Information Loss Paradox in Black Holes

by

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Dedicated to my loving sister

Sheshe Baji

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Abstract

In this dissertation a brief review of the information loss paradox has been discussed. Information loss was an old problem in physics which puzzled physicists for decades. This paradox was first raised by Hawking and then finally solved by Susskind using quantum mechanics. It may be possible to solve the problem using general relativity only. For this purpose I have reviewed the work of Qadir and Wheeler, i.e, “the suture model”. Foliation of this model by hypersurfaces of constant mean extrinsic curvature showed that the distance between denser and rare Friedmann regions increased to infinity. The asymptotic behaviour of length and volume has been discussed which showed that as the length increases to infinity the volume shrinks to zero. The asymptotic behaviour of a Schwarzschild black hole has been discussed separately without referring to the suture model, which showed that the length increases to infinity more rapidly than the suture model.

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Chapter 1

Introduction

General relativity (GR) is Einstein's theory of gravity in which the idea of the Newtonian gravitational force was replaced by the curvature of the spacetime. Newton believed that every body in the Universe attracts every other body with some force, **F**, which is directly proportional to the product of their masses and inversely proportional to the square of the distance from their centers. It can be defined as

$$\mathbf{F} = -G\frac{m_1 m_2}{r^3}\mathbf{r}, \tag{1.1}$$

where **F** is the force, G is the gravitational constant, m_1 is the first mass and m_2 is the second mass, **r** is the position vector of the center of one body relative to the other and r is the distance between their centers.

Einstein presented his theory in 1915, over a hundred years ago. Before the idea of relativity theory the space and time was considered to be the two separate entities. The concept about the time was as an absolute, i.e, time is the same for every observer. Einstein said it is not absolute, but relative, varying for different observers. Further he explained that space and time are not separate entities and he merged the space and time into a single entity known as spacetime. The spacetime is a 4-dimensional object in which there are three space and one time directions. Time only flows in forward direction, which means that one cannot go backward in time. The theory mainly describes the motion of the macroscopic objects with more accuracy than the Newtonian theory. A massive body bends the spacetime around it and consequently the spacetime becomes curved there, i.e, it produces curvature in the spacetime. So the bodies move on the curved spacetime and their motion is described by its curvature. GR is described by the geometry of spacetime. Some predictions of the theory were then tested experimentally, which verified that it works perfectly well. One of the predictions of GR was that there are regions in the spacetime where due to the gravitational collapse of a massive body leads to an object from which even light can not escape. These regions of spacetime are commonly known as black holes. This name was coined by John Wheeler [1]. Since the physics of a black hole is described by GR which is the theory of gravity. The language of GR is the language of tensors. In order to understand the theory itself and the Einstein Field Equations one must have a knowledge of tensors.

In this chapter, I will discuss some basic definitions of GR and then I will discuss Friedmann universe. I will finish chapter 1 with basic terminologies of thermodynamics. In chapter 2, I will briefly review the background of information loss paradox with Hawking's argument of the information loss paradox and then resolution to the paradox in the end. In chapter 3, a detail procedure of the foliation will be discussed. After that in the same chapter mathematical procedure for the foliating hypersurfaces will be discussed. I will conclude the thesis in chapter 4 with a brief review of asymptotic behaviour of the length and the volume of the "suture model" with hypersurfaces of constant mean extrinsic curvature and then asymptotic behaviour of the length of Schwarzschild singularity with hypersurfaces of constant mean extrinsic curvature.

1.1 Curvature Tensors and Scalars

On a flat space, if a vector is transported parallely to itself on some closed curve then on returning to the starting point it will match with the original vector. But in a curved space, if the same procedure is repeated then we get two different values at the starting point of the parallely transported vector. This difference suggests that the space has some curvature in it. Riemann generalized Gauss's invariant intrinsic curvature to higher dimensional spaces by carrying a basis vector along two different directions in opposite order and taking the difference of the two results [2].

$$\begin{aligned} A^a{}_{;c;d} - A^a{}_{;d;c} &= (A^a{}_{,c} + \Gamma^a_{bc}A^b)_{;d} - (A^a{}_{,c} + \Gamma^a_{db}A^b)_{;c}, \qquad (1.2) \\ &= (\Gamma^a_{bc,d} - \Gamma^a_{db,c})A^b + (\Gamma^a_{cf}\Gamma^f_{db} - \Gamma^a_{df}\Gamma^f_{cb})A^b, \\ &= 2[\Gamma^a_{b[c,d]} + \Gamma^a_{f[c}\Gamma^f_{d]b}]A^b, \\ A^a{}_{;c;d} - A^a{}_{;d;c} &:= R^a{}_{bcd}A^b. \qquad (1.3) \end{aligned}$$

Since the left hand side is a tensor, so right hand must also be a tensor,
where

$$R^a{}_{bcd} = 2[\Gamma^a_{b[c,d]} + \Gamma^a_{f[c}\Gamma^f_{d]b}], \tag{1.4}$$

or

$$R^a{}_{bcd} = \Gamma^a_{bc,d} - \Gamma^a_{db,c} + \Gamma^a_{cf}\Gamma^f_{db} - \Gamma^a_{df}\Gamma^f_{cb}, \tag{1.5}$$

is called the Riemann-curvature (or Riemann Christoffel) tensor. The Christoffel symbols appeared in the above expression are given by the following formula

$$\Gamma^a_{bc} = \frac{1}{2}g^{ad}\big(g_{bd,c} + g_{cd,b} - g_{bc,d}\big). \tag{1.6}$$

Thus curvature tensor measures the curvature of the space. A manifold is flat if $R^a{}_{bcd} = 0$ or it is locally flat in some region if curvature tensor is zero there.
$R^a{}_{bcd}$ can also be written as

$$R_{abcd} = g_{ae}R^e{}_{bcd}. \tag{1.7}$$

It is skew-symmetric in the first two indices, i.e.

$$R_{abcd} = -R_{bacd}. \tag{1.8}$$

It is also skew-symmetric in the last two indices, i.e.

$$R_{abcd} = -R_{abdc}. \tag{1.9}$$

If two pairs of indices are interchanged then it is symmetric, i.e.

$$R_{abcd} = R_{cdab}. \tag{1.10}$$

It satisfies the following identity

$$R_{abcd} + R_{acdb} + R_{adcb} = 0, \tag{1.11}$$

which is also known as the first Bianchi identity. The second Bianchi identity is given by the following form

$$R^a{}_{bcd;e} + R^a{}_{bec;d} + R^a{}_{bde;c} = 0. \tag{1.12}$$

The trace of the Riemann tensor gives the Ricci tensor which is obtained by contracting the two indices of Riemann tensor, i.e.

$$R_{ab} = R^c{}_{acb}, \tag{1.13}$$

where R_{ab} is a symmetric tensor, i.e,

$$R_{ab} = R_{ba}. \tag{1.14}$$

By contracting the Ricci tensor, we get the Ricci Scalar as

$$R = R^a{}_a = g^{ab}R_{ab}. \tag{1.15}$$

1.2 The Geodesic Equation and Geodesic Deviation

In flat spacetime the shortest path between two points is the straightest path. On the curved spacetime, the shortest path between two points is called the "geodesic". On such a spacetime the tangent vector on a curve maintain the same direction. This can be assured by transported parallely the tangent vector along the curve, i.e,

$$t^b t^a{}_{;b} = 0, \tag{1.16}$$

where

$$t^b = \dot{x}^b = \frac{dx^b}{ds}. \tag{1.17}$$

Above equation can be written as

$$t^b t^a{}_{;b} = t^b t^a{}_{,b} + \Gamma^a_{bc} t^b t^c, \tag{1.18}$$

this implies that

$$\ddot{x}^a + \Gamma^a_{bc} \dot{x}^b \dot{x}^c = 0, \tag{1.19}$$

which is called the geodesic equation. The solution of this equation is known as geodesic.

Let us consider two neighbouring geodesics with tangent vectors **t** and these two geodesics are connected by a separation vector **p**. For a separation vector **p** to be transported parallely along the curve, we require that

$$\pounds_t p = 0, \tag{1.20}$$

$$t^d p^a{}_{;d} - p^d t^a{}_{;d} = 0, \tag{1.21}$$

or

$$t^d p^a{}_{;d} = p^d t^a{}_{;d}. \tag{1.22}$$

The geodesic deviation is given by the acceleration vector, **A**, as

$$A^a = \frac{d^2 p^a}{ds^2} = t^c [t^d p^a{}_{;d}]_{;c}. \tag{1.23}$$

By using Eq. (1.22), we get

$$A^a = t^c [p^d t^a{}_{;d}]_{;c}, \tag{1.24}$$

$$A^a = t^c p^d{}_{;c} t^a{}_{;d} + t^c p^d t^a{}_{;d;c}, \tag{1.25}$$

$$A^a = p^c t^d{}_{;c} t^a{}_{;d} + t^c p^d t^a{}_{;d;c}, \tag{1.26}$$

$$A^a = p^c (t^d{}_{;c} t^a)_{;d} - p^c t^d t^a{}_{;d;c} + t^c p^d t^a{}_{;d;c}, \tag{1.27}$$

$$A^a = -p^c t^d t^a{}_{;d;c} + t^c p^d t^a{}_{;d;c}, \tag{1.28}$$

or

$$A^a = t^c p^d t^a{}_{;d;c} + p^c t^d t^a{}_{;d;c}. \tag{1.29}$$

Interchanging *c* and *d* in first term of above equation, we get

$$A^a = t^d p^c t^a{}_{;c;d} + p^c t^d t^a{}_{;d;c}, \tag{1.30}$$

$$A^a = R^a{}_{bcd} t^b p^c t^d. \tag{1.31}$$

This equation gives the acceleration which arises due to the curvature.

1.3 The Einstein Field Equations

GR deals with the gravitation field, this field depends upon the distribution of matter and its evolution [2]. Energy can be stored in a field or it can be carried with a matter . If matter is present in a spacetime then it would not be flat. All stresses present in the field and total energy and momentum can be represented by the following tensor.

$$T^{ab} = \rho u^a u^b + \sigma^{ij}\delta^a_i \delta^b_j, \tag{1.32}$$

where T^{ab} is the stress-energy momentum tensor, ρ is the density, $u^a = \dot{x}^a$ is 4-velocity and σ^{ij} is called the stress-tensor. The laws of conservation of mass-energy and momentum required that the stress-energy tensor be divergence free, i.e, its divergence is zero.

$$T^{ab}{}_{;c} = 0. \tag{1.33}$$

As matter wraps the spacetime around it and produces the curvature. To find relationship between matter (and energy) and curvature, consider the relationship is expressed by

$$\varepsilon^{ab}[g_{\alpha\beta}, R^{\alpha}_{\beta rs}] = \kappa T^{ab}, \tag{1.34}$$

where ε^{ab} is a tensor function of the metric tensor and the curvature tensor, κ is the constant of proportionality. Here we will find ε^{ab} which must be divergence free so that stress-energy momentum is conserved. The simplest function which we require is $\varepsilon^{ab} = g_{ab}$. This function is not only divergence free but also gradient free. For the function to be divergence free we consider the contraction of Eq. (1.12) over *a* and *c*.

$$R_{bd;e} + R_{be;d} + R^a_{bde;c} = 0, \tag{1.35}$$

multiply by g^{ad} the above equation takes the form

$$R_{;e} + 2R^a_{e;a} = 0, \tag{1.36}$$

which reduces to

$$\left(R^{ab} - \frac{1}{2}g^{ab}R\right)_{;a} = 0. \tag{1.37}$$

For 4-dimensional spacetime the linear, symmetric, divergence free function of the curvature is

$$\varepsilon^{ab} = R^{ab} - \frac{1}{2}Rg^{ab}, \tag{1.38}$$

where ε^{ab} is called Einstein tensor. Eq. (1.34) becomes

$$R^{ab} - \frac{1}{2}Rg^{ab} = \kappa T^{ab}, \tag{1.39}$$

which is called the Einstein field equations. Later, Einstein introduced a constant of integration to his equations, because when he applied GR to cosmology he could not get the satisfactory solution.

$$R^{ab} - \frac{1}{2}Rg^{ab} - \Lambda g^{ab} = \kappa T^{ab}, \tag{1.40}$$

where Λ is called the cosmological constant.

1.3.1 Schwarzschild Solution for the Einstein Field Equations

Karl Schwarzschild was a German physicist and an astronomer. He was one who gave the first exact solution of the Einstein's field equations (EFEs) soon after the discovery of GR. Since it was difficult to solve these equations directly so he took some assumptions. Schwarzschild's solution describes the vacuum solution of EFEs which is spherically symmetric and static for a point mass m with no charge. Static means that metric tensor $g_{\mu\nu}$ is independent of time and there exist a time reversal symmetry, i.e, the metric tensor is invariant under the time transformation $t \rightarrow -t$. Spherically symmetric spacetime means that it is invariant under rotation and vacuum solution is the one which satisfies the equation $T_{ab} = 0$, which shows that there are no stresses or energy in that region. The most general static and spherically symmetric metric in spherical polar coordinates (t, r, θ, φ) can be written as

$$ds^2 = -e^{v(t,r)}dt^2 + e^{\lambda(t,r)}dr^2 + r^2 d\theta^2 + R^2(t,r)d\Omega^2, \tag{1.41}$$

where v, λ and R are the functions of the time coordinate, t, and the radial coordinate, r, and

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2, \tag{1.42}$$

is the metric of a unit sphere. The gravitational field for a point mass do not vary with time and hence remains the same. Consequently, there would be no time dependence and Eq. (1.41) reduces to

$$ds^2 = -e^{v(r)}dt^2 + e^{\lambda(r)}dr^2 + R^2 d\Omega^2. \tag{1.43}$$

Here we have two cases for $R^2(r)$, either it is a constant function or a varying function. In case of constant function the area subtended by a given solid angle will be independent of r which is not realistic. Whereas, in the case of varying function the radial coordinate can be considered to be R instead of r. Taking new coordinate as r, Eq. (1.43) becomes

$$ds^2 = -e^{v(r)}dt^2 + e^{\lambda(r)}dr^2 + r^2 d\Omega^2. \tag{1.44}$$

The metric tensor for the above line element is given by

$$g_{ab} = \begin{pmatrix} -e^{v(r)} & 0 & 0 & 0 \\ 0 & e^{\lambda(r)} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}. \tag{1.45}$$

And the inverse metric tensor is

$$g^{ab} = \begin{pmatrix} -e^{-v(r)} & 0 & 0 & 0 \\ 0 & e^{-\lambda(r)} & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{pmatrix}. \tag{1.46}$$

For vacuum solution Eq. (1.39) can be written as

$$R_{ab} - \frac{1}{2}g_{ab}R = 0, \tag{1.47}$$
$$g^{ab}\big(R_{ab} - \frac{1}{2}g_{ab}R\big) = 0,$$
$$g^{ab}R_{ab} - \frac{1}{2}g^{ab}g_{ab}R = 0,$$
$$R - \frac{1}{2}R = 0,$$
$$R = 0. \tag{1.48}$$

Equation (1.47) reduces to

$$R_{ab} = 0, \tag{1.49}$$

which are the vacuum Einstein's field equations. The non-zero Christoffel symbols are

$$\begin{aligned}
\Gamma^0_{01} &= \Gamma^0_{10} = \frac{1}{2v'}, \qquad \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r},\\
\Gamma^3_{13} &= \Gamma^3_{31} = \frac{1}{r}, \qquad \Gamma^3_{23} = \Gamma^3_{32} = \cot\theta,\\
\Gamma^1_{11} &= \frac{1}{2\lambda'}, \qquad \Gamma^1_{00} = \frac{1}{2v'}e^{v-\lambda},\\
\Gamma^1_{33} &= -re^{-\lambda}\sin\theta, \qquad \Gamma^2_{33} = -\sin\theta\cos\theta.
\end{aligned} \tag{1.50}$$

The surviving Einstein's vacuum field equations are

$$R_{00} = 0 \Rightarrow v'' + \frac{1}{2v'}\left(v' - \lambda' + \frac{2v'}{r}\right) = 0, \tag{1.51}$$
$$R_{11} = 0 \Rightarrow -v'' - \frac{1}{2v'}\left(v' - \lambda' - \frac{2\lambda'}{r}\right) = 0, \tag{1.52}$$
$$R_{22} = 0 \Rightarrow 1 - e^{-\lambda} + \frac{1}{2r}\big(\lambda' - v'\big)e^{-\lambda} = 0, \tag{1.53}$$
$$R_{33} = 0 \Rightarrow R_{22}\sin^2\theta = 0. \tag{1.54}$$

Adding Eqs. (1.51) and (1.52), we get

$$\frac{2v'}{r} + \frac{2\lambda'}{r} = 0, \tag{1.55}$$
$$v' + \lambda' = 0. \tag{1.56}$$

Integrating, we get

$$v + \lambda = constant. \tag{1.57}$$

Setting $v = -\lambda$, and using this value in Eq. (1.55), we get

$$-e^{v} - re^{v}v' + 1 = 0, \tag{1.58}$$

or

$$v' - \frac{1}{re^v} \;=\; -\frac{1}{r}. \tag{1.59}$$

By solving Eq. (1.59), we get

$$e^v = 1 + \frac{\alpha}{r}, \tag{1.60}$$

since $v = -\lambda$, so we have

$$e^{-\lambda} = 1 + \frac{\alpha}{r}, \tag{1.61}$$

where

$$\alpha = -2m. \tag{1.62}$$

For this value of α, Eqs. (1.60) and (1.61) become

$$e^v = 1 - \frac{2m}{r}, \tag{1.63}$$

$$e^{-\lambda} = 1 - \frac{2m}{r}. \tag{1.64}$$

Putting the values of e^v and $e^{-\lambda}$ in Eq. (1.44)

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \tag{1.65}$$

where

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2. \tag{1.66}$$

Eq. (1.65) is known as Schwarzschild solution of EFEs and also known as Schwarzschild metric. $r_s = 2m$ is called the Schwarzschild radius. As $r \to \infty$ the metric reduces to flat spacetime.

