Fractional Differential Equations

Oscillation Theory and Numerical Solutions

by

Arjumand Seemab

Supervised by

Dr. Mujeeb ur Rehman

School of Natural Sciences,

National University of Sciences and Technology, Islamabad

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To my Parents

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Abstract

Since the 1980s the oscillation theory has been developing quite rapidly. Investigation of the oscillation and non oscillation of delay differential equations gained tremendous success.

In this thesis we discuss the theory of oscillation of differential equations. We have given basic definitions and important results for the oscillation of ordinary differential equations. Necessary and sufficient conditions for the oscillation of the solutions of delay differential equations have been given. Moreover, we have pointed out some major differences between ordinary and delay differential equations. Some applications of delay differential equations are also given and the method of steps for solving delay differential equations has been discussed.

We have also discussed is the oscillatory behavior of solutions of impulsive differential equations of first and second order. In this thesis, the original contribution of the author is the oscillation of class of fractional differential equations with impulsive conditions and fractional-order delay differential equations with constant coefficients.

The iterative methods such as Daftardar and Jafari method [DJ-method] and Iterative Laplace Transform method [ILTM] for solving fractional differential equations with initial and boundary conditions have been discussed.

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Chapter 1

Introduction

Fractional Calculus is a field of mathematics that came out of the traditional definitions of the calculus integral and derivative operators in almost the similar way as fractional exponents is the concept coming from integer order exponents. When one thinks about the physical meaning of the exponent, according to general concept exponents are short notation for repeated multiplication of a numerical value. This concept in itself is easy to understand. However, this physical definition can become confused when we take non integer value of exponents. It is easy to verify that $u^4 = u.u.u.u$, how can we describe the physical meaning of $u^{3/5}$, or of the transcendental exponent $u^{\pi/2}$. One cannot conceive what it might be like to multiply a number or a value by itself 3/5 times, or $\pi/2$ times, while these expressions have a definite value for any value u, can be calculated by infinite series expansion, or more easily, by calculator. Now, similarly consider the integral and derivative of non-integer order. Although they are more complex concepts, it is still easy enough to physically explain their meanings. Given the satisfaction of some conditions (e.g. function continuity), evaluating n integrations can become as logical as multiplication. But an interested mind can ask, ”what if n were not restricted to an integer value?”. Again, the physical meaning looks complicated, but this report will show, fractional calculus comes out naturally from the traditional definitions of integrals and derivatives. And just as exponents of non integer values, such as the square root can find their way into numerous equations and applications, it will become understandable that integrations of order 1/2 and beyond have practical importance in many real life problems.

History and mathematical background: Most of the authors will give a fixed date for the birthday of ”Fractional Calculus”. In a letter (dated September 30th, 1695) L’Hopital asked Leibniz about a particular notation he had used in his publications for the nth-derivative of the linear function $f(x) = x$, $\frac{D^n x}{Dx^n}$. L’Hopital asked Leibniz, "what would the result be if $n = 1/2$." Leibniz’s response: "An apparent paradox, from which one day useful consequences will be drawn." This was the birth of fractional calculus.

Following L’Hopital’s and Liebniz’s first investigation, fractional calculus was mainly a study for the brilliant minds in mathematics. Fourier(21 March 1768 - 16 May 1830), Euler(15 April 1707 - 18 September 1783), Laplace(23 March 1749 - 5 March 1827) are among the many that played their active role in fractional calculus and the mathematical consequences. Many mathematicians using their own notation and methodology found the definitions that gives a justification for the concept of a non-integer order integrals and derivatives. The most popular definitions that have been used in the world of fractional calculus are the Riemann-Liouville and Caputo definitions. We have discussed Riemann-Liouville and caputo definition in detail in this thesis. Most of the mathematical theory used for the study of fractional calculus was developed before the 20th century. However, in the past 100 years most interesting fields in engineering and scientific applications have been found. Caputo redeveloped the more classic definition of the Riemann-Liouville fractional derivative in order to use initial conditions of integer order to solve his fractional order differential equations. Recently in 1996, Kolowankar redeveloped again, the Riemann-Liouville fractional derivative to differentiate nowhere differentiable fractal functions. Leibniz's response has proven almost half right. A large number of applications and physical use of fractional calculus within the 20th century have been discovered. However, these applications and the mathematical background associated with fractional calculus are very different from paradoxical. While the physical meaning may be impossible to understand, the accuracy of these definitions is no more accurate than of integer order analogue.

