Some inequalities and bounds for Stanley depth

by

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Dedicated to

My loving Parents.

Abstract

In this thesis we have discussed a few results for depth and Stanley depth of ideals with disjoint support. We have also discussed some bounds of Stanley depth of addition and of ideals. For any ideals we have lower bounds of Stanley depth for addition and intersection of ideals. We have also given some equivalent forms of Stanley conjecture for a monomial ideal and quotient of a monomial ideal.

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Introduction

Stanley gave the idea of what we call stanley depth of $\mathbb{Z}^n$-graded module over a commutative polynomial ring S ([18]).He gave the inequality that $\operatorname{depth} M \leq \operatorname{sdepth} M$ but it is still largely open. According to [8] the Stanely depth of a monomial ideal is computed in a finite number of steps by partitioning a finite poset associated to the monomial ideal into intervals. But it is still very difficult to compute the Stanely depth of monomial ideals and even more difficult for modules, so we find it difficult to verify the Stanely conjecture.

The thesis has three chapters. Chapter one discusses some basic definitions and concepts of abstract algebra in detail with examples and some related results.

Chapter two provides the concept of Stanely depth, Stanely decomposition and states Stanely conjecture. Next we recall some fundamental results related to depth and Stanely depth of $\mathbb{Z}^n$ graded S-modules where S is a polynomial ring in m variables over a field K.

Chapter three gives a review of [3] which provides some upper and lower bounds for $\operatorname{sdepth}_S(I_1S + I_2S)$ and $\operatorname{sdepth}_S(S/(I_1S \cap I_2S))$ where I_1 and I_2 are ideals with disjoint supports. It also discusses some lower bounds for $\operatorname{sdepth}_S(I_1S \cap I_2S)$ and $\operatorname{sdepth}_S(S/(I_1S + I_2S))$. We will also use the equalities $\operatorname{sdepth}_S(I_1S) = \operatorname{sdepth}_{S_1}(I_1)+n-m$ and $\operatorname{depth}_S(I_1S) = \operatorname{depth}_{S_1}(I_1)+n-m$ for the above obtained lower and upper bounds. For the specific case $\operatorname{sdepth}_S(J) = \operatorname{sdepth}_{S[y_1]}(J, y_1)$ we will prove Asia Rauf conjecture that $\operatorname{sdepth}_S(J) \geq \operatorname{sdepth}_S(S/J) + 1$. Next we will prove a few corollaries in third section.

This section gives lower bounds for $\operatorname{sdepth}_S(I_1+I_2)$, $\operatorname{sdepth}_S(I_1\cap I_2)$, $\operatorname{sdepth}_S(S/(I_1+I_2))$ and $\operatorname{sdepth}_S(S/(I_1\cap I_2))$ where $I_1, I_2 \subset S$ are arbitrary monomial ideals. It is proved that $\operatorname{sdepth}_S(S/(I_1 : v)) \geq \operatorname{sdepth}_S(S/I_1)$ for $I_1 \subset S$ and $v \in S$ which gives the lower bounds for $\operatorname{sdepth}_S(I_1 : I_2)$ and $\operatorname{sdepth}_S(S/(I_1 : I_2))$ where $I_1, I_2 \subset S$ are any monomial ideals. For the monomial ideal $I_1 \subset S$ we have given some bounds for $\operatorname{sdepth}_S(I_1)$ and $\operatorname{sdepth}_S(S/I_1)$ in the form of irreducible irredundant decomposition of I_1.

In the third section we have discussed a few equivalent forms of Stanely conjecture for (I_1) and S/I_1 where $I_1 \subset S$ is a monomial ideal.

Contents

Chapter 1

Preliminaries

In this chapter basic definitions and concepts of abstract algebra are discussed which will be used further. It states definitions, examples and some related results in detail.

1.1 Rings and Fields

Definition 1.1.1. A ring R is a non-empty set with two binary operations " + " (addition) and " $\cdot$ "(multiplication) which holds the following axioms:

(1) R is an abelian group with respect to " + ",

(2) R is associative with respect to " $\cdot$ " and

(3) For any $x, y, z \in R$, the left distributive law, $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$ and the right distributive law $(x + y) \cdot z = (x \cdot z) + (y \cdot z)$ are satisfied.

In this section are rings are assumed to have identity unless stated otherwise

Definition 1.1.2. A ring R is called commutative if R is commutative with respect to multiplication that is for all $a, b \in R$, $a \cdot b = b \cdot a$.

Definition 1.1.3. A ring R is considered to have an identity, when it has an identity with respect to multiplication.

Proposition 1.1.1 ([6])**.** *Consider a ring R with additive identity 0, then $\forall\, a, b \in R$, the following hold*

(1) $0a = a0 = 0$,

(2) $a(-b) = (-a)b = -(ab)$,

(3) $(-a)(-b) = +ab$.

Example 1.1.1. (1) Rational numbers $\mathbb{Q}$, real numbers $\mathbb{R}$, integers $\mathbb{Z}$ and complex numbers $\mathbb{C}$ are rings under the usual operations of addition and multiplication.
(2) $\mathbb{R}^3$ is a non-commutative ring without identity with the usual addition and cross product of vectors.

Definition 1.1.4. A commutative ring R with identity $1 \neq 0$ is called as division ring, if for all $a(\neq 0) \in R$, there exists $b \in R$ such that $ab = ba = 1$.

Example 1.1.2. Rational numbers $\mathbb{Q}$ form a division ring.

Definition 1.1.5. Consider a ring R having unity $1 \neq 0$. Any $a \in R$ is a unit in R if there exists $b \in R$ such that $ab = ba = 1$.

Theorem 1.1.2 ([6])**.** *In the ring $\mathbb{Z}/n\mathbb{Z}$ all those elements which are relatively prime to n are units in $\mathbb{Z}/n\mathbb{Z}$.*

Example 1.1.3. All non-zero elements in $\mathbb{Z}_{13}$ are units because they are relatively prime to 13.

Example 1.1.4. The set of integers $\mathbb{Z}$ has units ± 1.

Definition 1.1.6. A commutative division ring is called a field.

Example 1.1.5. $(\mathbb{Q}, +, .)$ and $(\mathbb{R}, +, .)$ are fields.

Definition 1.1.7. Consider a ring R, any non-zero element $a \in R$ is called a zero divisor if there exists $b\ (\neq 0) \in R$ such that $ab = 0$ or $ba = 0$.

Theorem 1.1.3 ([6])**.** *In the ring $\mathbb{Z}/n\mathbb{Z}$ all those non-zero elements which are not relatively prime to n are zero divisors in $\mathbb{Z}/n\mathbb{Z}$.*

Corollary 1.1.4 ([6])**.** *If p is prime then $\mathbb{Z}/p\mathbb{Z}$ has no zero divisors.*

Definition 1.1.8. A commutative ring R with unity $1 \neq 0$ without zero divisors is called an integral domain.

Proposition 1.1.5 ([6])**.** *Consider a, b and c of any ring R and a is not a zero divisor. If $ab = ac \Rightarrow$ either $a = 0$ or $b = c$. Particularly for any a, b and c in any integral domain if $ab = ac$ then either $a = 0$ or $b = c$.*

Theorem 1.1.6 ([6])**.** *Every field ia an integral domain.*

Theorem 1.1.7 ([6])**.** *A finite integral domain is a field.*

Corollary 1.1.8. *For any prime p, $\mathbb{Z}_p$ is a field.*

Definition 1.1.9. A subring S of a ring R is a subset of R which becomes a ring under the same operations as R.

Example 1.1.6. (1) $\mathbb{Z}$ is a subring of $\mathbb{R}$.

(2) $2\mathbb{Z}$ is a subring of $\mathbb{Z}$.

(3) Suppose D is a squarefree integer. If $D \equiv 1\ mod\ 4$ then the set

$$\mathbb{Z}[(1+\sqrt{D})/2] = \{x + y(1+\sqrt{D})/2 : x, y \in \mathbb{Z}\}$$

is a subring of $\mathbb{Q}(\sqrt{D}) = \{q_1 + q_2\sqrt{D} : q_1, q_2 \in \mathbb{Q}\}$.

Definition 1.1.10. Let R be a commutative ring with unity. Then the polynomial in variable x with coefficients from R is

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

with $n \geq 0$ and $a_i \in R$.
When $a_n \neq 0$, then the polynomial is said to have degree n, $a_n x^n$ is the *leading term* and a_n is the *leading coefficient*. The set containing all such polynomials is termed as *ring of polynomials in x with coefficients in R* and is denoted as $R[x]$. The polynomial ring in two variables x_1, x_2 with coefficients from R is $R[x_1, x_2] = R[x_1][x_2]$.
By induction the polynomial ring for n variables $x_1, x_2, \cdots, x_n$ with coefficients from R is defined as

$$R[x_1, x_2, \cdots, x_n] = R[x_1, x_2, \cdots, x_{n-1}][x_n]$$

which means that we now consider n variables with coefficients in R, as polynomials in just one variable x_n with the cofficients which are themselves polynomials in $n-1$ variables.

The operations of addition is componentwise i.e.

$$(a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0) + (b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0)$$

$$= (a_n + b_n) x^n + (a_{n-1} + b_{n-1}) x^{n-1} + \cdots + (a_1 + b_1) x + (a_0 + b_0).$$

The multiplication is done by first defining $(ax^i)(bx^j) = (abx^{i+j})$ for only one non-zero term polynomials and then further elongating to all polynomials

$$(a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0) \times (b_n x^n + b_{n-1} x^{n-1} + \cdots . + b_1 x + b_0)$$

$$= a_0 b_0 + (a_0 b_1 + a_1 b_0) x + (a_0 b_2 + a_1 b_1 + a_2 b_0) x^2 + \cdots$$

the coefficient of x^k in the product will be $\sum_{j=0}^{k} a_j b_{k-j}$.
By definition of multiplication $R[x]$ is a commuatative ring with the same unity as that of R.

Proposition 1.1.9 ([6])**.** Let R be an integral domain and let $p(y), q(y)$ be non-zero elements of $R[y]$. Then

(1) degree $(p(y)q(y))$ = degree$(p(y))$+degree $(q(y))$,

(2) The units of $R[y]$ are just the units of R and

(3) $R[y]$ is an integral domain.

Remark 1.1.1. *If S is a subring of R then $S[y]$ is a subring of $R[y]$.*

Example 1.1.7. $\mathbb{Z}$ is a subring of $\mathbb{R}$ so $\mathbb{Z}[y]$ is a subring of $\mathbb{R}[y]$.

Definition 1.1.11. Let S_1 and S_2 be two rings. A map $\alpha : S_1 \to S_2$ satisfying

(1) $\alpha(x+y) = \alpha(x) + \alpha(y) \quad \forall\ x, y \in S_1$ and

(2) $\alpha(xy) = \alpha(x)\alpha(y) \quad \forall\ x, y \in S_1$
is called a ring homomorphism.

(i) The kernal of α, denoted as $ker(\alpha)$, is the set

$$ker(\alpha) = \{x \in S_1 : \alpha(x) = 0_{S_2}\}.$$

(ii) A bijective one to one and onto homomorphism is called a ring isomorphism.

Example 1.1.8. (i) $\phi : \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ defined as

$$\phi(y) = \begin{cases} \overline{0}, & \text{if } y \text{ is even;} \\ \overline{1}, & \text{if } y \text{ is odd.} \end{cases}$$

is a ring homomorphism.

(ii) Consider $\phi : \mathbb{Q}[y] \to \mathbb{Q}$ to be the map from the ring of polynomials in y with rational coefficients to the rationals defined as $\phi(p(y)) = p(0)$, then ϕ is a ring homomorphism.

(iii) For $n \in \mathbb{Z}$, the map $\phi_n : \mathbb{Z} \to \mathbb{Z}$, defined by $\phi_n(y) = ny$ is not a ring homomorphism when $n \neq 0, 1$.

Proposition 1.1.10. *Let S_1 and S_2 be two rings and $\theta : S_1 \to S_2$ be a homomorphism*

(i) $Im(\theta)$ is a subring of S_2.

(ii) $Ker(\theta)$ is a subring of S_1. If $\alpha \in Ker(\theta)$, then $r_1\alpha, \alpha r_1 \in Ker(\theta)$ *for all* $r_1 \in S_1$.

Definition 1.1.12. Consider a subset I of the ring R and the following three properties

(i) I is an additive subgroup of R.

(ii) For $a \in I$ and $r \in R$, $ra \in I$ or we can write $rI \subseteq I$ for every $r \in R$.

(iii) For $a \in I$ and $r \in R$, $ar \in I$ or we can write $Ir \subseteq I$ for every $r \in R$.

If (i) and (ii) are satisfied, then I is a left ideal of R and if (iii) and (i) are satisfied, then I is a right ideal of R. If all three properties hold, then I is said to be an ideal (two sided) of R.

Definition 1.1.13. I is called a proper ideal if $I \neq R$.

Remark 1.1.2. (1) Every ideal is a subring.

(2) A subring may not be an ideal in general e.g $\mathbb{Z}$ is a subring of $\mathbb{R}$ but $\mathbb{Z}$ is not an ideal (for $5 \in \mathbb{Z}$ and $5/9 \in \mathbb{R}$ $(5)(5/9) = (25/9) \notin \mathbb{Z}$).

Proposition 1.1.11 ([6])**.** Consider a ring R. A subset I $(\neq \emptyset)$ of R is an ideal if and only if for all $a, b \in I$ and $r \in R$, $a - b \in I$, $ar \in I$ and $ra \in I$.

Remark 1.1.3. Let $\lambda : R_1 \to R_2$ be a ring homomorphism then $ker(\lambda)$ is an ideal of R_1.

Definition 1.1.14. For any proper ideal I of the ring R, since I is a subgroup of the additive group R, so we can form the quotient group R/I, containing cosets $r+I$ for $r \in R$. The addition and multiplication are defined in the natural way as

$$(s_1 + I) + (s_2 + I) = (s_1 + s_2) + I$$

$$(s_1 + I)(s_2 + I) = (s_1 s_2) + I.$$

Example 1.1.9. (1) For any $n \in \mathbb{Z}$, $n\mathbb{Z}$ is an ideal of $\mathbb{Z}$ so the associated quotient ring is $\mathbb{Z}/n\mathbb{Z}$.

(2) Consider $R = \mathbb{Z}[x]$ to be the ring of polynomials in x with integer coefficients. Let I be the collection of polynomials whose terms are of degree atleast 2. The associated quotient ring R/I is given by the polynomials $a + bx$ of degree at most 1.

Theorem 1.1.12 ([6])**.** (First Isomorphism Theorem for Rings). Let $\phi : R_1 \to R_2$ be a ring homomorphism, then $ker(\phi)$ is an ideal of R_1 and image of ϕ is a subring of R_2, and $Im(\phi)$ is isomorphic to $R_1/Ker(\phi)$, that is

$$Im(\phi) \cong R_1/Ker(\phi).$$

Theorem 1.1.13 ([6])**.** (Second Isomorphism Theorem for Rings). Consider I to be an ideal of R_1 and S be a subring of R_1. Then $S + I = \{x + y \mid x \in S,\ y \in I\}$ is a subring of R_1, $S \cap I$ is an ideal of S and we have

$$S/(S \cap I) \cong (S + I)/I.$$

Theorem 1.1.14 ([6])**.** (Third Isomorphism Theorem for Rings). Suppose I_1 and I_2 are ideals of R such that $I_2 \subseteq I_1$. Then I_1/I_2 is an ideal of R/I_2, we have

$$(R/I_2)/(I_1/I_2) \cong R/I_1.$$

Proposition 1.1.15. Every proper ideal is the kernal of a ring homomorphism.

Definition 1.1.15. Consider I and J to be ideals of a ring R then we define

(1) The sum of I and J is given by

$$I + J = \{ c_1 + c_2 \mid c_1 \in I, \ c_2 \ \in J \}.$$

(2) The product of I and J, written as IJ, as the set of all finite sums of the elements of the form c_1c_2, where $c_1 \in I$ and $c_2 \in J$.

Example 1.1.10. Consider an ideal I in $\mathbb{Z}[x]$ having polynomials with integer coefficients with even constant terms . The polynomials 4 and y are contained in I.
$16 = 4.4 \in I^2 = I.I$ and $y^2 = y.y \in I^2 = I.I$
Their sum $y^2 + 16$ is also contained in I. But $y^2 + 16$ cannot be expressed as the product of two polynomials of I like $p(y)q(y)$.

Definition 1.1.16. Suppose X is any subset of a commutative ring R

(1) The smallest ideal of R containing X, denoted as (X), is called the ideal generated by X.

(2) The set of all possible finite sums, denoted by RX, is of the form rx such that $r \in R$ and $x \in X$ is given as

$$RX = \{ r_1x_1 + r_2x_2 + ... + r_nx_n \mid r_i \in R, \ x_i \in X, \ n \in Z^+ \}.$$

(3) Any ideal which is generated by a finite set is a finitely generated ideal.

Definition 1.1.17. An ideal I of any ring R is called principal if it is generated by a single element.

Example 1.1.11. (1) The ideals 0 and R are both principle

$$0 = (0) \qquad \textit{and} \qquad R = (1).$$

(2) For the ring $\mathbb{Z}$ the principal ideals are $k\mathbb{Z}$ where $k \in \mathbb{Z}$. These are the ideals generated by k and denoted as (k).

Proposition 1.1.16. *Consider an ideal I of a ring R then*

(1) $I = R$ if and only if I contains a unit.

(2) Suppose R is commutative. Then R is a field if and only if 0 and R are its only ideals.

Definition 1.1.18. In any ring R any ideal M is called a maximal ideal if $M \neq R$ and M and R are the only ideals that contain M.

Proposition 1.1.17 ([6])**.** Every proper ideal, in a ring with identity, is contained in a maximal ideal.

Proposition 1.1.18. Suppose S is a commutative ring. The ideal M is maximal if and only if the quotient ring S/M is a field.

Example 1.1.12. $(2, y)$ is a maximal ideal in $\mathbb{Z}[y]$ as the quotient ring $\mathbb{Z}[y]/(2, y) \cong \mathbb{Z}/2\mathbb{Z}$ is a field.

Definition 1.1.19. Suppose a commutative ring R. An ideal $I \neq R$ in R is called a prime ideal for $x, y \in R$ such that $xy \in I$, either $x \in I$ or $y \in I$.

Proposition 1.1.19 ([6])**.** *An ideal P, in a commutative ring R, is a prime ideal in R if and only if the quotient ring R/P is an integral domain.*

Corollary 1.1.20 ([6])**.** *Every maximal ideal in a commutative ring R is a prime ideal.*

The converse of Corollary 1.1.20 is not true.

Example 1.1.13. (1) The 0 ideal in $\mathbb{Z}$ is prime but is not maximal.

(2) $p\mathbb{Z}$ are the principal ideals, which are generated by primes in $\mathbb{Z}$ are prime as well as maximal.

Definition 1.1.20. Suppose Q is a proper ideal of a commutative R, if $ax \in Q$ then either $a \in Q$ or $x^k \in Q$ for some $k > 0$ then Q is called a primary ideal.

Remark 1.1.4. All prime ideals are primary.

Definition 1.1.21. A binary relation over a set Q which satisfies for all $x, y, z \in Q$

(a) $x \leq x$. (reflexive)

(b) $x \leq y$ and $y \leq x$ then $x = y$. (antisymmetric)

(c) $x \leq y$ and $y \leq z$ then $x \leq z$. (transitive)

is called a (non-strict) partial order and Q is called partially ordered st.
For any elements $p, q \in Q$ if $p \leq q$ or $q \leq p$ then p and q are called comparable.

Example 1.1.14. (1) The real numbers are ordered by $\leq$.

(2) The power set of a set is ordered by inclusion.

Definition 1.1.22. A partial order set S which satisfies the condition for any $x, y \in S$, either $x \leq y$ or $y \leq x$ is called totally ordered set.

Example 1.1.15. Real numbers ordered by $\leq$ is a totally ordered set.

Lemma 1.1.21. *Suppose N^n is the set of n-tuples of natural numbers where n is a fixed constant. The tuples can be assigned a product order such that*

$$(x_1, x_2, \cdots, x_n) \leq (y_1, y_2, \cdots, y_n)$$

if and only if $x_j \leq y_j$ *for each* j.

Definition 1.1.23. A partially ordered set is said to satisfy the ascending chain condition if every strictly increasing sequence of elements eventually stops

$$e_1 \leq e_2 \leq e_3 \leq \cdots \leq e_n = e_{n+1} = e_{n+2} = \cdots$$

Definition 1.1.24. A partially ordered set is said to satisfy the descending chain condition if every strictly decreasing sequence of elements eventually stops

$$e_1 \geq e_2 \geq e_3 \geq \cdots \geq e_n = e_{n+1} = e_{n+2} = \cdots$$

Definition 1.1.25. For a commutative ring R to be notherian if any infinite sequence of ideals

$$S_1 \subset S_2 \subset S_3 \subset \cdots$$

in S eventually stops for some value of n that is $S_n = S_{n+1} = S_{n+2} = \cdots$ or we can say it fulfills the ascending chain condition with respect to inclusion on ideals.

Proposition 1.1.22. *(1)* For a notherian ring R and an ideal J of R the quotient ring R/J is notherian.

(2) A homomorphic image of a notherian ring is notherian.

Theorem 1.1.23 ([6])**.** For a commutative ring R the following are equivalent

(a) R is a notherian ring.

(b) Every non-empty set of ideals of R has a maximal element with respect to inclusion.

(c) Every ideal of R is finitely generated.

Example 1.1.16. $\mathbb{Z}$ is notherian.

Example 1.1.17. $\mathbb{Z}[y_1, y_2, \cdots]$ is not notherian.

Theorem 1.1.24. *If R is a notherian then $R[y]$ is also notherian.*

Corollary 1.1.25. *For a notherian ring $R[y_1][y_2] = R[y_1, y_2]$ is also notherian. We can say by induction that $R[y_1, y_2, \cdots, y_n]$ is also notherian ring.*

Definition 1.1.26. If a commutative ring R has a unique maximal ideal then R is called a local ring.

