaves adjacent to it. Let $N(w) = \{y_1, y_2, \ldots, y_j\}$ and let $l = \min\{l_1, l_2\}$. Without loss of generality, let $l = l_2 \geq 2$. For $k = 3$ the result is true. Assume $k \geq 4$. For a short exact sequence

$$0 \longrightarrow\ S/(I^k : w) \longrightarrow\ S/I^k \longrightarrow\ S/(I^k, w) \longrightarrow\ 0$$

$(I^k, w) = (I^k(\mathcal{S}_{l_1+1}), w)$, and

$$\begin{aligned} \operatorname{depth}(S/(I^k, w)) &= \operatorname{depth}(S'/I^k(\mathcal{S}_{l_1+1})) + |\mathcal{L}(\mathcal{S}_l)| \\ &\geq 1 + (l-1) \\ &= l, \end{aligned}$$

where $S' = S\backslash V(\mathcal{S}_l)$. Now for $\operatorname{depth} S/(I^k : w)$, consider a short exact sequence

$$0 \longrightarrow\ S/(I^k : wy_1) \longrightarrow\ S/(I^k : w) \longrightarrow\ S/((I^k : w), y_1) \longrightarrow\ 0$$

Then by Lemma 3.1.4 and induction on k

$$\operatorname{depth}(S/I^k : wy_1) = \operatorname{depth}(S/I^{k-1}) \geq 1.$$

Now for depth of $S/((I^k : w), y_1)$ consider another short exact sequence

$$0 \longrightarrow\ S_1/(I^k_1 : wy_2) \longrightarrow\ S_1/(I^k_1 : w) \longrightarrow\ S_1/((I^k_1 : w), y_2) \longrightarrow\ 0$$

Again by Lemma 3.1.4 and induction on k

$$\operatorname{depth}(S_1/I_1^k : wy_2) = \operatorname{depth}(S_1/I_1^{k-1}) \geq 1.$$

$$\vdots$$

Proceeding in the same manner, for the sequence

$$0 \longrightarrow S_{j-1}/(I_{j-1}^k : wy_l) \longrightarrow S_{j-1}/(I_{j-1}^k : w) \longrightarrow S_{j-1}/((I_{j-1}^k : w), y_1) \longrightarrow 0$$

$$\operatorname{depth}(S_{j-1}/I_{j-1}^k : wy_l) = \operatorname{depth}(S_{j-1}/I_{j-1}^{k-1}) \geq 1.$$

and

$$\operatorname{depth}(S_j/I_j^k : w) = \operatorname{depth}(S_j/I^k(\mathcal{S}_{l_1})) + 1 \geq 2$$

So, Lemma 5.1.3 implies

$$\operatorname{depth}(S/I^k : w) \geq 1,$$

and by Depth Lemma

$$\operatorname{depth}(S/I^k) \geq 1.$$

When $j = 1$, w has a unique neighbor $y_1 = e$. Therefore, $(I^k, w) = (I^k(\mathcal{S}_{l_1+1}), w)$ and

$$\begin{aligned} \operatorname{depth}(S/(I^k, w)) &= \operatorname{depth}(S''/(I^k(\mathcal{S}_{l_1+1}))) \\ &\geq 1, \end{aligned}$$

where $S'' = S \backslash \{w\}$. Next by induction on k

$$\operatorname{depth}(S/(I^k : wy_1)) = \operatorname{depth}(S/(I^{k-1})) \geq 1,$$

and

$$\operatorname{depth}(S_1/I_1^k : w) \geq 1.$$

So, by Lemma 5.1.3

$$\operatorname{depth}(S/I^k : w) \geq 1.$$

Hence $\operatorname{depth}(S/I^k) \geq 1$. $\square$

Theorem 5.1.5. *Let $I = I(S(r; l_1, l_2, \ldots, l_{r-1}))$ and consider $l = \min\{l_1, l_2, \ldots, l_{r-1}\}$, then for $r \geq 2$, and $k \geq 1$*

$$\operatorname{depth}(S/I^k) \geq \begin{cases} \max\{1, r-k-1\}, & \textit{when } l = 1; \\ \max\{1, r-k\}, & \textit{otherwise.} \end{cases}$$