In this chapter, we remind some basic definitions and results. The Gamma function, fractional integrals and derivatives, properties of fractional derivatives and integrals, The Laplace Transform and useful results for delay differential equations are discussed. We will provide examples to understand the given results where necessary.

In Chapter 2, we study the oscillation criteria of ordinary differential equations. In Chapter 3, we discuss some results for the oscillatory behavior of the delay differential equations. In Chapter 4, we will study oscillation of solution of impulsive differential equations. In Chapter 5, we will study oscillation of solution of fractional differential equations and will establish some new results. In chapter 6, we shall use Daftardar and Jafari method [DJ-method] for solving differential equations, fractional order differential equations and integral equations.

1.1 The Gamma function

Factorial function is defined as $m! = m(m-1)(m-2)\dots(2)(1)$ for all $m \in \mathbb{Z}^+$. But to include non-integer values, how can we extend factorial function? This problem is originally posed by Daniel Bernoulli and Christian Goldbach in the 1720 and it was eventually solved by Leonhard Euler in 1729. $m!$ is communicated by Euler as both an infinite sum and an integral, in his famous paper [1] "De progressionibus transcendentibus seu quarum termini generales algebraice dari nequeunt".

After a year, in 1730, James Stirling discovered a formula to find approximate value of $m!$ as m become very large. Which is known as Stirling's Formula and was purified right through the years.

From 1730 to the present day, there have been made significant progress. Nearby 1812, Carl Friedrich Gauss recast Euler's product formula as a limit of a function, also he considered the factorial of a complex number. After a few years, Karl Weierstras gave another representation, which is now known as the gamma function.

The gamma function $\Gamma(x)$ [2] plays an important role in the theory of differentiation and integration. $\Gamma(x) : (0, \infty) \to \mathbb{R}$ is defined by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt, \ x > 0. \tag{1.1.1}$$

The integral defining Gamma function is uniformly convergent for all x in $[a,b]$ where $0 < a \leq b < \infty$, and hence $\Gamma(x)$ is a continuous function for all $x > 0$.

Relation between Gamma function and factorial

$$\Gamma(n) = (n-1)!, \quad n > 0.$$

Integration of (1.1.1) by parts yields

$$\Gamma(x+1) = x\Gamma(x), \; x > 0. \tag{1.1.2}$$

From (1.1.2) we have

$$\Gamma(x) = \frac{\Gamma(x+1)}{x}, \; x \neq 0, \; x > -1.$$

Similarly

$$\Gamma(x) = \frac{\Gamma(x+n)}{x(x+1)(x+2)\dots(x+n-1)}, \qquad n \in \mathbb{N}.$$

Thus $\Gamma(x)$ is defined for all $x \in \mathbb{R}$ except $x = 0, -1, -2, \dots$

For $x = 1$, (1.1.1) implies

$$\Gamma(1) = \int_0^\infty e^{-t} dt = 1$$

and

$$\begin{aligned}
&\Gamma(2) = 1\Gamma(1) = 1!, \\
&\Gamma(3) = 2\Gamma(2) = 2.1 = 2!, \\
&\Gamma(4) = 3\Gamma(3) = 3.2! = 3!, \\
&\vdots \\
&\Gamma(n) = (n-1)!.
\end{aligned}$$

So, the Gamma function generalizes the factorial function.

In particular for the Gamma of rational numbers, we put $t = u^2$ in (1.1.1)

$$\Gamma(x) = 2\int_0^\infty e^{-u^2} u^{2x-1} du, \; Re(x) > 0.$$

Let $x = \frac{1}{2}$

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{-1/2} e^{-t} dt.$$

Substitute $t = u^2$, then $dt = 2udu$, as $\; t \to 0, \; u \to 0$, and as $\; t \to \infty, \; u \to \infty$.