Example 1.1.18. (1) All fields are local rings since 0 is the only maximal ideal.

(2) A nonzero ring in which every element is either a unit or nilpotent is a local ring.

1.2 Modules over commutative rings

Definition 1.2.1. Suppose a commutative ring R then a non-empty set M is said to be an R-module M if

(i) M is an abelian group under addition.

(ii) A map $R \times M \to M$ which satisfies for all $r, s \in R$ and $l, m \in M$, the following

(1) $(r+s)m = rm + sm$

(2) $(rs)m = r(sm)$

(3) $r(l+m) = rl + rm$

(4) $1 \cdot m = m$

Example 1.2.1. (a) Every vector space over a field K is a K-module.

(b) For a ring R, R is a module over itself under usual multiplication.

Definition 1.2.2. For a ring R, the additive subgroup M' of a module M over R is a submodule of M' if $mb \in M'$ for all $r \in R$ and $m \in M'$.

Remark 1.2.1. Submodules of modules are just subsets which are modules under the specified operations.

Proposition 1.2.1 ([6])**.** Suppose a module M over a commutative ring R. Then a subset M_1 of a module M is a submodule of M if and only if

(a) *M_1 is non-empty.*

(b) *For any $m_1, m_2 \in M_1$ and $r \in R$, $m_1 - m_2 \in M_1$ and $m_1 r \in M_1$.*

Definition 1.2.3. Suppose a ring R and let M_1 and M_2 be two R-modules. Then a map $\gamma : M_1 \to M_2$ is an R-module homomorphism when it satisfies the following conditions

(a) $\gamma\ (x + y) = \gamma\ (x) + \gamma\ (y) \quad \forall\ x, y \in M_1$.

(b) $\gamma\ (rz) = r\gamma\ (z) \quad \forall\ z \in M_1$ and $r \in R$.

Definition 1.2.4. A module homomorphism is called a module isomorphism if it is one-to-one and onto. If there is a module isomorphism for two modules M_1 and M_2, such as $\pi : M_1 \to M_2$ then M_1 and M_2 are isomorphic and written as $M_1 \cong M_2$.

Definition 1.2.5. Suppose $\pi : M_1 \to M_2$ is an R-module homomorphism then

(a) The kernal of π is

$$\ker(\pi) = \{\ m \in M_1 :\ \pi(n) = 0_{M_2}\ \}.$$

(b) The image of π is

$$\pi(M_1) = \{\ m_2 \in M_2 :\ m_2 = \pi(m_1) \ \text{ for some } m_1 \in M_1\ \}.$$

(c) The set of all R-module homomorphisms from M_1 to M_2 is denoted as $Hom_R(M_1, M_2)$.

Example 1.2.2. For a ring R and a positive integer n the map $\alpha_j : R^n \to R$ defined as

$$\alpha_j(y_1, y_2, \cdots, y_n) = y_j$$

is an onto R-module homomorphism and its kernal is given as the submodule of n-tuples having a zero in j^{th} position.

Theorem 1.2.2 ([6])**.** (First Isomorphism Theorem for Modules)*. Suppose M and M' are two R modules and $\beta : M \to M'$ be R-module homomorphism then ker β is a submodule of M with*

$$M/ker\beta \cong \beta(M).$$

Theorem 1.2.3 ([6])**.** (Second Isomorphism Theorem for Modules)*. Suppose M and M' are two submodules of R-module N then*

$$(M + M')/M' \cong M/(M \cap M').$$

Theorem 1.2.4 ([6])**.** (Third Isomorphism Theorem for Modules)*. Suppose M and M' are two submodules of an R module M_1 with $M \subseteq M'$ then*

$$(M_1/M)/(M'/M) \cong (M_1/M').$$

Definition 1.2.6. An R-module M is finitely generated if any $s \in M$ can be expressed as $s = r_1s_1 + \cdots + r_ms_m$, when r_i's $\in R$ and s_i's $\in M$ then $\{s_1, \cdots, s_m\}$ is the generating set for M.

Definition 1.2.7. Let M be an R-module and $M_1, \cdots, M_t, \cdots$ be submodules in M then for notherian modules any infinite increasing sequence of these submodules finally stops that is

$$M_b = M_{b+1},$$

for a positive integer b.

Definition 1.2.8. Suppose M is a module over a commutative ring R. An element $s(\neq 0) \in R$ is a zero divisor in module M if there exists $r \in M$ then $sr = 0$.

Definition 1.2.9. For an R-module M and $M_1, M_2, \cdots, M_t$ as submodules of M then M can be written as a (internal)direct sum of $M_1, M_2, \cdots, M_t$ if every $r \in R$ can be given uniquely as $r = m_1 + m_2 + \cdots m_t$ for $m_j \in R_j$ that is

$$M = M_1 \bigoplus M_2 \bigoplus \cdots \bigoplus M_t.$$

1.3 Monomials and monomial ideals

Definition 1.3.1. Consider $S = K[y_1, y_2, \cdots, y_n]$ a polynomial ring in n variables over the field K. The set of vectors $c = (c_1, \cdots, c_n) \in \mathbb{R}_n$ such that $c_i \geq 0$. The product $y_1^{c_1} \cdots y_n^{c_n}$ such that $c_i \in \mathbb{Z}^+$ is a monomial. Another way of writing the monomial $z = y_1^{c_1} ... y_n^{c_n}$ is the form $z = y^{\mathbf{c}}$ where $\mathbf{c} = (c_1, c_2, \cdots, c_n) \in \mathbb{Z}_+^n$.
If the set of all monomials of S is denoted by Q then Q forms a K-basis of S. We can express any polynomial $h \in S$ as a unique linear combination of monomials having coefficients in K

$$h = \sum c_z z.$$

such that c_z is an element of K.
The support of h, denoted as supp(h), is defined as

$$\text{supp}(h) = \{\ q \in Q : c_q \neq 0\ \}.$$

The support of a monomial q is given as

$$\text{supp}(q) = \{\ y_i : y_i | q\ \}.$$

Definition 1.3.2. If an ideal $J \subset S$ is generated by monomials then J is called a monomial ideal.

Theorem 1.3.1. *The set Q of monomials belonging to $J \subset S$ is a K-basis of J.*

Corollary 1.3.2 ([7])**.** *Consider an ideal $J \subset S$ then the following are equivalent*

(1) J is a monomial ideal.

(2) For any $h \in S$, we have $h \in J$ iff $\operatorname{supp}(h) \subset J$.

Corollary 1.3.3 ([7])**.** *Consider a monomial ideal J. The residue classes of the monomials not belonging to J form a K-basis of the residue class ring S/J.*

Proposition 1.3.4 ([7])**.** *Consider $\{w_1, w_2, \cdots, w_n\}$ be the monomial system of the generators of the monomial ideal $J \subset S$ then the monomial $y \in J$ iff there exists a monomial z such that $y = zw_i$ for some i.*

Proposition 1.3.5 ([7])**.** *Every monomial ideal has a unique minimal set of monomial generators. For a monomial ideal J the unique minimal set of monomial generators is commonly denoted as $G(J)$.*

Example 1.3.1. Let $S = K[\ y_1, y_2, y_3\ ]$ then the ideal $J = (y_1^2y_2^2, y_2^2y_3^2, y_2^3y_3^2, y_1^2y_2^3)$ has $G(J) = \{\ y_1^2y_2^2, y_2^2y_3^2\ \}$.

Definition 1.3.3. A monomial $y^{\mathbf{c}} = y_1^{c_1}y_2^{c_2}\cdots y_n^{c_n}$ is called squarefree if c_i's that is the components of $\mathbf{c}$ are 0 or 1.

Definition 1.3.4. An ideal $J \subset S$ generated by squarefree monomials is called a squarefree monomial ideal.

Example 1.3.2. Let $S = K[\ y_1, y_2, y_3\ ]$ then the ideal $J = (y_1y_2, y_2y_3, y_3y_1)$ is squarefree monomial ideal.

Definition 1.3.5. If an ideal $J \subset S$ generated by variables not by its power then J is called monomial prime ideal.

Corollary 1.3.6 ([7])**.** *A squarefree monomial ideal is intersection of prime monomial ideals.*

Definition 1.3.6. Radical of an ideal $I \subset S$ of a commutative ring R, denoted as $\text{rad}(I)$ or $\sqrt{I}$, is given as

$$\text{rad}(I) = \{ \ a \in R \ \mid a^n \in I, \ for \ n > 0 \ \}.$$

Definition 1.3.7. If I =rad(I), then I is called a radical ideal.

Proposition 1.3.7 ([7])**.** *Let M be a monomial ideal then the set of generators of rad(M) is given as*

$$rad(M) = \ \{ \ \sqrt{m} : m \in G(M) \ \}.$$

Corollary 1.3.8 ([7])**.** *A monomial ideal N is called radical monomial ideal that is $N = \sqrt{N}$ iff N is squarefree monomial ideal.*

Example 1.3.3. (1) $9\mathbb{Z}$ has radical $3\mathbb{Z}$.

(2) Let $S = K[y_1, y_2, y_3 \ ]$ then the radical of ideal $J = (y_1^2 y_2^2, y_2^2 y_3^2, y_3^2 y_1^2)$ is $\text{rad}(J) = (y_1 y_2, y_2 y_3, y_3 y_1)$.

Definition 1.3.8. A monomial ideal J is called p-primary if its radical equals prime ideal p.

Definition 1.3.9. An ideal J is irreducible if it cannot be expressed as the intersection of two bigger ideals i.e

$$J = U \cap V \text{ then either } J = U \text{ or } J = V.$$

Definition 1.3.10. A monomial ideal that is not irreducible is called reducible monomial ideal.

Corollary 1.3.9 ([7])**.** *A monomial ideal is irreducible if and only if it is generated from pure powers of variables.*

Theorem 1.3.10. *Every irreducible ideal is primary in a notherian ring.*

Proposition 1.3.11 ([7])**.** The irreducible monomial ideal $(y_{c_1}^{a_1}, y_{c_2}^{a_2}, \cdots, y_{c_r}^{a_r})$ is $(y_{c_1}, y_{c_2}, \cdots, y_{c_r})$-primary.

1.4 Algebric operations on ideals

Addition and multiplication of monomial ideals gives monomial ideals. Also $G(I_1 + I_2) \subset G(I_1) \cup G(I_2)$ and $G(I_1I_2) \subset G(I_1)G(I_2)$.

Proposition 1.4.1 ([7])**.** Let I_1 nd I_2 be monomial ideals then the set of generators for the monomial ideal $I_1 \cap I_2$ is given as $\{lcm(w_1, w_2) : w_1 \in G(I_1),\ w_2 \in G(I_2)\}$.

Definition 1.4.1. Let I_1,$I_2 \subset S$ be monomial ideals then the set

$$I_1 : I_2 = \{\ g \in S :\ \ gh \in I_1 \text{ for all }\ h \in I_2\ \}.$$

is called the colon ideal of I_1 with respect to I_2.

Proposition 1.4.2 ([7])**.** Let I_1,I_2 be monomial ideals then $I_1 : I_2$ is a monomial ideal and

$$I_1 : I_2 = \bigcap \quad I_1 : (x).$$

Where the set of generators of $I_1 : (x)$ *is* $\{\ y/gcd(y, x) \mid y \in G(I_1)\ \}$.

Example 1.4.1. Consider $S = K[\ y_1, y_2, y_3\ ]$ then the ideal $J = (y_1, y_2)$ is monomial prime ideal.

Definition 1.4.2. Consider an ideal $J \subset R$ of a ring R. A prime ideal P is called a minimal prime ideal of J, denoted as $Min(J)$, if $J \subset P$ and there is no prime ideal containing J which is properly contained in P.

Definition 1.4.3. An irredundant presentation of an ideal I is an intersection $I = \bigcap_{i=1}^{m} I_i$ such that none of the ideals I_i can be omitted in the presentation.

Theorem 1.4.3 ([7])**.** *Suppose an ideal* $J \subset S = K[y_1, \cdots, y_n]$ *is a monomial ideal. Then we have* $J = \bigcap_{i=1}^{m} I_i$ *where each* I_i *is generated by pure powers of the variables or we can say each* I_i *is of the form* $(y_{c_1}^{a_1}, \cdots, y_{c_r}^{a_r})$*. Moreover, an irredundant presentation of this form is unique.*

Lemma 1.4.4 ([7])**.** *Let J has irredundant presentation as $J = Q_1 \cap Q_2 \cap \cdots \cap Q_m$ as the intersection of prime ideals. Then $Min(J) = \{Q_1, Q_2, \cdots, Q_m\}$.*

Corollary 1.4.5 ([7])**.** *Consider $M \subset S$ be a squarefree monomial ideal. Then*

$$M = \bigcap_{P \in Min(M)} P,$$

and each $P \in Min(M)$ is a monomial prime ideal.

Definition 1.4.4. Suppose $J \subset S$ is a monomial ideal with $G(J) = \{w_1, w_2, \cdots, w_n\}$ then J is called complete intersection ideal if and only if

$$\text{supp}(w_l) \cap \text{supp}(w_m) = \emptyset, \quad for\ l \neq m$$

Example 1.4.2. Consider $S = K[y_1, y_2, y_3]$ then the ideal $J = (y_1^2 y_3^2, y_2^2)$ is a complete intersection ideal.

Definition 1.4.5. Presentation of an ideal J as the intersection $J = \bigcap_{i=1}^{m} I_i$, where each ideal I_i is a primary ideal, is called primary decomposition.

Theorem 1.4.6. In a notherian ring every ideal can be presented as the intersection of finite number of primary ideals.

Definition 1.4.6. Consider a notherian ring R and M a finitely generated $R-$ module. For a prime ideal ideal $P \subset R$ if there exists an element $s \in M$ such that $P = Ann(s)$, then P is called an associated prime ideal of M.

Corollary 1.4.7 ([7])**.** The associated prime ideals of a monomial ideal are monomial prime ideals.

Example 1.4.3. For an ideal $I = (y_1^2 y_3, y_1^2 y_4, y_2^2)$ of a polynomial ring $S = K[y_1, y_2, y_3, y_4]$ over the field K, the primary decomposition of I is given as $(y_1^2, y_2^2) \cap (y_3, y_4, y_2^2)$ and $\text{Ass}(S/I) = \{(y_1, y_2), (y_3, y_4, y_2)\}$.

Corollary 1.4.8 ([7]). *Consider $J \subset S$ and let $Q \in Ass(J)$. Then there exists a monomial w such that $Q = J : w$.*

Definition 1.4.7. Consider an abelian semigroup $(G, +)$. The commutative ring R is called G-graded if R has a family of subgroups, $\{R_p\}$, such that

(1) $R = \bigoplus_{p \in G} R_p$ and

(2) $R_p R_q \subset R_{p+q}$, for all $p, q \in G$.

Example 1.4.4. For $h \in S$, with $b \in \mathbb{Z}^n$, if h is of the form xy^b with $x \in K$ it is called homogenous of degree b. Then S is $\mathbb{Z}^n$ -graded with graded components

$$S_b = \begin{cases} Ky^b, & \text{if } b \in \mathbb{Z}^n_+; \\ 0, & \text{otherwise.} \end{cases}$$

Definition 1.4.8. An S-module M is called Z_n-graded module when

(a) $M = \bigoplus_{\mathbf{c} \in Z_n} M_{\mathbf{c}}$ and

(b) $S_c M_d \subset M_{c+d}$ for $c, d \in Z_n$.

Definition 1.4.9. Suppose M is a G-graded R module. An element $m \in M$ is called homogenous if there exists $k \in G$ such that $m \in M_k$ where k is called degree of M.
Every M_k, for $k \in G$, is a homogenous component of M of degree k.

Definition 1.4.10. A module M is called G-graded over a G-graded ring R if the following conditions hold

(1) $M = \bigoplus_{f \in G} M_f$ and

(2) $R_h M_f \subset M_{h+f}$ for all $h, f \in G$.

Example 1.4.5. *(1)* $\mathbb{Z}$- grading for $K[y]$

$$K[y] = \underbrace{K}_{deg\ 0} \bigoplus \underbrace{Ky}_{deg\ 1} \bigoplus \underbrace{Ky^2}_{deg\ 2} \bigoplus \underbrace{Ky^3}_{deg\ 3} \bigoplus \cdots$$

(2) $\mathbb{Z}$ - grading for $K[y_1, y_2, \cdots, y_n]$

$$K[y_1, y_2, \cdots, y_n] = \underbrace{K}_{deg\ 0} \bigoplus \underbrace{(Ky_1 + Ky_2 + \cdots + Ky_n)}_{deg\ 1}$$
$$\bigoplus \underbrace{(Ky_1^2 + Ky_2^2 + \cdots + Ky_n^2 + Ky_1y_2 + \cdots)}_{deg\ 2} \bigoplus \cdots$$

(3) $\mathbb{Z}^n$ - grading for $K[y_1, y_2, \cdots, y_n]$

$$K[y_1, y_2, \cdots, y_n] = \bigoplus_{\mu \in Z^n} R_\mu \ = \bigoplus_{\mu \in Z^n} KY^\mu$$

where $\mu = (c_1, c_2, \cdots, c_n)$ and $Y^\mu = (y_1^{c_1} y_2^{c_2} \cdots y_n^{c_n})$.

Definition 1.4.11. An ideal J, of a G-graded ring R, is G-graded when

$$J = \bigoplus_{i \in G} (J \cap R_i).$$

Definition 1.4.12. Suppose M is a module over a ring R then the annihilator of M over the ring R is given as

$$Ann_R(M) = \ \{z \in R \ : zM = 0\ \}.$$

Definition 1.4.13. Krull dimension is usually considered as the dimension of a ring. For a commutative ring it is given as supremum of the length of all chains of prime ideals

$$\mathbf{P}_0 \subsetneq \mathbf{P}_1 \subsetneq \cdots \subsetneq \mathbf{P}_k.$$

When there is no upper bound on the length we call dimension to be infinite.

Example 1.4.6. (1) The ring $\mathbb{Z}$ has dimension 1.

(2) $K[y_1, y_2, \cdots, y_p]$ has dimension p.

(3) $K[y_1, y_2, \cdots]$ has infinite dimension.

Corollary 1.4.9. *If we have a notherian ring R then,*

$$dim\ R[y_1, y_2, \cdots, y_n] = \ n{+}dimR.$$

If K is a field then,

$$dim\ K[y_1, y_2, \cdots, y_n] = n.$$

Definition 1.4.14. For a module M the krull dimension, denoted as $\dim(M)$, is supremum over the strictly increasing chains of length r

$$\mathbf{P}_0 \subsetneq \mathbf{P}_1 \subsetneq \cdots \subsetneq \mathbf{P}_r \qquad where\ \ \mathbf{P}_l \in \operatorname{supp}(M)$$

that is $\dim(M) = \dim(R/\operatorname{ann}M)$ when $M \neq 0$.

Definition 1.4.15. If an element of a ring is not a zero divisor, then it is called regular element.

Definition 1.4.16. Suppose M is a module over a commutative ring R. An element $s\ (\neq 0) \in R$ is M-regular only if s is not a zero divisor in M that is if $r \in M$ and $rs = 0$ then $r = 0$.

Definition 1.4.17. The sequence $z = (z_1, \cdots, z_m)$ of a ring R is called M-regular sequence if

(i) z_j is $M/(z_1, \cdots, z_{j-1})M$ regular element for $j = 1, \cdots, m$.

(ii) $M/zM \neq 0$.

Example 1.4.7. As $S = K[y_1, \cdots, y_m]$ is a module over itself so we have $y_1, \cdots, y_m$ a regular sequence on S.

Definition 1.4.18. Suppose a graded ring R having a graded maximal ideal $\mathbf{m}$ and $M(\neq 0)$ a finitely generated graded R-module. The depth of M is the common length of the maximal regular sequence on M in m. It is denoted as $depth(M)$.

Example 1.4.8. Suppose $S = K[y_1, \cdots, y_b]$ then $y = y_1, \cdots, y_b$ is a maximal regular sequence on S as $S/(y) \cong K$. So $depth(S) = b$.

Proposition 1.4.10. *[20] Suppose a G-graded ring R with graded maximal ideal m and a finitely generated graded R-module* $M(\neq 0)$. *Then* $depth(M) \leq dim(R/P)$ *for* $P \in Ass(M)$. *Particularly* $depth(M) \leq dim(M)$.

Definition 1.4.19. In particular if $v \in M$ then the annihilator of v is given as

$$Ann_R(v) = \{b \in R \ : bv = 0 \}.$$

Example 1.4.9. For an ideal $I = (x_1^2x_2, x_1^3x_4, x_3x_4^2, x_5)$ then two annihilators of $\overline{x}_1 = x_1 + I \in S/I$ are x_1x_2 and $x_1^2x_4$.

Definition 1.4.20. A finite or infinite sequence of modules and module homomorphisms

$$\cdots N_0 \xrightarrow{g_1} N_1 \xrightarrow{g_2} N_2 \xrightarrow{g_3} \cdots \xrightarrow{g_m} N_m \cdots$$

is called exact sequence if the image (range) of each homomorphism equals the kernel of the next that is

$$Im(g_n) = Ker(g_{n+1}).$$

Definition 1.4.21. A finite exact sequence is called a short exact sequence when it is of the form

$$0 \to N_0 \xrightarrow{g} N_1 \xrightarrow{h} N_2 \to 0.$$

Hence g is monomorphism and h is epimorphism.

Example 1.4.10.

$$0 \to \mathbb{Z} \xrightarrow{.2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

is a short exact sequence.

Chapter 2

Stanley Decomposition and the Stanley Depth

In this chapter Stanley decomposition, Stanley depth of $\mathbb{Z}^n$-graded S-modules are discussed where S is the polynomial ring over a field. R. P. Stanley's conjecture is also stated. Some results regarding to depth, Stanley depth and Stanley's conjecture are discussed which will be helpful in further chapters.

