# Study of Blasius Problem

by

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**Dedicated to**

my mother and my father(late).

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# Abstract

Blasius problem deals with the simplest third order non-linear equation of boundary layer flow. Any approach developed for this problem may be extended to more difficult hydrodynamics problems. With this motivation current work reviewed some techniques that solve Blasius problem and extended some techniques. Herman Weyl's solution is expanded and applied to Blasius problem by making the use of symmetry properties of the differential equation. Domb-Sykes method is used to find the location of the singularity for the series solution of Blasius problem.

Power series solution of Blasius problem is examined and its approximate solution is analyzed with MATHEMATICA. Approximations for second order boundary condition of Blasius problem is obtained by method of asymptotic approximants.

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# Chapter 1

# Introduction

## 1.1 Background of Blasius Problem

The Blasius problem is one of the most illustrative and celebrated problems of applied mathematics. It is a simple third order, non-linear ordinary differential equation subject to boundary conditions. Blasius problem is found in most undergraduate fluid mechanics books, representing two dimensional, steady, laminar viscous flow over a semi infinite plate. The solution to Blasius problem is called Blasius function denoted by $f$, named after the German fluid dynamicist Paul Richard Heinrich Blasius. He showed that two PDEs, namely Navier-Stokes equations could be converted to a single third order ODE by introducing a stream function and using similarity transformation with appropriate boundary conditions [17].

The reason why it is not easy to find analytic solution of Blasius problem is the absence of second order boundary condition. In 1908 Blasius found power series solution of the equation and combined it with asymptotic expansions at finite $x$ to get an approximation [8]. Numerical solution of Blasius problem can also be adopted in this direction. The first to provide a highly accurate numerical solution to Blasius problem was Topfer [20] in 1912. By using symmetry principles and similarity reduction with Runge-Kutta method for integrating the ordinary differential equation (ODE) he solved this problem.

Blasius Problem is foundational to study fluid behaviour. Flows past a solid body like air rushing past an airplane, ocean currents streaming past an undersea mountain and even the blood and breath flowing through our own bodies all have thin boundary layers similar to Blasius flow.

Blasius Problem is now more than a century old and it has cultivated a prodigious reference list. It has continued to fascinate giants like John P. Boyd [9], Herman Weyl [21], Von Neuman and many more. As usual the initial interest was to obtain the qualitative behaviour of the solution of Blasius problem. But in recent years people have become more interested in the improvement of the analytic or numerical accuracy, see Klamin [15] and Parlage, Braddock and Sander [19]. Usually in order to obtain an idea of accuracy involved, the value of wall shear parameter i.e second order derivative of Blasius function at boundary is quoted. That value is of physical interest since it defines the skin friction of the plate.

Liao [16] analytical solution by applying a method to the Blasius problem called homotopy analysis method (HAM). An improved version of Adomian decomposition method was used by Abbasbandy [1] to find the numerical solution. Wang [4] used a transformation made by Crocco in 1940 which later helped to approach Blasius problem from a different perspective.

K Parand, et al [18] found a comparatively accurate solution of the Problem by Sinc-

Collocation method in 2009. Beong In Yun [22] has found an analytical solution of Blasius problem in terms of alogarithm of geometric functions in 2010. S Ghorbani, et al [13] 2015 combined the Green's function method the best approximation theorem to get an approximate analytic solution of the problem.

## 1.2 Objectives of this Research

In this thesis we have reviewed and discussed papers by F Ahmad [2], M.V Dyke [11], John P. Boyd [10], H. Weyl [21], N.S Barlow et al [7] from a variety of perspectives. Main purpose of taking up techniques to bring out various features of the Blasius problem is to emphasize the fact that such techniques are often useful in dealing with other nonlinear problems.

**Our work does not report any original results. We have only reviewed several important techniques which have been used to solve the problem. Such techniques may find some use in solution of other non-linear problems.**

### 1.2.1 Chapters Outline

Chapter 2 consists of some basic concepts and definitions such as boundary layer, stream function and similarity transformation. We derive an ordinary non-linear differential equation with suitable boundary conditions i.e Blasius equation from two partial differential equations of flow with the help of stream functions and using similarity transformation of differential equations.

In Chapter 3, we convert third order Blasius problem into a second order initial value problem by using dilational symmetry property of differential equations by making some assumptions. Estimated values of second order derivative of Blasius function for large values of independent variable $x$ are obtained by using Runge-Kutta method of order four in MATLAB. Our calculated value matches the value of second order derivative of Blasius function by sixteen digits. In Section 3.1 we examine the solution of Blasius problem from Herman Weyl [21] and reviewed it in detail and later applied Herman Weyl's method to our Blasius differential equation which we had derived in Chapter 2. The approximate value for second order derivative of Blasius function obtained is close enough to the previous values calculated in the literature.

In Chapter 4, power series solution for the problem is examined with the help of techniques discussed by Van Dyke [11] "*Analysis and Improvement of Perturbation Series*". A simple pole of the function is located by using Domb-Sykes method [11]. This method is illustrated first by a simple example then applied to Blasius Problem.

Chapter 5 is devoted to application of Crocco-Wang transformation [2], in its first section the third order Blasius equation is transformed into second order equation based on transformation introduced by Crocco and used by Wang. The transformation applies to boundary conditions as well. Later the method of power series solution is applied to transformed equation and a large number of coefficients are found for the series solution of the problem by using MATHEMATICA. We have found an increasing sequence and a decreasing sequence which give bounds for the parameter $\kappa$ i.e the second order boundary condition of Blasius problem.

Chapter 6 consists of a review of "*method of asymptotic approximants*" [7] for Blasius equation. In this review we have devised a suitable approximant for power series solution of Blasius problem. By equating finite terms of power series solution and finite approximant a table of values of practically useful parameters such as wall shear parameter calculated. This value matches with previously calculated approximations in this thesis upto four digits.

# Chapter 2

# Preliminaries

**Boundary Layer:**

As a fluid moves past a solid body a thin layer of fluid near the surface of solid body is formed. This layer is called the boundary layer.

**Stream Function:**

If $u$ and $v$ are velocities of the flow field, the stream function is defined by:

$$\frac{\partial \psi(x,y)}{\partial x} := -v,$$
$$\frac{\partial \psi(x,y)}{\partial y} := u.$$

## 2.1 Blaisus Problem

Let us consider a uniform flow over a flat semi infinite plate. From [17] governing equations of flow in this case are:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2.1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}, \tag{2.2}$$

where $u$,$v$ are components of the velocity vector and $\nu$ is the viscosity of the fluid. Boundary conditions are given

$$u(x,0) = 0, \qquad v(x,0) = 0, \tag{2.3}$$

$$y \to \infty \Rightarrow u(x,y) \to U. \tag{2.4}$$

$U$ is constant speed of fluid outside the boundary layer. Let $\psi(x,y)$ is some stream function

$$u = \frac{\partial \psi}{\partial y}, \tag{2.5}$$

$$v = -\frac{\partial \psi}{\partial x}. \tag{2.6}$$

This implies

$$\frac{\partial u}{\partial x} = \frac{\partial^2 \psi}{\partial x \partial y},$$

$$\frac{\partial v}{\partial y} = -\frac{\partial^2 \psi}{\partial x \partial y},$$

and putting above two equations in Eq. (2.1),

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial x \partial y} = 0.$$

Eq. (2.2) becomes

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial x \partial y} = v\frac{\partial^3 \psi}{\partial y^3}. \tag{2.7}$$

A partial differential equation can be converted into an ordinary differential by similarity transformation. Furthermore the method of similarity transformation can be used to reduce order of diiferential equations. Blasius introduced following similarity transformation to transform Eq. (2.7) into and ordinary differential equation.