So

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-u^2} du.$$

But $\displaystyle\int_0^\infty e^{-u^2} du = \int_0^\infty e^{-v^2} dv.$

$$\left(\Gamma\left(\frac{1}{2}\right)\right)^2 = 4\int_0^\infty \int_0^\infty e^{-(u^2+v^2)} dudv.$$

Let $u = r\cos\theta,\ v = r\sin\theta$.
Then $\left(\Gamma\left(\frac{1}{2}\right)\right)^2 = 4\int_0^{\pi/2}\int_0^{\infty} e^{-r^2} r dr d\theta = 2\int_0^{\pi/2} d\theta = \pi$. So $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.
Similarly

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}/2$$

and

$$\Gamma(-3/2) = \Gamma(-1/2)/(-3/2) = \left[\Gamma\left(\frac{1}{2}\right)/\left(\frac{-1}{2}\right)\right]/(\frac{-3}{2}) = \frac{4}{3}\sqrt{\pi}.$$

1.2 Fractional derivatives and integrals

In this section we give some definitions and important results for fractional derivatives and integrals [3] as follows:

Riemann-Liouville fractional integral

Definition 1.2.1. The Riemann-Liouville integral I_a^α with fractional order $\alpha \in \mathbb{R}^+$ of function $f \in L_1[a,b]$, for $a \le x \le b$, is defined as

$$I_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-s)^{\alpha-1} f(s) ds.$$

For $\alpha = 0$, we set $I_a^0 := I$, identity operator.

For some $\beta > -1$ and $\alpha > 0$, the Riemann Liouville integral of $f(x) = (x-a)^\beta$ is given by

$$I_a^\alpha (x-a)^\beta = \int_a^x \frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)}(s-a)^\beta ds.$$

Let $t = \frac{s-a}{x-a}$, as $s \to a,\ t \to 0$, and as $s \to x,\ t \to 1$, so $(x-a)dt = ds$ and $s = a + t(x-a)$

$$I_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_0^1 [(x-a)(1-t)]^{\alpha-1}[t(x-a)]^\beta (x-a)dt,$$

$$I_a^\alpha f(x) = \frac{(x-a)^{\alpha+\beta}}{\Gamma(\alpha)}\int_0^1 (1-t)^{\alpha-1} t^\beta dt.$$

Since $\int_0^1 t^{\alpha-1}(1-t)^{\beta-1} dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$

$$I_a^\alpha f(x) = \frac{\Gamma(\beta+1)\Gamma(\alpha)}{\Gamma(\alpha+\beta+1)}\frac{(x-a)^{\alpha+\beta}}{\Gamma(\alpha)},$$

$$= \frac{\Gamma(\beta+1)}{(\alpha+\beta+1)}(x-a)^{\alpha+\beta}.$$

In particular $I^{\frac{1}{2}} x^{\frac{3}{2}} = \frac{\Gamma(1+\frac{3}{2})}{\Gamma(1+\frac{3}{2}+\frac{1}{2})} x^{\frac{1}{2}+\frac{3}{2}} = \frac{3\sqrt{\pi}}{8}$.
Riemann Liouville fractional integral has the following properties:

(i) (Semigroup property) Let $\alpha, \beta \geq 0$ and $f \in L_1[a,b]$, then,

$$I_a^\alpha I_a^\beta f = I_a^\beta I_a^\alpha f = I_a^{\alpha+\beta} f$$

holds almost everywhere on $[a,b]$. If additionally $f \in \mathcal{C}[a,b]$ or $\alpha + \beta \geq 1$, then the identity holds everywhere on $[a,b]$.

(ii) Let f and g be two functions defined on $[a,b]$ such that $I_a^\alpha f and I_a^\alpha g$ exist almost everywhere. Fractional Riemann Liouville integrals are linear almost everywhere i.e. for $c, d \in \mathbb{R}$,

$$I_a^\alpha (cf(x) + dg(x)) = cI_a^\alpha f(x) + dI_a^\alpha g(x).$$

Another important property of Riemann Liouville integral is given in following theorem.