2.1 Stanley Decomposition and Stanley depth

Definition 2.1.1. Consider a field K and a polynomial ring $S = K[x_1, \cdots, x_m]$ in m variables with coefficients in the field K. Consider M a finitely generated multigraded $\mathbb{Z}^n$-graded S-module. Suppose $u \in M$ is homogeneous in M and $Z \subseteq \{x_1, \cdots, x_m\}$. The K-subspace of M, denoted as $uK[Z]$, is generated by all elements ux where x is a monomial in $K[Z]$. The $\mathbb{Z}^n$ graded K sub-space $uK[Z] \subset M$ is called *Stanley space of dimension* $|Z|$ and $uK[Z]$ is a free $K[Z]$-module, where $|Z|$ is the cardinality of Z. The presentation of the K-vector space M as finite direct sum of

Stanley spaces is called *Stanley decomposition of M* i.e

$$D : M = \bigoplus_{i=1}^{m} u_i K[Z_i].$$

Then the Stanley depth of the decomposition is given by

$$\operatorname{sdepth} D = \min\{|Z_i| : i = 1, \cdots, m\}$$

The Stanley depth of M is given as

$$\operatorname{sdepth} M := \max\{\operatorname{sdepth} D : D \text{ is a Stanley decomposition of } M\}$$

2.2 Method for finding Stanley Depth

The method for finding the Stanley depth of a module I/J where $J \subset I \subset S = K[x_1, \cdots, x_n]$ are monomial ideals is given in [8]. We define a natural partial order on $\mathbb{N}^n$ as $x \leq y$ if and only if $x_i \leq y_i$ for $i = 1, \cdots, n$. Suppose we have $x^{\mathbf{c}} = x^{c_1} \cdots x^{c_n}$ for $c \in \mathbb{N}^n$. Suppose I is generated by monomials $x_1^{c_1}, \cdots, x_n^{c_n}$ and J is generated by $x_1^{d_1}, \cdots, x_n^{d_n}$. Choose $g \in \mathbb{N}^n$ such that $x_i \leq g$ and $y_j \leq g$ for all i and j. Let $P^g_{I/J}$ be the set of all $s \in \mathbb{N}^n$ with $s \leq g$ such that $x_i \leq s$ for some i and $s \ngeq y_j$ for all j. It is called the characteristic poset of I/J with respect to g and is viewed as a finite subposet of $\mathbb{N}^n$. For any poset P and $x, y \in P$ we set an interval $[x, y]$ for $x \leq v \leq y$ and $v \in P$. A partition of of a finite poset P is a disjoint union

$$\mathsf{P} : P = \bigcup_{a=1}^{r} [x_a, y_a].$$

Each partition of $P^g_{I/J}$ gives rise to a Stanley decomposition of I/J.

Example 2.2.1. For an ideal $I = (x_1x_3, x_1x_4, x_2x_3, x_2x_4) \subset S = K[x_1, x_2, x_3, x_4]$
$D_1 := x_1x_3K[x_1, x_2, x_3, x_4] \oplus x_1x_4K[x_1, x_2, x_4] \oplus x_2x_4K[x_2, x_3, x_4] \oplus x_2x_3K[x_2, x_3]$.
$\operatorname{sdepth}(D_1) = 2$.
$D_2 := x_1x_2x_3K[x_1, x_2, x_3, x_4] \oplus x_2x_3K[x_2, x_3, x_4] \oplus x_2x_4K[x_1, x_2, x_4] \oplus x_1x_3K[x_1, x_3, x_4]$.

sdepth$(D_2) = 3$.
sdepth$(I) \geq \max\{\text{sdepth}(D_1), \text{sdepth}(D_2)\} = \max\{2, 3\} = 3$
$D_3 \quad : S/I = \quad K[x_1, x_2] \oplus x_3K[x_3, x_4]$.
sdepth$(D_3) = 2$.
$D_4 \quad : S/I = \quad K[x_4] \oplus x_3K[x_3, x_4] \oplus x_1K[x_1, x_2] \oplus x_2K[x_2]$.
sdepth$(D_4) = 1$.
sdepth$(S/I) \geq \max\{\text{sdepth}(D_3), \text{sdepth}(D_4)\} = \max\{2, 1\} = 2$

Conjecture 2.2.1. *Richard P. Stanely [18], in 1982, conjectured that for all finitely generated $\mathbb{Z}^n$-graded S-modules M [18]*

$$\text{depth}\, M \leq \text{sdepth}\, M.$$

The conjecture is disproved on 16 April 2015 by Art M. Duval, Bennet Goeckner, Caroline J. Klivans, Jeremy L. Martin.

2.3 Some known values, equalities and bounds for Stanley depth

Theorem 2.3.1. [1, Theorem 2.2]. *Let $\boldsymbol{m} = (x_1, \cdots, x_n) \subset S$ then*

$$\text{sdepth}(\boldsymbol{m}) = \lceil \frac{n}{2} \rceil.$$

Lemma 2.3.2. [4, Lemma 1.1]. *Suppose $u_1, \cdots, u_n \in K[x_2, \cdots, x_n]$ be some monomial and let c be a positive integer. Let $I = (x_1^c u_1, u_2, \cdots, u_n)$ and $J = (x_1^{c+1} u_1, u_2, \cdots, u_n)$ then* sdepth$(I) =$ sdepth(J).

Theorem 2.3.3. [4, Theorem 2.1]. *Let I be a complete intersection monomial ideal in S and $\sqrt{I}$ be its radical. Then*

$$\text{sdepth}(I) = \text{sdepth}(\sqrt{I}).$$

Corollary 2.3.4. [10, Corollary 2.2]. *Suppose $J \subset S$ is a monomial ideal and $\sqrt{J}$ is its radical. Then* $\operatorname{sdepth}(S/J) \leq \operatorname{sdepth}(S/\sqrt{J})$ *and* $\operatorname{sdepth}(J) \leq \operatorname{sdepth}(\sqrt{J})$.

Corollary 2.3.5. [10, Corollary 2.3]. *Suppose I and J are two monomial ideals of S such that $I \subset J$ and the radical ideals of I and J are $\sqrt{I}$ and $\sqrt{J}$ respectively. If* $\operatorname{sdepth}(J/I) = \dim(J/I)$. *Then* $\operatorname{sdepth}(\sqrt{J}/\sqrt{I}) = \dim(\sqrt{J}/\sqrt{I})$.

Theorem 2.3.6. *[2] Suppose $I \subset S = K[x_1, \cdots, x_n]$ is a monomial ideal which is minimally generated by q elements then*

$$\operatorname{sdepth}(S/I) \geq n - q.$$

Theorem 2.3.7. *[19] Suppose $I \subset S = K[x_1, \cdots, x_n]$ is a monomial ideal which is minimally generated by q elements then*

$$\operatorname{sdepth}(I) \leq n - \lfloor \frac{q}{2} \rfloor.$$

Proposition 2.3.8. [13, Proposition 1.3]. *Suppose a monomial ideal I in S. Then for each monomial $u \notin I$* $\operatorname{sdepth}_S(I) \leq \operatorname{sdepth}_S(I : u)$.

Corollary 2.3.9. [13, Corollary 1.2]. *Suppose a monomial ideal I in S. Then for each monomial $v \notin I$*

$$\operatorname{depth}_S(S/(I : v)) \geq \operatorname{depth}_S(S/I).$$

Theorem 2.3.10. [8, Theorem 1.4]. *Let $J \subset S$ be a monomial ideal (minimally) generated by m monomials. Then:*

$$\operatorname{sdepth}(J) \geq \max\{1, n - \lceil \frac{m}{2} \rceil\}.$$

Corollary 2.3.11. [5, Corollary 2.6]. *If $J \subsetneq I \subset S$ are two monomial complete intersection ideals, then*

$$\operatorname{sdepth}_S(J/I) \geq \operatorname{depth}_S(J/I).$$

Theorem 2.3.12. [9, Theorem 1.1]. *Suppose $I \subset S$ is a monomial ideal such that* $\operatorname{Ass}(S/I) = \{P_1, \cdots, P_r\}$ *then*

$$\operatorname{sdepth}(I) \leq \min\{\operatorname{sdepth}(P_j) : 1 \leq j \leq r\}.$$

Lemma 2.3.13. [4, Lemma 2.2.]. *Suppose $I' \subset S[x_{n+1}]$ be a monomial ideal. We consider the homomorphism $\alpha : S[x_{n+1}] \to S$, $\alpha(x_i) = x_i$ for $i \leq n$ and $\alpha(x_{n+1}) = 1$. Let $I = \alpha(I')$. Then* $\operatorname{sdepth}(I') \leq \operatorname{sdepth}(I) + 1$.

Proposition 2.3.14. [14, Proposition 2.2]. *Suppose P and P' are two non-zero monomial primary ideals of S with different associated prime ideals. Suppose that* $\dim(S/(P + P')) = 0$. *Then* $\operatorname{sdepth}(S/(P \cap P')) \leq$

$$\max\{\min\{\dim(S/P'), \lceil\frac{\dim(S/P)}{2}\rceil\}, \min\{\dim(S/P), \lceil\frac{\dim(S/P')}{2}\rceil\}\}.$$

Theorem 2.3.15. [14, Theorem 4.5]. *If P and P' are irreducible monomial ideals and $P + P'$ is the maximal ideal of S then*

$$\operatorname{sdepth}(P \cap P') \geq \lceil\frac{\dim(S/P)}{2}\rceil + \lceil\frac{\dim(S/P')}{2}\rceil.$$

Theorem 2.3.16. [10, Theorem 2.8]. *Suppose Q and Q' are two primary monomial ideals with $Q = (x_1, \cdots, x_p)$ and $Q' = (x_{p+1}, \cdots, x_q)$, where $p \geq 2$ and $q \geq 4$. Then*

$$\operatorname{sdepth}(Q \cap Q') \leq \frac{q+2}{2}.$$

Corollary 2.3.17. [10, Corollary 2.9]. *Suppose P and P' are two irreducible monomial ideals such that $\sqrt{P} = (x_1, \cdots, x_s)$ and $\sqrt{P'} = (x_{s+1}, \cdots, x_n)$. Suppose that n is odd. Then* $\operatorname{sdepth}(P \cap P') = \lceil\frac{n}{2}\rceil$.

Lemma 2.3.18. [10, Lemma 2.6]. *Suppose P and P' are two monomial primary ideals with $\sqrt{P} = (x_1)$ and $\sqrt{P'} = (x_2, \cdots, x_n)$. Then*

$$\operatorname{sdepth}(P \cap P') \leq 1 + \lceil\frac{n-1}{2}\rceil.$$

Lemma 2.3.19. [10, Lemma 2.15]. *Suppose P and P' are two primary monomial ideals with $\sqrt{P} = (x_1, \cdots, x_{n-1})$ and $P' = (x_2, \cdots, x_n)$. Then*

$$\text{sdepth}(P \cap P') \leq n - \lfloor \frac{n-1}{2} \rfloor.$$

Proposition 2.3.20. [10, Proposition 2.16]. *Suppose P and P' are two monomial primary ideals with $\sqrt{P} = (x_1, \cdots, x_l)$ and $\sqrt{P'} = (x_{s+1}, \cdots, x_n)$ where $1 < s \leq l < n$. Then*

$$\text{sdepth}(P \cap P') \leq \min\{n - \lfloor \frac{l}{2} \rfloor, n - \lfloor \frac{n-l}{2} \rfloor\}.$$

Lemma 2.3.21. [14, Lemma 4.1]. *Suppose $Q, Q' \subset S = K[x_1, \cdots, x_n]$ are two non-zero irreducible monomial ideals such that $\sqrt{Q} = \{x_1, \cdots, x_m\}, \sqrt{Q'} = \{x_{m+1}, \cdots, x_n\}$ for some integer m with $1 \leq m \leq n$. Then*

$$\text{sdepth}(Q \cap Q') \geq \lceil \frac{m}{2} \rceil + \lceil \frac{n-m}{2} \rceil \geq \frac{n}{2}.$$

Lemma 2.3.22. [14, Lemma 1.3]. *Suppose P and Q are two monomial ideals of $S = K[x_1, \cdots, x_n]$. Then*

$$\text{sdepth}(S/(P \cap Q)) \geq \max\{min\{\text{sdepth}(S/P), \text{sdepth}(P/(P \cap Q))\},$$
$$\min\{\text{sdepth}(S/Q), \text{sdepth}(Q/(P \cap Q))\}\}.$$

Lemma 2.3.23. [14, Lemma 1.5]. *Suppose P and P' are two irreducible monomial ideals of S. Then*

$$\text{sdepth}(S/(P \cap P')) \geq \max\{\min\{\dim(S/P'), \lceil \frac{\dim(S/P) + \dim(S/(P + P'))}{2} \rceil\},$$
$$\min\{\dim(S/P), \lceil \frac{\dim(S/P') + \dim(S/(P + P'))}{2} \rceil\}\}.$$

Theorem 2.3.24. [14, Theorem 2.3]. *Suppose P and P' are two non-zero monomial primary ideals of S with different associated prime ideals. Then*

$$\text{sdepth}(S/(P \cap P')) \leq \max\{\min\{\dim(S/P'), \lceil \frac{\dim(S/P) + \dim(S/(P + P'))}{2} \rceil\},$$
$$\min\{\dim(S/P), \lceil \frac{\dim(S/P') + \dim(S/(P + P'))}{2} \rceil\}\}.$$

Theorem 2.3.25. [14, Theorem 4.5]. *Suppose P and P' are two non-zero irreducible monomial ideals of S. Then*

$$\operatorname{sdepth}(P\cap P') \geq \dim(S/(P+P')) + \lceil\frac{\dim(S/P')-\dim(S/(P+P'))}{2}\rceil$$
$$+\lceil\frac{\dim(S/P)-\dim(S/(P+P'))}{2}\rceil \geq \lceil\frac{\dim(S/P')+\dim(S/P)}{2}\rceil.$$

Lemma 2.3.26. [14, Lemma 5.7]. *Suppose P_1, P_2, P_3 are three non-zero irreducible monomial ideals of $S = K[x_1, \cdots, x_n]$. Then* $\operatorname{sdepth}((P_2\cap P_3)/(P_1\cap P_2\cap P_3)) \geq$

$$\dim(S/(P_1+P_2+P_3)) + \lceil\frac{\dim(S/(P_1+P_2))-\dim(S/(P_1+P_2+P_3))}{2}\rceil$$
$$+\lceil\frac{\dim(S/(P_1+P_3))-\dim(S/(P_1+P_2+P_3))}{2}\rceil$$
$$\geq \lceil\frac{\dim(S/(P_1+P_2))+\dim(S/(P_1+P_3))}{2}\rceil.$$

Proposition 2.3.27. [14, Proposition 5.8]. *Suppose P_1, P_2, P_3 are three non-zero irreducible ideals of S and $\dim S/(P_1+P_2+P_3) = 0$. Then* $\operatorname{sdepth}(S/(P_1\cap P_2\cap P_3)) \geq$

$$\max\{\min\{\operatorname{sdepth}\ S/(P_2\cap P_3), \lceil\frac{\dim(S/(P_1+P_2))}{2}\rceil + \lceil\frac{\dim(S/(P_1+P_3))}{2}\rceil\}$$
$$\min\{\operatorname{sdepth}\ S/(P_1\cap P_3), \lceil\frac{\dim(S/(P_1+P_2))}{2}\rceil + \lceil\frac{\dim(S/(P_2+P_3))}{2}\rceil\},$$
$$\min\{\operatorname{sdepth}\ S/(P_1\cap P_2), \lceil\frac{\dim(S/(P_3+P_2))}{2}\rceil + \lceil\frac{\dim(S/(P_1+P_3))}{2}\rceil\}\}.$$

Corollary 2.3.28. [10, Corollary 1.5]. *Let a monomial ideal $J \subset S$ and $P_j \in Ass(S/J) = \{P_1, \cdots, P_t\}$, $d_j = ht(P_j)$, $J' := (J, x_{n+1}, x_{n+2}) \subset S' := S[x_{n+1}, x_{n+2}]$. Then*

$$\operatorname{sdepth}_{S'}(J') = \min\{n+1-\lceil\frac{d_j}{2}\rceil, 1 \leq j \leq t\}.$$

Corollary 2.3.29. [10, Corollary 1.12]. *Let $J \subset S$ is a monomial ideal, $Ass(S/J)= \{P_1, \cdots, P_t\}$ and suppose that $G(P_l)\cap G(P_m) = \emptyset$ for $l \neq m$. Then* $\operatorname{sdepth}(J) = t$. *In particular Stanleys conjecture holds for J.*

Lemma 2.3.30. [10, Lemma 3.1]. *Let $J' \subset S' = S[x_{n+1}]$ be a monomial ideal, x_{n+1} being a new variable. If $J' \cap S \neq (0)$, then* $\operatorname{sdepth}_S(J' \cap S) = \operatorname{sdepth}_{S[x_{n+1}]} J' - 1$.

Theorem 2.3.31. *[8] For $S = K[x_1, \cdots, x_n]$ we suppose $I \subset S' = K[x_1, \cdots, x_r]$ then we have*

$$\operatorname{sdepth}_S(IS) = \operatorname{sdepth}_{S'}(I) + n - r.$$

Theorem 2.3.32. *[8] For $S = K[x_1, \cdots, x_n]$ we suppose $J \subset S' = K[x_1, \cdots, x_r]$ then*

$$depth_S(JS) = depth_{S'}(J) + n - r.$$

Theorem 2.3.33. *[2] Suppose a monomial ideal $I \subset S$ such that $I = u(I : u)$ for monomial $u \in S$. Then*

(1) $\operatorname{sdepth}_S(I) = \operatorname{sdepth}_S(I : u)$.

(2) $\operatorname{sdepth}_S(S/I) = \operatorname{sdepth}_S(S/(I : u)$.

Lemma 2.3.34. *[16] Suppose $I \subset S = K[x_1, \cdots, x_n]$ is a monomial complete intersection then* $\operatorname{sdepth}(I) = \operatorname{sdepth}(S/I) + 1$ *and*

$$\operatorname{depth}(I) \geq \operatorname{depth}(S/I) + 1.$$

Conjecture 2.3.35. *[13] Suppose a monomial square free ideal $I \subset S$. Then*

$$\operatorname{sdepth}(I) \geq \operatorname{depth}(I).$$

Theorem 2.3.36. *[15] Suppose a monomial ideal $I \subset S$. If $n = 5$ then I is a Stanely ideal and* $\operatorname{sdepth}(S/I) \geq \operatorname{depth}(S/I)$.

Lemma 2.3.37. *[14] Suppose a primary monomial ideal $Q \subset S = K[x_1, \cdots, x_n]$. Then*

$$\operatorname{sdepth}(S/Q) = \dim(S/Q).$$

Lemma 2.3.38. *Suppose an ideal P of $S = K = [x_1, \cdots, x_n]$ such that $P = \sqrt{Q}$. Then*

$$\dim(S/P) = dim(S/Q).$$

Lemma 2.3.39. *[11] Suppose N is a $\mathbb{Z}^n$-graded R module and N_1 and N_2 are its two submodules. Let*

$$0 \to N_1 \to N \to N_2 \to 0$$

is an exact sequence. Then

$$\operatorname{sdepth} N \geq \min\{\operatorname{sdepth} N_1, \operatorname{sdepth} N_2\}.$$

Lemma 2.3.40. *[20] Suppose a short exact sequence of finitely generated graded modules over a graded ring S*

$$0 \to N_0 \to N_1 \to N_2 \to 0$$

Then

(1) For $\operatorname{depth} N_1 < \operatorname{depth} N_2,\ \operatorname{depth} N_0 = \operatorname{depth} N_1.$

(2) For $\operatorname{depth} N_1 = \operatorname{depth} N_2,\ \operatorname{depth} N_0 \geq \operatorname{depth} N_1.$

(3) For $\operatorname{depth} N_1 > \operatorname{depth} N_2,\ \operatorname{depth} N_0 = \operatorname{depth} N_2 + 1.$

Chapter 3

Inequalities for Stanley Depth

In this chapter we will review the paper [3]. We will discuss bounds of Stanley depths for ideals with disjoint support and will also consider the general case. A few equivalent forms of Stanley inequality are discussed in detail.

3.1 The case of ideals with disjoint support

For $S = K[x_1, \cdots, x_n]$ and $n \geq 2$ the support of a monomial $v \in S$ is denoted as $\operatorname{supp}(v) = \{x_j : x_j | v\}$.

Proposition 3.1.1. *Suppose two monomial ideals $I_1 \subset S_1 = K[x_1, \cdots, x_m]$ and $I_2 \subset S_2 = K[x_{m+1}, \cdots, x_n]$ for $1 \leq m \leq n$. Then we have*

(1) $\operatorname{sdepth}_S(I_1 S \cap I_2 S) \geq \operatorname{sdepth}_{S_1}(I_1) + \operatorname{sdepth}_{S_2}(I_2)$.

(2) $\operatorname{sdepth}_S(S/(I_1 S + I_2 S)) \geq \operatorname{sdepth}_{S_1}(S_1/I_1) + \operatorname{sdepth}_{S_2}(S_2/I_2)$.

(3) $\operatorname{depth}_S(S/(I_1 S \cap I_2 S)) - 1 = \operatorname{depth}_S(S/(I_1 S + I_2 S)) = \operatorname{depth}_{S_1}(S_1/I_1) + \operatorname{depth}_{S_2}(S_2/I_2)$.