Proof. The proof is done by induction on k and r. For $r \geq 2$ and $k = 1$, let e be the center of the graph. For a short exact sequence

$$0 \longrightarrow S/(I : e) \longrightarrow S/I \longrightarrow S/(I, e) \longrightarrow 0$$

$$(I : e) = (N(e))$$

Thus

$$\begin{aligned} \operatorname{depth}(S/(I : e)) &= (l_1 - 1) + (l_2 - 1) + \cdots + (l_{r-1} - 1) + 1 \\ &= l_1 + l_2 + \cdots + l_{r-1} - r + 2 \\ &\geq r - 1, \end{aligned}$$

and

$$(I, e) = (I(\mathcal{S}_{l_1}), \ldots, I(\mathcal{S}_{l_{r-1}}), e)$$

Consequently, from Proposition 4.1.2

$$\begin{aligned} \operatorname{depth}(S/(I, e)) &= 1 + 1 + \cdots + 1 \\ &= r - 1. \end{aligned}$$

So by applying Depth Lemma,

$$\operatorname{depth}(S/I) \geq r - 1.$$

Now for $r = 2$ and for all k, since given graph is a bipartite, so the result follows from Lemma 3.0.15. Assume $r \geq 3$ and $k \geq 2$. Let w be the neighbor of e, with $l - 1$ leaves adjacent to it. Let $N(w) = \{y_1, y_2, \ldots, y_j\}$. Without loss of generality, let $l = l_{r-1} \geq 2$. For a short exact sequence

$$0 \longrightarrow S/(I^k : w) \longrightarrow S/I^k \longrightarrow S/(I^k, w) \longrightarrow 0$$

$(I^k, w) = (I^k(S(r-1; l_1, \dots, l_{r-2})), w)$, therefore

$$\begin{aligned}\operatorname{depth}(S/(I^k, w)) &= \operatorname{depth}(S'/I^k(S(r-1; l_1, \dots, l_{r-2}))) + |\mathcal{L}(\mathcal{S}_l)| \\ &\geq (r-1-k) + (l-1) \\ &= l + r - k - 2,\end{aligned}$$

where $S' = S \backslash V(\mathcal{S}_l)$. Now for depth $S/(I^k : w)$ consider a short exact sequence

$$0 \longrightarrow S/(I^k : wy_1) \longrightarrow S/(I^k : w) \longrightarrow S/((I^k : w), y_1) \longrightarrow 0$$

Then by Lemma 3.1.4 and induction on k

$$\operatorname{depth}(S/I^k : wy_1) = \operatorname{depth}(S/I^{k-1}) \geq r - (k-1) = r - k + 1.$$

Now for depth of $S/((I^k : w), y_1)$ consider another short exact sequence

$$0 \longrightarrow S_1/(I_1^k : wy_2) \longrightarrow S_1/(I_1^k : w) \longrightarrow S_1/((I_1^k : w), y_2) \longrightarrow 0$$

Again by Lemma 3.1.4 and induction on k

$$\operatorname{depth}(S_1/I_1^k : wy_2) = \operatorname{depth}(S_1/I_1^{k-1}) \geq r - (k-1) = r - k + 1.$$

$$\vdots$$

Proceeding in the same manner, for the sequence

$$0 \longrightarrow S_{j-1}/(I_{j-1}^k : wy_l) \longrightarrow S_{j-1}/(I_{j-1}^k : w) \longrightarrow S_{j-1}/((I_{j-1}^k : w), y_1) \longrightarrow 0$$

$$\operatorname{depth}(S_{j-1}/I_{j-1}^k : wy_l) = \operatorname{depth}(S_{j-1}/I_{j-1}^{k-1}) \geq r - (k-1) - 1 = r - k.$$

and $(I_l^k : w) = H^k$, where H is a forest consisting of $r-2$ components and each component is a star. Then by Theorem 3.2.4

$$\begin{aligned}\operatorname{depth}(S_j/I_j^k : w) &= \operatorname{depth}(S_j/H^k) + 1 \\ &\geq \max\{\lceil\frac{2-t+2}{3}\rceil + (r-2) - 1, r-2\} \\ &= (r-2) + 1 \\ &= r - 1 \\ &\geq r - k.\end{aligned}$$