Theorem 1.2.2. *Let $f \in L_1[a,b]$ and $\alpha > 0$. Then the integral $I_a^\alpha f(x)$ exists for almost every $x \in [a,b]$. Moreover, the function $I_a^\alpha f$ itself is also an element of $L_1[a,b]$.*

Riemann-Liouville fractional derivative: Let f be continuous on $[a,b]$ and define $F : [a,b] \to \mathbb{R}$ as

$$F(t) = \int_a^t f(s)ds.$$

Then, F is differentiable and

$$\frac{d}{dt} F(t) = f(t),$$

or

$$f(t) = \frac{d}{dt}\Big(\int_a^x f(s)ds\Big),$$

or

$$f = DI_a f,$$

where $D = \frac{d}{dt}$. Repeated application give $f = D^n I_a^n f$, where $D^n = \frac{d^n}{dt^n},\ n = 0, 1, 2, \ldots$
Replacing n by $m - n$, with $n < m$ and applying D^n on both sides, we have

$$D^n f = D^n D^{m-n} I_a^{m-n} f = D^m I_a^{m-n} f.$$

This relation is still valid if n is replaced by $\alpha \in \mathbb{R}^+$, provided $m - \alpha > 0$, that is,

$$D_a^\alpha f = D^m I_a^{m-\alpha} f = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_a^t f(s)(t-s)^{m-\alpha-1} ds,$$

where $m - 1 \leq \alpha < m \in \mathbb{Z}^+$.

Definition 1.2.3. The Riemann Liouville derivative D_a^α with fractional order $\alpha \in \mathbb{R}^+$ of f with an integer m such that $m - 1 < \alpha \leq m$ is defined as

$$D_a^\alpha f(x) = \begin{cases} \frac{d^m}{dt^m}\left[\frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f(t)dt\right], & m-1 < \alpha < m, \\ \frac{d^m}{dt^m} f(x), \quad \alpha = m. \end{cases}$$

Example 1.2.4. The fractional derivative of order $\alpha > 0$ of the function $f(x) = x^{\beta}$, $\beta > -1$ is given by

$$D_a^{\alpha} x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} x^{\beta-\alpha}.$$

In particular,

$$D^{\frac{1}{2}} x^{\frac{3}{2}} = \frac{3\sqrt{\pi}}{4} x.$$

Definition 1.2.5. By A^m or $A^m[a,b]$ we denote the set of functions with absolutely continuous $(m-1)^{st}$ derivative, i.e, the functions f for which there exists (almost everywhere) a function $g \in L_1[a,b]$ such that

$$f^{m-1}(x) = f^{m-1}(a) + \int_a^x g(t)dt.$$

In this case, we call g the (generalized) n^{th} derivative of f, and we simply write $g = f^m$.

In general, the fractional integral operator do not commute with fractional derivative.

Theorem 1.2.6. [4] *Let $\alpha > 0$ and $m = \lfloor \alpha \rfloor + 1$. Assume that f is such that $I_a^{m-\alpha} f \in A^m[a,b]$. Then,*

$$I_a^{\alpha} D_a^{\alpha} f(x) = f(x) - \sum_{k=0}^{m-1} \frac{(x-a)^{\alpha-k-1}}{\Gamma(\alpha-k)} \lim_{s\to a^+} D^{m-k-1} I_a^{m-\alpha} f(s).$$

Specifically for $0 < \alpha < 1$ we have,

$$I_a^{\alpha} D_a^{\alpha} f(x) = f(x) - \frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)} \lim_{s\to a^+} I_a^{1-\alpha} f(s).$$

Theorem 1.2.7. [5] *Let $\alpha, \beta \geq 0$ and $f \in L_1[a,b]$. Then, $D_a^{\beta} I_a^{\alpha} f = I_a^{\alpha-\beta} f$. In particular, $D_a^{\alpha} I_a^{\alpha} f = f$.*