$$\eta = a\frac{y}{\sqrt{x}}, \tag{2.8}$$

$$\psi(x,y) = b\sqrt{x}f(\eta), \tag{2.9}$$

where $a$, $b$ are constants chosen such that $f(\eta)$ is dimensionless i.e

$$a = \sqrt{\frac{U}{\nu}}, \qquad b = \sqrt{\nu U}, \tag{2.10}$$

where $\eta$ is a dimensionless similarity variable. By taking derivative of $\psi(x,y)$ with respect to $x$,

$$\frac{\partial\psi}{\partial x} = -\frac{U}{2}\frac{y}{x}f'(\eta) + \frac{1}{2}\sqrt{\nu U}\frac{f(\eta)}{\sqrt{x}},$$

derivatives of $\psi(x,y)$ with respect to $y$,

$$\begin{aligned}
\frac{\partial\psi}{\partial y} &= Uf'(\eta),\\
\frac{\partial^2\psi}{\partial y^2} &= Uf''(\eta)\frac{a}{\sqrt{x}},\\
\frac{\partial^2\psi}{\partial x\partial y} &= -\frac{U}{2}\sqrt{\frac{U}{\nu}}\frac{y}{x^{\frac{3}{2}}}f''(\eta) = -\frac{U}{2x}\eta f''(\eta),\\
\frac{\partial^3\psi}{\partial y^3} &= \frac{U^2}{\nu x}f'''(\eta),
\end{aligned}$$

substituting above derivatives in Eq. (2.7) we get

$$-\frac{U}{2}\frac{ay}{x\sqrt{x}}f'(\eta)f''(\eta) + \frac{U}{2}\frac{ay}{x\sqrt{x}}f'(\eta)f''(\eta), -\frac{1}{2}\sqrt{U\nu}\frac{a}{x}f(\eta)f''(\eta) = \frac{U}{x}f'''(\eta),$$

which further simplifies as

$$f'''(\eta) + \frac{1}{2}f(\eta)f''(\eta) = 0. \tag{2.11}$$

Eq. (2.11) is called Blasius equation. Also

$$\begin{aligned} \eta(x,0) &= 0, \\ u(x,0) &= Uf'(\eta(x,0)), \\ 0 &= Uf'(0) \Rightarrow f'(0) = 0. \end{aligned} \tag{2.12}$$

This is transformed boundary condition of Eq. (2.3). Now

$$\begin{aligned} v(x,0) &= -\frac{U}{2}\frac{y}{x}f'(0) + \frac{1}{2}\sqrt{U\nu}\frac{f(0)}{\sqrt{x}}, \\ \Rightarrow f(0) &= 0. \end{aligned} \tag{2.13}$$

Likewise $y \to \infty \Rightarrow \eta \to \infty$

$$f'(\eta) = 1, \tag{2.14}$$

as $\eta \to \infty$.

With the help of stream function and using similarity transformation method partial differential equations of flow over a semi infinite plate are transformed into third order non-linear ordinary differential equation.

$$f'''(\eta) + \frac{1}{2}f(\eta)f''(\eta) = 0, \tag{2.15}$$

with boundary conditions

$$f(0) = f'(0) = 0, \qquad f'(\infty) = 1.$$

# Chapter 3

# Topfer and Weyl's Solutions

This chapter includes power series solution, Topfer's [20] method to find value of $f''(0)$ and Weyl's [21] solution of Blasius Problem. Power series solution is of limited usefulness because its radius of convergence is finite and is found to be approximately 5.69. Therefor the solution is useless for larger values of $\eta$ and to overcome this problem one has to adopt some other means. Topfer's Method (1912) overcomes the absence of $f''(0)$ by replacing the boundary value problem by an initial value problem and exploiting its symmetry. Topfer's method finds $\kappa = f''(0)$ by relying on a numerical solution of modified problem. Weyl solved the problem analytically which we present in third section of this chapter, however the accuracy achieved is not of the same order.

## 3.1 Power Series Solution of Blasius Problem

For a power series solution of Blasius problem Eq. (2.15) with boundary conditions, we assume

$$f(\eta) = \sum_{n=0}^{\infty} a_n \eta^n, \tag{3.1}$$

and we assume $f''(0) = \kappa$.

We can see from boundary conditions

$$\begin{aligned} a_0 &= 0, \\ a_1 &= 0, \\ a_2 &= \frac{1}{2}\kappa. \end{aligned}$$

By differentiating Eq. (3.1) w.r.t $\eta$, we get

$$\begin{aligned} f'(\eta) &= \sum_{n=1}^{\infty} n a_n \eta^{n-1}, \\ f''(\eta) &= \sum_{n=2}^{\infty} n(n-1) a_n \eta^{n-2}, \\ f'''(\eta) &= \sum_{n=3}^{\infty} n(n-1)(n-2) a_n \eta^{n-3}. \end{aligned}$$

Putting in (2.15)

$$\sum_{n=3}^{\infty} n(n-1)(n-2) a_n \eta^{n-3} + \frac{1}{2} \sum_{n=0}^{\infty} a_n \eta^n \sum_{n=2}^{\infty} n(n-1) a_n \eta^{n-2} = 0. \tag{3.2}$$

Let $k = n-3$ and $m = n-2$, we obtain

$$\sum_{k=0}^{\infty} (k+3)(k+2)(k+1) a_{k+3} \eta^k = -\frac{1}{2} \sum_{n=0}^{\infty} a_n \eta^n \sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} \eta^m.$$

Above equation can be written as

$$\begin{aligned}
\sum_{n=0}^{\infty}(n+3)(n+2)(n+1)a_{n+3}\eta^n &= -\frac{1}{2}\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\eta^n\sum_{n=0}^{\infty}a_n\eta^n,\\
\sum_{n=0}^{\infty}(n+3)(n+2)(n+1)a_{n+3}\eta^n &= -\frac{1}{2}\sum_{n=0}^{\infty}\sum_{i=0}^{n}a_{n-i}(n+2)(n+1)a_{n+2}\eta^n,\\
(n+3)(n+2)(n+1)a_{n+3} &= -\frac{1}{2}\sum_{i=0}^{n}a_{n-i}(n+2)(n+1)a_{n+2}.
\end{aligned}$$

Few coefficients are

$$\begin{aligned}
a_3 &= 0, \quad a_4 = 0, \quad a_5 = -\frac{1}{240}, \quad a_6 = 0,\\
a_7 &= 0, \quad a_8 = \frac{11}{161280}.
\end{aligned}$$

We get a non-zero coefficient after every two terms and negative sign with alternating terms, rest of the series follow the same pattern. The recurrence relation for series solution can be written as

$$a_{n+3} = \frac{-\sum_{i=0}^{n}a_{n-i}a_{n+2}(n+2)(n+1)}{2(n+3)(n+2)(n+1)}.$$

Thus we obtain

$$f(\eta) = \frac{1}{2}\kappa\eta^2 - \frac{1}{240}\kappa^2\eta^5 + \frac{11}{161280}\kappa^3\eta^8 - \frac{5}{4257792}\kappa^4\eta^{11} + ... \tag{3.3}$$

A number of coefficients can be found with MATHEMATICA

$$\begin{aligned}
a_{11} &= -\frac{5}{4257792}\kappa^4, a_{14} = \frac{9299}{464950886400}\kappa^5,\\
a_{17} &= -\frac{1272379}{3793999233024000}\kappa^6,\\
a_{20} &= \frac{19241647}{346012730051788800}\kappa^7,\\
a_{29} &= -\frac{1375703592341009}{558886684048731777867448320000000}\kappa^{10}.
\end{aligned}$$

## 3.2 Topfer's Solution

Blasius problem possesses two continuous symmetries, translational symmetry and dilational symmetry.