So, Lemma 5.1.3 implies

$$\operatorname{depth}(S/I^k : w) \geq r - k,$$

and by Depth Lemma

$$\operatorname{depth}(S/I^k) \geq r - k.$$

When $j = 1$, w has a unique neighbor $y_1 = e$. Therefore,

$$(I^k, w) = (I^k(S(r-1; l_1, \ldots, l_{r-2})), w),$$

and

$$\begin{aligned} \operatorname{depth}(S/(I^k, w)) &= \operatorname{depth}(S''/I^k(S(r-1; l_1, \ldots, l_{r-2}))) \\ &\geq (r-1) - k, \end{aligned}$$

where $S'' = S \backslash \{w\}$. Next by induction on k

$$\begin{aligned} \operatorname{depth}(S/(I^k : wy_1)) &= \operatorname{depth}(S/(I^{k-1})) \\ &\geq r - (k-1) - 1 \\ &= r - k, \end{aligned}$$

and

$$\begin{aligned} \operatorname{depth}(S_1/I_1^k : w) &\geq (r-2) + 1 \\ &= r - 1 \\ &> r - k - 1. \end{aligned}$$

So, by Lemma 5.1.3

$$\operatorname{depth}(S/I^k : w) \geq r - k - 1,$$

and hence the result holds by Depth Lemma. □

On the same lines, the following result can be proved by using Lemma 2.2.11, Lemma 3.1.2 and Corollary 3.3.8.

Proposition 5.1.6. *Let $I = I(S(r; l_1, l_2, \ldots, l_{r-1}))$ be the edge ideal, for $r \geq 2$, and $k \geq 1$*

$$\operatorname{sdepth}(S/I^k) \geq \begin{cases} \max\{1, r-k-1\}, & \textit{when } l = 1; \\ \max\{1, r-k\}, & \textit{otherwise.} \end{cases}$$

5.2 Generalized Banana Tree Graph

Definition 5.2.1. An (γ, z)-banana tree is a graph obtained by connecting one leaf of each of γ copies of an z-star graph with a single vertex that is distinct from all the stars. It is denoted by $B(\gamma, z)$.

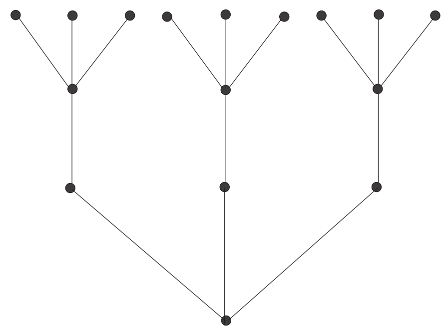

Figure 5.2: $B(3, 5)$

Definition 5.2.2. A generalized banana tree graph $B(\gamma; z_1, \ldots, z_\gamma)$ is a graph in which γ number of z_i-star graphs are connected to a single vertex that is distinct from all the stars, where at most one $z_i = 1$ or at most one $z_i = 2$ and remaining all $z_i \geq 3$. Banana tree graphs are the subclass of these graphs, i.e., for all $z_i = z$ for some $z \geq 2$, $B(\gamma; z_1, \ldots, z_\gamma) = B(\gamma, z)$.

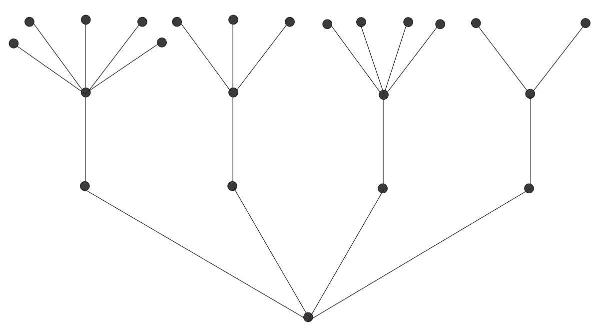

Figure 5.3: Banana tree $B(4; 7, 5, 6, 4)$

Theorem 5.2.3. *Let G be a $(r;\gamma_1,\dots,\gamma_r)$-star graph, where γ_j is the length of P_{γ_j} and assume $I = I(G)$. For $k \geq 1$*

$$\operatorname{depth}(S/I^k) \geq \max\{\sum_{j=1}^{r}\lceil\frac{\gamma_j-1}{3}\rceil, 1\}.$$

Proposition 5.2.4. *Let $G = B(\gamma; z_1,\dots,z_\gamma)$, $q = \min\{z_i : 1 \leq i \leq \gamma\}$ and $I = I(G)$. Then for $\gamma \geq 2$*

$$\operatorname{depth}(S/I) \geq \gamma.$$

Proof. Let v be the root vertex of G and e be the central vertex of star $\mathcal{S}_q$ in G. Without loss of generality, let $q = z_\gamma$. Consider a short exact sequence

$$0 \longrightarrow S/(I:v) \longrightarrow S/I \longrightarrow S/(I,v) \longrightarrow 0$$