1.4 Singularities of the Schwarzschild Metric

There are two types of singularities, i.e, the essential singularity and the coordinate singularity. The singularity which is actually present in the manifold and can not be removed by any choice of the coordinate system is called an essential singularity. At this point, all the spacelike, null, or timelike geodesics terminate after a lapse of finite proper time [3]. The singularity which arises due to the bad choice of the coordinates and which can be removable by an appropriate choice of the coordinates is known as the coordinate singularity. The nature of the singularity can be determined by the curvature invariants as

$$R_1 = R = R^a{}_a, \tag{1.67}$$
$$R_2 = R^{ab}{}_{cd}R^{cd}{}_{ab}, \tag{1.68}$$
$$R_3 = R^{ab}{}_{cd}R^{cd}{}_{ef}R^{ef}{}_{ab}. \tag{1.69}$$

One can check these curvature invariants which are regular at $r_s = 2m$, and are undefined at $r = 0$. The Schwarzschild solution shows that it has two singularities. The singularity which is located at $r = 0$ is the essential singularity whereas at $r = r_s$ is the coordinate singularity. The values of the curvature invariants obtained for the Schwarzschild solution are given as under [2].

$$R_1 = 0, \tag{1.70}$$
$$R_2 = \frac{12r_s^2}{r^6}, \tag{1.71}$$
$$R_3 = \frac{8r_s^3}{r^9}. \tag{1.72}$$

This shows that $r = r_s$ is not the actual or the physical singularity. It appears due to inappropriate choice of the coordinates. For $r > r_s$, the metric (1.65) gives a static vacuum solution but for $r < r_s$, the metric does not give any physical information since it is singular at $r = r_s$. So the Schwarzschild's (t, r) coordinates system becomes singular at $r = r_s$. This mathematical boundary is known as the event horizon for the Schwarzschild metric. We need some appropriate coordinates to solve this which will be discuss in the coming section.

1.5 The Eddington-Finkelstein Coordinates

Eddington [4] constructed a new coordinate system which was then rediscovered by Finkelstein [5]. They used the following transformations

$$(t, r, \theta, \phi) \to (v, u, \theta, \phi) \tag{1.73}$$

To avoid the singularity they defined a new radial coordinate such that the singularity at $r_s = 2m$ disappears.

$$r^* = \int \frac{dr}{1 - \frac{2m}{r}}. \tag{1.74}$$

After integrating, we get

$$r^* = r + 2m \ln |\frac{r - 2m}{2m}|. \tag{1.75}$$

The constant of integration is chosen so that to make the argument of the logarithm dimensionless. They also introduced two retarded and advanced coordinates as

$$u = t - r^*, \qquad v = t + r^*, \tag{1.76}$$

where u and v play the role of retarded and advanced time respectively.

Using (1.76), the advanced time can be written as

$$dv = dt + dr^*. \tag{1.77}$$

Using Eq. (1.74), we can write

$$dv = dt + \frac{dr}{1 - \frac{2m}{r}}. \tag{1.78}$$

Eq. (1.78) can be written as

$$dt = dv - \frac{dr}{1 - \frac{2m}{r}}. \tag{1.79}$$

After squaring Eq. (1.79), we get

$$dt^2 = dv^2 + \frac{dr^2}{\left(1 - \frac{2m}{r}\right)^2} - \frac{2dvdr}{1 - \frac{2m}{r}}. \tag{1.80}$$

Substituting Eq. (1.80) into Eq. (1.65), we get

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dv^2 + 2dvdr + r^2 d\Omega^2. \tag{1.81}$$

This is the Schwarzschild metric in advanced coordinates. The metric tensor for this line element is given by

$$g_{ab} = \begin{pmatrix} -(1 - \frac{2m}{r}) & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix}. \tag{1.82}$$

For retarded time u, Eq. (1.76) can be written as

$$du = dt - dr^*, \tag{1.83}$$

$$du = dt - \frac{dr}{1 - \frac{2m}{r}}. \tag{1.84}$$

Squaring and re-arranging, we get

$$dt^2 = du^2 + \frac{dr^2}{\left(1 - \frac{2m}{r}\right)^2} + \frac{2dvdr}{1 - \frac{2m}{r}}. \tag{1.85}$$

Substituting Eq. (1.85) into Eq. (1.65), we get

$$ds^2 \quad = \quad -\left(1-\frac{2m}{r}\right)dv^2 - 2vdvdr + r^2 d\Omega^2. \tag{1.86}$$

This is the Schwarzschild metric in the retarded coordinates, and the metric tensor is given by

$$g_{ab} = \begin{pmatrix} -(1-\frac{2m}{r}) & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}. \tag{1.87}$$

Adding and subtracting Eqs. (1.77) and (1.83), we get

$$dt = \frac{1}{2}(dv + du), \tag{1.88}$$

$$\frac{dr}{1-\frac{2m}{r}} = \frac{1}{2}(dv - du). \tag{1.89}$$

After squaring Eq. (1.88) and (1.89), we get

$$dt^2 = \frac{1}{4}(dv^2 + du^2 + 2dvdu), \tag{1.90}$$

$$\frac{dr^2}{\left(1-\frac{2m}{r}\right)^2} = \frac{1}{2}(dv^2 + du^2 - 2dvdu). \tag{1.91}$$

Inserting Eqs. (1.90) and (1.91) into Eq. (1.65) and simplifying, we get

$$ds^2 = -\left(1-\frac{2m}{r}\right)dvdu + r^2 d\Omega^2. \tag{1.92}$$

This form of the metric is called double null form. The metric tensor for this line element is given by

$$g_{ab} = \begin{pmatrix} 0 & -\frac{1}{2}(1-\frac{2m}{r}) & 0 & 0 \\ -\frac{1}{2}(1-\frac{2m}{r}) & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}. \tag{1.93}$$

These coordinates are good to study the value of $r \geq 2m$ but not convenient for $r < 2m$. To study the behaviour inside the surface $r = 2m$, we required another coordinate system.

1.6 Kruskal Coordinates

The above coordinate system seems asymmetrical in that the diagonal term in the radial part of the metric tensor has been replaced by an off diagonal term, which is singular at $r = 2m$. Since determinant of the metric tensor is zero there. For this purpose we need a new coordinate system which is symmetrical looking and is non-singular. As singularity arises due to zero in the metric. To describe the interior region of the Schwarzschild black hole, Kruskal coordinates can be used [4]. Kruskal exponentiated the advanced and retarded (v, u) coordinates and A.Qadir [2] introduced two new constants α and β such that the entire manifold of the maximally extended Schwarzschild solution is covered by a single coordinate patch. Thus

$$V = \alpha e^{v/\beta}, \tag{1.94}$$
$$U = -\alpha e^{-u/\beta}. \tag{1.95}$$

Hence VU can be written as

$$VU = -\alpha^2 e^{\frac{v-u}{\beta}}, \tag{1.96}$$
$$VU = -\alpha^2 e^{\frac{2r^*}{\beta}}, \tag{1.97}$$
$$VU = -\alpha^2 e^{\frac{2}{\beta}(r+2m\ln|\frac{r}{2m}-1|)}. \tag{1.98}$$

After simplifying Eq. (1.98), we get

$$VU = -\alpha^2 \left(\frac{r}{2m} - 1\right)^{4m/\beta} e^{2r/\beta}. \tag{1.99}$$

And

$$dV = \frac{\alpha}{\beta} e^{v/\beta} dv = \frac{V}{\beta} dv, \tag{1.100}$$
$$dU = \frac{\alpha}{\beta} e^{-u/\beta} du = -\frac{U}{\beta} du. \tag{1.101}$$

So

$$dVdU = -\frac{VU}{\beta^2} dvdu, \tag{1.102}$$
$$dvdu = -\frac{vu}{\beta^2} dVdU. \tag{1.103}$$

Inserting Eq. (1.103) into Eq. (1.92) and simplifying, we get

$$ds^2 = -\frac{32m^3}{re^{r/2m}} dVdU + r^2 d\Omega^2, \tag{1.104}$$

where

$$VU = \left(\frac{r}{2m} - 1\right) e^{r/2m}. \tag{1.105}$$

If we put $r = 2m$, then we get the coefficient $16m^2/e$ which is regular there. This is the Schwarzschild metric in Kruskal coordinates. Hence there is no singularity at $r = 2m$ and $g \neq 0$. Normally α is chosen to be unity but if we want U, V to have units of length it is more convenient to take $\alpha = 2m$, in which case $g_{01} = 2m = g_{10}$.

1.7 Kruskal-Szekeres Coordinates

A convenient coordinate system which can be obtained from Kruskal coordinates is called Kruskal-Szekeres coordinates and the metric in these coordinates is in diagonal form. Kruskal-Szekeres coordinates have a timelike coordinate T and spacelike coordinate R. One question is solved that at $r = 2m$, the coefficient is not zero,

$$T = V - U \Rightarrow dT = dV - dU, \tag{1.106}$$

$$R = V + U \Rightarrow dR = dV + dU. \tag{1.107}$$

Adding and subtracting we have,

$$dV = \frac{1}{2}(dT + dR), \tag{1.108}$$

$$dU = -\frac{1}{2}(dT - dR). \tag{1.109}$$

And

$$dVdU = \frac{1}{4}(-dT^2 + dR^2). \tag{1.110}$$

Since

$$R = V - U \Rightarrow R^2 = V^2 + U^2 - 2VU, \tag{1.111}$$

$$T = V + U \Rightarrow T^2 = V^2 + U^2 + 2VU. \tag{1.112}$$

And

$$R^2 - T^2 = -4VU. \tag{1.113}$$

Inserting in metric, we get

$$ds^2 = \frac{32m^3}{re^{r/2m}}(-dT^2 + dR^2) + r^2 d\Omega^2, \tag{1.114}$$

where

$$R^2 - T^2 = 4\left(\frac{r}{2m} - 1\right)e^{r/2m}. \tag{1.115}$$

1.7.1 Compactified Kruskal-Szekeres (CSK) Coordinates

The Kruskal-Szekeres coordinates can be converted into a Compactified Kruskal-Szekeres (CKS) coordinates by the following transformations [6].

$$v + u = \tan\frac{1}{2}(\psi + \xi), \tag{1.116}$$

$$v - u = \tan\frac{1}{2}(\psi - \xi). \tag{1.117}$$

The above transformation becomes

$$dv + du = \frac{1}{2} sec^2 \frac{1}{2}(\psi + \xi)(d\psi + d\xi), \tag{1.118}$$

$$dv - du = \frac{1}{2} sec^2 \frac{1}{2}(\psi - \xi)(d\psi - d\xi). \tag{1.119}$$

The Schwarzschild line element can be written in these coordinates as

$$ds^2 = f^2(r)\left[\frac{-d\psi^2 + d\xi^2}{4\cos^2\frac{1}{2}(\psi+\xi)\cos^2\frac{1}{2}(\psi-\xi)}\right] + r^2 d\Omega^2. \tag{1.120}$$

This line element is singular at $r = 0$, and at the coordinate singularities (ψ, ξ) ranges from -$\pi/2$ to $\pi/2$ with end points not included. These coordinates can be compactified to $[-\pi/2, \pi/2]$. Carter and Penrose developed a method to represent such a spacetime on a two dimensional plane and is known as Carter-Penrose diagrams.

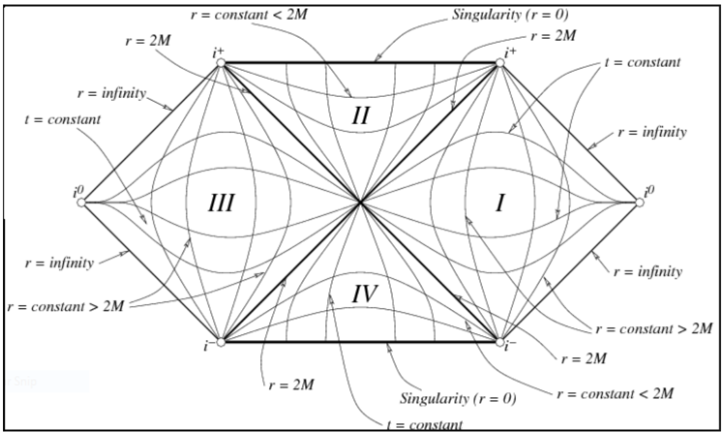

Figure 1.1: The Carter-Penrose diagram of a Schwarzschild black hole (taken from [7]).

1.8 The Friedmann Model of the Universe

Friedmann was a Russian mathematician and physicist and was famous for his idea about the expanding universe. Friedmann gave a homogeneous, isotropic model for the universe as solution to Einstein's field equations [6]. The model is given by the following metric

$$ds^2 = -d\eta^2 + a^2(\eta)[d\chi^2 + f_k^2(\chi)d\Omega^2], \tag{1.121}$$

where

$$f_{+1} = \sin\chi, \quad (0 \le \chi \le \pi), \tag{1.122}$$

$$f_0 = \chi, \quad (0 \le \chi \le \infty), \tag{1.123}$$

$$f_{-1} = \sinh\chi, \quad (0 \le \chi \le \infty). \tag{1.124}$$

k = +1, shows that the model has a positive curvature, k = 0, for zero curvature and k = -1, for the negative curvature. The term $a(\eta)$ is known as the expansion factor and it depends upon time parameter t. $a(\eta)$ can be expressed parametrically by the following relations.

For k = +1, we have

$$a(\eta) = \frac{a_0}{2}(1+\cos\theta), \qquad (-\pi \leq \eta \leq \pi), \tag{1.125}$$

$$t(\eta) = \frac{a_0}{2}(\pi+\eta+\sin\theta), \quad (-\pi \leq \eta \leq \pi). \tag{1.126}$$

For k=0, we have

$$a(\eta) = \frac{a_0}{2}(\pi+\eta)^2, \qquad (-\pi \leq \eta \leq \infty), \tag{1.127}$$

$$t(\eta) = \frac{a_0}{2}(\pi+\eta)^3, \qquad (-\pi \leq \eta \leq \infty). \tag{1.128}$$

For k = -1, we have

$$a(\eta) = \frac{a_0}{2}[\cosh(\eta+\pi)-1], \qquad (-\pi \leq \eta \leq \infty), \tag{1.129}$$

$$t(\eta) = \frac{a_0}{2}[\sinh(\eta+\pi)-\eta], \qquad (-\pi \leq \eta \leq \infty). \tag{1.130}$$

In the case k = 0, we have a flat space but not flat spacetime. The entire curvature then comes from the factor $a^2(\eta)$. For k = 0, the geometry corresponds to a 3-cone (hypercone). For k = +1, the corresponding to a 3-sphere or S^3 whereas for k =-1, represents the geometry of 3-hyperboloid or H^3.

In all the cases, the model universe starts at $\eta = -\pi$ and then expand. The 3-sphere which is also known as the closed Friedmann model universe starts from the big bang at $\eta = -\pi$, and expand to the maximum size at $\eta = 0$. It then shrinks and and collapse to a big crunch at $\eta = \pi$. For k = 0, represents an eternal universe expanding forever and referred to as a flat universe. For k = -1, the model universe starts from the big bang and does not have an end referred to as an open model universe. The three cases are shown in the figure 1.2.

1.9 Thermodynamics

Thermodynamics deals with the study of relationship between heat and temperature and their relation to energy and work [8]. It defines the properties of matter such as volume, temperature and pressure. These are the properties of matter in bulk rather than that of individual isolated molecules and are, therefore, called macroscopic properties as opposed to microscopic properties [8]. It provides the most general and efficient methods for studying and understanding complex physical and chemical phenomena. Thermodynamics does not provide the microscopic view of the physical and chemical properties of the matter. On the other hand statistical mechanics deals with such microscopic properties of the matter. Classical thermodynamics is based on four laws. Here I shall discuss them in brief.

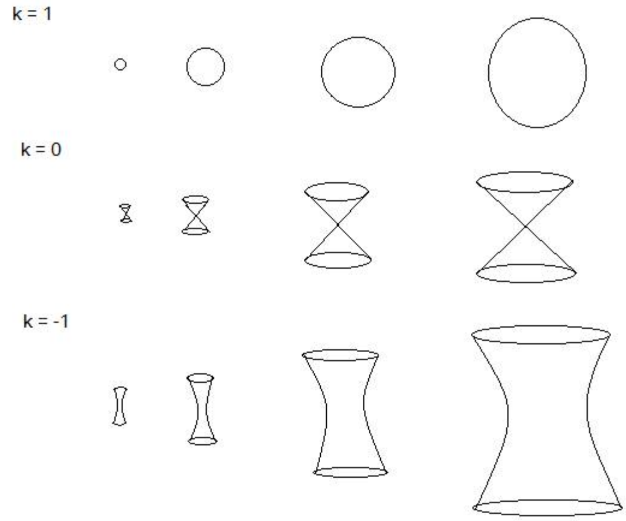

Figure 1.2: The Friedmann model universe for different values of k (taken from [9]).

1.9.1 The Zeroth Law of Thermodynamics

The zeroth law defines the relationship between two systems in thermal equilibrium. A system is in thermal equilibrium if the properties of the system remain unchanged with time. It states that "Two systems which are both in thermal equilibrium with a third system are in thermal equilibrium with each other". Two bodies are said to be in thermal equilibrium if heat does not flow from one body to the other when they are in thermal contact.