Some other properties of Riemann fractional derivative are as under:

(i) Let $\alpha, \beta \geq 0$ and $\phi \in L_1[a,b]$ and $f = I_a^{\alpha+\beta}\phi$. Then,

$$D_a^{\alpha} D_a^{\beta} f = D_a^{\alpha+\beta} f.$$

Note that an unconditional semigroup property of fractional differentiation in the Riemann Liouville sense does not hold.
It is possible to have

$$D_a^{\alpha} D_a^{\beta} f = D_a^{\beta} D_a^{\alpha} f \neq D_a^{\alpha+\beta} f,$$
$$D_a^{\alpha} D_a^{\beta} f \neq D_a^{\beta} D_a^{\alpha} f = D_a^{\alpha+\beta} f.$$

For example, let $f(x) = x^{-1/2}$ and $\alpha = \beta = \frac{1}{2}$. Then $D_0^{\alpha} f(x) = D_0^{\beta} f(x) = 0$, and hence also $D_0^{\alpha} D_0^{\beta} f(x) = 0$, but $D_0^{\alpha+\beta} f(x) = D^1 f(x) = -(2x^{3/2})^{-1}$.
And if we let $f(x) = x^{1/2}$, $\alpha = 1/2$ and $\beta = 3/2$. Then $D_0^{\alpha} f(x) = \sqrt{\pi/2}$ and $D_0^{\beta} f(x) = 0$. This implies $D_0^{\alpha} D_0^{\beta} f(x) = 0$ but $D_0^{\beta} D_0^{\alpha} f(x) = -x^{-3/2}/4 = D^2 f(x) = D_0^{\alpha+\beta} f(x)$.

(ii) Let f and g be two functions defined on $[a,b]$ such that $D_a^{\alpha} f$, $D_a^{\alpha} g$ exist almost everywhere. Fractional derivatives are linear almost everywhere i.e. for $c, d \in \mathbb{R}$,

$$D_a^{\alpha}(cf(x) + dg(x)) = cD_0^{\alpha} f(x) + dD_0^{\alpha} g(x).$$

1.3 The Caputo fractional derivative

The Riemann-Liouville derivatives have many complications when one try to model real-world problems with fractional differential equations. So, we shall therefore discuss Caputo derivative. As when these two ideas are compared, Caputo derivative become very suitable to such tasks. It was introduced by M. Caputo in his 1967 paper. In contrast to the Riemann Liouville fractional derivative, when solving differential equations using Caputo's definition, it is not necessary to define the fractional order initial conditions.

Definition 1.3.1. [6] The Caputo derivative ${}^cD_a^\alpha$ of fractional order $\alpha \in \mathbb{R}^+$ of function $f \in C^{m+1}[a,b]$, is given as

$${}^cD_a^\alpha f(t) = I^{m-\alpha}D^m f(x) = \frac{1}{\Gamma(m-\alpha)}\int_a^t (t-s)^{m-\alpha-1} f^{(m)}(s)(t-s)^{1+\alpha-m} ds,$$

where $m-1 < \alpha \leq m \in \mathbb{Z}^+$.

Definition 1.3.2. The polynomial

$$T_n[f(x), x_0] = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

is called the nth degree taylor polynomial of $f(x)$, centered at $x = x_0$.

Theorem 1.3.3. [4] *Let $\alpha \geq 0$ and $m = \lceil \alpha \rceil$. Also suppose that $f \in A^m[a,b]$. Then,*

$${}^cD_a^\alpha f = D_a^\alpha [f - T_{m-1}[f;a]] \tag{1.3.1}$$

almost everywhere. Here $T_{m-1}[f;a]$ denotes the Taylor polynomial of degree $m-1$ for the function f, centered at a, in the case $m = 0$, we define $T_{m-1}[f;a] = 0$.