Case 1: When $q = 1$, then $(I:v) = (N(v), I(\mathcal{S}_{z_1-1}),\dots,I(\mathcal{S}_{z_{\gamma-1}-1}))$, therefore

$$\operatorname{depth}(S/(I:v)) = 1 + \operatorname{depth}(S'/(I(\mathcal{S}_{z_1-1}),\dots,I(\mathcal{S}_{z_{\gamma-1}-1}))) = 1 + (\gamma-1) = \gamma.$$

Also $(I,v) = (v, I(\mathcal{S}_{z_1}),\dots,I(\mathcal{S}_{z_{n\gamma-1}}))$ so

$$\operatorname{depth}(S/(I,v)) = \operatorname{depth}(S'[e]/(I(\mathcal{S}_{z_1}),\dots,I(\mathcal{S}_{z_{\gamma-1}}))) = 1 + (\gamma-1) = \gamma,$$

where $S' = S\backslash\{v\}$. By Depth Lemma $\operatorname{depth}(S/I) \geq \gamma$.

Case 2: When $q = 2$, then $(I:v) = (N(v), I(\mathcal{S}_{z_1-1}),\dots,I(\mathcal{S}_{z_{\gamma-1}-1}))$ and

$$\operatorname{depth}(S/(I:v)) = 1+\operatorname{depth}(S'[e]/(I(\mathcal{S}_{z_1-1}),\dots,I(\mathcal{S}_{z_{\gamma-1}-1}))) = 1+1+(\gamma-1) = \gamma+1.$$

Moreover, $(I,v) = (v, I(\mathcal{S}_{z_1}),\dots,I(\mathcal{S}_{z_{\gamma-1}}), I(\mathcal{S}_q))$ so

$$\operatorname{depth}(S/(I,v)) = \operatorname{depth}(S'[e]/(I(\mathcal{S}_{z_1}),\dots,I(\mathcal{S}_{z_{\gamma-1}}), I(\mathcal{S}_q))) = \gamma.$$

By Depth Lemma $\operatorname{depth}(S/I) \geq \gamma$.

Case 3: For $q \geq 3$, notice that $(I:v) = (N(v), I(\mathcal{S}_{z_1-1}),\dots,I(\mathcal{S}_{z_{\gamma-1}-1}), I(\mathcal{S}_{q-1}))$, where $N(v)$ be the neighbors of v in G. Then by using Proposition 4.1.2,

$$\begin{aligned}\operatorname{depth}(S/(I:v)) &= \operatorname{depth}(S^*/(I(\mathcal{S}_{z_1-1}),\dots,I(\mathcal{S}_{z_{\gamma-1}-1}), I(\mathcal{S}_{q-1}))) + 1\\ &= (1+\cdots+1)+1 = \gamma+1,\end{aligned}$$

where $S^* = S\backslash N(v) \cup \{v\}$. Next $(I, v) = (v, I(\mathcal{S}_{z_1}), \dots, I(\mathcal{S}_{z_{\gamma-1}}), I(\mathcal{S}_q))$, so by applying Proposition 4.1.2

$$\begin{aligned} \operatorname{depth}(S/(I, v)) &= \operatorname{depth}(S/(v, , I(\mathcal{S}_{z_1}), \dots, I(\mathcal{S}_{z_{\gamma-1}}), I(\mathcal{S}_q))) \\ &= 1 + \cdots + 1 = \gamma. \end{aligned}$$

Consequently, by Depth Lemma

$$\operatorname{depth}(S/I) \geq \gamma. \tag{5.1}$$

□

Proposition 5.2.5. *For $\gamma = 2$, let $G = B(2; z_1, z_2)$, $q = \min\{z_1, z_2\}$ and $I = I(G)$. Then for $k \geq 2$*

$$\operatorname{depth}(S/I^k) \geq 1.$$

Proof. Let e be the central vertex of last star in G and let f be the neighbor of e which is adjacent to v. The proof is done by induction on k. Suppose $k = 2$. For $q = 1$, the result follows from Theorem 5.2.3. For $q \geq 2$, consider a short exact sequence

$$0 \longrightarrow S/(I^2 : e) \longrightarrow S/I^2 \longrightarrow S/(I^2, e) \longrightarrow 0 \tag{5.2}$$