To understand the zeroth law let we consider three systems A, B and C. If system A is in thermal contact with system C and system B is in thermal contact with system C. After some time this whole system will be in thermal equilibrium with each other by exchanging heat energy. So A is in thermal equilibrium with C and C is in thermal equilibrium with B. When we brought both A and B in thermal contact we see they do not exchange any heat energy. So they too are in thermal equilibrium with each other. It is known as the zeroth law of thermodynamics. It also gives the equivalence relation, as two bodies are in thermal equilibrium if they both have the same temperatures. If $T_A = T_C$ and $T_B = T_C$ then $T_A = T_B$. Zeroth law is therefore gives us the definition of temperature which does not depend on physical feelings of hotness or coldness. It is based on the observations and experience that systems in thermal contact are not in complete equilibrium with one another until they have the same measure of sensation of heat, i.e. the same temperature. This law was developed after the first three laws of thermodynamics and the importance of this law was not completely realized until the other important properties of thermodynamics had not developed. Due to its more fundamental nature it was then ranked as a basic law of thermodynamics, i.e, the zeroth law.

1.9.2 The First Law of Thermodynamics

The first law of thermodynamics is a form of the law of conservation of energy, which states that the total energy of an isolated system remains conserved, energy can never be created nor be destroyed but it can be transformed from one form to another. It deals with the macroscopic properties of the system that is heat, work and internal energy. In a more formal way this law can be stated as follow: "In any thermodynamical process the change in internal energy ΔE of the system is equal to the amount of heat Q added to the system minus work W done by the system " [8]. It can be written as

$$\Delta E = Q - W, \tag{1.131}$$

where ΔE is the change in the internal energy of the system, Q is the heat added to the system and W is the work done by the the system. The sum of all the energies of all the particles of a system is called the internal energy of the system. The sign of Q is positive if heat is added to the system and is negative if its is given out of the system. Similarly the sign of W is positive if it is done on the system and is negative if it is done by the system. The above equation for the first law can be define in the form of differentials

$$dE = dQ - dW, \tag{1.132}$$

where dQ and dW are the non exact differentials because they are path dependent. Because they do not depend on the initial and final states of the system rather they depend on the path that followed by the system between the initial and final states. Thus dQ and dW have infinitely many values between initial and final states depending upon the path followed by the system, but their sum dE is path independent and only depends upon the initial and final states of the system. The first law of thermodynamics enables to calculate the energy changes during a chemical reaction. But it does not consider whether the reaction occurs or not. For example heat spontaneously flows from a hot body to a cold one until a uniform temperature is reached and the energy is conserved during the flow. One can observe, in the reverse process energy would still remain conserved but naturally we have no such observations in which heat flow from hot to cold without external work. It seems that Nature goes in preferred direction. To understand this natural flow of heat we need another law to explain this phenomenon.

1.9.3 The Second Law of Thermodynamics and the Concept of Entropy

The first law of thermodynamics tells us about the internal energy of the system, whereas the second law of thermodynamics describes another important thermodynamical property called "entropy" which further help us to understand the flow of heat. The thermodynamical definition of entropy is the change in entropy dS of a system which exchange a small amount of heat dQ with its surroundings at temperature T given by the following relation

$$dS = dQ/T. \tag{1.133}$$

The above relation tells us that the entropy and heat are directly related to each other. Entropy changes with change in heat. The second law of thermodynamics states that "In any thermodynamic process that proceeds from one equilibrium state to another the entropy of the system and that of the environment either remains unchanged or increases" [8]. If dS_{sys} is changed in the entropy of the system and dS_{env} is changed in the entropy of the environment then the total entropy according to the second law of thermodynamics would be

$$dS_{sys} + dS_{env} \geq 0. \tag{1.134}$$

This expression describes that if we decrease the entropy of a system the entropy of the environment would increase in greater amount than it decrease. But the total sum is always greater than zero and the equality only holds for a reversible process. The entropy increases when heat is added into the system and decreases when it is taken out from the system.

As we know all the spontaneous precesses that occur in nature are irreversible [8] and we can only make some of them reversible by using energy. Water flows from uphill to the downhill, heat flows from hot body to the cold one and it is never observed that water spontaneously flows from downhill to the uphill or the heat flows from cold to the hot body. Therefore no spontaneous process can take place that results in the decrease in the total entropy of the system and environment. We can decrease the entropy of a system by doing some work on the system but at the same time the entropy of the environment increases more than the decrease of the entropy of the system, so the net change in the entropy of the system is positive. The second law of thermodynamics does not allow those processes in which the entropy of the system plus the entropy of the environment decreases. That is the reason why water does not flow to uphill by its own. We can use the definition of entropy and re-write the first law of thermodynamics as follows

$$dE = TdS - dW. \tag{1.135}$$

The entropy dS is a state variable, it does not depend on the path that is taken between the states of a system rather it only depends on the initial and final states of a system. Statistical thermodynamics gives the better way to understand the concept of entropy. It links the information about a system to the entropy, i.e, the measure of disorder in a system. The more the entropy of a system the less we have the information about the system means we are ignorant about the system. Lack of information about a system results greater entropy of the system. If the known information about the internal configuration of a system is that it may be found in any of a number of available states. If p_n is the probability of the n^{th} state, then the entropy associated with the system is given by the following relation.

$$S = -\sum p_n \ln p_n. \tag{1.136}$$

This is the statistical definition of the entropy.

1.9.4 The Third Law of Thermodynamics

The third law of thermodynamics can be stated as " by no finite series of processes the absolute zero is attainable" [10] . Experiment shows that the fundamental feature of all the cooling processes is that, the lower the temperature attained, the more difficult it is to cool further. For example, the colder a liquid is the lower the vapour pressure, and harder it is to produce further cooling by pumping away the vapour. The above statement of third law of thermodynamics is also known as the principle of unattainability.

Chapter 2

Information Loss Paradox

Quantum mechanics (QM) and GR are two fundamental theories of nature. Both theories have been developed in the 20th century. Einstein was the father of GR and one of the founders of QM. The latter one deals with the motion of macroscopic objects and describes phenomena with great accuracy, whereas, the former one deals with the motion of objects at atomic and subatomic level perfectly. Both theories have their limitations and are incompatible with each other. There have been numerous attempts to combine them but they did not succeed. It is hoped that there will be a theory of quantum gravity which can possibly solve these issues. Stephen A. Fulling [11] tried to combine these two but he did not consider gravity, rather he quantized scalar fields in a linearly accelerating frame. The expectation value he obtained for the number operator has a fractional value which was defined in the Schwarzschild background in the Minkowski space vacuum states. There was an ambiguity in the quantization procedure for gravity. Hawking repeated the procedure for curved spacetime, i.e, for the Schwarzschild black hole. Surprisingly, his result gave the Planck spectrum. Further, by using the Feynman path integral he calculated the temperature of the black hole.

2.1 Black Holes and Entropy

Apparently the concepts of entropy and black holes have nothing to do with each other. Around 1967, Penrose constructed a paradox regarding entropy and black holes [12]. In a thought experiment, he visualized a civilization living around a black hole whose means of producing energy for themselves is by lowering a box full of thermal radiation into the vicinity of a black hole with the help of a spring and throwing the radiation in. As the radiation has an equivalent mass, when the empty box is brought back up, the spring has stored more energy which can be used. In this process, the civilization will not only get the free energy but will also reduce the thermal pollution in the surroundings. It seems that entropy around the black hole decreases which is a violation of the second law of thermodynamics. Penrose argued that the only way to save the second law of thermodynamics is to require that the black hole has an entropy that rises more in the above process than is lost by the surroundings. This was the first connection made of entropy with black holes. A black hole's surface area increases when it undergoes any transformation [13]. Hawking proved that the surface area of a black hole cannot decrease by any process. When two black holes merge, the area of the resultant black hole can never be smaller than the sum of the areas of the two. Let A_1 be the area of a first black hole and A_2 be the area of second black hole. When both of these black holes merge, the area A_3 of the final would be greater than the sum of the individual areas [14].

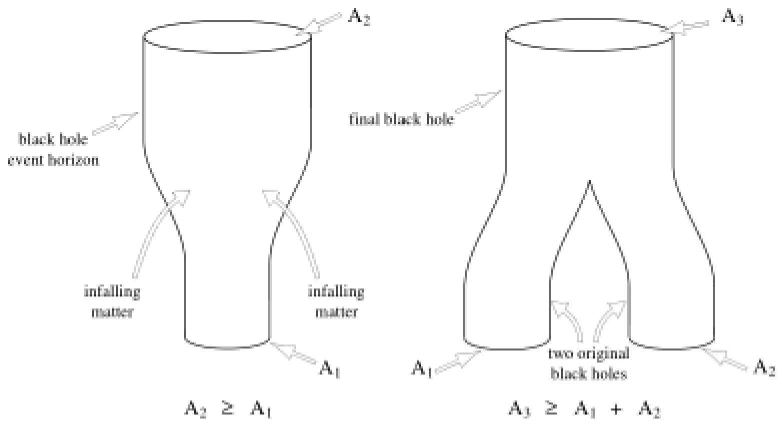

Figure 2.1: Merging of two black holes (taken from [14]).

$$A_3 \geq A_1 + A_2. \tag{2.1}$$

In the recent detection of gravitational waves, two black holes of 35 and 29 solar masses merged to give a black hole of 61 solar masses [15]. There is a difference of 3 solar masses between final black hole and its constituents that was radiated away as gravitational waves. As such, $M_3 < M_1 + M_2$ but it is easy to see that $M_3^2 > M_1^2 + M_2^2$ and hence the area theorem holds as for the Schwarzschild black hole the area is given by

$$A = 4\pi r_s^2. \tag{2.2}$$

Penrose and his student Floyd [2] developed a mechanism to extract energy from a Kerr black hole. This procedure is now known as the Penrose process. There is a region outside the Kerr black hole which is called the "ergosphere". As the Kerr black hole rotates, anything inside the ergosphere will also rotate with it. This is called inertial frame dragging. If a particle splits into two pieces inside the ergosphere of the Kerr black hole in such a way that one piece falls into the hole and other goes to infinity from the ergosphere with an energy greater than the rest energy of the original particle. One can extract energy in this way from a rotating black hole. Independently, Christodoulou had shown that no process whose ultimate outcome is capture of a particle by a Kerr black hole can result in the decrease of a quantity which he named the irreducible mass of the black hole [13]. In most processes the irreducible mass, m_{ir}, increases but for a class of processes called reversible processes in which m_{ir} remains unchanged. The irreducible mass is related to the area by

$$m_{ir} = \sqrt{A/16\pi}, \tag{2.3}$$

where A is the area of the Kerr black hole given by

$$A = 4\pi[r_+^2 + a^2/c^2], \tag{2.4}$$

where r_+ is the outer horizon of the Kerr black hole, a is the angular momentum and c is the speed of light. Christodoulou's conclusion also supports Penrose and Floyd conjuncture.

2.2 Energy and Entropy

Energy changes its forms. It can be converted from one form to another. Heat is the form that most of the energy is converted into. But the sum of all of its forms never changes and hence remains conserved. Heat is a form of energy which is not easily understandable. Earlier, physicists and chemists thought that it was a substance and behaved like a fluid and called it *phlogiston* [16], but heat is not a new substance, it is one of the forms of energy. If we heat a box filled with gas, first its molecules start moving in a random motion. If we add more heat into the box, their motion become more random and chaotic. At this stage, it is difficult to find the velocity or position of a molecule because of random motion. This system would be highly disordered. Entropy is a measure of the disorder in a system. The order in a system is considered to be the information stored in it. According to the second law of thermodynamics, entropy of a system always increases, which means that with the passage of time, information is lost about the internal configurations of a system. Thus, entropy is microscopic hidden information about the system. The information is measured in bits. Heat and entropy has a deep connection. Heat is the energy of random chaotic motion and entropy is the amount of hidden microscopic information [16]. To understand the connection between heat and entropy, let us consider a bath tub full of hot water. If we remove all of the heat from the tub at absolute zero, then all the atoms would be in a perfect order. At this moment, one can find the position of each atom precisely and there would be no hidden information. Temperature, energy and entropy are all zero at this particular moment. On the contrary, if a small amount of heat is added, the molecules will start to move and there configuration will break down and as a result, a small amount of information will be lost. So by adding heat information is lost, which means that as energy increases, entropy also increases.

2.3 Bekenstein Entropy of Black Holes

Soon after the discovery of Hawking, a doctoral student of John Wheeler named Jakob Bekenstein discovered [13] the entropy associated with a black hole which is directly proportional to the change in surface area of a black hole in Planck units. He was not interested in calculating how much information can be stored into a black hole. Instead, he calculated the change in entropy of a black hole when a single bit of information is thrown into it. The expectation was that the change in entropy of a black hole must be proportional to the volume of the black hole, similar to the procedure if someone wants to calculate how many drops of water there are in a bathtub. The number of drops is directly proportional to the volume of the bath tub. If the volume is doubled then the number of drops becomes double. But surprisingly, it comes out directly proportional to the change in the area of the horizon of the black hole.

Bekenstein considered the simplest (Schwarzschild) black hole of a certain size. Now the problem was how to choose a single bit of information. He considered a photon of appropriate wavelength such that it can fit into the black hole. Otherwise, if its wavelength is greater than the diameter of the black hole, it will not fit into the hole. So, he considered the wavelength equal to the Schwarzschild radius, i.e, $\lambda = r_s$. Since a photon has energy, and according to Einstein's mass-energy relationship, a photon has an equivalent mass. So when a photon is thrown into the hole, it increases the black hole mass, which results in an increase in the area. Since area of a black hole is directly proportional to its Schwarzschild radius, i.e, $A = 4\pi r_s^2$. He calculated the change in area of the horizon which comes out, i.e,

$$dA = 16\pi^2 l_p^2, \tag{2.5}$$

where $l_p = \sqrt{G\hbar/c^3}$ is the Planck's length and $l_p \sim 10^{33} m$.

When Hawking came to know about the discovery of Bekenstein, it did not make any sense to him. He described in his book *The theory of everything* how Bekenstein's discovery irritated him and that Bekenstein has mis-applied his area theorem, but he finally accepted that Bekenstein was correct [17]. However, Bekenstein was unable to find the complete formula for the entropy of the black holes. It was Hawking who managed to find an exact relation between entropy and area of the black hole which is defined as

$$S = \frac{A}{4}, \tag{2.6}$$

where S is the entropy and A is the surface area of the black hole.

Later, Hawking was studying black holes in the context of quantum field theory in a curved background [13]. Before the discovery of Hawking, everyone thought that black holes are black bodies but at absolute zero, i.e, they have no temperature at all and they were considered to be the coldest objects in the Universe. After the discovery of Hawking, everything has changed about black holes. I will not go into details of his discovery but he concluded that black holes are not as black as we once thought. He further explained that every black hole has a temperature which is inversely proportional to its mass. Massive black holes have a very low temperature and smaller black holes have a very high temperature. The mathematical expression of his formula about the temperature of a black hole is given below

$$T_H = \frac{\hbar c^3}{16\pi^2 Gmk}, \tag{2.7}$$

where T_H is the Hawking temperature, $\hbar$ is the Plank's constant, c is the speed of light, G is the gravitational constant, m is the mass of the black hole and k is the Boltzmann constant.

2.4 Black Hole Thermodynamics

From these developments, it was recognized that there is a deep connection between thermodynamics and black hole physics. Bardeen, Carter and Hawking formulated the laws of black hole mechanics. They considered the analogy between black holes and thermodynamics as suggestive but purely formal, reflecting no deep relationship of the two subjects and being disconnected with the quantum [18].

2.4.1 The Zeroth Law of Black Hole Mechanics

The zeroth law of thermodynamics states that if the system is in thermal equilibrium then its temperature is constant. The corresponding statement for the zeroth law of black hole mechanics is "the surface gravity k, of a stationary black hole is constant over the horizon". So, the law suggests that there is an analogy between the statements that "the temperature of different parts of a system in thermal equilibrium is the same and the surface gravity is constant over the horizon of a stationary black hole".

2.4.2 The First Law of Black Hole Mechanics

According to the first law of black hole mechanics when a particle is thrown into a stationary black hole then it changes from one stationary state to another. The change in the mass of the black hole will be defined as

$$dM = \frac{\kappa}{8\pi} dA + \text{“\textit{work terms}”}. \tag{2.8}$$

The work terms could be different for different black holes. However, for the Schwarzschild black hole, the work term would be zero and the expression becomes

$$dM = \frac{\kappa}{8\pi} dA. \tag{2.9}$$

As energy is directly proportional to the mass, the area of a black hole will only change when the mass of the black hole changes. From the first law of thermodynamics, the expression for the law when work is zero can be written as

$$dE = TdS. \tag{2.10}$$

By comparing the last two equations, we also get the expression for the entropy of the black hole, i.e,

$$S = \frac{A}{4}. \tag{2.11}$$

The expression for the first law of black hole thermodynamics for the Kerr-Newman black hole would be

$$dM = \frac{\kappa}{8\pi} dA + \Omega dJ + \Phi dQ, \tag{2.12}$$

where Ω is the angular velocity, J is the angular momentum, Φ is the electrostatic potential and Q is the electric charge.

The first law of thermodynamics gives the energy conservation and in the case of black hole, it tells us that if a particle is thrown into the black hole then the change in the energy of the black hole is equal to the energy of the particle minus the energy that is radiated away from the infalling particle.

2.4.3 The Second Law of Black Hole Mechanics

According to the second law of black hole thermodynamics, the area A of an event horizon never decreases with time. It either increases or still remains the same, i.e,

$$dA \geq 0. \tag{2.13}$$

Classically, since nothing can come out from the black hole so it does not lose its mass, therefore, the black hole area A never decreases by any process. In thermodynamics, the second law ensures that in any thermodynamical process, the entropy is always non-decreasing. This establishes the analogy between the area of event horizon and the entropy. The statement of the second law of black hole mechanics is a little stronger than the corresponding thermodynamic law. Thermodynamically, entropy can be transferred from one system to another in such a way that the total entropy must not decrease. However, one cannot transfer area from one black hole to another since black holes cannot bifurcate [10]. So the area of individual black hole must not decrease as required by the second law of black hole mechanics.