**Dilational symmetry:** If $f(x)$ is a solution of Blasius problem then $g(x) = \lambda f(\lambda x)$ is also a solution of it. To see this, we have

$$\begin{aligned} g'(x) &= \lambda^2 f'(\lambda x), \\ g''(x) &= \lambda^3 f''(\lambda x), \\ g'''(x) &= \lambda^4 f'''(\lambda x). \end{aligned}$$

Hence

$$g'''(x) + \frac{1}{2} g(x) g''(x) = \lambda^4 [f'''(\lambda x) + \frac{1}{2} f(\lambda x) f''(\lambda x)] = 0.$$

Also

$$\begin{aligned} g(0) &= \lambda f(0) = 0, \\ g'(0) &= \lambda^2 f'(0) = 0, \\ g''(0) &= \lambda^3 f''(0) = \lambda^3 \kappa, \\ g'(\infty) &= \lim_{x \to \infty} \lambda^2 g'(\lambda x) = \lambda^2. \end{aligned} \tag{3.4}$$

Now let $g''(0) = 1$. From (3.4)

$$\lambda^3 \kappa = 1 \Rightarrow \lambda = \kappa^{-1/3}.$$

Also let $g'(\infty) = A$, then

$$g'(\infty) = \lambda^2 f'(\infty),$$

which implies

$$A = \lambda^2,$$

which further implies

$$\kappa^{-1/3} = A^{1/2}, \quad and \quad \kappa = A^{-3/2}.$$

For large values of $x$, $A$ can give us an approximate value of $\kappa$. In order to find $A$ we need to solve following initial value problem.

$$g'''(x) + \frac{1}{2}g(x)g''(x) = 0,$$
$$g(0) = 0, g'(0) = 0, g''(0) = 1.$$

On a sufficiently large interval until $g'(x)$ becomes a constant. We solve this problem on $[0, 25]$ and find that $g'(x)$ will be practically constant for $x \geq 10$. We use Runge-Kutta method of order four for third order differential equation with the help of MATLAB. Following is table(Table.3.1) of values of $\kappa$ for different values of $x$.

Numerical value of $\kappa$ matches in the first decimal position for $x = 3.040$. However first three digits of $\kappa$ match for $x$ on the interval $4.028 - 5.021$. Accuracy increases rapidly from $x$ equals to 6 and onwards. In interval $7.006 - 7.809$ first nine digits of $\kappa$ are same, this number increases in next interval from 8.023 to 9.895 to fifteen.

## 3.3 Solution of Blasius Problem by Herman Weyl

Herman Weyl made use of initial value problem discussed in previous section to get an approximate analytical solution of Blasius problem. Let us consider the problem

$$\omega''' + 2\omega\omega'' = 0. \tag{3.5}$$

| Sr. no | $x$ | $A$ | $\kappa$ |
|---|---|---|---|
| 1 | 2.007 | 1.728794769381651 | 0.439931277863228 |
| 2 | 3.040 | 2.034331480130082 | 0.344641379982161 |
| 3 | 4.028 | 2.082146271755432 | 0.332838186111262 |
| 4 | 5.021 | 2.085329605553927 | 0.332076342093992 |
| 5 | 6.014 | 2.085408443437374 | 0.332057511287537 |
| 6 | 6.900 | 2.085409171264272 | 0.332057336217753 |
| 7 | 7.006 | 2.085409173939981 | 0.332057336811808 |
| 8 | 7.597 | 2.085409176384847 | 0.332057336227868 |
| 9 | 7.809 | 2.085409176425792 | 0.332057336218089 |
| 10 | 8.023 | 2.085409176435273 | 0.332057336215825 |
| 11 | 8.565 | 2.085409176437859 | 0.332057336215207 |
| 12 | 8.895 | 2.085409176437900 | 0.332057336215197 |
| 13 | 9.015 | 2.085409176437902 | 0.332057336215197 |
| 14 | 10.01 | 2.085409176437902 | 0.332057336215197 |

**Table 3.1: Values of parameter $\kappa$ and $A$**

with initial conditions,

$$\omega(0) = \omega'(0) = 0, \qquad \omega''(0) = 1. \tag{3.6}$$

For $\omega(z)$ to be the solution of equation (3.5), $m\omega(mz)$ will also satisfy the equation (3.5) by [21], where $m$ is any constant. Let $\omega = f(z)$ be the solution determined by initial conditions Eq. (3.6). Since $f(z)$ is defined over the whole interval $[0, \infty)$ and with $z \to \infty$ we can define

$$\xi = \int_0^\infty f''(z) > 0. \tag{3.7}$$

We can define our constant $m$ in such a manner that derivative of $\omega = mf(mz)$ approaches to $\kappa$ at infinity i.e

$$\begin{aligned} m^2\xi &= \kappa, \\ m &= (\kappa/\xi)^{1/2}. \end{aligned} \tag{3.8}$$

Hence,

$$\omega''(0) = m^3 = \gamma\kappa^{3/2}, \qquad \gamma = \xi^{-3/2}.$$

In the equation,

$$\frac{df''}{dz} + 2ff'' = 0, \tag{3.9}$$

by considering $f$ and $f''$ as two distinct functions,

$$\frac{df''}{dz} = -2ff'',$$

$$\begin{aligned}
\frac{df''/dz}{f''} &= -2f, \\
\int_0^z \frac{df''/dz}{f''} &= \int_0^z -2f, \\
\ln f''(z) &= \int_0^z -2f + C,
\end{aligned}$$

where $C$ being the constant of integration and using initial condition $f''(0) = 1$, one then obtain

$$f''(z) = \exp(-2\int_0^z f(\tau)d\tau). \tag{3.10}$$

Sice $f(0) = f'(0) = 0$ and partial integration gives,

$$\begin{aligned}
\int_0^z (z-\tau)f''(\tau)d\tau &= (z-\tau)^2 f'(\tau)|_0^z + 2\int_0^z (z-\tau)f'(\tau)d\tau, \\
\int_0^z (z-\tau)f''(\tau)d\tau &= 0 + 2\int_0^z (z-\tau)f'(\tau)d\tau, \\
2\int_0^z (z-\tau)f'(\tau)d\tau &= 2((z-\tau)f(\tau)|_0^z + \int_0^z f(\tau)d\tau),
\end{aligned}$$

which implies,

$$2\int_0^z f(\tau)d\tau = \int_0^z (z-\tau)^2 f''(\tau)d\tau. \tag{3.11}$$

Eq. (3.9) thus becomes

$$f''(z) = \exp(-\int_0^z (z-\tau)^2 f''(\tau)d\tau).$$

Introducing $f'' = g$ as the unknown function, above expression can be written as

$$g = \Phi\{g\}. \tag{3.12}$$

With the operator,

$$\Phi\{g\} = \exp(-\int_0^z (z-\tau)^2 g(\tau)d\tau). \tag{3.13}$$

From the definition of operator following properties of operator are obvious

(i) $\Phi\{g\} \geq 0,$

(ii) $\Phi\{g\} \geq \Phi\{g*\}, \qquad \text{if } g \leq g*.$

For successive approximation $g_n$, a sequence can be defined as

$$g_{n+1} = \Phi\{g_n\}, \qquad (n = 0, 1, 2, ...), \tag{3.14}$$

with first approximation be $g_0(z) = 0$, which implies

$$\begin{aligned} g_1 &= \Phi\{g_0\} = \exp(-\int_0^z (z-\tau)^2 g_0(\tau) d\tau), \\ g_1 &= 1. \end{aligned}$$

Relations $g_0 \leq g_1$ and $g_0 \leq g_2$ are trivial.

Also,

$$g_0 \leq g_1 \qquad \Rightarrow \qquad \Phi\{g_0\} \geq \Phi\{g_1\}, \qquad g_1 \geq g_2$$

Similarly following arguments can be constructed.

$$g_2 \leq g_3, \qquad g_3 \geq g_4, \qquad g_4 \leq g_5, \qquad g_5 \geq g_6 \qquad g_6 \leq g_7, ... \tag{3.15}$$

and,

$$g_0 \leq g_2, \qquad g_1 \geq g_3, \qquad g_2 \leq g_4, \qquad g_3 \geq g_5, \qquad g_6 \leq g_8 ... \tag{3.16}$$

Rearrange,

$$g_0 \leq g_2 \leq g_4 \leq g_6 \leq, ... \tag{3.17}$$

and

$$g_1 \geq g_3 \geq g_5 \geq g_7 \geq, ... \tag{3.18}$$

With a little more explanation it can be seen that decreasing sequence of the odd approximations exceeds the increasing sequence of even approximations. For instance $g_7 \geq g_{12}$, from (3.18) $g_7 \geq g_9 \geq g_{11}$ implies $g_7 \geq g_{11}$ and from Eq. (3.15) $g_{11} \geq g_{12}$, hence $g_7 \geq g_{12}$ is proved. Again $g_7 \geq g_{100}$, from (3.18) $g_7 \geq g_9 \geq ... \geq g_{99}$ implies $g_7 \geq g_{99}$ and from Eq. (3.15) $g_{99} \geq g_{100}$ thus $g_7 \geq g_{100}$. Similarly any odd $g_n$ of arbitrary order can be proved to be greater than even $g_n$ of arbitrary order. Lets define an abbreviation,

$$M(z) = \int_0^z (z-\tau)^2 g(\tau) d\tau \tag{3.19}$$

For $0 \leq g(z) \leq g^*(z)$,

$$\Delta g = g^*(z) - g(z), \qquad \textit{and} \qquad \Delta\Phi\{g\} = \Phi\{g^*\} - \Phi\{g\}$$

**Mean Value Theorem**

If $f$ is continuous on $[a, b]$, differentiable on $(a, b)$ then

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

for some $c \in (a, b)$.