Proof. (1) For $1 \leq m \leq n$ suppose $a \in I_1 \cap K[x_1, \cdots, x_m]$, $b \in I_2 \cap K[x_{m+1}, \cdots, x_n]$ and $P(J)$ are the monomials from ideal I_2. Define a map $\gamma : P(I_1 \cap K[x_1, \cdots, x_m]) \times$

$P(I_2 \cap K[x_{m+1}, \cdots, x_n]) \to P(I_1 \cap I_2)$, it is injective. Let u be a monomial of $I_1 \cap I_2$ and $u = ab$ for some monomials $a \in K[x_1, \cdots, x_m]$, $b \in K[x_{m+1}, \cdots, x_n]$ which implies $ab \in I_1$ and we have $a \in I_1$ since x_j, $j > m$ are regular on S/I_1. Likewise $b \in I_2$ then $u = \gamma((a, b))$, which means γ is surjective. Suppose $\boldsymbol{D}_1$ is a Stanley decomposition of $I_1 \cap K[x_1, \cdots, x_m]$

$$\boldsymbol{D}_1 : I_1 \cap K[x_1, \cdots, x_m] = \bigoplus_{l=1}^{p} c_l K[Z_l],$$

such that sdepth $\boldsymbol{D}_1 = \text{sdepth}(I_1 \cap K[x_1, \cdots, x_m])$ and $\boldsymbol{D}_2$ is a Stanley decomposition of $I_2 \cap K[x_{m+1}, \cdots, x_n]$

$$\boldsymbol{D}_2 : I_2 \cap K[x_{m+1}, \cdots, x_n] = \bigoplus_{m=1}^{q} d_m K[Z_m],$$

such that sdepth $\boldsymbol{D}_2 = \text{sdepth}(I_2 \cap K[x_{m+1}, \cdots, x_n])$. From bijection they induce a Stanley decomposition

$$\boldsymbol{D} : I_1 \cap I_2 = \bigoplus_{l=1}^{p} \bigoplus_{m=1}^{q} c_l d_m K[Z_l \cup Z_m].$$

So $\text{sdepth}(I_1 \cap I_2) \geq \text{sdepth}\boldsymbol{D} = \min_{a,b} |Z_a| + |Z_b| \geq \min_a |Z_a| + \min_b |Z_b| =$ sdepth $\boldsymbol{D}_1 +$ sdepth $\boldsymbol{D}_2$.

(2) Suppose a Stanley decomposition

$$\boldsymbol{D}_1 : S_1/I_1 = \bigoplus_{x=1}^{p} u_x K[Z_x]$$

of S_1/I_1 such that sdepth $\boldsymbol{D}_1 = \text{sdepth}S_1/I_1$ and

$$\boldsymbol{D}_2 : S_2/I_2 = \bigoplus_{y=1}^{q} v_y K[X_y]$$

of S_2/I_2 such that sdepth $\boldsymbol{D}_2 = \text{sdepth}S_2/I_2$. So $S/I_1S = S_1[x_{m+1}, \cdots, x_n]/I_1S = (S_1/I_1)[x_{m+1}, \cdots, x_n] = \bigoplus_{x=1}^{p} u_x K[Z_x][x_{m+1}, \cdots, x_n] = \bigoplus_{x=1}^{p} u_x K[Z_x, x_{m+1}, \cdots, x_n]$. Likewise $S/I_2S = S_2[x_1, \cdots, x_m]/I_2S = (S_2/I_2)[x_1, \cdots, x_m] = \bigoplus_{y=1}^{q} v_y K[X_y][x_1, \cdots, x_m] = \bigoplus_{y=1}^{q} v_y K[X_y, x_1, \cdots, x_m]$.

For claiming $S/(I_1 + I_2) = \bigoplus_{x,y} u_x v_y K[Z_x, X_y]$ suppose a monomoial $r \in (I_1S \cap I_2S)^c = S/(I_1S \cap I_2S)$ i.e $r \in S$ and $r \notin (I_1S \cap I_2S)$. It implies $r \notin (I_1S)$ and $r \notin (I_2S)$ but $r \in (I_1S)^c$ and $r \in (I_2S)^c$ so we have x, y such that $r \in$

$u_xK[Z_x, x_{m+1}, \cdots, x_n]$ and $r \in v_yK[X_y, x_1, \cdots, x_m]$. Now $r \in u_xK[Z_x, x_{m+1}, \cdots, x_n]$ $\cap v_yK[X_y, x_1, \cdots, x_m]$. As $u_x \in S_1$ and $v_y \in S_2$ so $u_xK[Z_x, x_{m+1}, \cdots, x_n] \cap v_yK[X_y,$ $x_1, \cdots, x_m] = u_xv_yK[Z_x \cap X_y]$.

For the other inclusion suppose a monomial $s \in u_xv_yK[Z_x \cap X_y]$. So $s \in u_xK[Z_x, x_{m+1}$ $, \cdots, x_n] \subset (I_1S)^c$ and $s \in v_yK[X_y, x_1, \cdots, x_m] \subset (I_2S)^c$. Therefore $s \in (I_1S \cap I_2S)^c$. For this sum to be direct suppose $x_1, x_2 \in [x]$ and $y_1, y_2 \in [y]$ such that $(x_1, y_1) \neq (x_2, y_2)$ for say $x_1 \neq x_2$. Now $u_{x_1}v_{y_1}K[Z_{x_1}, X_{y_1}] \cap u_{x_2}v_{y_2}K[Z_{x_2}, X_{y_2}] \subset u_{x_1}K[Z_{x_1}, x_{m+1}$ $, \cdots, x_n] \cap v_{y_2}K[Z_{x_2}, x_{m+1}, \cdots, x_n] = \{0\}$ which shows the claim holds and hence concludes the proof.

(3) Consider the exact sequence of S-modules: $0 \to S/(I_1S \cap I_2S) \to S/I_1S \oplus S/I_2S \to (S/(I_1S+I_2S)) \to 0$. From (2) and depth Lemma we have $\operatorname{depth}_S(S/(I_1S \cap I_2S)) = \operatorname{depth}_S(S/(I_1S + I_2S)) + 1 = \operatorname{depth}_{S_1}(S_1/I_1) + \operatorname{depth}_{S_2}(S_2/I_2) + 1$. □

Lemma 3.1.2. *Suppose* $v_1, v_2 \in S$ *are two monomials and* $P, Q \subset \{x_1, x_2, \cdots, x_n\}$. *Then* $v_1K[P] \cap v_2K[Q] = lcm(v_1, v_2)K[P \cap Q]$ *or* $v_1K[P] \cap v_2K[Q] = (0)$.

Proof. Suppose $v_1K[P] \cap v_2K[Q] \neq (0)$ and a monomial $l(\neq 0) \in v_1K[P] \cap v_2K[Q]$. Then for two monomials $x \in K[P]$ and $y \in K[Q]$, l can be given as $l = v_1x$ and $l = v_2y$. As $v_1|l$ and $v_2|l$, we have $\operatorname{lcm}(v_1, v_2)|l$ so for $z \in K[P] \cap K[Q] = K[P \cap Q]$ we get $l = \operatorname{lcm}(v_1, v_2)z$. Particularly $\operatorname{lcm}(v_1, v_2) = v_1/z \in v_1K[P] \cap v_2K[Q]$ so $\operatorname{lcm}(v_1, v_2)K[P \cap Q] \subset v_1K[P] \cap v_2K[Q]$. As z was randomly selected we have $v_1K[P] \cap v_2K[Q] \subset \operatorname{lcm}(v_1, v_2)K[P \cap Q]$. □

Theorem 3.1.3. *Suppose two monomial ideals* $I_1 \subset S_1 = K[x_1, \cdots, x_m]$ *and* $I_2 \subset S_2 = K[x_{m+1}, \cdots, x_n]$ *for* $1 \leq m \leq n$. *Then we have*

(1) $\operatorname{sdepth}_S(I_1S) > \operatorname{sdepth}_S(I_1S+I_2S) \geq \min\{\operatorname{sdepth}_S(I_1S), \operatorname{sdepth}_{S_2}(I_2)+\operatorname{sdepth}_{S_1}(S_1/I_1)\}$.

(2) $\operatorname{sdepth}_S(S/I_1S) \geq \operatorname{sdepth}_S(S/(I_1S \cap I_2S)) \geq \min\{\operatorname{sdepth}_S(S/I_1S), \operatorname{sdepth}_{S_2}(S_2/I_2) + \operatorname{sdepth}_{S_1}(I_1)\}$.

Proof. (1) Firstly we will prove the 1st inequality. For the ideal $I_1S + I_2S \subset S$ we suppose a Stanley decomposition

$$I_1S + I_2S \ = \bigoplus_{j=1}^{t} l_jK[L_j] \tag{3.1.1}$$

Now take the intersection of this decomposition with S_1

$$(I_1S + I_2S) \cap S_1 \ = (\bigoplus_{j=1}^{t} l_jK[L_j]) \cap S_1 \tag{3.1.2}$$

As $I_1S \cap S_1 = I_1$ and $I_2S \cap S_1 = (0)$. We get $(I_1S \cap S_1 + I_2S \cap S_1) \ = I_1$ so equation 3.1.2 becomes $\qquad I_1 + (0) = (\bigoplus_{j=1}^{t} l_jK[L_j] \cap S_1)$

$$\Rightarrow \quad I_1 = (\bigoplus_{j=1}^{t} l_jK[L_j] \cap S_1) \tag{3.1.3}$$

The L.H.S of equation 3.1.3 has two cases for l_j

(1) If $l_j \in S_1$ then by lemma 3.1.2

$$l_iK[L_i] \cap S_1 = l_iK[L_i \cap \{x_1, \cdots, x_m\}]$$

(2) If $l_j \notin S_1$, then $l_iK[L_i] \cap S_1 = (0)$

Therefore $\qquad I_1 = \bigoplus_{l_i \in S_1} l_jK[L_j \cap \{x_1, \cdots, x_m\}]$

$$\Rightarrow I_1S = \bigoplus_{l_i \in S_1} l_jK[L_j \cup \{x_{m+1}, \cdots, x_n\}] \tag{3.1.4}$$

From equation 3.1.1 and 3.1.4 we have

$$\text{sdepth}_S(I_1S + I_2S) < \text{sdepth}_S(I_1S).$$

For the second inequality suppose two Stanley decompositions

$$I_2 = \bigoplus_{c=1}^{q} y_cK[Y_c]. \tag{3.1.5}$$

and $S_1/I_1 = \bigoplus_{k=1}^{p} x_k K[X_k]$.
We have $S_1/I_1S = \bigoplus_{k=1}^{p} x_k K[X_k \cup \{x_{m+1}, \cdots, x_n\}]$ and $I_2S = \bigoplus_{c=1}^{q} y_c K[Y_c \cup \{x_1, \cdots, x_m\}]$. Now consider K-vector spaces decomposition $I_1S + I_2S = ((I_1S + I_2S) \cap I_1S) \oplus ((I_1S + I_2S) \cap S/I_1S) = I_1S \oplus (I_2S \cap S/I_1S)$. We get

$$\text{sdepth}(I_1S + I_2S) \geq \min\{\text{sdepth}(I_1S), \text{sdepth}(I_2S \cap S/I_1S)\}. \tag{3.1.6}$$

As $I_2S \cap (S/I_1S) \cong (I_2S + I_1S)/I_1S$ so we get
$I_2S \cap S/I_1S = \bigoplus_{k=1}^{p} \bigoplus_{c=1}^{q} x_k K[X_k \cup \{x_1, \cdots, x_m\}] \cap y_c K[Y_c \cup \{x_{m+1}, \cdots, x_n\}]$.
We now apply lemma 3.1.2 as $x_k \in S_1$ and $y_c \in S_2$ for $(k,c)'s$

$$I_2S \cap S/I_1S = \bigoplus_{k=1}^{p} \bigoplus_{c=1}^{q} x_k y_c K[X_k \cup Y_c]. \tag{3.1.7}$$

From equation 3.1.5 and equation 3.1.7

$$\text{sdepth}_S(I_2) \leq \text{sdepth}_{S_2}(I_2S \cap S/I_1S). \tag{3.1.8}$$

Now from equation 3.1.6 and equation 3.1.8 we get second inequality.
(2) Firstly we will prove the 1st inequality. For $S/(I_1S + I_2S)$ suppose a Stanley decomposition

$$S/(I_1S + I_2S) = \bigoplus_{j=1}^{l} r_j K[R_j]. \tag{3.1.9}$$

From proof of (1) we have

$$S/I_1S = \bigoplus_{j=1}^{l} r_j K[R_j \cup \{x_{m+1}, \cdots, x_n\}]. \tag{3.1.10}$$

so $\text{sdepth}_S(S/I_1S) \geq \text{sdepth}_S(S/(I_1S + I_2S))$.
For the 2nd inequality suppose the decomposition
$S/(I_1S \cap I_2S) = (S/(I_1S \cap I_2S) \cap S/I_1S) \oplus (S/(I_1S \cap I_2S) \cap I_1S) = S/I_1S \oplus ((S/I_2S) \cap I_1S)$. From lemma 2.3.39 we get

$$\text{sdepth}_S(S/(I_1S \cap I_2S)) \geq \min\{\text{sdepth}(S/I_1S), \text{sdepth}((S/I_2S) \cap I_1S)\}. \tag{3.1.11}$$

From Proposition 3.1.1(1)

$$\operatorname{sdepth}((S/I_2S) \cap I_1S) \geq \operatorname{sdepth}_{S_1}(I_1) + \operatorname{sdepth}_{S_2}(S_2/I_2). \tag{3.1.12}$$

From equation 3.1.11 and equation 3.1.12 we get the desired inequality. □

Lemma 3.1.4. *Suppose two monomial ideals* $I_1 \subset S_1 = K[x_1, \cdots, x_m]$ *and* $I_2 \subset S_2 = K[x_{m+1}, \cdots, x_n]$ *for* $1 \leq m \leq n$*. Then we have* $\operatorname{depth}_S(I_1S \cap I_2S) = \operatorname{depth}_S(I_1S + I_2S) + 1 = \operatorname{depth}_{S_1}(I_1) + \operatorname{depth}_{S_2}(I_2)$ *and* $\operatorname{depth}_S((I_1S + I_2S)/I_1S) = \operatorname{depth}_S(I_1S + I_2S)$.

Proof. The 1st equality follows directly from Proposition 3.1.1(3). For the 2nd equality we consider the short exact sequence

$$0 \to I_1 \to I_1 + I_2 \to (I_1 + I_2)/I_1 \to 0.$$

Now from lemma 2.3.40(2) we get the required equality. □

Remark 3.1.1. If $I_1 \subset S$ is a monomial ideal, the support of I_1 is defined to be the set $\operatorname{supp}(I_1) = \bigcup_{z \in G(I_1)} \operatorname{supp}(z)$ where $G(I_1)$ is the set on minimal monomial generators of I_1. For two monomial ideals $I_1, I_2 \subset S$ with $\operatorname{supp}(I_1) \cap \operatorname{supp}(I_2) = \emptyset$ and notations stated above notation we reformulate and modify Proposition 3.1.1 and Theorem 3.1.3 as:

$$\operatorname{sdepth}_S(I_1 \cap I_2) \geq \operatorname{sdepth}_S(I_1) + \operatorname{sdepth}_S(I_2) - n.$$

By 2nd isomorphism theorem $I_1/(I_1 \cap I_2) \cong (I_1 + I_2)/I_1$. Now we consider the short exact sequences $0 \to I_1 \to I_1 + I_2 \to (I_1 + I_2)/I_1 \to 0$ so we have $0 \to I_1/(I_1 \cap I_2) \cong (I_1 + I_2)/I_1 \to S/(I_1 \cap I_2) \to (S/I_1) \to 0$. Then clearly $\operatorname{sdepth}_S(I_1 + I_2) \geq \min\{\operatorname{sdepth}_S(I_1), \operatorname{sdepth}_S((I_1+I_2)/I_1\}$ and $\operatorname{sdepth}_S(S/(I_1 \cap I_2)) \geq \min\{\operatorname{sdepth}_S(S/I_1), \operatorname{sdepth}_S((I_1 + I_2)/I_1)\}$.

Observe that $(I_1 + I_2)/I_1 = I_2 \cap (S/I_1)$. Using Theorem 3.1.3(1) when $\operatorname{supp}(I_1) \cap \operatorname{supp}(I_2) = \emptyset$ we obtain

$$\operatorname{sdepth}((I_1 + I_2)/I_1) \geq \{\operatorname{sdepth}_S(I_2) + \operatorname{sdepth}_S(S/I_1) - n\}. \tag{3.1.13}$$

We get equation (3.1.13) again for $(I_1 + I_2)/I_2 = I_1 \cap (S/I_2)$.

For an ideal $I_1 = (v_1, \cdots, v_r) \subset S$ if I_1 is a monomial complete intersection then we have $\operatorname{sdepth}_S(I_1) = n - \lfloor \frac{r}{2} \rfloor$ *[4, Theorem 2.4]* and $\operatorname{sdepth}_S(S/I_1) = n - r$ *[17, Theorem 2.4]*.
If I_1 is any arbitrary monomial ideal then $\operatorname{sdepth}_S(I_1) \geq n - \lfloor \frac{r}{2} \rfloor$*[11, Theorem 2.3]* and $\operatorname{sdepth}_S(S/I_1) \geq n - r$ *[2, Proposition 1.2]*.

Corollary 3.1.5. *Suppose a monomial ideals $I_1 \subset S_1 = K[x_1, \cdots, x_m]$ and $I_2 = (v_1, \cdots, v_r) \subset S_2 = K[x_{m+1}, \cdots, x_n]$ then:*

(1) $\operatorname{sdepth}_S(I_1S) > \operatorname{sdepth}_S(I_1S + I_2S) \geq \min\{\operatorname{sdepth}_S(I_1S), \operatorname{sdepth}_S(S/SI_1) - \lfloor \frac{r}{2} \rfloor\}$.

(2) $\operatorname{sdepth}_S(I_1S \cap I_2S) \geq \operatorname{sdepth}_S(I_1S) - \lfloor \frac{r}{2} \rfloor$.

(3) $\operatorname{sdepth}_S(S/I_1S) \geq \operatorname{sdepth}_S(S/(I_1S \cap I_2S)) \geq \min\{\operatorname{sdepth}_S(S/I_1S), \operatorname{sdepth}_S(I_1 S) - r\}$.

(4) $\operatorname{sdepth}(S/(I_1S + I_2S)) \geq \operatorname{sdepth}_S(S/I_1S) - r$.

(5) *If I_2 is complete intersection ideal then:*
$\operatorname{depth}_S(S/(I_1S \cap I_2S)) - 1 = \operatorname{depth}_S(S/(I_1S + I_2S)) = \operatorname{depth}_S(S/I_1S) - r$.

Proof. (1) As we have

$$\operatorname{sdepth}_{S_2}(I_2) = \operatorname{sdepth}_S(I_2S) - m \geq n - \lfloor \frac{r}{2} \rfloor - m. \tag{3.1.14}$$

$$\operatorname{sdepth}_{S_1}(S_1/I_1) = \operatorname{sdepth}_S(S/I_1S) - (n - m). \tag{3.1.15}$$

Using equations (3.1.14) and (3.1.15) in *[3, Theorem 1.3(1)]* we have
$\operatorname{sdepth}_S(I_1S) \geq \operatorname{sdepth}_S(I_1S+I_2S) \geq \min\{\operatorname{sdepth}_S(I_1S), n-\lfloor \frac{r}{2} \rfloor - m + \operatorname{sdepth}_S(S/I_1 S) - n + m\}$.
$\operatorname{sdepth}_S(I_1S) \geq \operatorname{sdepth}_S(I_1S + I_2S) \geq \min\{\operatorname{sdepth}_S(I_1S), \operatorname{sdepth}_S(S/I_1S) - \lfloor \frac{r}{2} \rfloor\}$.
(2)

$$\operatorname{sdepth}_{S_1}(I_1) = \operatorname{sdepth}_S(I_1S) - (n - m). \tag{3.1.16}$$

Using equations (3.1.14) and (3.1.16) in *[3, Proposition 1.1(1)]* we have
$\operatorname{sdepth}(I_1S \cap I_2S) \geq \min \operatorname{sdepth}_S(I_1S) - (n-m) + n - \lfloor \frac{r}{2} \rfloor - m$
$\operatorname{sdepth}(I_1S \cap I_2S) \geq \min \operatorname{sdepth}_S(I_1S) - \lfloor \frac{r}{2} \rfloor$.

(3)
$$\operatorname{sdepth}_{S_2}(S_2/I_2) = \operatorname{sdepth}_S(S/I_2S) - m \geq (n-r) - m. \quad (3.1.17)$$

$$\operatorname{sdepth}_{S_1}(I_1) = \operatorname{sdepth}_S(I_1S) - (n-m). \quad (3.1.18)$$

Using equations (3.1.18) and (3.1.17) in *[3, Theorem 1.3(2)]* we get
$\operatorname{sdepth}_S(S/I_1S) \geq \operatorname{sdepth}_S(S/(I_1S \cap I_2S)) \geq \min\{\operatorname{sdepth}_S(S/I_1S), (n-r) - m + \operatorname{sdepth}_S(I_1S) - (n-m)\}$.
$\operatorname{sdepth}_S(S/I_1S) \geq \operatorname{sdepth}_S(S/(I_1S \cap I_2S)) \geq \min\{\operatorname{sdepth}_S(S/I_1S), \operatorname{sdepth}_S(I_1S) - r\}$.
(4) Using equations (3.1.15)and (3.1.17) in *[3, Proposition 1.1(2)]* we get
$\operatorname{sdepth}(S/(I_1S + I_2S)) \geq \operatorname{sdepth}_S(S/I_1S) - (n-m) + (n-r) - m$
$\operatorname{sdepth}(S/(I_1S + I_2S)) \geq \operatorname{sdepth}_S(S/I_1S) - r$. (5)

$$\operatorname{depth}_{S_1}(S_1/I_1) = \operatorname{depth}_S(S/I_1S) - (n-m). \quad (3.1.19)$$

$$\operatorname{depth}_{S_2}(S_2/I_2) = \operatorname{depth}_S(S/I_2S) - m = (n-r) - m. \quad (3.1.20)$$

Using equations(3.1.19) and (3.1.20) in *[3, Proposition 1.1(3)]* we have
$\operatorname{depth}_S(S/(I_1S \cap I_2S)) - 1 = \operatorname{depth}_S(S/(I_1S + I_2S)) = \operatorname{depth}_S(S/I_1S) - (n-m) + (n-r) - m$.
$\operatorname{depth}_S(S/(I_1S \cap I_2S)) - 1 = \operatorname{depth}_S(S/(I_1S + I_2S)) = \operatorname{depth}_S(S/I_1S) - r$. □

Remark 3.1.2. Suppose a monomial ideal $J \subset S = K[x_1, \cdots, x_p]$. If $S^* = S[y_1, \cdots, y_q]$ then according to *[3, Corollary 1.6(1)]*
$\operatorname{sdepth}_S(J) + q \geq \operatorname{sdepth}_{S^*}(J, y_1, \cdots, y_q) \geq \min\{\operatorname{sdepth}_S(J) + q, \operatorname{sdepth}_S(S/J) + \lceil \frac{q}{2} \rceil\}$.