Then $(I^2, e) = (W^2, e) = W^2$, here W is the edge ideal of deletion minor obtained by eliminating e and variables in $N' = N(e)\backslash\{f\}$.

$$\begin{aligned} \operatorname{depth}(S/(I^2, e)) &= \operatorname{depth}(S'/W^2) + |N'| \\ &= \operatorname{depth}(S'/W^2) + q - 2, \end{aligned}$$

with $S' = S\backslash N' \cup \{e\}$. Then by Theorem 5.2.3, $\operatorname{depth}(S'/W^2) \geq 1$. Therefore, $\operatorname{depth}(S/(I^2, e)) \geq q - 1$. Now let $N(e) = \{y_1, \dots, y_l = f\}$ and for depth of $S/(I^2 : e)$, consider a short exact sequence

$$0 \longrightarrow S/(I^2 : ey_1) \longrightarrow S/(I^2 : e) \longrightarrow S/((I^2 : e), y_1) \longrightarrow 0$$

By Lemma 3.1.4 and Eq. (5.1)

$$\operatorname{depth}(S/(I^2 : ey_1)) = \operatorname{depth}(S/I) \geq 2.$$

For the depth of $S/((I^2 : e), y_1)$, consider another short exact sequence

$$0 \longrightarrow S_1/(I_1^2 : ey_2) \longrightarrow S_1/(I_1^2 : e) \longrightarrow S_1/((I_1^2 : e), y_2) \longrightarrow 0$$

By Lemma 3.1.4 and Eq. (5.1)

$$\operatorname{depth}(S_1/(I_1^2 : ey_2)) = \operatorname{depth}(S_1/I_1) \geq 2.$$

$$\vdots$$

Proceeding in the same manner, for the sequence:

$$0 \longrightarrow S_{l-1}/(I_{l-1}^2 : ey_l) \longrightarrow S_{l-1}/(I_{l-1}^2 : e) \longrightarrow S_{l-1}/((I_{l-1}^2 : e), y_1) \longrightarrow 0$$

Then by Lemma 3.1.4 and Theorem 5.2.3

$$\operatorname{depth}(S_{l-1}/(I_{l-1}^2 : ey_l)) = \operatorname{depth}(S_{l-1}/(I_{l-1}) \geq 1,$$

and by Lemma 2.2.8 and Theorem 5.2.3

$$\operatorname{depth}(S_l/I_l^2 : e) = \operatorname{depth}(S_l'/T^2) + 1 \geq 2,$$

where T is the edge ideal corresponding to contraction minor, obtained by removing $\{e\} \cup N(e)$ from G and $S_l' = S_l \backslash \{e\}$. Therefore from Lemma 5.1.3, $\operatorname{depth}(S/I^2 : e) \geq 1$ and from Equation 5.2, $\operatorname{depth}(S/I^2) \geq 1$. Now assume $k \geq 3$. For $q = 1$, the result follows from Theorem 5.2.3. When $q \geq 2$, assume a short exact sequence

$$0 \longrightarrow S/(I^k : e) \longrightarrow S/(I^k) \longrightarrow S/((I^k, e)) \longrightarrow 0$$

Then $\operatorname{depth}(S/I^k, e) = \operatorname{depth}(S'/W^k) + |N'| = \operatorname{depth}(S'/W^k) + q - 2$. Since by Theorem 5.2.3 $\operatorname{depth}(S'/W^k) \geq 1$, it implies

$$\operatorname{depth}(S/I^k, e) \geq q - 1.$$

For depth of $S/(I^k : e)$, consider a short exact sequence

$$0 \longrightarrow S/(I^k : ey_1) \longrightarrow S/(I^k : e) \longrightarrow S/((I^k : e), y_1) \longrightarrow 0$$

By Lemma 3.1.4 and induction on k

$$\operatorname{depth}(S/(I^k : ey_1)) = \operatorname{depth}(S/I^{k-1}) \geq 1.$$

For the depth of $S/((I^k : e), y_1)$, consider another short exact sequence

$$0 \longrightarrow S_1/(I_1^k : ey_2) \longrightarrow S_1/(I_1^k : e) \longrightarrow S_1/((I_1^k : e), y_2) \longrightarrow 0$$