Consider an arbitrary function $g(x) = e^{-x}$ defined on interval $(x, y)$, from mean value theorem

$$\frac{e^{-y} - e^{-x}}{y - x} = -e^c$$

or,

$$\frac{e^{-x} - e^{-y}}{y - x} = e^c \leq 1$$

From here we are led to the result:

$0 \leq \exp(-x) - \exp(-y) \leq y - x$ if $x \leq y$

By using this result we can see

$$\begin{aligned} 0 \leq \exp(-\int_0^z (z-\tau)^2 g(\tau) d\tau) - \exp(-\int_0^z (z-\tau)^2 g^*(\tau) d\tau) \\ \leq \int_0^z (z-\tau)^2 g^*(\tau) d\tau - \int_0^z (z-\tau)^2 g(\tau) d\tau \end{aligned}$$

Implies,

$$\begin{aligned} 0 \leq -(\exp(-\int_0^z (z-\tau)^2 g^*(\tau) d\tau - \exp(-\int_0^z (z-\tau)^2 g(\tau) d\tau)) \\ \leq M^*(z) - M(z) = \Delta M \end{aligned}$$

Finally,

$$0 \leq -\Delta\Phi\{g\} \leq \Delta M.$$

Herman deduced if we integrate $2.\Delta g$ three times from 0 to $z$, we can obtain the increment $\Delta M$ and the inequality

$$|g_{n+1}(z) - g_n(z)| \leq (2z^3)^n/(3n)! \tag{3.20}$$

Inequality can be proved by induction i.e,

For $n = 1$, it is true.

$$|g_1(z) - g_0(z)| = |1 - 0| = 1.$$

Assume for $n = k$ we have

$$|g_{k+1}(z) - g_k(z)| \leq (2z^3)^k/(3k)! \tag{3.21}$$

Now, for $n = k + 1$,

$$\begin{aligned}
|g_{k+2}(z) - g_{k+1}(z)| &= |\exp(-\int_0^z (z-\tau)^2 g_{k+1}(\tau)d\tau - \exp(-\int_0^z (z-\tau)^2 g_k(\tau)d\tau)| \\
&\leq |\int_0^z (z-\tau)^2 (g_k - g_{k+1})d\tau| \\
&\leq |\int_0^z (z-\tau)^2 \frac{(2(\tau)^3)^k}{(3k)!} d\tau| \\
&= |\frac{2^k}{(3k)!} \int_0^z (z^2 + \tau^2 - 2z\tau)(\tau^{3k})d\tau| \\
&= |\frac{2^k}{(3k)!} \int_0^z (\tau^{3k}z^2 + \tau^{3k+2} - 2z\tau^{3k+1})d\tau| \\
&= |\frac{2^k}{(3k)!} (\frac{\tau^{3k+1}z^2}{3k+1} + \frac{\tau^{3k+3}}{3k+3} - \frac{2z\tau^{3k+2}}{3k+2})|_0^z| \\
&= |\frac{2^k}{(3k)!} (\frac{\tau^{3k+3}}{3k+1} + \frac{\tau^{3k+3}}{3k+3} - \frac{2\tau^{3k+3}}{3k+2})| \\
&= |\frac{2^k}{(3k)!} (\frac{2\tau^{3k+3}}{(3k+1)(3k+3)(3k+2)})| \\
&= \frac{(2\tau^3)^{k+1}}{(3(k+1))!}.
\end{aligned} \tag{3.22}$$

Thus it is true for $n = k + 1$ which proves the inequality. Relation (3.21) suffices to guarantee the convergence of the sequence $g_n(z)$

$$g(z) = \lim_{n\to\infty} g_n(z)$$

Since series $g_n$ is a combination of ascending sequence of odd $g_n$ and descending sequence of even $g_n$, limit function $g(z)$ would be greater than the even and less than the odd $g_n$. Furthermore limit of a bounded convergent sequence is unique, uniqueness can be proved by inequalities

$$g \geq g_0, \qquad g \leq g_1, \qquad g \geq g_2, \qquad g \leq g_3, ...$$

Thus $g(z)$ is certainly between even and odd approximations of $g_n$.

Next we will examine the asymptotic behaviour of $g(z)$ as $z \to \infty$.

Let for any $z_0 > 0$ we have

$$\int_0^{z_0} g_2(\tau)d\tau = c > 0 \tag{3.23}$$

From Eq. (3.19) $M_2$ is $\int_0^{z_0}(z-\tau)^2 g_2(\tau)d\tau$. Partial integration of the expression gives,

$$(z-\tau)^2 \int_0^{z_0} g_2(\tau)d\tau + 2\int_0^{z_0}((z-\tau)\int_0^{z_0} g_2(\tau))d\tau \geq (z-\tau)^2 \int_0^{z_0} g_2(\tau)d\tau$$

Using Eq. (3.23) we get,

$$M_2 \geq c(z-z_0)^2$$

Multiply both sides with -1

$$-M_2 \leq -c(z-z_0)^2 \qquad \Rightarrow \qquad e^{-M_2} \leq e^{-c(z-z_0)^2}$$

$$g(z) \leq g_3(z) \leq e^{-c(z-z_0)^2}$$

Consequently

$$\int_0^\infty g(z)dz = \xi > 0$$

and,

$$\int_0^\infty zg(z)dz = \xi' > 0$$

So,

$$2\int_0^z f(\tau)d\tau \;\; = \;\; \int_0^z (z-\tau)^2 g(\tau)d\tau.$$

By partial integration,

$$\begin{aligned} \int_0^z f(\tau)d\tau &= (z-\tau)^2|_0^z \int_0^z g(\tau)d\tau + 2\int_0^z (\int_0^z g(\tau))(z-\tau)d\tau \\ f(\tau)d\tau &= \int_0^z (z-\tau)g(\tau)d\tau. \end{aligned}$$

At infinity

$$\begin{aligned} f(z) \sim \int_0^\infty (z-\tau)g(\tau)d\tau &= \int_0^\infty zg(\tau) - \tau g(\tau)d\tau \\ &= z\xi - \xi'. \end{aligned}$$

Hence

$$\omega(z) \sim mf(mz) = m^2\xi z - m\xi'.$$

From Eq. (3.8) $m = (\kappa/\xi)^{1/2}$

$$w(z) = z\kappa - \xi'.(\frac{\kappa}{\xi})^{1/2}.$$

Let,

$$B = \int_0^\infty g_2(z)dz = \int_0^\infty e^{-z^3/3}dz$$

$g(z) \geq g_2(z)$ implies

$$\begin{aligned}
\int_0^\infty g(z)dz \geq \int_0^\infty g_2(z)dz & \\
\xi \geq B &= \int_0^\infty g_2(z)dz \\
&= \int_0^\infty \exp(\int_0^z (z-\tau)^2 g_1(z)dz)dz \\
&= \int_0^\infty \exp(\int_0^z (z-\tau)^2 dz)dz \\
&= \int_0^\infty \exp(-\frac{1}{3}z^3)dz \qquad (3.24)
\end{aligned}$$

Substitution $t = \frac{1}{3}z^3$ implies $z = (3t)^{\frac{1}{3}}$, therefore,

$$dz = \frac{1}{3}3^{\frac{1}{3}}t^{-\frac{2}{3}}.$$

Substituting these in Eq.(3.24) we get,

$$\begin{aligned}
B &= \frac{3^{\frac{1}{3}}}{3}\int_0^\infty e^{-t}t^{-2/3}dt \\
&= \frac{3^{\frac{1}{3}}}{3}\int_0^\infty e^{-t}t^{\frac{1}{3}-1}dt \\
&= \frac{3^{\frac{1}{3}}}{3}\Gamma(\frac{1}{3}) \qquad (3.25)
\end{aligned}$$

Using property of gamma function, $\Gamma(x+1) = x\Gamma(x)$.