Suppose $\operatorname{sdepth}_S(J) + q > \operatorname{sdepth}_{S^*}(J, y_1, \cdots, y_q)$ then $\operatorname{sdepth}_S(J) + q > \operatorname{sdepth}_S(S/J) + \lceil \frac{q}{2} \rceil$ so $\operatorname{sdepth}_S(J) \geq \operatorname{sdepth}(S/J) + \lfloor \frac{q}{2} \rfloor + 1$.
If $q = 1$ and $\operatorname{sdepth}_{S^*}(J, y_1) = \operatorname{sdepth}_S(J)$ then remark 3.1.2 gives $\operatorname{sdepth}_S(J) = \operatorname{sdepth}_{S^*}(J, y_1) \geq \operatorname{sdepth}_S(S/J) + 1$ which implies $\operatorname{sdepth}_S(J) \geq \operatorname{sdepth}_S(S/J) + 1$.

Corollary 3.1.6. *The following hold with notations of Theorem 3.1.3*

1. *(1) If Stanely inequality hold for I_1 and I_2 then Stanely inequality holds for $(I_1S \cap I_2S)$.*

2. *(2) If the Stanley inequality hold for S_1/I_1 and S_2/I_2, then the Stanley inequality holds for $S/(I_1S + I_2S)$.*

3. *(3) If the Stanley inequality hold for I_1, I_2 and S_1/I_1 or for I_1, I_2 and S_2/I_2, then the Stanley inequality holds for $(I_1S + I_2S)$.*

4. *(4) If the Stanley inequality hold for S_1/I_1,S_2/I_2 and I_1 or S_1/I_1, S_2/I_2 and I_1 and I_2, then the Stanley inequality holds for $S/(I_1S \cap I_2S)$.*

Proof. (1) In Proposition 3.1.1(1) use Stanley inequality for I_1 and I_2 then apply Lemma 3.1.4(2) we get the required result.
(2) In Proposition 3.1.1(2) use Stanley inequality for S_1/I_1 and S_2/I_2 then apply Proposition 3.1.1(3) we get the required result.
(3) Suppose Stanley inequality satisfies for I_2 and S_1/I_1 consider the case when $\text{sdepth}_S(I_1S + I_2S) = \text{sdepth}_S(I_1S)$ then Theorem 3.1.3(1) gives us $\text{sdepth}_S(I_1S + I_2S) \geq \text{sdepth}_S(I_1S) = \text{depth}_{S_1}(I_1) + n - m \geq \text{depth}_{S_1}(I_1) + \text{depth}_{S_2}(I_2)$ then by Lemma(3.1.4) $\text{depth}_{S_1}(I_1) + \text{depth}_{S_2}(I_2) > \text{depth}_S(I_1S + I_2S)$ so we finally have $\text{sdepth}_S(I_1S + I_2S) > \text{depth}_S(I_1S + I_2S)$.
When $\text{sdepth}_S(I_1S + I_2S) < \text{sdepth}_S(I_1S)$ Theorem 3.1.3(1) gives $\text{sdepth}_S(I_1S + I_2S) \geq \text{sdepth}_{S_2}(I_2) + \text{sdepth}_{S_1}(S_1/I_1)$ from Stanley inequalities of I_2 and S_1/I_1 we get $\text{sdepth}_S(I_1S + I_2S) \geq \text{depth}_{S_2}(I_2) + \text{depth}_{S_1}(S_1/I_1)$ then by Lemma (3.1.4) $\text{depth}_{S_2}(I_2) + \text{depth}_{S_1}(S_1/I_1) = \text{depth}_S(I_1S + I_2S)$ so we get the form $\text{sdepth}_S(I_1S + I_2S) \geq \text{depth}_S(I_1S + I_2S)$.
Likewise for I_1 and S_2/I_2 when $\text{sdepth}_S(I_1S + I_2S) = \text{sdepth}_S(I_2S)$ by Theorem 3.1.3(1) we have $\text{sdepth}_S(I_1S + I_2S) \geq \text{sdepth}_S(I_2S) = \text{depth}_{S_1}(I_2) + m \geq \text{depth}_{S_1}(I_2) + \text{depth}_{S_2}(I_1)$ then by Lemma(3.1.4) $\text{depth}_{S_1}(I_1) + \text{depth}_{S_2}(I_2) > \text{depth}_S(I_1$

$S + I_2S)$ so we finally have $\text{sdepth}_S(I_1S + I_2S) > \text{depth}_S(I_1S + I_2S)$.
When $\text{sdepth}_S(I_1S + I_2S) < \text{sdepth}_S(I_2S)$ Theorem 3.1.3(1) gives $\text{sdepth}_S(I_1S + I_2S) \geq \text{sdepth}_{S_1}(I_1) + \text{sdepth}_{S_2}(S_2/I)$ from Stanley inequalities of I_1 and S_2/I_2 we have $\text{sdepth}_S(I_1S+I_2S) \geq \text{depth}_{S_1}(I_1)+\text{depth}_{S_2}(S_2/I_2) = \text{depth}_{S_1}(I_1)+\text{depth}_{S_2}(I_2)$ then using Lemma (3.1.4) $\text{depth}_{S_1}(I_1) + \text{depth}_{S_2}(I_2) - 1 = \text{depth}_S(I_1S + I_2S)$ so we have $\text{sdepth}_S(I_1S + I_2S) \geq \text{depth}_S(I_1S + I_2S)$.
(4) From Theorem 3.1.3(2) $\text{sdepth}_S(S/I_1S \cap I_2S) \geq \text{sdepth}_S(S/I_1S)$. Also $\text{sdepth}_S(S/I_1S) = \text{sdepth}_{S_1}(S_1/I_1) + n - m$. As Stanley inequality holds for S_1/I_1 so $\text{sdepth}_{S_1}(S_1/I_1) + n - m \geq \text{depth}_{S_1}(S_1/I_1) + n - m$ then $\text{depth}_{S_1}(S_1/I_1) + n - m \geq \text{depth}_{S_1}(S_1/I_1) + \text{depth}_{S_2}(I_2)$. Using Lemma 2.3.34 $\text{depth}_{S_1}(S_1/I_1) + \text{depth}_{S_2}(I_2) \geq \text{depth}_{S_1}(S_1/I_1) + \text{depth}_{S_2}(S_2/I_2) + 1$ then from Proposition 1.1(3) $\text{depth}_{S_1}(S_1/I_1) + \text{depth}_{S_2}(S_2/I_2) + 1 = \text{depth}(S/(I_1S \cap I_2S)) - 1 + 1 = \text{depth}(S/(I_1S \cap I_2S))$.

Theorem 3.1.3(2) gives $\text{sdepth}_S(S/I_1S \cap I_2S) \geq \text{sdepth}_S(S/I_2S)$. As $\text{sdepth}_S(S/I_2S) = \text{sdepth}_{S_2}(S_2/I_2) + m$. from Stanley inequality of S_2/I_2 we get $\text{sdepth}_{S_2}(S_2/I_2) + m \geq \text{depth}_{S_2}(S_2/I_2) + m$ then $\text{depth}_{S_2}(S_2/I_2) + m \geq \text{depth}_{S_2}(S_2/I_2) + \text{depth}_{S_1}(I_1)$. From Lemma 2.2.13 $\text{depth}_{S_2}(S_2/I_2) + \text{depth}_{S_1}(I_1) \geq \text{depth}_{S_2}(S_2/I_2) + \text{depth}_{S_1}(S_1/I_1) + 1$ then using Proposition 1.1(3) $\text{depth}_{S_2}(S_2/I_2) + \text{depth}_{S_1}(S_1/I_1) + 1 = \text{depth}(S/(I_1S \cap I_2S)) - 1 + 1 = \text{depth}(S/(I_1S \cap I_2S))$. □

From the above corollary if $\text{sdepth}_S(I_1S + I_2S) = \text{sdepth}_S(I_1S)$ and Stanley inequality holds for I_1 i.e $\text{sdepth}_S(I_1) \geq \text{depth}_S(I_1)$, then $\text{sdepth}_S(I_1S + I_2S) \geq \text{depth}_S(I_1S + I_2S) + n - m - \text{depth}_{S_2}(S_2/I_2)$. Likewise if $\text{sdepth}_S(S/(I_1S \cap I_2S)) = \text{sdepth}_S(I_1S)$ and Stanley inequality holds for S_1/I_1 i.e $\text{sdepth}_S(S_1/I_1) \geq \text{depth}_S(S_1/I_1)$, then $\text{sdepth}_S(S/(IS \cap I_2S)) \geq \text{depth}_S(S/(I_1S \cap I_2S)) + n - m - \text{depth}_{S_2}(S_2/I_2)$.

Corollary 3.1.7. *Consider the monomial ideals $I_t \subset S_t = K[x_{t1}, \cdots, x_{tm_t}]$ for $q \geq 2$, $m_t \geq 1$ and $1 \leq t \leq q$ and $S = K[x_{ti} : 1 \leq t \leq q, 1 \leq i \leq m_t]$ then:*

(1) $\text{sdepth}_S(I_1S \cap \cdots \cap I_qS) \geq \text{sdepth}_{S_1}(I_1) + \cdots + \text{sdepth}_{S_q}(I_q)$.

(2) $\text{sdepth}_S(I_1S+\cdots+I_qS) \geq \min\{\text{sdepth}_{S_1}(I_1)+m_2+\cdots+m_q, \text{sdepth}_{S_2}(I_2)+\text{sdepth}_{S_1}(S_1/I_1)+m_3+\cdots+m_q, \cdots, \text{sdepth}_{S_q}(I_q)+\text{sdepth}_{S_{q-1}}(S_{q-1}/I_{q-1})+\cdots+\text{sdepth}_{S_1}(S_1/I_1)\}$.
$\text{sdepth}_S(I_1S+\cdots+I_qS) \leq \min\{\text{sdepth}_S(I_tS) : t=1,\cdots,q\}$

(3) $\text{sdepth}_S(S/(I_1S\cap\cdots\cap I_qS) \geq \min\{\text{sdepth}_{S_1}(S_1/I_1)+m_2+\cdots+m_q, \text{sdepth}_{S_2}(S_2/I_2)+\text{sdepth}_{S_1}(I_1)+m_3+\cdots+m_q, \cdots, \text{sdepth}_{S_q}(S_q/I_q)+\text{sdepth}_{S_{q-1}}(I_{q-1})+\cdots+\text{sdepth}_{S_1}(I_1)\}$.
$\text{sdepth}_S(S/(I_1S\cap\cdots\cap I_qS) \leq \min\{\text{sdepth}_S(S/I_tS) : t=1,\cdots,q\}$

(4) $\text{sdepth}_S(S/(I_1S+\cdots+I_qS) \geq \text{sdepth}_{S_1}(S_1/I_1)+\cdots+\text{sdepth}_{S_q}(S_q/I_q)$.

(5) $\text{depth}_S(I_1S\cap\cdots\cap I_qS) = \text{depth}_S(I_1S+\cdots+I_qS)+(q-1) = \text{depth}_{S_1}(I_1)+\cdots+\text{depth}_{S_q}(I_q)$.

Proof. (1) We will use induction. $q=2$ satisfies by Proposition 3.1.1(1). Consider $S^* = m_1+\cdots+m_{q-1}$ and Suppose (1) holds for $q-1$ i.e
$\text{sdepth}_{S^*}(I_1S\cap\cdots\cap I_{q-1}S) \geq \text{sdepth}_{S_1}(I_1)+\cdots+\text{sdepth}_{S_{q-1}}(I_{q-1})$.
Now we need to proof it holds for q then
$\text{sdepth}_S((I_1S\cap\cdots\cap I_{q-1}S)\cap I_qS) \geq \text{sdepth}_{S^*}(I_1S\cap\cdots\cap I_{q-1}S)\cap\text{sdepth}_{S_q}(I_qS) \geq (\text{sdepth}_{S_1}(I_1)+\cdots+\text{sdepth}_{S_{q-1}}(I_{q-1}))+\text{sdepth}_{S_q}(I_q)$ by using Proposition 3.1.1(1).
$\text{sdepth}_S((I_1S\cap\cdots\cap I_{q-1}S)\cap I_qS) \geq \text{sdepth}_{S_1}(I_1)+\cdots+\text{sdepth}_{S_{q-1}}(I_{q-1})+\text{sdepth}_{S_q}(I_q)$.
(2) $q=2$ holds according to Theorem 3.1.3(1). Consider $S^* = m_1+\cdots+m_{q-1}$ and assume (2) holds for $q-1$ i.e $\text{sdepth}_{S^*}(I_1S+\cdots+I_{q-1}S) \geq \min\{\text{sdepth}_{S_1}(I_1)+m_2+\cdots+m_{q-1}, \text{sdepth}_{S_2}(I_2)+\text{sdepth}_{S_1}(S_1/I_1)+m_3+\cdots+m_{q-1}, \cdots, \text{sdepth}_{S_{q-1}}(I_{q-1})+\text{sdepth}_{S_{q-2}}(S_{q-2}/I_{q-2})+\cdots+\text{sdepth}_{S_1}(S_1/I_1)\}$.
Then $\text{sdepth}_S(I_1S+\cdots+I_{q-1}S+(I_qS)) \geq (\text{sdepth}_{S^*}(I_1S+\cdots+I_{q-1}S))+\text{sdepth}_{S_q}(I_qS)$
$\geq \min\{\text{sdepth}_{S_1}(I_1)+m_2+\cdots+m_{q-1}+m_q, \text{sdepth}_{S_2}(I_2)$
$+\text{sdepth}_{S_1}(S_1/I_1)+m_3+\cdots+m_{q-1}+m_q, \cdots, (\text{sdepth}_{S_q}(I_q)+\text{sdepth}_{S_{q-1}}(S_{q-1}/I_{q-1}))+\text{sdepth}_{S_{q-1}}(I_{q-1})+\text{sdepth}_{S_{q-2}}(S_{q-2}/I_{q-2})+\cdots+\text{sdepth}_{S_1}(S_1/I_1\}$ holds for q using Theorem 3.1.3(1).

(3) $q = 2$ is satisfied according to Theorem 3.1.3(2). Consider $S^* = m_1 + \cdots + m_{q-1}$ and assume (3) satisfies for $q-1$ i.e
$\operatorname{sdepth}_{S^*}(S/(I_1S \cap \cdots \cap I_{q-1}S) \geq \min\{\operatorname{sdepth}_{S_1}(S_1/I_1) + m_2 + \cdots + m_{q-1}, \operatorname{sdepth}_{S_2}(S_2/I_2) + \operatorname{sdepth}_{S_1}(I_1) + m_3 + \cdots + m_{q-1}, \cdots, \operatorname{sdepth}_{S_{q-1}}(S_{q-1}/I_{q-1}) + \operatorname{sdepth}_{S_{q-2}}(I_{q-2}) + \cdots + \operatorname{sdepth}_{S_1}(I_1)\}$.
$\operatorname{sdepth}_S(S/(I_1S \cap \cdots \cap I_{q-1}S \cap (I_qS)) \geq \min\{\operatorname{sdepth}_{S_1}(S_1/I_1) + m_2 + \cdots + m_{q-1} + m_q, \operatorname{sdepth}_{S_2}(S_2/I_2) + \operatorname{sdepth}_{S_1}(I_1) + m_3 + \cdots + m_{q-1} + m_q, \cdots, (\operatorname{sdepth}_{S_q}(S_q/I_q) + \operatorname{sdepth}_{S_{q-1}}(I_{q-1})) + \operatorname{sdepth}_{S_{q-1}}(S_{q-1}/I_{q-1}) + \operatorname{sdepth}_{S_{q-2}}(I_{q-2}) + \cdots + \operatorname{sdepth}_{S_1}(I_1)\}$ is satisfied for q applying Theorem 3.1.3(2).
(4) It holds for $q = 2$ according to Proposition 3.1.1(2). We have $S^* = m_1 + \cdots + m_{q-1}$ and consider (4) holds for $q-1$ i.e $\operatorname{sdepth}_{S^*}(S/(I_1S + \cdots + I_{q-1}S)) \geq \operatorname{sdepth}_{S_1}(S/I_1) + \cdots + \operatorname{sdepth}_{S_{q-1}}(S_{q-1}/I_{q-1})$
We have $\operatorname{sdepth}_S(S/(I_1S + \cdots + I_{q-1}S + (I_qS))) \geq \operatorname{sdepth}_{S^*}(S^*/(I_1S + \cdots + I_{q-1}S)) + \operatorname{sdepth}_{S_q}(S_q/I_q) \geq \operatorname{sdepth}_{S_1}(S_1/I_1) + \cdots + \operatorname{sdepth}_{S_{q-1}}(S_{q-1}/I_{q-1}) + (\operatorname{sdepth}_{S_q}(S_q/I_q))$ holds for q by using 3.1.1(2).
(5) $q = 2$ is satisfied according to Lemma 3.1.4. We have $S^* = m_1 + \cdots + m_{q-1}$ and consider (5) satisfies for $q-1$ i.e
$\operatorname{depth}_{S^*}(I_1S \cap \cdots \cap I_{q-1}S) = \operatorname{depth}_{S^*}(I_1S + \cdots + I_{q-1}S) + (q-2) = \operatorname{depth}_{S_1}(I_1) + \cdots + \operatorname{depth}_{S_{q-1}}(I_{q-1})$
Then we get by Lemma 3.1.4 $\operatorname{depth}_S((I_1S \cap \cdots \cap I_{q-1}S) \cap I_qS) = \operatorname{depth}_S((I_1S \cap \cdots \cap I_{q-1}S) + I_qS) + 1 = \operatorname{depth}_{S^*}(I_1S \cap \cdots \cap I_{q-1}S) + \operatorname{sdepth}_S(I_qS) = \operatorname{depth}_{S^*}(I_1S + \cdots + I_{q-1}S) + (q-2) + \operatorname{depth}_{S_q}(I_qS) = \operatorname{depth}_S(I_1S + \cdots + I_{q-1}S + I_qS) + 1 + (q-2) = \operatorname{depth}_S(I_1S + \cdots + I_{q-1}S + I_qS) + (q-1)$. By Lemma 3.1.4 $\operatorname{depth}_S(I_1S + \cdots + I_{q-1}S + I_qS) + (q-1) = \operatorname{depth}_{S^*}(I_1S + \cdots + I_{q-1}S) + \operatorname{depth}_{S_q}(Iq) = \operatorname{depth}_{S_1}(I_1) + \cdots + \operatorname{depth}_{S_{q-1}}(I_{q-1}) + \operatorname{depth}_{S_q}(I_q)$ is satisfied for q. □

Corollary 3.1.8. *With the notations of the above Corollary following hold:*

(1) *If $I_1, \cdots, I_q$ satisfy the Stanley inequality, then $I_1S \cap \cdots \cap I_qS$ satisfies the Stanley inequality.*

(2) *If* $S/I_1, \cdots, S/I_q$ *satisfy the Stanley inequality, then* $S/(I_1S+\cdots+I_qS)$ *satisfies the Stanley inequality.*

(3) *If* $1 \leq l \leq m$ *is an integer and the Stanley inequality hold for* I_t *for all* $1 \leq t \leq m$ *and for* S/I_t *for all* $t \neq l$ *then, the Stanley inequality holds for* $I_1S + \cdots + I_qS$.

(4) *If* $1 \leq l \leq m$ *is an integer and the Stanley inequality hold for* S_t/I_t *for all* $1 \leq t \leq m$ *and for* I_t *for all* $t \neq l$ *then, the Stanley inequality holds for* $S/(I_1S \cap \cdots \cap I_qS)$.

Proof. (1) We will apply induction on q. $q = 2$ holds according to Proposition 3.1.1(1). When $S^* = m_1 + \cdots + m_{q-1}$ suppose (1) is satisfied for $q-1$
$\operatorname{sdepth}_{S^*}(I_1S \cap \cdots \cap I_{q-1}S) \geq \operatorname{depth}_{S^*}(I_1S \cap \cdots \cap I_{q-1}S)$. We need to show (1) is satisfied for q then by Corollary 3.1.7(1)
$\operatorname{sdepth}_S(I_1S \cap \cdots \cap I_qS) \geq \operatorname{sdepth}_{S_1}(I_1S) + \cdots + \operatorname{sdepth}_{S_q}(I_qS)$. Since Stanley inequality satisfies for I_t's so we get $\operatorname{sdepth}_S(I_1S \cap \cdots \cap I_qS) \geq \operatorname{depth}_{S_1}(I_1S) + \cdots + \operatorname{depth}_{S_q}(I_qS)$. Now by using Lemma 3.1.4 $\operatorname{sdepth}_S(I_1S \cap \cdots \cap I_qS) \geq \operatorname{depth}_S(I_1S \cap \cdots \cap \operatorname{depth}_S I_qS)$.

(2) $q = 2$ is satisfied by Proposition 3.1.1(2). When $S^* = m_1 + \cdots + m_{q-1}$ suppose (2) is satisfied for $q-1$
$\operatorname{sdepth}_{S^*}(I_1S \cap \cdots \cap I_{q-1}S) \geq \operatorname{depth}_{S^*}(I_1S \cap \cdots \cap I_{q-1}S)$. For q we apply Corollary 3.1.7(4) $\operatorname{sdepth}_S S/(I_1S + \cdots + I_qS) \geq \operatorname{sdepth}_{S_1}(S_1/I_1) + \cdots + \operatorname{sdepth}_{S_q}(S_1/I_q)$. As Stanley inequality satisfies for S/I_t's so we have $\operatorname{sdepth}_S S/(I_1S + \cdots + I_qS) \geq \operatorname{depth}_{S_1}(S_1/I_1) + \cdots + \operatorname{depth}_{S_q}(S_1/I_q)$. Now by using Proposition 3.1.1(3) $\operatorname{sdepth}_S(I_1S \cap \cdots \cap I_qS) \geq \operatorname{depth}_S(I_1S \cap \cdots \cap I_qS)$.