Then by Lemma 3.1.4 and induction on k

$$\operatorname{depth}(S_1/(I_1^k : ey_2)) = \operatorname{depth}(S_1/I_1^{k-1}) \geq 1.$$

$$\vdots$$

Proceeding in the same manner, for the sequence:

$$0 \longrightarrow S_{l-1}/(I_{l-1}^k : ey_l) \longrightarrow S_{l-1}/(I_{l-1}^k : e) \longrightarrow S_{l-1}/((I_{l-1}^k : e), y_1) \longrightarrow 0$$

Then by Lemma 3.1.4 and Theorem 5.2.3

$$\operatorname{depth}(S_{l-1}/(I_{l-1}^k : ey_l)) = \operatorname{depth}(S_{l-1}/(I_{l-1}^{k-1}) \geq 1.$$

Moreover, Lemma 2.2.8 and Theorem 5.2.3 implies

$$\operatorname{depth}(S_l/I_l^k : e) = \operatorname{depth}(S_l'/T^k) + 1 \geq 2,$$

where T is the edge ideal corresponding to contraction minor, obtained by removing $\{e\} \cup N(e)$ from G and $S_l' = S_l \backslash \{e\}$. From Lemma 5.1.3, $\operatorname{depth}(S/I^k : e) \geq 1$ and by Equation 5.2, $\operatorname{depth}(S/I^k) \geq 1$. □

Theorem 5.2.6. *Let $G = B(\gamma; z_1, \ldots, z_\gamma)$ and let $q = \min\{z_i : 1 \leq i \leq \gamma\}$. Then for $\gamma \geq 2$ and $k \geq 1$*

$$\operatorname{depth}(S/I^k(G)) \geq \begin{cases} \max\{\gamma - k + 1, 1\}, & \text{when } q > 2; \\ \max\{\gamma - k, 1\}, & \text{when } q = 2. \end{cases}$$

Proof. Since G is a bipartite graph so $\operatorname{depth}(S/I^k) \geq 1$ for all k. Therefore, the focus of the proof is to prove the other part of the result. For $k = 1$ and $\gamma \geq 2$, the result follows from Proposition 5.2.4. Suppose $k \geq 2$. If $\gamma = 2$, then Proposition 5.2.5 implies the result. Assume $n \geq 3$. Let v be the root vertex in G and suppose e be the central vertex of last star in G and f be the neighbor of e which is adjacent to v. Without loss of generality, let $q = k_n$.

Case 1: When $q = 1$. Let W be the edge ideal corresponding to graph $B(\gamma; z_1, \ldots, z_{\gamma-1}, 1)$ and let e be the only vertex of star $\mathcal{S}_q$ in the graph. Now for the depth of S'/W^k, consider another short exact sequence

$$0 \longrightarrow S/(W^k : e) \longrightarrow S/W^k \longrightarrow S/(W^k, e) \longrightarrow 0 \tag{5.3}$$

Then by induction on γ

$$\operatorname{depth}(S/((W^t, e))) = \operatorname{depth}(S'/I^k(B(\gamma - 1, z))) \geq (\gamma - 1) - k + 1 = \gamma - k, \tag{5.4}$$

where $S' = S \backslash \{e\}$. Moreover, for the depth of $S'/(W^k : e)$, consider a short exact sequence

$$0 \longrightarrow S/(W^k : ev) \longrightarrow S/(W^k : e) \longrightarrow S/((W^k : e), v) \longrightarrow 0 \tag{5.5}$$

Then by Lemma 3.1.4 and induction on k,

$$\operatorname{depth}(S/(W^k : ev)) = \operatorname{depth}(S/W^{k-1}) \geq \gamma - (k-1) = \gamma - k + 1, \tag{5.6}$$

and by Lemma 2.2.8

$$\operatorname{depth}(S/((W^k : e), v)) = \operatorname{depth}(S'/H^k, v) + 1,$$

where H is the edge ideal corresponding to a forest with $\gamma - 1$ number of components and each component is a z-star graph. Therefore, by Theorem 3.2.4

$$\operatorname{depth}(S/((W^k : e), v)) \geq \max\{\lceil \frac{2-k+2}{3} \rceil + (\gamma-1) - 1, \gamma - 1\} + 1 = \gamma - 1 + 1 = \gamma \tag{5.7}$$