Eq.(3.25) becomes,

$$B = 3^{\frac{1}{3}}.\Gamma(\frac{4}{3}).$$

Let us consider our differential equation (2.11) with same boundary conditions as of Eq.(3.6)

$$f'''(\eta) + \frac{1}{2}f(\eta)f''(\eta) \;\; = \;\; 0,$$

$$f(0) = f'(0) = 0, \qquad f''(0) = 1,$$

which can be written as

$$\frac{df''}{dz} + \frac{1}{2}ff'' = 0,$$
$$\frac{\frac{df''}{dz}}{f''} = \frac{1}{2}f.$$

Integrating and treating $f(z)$ and $f''(z)$ as two distinct functions

$$\ln(f'') = \frac{1}{2}\int_0^z f(\tau)d\tau + C_1$$

$C_1$ is the constant of integration. Using initial condition $f''(0) = 1$,

$$f''(z) \;\; = \;\; \exp(-\frac{1}{2}\int_0^z f(\tau)d\tau). \tag{3.26}$$

Since partial integration gives

$$\begin{aligned} 2\int_0^z f(\tau)d\tau &= \int_0^z (z-\tau)^2 f''(\tau)d\tau, \\ \Longrightarrow \frac{1}{2}\int_0^z f(\tau)d\tau &= \frac{1}{4}\int_0^z (z-\tau)^2 f''(\tau)d\tau. \end{aligned}$$

Differential equation (3.26) thus becomes

$$f''(z) \;\; = \;\; \exp(-\frac{1}{4}\int_0^z (z-\tau)^2 f''(\tau)d\tau)$$

$f'' = g$ gives

$$g \;\; = \;\; \Phi\{g\},$$

with the operator

$$\Phi\{g\} \;=\; \exp(-\frac{1}{4}\int_0^z (z-\tau)^2 g(\tau)d\tau). \tag{3.27}$$

For successive approximation $g_n$, a sequence can be defined as

$$g_{n+1} = \Phi\{g_n\}, \qquad\qquad (n = 0, 1, 2, ...).$$

$$|g_{n+1}(z) - g_n(z)| \le (2z^3)^n/(3n)!$$

$$B = \int_0^\infty g_2(z)dz = \int_0^\infty e^{-z^3/4.3}dz$$

$g(z) \ge g_2(z)$ implies

$$\int_0^\infty g(z)dz \ge \int_0^\infty g_2(z)dz$$

$$\begin{aligned}\xi \ge B &= \int_0^\infty g_2(z)dz \\ &= \int_0^\infty \exp(\frac{1}{4}\int_0^z (z-\tau)^2 g_1(z)dz)dz \\ &= \int_0^\infty \exp(-\frac{1}{12}z^3)dz.\end{aligned}$$

Substitution $t = \frac{1}{12}z^3$ implies $z = (12t)^{\frac{1}{3}}$ therefore,

$$dz = \frac{1}{3}(12)^{\frac{1}{3}}t^{-\frac{2}{3}}.$$

Substituting these in Eq.(3.24), we get

$$\begin{aligned}B &= \frac{(12)^{\frac{1}{3}}}{3}\int_0^\infty e^{-t}t^{-2/3}dt \\ &= \frac{(12)^{\frac{1}{3}}}{3}\int_0^\infty e^{-t}t^{\frac{1}{3}-1}dt \\ &= \frac{(12)^{\frac{1}{3}}}{3}\Gamma(\frac{1}{3})\end{aligned}$$

Consequently

$$
\begin{aligned}
B &= (12)^{\frac{1}{3}}.\Gamma(\frac{4}{3}) \\
&\simeq 2.0445.
\end{aligned}
$$

From first section of this chapter we can infer that $\kappa = B^{-3/2}$, so that we can have an approximation for $\kappa$. It turns out to be $\kappa = 0.3420953217$.

# Chapter 4

# Domb-Sykes Method

We saw that the series solution has a finite radius of convergence which indicates existance of a singular point in the complex plane. Now we discuss a method for the location of such a singularity. This method is called Dom-Syke's method and is applicable to problematic series of type above.

## 4.1 Domb-Sykes Diagram

A series converges for $|x| < r$ having centre at zero, $r$ is distance from origin to nearest singularity. Let singularity at $x_0$ is of type $(x_0 - x)^\alpha$, $\alpha \neq 0, 1, 2...$
Now let

$$(x_0 - x)^\alpha = \sum_{n=0}^{\infty} c_n x^n \tag{4.1}$$

Implies,

$$\frac{c_n}{c_{n-1}} = \mp \frac{1}{x_0}(1 - \frac{1+\alpha}{n}) \tag{4.2}$$

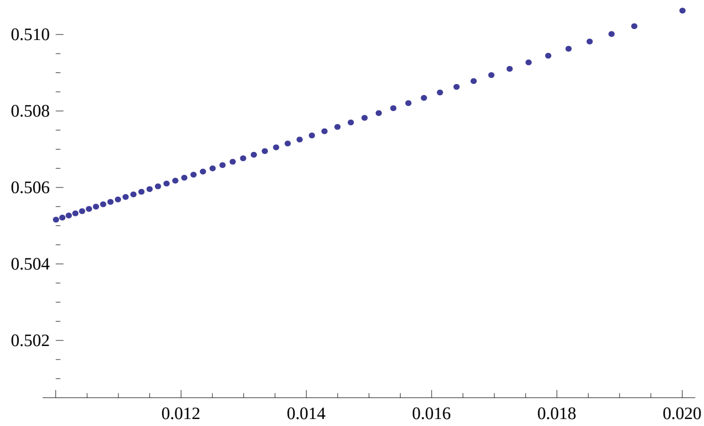


FIG. 4.1: Domb Sykes diagram for function in example

If we plot $\frac{c_n}{c_{n-1}}$ against $\frac{1}{n}$, the line will intersect the $\frac{c_n}{c_{n-1}}$ line at $\mp\frac{1}{x_0}$, which gives the location of singularity. Order of singularity which is $\alpha$ can be inferred from the slop $\mp\frac{1+\alpha}{x_0}$ of line. This idea has been used many times by Domb , Sykes and Punnis. Van Dyke [11] called the figure **Domb-Sykes Diagram**.

### 4.1.1 Example

$$f(x) = \frac{1 - x + 2x^3}{(2 - x)^2} \tag{4.3}$$

Maclauren series for $f(x)$ denotes by $S(x)$ is given by,

$$S(x) = 0.25 - 0.0625x^2 + 0.4375x^3 + 0.453125x^4 + 0.34375x^5 + ... \tag{4.4}$$

Function gives information that it has a pole of order 2 at $x = 2$. We shall recover this information from the series. Line cut the $\frac{c_n}{c_{n-1}}$ approximately at 0.5, thus

$$x_0 = 1/0.5 = 2 \qquad and \qquad -\frac{1}{x_0}(\alpha + 1) = 1/2,$$

produces $\alpha = -2$.

We have a pole of order 2 at $x = 2$. If we set

$$S(x) = p(x)/(x-2)^2,$$

then

$$p(x) = S(x)(x-2)^2 = 1 - x + 2x^3.$$

We indeed have recovered the function from the series.

## 4.2 Blasius Problem

Assuming the unknown second derivative at origin $f''(0)$, to be $\kappa$, successive coefficients $c_n$, in the series solution,

$$f(x) = \sum_{n=0}^{\infty} (-1)^n c_n x^{3n+2}, \tag{4.5}$$

are calculated from the recurrence relation,

$$3n(3n+1)(3n+2)c_n = \frac{1}{2}\sum_{i=0}^{n-1}(3i+1)(3i+2)c_i c_{n-1-i}. \tag{4.6}$$

Series obtained here are

$$\frac{1}{2}\kappa x^2 - \frac{1}{240}\kappa^2 x^5 + \frac{11}{161280}\kappa^3 x^8 - \frac{5}{4257792}\kappa^4 x^{11} + ... \tag{4.7}$$

Let us make a substitution,

$$\eta = \kappa x^3 \tag{4.8}$$

the series (4.7) becomes,

$$\kappa x^2 (\frac{1}{2} - \frac{1}{240}\eta + \frac{11}{161280}\eta^2 - \frac{5}{4257792}\eta^3 + ...) \tag{4.9}$$

The series in powers of $\eta$ has finite radius of convergence.