2.4.4 The Third Law of Black Hole Mechanics

The third law of black hole mechanics states that:
"It is impossible by any procedure, no matter how idealized, to reduce κ to zero by a finite sequence of operations [3]."

2.4.5 The Generalized Second Law

There may be some situations in which the second law of thermodynamics could have been violated. For example, if we throw an object having some entropy into the black hole then the entropy of the Universe decreases. Classically we have no longer any access to anything that has crossed the event horizon of a black hole. The entropy seems to decrease in this process as it has gone from the outside world. But an outside observer cannot make any measurement to show that the entropy of the total universe decreases. It seems contradictory to the second law of thermodynamics. It was then conjectured by Bekenstein that there should be a more general law [18]. The second law was then generalized to

"The common entropy in the black hole exterior plus the black hole entropy never decreases" [13]. If the black hole entropy is represented by S_{bh} and the exterior entropy is by S_e then the generalized second law of black hole mechanics can be written as

$$S_{bh} + S_e \geq 0. \tag{2.14}$$

Since the ordinary second law of thermodynamics was violated by the argument that when an object has fallen into the black hole the entropy of the Universe decreases. Thus the generalized second law of black hole mechanics makes the statement valid for the total entropy of the Universe would increase when something is thrown into the black hole.

2.5 Information Loss Paradox

In classical theory, the loss of information is not a problem. A classical black hole would last forever and information could be thought of as preserved inside it, but just not very accessible [19]. It is permanently lost from the outside world but it is still inside it. But Hawking's discovery has changed everything. Black holes are not so black as was once thought. Now, a black hole has a temperature T, so if a body has a temperature it must radiate its energy. According to Einstein's mass-energy relation, if a body reduces its energy then it also reduces its mass. So, quantum theory causes the black hole to radiate its energy. As a result, a black hole loses all of its mass and hence evaporates. All the information that was inside the black hole has gone with the radiations. Here the details of the discovery will not be presented instead his argument will be summarized in the following subsection.

2.5.1 Hawking's Argument of Information Loss

There are two types of fluctuations in nature, i.e, thermal fluctuations and quantum or vacuum fluctuations. Quantum field theory provides a way to understand both. Thermal fluctuations are due to excess energy and are produced due to the real pairs of photons. Quantum fluctuations, on the other hand, are produced due to virtual pairs of photons, which are created and quickly absorbed back into the vacuum [16]. This phenomenon takes place in empty space even at absolute zero. Hawking used the complex mathematics of quantum field theory and showed that a black hole emitted photons due to the vacuum fluctuations. These photons are known as Hawking radiation. In the Hawking formula Eq. (2.7), the temperature of the black hole is inversely proportional to its mass m. The smaller the black hole, the larger would be its temperature. By the time its mass is about Planck size, it would be the hottest object in the Universe. Finally, it would lose all its mass and evaporates away. Everything that was inside the black hole would go with Hawking radiation.

Hawking asked if a pure quantum state is thrown into the black hole then what would happen to it? Classically, nothing would happen, it would stay inside the black hole forever and nothing would come out of it. But, quantum mechanically, as a black hole radiates its energy so anything inside it must come out and would no longer be inside. The information which comes out with the

Hawking radiations is thermal which is totally mixed and completely different from the information that was thrown in. So it is lost. In other words, elementary quantum evolution is unitary and unitary evolution of a pure state should yield a pure state [12]. Contrarily, it comes out as a mixed state. Hawking concluded that in the presence of a strong gravitational field, QM violates its own principles, so it needs to be modified. On the other hand, Leonard Susskind a theoretical physicist from the USA and a Dutch Nobel Laureate t' Hooft did not accept the argument and came with their own idea. In the next section, I will explain their proposal to the resolution of the paradox.

2.6 Black Hole Complementarity

The information loss paradox puzzled physicists for decades. However, Susskind and 't Hooft believed that there must be something wrong with the argument and they started to figure it out. According to them, the information could not be lost. If it was lost, then it violates one of the fundamental principles of QM. Susskind argued that this principle is more fundamental than one of the basic laws of physics, i.e, the law of conservation of energy. How can it be violated? This principle is known as "reversibility" or "information conservation". It simply states that if one knows the present with full precision, one can predict the future at all time. The law also says that if one knows the present absolutely, then one can be very clear on the past as well. The law goes forward as well as backward in time. In QM, this reversibility is known as unitarity. The quantum logic would not hold without it [16], and if information is lost then all the fundamentals of physics would break down. So they started formulating, the counter arguments to fix the problem, but could not succeed at that time.

In 1993, Susskind came up with the "Black hole complementarity" idea. This idea is similar to Bohr's complementarity principle which says that in an experiment one can either see a particle or a wave but not both. Similarly, no observer can make an experiment from outside the black hole that can describe the physics of black hole from inside, and no observer from inside the black hole can make an experiment to show that the black hole can evaporate. Both the classes of observers are apparently contradictory but actually complementary to each other. In this situation, there would be no paradox. It still remained to answer for an outside observer that whether the information is lost or not. To answer this, they presented another principle.

2.6.1 The Holographic Principle

Bekenstein's argument says that the entropy of a black hole is proportional to the surface area of the horizon and not to the volume of the black hole. Susskind and 't Hooft put forward the idea of holographic principle, according to which the maximum amount of information that can be stored in a region of space cannot be greater than that which can be stored on the boundary of the region [16]. It was argued that just like the hologram can store a 3-dimensional image of an object on a 2-dimensional film, so all the information inside a black hole can be thought of as stored on the surface of the horizon, it was then known as the Maldacena conjecture [20]. Using a (2+1) dimensional black hole, this conjecture was proved in the context of superstrings [21]. The spacetime that was used for this purpose was “anti-de Sitter”. In this spacetime it can be demonstrated [16] that all the information inside the Universe can be stored on the outer horizon of the Universe. So in this context, information can never be lost.

Chapter 3

Foliation

The word foliation comes from the Latin word “folia” for leaf [2]. The spacetime geometry can be understood by slicing the spacetime into hypersurfaces having some constant value of “time” for each hypersurface. An n-dimensional manifold can be split or decomposed into sub-manifolds in such a way that the dimension of each sub-manifold remains the same. Thus each sub-manifold is the leaf of the foliation. If a manifold is foliated in such a way that the dimension of each slice is one less than the dimension of the original manifold then it is called foliation by hypersurfaces.

We can foliate a two-dimensional xy-plane by straight lines. Generally the foliation is not unique. If the foliation covers the entire manifold by sequence of non-intersecting sub-manifolds or slices then it is said to be complete.

Mean Extrinsic Curvature:

The ratio of the second fundamental form to the first fundamental form describes the normal curvature of the surface. On the surface, there are a large number of curves. The curvature of these curves is independent of the nature of the surface. The normal curvature depends which curve we choose on the surface. However, some quantities are invariant and independent of the choice of the curve. These quantities are called maximum K_+ and minimum K_- values of the normal curvature along two perpendicular directions and known as the *principle curvatures*. Their product K_+K_- is known as *Gaussian curvature* and is invariant under coordinate transformation. Also their average $(K_+ + K_-)/2$ is an invariant quantity and known as *mean curvature*.

The curvature which is described entirely by living within the surface is called *intrinsic curvature* and is given by the first fundamental form. Intrinsic means one do not need to go off from the surface and then measure the curvature of the surface. It can be measured within the surface. This fact is known as the *Gauss's theorem* [2]. Whereas the *extrinsic curvature* is measured when a curved surface is embedded in one higher dimension. It is extrinsic to the surface and given by the second fundamental form. The mean extrinsic curvature describes the curvature of a hypersurface (which is embedded in one higher dimension) relative to the enveloping geometry. It can be defined as

$$K = -div\mathbf{n} = -n^a_{;a}, \tag{3.1}$$

where n^a is the unit normal to the hypersurface.

3.1 Foliation by Spacelike Hypersurfaces

A spacetime manifold can be foliated by hypersurfaces which may be timelike, nulllike or spacelike. The first fundamental can be written as

$$ds^2 = g_{ab}n^a n^b. \tag{3.2}$$

These hypersurfaces can be classified uniquely by the metric **g** as:

- If $g_{ab}n^a n^b > 0$, then it will be a timelike hypersurface;
- If $g_{ab}n^a n^b = 0$, then it will be a lightlike hypersurface;
- If $g_{ab}n^a n^b < 0$, then it will be a spacelike hypersurface.

A *W-universe* (*W* stands for Wheeler) is a closed, compact, big bang and a big crunch model universe [22]. This universe has a positive spatial curvature. If we foliate such a model universe by spacelike hypersurfaces then every spacetime event lies on the unique spacelike hypersurface at a unique value of the time. A time parameter is then provided which vary with each hypersurface. Since this foliation is obtained by hypersurfaces of constant mean extrinsic curvature so hereafter it is referred to as K*-slicing*. Thus $tr\mathbf{K} = K$ has the same valve everywhere but differ from one slice to another. It was then referred to as the York time [6]. K varies with each hypersurface just like the time parameter. So there is a $1-1$ correspondence between different values of K and the time parameter.

3.2 York-Slicing or K-Slicing

In a closed universe, the difference between inside and outside of a black hole is not clear or in other words distinction between black hole singularity and final cosmological singularity is not clear. Penrose pointed out that by a conformal transformation the black hole singularity can be viewed as part of the final singularity or in an open universe part of the compactification of the spacetime at future infinity [23]. The picture he proposed was to view that black hole singularity as stalactites on the roof of a cave, and the cave which represented the final singularity. If roof can be straightened out appear smooth by some appropriate conformal transformation [24]. This also means that spacetime should be K-foliated by means of spacelike hypersurfaces which would

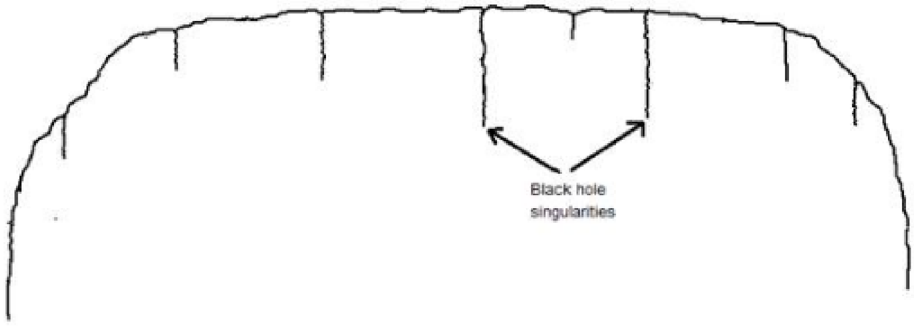

Figure 3.1: In the Penrose's picture the black hole singularity is to the final singularity as a stalactite is to the ceiling of the cave [24].

approach the singularity without cutting it anywhere. The entire spacetime would be foliated. The limit of some parameter going to some specific value should yield the entire singularity. The Penrose's idea can be depicted in the above figure.

Initial attempts founded on foliation by maximal slicing [23]. *Maximal slicing* means that foliation by hypersurfaces of *zero mean* extrinsic curvature. Some problems arising in maximally slicing a spacetime even the simplest of the spacetime, i.e, the Schwarzschild black hole could not be foliated completely. The hypersurface *ran out* into a boundary, i.e, it didn't cover the whole spacetime. This shows that either the Penrose conjecture was incorrect or the maximal slicing was not enough to foliate the whole spacetime [24]. It was argued that foliation by hypersurfaces of constant mean extrinsic curvature would be more appropriate [25]. These hypersurfaces foliated the entire spacetime as was expected.

3.3 K-Slicing of the Closed Model Universe

In a closed model universe the distinction of the principle between inside and outside of a black hole event horizon breaks down. This is done by foliation of two model W-universes by hypersurfaces of constant mean extrinsic curvature. One of the simplest examples of the W-universe is the *Schwarzschild lattice* [26]universe in which black hole represented by the Einstein-Rosen bridge [27]. The other model consists of a black hole part away through the evolution of the Friedman universe. By this foliation it was concluded that a W-universe has two singularities the *big bang* and the *big crunch*.

3.3.1 Foliation of the Schwarzschild Lattice Universe

Schwarzschild lattice universe is one of the simplest models of W-universe. Such model has 5, 16, 20, 120 or 600 black holes in it (they choose 120 for definiteness) having identical masses as shown in the figure below.

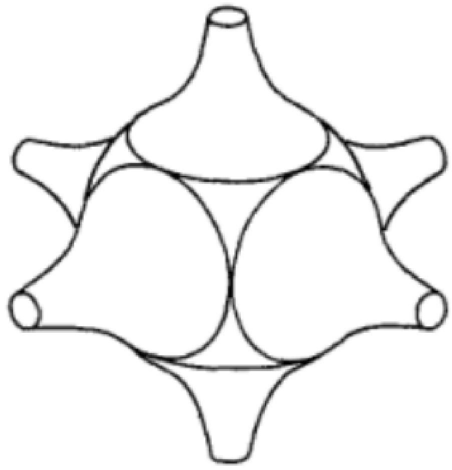

Figure 3.2: Different Schwarzschild regions are joined together to form a closed universe (taken from [26]).

The basic analysis required for this purpose has already been set out by Lindquist and Wheeler [26]. This model is foliated by a sequence of K = constant hypersurfaces by breaking each cell independently as a Schwarzschild geometry and requiring that the foliating hypersurfaces have zero derivative on the boundary, B. For the solution to be unique condition provided at B and the centre of the cell. As the hypersurface must be non-singular anywhere, so a proper treatment had to be made at the center of the lattice cell since the black hole singularity is located there. However by requiring that extrinsic curvature and hence the York time become infinity at the singularity no hypersurface of finite mean extrinsic curvature will hit the singularity. This foliation should have the character of many one-finger gloves applied one over the other to a finger. No glove except the last one should touch the finger, while the last one should fit on it [24]. More mathematically none of the sequence of the hypersurfaces should touch the singularity but their limit should coincide with Schwarzschild singularity which is located at the zero of Schwarzschild radial parameter, $r = 0$.

The foliation runs smoothly with no problem anywhere else, this demonstrating that K-slicing rather than the maximal slicing is more appropriate foliation procedure as noted by Brill, Cavollo and Insenberg [28]. The problem with this model is that in the process of formulating the initial condition the black hole singularity which was sheathed has been lost. In effect this model has "mass without mass" as the singularities has been removed from the universe and replaced by Einstein Rosen Bridges. In terms of Penrose's picture there is no stalactite on the 'cave roof' here. As there is no black hole in the model, so the problem is tackled by providing some mass in the form of dust. This can be done by replacing one of the lattice cells considered by a thin shell of

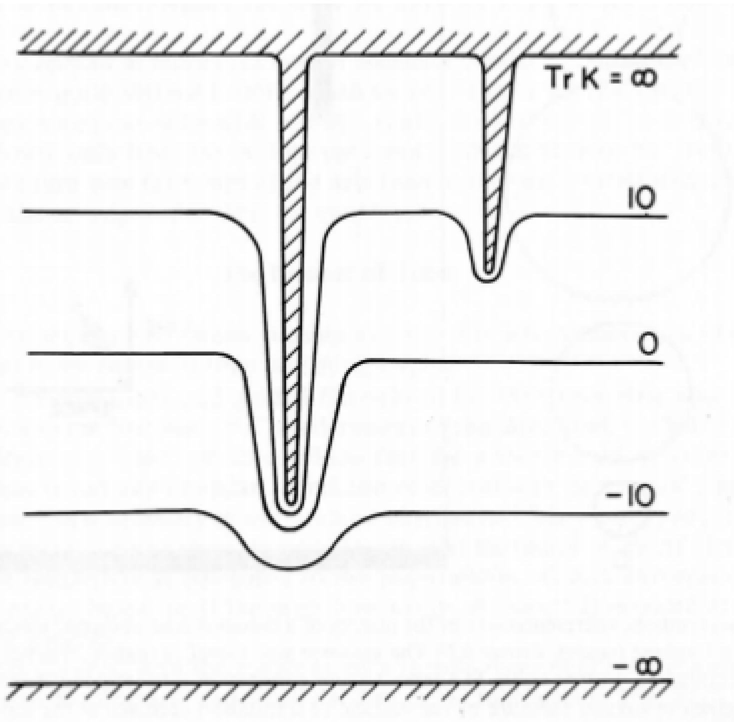

Figure 3.3: Expected gloving of a black hole singularity by a sequence of spacelike hypersurfaces of increasing mean extrinsic curvature, $tr\mathbf{K}$, shown highly schematically (taken from [6]).

dust [6]. The geometry exterior to the shell is the Schwarzschild geometry while the geometry interior to the cell is the Minkowski. The geometry of such a thin shell of dust in an asymptotically flat spacetime has been studied by Israel [29]. A jump condition connects the Minkowski geometry with the Schwarzschild due to the shell. The same foliation procedure was followed for this model and again proceeded without any problem. However, the model, while more realistic than the previous one, still lacked physical clarity. Then they developed a W-universe model in which the physics comes out more clearly [24].

3.4 The Sandwich Model

Friedmann universe is homogeneous and our assumption about the model is that it should have black hole in it. So cut and paste method is used to build such a model from the well understood Friedmann and Schwarzschild geometries. To provide such inhomogeneity two closed friedmann matter filled universes considered. One of the higher and the other of lower densities. A section cut out from the lower density model and replaced by the section of higher densities model in such a way that the mass of the model remains unchanged [24].