By applying Depth Lemma on exact sequence (5.5) along with using Eqs. (5.6) and (5.7)

$$\operatorname{depth}(S/(W^k : e)) \geq \gamma - k + 1, \tag{5.8}$$

and applying Depth Lemma on exact sequence (5.3) along with using Eqs. (5.4) and (5.8)

$$\operatorname{depth}(S/W^k) \geq \gamma - k. \tag{5.9}$$

Case 2: When $q \geq 2$. assume $I = I(G)$ and a short exact sequence

$$0 \longrightarrow S/(I^k : e) \longrightarrow S/(I^k) \longrightarrow S/(I^k, e) \longrightarrow 0 \tag{5.10}$$

Note that $(I^k, e) = (W^k, e) = W^k$, where W be the edge ideal of deletion minor G' obtained by eliminating e and variables in $N' = N(e)\backslash\{f\}$. Note that $q = 1$ in G'. Therefore

$$\operatorname{depth}(S/I^k, e) = \operatorname{depth}(S^*/W^k) + |N'| = \operatorname{depth}(S^*/W^k) + q - 2, \tag{5.11}$$

where $S^* = S\backslash N' \cup \{e\}$. Then Eq. (5.9) implies

$$\operatorname{depth}(S/(I^k, e)) \geq \gamma - k + q - 2. \tag{5.12}$$

Now let $N(e) = \{y_1, \ldots, y_l = f\}$ and for depth of $S/(I^k : e)$, consider a short exact sequence

$$0 \longrightarrow S/(I^k : ey_1) \longrightarrow S/(I^k : e) \longrightarrow S/((I^k : e), y_1) \longrightarrow 0$$

Then by Lemma 3.1.4 and induction on k

$$\operatorname{depth}(S/(I^k : ey_1)) = \operatorname{depth}(S/(I^{k-1}) \geq \gamma - (k-1) + 1 = \gamma - k + 2.$$

For the depth of $S/((I^k : e), y_1)$, consider another short exact sequence

$$0 \longrightarrow S_1/(I_1^k : ey_2) \longrightarrow S_1/(I_1^k : e) \longrightarrow S_1/((I_1^k : e), y_2) \longrightarrow 0$$

Then by Lemma 3.1.4 and induction on k

$$\operatorname{depth}(S_1/(I_1^k : ey_2)) = \operatorname{depth}(S_1/I_1^{k-1}) \geq \gamma - (k-1) + 1 = \gamma - k + 2.$$

$$\vdots$$

Proceeding in the same manner, for the exact sequence

$$0 \longrightarrow S_{l-1}/(I_{l-1}^k : ey_l) \longrightarrow S_{l-1}/(I_{l-1}^k : e) \longrightarrow S_{l-1}/((I_{l-1}^k : e), y_1) \longrightarrow 0$$

Then by Lemma 3.1.4 and induction on k

$$\operatorname{depth}(S_{l-1}/(I_{l-1}^k : ey_l)) = \operatorname{depth}(S_{l-1}/(I_{l-1}^{k-1}) \geq \gamma - (k-1) = \gamma - k + 1.$$

Moreover, Lemma 2.2.8 and induction on n implies

$$\operatorname{depth}(S_l/I_l^k : e) = \operatorname{depth}(S_l'/I^k(B(\gamma-1, z))) + 1 \geq (\gamma - 1) - k + 1 + 1 = \gamma - k + 1,$$

where $S_l' = S_l \backslash \{e\}$. Moreover, from Lemma 5.1.3

$$\operatorname{depth}(S/I^k : e) \geq \gamma - k + 1. \tag{5.13}$$

Hence applying Depth Lemma on exact sequence (5.10) along with using Eqs. (5.12) and (5.13) implies the result. □

Corollary 5.2.7. *For banana tree graph $B(\gamma, z)$, let $I = I(B(\gamma, z))$. Then for $\gamma \geq 2$, $z \geq 3$ and $k \geq 1$*

$$\operatorname{depth}(S/I^k) \geq \max\{\gamma - k + 1, 1\}.$$

Proposition 5.2.8. *Let $G = B(\gamma; z_1, \ldots, z_\gamma)$ and let $q = \min\{z_i : 1 \leq i \leq \gamma\}$. Then for $\gamma \geq 2$ and $k \geq 1$*

$$\operatorname{sdepth}(S/I^k(G)) \geq \begin{cases} \max\{\gamma - k + 1, 1\}, & \text{when } q > 2; \\ \max\{\gamma - k, 1\}, & \text{when } q = 2. \end{cases}$$

Proof. The proof is similar to proof of Theorem 5.2.6. Lemma 2.2.11, Lemma 3.1.2 and Corollary 3.3.8 implies the result. □

Bibliography

[1] Alipour, A., Tehranian, A. (2017). Depth and Stanley Depth of Edge Ideals of Star Graphs. *International Journal of Applied Mathematics and Statistics, 56*(4), 63-69.