(3) $q = 2$ holds by Theorem 3.1.3(1). For $q > 2$ we suppose $l = q$. We have $S^* = K[x_{ti} : 1 \geq t \geq q-1, 1 \geq i \geq m_t]$ and suppose an ideal $I^* = I_1S^* + \cdots + I_{q-1}S^* \subset S$. Now using Corollary 3.1.7(4) $\operatorname{sdepth}_{S^*}(S^*/I^*) = \operatorname{sdepth}_{S^*}(S^*/(I_1S^* + \cdots + I_{q-1}S^*)) \geq \operatorname{sdepth}_{S_1}(S_1/I_1) + \cdots + \operatorname{sdepth}_{S_{q-1}}(S_{q-1}/I_{q-1})$. Since Stanley inequality

satisfies for S/I_t's so we have $\text{sdepth}_{S^*}(S^*/(I_1S^*+\cdots+I_{q-1}S^*)) \geq \text{depth}_{S_1}(S_1/I_1) + \cdots + \text{depth}_{S_{q-1}}(S_{q-1}/I_{q-1})$. Using Proposition 3.1.1(3) $\text{sdepth}_{S^*}(S^*/(I_1S^* + \cdots + I_{q-1}S^*)) \geq \text{depth}_{S^*}(S^*/(I_1S^* + \cdots + I_{q-1}S^*))$. From induction Stanley inequality satisfies for I^*. For $I = I_1S + \cdots + I_qS = I^*S + I_qS$ as Stanley inequality holds for I^*, S^*/I^*, I_q so by the case of induction for $q = 2$ Stanley inequality holds for I.

(4) $q = 2$ holds by Theorem 3.1.3(2). For the case $q > 2$ we suppose $l = q$. We have $S^* = K[x_{ti} : 1 \geq t \geq q-1, 1 \geq i \geq m_t]$ and suppose an ideal $I^* = I_1S^* \cap \cdots \cap I_{q-1}S^* \subset S$. Using Corollary 3.1.7(1) $\text{sdepth}_{S^*}(I^*) = \text{sdepth}_{S^*}(I_1S^* \cap \cdots \cap I_{q-1}S^*) \geq \text{sdepth}_{S_1}(I_1) + \cdots + \text{sdepth}_{S_{q-1}}(I_{q-1})$. Stanley inequality satisfies for I_t's so we get $\text{sdepth}_{S^*}(I_1S^* \cap \cdots \cap I_{q-1}S^*) \geq \text{depth}_S(I_1S) + \cdots + \text{depth}_{S_{q-1}}(I_{q-1}S)$. By applying Lemma 3.1.4 $\text{sdepth}_{S^*}(I_1S^* \cap \cdots \cap I_{q-1}S^*) \geq \text{depth}_{S^*}(I_1S^* \cap \cdots \cap I_{q-1}S^*)$. From induction Stanley inequality holds for S^*/I^*. Now for $I = I_1S \cap \cdots \cap I_qS = I^*S \cap I_qS$ since Stanley inequality is satisfied for I^*, S^*/I^*, S_q/I_q so by the case of induction for $q = 2$ Stanley inequality holds for I. □

Corollary 3.1.9. *Following the notations of Corollary 3.1.7, if all $m_t \leq 5$ and I_t's are square free, then $I_1S \cap \cdots \cap I_qS$, $I_1S + \cdots + I_qS$, $S/(I_1S + \cdots + I_qS)$ and $S/(I_1S \cap \cdots \cap I_qS)$ satisfy the Stanley inequality.*

Proof. From Conjecture 2.3.35 and Theorem 2.3.36 for a squarefree monomial ideal $J \subset K[x_1, \cdots, x_m]$ and $m \leq 5$ we know J and S/J satisfy the Stanley inequality so J's and S/J's satisfy the Stanley inequality. Using Corollary 3.1.8(1,2,3,4) we directly get required results. □

Example 3.1.1. Suppose $J = (y_{11}, \cdots, y_{1m_1}) \cap (y_{21}, \cdots, y_{2m_2}) \cap \cdots \cap (y_{q1}, \cdots, y_{qm_q}) \subset S$ for $q \geq 2$,$1 \leq t \leq q$ and $S = K[x_{ti} : 1 \leq t \leq q, 1 \leq i \leq m_t]$. Now if $J = P_1 \cap \cdots \cap P_q$ when $G(P_r) \cap G(P_s) = \emptyset \ \ \forall\ r \neq s$ then $\text{depth}_S(S/J) = q - 1$ then from Lemma 2.3.34 $\text{depth}_S(J) = q$ and from Corollary 3.1.7(1) $\text{sdepth}_S(J) \geq \lceil m_1/2 \rceil + \cdots + \lceil m_q/2 \rceil$ so $\text{sdepth}_S(J) \geq \text{depth}_S(J)$. From *[9, Theorem 1.1]* we have $\text{sdepth}_S(J) \leq \min\{m - \lfloor m_t/2 \rfloor : 1 \leq t \leq q\}$. Now using Corollary 3.1.7(3)

$$\text{sdepth}_S(S/J) \geq \min\{m_2 + \cdots + m_q, \lceil m_1/2\rceil + m_3 + \cdots + m_q, \lceil m_1/2\rceil + \lceil m_2/2\rceil + m_4 + \cdots + m_q, \cdots, \lceil m_1/2\rceil + \cdots + \lceil m_{q-1}/2\rceil + m_q\}.$$

we have $\text{sdepth}_S(S/J) \geq \text{depth}_S(S/J) = q-1$. From Corollary 3.1.7(3) we have $\text{sdepth}_S(S/J) \leq \{m - m_t : 1 \leq t \leq q\}$.

3.2 The General case

Suppose $p, q, m \in \mathbb{Z}$ such that $1 \leq p \leq q+1 \leq m$ and $m \geq 2$. Denote $S = K[x_1, \cdots, x_n]$, $S_1 = K[x_1, \cdots, x_q]$ and $S_2 = K[x_p, \cdots, x_m]$ where $l = q - p + 1$.

Lemma 3.2.1. *Suppose two monomials $v_1 \in S_1$ and $v_2 \in S_2$, two set of variables $P \subset \{x_1, \cdots, x_q\}$ and $Q \subset \{x_p, \cdots, x_m\}$. Denote $\overline{P} = P \cup \{x_{q+1}, \cdots, x_m\}$ and $\overline{Q} = Q \cup \{x_1, \cdots, x_{p-1}\}$. When $M = v_1K[\overline{P}] \cap v_2K[\overline{Q}]$,then $M = (0)$ or $M = lcm(v_1, v_2)K[(P \cup Q) \setminus R]$ where $R \subset \{x_p, \cdots, x_q\}$ and $|(P \cup Q) \setminus R| \geq |P| + |Q| - l$*

Proof. $M = (0)$ is obvious. For $M \neq (0)$ from Lemma 3.1.2 we have $M = lcm(v_1, v_2)$ $K[\overline{P} \cap \overline{Q}]$. As $\overline{P} \cap \overline{Q} = (P \cup Q) \setminus R$ so we get $M = lcm(v_1, v_2)K[(P \cup Q) \setminus R]$ for $R \subset \{x_p, \cdots, x_q\}$. ☐

We will generalize some results for Proposition 1.1 and Theorem 1.3

Theorem 3.2.2. *Suppose two monomial ideals $I_1 \subset S_1$ and $I_2 \subset S_2$. Then we have*

(1) $\text{sdepth}_S(I_1S \cap I_2S) \geq \text{sdepth}_{S_1}(I_1) + \text{sdepth}_{S_1}(I_2) - l = \text{sdepth}_S(I_1S) + \text{sdepth}_S(I_2S) - m$.

(2) $\text{sdepth}_S(S/(I_1S + I_2S)) \geq \text{sdepth}_{S_1}(S_1/I_1) + \text{sdepth}_{S_2}(S_2/I_2) - l = \text{sdepth}_S(S/I_1S) + \text{sdepth}_S(S/I_2S) - m$.

(3) $\text{sdepth}_S(I_1S + I_2S) \geq \min\{\text{sdepth}_S(I_1S), \text{sdepth}_{S_2}(I_2) + \text{sdepth}_{S_1}(S_1/I_1) - l\} = \min\{\text{sdepth}_S(I_1S), \text{sdepth}_S(I_2S) + \text{sdepth}_S(S/I_1S) - m\}$.

(4) $\mathrm{sdepth}_S(S/(I_1S \cap I_2S)) \geq \min\{\mathrm{sdepth}_S(S/I_1S), \mathrm{sdepth}_{S_2}(S_2/I_2) + \mathrm{sdepth}_{S_1}(I_1) - l\} = \min\{\mathrm{sdepth}_S(S/I_1S), \mathrm{sdepth}_S(S/I_2S) + \mathrm{sdepth}_S(I_1S) - m\}$.

Proof. (1) Suppose two Stanley decompositions for I_1 and I_2 as $I_1 = \bigoplus_{a=1}^{i} v_aK[P_a]$ and $I_2 = \bigoplus_{b=1}^{j} u_bK[Q_b]$ respectively. If $\overline{P_a} = P_a \cup \{x_{q+1}, \cdots, x_m\}$ then $I_1S = \bigoplus_{a=1}^{i} v_aK[\overline{P_a}]$ and for $\overline{Q_b} = Q_b \cup \{x_1, \cdots, x_{p-1}\}$, $I_2S = \bigoplus_{b=1}^{j} v_bK[\overline{Q_b}]$. For $M_{ab} = v_aK[\overline{P_a}]u_bK[\overline{Q_b}]$ we get $I_1S \cap I_2S = \bigoplus_{a=1}^{i}\bigoplus_{b=1}^{j} M_{ab}$. Now Lemma 3.2.1 can be applied which gives $M_{ab} = \{0\}$ or $M_{ab} = lcm(v_a, u_b)K[(P_a \cup Q_b) \setminus R_{ab}]$ for $R_{ab} \subset \{x_p, \cdots, x_q\}$ and $|(P_a \cup Q_b) \setminus R_{ab}| \geq |P_a| + |Q_b| - l$.

(2) Suppose two Stanley decompositions for S_1/I_1 and S_2/I_2 as $S_1/I_1 = \bigoplus_{i=1}^{r} v_iK[P_i]$ and $S_2/I_2 = \bigoplus_{j=1}^{s} u_jK[Q_j]$ respectively. Then $S/(I_1S) = \bigoplus_{i=1}^{r} v_iK[\overline{P_i}]$ and for $\overline{P_i} = P_i \cup \{x_{q+1}, \cdots, x_m\}$ also $\overline{Q_j} = Q_j \cup \{x_1, \cdots, x_{p-1}\}$ for $S/(I_2S) = \bigoplus_{j=1}^{s} u_jK[\overline{Q_j}]$. We get $(S/I_1S) \cap (S/I_2S) = \bigoplus_{i=1}^{r}\bigoplus_{j=1}^{s} M_{ij}$ for $M_{ij} = v_iK[\overline{P_i}]u_jK[\overline{Q_j}]$. As $S/(I_1S + I_2S) = (S/I_1S) \cap (S/I_2S)$ so $\mathrm{sdepth}_S(S/(I_1S + I_2S)) = \mathrm{sdepth}_S((S/I_1S) \cap (S/I_2S)) = \bigoplus_{i=1}^{r}\bigoplus_{j=1}^{s} M_{ij}$. Now we apply Lemma 3.2.1, $M_{ij} = \{0\}$ or $M_{ij} = lcm(v_i, u_j)K[(P_i \cup Q_j) \setminus R_{ij}]$ for $R_{ij} \subset \{x_p, \cdots, x_q\}$ and $|(P_i \cup Q_j) \setminus R_{ij}| \geq |P_i| + |Q_j| - l$.

(3) Assume two Stanley decompositions $S_1/I_1 = \bigoplus_{a=1}^{i} v_aK[P_a]$ and $I_2 = \bigoplus_{b=1}^{j} u_bK[Q_b]$ for S_1/I_1 and I_2 respectively. We have $S_1/I_1 = \bigoplus_{a=1}^{i} v_aK[\overline{P_a}]$ where $\overline{P_a} = P_a \cup \{x_{q+1}, \cdots, x_m\}$ and $I_2S = \bigoplus_{b=1}^{j} v_bK[\overline{Q_b}]$ for $\overline{Q_b} = Q_b \cup \{x_1, \cdots, x_{p-1}\}$. As

$$I_1S + I_2S = ((I_1S + I_2S) \cap I_1S) \oplus ((I_1S + I_2S) \cap (S/I_1S)) = I_1S \oplus (I_2S \cap (S/I_1S)).$$

We get $\mathrm{sdepth}_S(I_1S + I_2S) \geq \min\{\mathrm{sdepth}_S(I_1S), \mathrm{sdepth}_S(I_2S \cap (S/I_1S))\}$. So we get a Stanley decomposition for $I_1S \cap I_2S$ as $I_2S \cap (S/I_1S) = \bigoplus_{a=1}^{i}\bigoplus_{b=1}^{j} M_{ab}$ where $M_{ab} = v_aK[\overline{P_a}]u_bK[\overline{Q_b}]$. From Lemma 3.2.1 $M_{ab} = \{0\}$ or $M_{ab} = lcm(v_a, u_b)K[(P_a \cup Q_b) \setminus R_{ab}]$ where $R_{ab} \subset \{x_p, \cdots, x_q\}$ and $|(P_a \cup Q_b) \setminus R_{ab}| \geq |P_a| + |Q_b| - l$ that gives $\mathrm{sdepth}_S(I_2S \cap (S/I_1S)) \geq \mathrm{sdepth}_{S_1}(S_1/I_1) + \mathrm{sdepth}_{S_2}(I_2)$.

(4) Suppose two Stanley decompositions for I_1 and S_2/I_2 as $I_1 = \bigoplus_{a=1}^{i} v_aK[P_a]$ and $S_1/I_2 = \bigoplus_{b=1}^{j} u_bK[Q_b]$ respectively. We have $I_1S = \bigoplus_{a=1}^{i} v_aK[\overline{P_a}]$ where $\overline{P_a} =$

$P_a \cup \{x_{q+1}, \cdots, x_m\}$ and $S/(I_2S) = \bigoplus_{b=1}^{j} v_b K[\overline{Q_b}]$ for $\overline{Q_b} = Q_b \cup \{x_1, \cdots, x_{p-1}\}$. We apply the decomposition

$$S/(I_1S \cap I_2S) = (S/(I_1S \cap I_2S) \cap I_1S) \oplus (S/(I_1S \cap I_2S) \cap (S/I_1S)) =$$
$$S/(I_1S) \oplus (I_1S \cap (S/I_2S)).$$

As $\text{sdepth}_S(I_1S \cap (S/I_2S)) \geq \min\{\text{sdepth}_S(S/I_1S) + \text{sdepth}_S((S/I_2S) \cap (I_1S)\}$ follows from above decomposition. We have $I_1S \cap (S/I_2S) = \bigoplus_{a=1}^{i} \bigoplus_{b=1}^{j} M_{ab}$ when $M_{ab} = v_a K[\overline{P_a}] u_b K[\overline{Q_b}]$. Using Lemma 3.2.1 $M_{ab} = \{0\}$ or $M_{ab} = lcm(v_a, u_b)K[(P_a \cup Q_b) \setminus R_{ab}]$ where $R_{ab} \subset \{x_p, \cdots, x_q\}$ and $|(P_a \cup Q_b) \setminus R_{ab}| \geq |P_a| + |Q_b| - l$ that follows $\text{sdepth}_S(I_1S \cap (S/I_2S)) \geq \{\text{sdepth}_{S_2}(S_2/I_2) + \text{sdepth}_{S_1}(I_1)\}$. □

Remark 3.2.1. Results of the previous theorem do not depend on the numbers p,q. So we can rewrite Theorem 3.2.2 as randomly chosen monomial ideals $I_1, I_2 \subset S$. If $I_1, I_2 \subset S$ are two monomial ideals, the minimal number l which can be selected by a reordering of the variables is $l = |\text{supp}(I_1) \cap \text{supp}(I_2)|$.
As observed in Remark 3.1.1 $(I_1 + I_2)/I_1 = I_2 \cap (S/I_1)$ so $\text{sdepth}_S(I_1 + I_2)/I_1 = \text{sdepth}_S(I_2 \cap (S/I_1)) \geq \text{sdepth}_{S_2}(I_2) + \text{sdepth}_{S_1}(S_1/I_1) \geq \text{sdepth}_S(I_2) + \text{sdepth}_S(S/I_1) - m$. Particularly if $I_1 \subset I_2$ then $\text{sdepth}_S(I_2/I_1) \geq \text{sdepth}_S(I_2) + \text{sdepth}_S(S/I_1) - m$.

Corollary 3.2.3. *If $I_1, I_2 \subset S$ are monomial ideals and $|G(I_1) = r|$ then:*

(1) $\text{sdepth}_S(I_1 \cap I_2) \geq \text{sdepth}_S(I_1) - \lfloor r/2 \rfloor$.

(2) $\text{sdepth}_S(I_1 + I_2) \geq \min\{\text{sdepth}_S(I_1), \text{sdepth}_S(S/I_1) - \lfloor r/2 \rfloor\}$.
$\text{sdepth}_S(I_1 + I_2) \geq \text{sdepth}_S(I_1) - r$.

(3) $\text{sdepth}_S(S/(I_1 + I_2)) \geq \text{sdepth}_S(S/I_1) - r$.

(4) $\text{sdepth}_S(S/(I_1 \cap I_2)) \geq \min\{\text{sdepth}_S(S/I_1), \text{sdepth}_S(I_1) - r\}$.
$\text{sdepth}_S(S/(I_1 \cap I_2)) \geq \min\{m - r, \text{sdepth}_S(S/I_1) - \lfloor r/2 \rfloor\}$.

(5) $\text{sdepth}_S((I_1 + I_2)/I_2) \geq \text{sdepth}_S(S/I_1) - \lfloor r/2 \rfloor$.
$\text{sdepth}_S((I_1 + I_2)/I_2) \geq \text{sdepth}_S(I_1) - r$.

Proof. (1) From theorem 3.2.2 (1) we have $\operatorname{sdepth}_S(I_1S \cap I_2S) \geq \operatorname{sdepth}_S(I_1S) + \operatorname{sdepth}_S(I_2S) - m = \operatorname{sdepth}_S(I_1S) + (m - \lfloor r/2 \rfloor) - m$ by using $\operatorname{sdepth}_S(I_2S) \geq m - \lfloor r/2 \rfloor$. Solving we get $\operatorname{sdepth}_S(I_1S \cap I_2S) \geq \operatorname{sdepth}_S(I_1S) - \lfloor r/2 \rfloor$.

(2) From Theorem 3.2.2 (3) $\operatorname{sdepth}_S(I_1S+I_2S) \geq \min\{\operatorname{sdepth}_S(I_1S), \operatorname{sdepth}_S(I_2S)+\operatorname{sdepth}_S(S/I_1S)-m\} \geq \min\{\operatorname{sdepth}_S(I_1S), (m-\lfloor r/2 \rfloor)+\operatorname{sdepth}_S(S/I_1S)-m\}$ using $\operatorname{sdepth}_S(I_2S) \geq m-\lfloor r/2 \rfloor$. We get $\operatorname{sdepth}_S(I_1S+I_2S) \geq \min\{\operatorname{sdepth}_S(I_1S), \operatorname{sdepth}_S(S/I_1S) - \lfloor r/2 \rfloor\}$ after solving. Also we have $\operatorname{sdepth}_S(I_1S + I_2S) \geq \operatorname{sdepth}_S(I_1) - r$.

(3) From Theorem 3.2.2 (2)$\operatorname{sdepth}_S(S/(I_1S+I_2S)) \geq \operatorname{sdepth}_S(S/I_1S)+\operatorname{sdepth}_S(S/I_2S) - m \geq \operatorname{sdepth}_S(S/I_1S) + (m - r) - m$ using $\operatorname{sdepth}_S(S/I_2) \geq m - r$. We finally have $\operatorname{sdepth}_S(S/(I_1S + I_2S)) \geq \operatorname{sdepth}_S(S/I_1S) - r$.

(4) From Theorem 3.2.2(4) $\operatorname{sdepth}_S(S/(I_1S \cap I_2S)) \geq \min\{\operatorname{sdepth}_S(S/I_1S), \operatorname{sdepth}_S(S/I_2S) + \operatorname{sdepth}_S(I_1S) - m\} \geq \min\{\operatorname{sdepth}_S(S/I_1S), (m - r) + \operatorname{sdepth}_S(I_1S) - m\}$ using $\operatorname{sdepth}_S(S/I_2) \geq m - r$. By solving we get $\operatorname{sdepth}_S(S/(I_1S \cap I_2S)) \geq \min\{\operatorname{sdepth}_S(S/I_1S), \operatorname{sdepth}_S(I_1S) - r\}$. We also have $\operatorname{sdepth}_S(S/(I_1S \cap I_2S)) \geq \min\{m - r, \operatorname{sdepth}_S(S/I_1) - \lfloor r/2 \rfloor\}$.

(5) As from Remark 3.2.1 $\operatorname{sdepth}_S(I_1 + I_2)/I_1 \geq \operatorname{sdepth}_S(I_2) + \operatorname{sdepth}_S(S/I_1) - m \geq (m - \lfloor r/2 \rfloor) + \operatorname{sdepth}_S(S/I_1) - m$. By solving we have $\operatorname{sdepth}_S(I_1 + I_2)/I_1 \geq \operatorname{sdepth}_S(S/I_1) - \lfloor r/2 \rfloor$. And we get $\operatorname{sdepth}_S(I_1 + I_2)/I_2 \geq \operatorname{sdepth}_S(I_1) - r$. □

Corollary 3.2.4. *If $I_1 \subset S$ is a monomial ideal and $v \in S$ is a monomial then:*

(1) $\operatorname{sdepth}_S(I_1 \cap (v)) \geq \operatorname{sdepth}_S(I_1)$.