The empty region between the two possessed the features of the Schwarzschild geometry. The model clearly has a picture of Schwarzschild geometry sandwich between the two slices of

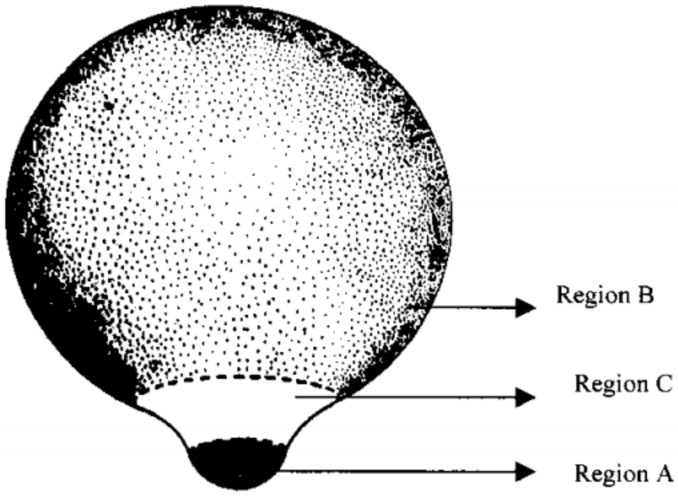

Figure 3.4: The ‘sandwich’ model (taken from ([24]).

Friedmann universes, one of the higher density and less mass (the enveloped region) and the other of lower density and greater mass (the enveloping region). To avoid the two regions ‘crossing into each’ as must occur before the Schwarzschild region appears. The two sections of the Friedmann models arrange in such a way that they fit perfectly together at the Big bang. The requirement is that the three regions join together so that no crack appears between them in the entire evolution of the model from the bang to crunch. The foliation procedure for such a model has a geometry of 3-spheres. This model is also called “The Suture Model” and will be discussed in the next section.

3.5 The Sandwich Model of a Black Hole in a Friedmann Universe

A W-model universe starts from a big bang, it then expands, this expansion does not go forever, it reaches at its maximum size, shrinks and ends in a big crunch. In the Friedmann universe, the big crunch may be everywhere at the same time. Whereas in a W-universe, the crunch happened earlier in some parts than in the others. As this model consists of a Friedmann universe whose $1/N$ sections has been cut out and replaced by a section of another Friedmann universe having the same mass but different density than that of the low density Friedmann universe. Due to the different densities the denser region collapse faster than the rare one and similarly the collapse occur earlier than that of the rare one. Therefore, the denser region has a shorter life than that of the rare one. Here time does not mean that the proper time, rather it means that the York time, K, as defined by the trace of the extrinsic curvature tensor.

3.5.1 Formation of the Suture Model

In a closed universe, the black hole singularity and the final big crunch singularity are the same. This has been discussed in the context of few inhomogeneous cosmological models by Bril, Cavallo and Isenberg [28]. Qadir and Wheeler studied the dynamics of inhomogeneous cosmological model by considering two closed matter filled universe model, one with high and the other with low density [30]. They studied this model and showed that there is no difference of principle between inside and outside of a black hole. The black hole singularity is a part of final singularity. Finally, they concluded that in such a model universe there are only two singularities, i.e, the big bang and the big crunch.

They used a ‘cut and paste’ method to build such a model from the well understood Friedmann and Schwarzschild geometries. The desired inhomogeneity was provided by considering two Friedmann universes of different densities. One is rare Friedmann and the other is dense Friedmann. $1/N$ sections from rare Friedmann universe was removed and replaced by denser Friedmann universe of same mass. These two sections were considered to be purely matter filled. Both of the portions are so arranged that the mass of the model remains unchanged. Higher density region has a short life-time than the lower density region. It will evolve at a much faster rate and collapse to less than its Schwarzschild radius as seen from the lower density region. Thus a black hole has formed in the lower dense Friedmann universe [30]. Comparison can be made with the Schwarzschild lattice universe model, they took the size of the section removed from the lower density model to be $1/120$ of the total. For definiteness the proper-life of the denser Friedmann model chosen to be $1/2$ that of the less dense Friedmann model. Other fractions can be chosen instead of $1/2$ to obtain results numerically different but qualitatively with same behaviour. The choice of $1/2$ has no further significance.

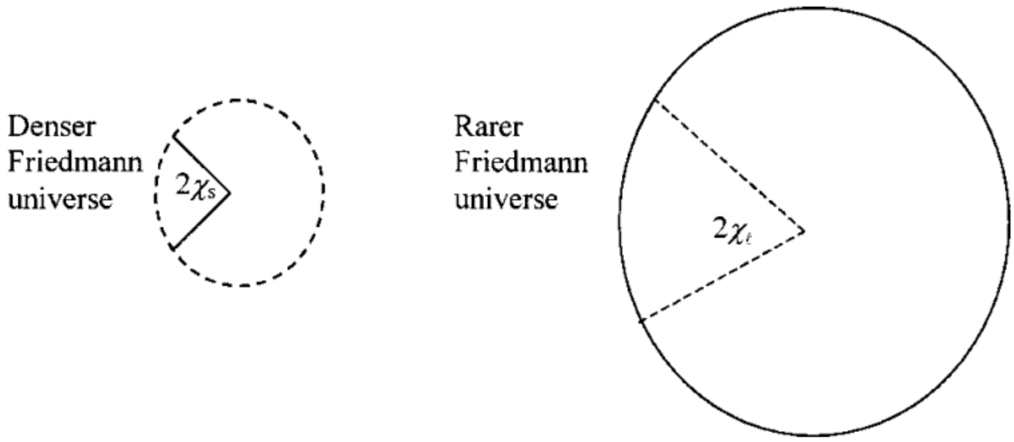

Figure 3.5: The ‘cut and past’ view of the model universe with two dimensions suppressed. The smaller and denser Friedmann universe contains the hyperspherical angle χ_s as the radial parameter, whereas χ_l stands as the radial parameter of the larger, the less dense domain (taken from [30]).

Clearly if the two models join together purely at an instant, due to their different densities and hence different evolutions they can not do before or afterwards. Once they start to separate off, they can be joined together by a region of pure Schwarzschild geometry. Thus the Schwarzschild geometry sandwich between the two slices of Friedmann universes. To avoid the two regions crossing into each other as must occur before the Schwarzschild region appears, so they arranged the two sections of the Friedmann models to fit perfectly together at the big bang. They required that the three regions fit together, so that no crack appears between them in the entire evolution of the model from big bang to the big crunch. Therefore, they have a well defined model. They foliated this model with a sequence of hypersurface of constant curvature K.

3.6 The Foliation Procedure and the Suture Model Parameters

The foliating spacelike hypersurfaces are spherically symmetric and independent of the angles θ and φ. Therefore, we are only left with two parameters in the Friedmann region η and χ, where η is the Friedmann angle-like parameter and χ is the spherical angle. Whereas in the Schwarzschild region r and t are the radial and time parameters respectively or in the Kruskal-Szekeres coordinates, the equivalent parameters are u and v or in the Penrose compactified coordinates the corresponding parameters are ξ and ψ. The evolution would proceed normally in the higher density Friedmann universe from the point of view of an observer which is located in the enveloped Friedmann universe given by the following metric [6].

$$ds^2 = a^2(\eta)[-d\eta^2 + d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\varphi^2)], \tag{3.3}$$

where $a(\eta)$ is given by

$$a(\eta) = (a_s/2)(1 + \cos\eta_s), \tag{3.4}$$

η_s being the 'time' parameter for the enveloped region, going from -π to π at the big bang, through 0 at the phase of maximum expansion, to π at the big crunch and a_s is the radius of the denser Friedmann universe at the phase of maximum expansion. The radial parameter, χ_s, goes from 0 at the center of the region to some χ_M at the boundary with the Schwarzschild region. For the section of the denser Friedmann region to remain unchanged χ_M has to be constant over the entire evolution of the model. The evolution proceeds normally in the less Friedmann region as shown by the Eq. (3.3) but with a_η given by

$$a(\eta) = (a_l/2)(1 + \cos\eta_l), \tag{3.5}$$

where η_l and a_l are the corresponding quantities for the enveloping less dense Friedmann region. The radial parameter η_l , here goes from some constant η_p to π as shown in the figure 3.6.

Now we have a model W-universe which satisfies all over requirements. It is fully satisfied from big bang to big crunch and has a black hole form in an essentially matter-filled closed Friedmann model universe. Our foliating hypersurfaces must start in the enveloped Friedmann region at $\chi_s = 0$. So that it is smooth when $\chi = 0$ is approached from any direction, i.e, for all values of θ and φ. In terms of an appropriate 'angle' parameter it must be flat at $\eta_s = 0$. The hypersurfaces then proceed out ward till it hits boundary between the enveloped Friedmann and the

Schwarzschild region. The requirement is now that it does not develop a 'kink' at the boundary. In other words, its 'angle' to the boundary remains the same on the either side of the boundary. The same requirement must be met at the other boundary where the hypersurface emerges into the enveloping Friedmann region. Finally, it must be smooth at $\chi_l = \pi$ regardless of which direction it is approached from. Thus it must be 'flat' also at $\chi_l = \pi$ in that the angle parameter again becomes zero.

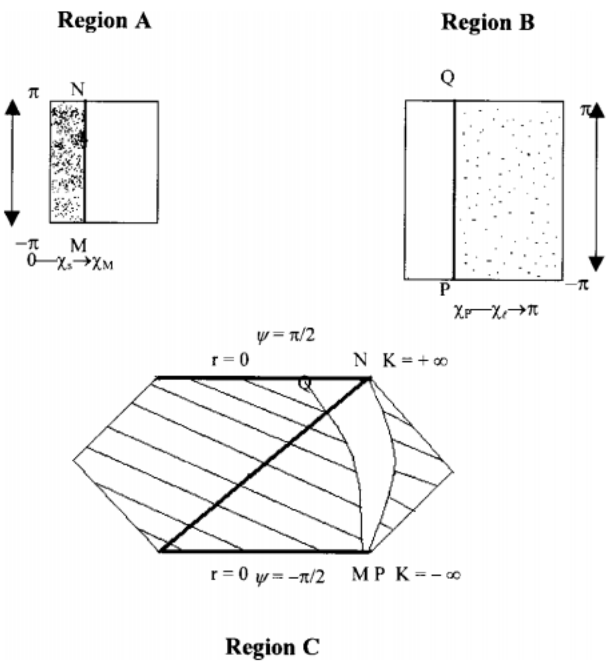

Figure 3.6: Matching the three regions of the 'sandwich', region A has a Friedmann geometry for 0$\leq \chi_s \leq \chi_M$ and $\pi \leq \eta_s \leq \pi$. A tracer particle at χ_M follows the 'boundary' MN as η_s varies. Region C in the Kruskal-Szekeres coordinate is shown bounded by two tracer particles one going from M to N, the other from P to Q. The two tracer particles coincide at the big bang so that M and P are identified. In terms of v and u the two tracers get two 'different places' at the big crunch. In region B, we have again a Friedmann universe, twice as long (in proper time) as A with $\chi_p \leq \chi_l \leq \pi, \pi \leq \eta_l \leq \pi$ and boundary PQ (taken from [30]).

Now to obtain the system of first order, first degree non-linear differential equations for the spacelike hypersurfaces of constant $tr\mathbf{K}$ in each of the three regions. They started with the given initial conditions at some guess value of η_s at $\chi_s = 0$, call it η_0. Now they solved the equations in the enveloped Friedmann region numerically till they hit the inner boundary at some value of η_l, call it η_1, and some angle. The value of η_1 and χ_M give some initial values of t and r in the Schwarzschild region. Using these values and angle already obtained they solved the equations and the Schwarzschild region numerically till they hit the outer boundary at some final values of t and r and the angle. Now they converted these values of t and r into the values of η_1 and χ_p, call it η_2. With these as the initial conditions they solved the equations in the enveloping Friedmann region till they reached $\chi_l = \pi$. The mathematical procedure will be discuss in the next section.

3.7 Mathematical Formulation for the Foliating Hypersurfaces

We need the mathematical equations for the suture model and then to solve them numerically and get results. We will only discuss the mathematical formulation of the foliating hypersurfaces in the Friedmann and the Schwarzschild regions of the suture model separately. The foliating hypersurfaces starts in the denser Friedmann region, A. But here we will discuss the mathematical procedure for foliating hypersurfaces in the Schwarzschild region, C, first in the following subsection.

3.7.1 Foliation in Region C

The mean extrinsic curvature of a hypersurface $tr\mathbf{K}$, is given by

$$K = -div\mathbf{n}, \tag{3.6}$$

where **n** is the unit normal to the hypersurface. As the hypersurface is spacelike so **n** is the timelike and a unit tangent to the hypersurface, **t**, is spacelike.

$$\mathbf{n.n} = -1, \tag{3.7}$$

$$\mathbf{t.t} = 1. \tag{3.8}$$

As this region consists of Schwarzschild geometry. Here we take $\theta = \pi/2$ and $\varphi = 0$ for the convenience, so that $d\theta$ and $d\varphi$ neglected throughout the equation, then the Schwarzschild metric becomes

$$ds^2 = -\big(1 - 2m/r\big)dt^2 + \big(1 - 2m/r\big)^2 dr^2. \tag{3.9}$$

The components of the tangent vector are

$$\mathbf{t}^\mu = (dt/ds, dr/ds, 0, 0). \tag{3.10}$$

The orthogonality between **t** and **n** require that 'time' and 'space' components of **n** must be like that

$$n^t = \big(1 - 2m/r\big)^{-1} dr/ds, \tag{3.11}$$
$$n^r = \big(1 - 2m/r\big) dt/ds. \tag{3.12}$$

The requirement is that **n** satisfy the equation (3.7) can be ensured by expressing the components in terms of an imaginary angle, β, so that

$$n^t = |1 - 2m/r|^{1/2} \cosh\beta, \tag{3.13}$$

$$n^r = |1 - 2m/r|^{1/2} \sinh\beta. \tag{3.14}$$

The expression for the mean extrinsic curvature can be expressed as

$$\begin{aligned} -K &= n^{\mu}{}_{;\mu}, &(3.15)\\ &= n^{\mu}{}_{,\mu} + [\ln\sqrt{|g|}]_{,\mu} n^{\mu}, &(3.16)\\ &= n^{t}{}_{,t} + n^{r}, r + [\ln\sqrt{|g|}]_{,t} n^{t}, &(3.17)\\ &= n^{t}{}_{,t} + n^{r}{}_{,r} + (2\ln r)_{,r} n^{r}. &(3.18)\end{aligned}$$

Now differentiating Eqs. (3.13) and (3.14) with respect to t and r respectively and using Eqs. (3.11), (3.12) and (3.13), (3.14) to eliminate n^t and n^r leads to

$$\frac{d\beta}{ds} = -K - \left(\frac{2}{r} - \frac{3m}{r^2}\right)\frac{dt}{ds}. \tag{3.19}$$

The initial conditions for the system of equations given by Eqs. (3.11) - (3.14) and (3.16) are obtained by choosing an 'initial time', r_0, at the throat of the Einstein-Rosen bridge $t = 0$. At this point the hypersurface must be 'flat', namely the 'time' coordinate must be unchanging and the angle parameter must vanish.

$$\begin{aligned} \frac{dt}{ds}|_{r=r_0} &= (2m/r_0 - 1)^{-1/2}, &(3.20)\\ \frac{dr}{ds}|_{r=r_0} &= 0, &(3.21)\\ \beta|_{r=r_0} &= 0, &(3.22)\\ \frac{d\beta}{ds}|_{r=r_0} &= -K + \big(2m/r_0 - 1\big)^{1/2}\big(3/2r_0 - 1/4m\big). &(3.23)\end{aligned}$$

Eqs. (3.11) - (3.14) and (3.16) subject to Eqs. (3.17) - (3.20) can be integrated till $r = 2m$ but they become singular there due to the Schwarzschild coordinate singularity.

To avoid this problem we have used Kruskal-Szekeres coordinates which are regular at the event horizon and hence the character of time and space rather than null coordinates.

$$u = \left(1 - \frac{2m}{r}\right)^{1/2} e^{r/4m}\sinh(t/4m), \tag{3.24}$$

$$v = \left(1 - \frac{2m}{r}\right)^{1/2} e^{r/4m}\cosh(t/4m), \tag{3.25}$$

$$u = \left(\frac{2m}{r} - 1\right)^{1/2} e^{r/4m}\cosh(t/4m), \tag{3.26}$$

$$v = \left(\frac{2m}{r} - 1\right)^{1/2} e^{r/4m}\sinh(t/4m), \tag{3.27}$$

where t has been assumed to be greater than zero. For $t < 0$ the signs of u and v have to be changed. In these coordinates the metric is of the form

$$ds^2 = f^2(-dv^2 + du^2) + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{3.28}$$

where

$$f^2 = \frac{32m^3}{r} e^{-r/2m}. \tag{3.29}$$

Corresponding to Eqs. (3.11)-(3.14) and (3.16) we now have

$$\begin{aligned} n^v &= f^{-1}\cosh\gamma = f^{-1}\frac{du}{ds}, &(3.30)\\ n^u &= f^{-1}\sinh\gamma = f^{-1}\frac{dv}{ds}, &(3.31)\\ \frac{d\gamma}{ds} &= -K - \left(\frac{6m}{r} - 1\right)\left(\frac{f}{4m}\right)^2\left(u\frac{dv}{ds} - v\frac{du}{ds}\right). &(3.32) \end{aligned}$$

In these coordinates the initial conditions are

$$\begin{aligned} \frac{dv}{ds}|_{r=r_0} &= 0, &(3.33)\\ \frac{du}{ds}|_{r=r_0} &= f^{-1}|_{r=r_0}, &(3.34)\\ \gamma|_{r=r_0} &= 0, &(3.35)\\ \frac{d\gamma}{ds}|_{r=r_0} &= -K - \left(\frac{6m}{r} - 1\right)\left(\frac{f}{4m}\right)^2. &(3.36) \end{aligned}$$

The final condition is that the hypersurface is orthogonal to the boundary where they meet. Thus **b** is the unit vector tangent to the geodesic representing the development of the position of the boundary between two cells, satisfies the equations

$$\mathbf{b.b} = 1, \mathbf{t.t}|_{boundary} = 0. \tag{3.37}$$

In Kruskal-Szekeres coordinates the latter equation is

$$(du/ds_B)b^\mu_B = (dv/ds_B)b^\nu_B. \tag{3.38}$$

The components of **b** are obtained by differentiating the equations for the boundary given by Lindquist and Wheeler respect to the proper time, to obtain

$$\begin{aligned} b^t &= \big(1 - 2m/R\big)^{1/2}\big(1 - 2m/r\big)^{-1}, &(3.39)\\ b^r &= \big(2m/r - 2m/R\big)^{1/2}, &(3.40) \end{aligned}$$

where R is the maximum value that r can take in the lattice cell. In the Kruskal-Szekres coordinates Eqs. (3.39) and (3.40) become

$$b^v = \frac{v\big(2m/r - 2m/R\big)^{1/2} - u\big(1 - 2m/r\big)^{1/2}}{4m\big(1 - 2m/r\big)}, \tag{3.41}$$

$$b^u = \frac{u\big(2m/r - 2m/R\big)^{1/2} - v\big(1 - 2m/r\big)^{1/2}}{4m\big(1 - 2m/r\big)}. \tag{3.42}$$

The subscript 'B' in Eq. (3.38) indicates that the relevant quantities are evaluated where the hypersurface meets the boundary.