[2] Anwar, I., Popescu, D. (2007). Stanley conjecture in small embedding dimension. *Journal of Algebra, 318*(2), 1027-1031.

[3] Apel, J. (2003). On a conjecture of RP Stanley; part II—quotients modulo monomial ideals. *Journal of Algebraic Combinatorics, 17*(1), 57-74.

[4] Bruns, W., Herzog, H. J. (1998). Cohen-macaulay rings. *Cambridge university press*.

[5] Cimpoeas, M. (2012). Several inequalities regarding Stanley depth. *Romanian Journal of Math. and Computer Science, 2*(1), 28-40.

[6] Cimpoeas, M. (2008). Some remarks on the Stanley's depth for multigraded modules. *arXiv preprint arXiv:0808.2657*.

[7] Cimpoeaş, M. (2008). Stanley depth of complete intersection monomial ideals. *Bulletin mathématique de la Société des Sciences Mathématiques de Roumanie,* 205-211.

[8] Dummit, D. S., Foote, R. M. (2004). *Abstract algebra* (Vol. 3). Hoboken: Wiley.

[9] Duval, A. M., Goeckner, B., Klivans, C. J., Martin, J. L. (2016). A non-partitionable Cohen–Macaulay simplicial complex. *Advances in Mathematics, 299*, 381-395.

[10] Fouli, L., Morey, S. (2015). A lower bound for depths of powers of edge ideals. *Journal of Algebraic Combinatorics, 42*(3), 829-848.

[11] Grayson, D. R., Stillman, M. E. Macaulay 2, a software system for research in algebraic geometry. http://www.math.uiuc.edu/Macaulay2/ Accessed September 20, 2010.

[12] Herzog, J., Hibi, T. (2011). Monomial ideals. In Monomial Ideals (pp. 3-22). *Springer, London.*

[13] Herzog, J., Vladoiu, M., Zheng, X. (2009). How to compute the Stanley depth of a monomial ideal. *Journal of Algebra, 322*(9), 3151-3169.

[14] Morey, S. (2010). Depths of powers of the edge ideal of a tree. *Communications in Algebra, 38*(11), 4042-4055.

[15] Okazaki, R. (2011). A lower bound of Stanley depth of monomial ideals. *Journal of Commutative Algebra, 3*(1), 83-88.

[16] Popescu, A. (2010). Special stanley decompositions. *Bulletin mathématique de la Société des Sciences Mathématiques de Roumanie,* 363-372.

[17] Popescu, D. (2009). An inequality between depth and Stanley depth. *Bulletin mathématique de la Société des Sciences Mathématiques de Roumanie,* 377-382.

[18] Pournaki, M., Seyed Fakhari, S. A., Yassemi, S. (2013). Stanley depth of powers of the edge ideal of a forest. *Proceedings of the American Mathematical Society, 141*(10), 3327-3336.

[19] Rauf, A. (2010). Depth and Stanley depth of multigraded modules. *Communications in Algebra, 38*(2), 773-784.

[20] Rauf, A. (2007). Stanley decompositions, pretty clean filtrations and reductions modulo regular elements. *Bulletin mathématique de la Société des Sciences Mathématiques de Roumanie,* 347-354.

[21] Shen, Y. H. (2009). Stanley depth of complete intersection monomial ideals and upper-discrete partitions. *Journal of Algebra, 321*(4), 1285-1292.

[22] Simis, A., Vasconcelos, W. V., Villarreal, R. H. (1994). On the ideal theory of graphs. *Journal of Algebra, 167*(2), 389-416.

[23] Stanley, R. P. (1982). Linear Diophantine equations and local cohomology. *Inventiones mathematicae, 68*(2), 175-193.

[24] Vasconcelos, W. V. (1994). Arithmetic of blowup algebras (Vol. 195). *Cambridge University Press.*

[25] Villarreal, R. H. (2001). Monomial Algebras, Monographs and Textbooks in Pure and Applied Mathematics, 238. *New York: Marcel Dekker, Inc.*

[26] West, D. B. (1996). Introduction to graph theory (Vol. 2). *Upper Saddle River, NJ: Prentice hall.*