(2) $\operatorname{sdepth}_S(I_1, v) \geq \min\{\operatorname{sdepth}_S(I_1), \operatorname{sdepth}_S(S/I_1)\}$.

(3) $\operatorname{sdepth}_S(S/(I_1, v)) \geq \operatorname{sdepth}_S(S/I_1) - 1$.

(4) $\operatorname{sdepth}_S(S/(I_1 \cap (v))) \geq \operatorname{sdepth}_S(S/I_1)$.

Proof. (1) Using Corollary 3.2.3(1) and substituting $|G(I_2)| = 1$ we have
$\operatorname{sdepth}_S(I_1 \cap (v)) \geq \operatorname{sdepth}_S(I_1) - \lfloor 1/2 \rfloor \geq \operatorname{sdepth}_S(I_1) - 0 \geq \operatorname{sdepth}_S(I_1)$.
which is the required result.

(2) Using Corollary 3.2.3(2) we get
$\operatorname{sdepth}_S(I_1,(v)) \geq \min\{\operatorname{sdepth}_S(I_1), \operatorname{sdepth}_S(S/I_1) - \lfloor 1/2 \rfloor\} \geq \min\{\operatorname{sdepth}_S(I_1), \operatorname{sdepth}_S(S/I_1)\}$.
(3) Using Corollary 3.2.3(3)
$\operatorname{sdepth}_S(S/(I_1,(v))) \geq \operatorname{sdepth}_S(S/I_1) - 1$.
(4) From Corollary 3.2.3(4)
$\operatorname{sdepth}_S(S/(I_1 \cap (v))) \geq \min\{\operatorname{sdepth}_S(S/I_1), \operatorname{sdepth}_S(I_1) - 1\}$. Using Theorem 3.1.3 $\operatorname{sdepth}_S(S/(I_1 \cap (v))) \geq \min\{\operatorname{sdepth}_S(S/I_1), \operatorname{sdepth}_S(S/I_1)\}$ so $\operatorname{sdepth}_S(S/(I_1 \cap (v))) \geq \operatorname{sdepth}_S(S/I_1)$. □

Theorem 3.2.5. *If $I_1 \subset S$ is a monomial ideal such that $I_1 = v(I_1 : v)$ for a monomial $v \in S$ then:*

(1) $\operatorname{sdepth}_S(I_1) = \operatorname{sdepth}_S(I_1 : v)$.

(2) $\operatorname{sdepth}_S(S/(I_1 : v)) \geq \operatorname{sdepth}_S(S/I_1)$.

Proposition 3.2.6. *If $I_1 \subset S$ is a monomial ideal and $v \in S$ is a monomial then:*

(1) $\operatorname{sdepth}_S(I_1 : v) \geq \operatorname{sdepth}_S(I_1)$.

(2) $\operatorname{sdepth}_S(S/I_1) \leq \operatorname{sdepth}_S(S/(I_1 : v))$.

Proof. (1) In Theorem 3.2.5(1) apply the fact $I_1 \cap (v) = v(I_1 : v)$ and then use Corollary 3.2.4(1) to get the desired result.
(2) By Theorem 3.2.5(2) and $I_1 \cap (v) = v(I_1 : v)$ and then using Corollary 3.2.4(4) follows the required proof. □

For an ideal $J \subset S$, if $P \in \operatorname{Ass}(S/J)$ is an associated prime then there exists a monomial $u \in S$ such that $P = (J : u)$.

Corollary 3.2.7. *For a monomial ideal $J \subset S$, with Ass$(S/J) = \{P_1, \cdots, P_t\}$, if we denote $d_j = $ ht(P_j) then:*

(1) $\text{sdepth}_S(J) \leq \min\{m - \lfloor d_j/2 \rfloor : \ j = 1, \cdots, t\}$.

(2) $\text{sdepth}_S(S/J) \leq \min\{m - d_j : \ j = 1, \cdots, t\}$.

Proof. (1) Notice that $\text{sdepth}_S(P_j) = m - d_j/2$. Now from Proposition 3.2.6(1) we get the required result.
(2) Notice that $\text{sdepth}_S(P_j) = m - d_j$. Then by Proposition 3.2.6(2) we are done. □

Corollary 3.2.8. *Suppose $J \subset S$ is a monomial ideal minimally generated by r monomials, such that there exists a prime ideal $P \in Ass(S/J)$ with $ht(P) = r$. Then* $\text{sdepth}_S(S/J) = m - r$.

Proof. It directly follows from Theorem 3.2.6(2) and Corollary 3.2.7(2). □

Remark 3.2.2. Assume a monomial ideal $J \subset S$. Then sdepth_S $(S/J) = m - 1$ iff J is principal. J is principal if and only if all the primes in $\text{Ass}(S/J)$ have height 1. Then by Corollary 3.2.7(2) we get desired form.

Corollary 3.2.9. *Suppose $q \geq 2$ be an integer, and let $I_j \subset S$ be some monomial ideals, when $1 \leq j \leq q$. Then:*

(1) $\text{sdepth}_S(I_1 \cap \cdots \cap I_q) \geq \text{sdepth}_S(I_1) + \cdots + \text{sdepth}_S(I_q) - m(q-1)$.

(2) $\text{sdepth}_S(I_1 + \cdots + I_q) \geq \min\{\text{sdepth}_S(I_1), \text{sdepth}_S(I_2) + \text{sdepth}_S(S/I_1) - m, \cdots, \text{sdepth}_S(I_q) + \text{sdepth}_S(S/I_{q-1}) + \cdots + \text{sdepth}_S(S/I_1) - m(q-1)\}$.

(3) $\text{sdepth}_S(S/(I_1 \cap \cdots \cap I_q)) \geq \min\{\text{sdepth}_S(S/I_1), \text{sdepth}_S(S/I_2) + \text{sdepth}_S(I_1) - m, \cdots, \text{sdepth}_S(S/I_q) + \text{sdepth}_S(I_{q-1}) + \cdots + \text{sdepth}_S(I_1) - m(q-1)\}$.

(4) $\text{sdepth}_S(S/(I_1 + \cdots + I_q)) \geq \text{sdepth}_S(S/I_1) + \cdots + \text{sdepth}_S(S/I_q) - m(q-1)$.

Proof. (1) Use induction on q.
If $q = 2$, then by Theorem 3.2.2(1), we have

$$\text{sdepth}_S(I_1 \cap I_2) \geq \text{sdepth}_S(I_1) + \text{sdepth}_S(I_2) - m$$

Now let $S^* = m_1 + \cdots + m_{q-1}$ then suppose by induction that (1) holds for $q-1$

$$\operatorname{sdepth}_{S^*}(I_1 \cap \cdots \cap I_{q-1}) \geq \operatorname{sdepth}_{S^*}(I_1) + \cdots + \operatorname{sdepth}_{S^*}(I_{q-1}) - m(q-2).$$

Then for q we have
$\operatorname{sdepth}_S((I_1 \cap \cdots \cap I_{q-1}) \cap I_q) = \operatorname{sdepth}_{S^*}(I_1 \cap \cdots \cap I_{q-1}) + \operatorname{sdepth}_{S_q}(I_q) - m \geq (\operatorname{sdepth}_S(I_1) + \cdots + \operatorname{sdepth}_S(I_{q-1}) - m(q-2)) + \operatorname{sdepth}_S(I_q) - m \geq \operatorname{sdepth}_S(I_1) + \cdots + \operatorname{sdepth}_S(I_{q-1}) + \operatorname{sdepth}_S(I_q) - m(q-1)$.
(2) Use induction on q.
If $q = 2$, then by Theorem 3.2.2(3), we get

$$\operatorname{sdepth}_S(I_1 + I_2) \geq \min\{\operatorname{sdepth}_S(I_1), \operatorname{sdepth}_S(I_2) + \operatorname{sdepth}_S(S/I_1) - m\}$$

Now let $S^* = m_1 + \cdots + m_{q-1}$ then suppose by induction that (2) holds for $q-1$

$$\operatorname{sdepth}_{S^*}(I_1 + \cdots + I_{q-1}) \geq \min\{\operatorname{sdepth}_S(I_1), \operatorname{sdepth}_S(I_2) + \operatorname{sdepth}_S(S/I_1) - m, \cdots, \operatorname{sdepth}_S(I_{q-1}) + \operatorname{sdepth}_S(S/I_{q-2}) + \cdots + \operatorname{sdepth}_S(S/I_1) - m(q-2)\}.$$

Now from Theorem 3.2.2(2) for q we have $\operatorname{sdepth}_S(I_1 + \cdots + I_{q-1} + I_q) = \operatorname{sdepth}_S((I_1 + \cdots + I_{q-1}) + I_q) \geq \min\{\operatorname{sdepth}_{S^*}(I_1 + \cdots + I_{q-1}), \operatorname{sdepth}_S(I_q) + \operatorname{sdepth}_{S^*}(S/(I_1 + \cdots + I_{q-1})S) - m\}$.
From Corollary 1.1.9(4), we get
$\operatorname{sdepth}_S((I_1 + \cdots + I_{q-1}) + I_q) \geq \min\{\min \operatorname{sdepth}_S(I_1), \operatorname{sdepth}_S(I_2) + \operatorname{sdepth}_S(S/I_1) - m, \cdots, \operatorname{sdepth}_S(I_{q-1}) + \operatorname{sdepth}_S(S/I_{q-2}) + \cdots + \operatorname{sdepth}_S(S/I_1) - m(q-2)\}, \operatorname{sdepth}_S(I_q) + \operatorname{sdepth}_S(S/I_1) + \operatorname{sdepth}_S(S/I_2) + \cdots + \operatorname{sdepth}_S(S/I_{q-1}) - m(q-2) - m\}$.
$\operatorname{sdepth}_S((I_1 + \cdots + I_{q-1}) + I_q) \geq \min\{\operatorname{sdepth}_S(I_1), \operatorname{sdepth}_S(I_2) + \operatorname{sdepth}_S(S/I_1) - m, \cdots, \operatorname{sdepth}_S(I_q) + \operatorname{sdepth}_S(S/I_{q-1}) + \cdots + \operatorname{sdepth}_S(S/I_1) - m(q-1)\}$.
(3) Use induction on q.
If $q = 2$, then by Theorem 3.2.2(4), we have

$$\operatorname{sdepth}_S(S/(I_1 \cap I_2)) \geq \min\{\operatorname{sdepth}_S(S/I_1), \operatorname{sdepth}_S(S/I_2) + \operatorname{sdepth}_S(I_1) - m\}$$

Now let $S^* = m_1 + \cdots + m_{q-1}$ then suppose by induction that (3) holds for $q-1$

$\mathrm{sdepth}_{S^*}(S/(I_1 \cap \cdots \cap I_{q-1})) \geq \min\{\mathrm{sdepth}_S(S/I_1), \mathrm{sdepth}_S(S/I_2) + \mathrm{sdepth}_S(I_1) - m, \cdots, \mathrm{sdepth}_S(S/I_{q-1}) + \mathrm{sdepth}_S(I_{q-2}) + \cdots + \mathrm{sdepth}_S(I_1) - m(q-2)\}$.

Now from Theorem 3.2.2(4) for q we have $\mathrm{sdepth}_S(S/(I_1 \cap \cdots \cap I_{q-1}) \cap I_q)$ $\geq \min\{\mathrm{sdepth}_{S^*}(S/(I_1 \cap \cdots \cap I_{q-1})), \mathrm{sdepth}_S(S/I_q) + \mathrm{sdepth}_{S^*}(I_1 \cap \cdots \cap I_{q-1}) - m\}$. Using Corollary 1.1.9(1) $\mathrm{sdepth}_S(S/(I_1 \cap \cdots \cap I_{q-1}) \cap I_q) \geq \min\{\min\{\mathrm{sdepth}_S(S/I_1),$ $\mathrm{sdepth}_S(S/I_2) + \mathrm{sdepth}_S(I_1) - m, \cdots, \mathrm{sdepth}_S(S/I_{q-1}) + \mathrm{sdepth}_S(I_{q-2}) + \cdots + \mathrm{sdepth}_S$ $(I_1) - m(q-2)\}, \mathrm{sdepth}_S(S/I_q) + \mathrm{sdepth}_S(I_1) + \cdots + \mathrm{sdepth}_S(I_{q-1}) - m(q-2) - m\} \geq \min\{\mathrm{sdepth}_S(S/I_1), \mathrm{sdepth}_S(S/I_2) + \mathrm{sdepth}_S(I_1) - m, \cdots, \mathrm{sdepth}_S(S/I_q) + \mathrm{sdepth}_S(I_{q-1}) + \cdots + \mathrm{sdepth}_S(I_1) - m(q-1)\}$. which is the required form for q.
(4) Use induction on q.
If $q = 2$, then by Theorem 3.2.2(2), we have

$$\mathrm{sdepth}_S(S/(I_1 + I_2)) \geq \mathrm{sdepth}_S(S/I_1) + \mathrm{sdepth}_S(S/I_2) - m$$

Now let $S^* = m_1 + \cdots + m_{q-1}$ then suppose by induction that (4) holds for $q - 1$

$$\mathrm{sdepth}_S(S/(I_1 + \cdots + I_{q-1})) \geq \mathrm{sdepth}_S(S/I_1) + \cdots + \mathrm{sdepth}_S(S/I_{q-1}) - m(q-2)$$

Now from Theorem 3.2.2(2) for q we have $\mathrm{sdepth}_S(S/(I_1 + \cdots + I_{q-1} + I_q)) \geq \mathrm{sdepth}_{S^*}(S/(I_1 + \cdots + I_{q-1})) + \mathrm{sdepth}_S(S/I_q) - m \geq \mathrm{sdepth}_S(S/I_1) + \cdots + \mathrm{sdepth}_S(S/I_{q-1}) - m(q-2) + \mathrm{sdepth}_S(S/I_q) - m \geq \mathrm{sdepth}_S(S/I_1) + \cdots + \mathrm{sdepth}_S(S/I_q) - m(q-1)$. □

Corollary 3.2.10. *Suppose I_1, I_2 are two monomial ideals such that $G(I_2) = \{v_1, \cdots, v_q\}$ is the set of minimal monomial generators of I_2. Then*

(1) $\mathrm{sdepth}_S(I_1 : I_2) \geq \mathrm{sdepth}_S(I_1 : v_1) + \mathrm{sdepth}_S(I_1 : v_2) + \cdots + \mathrm{sdepth}_S(I_1 : v_q) - m(q-1) \geq q\mathrm{sdepth}_S(I_1) - m(k-1)$.

(2) $\mathrm{sdepth}_S(S/(I_1 : I_2)) \geq \min\{\mathrm{sdepth}_S(S/(I_1 : v_1)), \mathrm{sdepth}_S(S(I_2 : v_2)) + \mathrm{sdepth}_S$ $(I_1 : v_1) - m, \cdots, \mathrm{sdepth}_S(S/(I_1 : v_q)) + \mathrm{sdepth}_S(I_1 : v_{q-1}) + \cdots + \mathrm{sdepth}_S(I_1 : v_1) - m(q-1) \geq \mathrm{sdepth}_S(S/I_1) + (q-1)\mathrm{sdepth}_S(I_1) - m(q-1)$.

Proof. (1) As we know $(I_1 : I_2) = (I_1 : v_1) \cap (I_1 : v_2) \cap \cdots \cap (I_1 : v_q)$ so from Corollary 3.2.9(1) $\text{sdepth}_S(I_1 : I_2) \geq \text{sdepth}(I_1 : v_1) + \text{sdepth}(I_1 : v_2) + \cdots + (I_1 : v_q) - m(q-1)$ and 2nd inequality follows from Proposition 3.2.6(1) $(I_1 : I_2) \geq \text{sdepth}(I_1) + \text{sdepth}(I_1) + \cdots + (I_1) - m(q-1) \geq q\text{sdepth}(I_1) - m(q-1)$.
(2) Likewise from Corollary 3.2.9(3) we get $\text{sdepth}_S(S/(I_1 : I_2)) \geq \{\text{sdepth}(S/(I_1 : v_1)), \text{sdepth}(S/(I_1 : v_2)) + \text{sdepth}(I_1 : v_1) - m, \cdots, \text{sdepth}(S/(I_1 : v_q)) + \text{sdepth}(I_1 : v_{q-1}) + \cdots + \text{sdepth}((I_1 : v_1) - m(q-1)$. Now using Proposition 3.2.6(2) $\text{sdepth}_S(S/(I_1 : I_2)) \geq \text{sdepth}_S(S/I_1) + (q-1)\text{sdepth}_S(I_1) - m(q-1)$. □

Assume $I_1 \subset S$ a monomial ideal and the irredundant minimal decomposition of I_1 is $I_1 = Q_1 \cap \cdots \cap Q_q$. Denote $P_i = \sqrt{Q_i}$ for $1 \leq i \leq q$ and $\text{Ass}(S/I_1) = \{P_1, \cdots, P_q\}$. Particularly when I_1 is squarefree then $Q_i = P_i$ for all i. Denote $d_i = ht(P_i)$, for $1 \leq i \leq q$. Suppose $d_1 \geq d_2 \geq \cdots \geq d_q$ then we have following bounds for $\text{sdepth}_S(I_1)$ and $\text{sdepth}_S(S/I_1)$.

Corollary 3.2.11. *(1)* $m - \lfloor d_1/2 \rfloor \geq \text{sdepth}_S(I_1) \geq m - \lfloor d_1/2 \rfloor - \cdots - \lfloor d_q/2 \rfloor$.

(2) $m - d_1 \geq \text{sdepth}_S(S/I_1) \geq m - \lfloor d_1/2 \rfloor - \cdots - \lfloor d_{q-1}/2 \rfloor - d_q$.

Proof. (1) For the 1st inequality use Corollary 3.2.7(1) $\text{sdepth}_S(I_1) \leq \min\{m - \lfloor d_i/2 \rfloor : i = 1, \cdots, s\} \leq m - \lfloor d_1/2 \rfloor$. From Corollary 3.2.9(1) $\text{sdepth}_S(I_1) \geq m - \lfloor d_i/2 \rfloor \geq m - \lfloor d_1/2 \rfloor - \cdots - \lfloor d_q/2 \rfloor - m(q-1) \geq m - \lfloor d_1/2 \rfloor - \cdots - \lfloor d_q/2 \rfloor$ which is the required 2nd inequality.
(2) The 1st inequality follows directly from Corollary 3.2.7(2). From Corollary 3.2.9(3) $\text{sdepth}_S(S/(I_1^* \cap \cdots \cap I_q)) \geq \min\{\text{sdepth}_S(S/I_1^*), \text{sdepth}_S(S/I_2) + \text{sdepth}_S(I_1^*) - m, \cdots, \text{sdepth}_S(S/I_q) + \text{sdepth}_S(I_{q-1}) + \cdots + \text{sdepth}_S(I_1^*) - m(q-1)\}$ now using Corollary 3.2.7(1) and Corollary 3.2.7(2) $\text{sdepth}_S(S/(I_1^* \cap \cdots \cap I_q)) \geq m - d_1, m - d_2 + m - \lfloor d_1/2 \rfloor - m, \cdots, m - d_q + m - \lfloor d_{q-1}/2 \rfloor + \cdots + m - \lfloor d_1/2 \rfloor - m(q-1) \geq m - \lfloor d_1/2 \rfloor - \cdots - \lfloor d_{q-1}/2 \rfloor - d_q$. □

More generally, assume the primary irredundant decomposition of I_1 is $I_1 = C_1 \cap \cdots \cap C_q$, $P_j = \sqrt{C_j}$. Denote $c_i = \text{sdepth}_S(C_i)$ and $d_i = ht(P_i)$. Suppose

$d_1 \geq d_2 \geq \cdots \geq d_q$. Observe that $c_i \leq m - d_i/2$, for a monomial $v_i \in S$ we have $P_i = (C_i : v_i)$ therefore by Proposition 3.2.6(1) $\operatorname{sdepth}_S(C_i) \leq \operatorname{sdepth}_S(P_i)$. We also have $\operatorname{sdepth}_S(S/C_i) = \operatorname{sdepth}_S(S/P_i)$. The following bounds for $\operatorname{sdepth}_S(I_1)$ and $\operatorname{sdepth}_S(S/I_1)$ are obtained by simple computations.

Corollary 3.2.12. *(1)* $m - \lfloor d_1/2 \rfloor \geq \operatorname{sdepth}_S(I_1) \geq c_1 + \cdots + c_q - m(q-1)$.

(2) $m - d_1 \geq \operatorname{sdepth}_S(S/I_1) \geq \min\{m - d_1, c_1 - d_2, c_1 + c_2 - d_3 - m, \cdots, c_1 + \cdots + c_{q-1} - d_q - m(q-2)$.