The procedure is now straightforward. With some guess value of r_0 for a given constant *K*, they integrated Eqs. (3.27), (3.28) and (3.29) with the initial conditions (3.30)-(3.33) and check whether the resulting values of (dv/ds) and (du/ds) satisfy Eqs. (3.35) and (3.38), (3.39). For some values of r_0 the left hand side of Eq. (3.35) will be greater than the the right, while for others it will be less. They iterated above till they found a good enough fit.

3.7.2 Foliation in Region A

In the Schwarzschild model we started in the Friedmann region. The equations for the foliating hypersurfaces for the constant, K, expressed in terms of an imaginary angle , analogous to Eqs. (3.11)-(3.14) and (3.16) are

$$n^{\eta} = \frac{d\chi}{s} = a^{-1}(\eta)\cosh\alpha, \tag{3.43}$$

$$n^{\chi} = \frac{d\eta}{s} = a^{-1}(\eta)\sinh\alpha, \tag{3.44}$$

$$\frac{d\alpha}{ds} = -K - 3\frac{1}{a(\eta)}\frac{da(\eta)}{ds} - 2\cot\chi\frac{d\eta}{ds}. \tag{3.45}$$

The initial conditions are stated at some guess value of η_0 at $\chi = 0$. Eq. (3.42) is singular at this point. However since the hypersurface is 'flat', $\alpha = 0$ at $\chi = 0$ and so the last term becomes indeterminate. We use the l'Hospital's rule to evaluate this indeterminate expression to obtain the initial conditions.

$$\frac{d\eta}{ds}|_{\chi=0} = 0, \tag{3.46}$$

$$\frac{d\chi}{ds}|_{\chi=0} = a^{-1}(\eta), \tag{3.47}$$

$$\alpha|_{\chi=0} = 0, \tag{3.48}$$

$$\frac{d\alpha}{ds}|_{\chi=0} = -\frac{1}{3}K - a^{-2}(\eta)\frac{da(\eta)}{d\eta}. \tag{3.49}$$

By integrating Eqs. (3.40)-(3.42) subject to the initial conditions (3.43)-(4.46) till we reach χ_M.

We now require that the hypersurface makes the same angle with the boundary between the Friedmann and Schwarzschild regions, on both sides of the boundary. Let the unit, timelike vector tangent to the boundary be **T**. Then we require that **t.T** be the same in both regions. There is no coordinate transformation available from the Friedmann to the Schwarzschild coordinates as the two geometries are essentially different. We are therefore forced to calculate the expression on both sides of the boundary and equate them. In the Friedmann region

$$\mathbf{T} = (a^1, 0, 0, 0), \tag{3.50}$$

as **T** has unit magnitude and is purely timelike in the rest-frame for the Friedmann universe model. Thus the invariant quantity is

$$\mathbf{t.T} = -a^2(\eta)(d\eta/ds)a^{-1}(\eta) = \sinh \alpha_M, \tag{3.51}$$

where α_M is the angle, worked out at the boundary.

Corresponding to Eqs. (3.40)-(3.43), in the Schwarzschild region the equations for the foliating hypersurface for a given constant *K* are already given by (3.11)-(3.14) and (3.16). Writing **T** in terms of new imaginary 'angle'

$$\mathbf{T}^\mu = \big((1 - 2m/r)^{1/2} \cosh \delta, (1 - 2m/r)^{/2} \sinh \delta, 0, 0\big). \tag{3.52}$$

We see that

$$\mathbf{t.T} = \sinh(\beta_M - \delta), \tag{3.53}$$

where

$$\delta = \tanh \left[\frac{dR/d\eta}{dt_b/d\eta} (1 - 2m/r)^{-1} \right], \tag{3.54}$$

$$\frac{dR}{d\eta} = -(R_s/2) \sin \eta, \tag{3.55}$$

$$\frac{dt_b}{d\eta} = (R_s/2m + 2)(R_s/2m - 1)^{1/2} + \frac{R_s}{2m}\big((R_s/2m - 1)^{1/2}\big) \sinh \eta + 2\frac{\big(R_s/2m - 1\big)^{1/2} \sec^2(\eta/2)}{(R_s/2m - 1)^{1/2} - \tanh^2(\eta/2)}. \tag{3.56}$$

Thus, knowing α_M we can compute β_M by using Eqs. (3.48) and (3.51)-(3.54).

We now solve numerically the Eqs. (3.48), (3.49) and (3.51) with the initial value of β, till we come to the outer boundary. Again converting back to the Friedmann geometry, setting $\chi = \chi_p$ at the boundary, we integrate numerically out till $\chi = \chi_f < \pi$. For the first guesses for η_0 for a given value of K, we keep χ_f small (say about 1 or 1.5) and use relatively large step sizes. Once a rough range of η_0 has been obtained we gradually increase χ_f till it reaches 3 and decrease step size till the results are sufficiently stable against changes in step size and in changes of η_0 over a sufficiently small range.

This procedure cannot be followed as stated here when $R_s < 2m$ as we would then have to integrate across the coordinate singularity at $r = 2m$. For this purpose we use Kruskal-Szekeres coordinates $r < 2m$. It is not convenient to use these coordinates for all values as they become too large to maintain the required accuracy. After crossing the horizon we therefore need to convert from the Kruskal-Szekeres coordinates Eqs. (3.27), (3.28) and (3.30)-(3.33) to the Schwarzschild coordinates equations. Here we can actually perform the coordinate transformation and compare the angle obtained by coordinate transformation from Schwarzschild to Kruskal-Szekeres coordinates with that obtained by computing throughout in the Kruskal-Szekeres coordinates.

This computation leads to

$$\gamma = \beta + t/4m. \tag{3.57}$$

Chapter 4

The Qadir-Wheeler Suture Model and the Information Loss Paradox

In this chapter we will discuss the connection between the Qadir-Wheeler suture model [24] and the information loss paradox. But, before going to discuss we will briefly review the "asymptotic behaviour of length and volume of suture model" [30] and "asymptotic behaviour of length and volume of Schwarzschild's singularity" [31].

4.1 Asymptotic Behaviour of Length and Volume of Suture Model

The Schwarzschild geometry between the Friedmann regions can be described by the following line element

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2. \tag{4.1}$$

We want to analyse how his geometry changes with the evolution of the model near the final singularity $r = 0$. Since the Schwarzschild (t, r) coordinates are not well behaved, so we convert them into Kruskal-Szekeres coordinates by using the transformations (3.24) and (3.25). The transformed evolution equation is given by

$$\frac{\mathrm{d}r}{\mathrm{d}s} = -K - \left(\frac{3r_s}{r} - 1\right)\left(\frac{f}{2r_s}\right)^2 \left(u\frac{\mathrm{d}v}{\mathrm{d}s} - v\frac{\mathrm{d}u}{\mathrm{d}s}\right). \tag{4.2}$$

The Kruskal-Szekeres coordinates can be converted into compactified form by using the following transformation

$$v + u = \tan\frac{1}{2}(\psi + \xi), \tag{4.3}$$

$$v - u = \tan\frac{1}{2}(\psi - \xi). \tag{4.4}$$

The Schwarzschild line element in compactified Kruskal-Szekeres coordinates can be written as

$$ds^2 = f^2(r)\left[\frac{-d\psi^2 + d\xi^2}{4\cos^2\frac{1}{2}(\psi+\xi)\cos^2\frac{1}{2}(\psi-\xi)}\right] + r^2 d\Omega^2. \tag{4.5}$$

Due to spherical symmetry, the expansion will be the same along any θ and ψ, so the line element becomes

$$ds^2 = f^2(r)\left[\frac{-d\psi^2 + d\xi^2}{4\cos^2\frac{1}{2}(\psi+\xi)\cos^2\frac{1}{2}(\psi-\xi)}\right]. \tag{4.6}$$

By simplifying the term in the denominator

$$\begin{aligned}
\cos\frac{1}{2}(\psi+\xi)\cos\frac{1}{2}(\psi-\xi) &= \cos^2\frac{\psi}{2}\cos^2\frac{\xi}{2} - \sin^2\frac{\psi}{2}\sin^2\frac{\xi}{2}, &(4.7)\\
2\cos\frac{1}{2}(\psi+\xi)\cos\frac{1}{2}(\psi-\xi) &= 2\big(\cos^2\frac{\psi}{2}\cos^2\frac{\xi}{2} - (1-\cos^2\frac{\psi}{2})\sin^2\frac{\psi}{2}\big), &(4.8)\\
&= 2\big(\cos^2\frac{\psi}{2}\cos^2\frac{\xi}{2} - \sin^2\frac{\xi}{2} + \cos^2\frac{\psi}{2}\sin^2\frac{\xi}{2}\big), &(4.9)\\
&= 2\big(\cos^2\frac{\psi}{2} - \sin^2\frac{\xi}{2}\big). &(4.10)
\end{aligned}$$

The proper distance between two Friedmann regions can be obtained by integrating the following equation

$$\Delta s = \int_{s_1}^{s_2} ds, \tag{4.11}$$

$$\Delta s = \int_{s_1}^{s_2} \frac{f(r)}{2\big(\cos^2\frac{\psi}{2} - \sin^2\frac{\xi}{2}\big)}\left[-\big(\frac{d\psi}{ds}\big)^2 + \big(\frac{d\xi}{ds}\big)^2\right]^{\frac{1}{2}} ds. \tag{4.12}$$

4.2 Asymptotic Behaviour of Δs with K

As K takes the higher values the hypersurfaces become flatter, so dr/ds tends to zero as final singularity approaches, so Eq. (4.2) becomes

$$K = \left(1 - \frac{3r_s}{r}\right)\left(\frac{f}{2r_s}\right)^2\left(u\frac{\mathrm{d}v}{\mathrm{d}s} - v\frac{\mathrm{d}u}{\mathrm{d}s}\right). \tag{4.13}$$

By using Eqs.(4.3) and (4.4), we can write

$$\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}s} &= \frac{1}{4}\left[\sec^2\frac{1}{2}(\psi+\xi)(\frac{\mathrm{d}\psi}{\mathrm{d}s} + \frac{\mathrm{d}\xi}{\mathrm{d}s}) - \sec^2\frac{1}{2}(\psi-\xi)(\frac{\mathrm{d}\psi}{\mathrm{d}s} - \frac{\mathrm{d}\xi}{\mathrm{d}s})\right], &(4.14)\\
\frac{\mathrm{d}v}{\mathrm{d}s} &= \frac{1}{4}\left[\sec^2\frac{1}{2}(\psi+\xi)(\frac{\mathrm{d}\psi}{\mathrm{d}s} + \frac{\mathrm{d}\xi}{\mathrm{d}s}) + \sec^2\frac{1}{2}(\psi-\xi)(\frac{\mathrm{d}\psi}{\mathrm{d}s} - \frac{\mathrm{d}\xi}{\mathrm{d}s})\right]. &(4.15)
\end{aligned}$$

So

$$u\frac{\mathrm{d}v}{\mathrm{d}s} - v\frac{\mathrm{d}u}{\mathrm{d}s} = \frac{1}{4}\Big[\tan\frac{1}{2}(\psi+\xi)\sec^2\frac{1}{2}(\psi-\xi)\big(\frac{\mathrm{d}\psi}{\mathrm{d}s} - \frac{\mathrm{d}\xi}{\mathrm{d}s}\big) - \tan\frac{1}{2}(\psi-\xi)\sec^2\frac{1}{2}(\psi+\xi)\big(\frac{\mathrm{d}\psi}{\mathrm{d}s} + \frac{\mathrm{d}\xi}{\mathrm{d}s}\big)\Big]. \tag{4.16}$$

Now

$$\tan\frac{1}{2}(\psi+\xi)\sec^2\frac{1}{2}(\psi-\xi) = \frac{\sin\frac{1}{2}(\psi+\xi)}{\cos\frac{1}{2}(\psi+\xi)}.\frac{1}{\cos^2\frac{1}{2}(\psi-\xi)}, \tag{4.17}$$

$$= \frac{\sin^2\frac{1}{2}(\psi+\xi)}{\sin\frac{1}{2}(\psi+\xi)\cos\frac{1}{2}(\psi+\xi)}.\frac{1}{\cos^2\frac{1}{2}(\psi-\xi)}, \tag{4.18}$$

$$= \frac{1-\cos(\psi+\xi)}{2\sin\frac{1}{2}(\psi+\xi)\cos\frac{1}{2}(\psi+\xi)}.\frac{2}{\cos^2\frac{1}{2}(\psi-\xi)}, \tag{4.19}$$

$$= \frac{1-\cos(\psi+\xi)}{\sin(\psi+\xi)}.\frac{2}{\cos^2\frac{1}{2}(\psi-\xi)}, \tag{4.20}$$

$$= \frac{2(1-\cos(\psi+\xi))\big(1+\cos(\psi+\xi)\big)}{\sin(\psi+\xi)\cos^2\frac{1}{2}(\psi-\xi)\big(1+\cos(\psi+\xi)\big)}, \tag{4.21}$$

$$= \frac{2\big(1-\cos^2(\psi+\xi)\big)}{\sin(\psi+\xi)\cos^2\frac{1}{2}(\psi-\xi)\big(1+\cos(\psi+\xi)\big)}, \tag{4.22}$$

$$= \frac{2\sin(\psi+\xi)}{\cos^2\frac{1}{2}(\psi-\xi).2\cos^2\frac{1}{2}(\psi+\xi)}, \tag{4.23}$$

$$= \frac{\sin(\psi+\xi)}{\big(\cos^2\frac{\psi}{2}-\sin^2\frac{\psi}{2}\big)^2}. \tag{4.24}$$

Also

$$\tan\frac{1}{2}(\psi+\xi)\sec^2\frac{1}{2}(\psi-\xi) = \frac{\sin(\psi+\xi)}{\big(\cos^2\frac{\psi}{2}-\sin^2\frac{\psi}{2}\big)^2}. \tag{4.25}$$

$$u\frac{\mathrm{d}v}{\mathrm{d}s} - v\frac{\mathrm{d}v}{\mathrm{d}s} = \frac{\sin(\psi+\xi)}{\big(\cos^2\frac{\psi}{2}-\sin^2\frac{\psi}{2}\big)^2}\big(\frac{\mathrm{d}\psi}{\mathrm{d}s} - \frac{\mathrm{d}\xi}{\mathrm{d}s}\big) - \frac{\sin(\psi-\xi)}{(\cos^2\frac{\psi}{2}-\sin^2\frac{\psi}{2})^2}\big(\frac{\mathrm{d}\psi}{\mathrm{d}s} + \frac{\mathrm{d}\xi}{\mathrm{d}s}\big), \tag{4.26}$$

which is simplified to

$$u\frac{\mathrm{d}v}{\mathrm{d}s} - v\frac{\mathrm{d}u}{\mathrm{d}s} = \frac{1}{\big(\cos^2\frac{\psi}{2}-\sin^2\frac{\psi}{2}\big)^2}\Big[\cos\psi\sin\xi\frac{d\psi}{ds} - \sin\psi\cos\xi\frac{d\xi}{ds}\Big]. \tag{4.27}$$

For higher values of K, we can approximate ψ by

$$\psi = \frac{\pi}{2} - \varepsilon, \quad 0 < \varepsilon \ll 1 \tag{4.28}$$

$$\frac{\mathrm{d}\psi}{\mathrm{d}s} = 0. \tag{4.29}$$

Eq. (4.27) becomes

$$\begin{aligned} u\frac{\mathrm{d}v}{\mathrm{d}s} - v\frac{\mathrm{d}u}{\mathrm{d}s} &= \frac{-\sin(\frac{\pi}{2}-\varepsilon)\cos\xi}{\big(\cos^2(\frac{\pi}{4}-\frac{\varepsilon}{2})\sin^2\frac{\xi}{2}\big)^2}\frac{d\xi}{ds} \\ &\quad -\frac{\cos\psi\cos\xi}{\big(\cos^2(\frac{\pi}{4}-\frac{\varepsilon}{2})\sin^2\frac{\xi}{2}\big)^2}\frac{d\xi}{ds}. \end{aligned} \tag{4.30}$$

The term in the denominator can be simplified as

$$\Big(\cos^2(\frac{\pi}{4}-\frac{\varepsilon}{2})\sin^2\frac{\xi}{2}\Big)^2 \;=\; \Big(\frac{1}{2}(1+\sin\varepsilon)-\frac{1}{2}(1-\cos\xi)\Big)^2. \tag{4.31}$$