### 4.2.1 Location of Singularities

If the signs of terms in a series follow a regular pattern it gives an indication of the location of a singularity. If there is a singularity on the positive real axis the series will eventually exhibit unchanged signs like the fuction

$$(1-a)^{-1} = 1 + a + a^2 + a^3 + ... \tag{4.10}$$

On the other hand a singularity on the negative real axis will give rise to alternating signs as in $(1+a)^{-1}$. The pattern in the Blasius series indicate likelihood of a singularity on the negative real axis. Coefficients in the series $\kappa x^2 \sum_{n=0}^{\infty} c_n \eta n$, are easily found by using MATHEMATICA. For example,

$$c_4 = 2 \times 10^{-8}, c_5 = -3.35366 \times 10^{-10}, c_6 = 5.5609 \times 10^{-12}, c_{99} = -3.97548 \times 10^{-178} \tag{4.11}$$

Plot of $\frac{c_n}{c_{n-1}}$ versus $1/n$ is as follows, The line has slope zero. Thus,

$$\alpha + 1 = 0, \alpha = -1$$

Corresponds to a simple pole. It is located approximately at,

$$\eta = \frac{c_{99}}{c_{98}} = -61.1729$$

By re-substitution $\eta = \kappa x^3$ and

$$\kappa = 0.33205$$

We find $x^3 = -184.224$ Hence there are three simple poles one of them located at $x = -\sqrt[3]{184.224} = -5.69004$. The other two are placed symmetrically in the complex plane. This is why radius of convergence of Blasius Series is 5.69004.

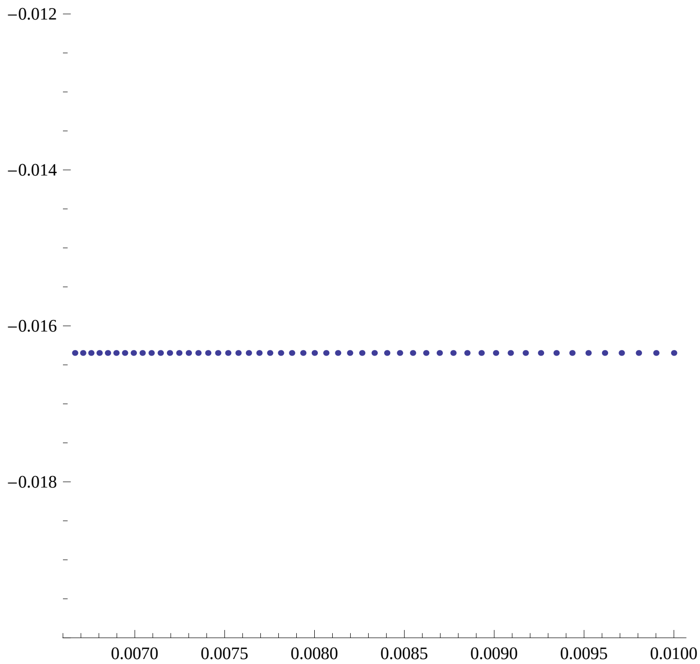


FIG. 4.2: Domb Sykes diagram for Blasius function

# Chapter 5

# Coroco-Wang Transformation

We shall discuss a method which transforms the Blasius problem with a differential equation of order two and a finite domain of $[0, 1]$. We find a series solution and use it to evaluate $f''(0)$. Two dimensional steady-state laminar viscous flow over a semi-infinite plate is modeled by following Blasius Equation,

$$f'''(\eta) + \beta_0 f''(\eta) f(\eta) = 0, \qquad \eta \in [0, \infty), \tag{5.1}$$

$$f(0) = f'(0) = 0, \qquad f'(\infty) = 1. \tag{5.2}$$

Boundary condition at second derivative of $f(\eta)$ is not obtained which is a constraint in finding solution of Blaisus Problem. Different scientists found approximate numerical values of $f''(0)$ by using or inventing various techniques. For example Howarth found $f''(0) = 0.33206$, Asaithambi [6] found $f''(0) = 0.469600$. Fang et al. [12] introduced a substitution

$$f(\eta) = \frac{1}{\beta_0} F(\sqrt{\beta_0}\eta). \tag{5.3}$$

Blaisus Equation transforms into,

$$F''' + FF'' = 0. \tag{5.4}$$

## 5.1 Transformation

Likewise Liao [16] and Abbasbandy [1] found solutions of Blasius Problem at high level of accuracy. Transformation introduced by Coroco [2] is,

$$\begin{aligned} x &= f'(\eta) && (5.5) \\ \Rightarrow \frac{dx}{d\eta} &= \frac{f'(\eta)}{d\eta} = f''(\eta) \\ y &= f''(\eta) && (5.6) \\ \Rightarrow \frac{dy}{dx} &= \frac{df''(\eta)}{dx} \\ &= \frac{df''(\eta)}{d\eta}\frac{d\eta}{dx} \\ &= f'''(\eta)\frac{d\eta}{dx} = f'''(\eta)/f''(\eta) && (5.7) \end{aligned}$$

Divide both sides of equation (5.1) with $f''(\eta)$ and using (5.5) and (5.6)

$$\begin{aligned} f'''(\eta)/f''(\eta) + \beta_0 f(\eta) = 0 \\ \frac{dy}{dx} + \beta_0 f(\eta) = 0 \end{aligned}$$

Taking Derivative and using (5.5)

$$\begin{aligned} \frac{d^2y}{dx^2} + \beta_0\frac{df(\eta)}{dx} &= 0, \\ \Rightarrow \frac{d^2y}{dx^2} + \beta_0\frac{df(\eta)}{d\eta}\frac{d\eta}{dx} &= 0, \end{aligned}$$

$$\Rightarrow \frac{d^2y}{dx^2} + \beta_0 \frac{f'(\eta)}{f''(\eta)} = 0,$$
$$\Rightarrow \frac{d^2y}{dx^2} + \beta_0 x/y = 0,$$

Blasius problem becomes,

$$\frac{d^2y}{dx^2} + \frac{x}{y} = 0, x \in [0, 1) \tag{5.8}$$

For $x = 0$, we can see from (5.5) that $\eta$ is zero,$x = 0 \Rightarrow f'(\eta) = 0$

$\eta = 0$,

$$y(0) = f''(0), \qquad y(0) = \alpha. \tag{5.9}$$

Since $x = 0 \Leftrightarrow \eta = 0$. From(5.7)

$$\begin{aligned} y'(0) &= f'''(0)/f''(0) = -x(0)/y(0) \\ y'(0) &= 0/\alpha = 0 \end{aligned} \tag{5.10}$$

Also

$$y(1) = f''(\infty) = 0.$$

So boundary conditions of Blasius problem transformed into

$$y(0) = f''(0), \qquad y'(0) = 0, \qquad \lim_{x \to 0} y(x) = 0. \tag{5.11}$$

Wang solved equation(57) by using Adomian decomposition method and found,

$$y(x) = \alpha - \frac{x^3}{6\alpha} - \frac{x^6}{180\alpha^3} - \frac{x^9}{2160\alpha^5} - \frac{x^{12}}{19008\alpha^7} \ldots \tag{5.12}$$

Here $\alpha$ denotes $f''(0)$. Further

$$y(1) = 0,$$

implies

$$\alpha = 0.453539.$$

by retaining six terms of series (5.12). Hashim used Adomian decomposition method and diagonal pade approximation found terms of series (5.12) up to $x^{24}$ and modified value of $\alpha$ is $\alpha = 0.466799$.