Note here the expression on the right of the equation (1.3.1) exists if $D_a^\alpha f$ exists and f possesses $m-1$ derivatives at a, the latter condition gives surety to existence of Taylor polynomial. The operator ${}^cD_a^\alpha$ is the Caputo differential operator of order α. Theorem 1.3.3 shows the relation of the Riemann Liouville and the Caputo derivatives.
We note that $m = n$ for $n \in \mathbb{N}$ and hence

$${}^cD_a^n f = D_a^n [f - T_{n-1}[f;a]] = D^n f - D^n(T_{n-1}[f;a]) = D^n f,$$

because $T_{n-1}[f;a]$ is a polynomial of degree $n-1$ that is annihilated by the classical operator D^n.

Example 1.3.4. Let $f(x) = (x-a)^\beta$ for some $\beta \geq 0$. Then,

$${}^cD_a^n f(x) = \begin{cases} 0, \quad \beta \in \{0,1,2,\ldots,m-1\}, \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-n)}(x-a)^{\beta-n}, \quad \beta \in \mathbb{N}, \beta \geq m, \\ \qquad\qquad\qquad\qquad \mathit{or} \ \ \beta \ \ \mathit{not} \ \ \mathit{in} \ \ \mathbb{N} \ \ \mathit{and} \ \ \beta > m-1. \end{cases}$$

Lemma 1.3.5. *Let $\alpha \geq 0$ and $m = \lceil \alpha \rceil$. Assume that f is such that both ${}^cD_a^\alpha$ and D_a^α exist. Then,*

$$ {}^cD_a^\alpha f(x) = D_a^\alpha f(x) - \sum_{k=0}^{m-1} \frac{D^k f(a)}{\Gamma(k-\alpha+1)}(x-a)^{k-\alpha}. $$

Theorem 1.3.6. *If f is continuous and $\alpha \geq 0$, then*

$$ {}^cD_a^\alpha I_a^\alpha f = f. $$

Theorem 1.3.7. *Assume that $\alpha \geq 0$, $m = \lceil \alpha \rceil$, and $f \in A^m[a,b]$. Then*

$$ I_a^{\alpha c}D_a^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} \frac{D^k f(a)}{k!}(x-a)^k. $$

Comparison of the Riemann Liouville and the Caputo derivatives:

(i) The Caputo derivative of a constant c is zero but Riemann Liouville derivative is not. That is,

$$ {}^cD_a^\alpha(c) = 0, \ D_a^\alpha(c) \neq 0. $$

(ii) Let $\alpha \geq 0$ and $m = \lceil \alpha \rceil$. Assume that f is such that both ${}^cD_a^\alpha$ and D_a^α exist. Then, ${}^cD_a^\alpha f = D_a^\alpha f$, holds if and only if f has an m-fold zero at a, i.e.
$D^k f(a) = 0$ for $k = 0, 1. \ldots, m-1$.

1.4 Mittage-Leffler function

During the last two decades Mittag-Leffler function [7] has come into prominence after about nine decades of its discovery by a Swedish Mathematician G.M. Mittag-Leffler, due vast potential of applications in solving the problems of physical, biological, engineering and earth sciences.

Definition 1.4.1. Let $\alpha > 0$ the one parameter Mittage Leffler function E_α is defined by

$$ E_\alpha(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(j\alpha+1)}, $$

whenever the series converges is called the Mittage Leffler function of order α.
We immediately notice that

$$ E_1(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(j+1)} = \exp(z) $$

is just the well known exponential function.

The two parameter Mittage Leffler function is defined as follows.

Definition 1.4.2. Let $\alpha, \beta > 0$. The function $E_{\alpha,\beta}$ defined by

$$ E_{\alpha,\beta} = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(j\alpha+\beta)}, $$

with parameters α and β.

Remark 1.4.3. It is evident that the one parameter Mittage Leffler functions may be defined in terms of their two parameter counterparts via the relation $E_\alpha(z) = E_{\alpha,1}(z)$.