Eq. (4.30) becomes

$$u\frac{\mathrm{d}v}{\mathrm{d}s} - v\frac{\mathrm{d}u}{\mathrm{d}s} = \frac{-\cos\psi\cos\xi}{\big(\frac{1}{2}(1+\sin\varepsilon)-\frac{1}{2}(1-\cos\xi)\big)^2}\frac{d\xi}{ds}, \tag{4.32}$$

$$= \frac{-\cos\psi\cos\xi}{\frac{1}{4}\big[(1+\sin\varepsilon-1-\cos\xi)^2\big]}\frac{d\xi}{ds}, \tag{4.33}$$

$$= \frac{-4\cos\psi\cos\xi}{\big(\sin\varepsilon+\cos\xi\big)^2}\frac{d\xi}{ds}. \tag{4.34}$$

Expanding '$\sin\varepsilon$' and '$\cos\varepsilon$' by using Taylor series and neglecting higher powers, we get

$$\sin\varepsilon = \varepsilon + O(\varepsilon^3), \tag{4.35}$$

$$\cos\varepsilon = 1 + O(\varepsilon^2). \tag{4.36}$$

From Eqs. (4.35) and (4.36), Eq. (4.34) becomes

$$u\frac{\mathrm{d}v}{\mathrm{d}s} - v\frac{\mathrm{d}u}{\mathrm{d}s} = \frac{-4\cos\xi}{\big(\varepsilon+\cos\xi\big)^2}\frac{d\xi}{ds} + O(\varepsilon^2), \tag{4.37}$$

$$= \frac{-4\cos\xi}{\big(\varepsilon^2+\cos^2\xi+2\varepsilon\cos\xi\big)}\frac{d\xi}{ds} + O(\varepsilon^2), \tag{4.38}$$

$$= \frac{-4\cos\xi}{\cos^2\xi\big(1+2\varepsilon\sec\xi+O(\varepsilon^2)\big)}\frac{d\xi}{ds} + O(\varepsilon^2), \tag{4.39}$$

$$= \frac{-4}{\cos\xi\big(1+2\varepsilon\sec\xi+O(\varepsilon^2)\big)}\frac{d\xi}{ds} + O(\varepsilon^2), \tag{4.40}$$

$$= \frac{-4\sec\xi}{\big(1+2\varepsilon\sec\xi\big)}\frac{d\xi}{ds} + O(\varepsilon^2), \tag{4.41}$$

$$= -4\sec\xi\big(1+2\varepsilon\sec\xi\big)^{-1}\frac{d\xi}{ds} + O(\varepsilon^2), \tag{4.42}$$

$$= -4\sec\xi\big(1-2\varepsilon\sec\xi\big)\frac{d\xi}{ds} + O(\varepsilon^2). \tag{4.43}$$

For the value of $\psi = \frac{\pi}{2} - \varepsilon$, Eq. (4.6) becomes

$$1 = \frac{f(r)^2}{\big(\varepsilon + \cos\xi\big)^2}\left(\frac{\mathrm{d}\xi}{\mathrm{d}s}\right)^2, \tag{4.44}$$

$$\left(\frac{\mathrm{d}\xi}{\mathrm{d}s}\right)^2 = \frac{\big(\varepsilon + \cos\xi\big)^2}{f(r)^2}, \tag{4.45}$$

$$\frac{\mathrm{d}\xi}{\mathrm{d}s} = \frac{\big(\varepsilon + \cos\xi\big)}{f(r)}, \tag{4.46}$$

$$\frac{\mathrm{d}\xi}{\mathrm{d}s} = \frac{\cos\xi\big(1 + \varepsilon\sec\xi\big)}{f(r)}. \tag{4.47}$$

Now

$$u = \big(1 - \frac{r}{r_s}\big)^{1/2} e^{r/2r_s}\sinh(t/2r_s), \tag{4.48}$$

$$v = \big(1 - \frac{r}{r_s}\big)^{1/2} e^{r/2r_s}\cosh(t/2r_s). \tag{4.49}$$

Squaring and subracting the above two equations, we get

$$v^2 - u^2 = \big(1 - \frac{r}{r_s}\big)e^{\frac{r}{r_s}}, \tag{4.50}$$

The asymptotic regions contain the values of r that are close to $r = 0$, so by taking $r = \varepsilon_1 r_s$, where ε_1 is a very small positive number, we get

$$v^2 - u^2 = \big(1 - \varepsilon_1\big)e^{\varepsilon_1}. \tag{4.51}$$

Expanding e^{ε_1} as a Taylor series, we get

$$v^2 - u^2 = 1 - \frac{\varepsilon_1^2}{2} - \frac{\varepsilon_1^3}{3} - \frac{\varepsilon_1^4}{8} + O(\varepsilon_1^5). \tag{4.52}$$

Putting $u = 0$ in the above equation, we get

$$v^2 = 1 - \frac{\varepsilon_1^2}{2} - \frac{\varepsilon_1^3}{3} - \frac{\varepsilon_1^4}{8} + O(\varepsilon_1^5). \tag{4.53}$$

Multiplying Eqs. (4.3) and (4.4), we get

$$v^2 - u^2 = \tan\frac{1}{2}(\psi + \xi)\tan\frac{1}{2}(\psi - \xi). \tag{4.54}$$

At $\xi = 0$ and $u = 0$, we get

$$v = \tan\frac{\psi}{2}. \tag{4.55}$$

As $\psi = \frac{\pi}{2} - \varepsilon$, thus from above equation we have

$$v = \tan\big(\frac{\pi}{4} - \frac{\varepsilon}{2}\big). \tag{4.56}$$

After simplifying, we get

$$v = 1 - \varepsilon + \frac{\varepsilon^2}{2} - \frac{\varepsilon^3}{3} + O(\varepsilon^4). \tag{4.57}$$

Squaring above equation and simplifying, we get

$$v^2 = \big[1 - 2\varepsilon + 2\varepsilon^2 + O(\varepsilon^3)\big]. \tag{4.58}$$

Comparing Eq. (4.53) with Eq. (5.58), we get

$$\varepsilon_1 = 2\varepsilon^{1/2}, \tag{4.59}$$

let

$$\varepsilon = \frac{\varepsilon_1^2}{4}(1 + \delta), \tag{4.60}$$

with this value of ε, solving Eqs. (4.53) and (4.58), we get

$$\delta = \frac{2}{3}\varepsilon_1 + O(\varepsilon_1^2). \tag{4.61}$$

Putting the value of δ in Eq. (4.60) and simplifying, we get

$$\varepsilon = \frac{\varepsilon_1^2}{4}\big(1 + \frac{4}{3}\varepsilon^{1/2} + O(\varepsilon)\big), \tag{4.62}$$

or

$$\varepsilon_1 = 2\varepsilon^{1/2}\big(1 - \frac{4}{3}\varepsilon + O(\varepsilon)\big). \tag{4.63}$$

From Eqs. (4.59) and (4.63), we can see

$$\varepsilon_1 \sim \varepsilon^{1/2}. \tag{4.64}$$

Now we again take $\psi = \frac{\pi}{2} - \varepsilon$, with the value of ξ chosen to be $\frac{\pi}{4}$ in the following equation.

$$v^2 - u^2 = \tan\frac{1}{2}(\psi + \xi)\tan\frac{1}{2}(\psi - \xi), \tag{4.65}$$

we get

$$v^2 - u^2 = 1 - 2^{2/3}\varepsilon + 4\varepsilon^2 - \frac{11}{3}2^{1/2}\varepsilon^3 + O(\varepsilon^4). \tag{4.66}$$

Comparing Eq. (4.66) with Eq. (4.52), we get

$$\varepsilon_1 = 2^{5/4}\varepsilon^{1/2}. \tag{4.67}$$

And for the second order approximation, we get

$$\varepsilon_1 = 2^{5/4}\varepsilon^{1/2}\big(1 - 2^{9/4}\frac{\varepsilon^{1/2}}{3} + O(\varepsilon)\big). \tag{4.68}$$

The second order approximation also supports Eq. (4.67), similarly this equation holds for other values of ξ and $\frac{\pi}{2}$. In general for $\psi = \frac{\pi}{2} - \varepsilon$, Eq. (4.65) yields

$$v^2 - u^2 = 1 - 2\varepsilon \sec\xi + 2\xi^2 \sec^2\xi + \xi^3 \sec^3\frac{\xi}{3} + O(\varepsilon^4). \tag{4.69}$$

Comparing Eq. (4.69) with Eq. (4.53) for any u, we get

$$\varepsilon_1 = 2\varepsilon^{1/2}\big(\sec\xi\big)^{1/2}. \tag{4.70}$$

Let us choose this ξ as ξ_0 whose value lies between ξ_1 and ξ_2, choosing $\xi_0 = (\xi_1 + \xi_2)/2$, i.e, the boundry values of ξ for the enveloped and the enveloping Friedmann regions.

By the equation

$$a_k = \left[R_k^3/r_s\right]^{1/2}, \tag{4.71}$$

we see that

$$r = R_s, \tag{4.72}$$

we get

$$r = R_s = a^{2/3} r_s^{1/3}. \tag{4.73}$$

For this value of r_s, we have

$$\xi = \xi_1. \tag{4.74}$$

Here

$$r = \varepsilon_1 r_s, \tag{4.75}$$

we get

$$r = 2r_s\big(\varepsilon \sec\xi\big)^{1/2}. \tag{4.76}$$

Comparing Eq. (4.75) with (4.72), we get

$$\cos\xi_1 = 4\varepsilon\left[\frac{a_s^{-4/3}}{r_s^{-4/3}}\right], \tag{4.77}$$

or

$$\xi_1 = \cos^{-1}\left[4\varepsilon\big(a_s/r_s\big)^{-4/3}\right]. \tag{4.78}$$

Similarly

$$\xi_2 = \cos^{-1}\left[4\varepsilon\big(a_l/r_s\big)^{-4/3}\right]. \tag{4.79}$$

Also

$$\cos\xi_2 = 4\varepsilon\big(\frac{a_l}{r_s}\big)^{-\frac{4}{3}}, \tag{4.80}$$

$$\varepsilon = \frac{1}{4}\frac{\cos\xi_2}{\big(\frac{a_l}{r_s}\big)^{-\frac{4}{3}}}, \tag{4.81}$$

$$\varepsilon = \frac{1}{4}\big(\frac{a_l}{r_s}\big)^{\frac{4}{3}}\cos\xi_2, \tag{4.82}$$

$$\cos\xi_1 = 4.\frac{1}{4}\big(\frac{a_l}{r_s}\big)^{\frac{4}{3}}\big(\frac{a_s}{r_s}\big)^{-\frac{4}{3}}\cos\xi_2, \tag{4.83}$$

$$\cos\xi_1 = \big(\frac{a_l}{a_s}\big)^{\frac{4}{3}}\cos\xi_2. \tag{4.84}$$

As

$$\frac{a_s}{a_l} = p^{-1}, \tag{4.85}$$

$$\frac{a_l}{a_s} = p. \tag{4.86}$$

Therefore

$$\cos\xi_1 = p^{\frac{4}{3}}\cos\xi_2, \tag{4.87}$$

$$\xi_1 = \cos^{-1}[p^{\frac{4}{3}}\cos\xi_2]. \tag{4.88}$$

This equation gives us the value of ξ_1 in terms of ξ_2, where ξ_2 can be calculated for a given η by matching the following two equations

$$v^2 - u^2 = \big(1 - \frac{r}{r_s}\big)e^{\frac{r}{r_s}}, \tag{4.89}$$

$$v^2 - u^2 = \tan\frac{1}{2}(\psi + \xi)\tan\frac{1}{2}(\psi - \xi), \tag{4.90}$$

and solving the boundary conditions,

$$R_k = a\sin\chi_k, \tag{4.91}$$

$$m = \frac{1}{2}a_k\sin^3\chi_k, \tag{4.92}$$

with

$$a(\eta) = \frac{a_0}{2}(1+\cos\eta), \tag{4.93}$$

$$t(\eta) = \frac{a_0}{2}(\pi + \eta + \sin\eta), \quad (-\pi \le \eta \le \pi). \tag{4.94}$$

The purpose to solve all these equations is to find ξ_2. Comparing Eq. (4.52) with Eq. (4.69), we get

$$1 - \frac{\varepsilon_1^2}{2} - \frac{\varepsilon_1^3}{3} - \frac{\varepsilon_1^4}{8} + O(\varepsilon_1^5) = 1 - 2\varepsilon\sec\xi + 2\xi^2\sec^2\xi + \xi^3\sec\frac{\xi}{3} + O(\varepsilon^4), \tag{4.95}$$

$$\frac{\varepsilon_1^2}{2} = 2\varepsilon\sec\xi, \tag{4.96}$$

$$\varepsilon_1 = 2\varepsilon^{\frac{1}{2}}\big(\sec\xi\big)^{\frac{1}{2}}, \tag{4.97}$$

$$\varepsilon_1 = b\varepsilon^{\frac{1}{2}}, \tag{4.98}$$

where b is a constant and its value is

$$b = \varepsilon^{\frac{1}{2}}\big(\sec\xi_0\big)^{\frac{1}{2}}. \tag{4.99}$$

Hence when $r = \varepsilon_1 r_s$, we get

$$r = br_s\varepsilon^{\frac{1}{2}}. \tag{4.100}$$

With this value of $r = br_s\varepsilon^{\frac{1}{2}}$, the following equation

$$f^2(r) = \frac{4r_s^3}{r}e^{\frac{-r}{r_s}}, \tag{4.101}$$

transformed into

$$\begin{aligned} f^2(r) &= \frac{4r_s^3}{br_s\varepsilon^{\frac{1}{2}}}e^{-b\varepsilon^{\frac{1}{2}}}, && (4.102)\\ f^2(r) &= 4r_s^2b^{-1}\varepsilon^{-\frac{1}{2}}\big[1+b\varepsilon^{\frac{1}{2}}+O(\varepsilon)\big], && (4.103)\\ f(r) &= 2r_sb^{-1}\varepsilon^{-\frac{1}{4}}\big[1+\frac{1}{2}b\varepsilon^{\frac{1}{2}}+O(\varepsilon)\big]. && (4.104)\end{aligned}$$

Inserting the value of $f(r)$ in the following equation

$$\frac{\mathrm{d}\xi}{\mathrm{d}s} = \frac{\cos\xi(1+\varepsilon\sec\xi)}{f(r)}, \tag{4.105}$$

we get

$$\begin{aligned} \frac{\mathrm{d}\xi}{\mathrm{d}s} &= \frac{\cos\xi(1+\varepsilon\sec\xi)}{2r_sb^{\frac{-1}{2}}\varepsilon^{-\frac{1}{4}}\big[1+\frac{1}{2}b\varepsilon^{\frac{1}{2}}+O(\varepsilon)\big]}, && (4.106)\\ \frac{\mathrm{d}\xi}{\mathrm{d}s} &= \frac{1}{2}r_s^{-1}b^{\frac{1}{2}}\varepsilon^{\frac{1}{4}}\cos\xi(1+\varepsilon\sec\xi)\big[1+\frac{1}{2}b\varepsilon^{\frac{1}{2}}+O(\varepsilon)\big]^{-1}, && (4.107)\\ \frac{\mathrm{d}\xi}{\mathrm{d}s} &= \frac{1}{2}r_s^{-1}b^{\frac{1}{2}}\varepsilon^{\frac{1}{4}}\big(\cos\xi+\cos\xi\varepsilon\sec\xi\big)\big[1-\frac{1}{2}b\varepsilon^{\frac{1}{2}}+O(\varepsilon)\big], && (4.108)\\ \frac{\mathrm{d}\xi}{\mathrm{d}s} &= \frac{1}{2}r_s^{-1}b^{\frac{1}{2}}\varepsilon^{\frac{1}{4}}\cos\xi\big(1-\frac{1}{2}b\varepsilon^{\frac{1}{2}}+O(\varepsilon)\big). && (4.109)\end{aligned}$$

Substituting Eq. (4.109) into Eq. (4.43), we get

$$u\frac{\mathrm{d}v}{\mathrm{d}s}-v\frac{\mathrm{d}v}{\mathrm{d}s} = -2r_s^{-1}\varepsilon^{\frac{1}{4}}b^{\frac{1}{2}}(1-\frac{1}{2}b\varepsilon^{\frac{1}{2}}+O(\varepsilon)). \tag{4.110}$$

Substituting Eq. (4.110) into Eq. (4.13), we get

$$\begin{aligned} K &= \left(1-\frac{3r_s}{r}\right)\left(\frac{f}{2r_s}\right)^2\left[-2r_s^{-1}\varepsilon^{\frac{1}{4}}b^{\frac{1}{2}}(1-\frac{1}{2}b\varepsilon^{\frac{1}{2}}+O(\varepsilon))\right], && (4.111)\\ K &= \frac{3}{2}r_s^{-3}\varepsilon^{-\frac{1}{4}}b^{-\frac{1}{2}}\left(1-\frac{5}{6}b\varepsilon^{\frac{1}{2}}+O(\varepsilon)\right)f^2. && (4.112)\end{aligned}$$

This equation gives us $f(r)$ as $f(\xi)$ which is

$$f^2 = \frac{2}{3} K r_s^3 \varepsilon^{\frac{1}{4}} b^{\frac{1}{2}} \big(1 - \frac{5}{6} b \varepsilon^{\frac{1}{2}} + O(\varepsilon)\big)^{-1}, \tag{4.113}$$

$$f(\xi) = \sqrt{\frac{2}{3}} K^{\frac{1}{2}} r_s^{\frac{3}{2}} \varepsilon^{\frac{1}{8}} b^{\frac{1}{4}} \big(1 + \frac{5}{12} b \varepsilon^{\frac{1}{2}} + O(\varepsilon)\big). \tag{4.114}$$