## 5.2 Power Series Solution

Equation(5.8) can be written as,

$$yy'' = -x \tag{5.13}$$

Let the power series solution solution of above problem is

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \tag{5.14}$$

From (5.6)

$$\begin{aligned} a_0 &= f''(0), \\ a_1 &= 0. \end{aligned}$$

Differentiate (5.14) twice and we get,

$$\begin{aligned} y'(x) &= \sum_{n=1}^{\infty} n a_n x^{n-1}, \\ y''(x) &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}. \end{aligned}$$

Put these two values in (5.13)

$$\sum_{k=0}^{\infty}\sum_{n=2}^{\infty} a_k a_n n(n-1)x^{n+k-2} = -x. \tag{5.15}$$

Expanding the above summation for first few terms

$$2\alpha a_2 + (2a_1a_2 + 6\alpha a_3)x + (12\alpha a_4 + 6a_1a_3 + 2a_2^2)x^2 + (20\alpha a_5 + 12a_1a_4 + 8a_2a_3)x^3 + ... = -x,$$

Which gives

$$a_2 = 0,$$
$$a_3 = -\frac{1}{6\alpha},$$
$$a_4 = 0,$$
$$a_5 = 0.$$

Let

$$m = n + k - 2,$$

equation (5.15) becomes

$$\sum_{m=0}^{\infty}\sum_{k=0}^{m} a_k a_{m-k+2}(m-k+2)(m-k+1)x^m = -x.$$

By comparing both sides of equation we can see that coefficients of each power of $x$ other than unity must vanish. This implies for $m \geq 2$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{m} a_k a_{m-k+2}(m-k+2)(m-k+1) = 0. \tag{5.16}$$

Which can be written as,

$$(m+2)(m+1)a_{m+2}a_0 = \sum_{m=0}^{\infty}\sum_{k=0}^{m} a_k a_{m-k+2}(m-k+2)(m-k+1). \tag{5.17}$$

The recurrence relation we obtained is

$$(m+2)(m+1)a_{m+2} = -\frac{1}{\alpha}\sum_{k=1}^{m} a_k a_{m-k+2}(m-k+2)(m-k+1). \tag{5.18}$$

$m \geq 2$. From above results we see that $a_0$ and $a_3$ are non-zero while $a_1$, $a_2$, $a_4$ and $a_5$. From Eq. (5.18) follows that $a_6$ is again non-zero but $a_7$ and $a_8$ are zeros. We can easily show that only those coefficients whose suffixes are multiple of 3 fail to vanish by mathematical induction. Equation (5.18) can be written as,

$$3n(3n-1)a_{3n} = -\frac{1}{\alpha}\sum_{k=1}^{n-1}(3n-3k)(3n-3k-1)a_{3k}a_{3n-3k} \tag{5.19}$$

$n \geq 2$. Starting with

$$a_0 = \alpha \tag{5.20}$$

and

$$a_3 = \frac{1}{6\alpha}$$

Equation(5.19) gives us

$$\begin{aligned}
a_6 &= -\frac{1}{180\alpha_3}, a_9 = -\frac{1}{2160\alpha_5}, a_{12} = -\frac{1}{119008\alpha_7}, \\
a_{15} &= -\frac{7.01125\times 10^{-6}}{\alpha_9}, a_{18} = -\frac{1.03002\times 10^{-9}}{\alpha_{11}}, \\
a_{21} &= -\frac{1.16614\times 10^{-7}}{\alpha_{13}}, a_{24} = -\frac{2.65906\times 10^{-8}}{\alpha_{15}}, \\
a_{27} &= -\frac{4.53471\times 10^{-9}}{\alpha_{17}}, a_{30} = -\frac{7.95335\times 10^{-10}}{\alpha_{19}},
\end{aligned}$$

Number of coefficients can found easily by using MATHEMATICA of any order with very little effort. For example

$$a_{1500} = -\frac{4.1809830\times 10^{-335}}{\alpha^{999}} \tag{5.21}$$

### 5.2.1 Increasing Sequence

To find $\alpha$ we solved equation $y(1) = 0$ where,

$$y(x) = \sum_{n=0}^{\infty} a_{3n} x^{3n} \tag{5.22}$$

This implies

$$\alpha + \sum_{n=1}^{\infty} \frac{a_{3n}}{\alpha^{2n-1}} = 0 \tag{5.23}$$

Hashim approximated $\alpha \approx 0.463662$ truncating the series after seven terms. He further improved this value to $\alpha \approx 0.466799$ by approximating the series for up to $x^{21}$ using Pade' approximation. We can attempt to further improve this value as we have found arbitrary number of terms. From Eq. (5.23) let

$$F_k(\alpha) = \alpha + \sum_{n=1}^{k} \frac{a_{3n}}{\alpha^{2n-1}} = 0, k = 1, 2, 3... \tag{5.24}$$

Let $\alpha_k$denotes the root of the equation

$$F_k(\alpha) = 0$$

Table below represents approximated value of $\alpha$ for different number of terms

### 5.2.2 Decreasing Sequence

Expression for $\frac{1}{y}$ found by Wang and Hashim [14] is

$$\frac{1}{y} = \frac{1}{\alpha} + \frac{x^3}{6\alpha^3} + \frac{x^6}{30\alpha^5} + \frac{x^9}{144\alpha^7} + \frac{2099}{1425600}\frac{x^{12}}{\alpha^9} + ... = \frac{1}{\alpha} + \sum_{n=1}^{\infty} c_{3n} \frac{x^{3n}}{\alpha^{2n+1}} \tag{5.25}$$

Solution $y(x)$ of Eq. (69) is related to Eq. (95) in following manner

$$y(x) = \alpha - \int_0^x \int_0^x \frac{x}{y} dx \tag{5.26}$$

| k | $\alpha_k$ | k | $\alpha_k$ |
|---|---|---|---|
| 1 | 0.408248 | 15 | 0.467293 |
| 2 | 0.441743 | 20 | 0.46798 |
| 3 | 0.452576 | 30 | 0.468611 |
| 4 | 0.457674 | 50 | 0.469066 |
| 5 | 0.460566 | 100 | 0.469365 |
| 7 | 0.463662 | 300 | 0.469535 |

**Table 5.1: Values of $\alpha_k$**

Using Eq. (5.25) in (5.26) we get,

$$y(x) = \alpha - \frac{x^3}{6\alpha} - \sum_{n=1}^{\infty} c_{3n} \frac{x^{3n+3}}{(3n+3)(3n+2)\alpha^{2n+1}} = \alpha - \sum_{n=0}^{\infty} \frac{c_{3n}x^{3n+3}}{(3n+3)(3n+2)\alpha^{2n+1}} \quad (5.27)$$

$$= \alpha - \sum_{n=1}^{\infty} \frac{c_{3n-3}x^{3n}}{3n(3n-1)\alpha^{2n-1}} \quad (5.28)$$

Comparison of Eq. (5.22) and (5.28) we get,

$$c_{3n-3} = -3n(3n-1)a_{3n} \quad (5.29)$$

$n \geq 1$ Putting $x = 1$ in (5.28) and truncating after $k$ terms we obtained

$$0 = y(1) = \alpha - \sum_{n=1}^{k} \frac{c_{3n-3}}{3n(3n-1)\alpha_{2n-1}} - \frac{c_{3k}}{(3k+3)(3k+2)\alpha^{2k+1}} \cdots \quad (5.30)$$

For large $k$ we have approximation,

$$\frac{c_{3k+3}}{c_{3k}} = \alpha^2 \frac{c_{3k+6}}{c_{3k}} = \alpha^4 \quad (5.31)$$

Hence equation (5.30) becomes

$$0 = \alpha - \sum_{n=1}^{k} \frac{c_{3n-3}}{3n(3n-1)\alpha^{2n-1}} - \frac{c_{3k}}{\alpha^{2k+1}} \sum_{n=k}^{\infty} \frac{1}{(3n+3)(3n+2)}.$$

This implies,

$$\sum_{n=k}^{\infty} \frac{1}{(3n+3)(3n+2)} = \frac{1}{(3k+2)},$$

$$0 = \alpha - \sum_{n=1}^{k} \frac{c_{3n-3}}{3n(3n-1)\alpha^{2n-1}} - \frac{c_{3k}}{(3k+2)\alpha^{2k+1}}.$$

Using equation (5.24) we get

$$F_k(\alpha) + \frac{3k+3}{\alpha^{2k+1}} a_{3k+3} = 0. \tag{5.32}$$

Left side of above equation

$$G_k(\alpha) = F_k(\alpha) + \frac{3k+3}{\alpha^{2k+1}} a_{3k+3}. \tag{5.33}$$

Let $\alpha^{(k)}$ denotes roots of above equation. Below is table for roots of above equation.

| k | $\alpha^{(k)}$ | k | $\alpha^{(k)}$ |
|---|---|---|---|
| 5 | 0.482463 | 500 | 0.469665 |
| 7 | 0.478413 | 900 | 0.469634 |
| 15 | 0.473268 | 1000 | 0.4696298 |
| 50 | 0.470505 | 1500 | 0.4696190 |
| 100 | 0.470006 | 1800 | 0.4696155 |
| 200 | 0.469784 | 2100 | 0.4696130 |
| 300 | 0.469716 | 4000 | 0.4696064 |

**Table 5.2: Values of $\alpha^{(k)}$**

From two tables, Table (5.1) and Table (5.2) it is clear that $0.469535 < \alpha < 0.469606$.