Theorem 1.4.4. *[4] Consider the two parameter Mittage Leffler function $E_{\alpha,\beta}$ for some $\alpha, \beta > 0$. The power series defining $E_{\alpha,\beta}(z)$ is convergent for all $z \in \mathbb{C}$. In other words $E_{\alpha,\beta}$ is an entire function.*

Example 1.4.5. Let $f(x) = \cos(x)$.Applying fractional derivative of order $\alpha > 0$, we have

$$\begin{aligned}
D_0^\alpha f(x) &= D_0^\alpha(\cos(x)) \\
&= D_0^\alpha \left[\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} \right] \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \frac{x^{2k-\alpha}(2k)!}{(2k-\alpha)!} \\
&= x^{-\alpha} \sum_{k=0}^{\infty} \frac{(-1)^k (x)^{2k}}{\Gamma(2k-\alpha+1)}.
\end{aligned}$$

By definition of Mittage Leffler function, we have

$$D_0^\alpha f(x) = x^{-\alpha} E_{2,1-\alpha}(-x^2).$$

1.5 The Laplace transform

Let $u : [0, \infty) \to \mathbb{R}$ be a real valued function. The Laplace transform of $u(t)$ denoted by $U(s)$ is given by the improper integral

$$U(s) = \int_0^\infty e^{-st} u(t) dt. \tag{1.5.1}$$

The Laplace transform of a function $u'(t)$ for $t \geq 0$ is

$$\mathcal{L}\{u'(t)\} = \int_0^\infty e^{-st} u'(t) dt = sU(s) - u_0.$$

Similarly, the Laplace transform of $u(t-\tau)$ is given by

$$\mathcal{L}\{u(t-\tau)\} = \int_0^\infty e^{-st} u(t-\tau) dt = e^{-s\tau} U(s) + e^{-s\tau} \int_{-\tau}^0 e^{-st} u(t) dt,$$

by substituting $x = t - \tau$.
Consider the Laplace transform of the two formulations of the fractional derivative for $0 < \alpha < 1$

$$\mathcal{L}(D_0^\alpha f(t)) = sF(s) - D_0 f(t)|_0,$$

$$\mathcal{L}({}^c D_0^\alpha f(t)) = sF(s) - f(t)|_0.$$

Definition 1.5.1. (**Abscissa of convergence**) For given function $u(t)$, the integral in (1.5.1) can behave one of the following three ways,

(i) it converges for all complex numbers s,

(ii) it diverges for all complex numbers s,

(iii) there exists a real number such that the integral (1.5.1) converges for all s with $\mathrm{Re}(s) > \sigma_0$ and diverges for all s with $Re\ s \leq \sigma_0$.

When (iii) holds, the number σ_0 is known as the abscissa of convergence of $U(s)$.
When (i) holds, the abscissa of convergence of $U(s)$ is $\sigma_0 = -\infty$, and if (ii) holds then the abscissa of convergence of $U(s)$ is $\sigma_0 = +\infty$.

Lemma 1.5.2. *Let $u \in \mathcal{C}[[0,\infty),\mathbb{R}]$, and let us assume that there exists positive constants M and μ such that*

$$|u(t)| \leq Me^{\mu t},$$

for $t \geq 0$. Then the abscissa of convergence σ_0 of the Laplace transform $U(s)$ of $u(t)$ satisfies, $\sigma_0 \leq \mu$. Furthermore, $U(s)$ exists and it is an analytic function of s for $Re(s) > \sigma_0$.

Theorem 1.5.3. [8] *Let $u \in \mathcal{C}[[0,\infty),\mathbb{R}]]$ and assume that the abscissa of convergence σ_0 of the Laplace transform $U(s)$ of $u(t)$ is finite. Then $U(s)$ has a singularity at the point $s = \sigma_0$. More precisely, there exists a sequence*

$$\alpha_n + i\beta_n$$

for $n = 1, 2, \ldots$ such that $\alpha_n \geq \alpha_0$ for $n \geq 1$, $\lim_{n\to\infty} \alpha_n = \sigma_0$, $\lim_{n\to\infty} \beta_n = 0$ and $\lim_{n\to\infty} |U(s_n)| = \infty$.