Proof. (1) From Proposition 3.2.7(1) $\operatorname{sdepth}_S(I_1) \leq \min\{m - \lfloor d_i/2 \rfloor : i = 1, \cdots, s\} \leq \min\{m - \lfloor d_1/2 \rfloor, \cdots, m - \lfloor d_q/2 \rfloor$ so $\operatorname{sdepth}_S(I_1) \leq m - \lfloor d_1/2 \rfloor$. Using Corollary 3.2.9 $\operatorname{sdepth}_S(I_1) \geq \operatorname{sdepth}_S(I_1^*) + \cdots + \operatorname{sdepth}_S(I_q) - m(q-1) \geq c_1 + \cdots + c_q - m(q-1)$.
(2) From Proposition 3.2.7(2) $\operatorname{sdepth}_S(S/I_1) \leq m - d_1$. For the 2nd inequality use Corollary 3.2.9(3) $\operatorname{sdepth}_S(S/(I_1^* \cap \cdots \cap I_q)) = \operatorname{sdepth}_S(S/I_1) \geq \min\{\operatorname{sdepth}_S(S/I_1^*), \operatorname{sdepth}_S(S/I_2) + \operatorname{sdepth}_S(I_1^*) - m, \cdots, \operatorname{sdepth}_S(S/I_q) + \operatorname{sdepth}_S(I_{q-1}) + \cdots + \operatorname{sdepth}_S(I_1^*) - m(q-1)\} \geq \min\{m - d_1, m - d_2 + c_1 - m, m - d_3 + c_2 + c_1 - 2m, \cdots, c_1, \cdots, c_{q-1} - d_q - m(q-1) \geq \min\{m - d_1, c_1 - d_2, c_1 + c_2 - d_3 - m, \cdots, c_1 + \cdots + c_{q-1} - d_q - m(q-2)\}$. □

Example 3.2.1. Suppose $I_1 = C_1 \cap C_2 \cap C_3 \subset S := K[y_1, \cdots, y_7]$, for $C_1 = (y_2^2, \cdots, y_5^2)$, $C_2 = (y_4^3, y_5^3, y_6^3)$ and $C_3 = (y_6^3, y_6 y_7, y_7^2)$. Denote $P_j = \sqrt{C_j}$. Observe $c_3 = \operatorname{sdepth}_S(C_3) = \operatorname{sdepth}_{K[y_6, y_7]}(C_3 \cap K[y_6, y_7]) + 5 = 1 + 5 = 6$. From *[4, Theorem 1.3]* $c_1 = 7 - \lfloor 5/2 \rfloor = 5$ and $c_2 = 7 - \lfloor 3/2 \rfloor = 6$.
Corollary 3.2.14(1) gives
$5 = 7 - \lfloor d_1/2 \rfloor \geq \operatorname{sdepth}_S(I_1) \geq c_1 + c_2 + c_3 - 14 = 3$.
From Corollary 3.2.12(2) we have
$2 = 7 - d_1 \geq \operatorname{sdepth}_S(S/I_1) \geq \min\{7 - d_1, c_1 - d_2, c_1 + c_2 - d_3 - 7\} = \min\{7 - 5, 5 - 3, 5 + 6 - 2 - 7\} = 2$.
Therefore $\operatorname{sdepth}_S(I_1) \in \{3, 4, 5\}$ and $\operatorname{sdepth}_S(S/I_1) = 2$.
Also $\operatorname{depth}_S(S/I_1) \geq \min\{m - \operatorname{depth}_S(S/P_i) : i = 1, 2, 3\} = 2$. We have $\operatorname{sdepth}_S(I_1) \geq$

$\operatorname{depth}_S(I_1)$ and $\operatorname{sdepth}_S(S/I_1) \geq \operatorname{depth}_S(S/I_1)$. Using CoCoA we have $\operatorname{depth}_S(S/I_1)$ $= 2$.

Proposition 3.2.13. *Suppose two monomial ideals $I_1 \subset I_2 \subset S = K[y_1, \cdots, y_n]$ and denote $S^* = S[x]$. Then:*

$\operatorname{sdepth}_S(I_2/I_1)+1 \geq \operatorname{sdepth}_{S^*}((I_2S^*+(x))/I_1S^*) \geq \min\{\operatorname{sdepth}_S(I_2/I_1), \operatorname{sdepth}_S(S/$

$I_1) + 1\}$.

Proof. For the 1st inequality consider a Stanley decomposition $(I_2S^* + (x))/I_1S^* = \bigoplus_{j=1}^{l} v_jK[Z_j]$ and $((I_2S^* + (x))/I_1S^*) \cap S = I_2/I_1$. So we have $S = I_2/I_1 = \bigoplus_{j=1}^{l} v_jK[Z_j] \cap S = \bigoplus_{x|v_j} v_jK[Z_j\{x\}]$ a Stanley decomposition. For the second inequality consider the short exact sequence

$$0 \to I_2/I_1 \to (I_2S^* + (x))/I_1S^* \to (S/I_1)[x] \to 0.$$

Thus we have $(I_2S^*+(x))/I_1S^* = I_2/I_1 \bigoplus (S/I_1)[x]$ and we obtain desired inequality. □

3.3 Equivalent forms of Stanley inequality

Proposition 3.3.1. *The following are equivalent:*

(1) For any integer $m \geq 1$ and any monomial ideal $I_1 \subset S = K[x_1, \cdots, x_m]$, Stanley inequality holds for I_1, i.e. $\operatorname{sdepth}_S(I_1) \geq \operatorname{depth}_S(I_1)$.

(2) For any integer $m \geq 1$ and any monomial ideals $I_1, I_2 \subset S$, if $\operatorname{sdepth}_S(I_1 + I_2) \geq \operatorname{depth}_S(I_1 + I_2)$, *then* $\operatorname{sdepth}_S(I_1) \geq \operatorname{depth}_S(I_1)$.

(3) For any integers $m, k \geq 1$, any monomial ideal $I_1 \subset S = K[x_1, \cdots, x_m]$, if $v_1, \cdots, v_k \in S$ is a regular sequence on S/I_1 and $I_2 = (v_1, \cdots, v_k)$, then if:

$$\operatorname{sdepth}_S(I_1 + I_2) \geq \operatorname{depth}_S(I_1 + I_2) \Rightarrow \operatorname{sdepth}_S(I_1) \geq \operatorname{depth}_S(I_1).$$

(4) For any integers $m, k \geq 1$,any monomial ideal $I_1 \subset S = K[x_1, \cdots, x_m]$, if $v_1, \cdots, v_k \in S$ is a regular sequence on S/I_1 and $I_2 = (v_1, \cdots, v_k)$, then if:

$$\operatorname{sdepth}_S(I_1 + I_2) = \operatorname{depth}_S(I_1 + I_2) \Rightarrow \operatorname{sdepth}_S(I_1) = \operatorname{depth}_S(I_1).$$

(5) For any integer $m \geq 1$, any monomial ideal $I_1 \subset S = K[x_1, \cdots, x_m]$, if $S^* = S[z]$, then:

$$\operatorname{sdepth}_{S^*}(I_1, z) = \operatorname{depth}_S(I_1) \Rightarrow \operatorname{sdepth}_S(I_1) = \operatorname{depth}_S(I_1).$$

Proof. (1)⇒(2)⇒(3) is clearly obvious.
(3)⇒(4). Suppose $\operatorname{sdepth}_S(I_1 + I_2) = \operatorname{depth}_S(I_1 + I_2)$. Observe $\operatorname{depth}_S(I_1 + I_2) = \operatorname{depth}_S(I_1) - k$, because $v_1, \cdots, v_k \in S$ is a regular sequence on S/I_1. Using Corollary 3.2.3(2) we have $\operatorname{depth}_S(I_1 + I_2) \geq \operatorname{depth}_S(I_1) - k$. As $\operatorname{sdepth}_S(I_1) \geq \operatorname{depth}_S(I_1)$ by (3) $\operatorname{sdepth}_S(I_1) \leq \operatorname{depth}_S(I_1)$, so $\operatorname{sdepth}_S(I_1) = \operatorname{depth}_S(I_1)$.
(4)⇒(5). As z is regular on S^*/I_1S^* then by (4) for I_1S we have $\operatorname{sdepth}_S(I_1) \geq \operatorname{depth}_S(I_1)$. Assume $\operatorname{sdepth}_S(I_1, z) = \operatorname{depth}_S(I_1)$ also $\operatorname{sdepth}_S(I_1, z) \geq \operatorname{sdepth}_S(I_1) + 1 = \operatorname{depth}_S(I_1) \Rightarrow \operatorname{sdepth}_S(I_1) + 1 = \operatorname{depth}_S(I_1)$ so $\operatorname{sdepth}_S(I_1) = \operatorname{depth}_S(I_1)$.
(5)⇒(1). Suppose a monomial ideal $I_1 \subset S$. For an integer $q \geq 1$, denote $I_q = (I_1, z_1, \cdots, z_q) \subset S_q := S[z_1, \cdots, z_q]$. As $z_1, \cdots, z_q$ is a regular sequence on S_q/I_q so $\operatorname{depth}_{S_q}(I_q) = \operatorname{depth}_S(I_1)$. By Corollary 3.1.5(1), $\operatorname{sdepth}_{S_q}(I_q) \geq \min\{\operatorname{sdepth}_S(I_1) + q, \operatorname{sdepth}_S(S/I_1) + \lceil q/2 \rceil$ then there exist $q_0 \geq 1$ such that for $q \geq q_0$ $\operatorname{sdepth}_{S_q}(I_q) \geq \operatorname{depth}_S(I_(1))$. Choose q_0 minimal such that we can claim that $\operatorname{sdepth}_{S_{q_0}}(I_{q_0}) = \operatorname{depth}_S(I_1)$. Observe that $\operatorname{sdepth}_{S_q}(I_q) \leq \operatorname{sdepth}_{S_{q-1}}(I_{q-1}) + 1$. By using (5) inductively we have $\operatorname{sdepth}_S(I_1) = \operatorname{depth}_S(I_1)$. □

Remark 3.3.1. Suppose a monomial ideal $I_1 \subset S = K[y_1, \cdots, y_m]$ such that $\operatorname{sdepth}_S(I_1) \geq \operatorname{depth}_S(I_1)$. Suppose $v_1, \cdots, v_k \in S$ is a regular sequence on S/I_1 and $I_2 = (v_1, \cdots, v_k)$. By Proposition 3.1.1(3), $\operatorname{depth}_S(I_1 \cap I_2) = \operatorname{depth}_S(I_1 + I_2) + 1 = \operatorname{depth}_S(I_1) - k + 1$. By Corollary 3.2.3(1) $\operatorname{sdepth}_S(I_1 \cap I_2) \geq \operatorname{sdepth}_S(I_1) - \lfloor k/2 \rfloor$. Suppose $\operatorname{sdepth}_S(I_1 \cap I_2) = \operatorname{depth}_S(I_1 \cap I_2)$ then $\operatorname{sdepth}_S(I_1) - k + 1 \geq \operatorname{depth}_S(I_1) -$

$k+1 \Rightarrow \operatorname{sdepth}_S(I_1) = \operatorname{depth}_S(I_1)$, $\lfloor k/2 \rfloor = k-1$ and $k \le 2$. We can contradict the Stanley inequality for an ideal $I_1 \subset S$ for $S^* = S[x_1, x_2, x_3]$ such that $\operatorname{sdepth}_{S^*}(I_1 S^* \cap (x_1, x_2, x_3)) = \operatorname{depth}_S(I_1)$.

Proposition 3.3.2. *The following are equivalent:*

(1) For any integer $m \ge 1$ and any monomial ideal $I_1 \subset S = K[x_1, \cdots, x_m]$, Stanley inequality holds for I_1, i.e. $\operatorname{sdepth}_S(I_1) \ge \operatorname{depth}_S(I_1)$.

(2) For any integer $m \ge 1$ and any monomial ideals $I_1, I_2 \subset S$, if $\operatorname{sdepth}_S(I_1 \cap I_2) \ge \operatorname{depth}_S(I_1 \cap I_2)$*, then* $\operatorname{sdepth}_S(I_1) \ge \operatorname{depth}_S(I_1)$.

(3) For any integers $m, k \ge 1$, any monomial ideal $I_1 \subset S = K[x_1, \cdots, x_m]$, if $v_1, \cdots, v_k \in S$ is a regular sequence on S/I_1 and $I_2 = (v_1, \cdots, v_k)$, then if:

$$\operatorname{sdepth}_S(I_1 \cap I_2) \ge \operatorname{depth}_S(I_1 \cap I_2) \Rightarrow \operatorname{sdepth}_S(I_1) \ge \operatorname{depth}_S(I_1).$$

Proof. (1)⇒(2) and (2)⇒ (3).It is obvious.
(3)⇒(1). Suppose a monomial ideal $I_1 \subset S$. For an integer $q \ge 1$, define $I_q = I_1 \cap (z_1, \cdots, z_q) \subset S_q := S[z_1, \cdots, z_q]$. We denote $I_2 = (z_1, \cdots, z_q) \subset S_q$. As $z_1, \cdots, z_q$ is a regular sequence on S_q/I_qS using Corollary 3.2.3(1) $\operatorname{sdepth}_{S_q}(I_q) \ge \operatorname{sdepth}_S(I_1) + \lceil q/2 \rceil$. By Corollary 3.1.5(5) $\operatorname{depth}_{S_q}(I_q) = \operatorname{depth}_S(I_1) + 1$ then there exist $q_0 \ge 1$ such that $\operatorname{sdepth}_{S_q}(I_q) \ge \operatorname{depth}_{S_q}(I_q)$ for $q \ge q_0$. By (3) $\operatorname{sdepth}_S(I_1) \ge \operatorname{depth}_S(I_1)$. □

Proposition 3.3.3. *The following are equivalent:*

(1) For any integer $m \ge 1$ and any monomial ideal $I_1 \subset S = K[x_1, \cdots, x_m]$, Stanley inequality holds for S/I_1, i.e. $\operatorname{sdepth}_S(S/I_1) \ge \operatorname{depth}_S(S/I_1)$.

(2) For any integer $m \ge 1$ and any monomial ideals $I_1, I_2 \subset S$, if $\operatorname{sdepth}_S(S/(I_1 \cap I_2)) \ge \operatorname{depth}_S(S/(I_1 \cap I_2))$*, then* $\operatorname{sdepth}_S(S/I_1) \ge \operatorname{depth}_S(S/I_1)$.

(3) *For any integers* $m, k \geq 1$*, any monomial ideal* $I_1 \subset S = K[x_1, \cdots, x_m]$*, if* $v_1, \cdots, v_k \in S$ *is a regular sequence on* S/I_1 *and* $I_2 = (v_1, \cdots, v_k)$*, then if:*

$$\operatorname{sdepth}_S(S/(I_1 \cap I_2)) \geq \operatorname{depth}_S(S/(I_1 \cap I_2)) \Rightarrow \operatorname{sdepth}_S(S/I_1) \geq \operatorname{depth}_S(S/I_1).$$

Proof. (1)⇒(2) and (2)⇒ (3).It is obvious.
(3)⇒(1). Suppose a monomial ideal $I_1 \subset S$. For an integer $q \geq 1$, define $I_q = I_1 \cap (z_1, \cdots, z_q) \subset S_q := S[z_1, \cdots, z_q]$. We denote $I_2 = (z_1, \cdots, z_q) \subset S_q$. As $z_1, \cdots, z_q$ is a regular sequence on S_q/I_qS using Corollary 3.2.3(4) $\operatorname{sdepth}_{S_q}(S_q/I_q) \geq \min\{m, \operatorname{sdepth}_S(S/I_1)+\lceil q/2\rceil\}$. By Corollary 3.1.5(5) $\operatorname{depth}_{S_q}(S_q/I_q) = \operatorname{depth}_S(S/I_1)$ then there exist $q_0 \geq 1$ such that $\operatorname{sdepth}_{S_q}(S_q/I_q) \geq \operatorname{depth}_{S_q}(S_q/I_q)$ for $q \geq q_0$. By (3) $\operatorname{sdepth}_S(S/I_1) \geq \operatorname{depth}_S(S/I_1)$. □

Remark 3.3.2. Suppose a monomial ideal $I_1 \subset S = K[y_1, \cdots, y_m]$ such that $\operatorname{sdepth}_S(S/I_1) \geq \operatorname{depth}_S(S/I_1)$. Suppose $v_1, \cdots, v_k \in S$ is a regular sequence on S/I_1 and $I_2 = (v_1, \cdots, v_k)$. Observe $\operatorname{depth}_S(S/(I_1 \cap I_2)) = \operatorname{depth}_S(S/(I_1+I_2))+1 = \operatorname{depth}_S(S/I_1) - k + 1$. By Corollary 3.2.3(4) $\operatorname{sdepth}_S(S/(I_1 \cap I_2)) \geq \min\{m - k, \operatorname{sdepth}_S(S/I_1) - \lfloor k/2\rfloor\}$. Suppose $\operatorname{sdepth}_S(S/(I_1 \cap I_2)) = \operatorname{depth}_S(S/(I_1 \cap I_2))$ then $\operatorname{depth}_S(S/I_1) - k + 1 \geq \min\{m - k, \operatorname{sdepth}_S(S/I_1) - \lfloor k/2\rfloor\} \geq \min\{m - k, \operatorname{depth}_S(S/I_1) - \lfloor k/2\rfloor\} \geq \min\{m-k, \operatorname{depth}_S(S/I_1)-k+1\} = \operatorname{depth}_S(S/I_1)-k+1$. For principal ideal $\operatorname{depth}_S(S/I_1) = m - 1$ then

$$\min\{m - k, \operatorname{depth}_S(S/I_1) - \lfloor k/2\rfloor\} = m - k$$

$$\Rightarrow \operatorname{depth}_S(S/I_1) - \lfloor k/2\rfloor = m - 1 - \lfloor k/2\rfloor \geq m - k$$

If ideal is not principal, then $\operatorname{depth}_S(S/I_1) \leq m - 2$ then by Remark 3.2.2.

$$\min\{m-k, \operatorname{sdepth}_S(S/I_1)-\lfloor k/2\rfloor\} = \operatorname{depth}_S(S/I_1)-\lfloor k/2\rfloor = \operatorname{depth}_S(S/I_1)-m+1.$$

Hence $\operatorname{sdepth}_S(S/I_1) = \operatorname{depth}_S(S/I_1)$ and $k \leq 2$. We can contradict the Stanley inequality if we find an ideal $I_1 \subset S$ which is not principal for $S^* = S[x_1, x_2, x_3]$ such that $\operatorname{sdepth}_{S^*}(S^*/(I_1S^* \cap (x_1, x_2, x_3))) = \operatorname{depth}_S(S/I_1)$.

The following Lemma is a specific case of Example 3.1.1

Lemma 3.3.4. *Suppose* $I_2 = (x_1, \cdots, x_m) \cap (y_1, \cdots, y_k) \subset S^* = K[x_1, \cdots, x_m, y_1, \cdots, y_k]$ *with* $m \geq k$. *Then*

(1) $k \geq \mathrm{sdepth}_{S^*}(S^*/I_2) \geq \min\{k, \lceil m/2 \rceil\}$.

(2) $\mathrm{depth}_{S^*}(S^*/I_2) = 1$.

Particularly if $m \geq 2k-1$ *then* $\mathrm{sdepth}_{S^*}(S^*/I_2) = k$.

Proposition 3.3.5. *The following are equivalent:*

(1) For any integer $m \geq 1$ *and any monomial ideal* $I_1 \subset S = K[x_1, \cdots, x_m]$, *Stanley inequality holds for* S/I_1 *and* I_1

(2) For any integer $m \geq 1$ *and any monomial ideals* $I_1, I_2 \subset S$ *with* $\mathrm{supp}(I_1) \cap \mathrm{supp}(I_2) = \emptyset$ *we have* $\mathrm{sdepth}_S((I_1 + I_2)/I_1) \geq \mathrm{depth}_S((I_1 + I_2)/I_1)$, *then* $\mathrm{sdepth}_S(S/I_1) \geq \mathrm{depth}_S(S/I_1)$ *and* $\mathrm{sdepth}_S(I_2) \geq \mathrm{depth}_S(I_2)$.

Proof. (1)⇒(2). Suppose two monomial ideals $I_1, I_2 \subset S$ with $\mathrm{supp}(I_1) \cap \mathrm{supp}(I_2) = \emptyset$. From Lemma 3.1.4 $\mathrm{depth}_S((I_1 + I_2)/I_1) = \mathrm{depth}_S(I_1 + I_2)) = \mathrm{depth}_S(S/I_1) + \mathrm{depth}_S(I_2) - m$. By Remark 3.1.1 and (1) $\mathrm{sdepth}_S((I_1 + I_2)/I_1) \geq \mathrm{sdepth}_S(S/I_1) + \mathrm{sdepth}_S(I_2) - m \geq \mathrm{depth}_S(S/I_1) + \mathrm{depth}_S(I_2) - m = \mathrm{depth}_S((I_1 + I_2)/I_1)$.
(2)⇒(1). Suppose $I_1 \subset S$ is a monomial ideal. For any positive integer q, denote $S_q = S[z_1, \cdots, z_q]$ and $I_q = (I_1, z_1, \cdots, z_q) \subset S_q$. Suppose $\mathrm{sdepth}_S(S/I_1) < \mathrm{depth}_S(S/I_1)$. By Remark 3.1.1 $\mathrm{sdepth}_{S_q}(I_q/I_1S_q) \geq \mathrm{sdepth}_S(S/I_1) + \lfloor q/2 \rfloor$ and as for all q $\mathrm{depth}_{S_q}(I_q/I_1S_q) = \mathrm{depth}_S(S/I_1) + 1$ so there exists a positive integer q_0 such that $\mathrm{sdepth}_{S_q}(I_q/I_1S_q) \geq \mathrm{depth}_{S_q}(I_q/I_1S_q)$ for all $q \geq q_0$. From (2) $\mathrm{sdepth}_S(S/I_1) \geq \mathrm{depth}_S(S/I_1)$ which is a contradiction.
Suppose $\mathrm{sdepth}_S(I_1) < \mathrm{depth}_S(I_1)$ and denote $I_{2_q} = (z_1, \cdots, z_{2q-1}) \cap (z_{2q}, \cdots, z_{3q-1}) \subset S_{3q-1} := S[z_1, \cdots, z_{3q-1}]$. From Lemma 3.1.4 $\mathrm{sdepth}_{S_{3q-1}}(S_{3q-1}/I_{2_q}) = m + q$ and $\mathrm{depth}_{S_{3q-1}}(S_{3q-1}/I_{2_q}) = m + 1$. Suppose $I_q := I_1S_{3q-1} + I_{2_q}$. From Remark 3.1.1

$\mathrm{sdepth}_{S_{3q-1}}(I_{1_q}/I_{2_q}) \geq \mathrm{sdepth}_S(I_1) + q$ and $\mathrm{depth}_{S_{3q-1}}((I_{1_q}/I_{2_q})) = \mathrm{depth}_S(I_1) + \mathrm{depth}_{S_{3q-1}}(S_{3q-1}/I_{2_q}) - m = \mathrm{depth}_S(I_1) + 1$. So there exists a positive integer q_0 such that for any $q \geq q_0$ we have $\mathrm{sdepth}_{S_{3q-1}}((I_{1_q}/I_{2_q})) \geq \mathrm{depth}_{S_{3q-1}}(I_{1_q}/I_{2_q})$. Therefore from (2) we get a contradiction $\mathrm{sdepth}_S(I_1) \geq \mathrm{depth}_S(I_1)$. $\square$

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