# Chapter 6

# Method of Asymptotic Approximants

When the infinite power series solution of a problem diverges and/or we only know a finite number of power series coefficients, a method is devised to find solution of such problems called Method of Asymptotic Approximants [7]. Many scientists have developed techniques to accelerate the convergence of divergent or slowly converging series. For instance Euler's Transformation,Conformal Mapping and Pade Approximation. Among all, Pade Approximation is the most frequently used method to approximate functions. Pade approximation is a method in which we approximate a function by a rational function of a given order. It's a quotient of two polynomials $P_N(x)$ and $Q_M(x)$ of degrees N and M on a small interval $[a, b]$. The method of asymptotic approximant has resemblance with pade approximation technique.

**Definition** Given a power series representation of some function $f(x)$:

$$f = \sum_{n=0}^{\infty} a_n (x - x_0)^n. \tag{6.1}$$

and an asymptotic behaviour,

$$f \sim C f_a(x) \quad as \quad x \to x_a,$$

where C is a constant, an *asymptotic approximant* is any function $F_A(x)$ that may be expressed analytically in closed form and that satisfies the following three properties:

1. *The* $N-$ *term Taylor expansion of* $f_A$ *about* $x_0$ *is identical to the N-term truncation of* (6.1).

2. $\lim_{x \to x_a}(F_A/f_a) = constatnt\ for\ any\ N.$

3. *The sequence of approximants converges for increasing* $N$.

Choosing an approximant wisely according to the above conditions of definition will give us a uniformly converging sequence incorporating the correct asymptotic behaviour. Approximants with trivially generated unknown coefficients are considered preferable in order to simplify the process, which is not the case in *Pade Approximation.* This technique deals with the simple asymptotic forms possible while preserving the desired accuracy and precision of the solution. Approximants being the closed form functions lead us to the prediction of physically useful properties. This feature of approximant makes it a lot easier to solve numerous physical problems i.e *Sakiadis Boundry Layer problem*, *The Blaisus Boundry Layer Problem* and *The Flierl Petviashvili Monopole.*

As shown in previous section by similarity transformation the equations of flow and continuity equation lead to the nonlinear boundary value problem Eq.(5.1) in $f(\eta)$ with boundary conditions Eq. (5.2).

Let us consider a power series solution to *Blaisus Problem*

$$y(x) = \sum_{n=0}^{\infty} a_n \eta^n. \tag{6.2}$$

Substituting Eq.(6.2) into Eq.(5.1) using boundary conditions Eq.(5.2), we get the following recurrence relation,

$$a_{m+3} = -\frac{\sum_{k=0}^{m} a_{k+2} a_{m-k}(k+1)(k+2)}{2(m+1)(m+2)(m+3)} \tag{6.3}$$

First two terms of the power series solution are zero due to first two boundary conditions at $\eta = 0$, while third term is non-zero. Next two terms are again zero and this pattern repeats throughout the solution:

$$\frac{1}{2}\kappa\eta^2 - \frac{1}{240}\kappa^2\eta^5 + \frac{11}{161280}\kappa^3\eta^8 - \frac{5}{4257792}\kappa^4\eta^{11} + ... \tag{6.4}$$

Where $\kappa = f''(0)$ is wall shear parameter can be obtained from the solution of Eq.(5.1). In this paper, $\kappa$ is predicted by asymptotic approximants.

According to Blaisus's [10] observation

$$f \sim \eta\text{+B} \qquad \text{x}\rightarrow \infty$$

$$f - \eta \sim\text{B} \qquad \text{x}\rightarrow \infty$$

$$\lim_{x\to\infty} (f - \eta) \equiv B \tag{6.5}$$

The Approximant below:

$$f_A = \eta + B - B(1 + \sum_{n=1}^{N} A_n \eta^n)^{-1} \tag{6.6}$$

is quite appropriate for *Blaisus Problem*, as it agrees with boundary conditions Eq.(5.2) and first order asymptotic expansion Eq.(6.5).

As mentioned above unknown physical properties can be predicted by approximants. The wall shear parameter $\kappa$ and other unknown coefficients $A_0, A_1 ... A_N$ and $B$, by equating the finite order Taylor Expansion of Approximant Eq.(6.6) about $\eta = 0$ and $N$-*term* truncation of assumed power series solution Eq.(6.2) i.e

$$\eta + B - B(1 + \sum_{n=1}^{N} A_n \eta^n)^{-1} = \sum_{n=0}^{N} a_n \eta^n$$

Rearrange and simplifly,

$$\begin{aligned} -B(1 + \sum_{n=1}^{N} A_n \eta^n)^{-1} &= -\eta - B + \sum_{n=0}^{N} a_n \eta^n, \\ \Longrightarrow (1 + \sum_{n=1}^{N} A_n \eta^n)^{-1} &= 1 + \frac{\eta}{B} - \frac{1}{B}(\sum_{n=0}^{N} a_n \eta^n), \\ \Longrightarrow \frac{1}{(1 + A_1\eta + A_2\eta^2 + ... + A_N\eta^N)} &= 1 + \frac{\eta}{B} - \frac{1}{B}(\sum_{n=0}^{N} a_n \eta^n), \end{aligned}$$

$$\Longrightarrow 1 = (1 + A_1\eta + A_2\eta^2 + ... + A_N\eta^N)(1 + \frac{\eta}{B} - \frac{1}{B}(\sum_{n=0}^{N} a_n \eta^n)),$$

By equating coefficients we get,

$$A_{n>0} = \frac{1}{B} \sum_{j=1}^{n} \tilde{a}_j A_{n-j}.$$

where $\tilde{a}_1 = -1$ , $\tilde{a}_{j>1} = a_j$ and $A_0 = 1$

Let

$$A_N = A_{N-1} = 0$$

Implies

$$\sum_{j=1}^{N} a_j A_{N-j} = 0,$$
$$\sum_{j=1}^{N-1} a_j A_{N-1-j} = 0$$

Which further leads to two nonlinear equations. These equations are solved simultaneously to find roots $\kappa$ and $B$. For various values of $N$, corresponding values of $\kappa$ and $B$ are given below.

| N | $\kappa$ | B |
|---|---|---|
| 5 | 0.125451 | -6.0895 |
| 8 | 0.265756 | -2.36178 |
| 11 | 0.309679 | -1.91379 |
| 14 | 0.323983 | -1.79089 |
| 17 | 0.329037 | -1.74806 |
| 20 | 0.330902 | -1.73173 |
| 23 | 0.331611 | -1.72523 |
| 26 | 0.331884 | -1.7226 |
| 30 | 0.332008 | -1.72134 |

**Table 6.1: Values of parameter $k$ and constant $B$**

# Chapter 7

# Conclusions

We have seen that by using symmetry of differential equation, we can solve a related boundary value problem in which we assume $f''(0) = 1$, and use its solution to accurately estimate $f''(0)$ for the Blasius problem. The technique to find Weyl's solution for large values of $\eta$ maybe helpful for finding asymptotic expressions for similar non-linear boundary value problems. Series solution for the Blasius problem diverges. This indicates existance of a singularity on the negative real axis. The sigularity can be located by using the Domb-Sykes technique [11]. This technique is fairly general and has been applied to several important problems. Method of asymptotic approximants is very useful in problems where more than one parameters need to be evaluated.

Methods discussed in this research work can be further extended to similar non-linear boundary value problems such as Sakiadis problem. These problems are broadly used in fluid dynamics. Techniques developed here can be used to elucidate various features of fluid flow.

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