Taking $\sqrt{\frac{2}{3}} b^{\frac{1}{4}} r_s = A_0$, we get

$$f(\xi) = A_0 (K r_s)^{\frac{1}{2}} \varepsilon^{\frac{1}{8}} \big(1 + \frac{5}{12} b \varepsilon^{\frac{1}{2}} + O(\varepsilon)\big). \tag{4.115}$$

Note that as $\psi \to \frac{\pi}{2}$ the time parameter for the Friedmann region $\eta \to \pi$, therefore at $\psi = \frac{\pi}{2} - \varepsilon$, we take $\eta = \pi - \lambda$. Obviously $0 < \lambda < 1$. For this value of η, the following equation

$$a(\eta) = \frac{a_0}{2}(1 + \cos\eta), \tag{4.116}$$

becomes

$$a(\eta) = \frac{a_0}{2}(1 + \cos(\pi - \lambda)), \tag{4.117}$$

$$a(\eta) = \frac{a_0}{4}\lambda^2 + O(\lambda^4). \tag{4.118}$$

And its derivative

$$\frac{\mathrm{d}a}{\mathrm{d}\eta} = -\frac{a_0}{2}\sin\eta, \tag{4.119}$$

$$= -\frac{a_0}{2}\sin(\pi - \lambda), \tag{4.120}$$

$$= -\frac{a_0}{2}\sin\lambda, \tag{4.121}$$

$$\frac{\mathrm{d}a}{\mathrm{d}\eta} = -\frac{a_0}{2}\lambda + O(\lambda^3). \tag{4.122}$$

By substituting Eq. (4.122) into the following equation (which was discussed in last chapter), we get

$$\left[\frac{\mathrm{d}\alpha}{\mathrm{d}s}\right]_{\chi=0} = -\frac{K}{4} - \frac{3}{4a^2}\frac{\mathrm{d}a}{\mathrm{d}\eta}, \tag{4.123}$$

we get

$$\left[\frac{\mathrm{d}\alpha}{\mathrm{d}s}\right]_{\chi=0} = -\frac{K}{4} - \frac{3}{4(\frac{a_0^2\lambda^4}{16})}\big(-\frac{a_0}{2}\lambda + O(\lambda^3)\big), \tag{4.124}$$

$$= -\frac{K}{4} + \frac{6}{a_0\lambda^3}. \tag{4.125}$$

The initial conditions imply that as the hyperspherical angle $\chi \to 0$, the inclination parameter α also approaches to zero. Then the above equation gives

$$0 = -\frac{K}{4} + \frac{6}{a_0\lambda^3}, \tag{4.126}$$
$$K = \frac{24}{a_0\lambda^3}, \tag{4.127}$$

or

$$\lambda = 2\left(\frac{a_0}{3}\right)^{-\frac{1}{3}} K^{-\frac{1}{3}}. \tag{4.128}$$

Now the matching conditions between the Friedmann and Schwarzschild regions give the "r" value of the hypersurface from equation ($R_k = a \sin \chi_k$).
Inserting the value of $a(\eta)$ and λ in equation ($R_k = a \sin \chi_k$) by taking $a_0 = a_k$, we get

$$r = R_k = 3^{\frac{2}{3}} a_k^{\frac{1}{3}} K^{-\frac{2}{3}} \sin \chi_k. \tag{4.129}$$

Now we have the equation

$$m = \frac{1}{2} a_k \sin^3 \chi_k, \tag{4.130}$$
$$2m = a_k \sin^3 \chi_k, \tag{4.131}$$
$$r_s = a_k \sin^3 \chi_k, \tag{4.132}$$
$$\sin \chi_k = \left(\frac{r_s}{a_k}\right)^{\frac{1}{3}}. \tag{4.133}$$

Now Eq. (4.129) becomes

$$r = R_k = 3^{\frac{2}{3}} a_k^{\frac{1}{3}} K^{-\frac{2}{3}} \sin \chi_k, \tag{4.134}$$
$$= 3^{\frac{2}{3}} a_k^{\frac{1}{3}} k^{-\frac{2}{3}} \left(\frac{r_s}{a_k}\right)^{\frac{1}{3}}, \tag{4.135}$$
$$r = 3^{\frac{2}{3}} r_s \big(K r_s\big)^{-\frac{2}{3}}. \tag{4.136}$$

This equation matches with

$$r = \varepsilon_1 r_s, \tag{4.137}$$

if and only if

$$\varepsilon_1 = 3^{\frac{2}{3}} \big(K r_s\big)^{-\frac{2}{3}}, \tag{4.138}$$

we insert this equation in the following equation

$$\varepsilon_1 = b\varepsilon^{\frac{1}{2}}, \tag{4.139}$$

and get

$$3^{\frac{2}{3}}\big(Kr_s\big)^{-\frac{2}{3}} = b\varepsilon^{\frac{1}{2}}. \tag{4.140}$$

Inserting the above equation into

$$\begin{aligned} f(\xi) &= A_0\big(Kr_s\big)^{\frac{1}{2}}\varepsilon^{\frac{1}{8}}\big[1 + \frac{5}{12}b\varepsilon^{\frac{1}{2}} + O(\varepsilon)\big], & (4.141)\\ &= A_0 b^{-\frac{1}{4}}\big(Kr_s\big)^{\frac{1}{3}}\big[1 + \frac{5}{12}3^{\frac{2}{3}}\big(Kr_s\big)^{-\frac{2}{3}} + O\big(Kr_s\big)^{-\frac{4}{3}}\big], & (4.142)\\ f(\xi) &= A_1\big(Kr_s\big)^{\frac{1}{3}}\big[1 + O\big(Kr_s\big)^{-\frac{2}{3}}\big], & (4.143)\end{aligned}$$

where

$$A_1 = A_0 b^{-\frac{1}{4}} 3^{\frac{1}{6}} = \frac{\sqrt{2}}{3^{\frac{1}{3}}} r_s. \tag{4.144}$$

Now we have to put the value of $f(\xi)$ from Eq. (4.143) and using Eqs. (4.28) and (4.29) into Eq. (4.12), we get

$$\begin{aligned} \Delta s &= \int_{s_1}^{s_2} \frac{f(\xi)}{2\big(\cos^2(\frac{\pi}{4} - \frac{\varepsilon}{2}) - \sin^2\frac{\xi}{2}\big)}\frac{\mathrm{d}\xi}{\mathrm{d}s}, & (4.145)\\ &= \frac{1}{2}A_1\big(Kr_s\big)^{\frac{1}{3}}\big[1 + O(Kr_s)^{-\frac{2}{3}}\big]\int_{\xi_1}^{\xi_2} \frac{d\xi}{\big(\cos^2(\frac{\pi}{4} - \frac{\varepsilon}{2}) - \sin^2\frac{\xi}{2}\big)}. & (4.146)\end{aligned}$$

Simplifying the above equation, we get

$$\Delta s = \frac{1}{3^{\frac{1}{3}}\sqrt{2}} r_s \ln\left[\frac{\tan\xi_2 + \sec\xi_2}{\tan\xi_1 + \sec\xi_1}\right]\big(Kr_s\big)^{\frac{1}{3}}\big(1 + O(Kr_s)^{-\frac{4}{3}}\big), \tag{4.147}$$

where ξ_1 can be obtained from ξ_2 by the following equation

$$\xi_1 = \cos^{-1}\big[p^{\frac{4}{3}}\cos\xi_2\big]. \tag{4.148}$$

The above equation is shown explicitly.

$$\Delta s \sim K^{\frac{1}{3}}. \tag{4.149}$$

Therefore one can say rigorously that as K goes to ∞, Δs also increases without bound and tends to a line of infinite length as the cube root of K [30].
The two Friedmann universes which have now collapsed to a point are infinitely separated from each other. This gives the first part of the proof of Qadir's conjecture [12] but with the separation varying as one-third power of K instead of one-half.

4.3 Asymptotic Behaviour of Volume of the Suture with K

To workout an expression for the volume of the Schwarzschild region of the model, we need the full Schwarzschild metric in compactified coordinates by the following equation

$$
\begin{aligned}
ds^2 &= f^2(r)\left[\frac{-d\psi^2+d\xi^2}{4\cos^2\frac{1}{2}(\psi+\xi)\cos^2\frac{1}{2}(\psi-\xi)}\right]+r^2d\Omega^2, && (4.150)\\
g &= \frac{-f^4(r)r^4\sin^2\theta}{16\cos^4\frac{1}{2}(\psi+\xi)\cos^4\frac{1}{2}(\psi-\xi)}, && (4.151)\\
|g| &= \frac{f^4(r)r^4\sin^2\theta}{16\cos^4\frac{1}{2}(\psi+\xi)\cos^4\frac{1}{2}(\psi-\xi)}, && (4.152)\\
\sqrt{|g|} &= \frac{f^2(r)r^2\sin\theta}{\cos^2\xi(1+O(\varepsilon))}. && (4.153)
\end{aligned}
$$

By substituting the value of f and r, we get

$$
\begin{aligned}
\sqrt{|g|} &= \frac{A_1^2\big(Kr_s\big)^{\frac{2}{3}}\big[1+O(Kr_s)^{-\frac{2}{3}}\big]^2 3^{\frac{4}{3}}r_s^2\big(Kr_s\big)^{-\frac{4}{3}}\sin\theta}{\cos^2\xi\big(1+O(\varepsilon)\big)}, && (4.154)\\
\sqrt{|g|} &= A_1^2 3^{\frac{4}{3}}\big(Kr_s\big)^{-\frac{2}{3}}\big[1+O(Kr_s)^{-\frac{2}{3}}\big]\sin\theta\sec^2\xi. && (4.155)
\end{aligned}
$$

Now we can use this equation in the expression for the volume of the schwarzschild region, which is

$$
\begin{aligned}
V &= \int_{\xi_1}^{\xi_1}\int_0^{\pi}\int_0^{2\pi}\sqrt{|g|}d\psi d\theta d\xi, && (4.156)\\
&= A_1^2 3^{\frac{4}{3}}\big(Kr_s\big)^{-\frac{2}{3}}\big[1+O\big(Kr_s\big)^{-\frac{2}{3}}\big]\int_{\xi_1}^{\xi_1}\int_0^{\pi}\int_0^{2\pi}(\sin\theta\sec^2\xi)d\psi d\theta d\xi, && (4.157)\\
&= 4\pi A_1^2 3^{4/3}r_s^2\big(Kr_s\big)^{-2/3}\big(1+O(Kr_s)\big)^{-2/3}\tan\xi|_{\xi_1}^{\xi_2}. && (4.158)
\end{aligned}
$$

Since

$$A_1=\frac{\sqrt{2}}{3^{\frac{1}{3}}}r_s. \qquad (4.159)$$

Therefore

$$A_1^2=\frac{2}{3^{\frac{1}{3}}}r_s^2. \qquad (4.160)$$

Putting the above value in Eq. (4.158) and simplification gives

$$V=8\pi r_s^4\big(3^2\big)^{1/3}\big(Kr_s\big)^{-2/3}[\tan\xi_2-\tan\xi_1]\big[1+O(Kr_s)\big]^{-2/3}. \qquad (4.161)$$

Which shows that

$$V\backsim K^{-2/3}. \qquad (4.162)$$

Therefore the volume of the Schwarzschild region goes to zero as the inverse two third power of K.

4.3.1 Second Order Approximation

Here I will not go into the detail derivation of second order approximation for length and volume. The second order approximation can easily be calculated by the same procedure as discussed above and it comes out to be

$$\Delta s = \frac{\sqrt{2}}{3^{1/3}} r_s \sec\xi_0(\xi_2 - \xi_1)(Kr_s)^{1/3}\big[1 + \frac{5}{12}3^{2/3}(Kr_s)^{-2/3} + O(Kr_s)^{-4/3}\big]. \qquad (4.163)$$

Similarly the expression for volume can easily be calculated as

$$V = \frac{2}{3^{-2/3}} r_s^4 \sec^2\xi_0(\xi_2 - \xi_1)(Kr_s)^{1/3}\big[1 + \frac{5}{12}3^{2/3}(Kr_s)^{-2/3} + O(Kr_s)^{-4/3}\big]. \qquad (4.164)$$

Both these above equations agree with the first order approximation.

4.4 Asymptotic Behaviour of the Proper Length and Volume of the Schwarzschild Singularity

Since we have discussed the proper length and volume of the suture model with increasing values of *K*. Here we will review the behaviour of the proper length and volume of the Schwarzschild singularity separately, apart from the suture model. As it is clear to everyone that the Schwarzschild singularity is spacelike and not timelike. Hence it is not a point, rather it is a line. If it is a line then what is its length? Since it is singular it must have an infinite or zero length. As the singularity is not a point so it cannot have a zero length, hence its length must be infinite. Here we have two questions. First, how does the collapse occur if the length is infinite? Second, [31] how does the length approach infinity as the singularity is approached? The answer to the first question is that the volume approaches to zero even though the length approaches to infinity. The second question cannot be answered without the foliation of the spacetime. Qadir and Siddiqui had been obtained earlier [32] the foliation of the Schwarzschild spacetime by K-slices and then proved to be complete [33]. The second question will be discussed in the next section.

4.4.1 Asymptotic Behaviour of the Proper Length of the Schwarzschild Singularity in CSK

To find the proper length of the Schwarzschild singularity we will use the following metric in the CSK coordinates

$$\Delta s = \int_{s_1}^{s_2} \frac{f(r)}{2\big(\cos^2\frac{\psi}{2} - \sin^2\frac{\xi}{2}\big)}\Big[-\big(\frac{d\psi}{ds}\big)^2 + \big(\frac{d\xi}{ds}\big)^2\Big]^{\frac{1}{2}} ds, \qquad (4.165)$$

where

$$f^2(r) = \frac{4r_s^3}{r} e^{\frac{-r}{r_s}}. \qquad (4.166)$$

Consider the hypersurface close to $\psi = \frac{\pi}{2}$, thus putting $\psi = \frac{\pi}{2} - \epsilon$ in the denominator of the above expression, we get

$$2\big(\cos^2\frac{\psi}{2} - \sin^2\frac{\xi}{2}\big) = (\sin\epsilon + \cos\xi), \tag{4.167}$$

$$\Delta s = \int_{-\pi/2-\epsilon}^{\pi/2+\epsilon} \frac{f(r)}{(\sin\epsilon + \cos\xi)} d\xi, \tag{4.168}$$

and $f^2(r)$ can be calculated as

$$f^2(r) = 4r_s^2 c^2 \epsilon^{-1/4}\big[1 + \epsilon^{1/2} + O(\epsilon)\big], \tag{4.169}$$

where c is a constant. Solving the Eqs. (4.168) and (4.169), we get

$$\Delta s = 2cr_s\epsilon^{-1/8}\ln\left(\frac{2}{\epsilon}\right)\big[1 + \epsilon^{1/2} + O(\epsilon)\big]. \tag{4.170}$$

4.4.2 Length and Volume of Schwarzschild Singularity Along *K*

The basic analysis of the proper length and volume for the suture model has been discussed in the previous section. There we have used the boundaries of the Schwarzschild region so that spacelike coordinate ξ to lie between a certain given ξ_1 and ξ_2. For our present purpose $-\pi \leq \xi \leq \pi$ at $r = 0$. By taking $\xi_1 = 0$ and $\xi_2 = \psi - 2\delta$, with $\psi = \frac{\pi}{2} - \epsilon$, and using the extreme value of δ i.e. $\delta = \epsilon$, we obtain

$$\Delta s = 3^{-1/6}8\sqrt{2}r_s\big(Kr_s\big)^{1/3}\ln\big(Kr_s\big)\big[1 + \frac{5}{4}3^{-1/3}\big(Kr_s\big)^{-2/3} + O(Kr_s)^{-4/3}\big], \tag{4.171}$$

and the volume V is given by the following equation.

$$V = 3^{-1/6}8\sqrt{2}\pi r_s\big(Kr_s\big)^{-1}\ln\big(Kr_s\big)\big[1 + \frac{5}{4}3^{-1/3}\big(Kr_s\big)^{-2/3} + O(Kr_s)^{-4/3}\big]. \tag{4.172}$$

4.5 Conclusion

Qadir-Wheeler Suture Model and Information Loss:
Since the suture model is based on a closed Friedmann universe, whereas in the black hole complementarity principle and information loss paradox there is a concept of inside and outside of a black hole for all time, an open universe is needed. For an open universe, the suture model can be modified by taking the rare part of the model to be rare enough to be open. For this model, the foliation would proceed in the same way as for a closed model but here we would lose the point of principle that was proved by the suture model. On the other hand, foliation of open universe by York time would conclude that the black hole singularity never forms [12]. As in the case of an

open universe, if the outside universe goes on forever then there would not be enough York time for the black hole singularity to form. By a conformal transformation the outside universe can be made compact. Now, by using the result of the suture model one can prove that the big crunch and the black hole singularity are simultaneous. On the other hand, if we reverse the conformal transformation then there would be no end of the universe and hence the black hole singularity never forms.

Foliation of the Schwarzschild spacetime by K-slices proved that the proper length of the Schwarzschild region increases as $K^{4/3}$ during which the area decreases as K^2, consequently the volume decreases as $K^{-2/3}$. As the radius of the foliating hypersurface contracts, the cross-sectional area reduces but it can never go less than 1 Planck unit. So there would be a point at which the foliation cannot go further. As relativity is invariant under time reversal, so when we get to the lowest value the only possibility for the hypersurfaces to emerge again. The black hole at this point of the York time would turn into a white hole and pour out all the stuff. Never would any information that had gone in be lost [12]. There is a notable similarity between this view of the end result of the black hole and the information it holds with Susskind's black hole complementarity principle. Besides, there is a radical difference between these two views. That principle concludes that the final result depends upon the view of the observer, here we have reached a definite answer for all observers. Also there is a notable resemblance between this notion to the Hawking radiation idea. Both agree on that the black hole will finally come to an end. According to both the views, a massive black hole will take longer time to an end. However, Hawking's approach was totally different than this one.

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