Chapter 2

Oscillation of solutions of differential equations

In this chapter, we will discuss the oscillation criteria of differential equations. Oscillation theory was produced by Jacques Charles Francois Sturm in 1836 while investigating the Sturm-Liouville problems.
Oscillation is the recurring variation, usually in time, of some measure about a central value (often a point of equilibrium) or between two or more different states. Most common examples include a swinging pendulum and alternating current power. The term vibration is sometimes used to mean a mechanical oscillation but is sometimes used in place of "oscillation". Oscillations occur not only in mechanical systems but also in dynamic systems in almost every area of science: for example the beating human heart, business cycles in economics, predator-prey population cycles in ecology, geothermal geysers in geology, vibrating strings in musical instruments and periodic firing of nerve cells in the brain.
In mathematics, oscillation quantifies the amount that a sequence or function tends to move between extremes. There are several related ideas: oscillation of a sequence of real numbers, oscillation of a real valued function at a point, and oscillation of a function on an open set [9].
Oscillating Sequences: Oscillating sequences are the sequences in which the values of the terms neither converge nor diverge, they move around. Common types are those which oscillate in sign,for example
$(-1)^n n$ begins with $-1, 2, -3, 4, -5$ etc.
$(-1)^{n-1}(2)^{n-1}$ begins with $1, -2, 4, -8, 16, -32$ etc.
Note that periodic sequences are also oscillating, since they neither converge nor diverge.
If (b_n) is a sequence of real numbers, then the oscillation of (b_n) is defined as the difference (possibly ∞) between the limit superior and limit inferior of

$$b_n : \omega(b_n) = \limsup b_n - \liminf b_n.$$

It is undefined if both are $+\infty$ or both are $-\infty$, that is, if the sequence tends to $+\infty$ or to $-\infty$. The oscillation is zero if and only if the sequence converges.
Oscillation of a function on an open set: Let g be a real-valued function of a real variable. The oscillation of g on an interval J in its domain is the difference between the supremum and infimum of

$$g : \omega_g(J) = \sup_{t \in J} g(x) - \inf_{t \in J} g(t).$$

More generally, if $g : X \to \mathbb{R}$ is a function on a topological space X (such as a metric space), then the oscillation of g on an open set U is $\omega_g(U) = \sup_{t\in U} g(t) - \inf_{t\in U} g(t)$.

Oscillation of a function at a point: The oscillation of a function g of a real variable at a point t_0 is defined as the limit as $\epsilon \to 0$ of the oscillation of g on an ϵ-neighborhood of

$$t_0 : \omega_g(t_0) = \lim_{\epsilon\to 0} \omega_g(t_0 - \epsilon, t_0 + \epsilon).$$

This is the same as the difference between the limit superior and limit inferior of the function at t_0, provided the point t_0 is not excluded from the limits.

Oscillation of solutions of differential equations:

A non-trivial solution to an ordinary differential equation

$$u^n = F(t, u, u', \ldots, u^{n-1}), \qquad t \in [0, +\infty), \tag{2.0.1}$$

is called oscillatory if it has an infinite number of roots, if not then is called non-oscillatory.

Formally, we give definition for the oscillation of solutions of ordinary differential equation [10] as follows:

Definition 2.0.4. A nontrivial solution u (implying a regular solution always) is said to be oscillatory if it has arbitrarily large zeros for $t \geq t_0$, that is, there exists a sequence of zeros t_n (i.e., $u(t_n) = 0$) of u such that $\lim_{n\to\infty} t_n = \infty$. Otherwise, u is said to be non-oscillatory.

For non-oscillatory solutions there exists t_1 such that $u(t) \neq 0$ for all $t \geq t_1$.

More precisely a solution u is oscillatory if it is neither eventually positive nor eventually negative, if not, is non oscillatory.

2.1 Oscillation and the Sturm separation theorem

Consider the linear homogeneous differential equation

$$u'' + P(t)u' + Q(t)u = 0. \tag{2.1.1}$$

It is tedious to solve this equation in general. However, by the properties of the coefficient functions, we can say something about the behavior of the solutions. A necessary characteristic that is the number of zeros of a solution to (2.1.1). A function having an infinite number of zeros in an interval $[a, \infty)$, is called oscillatory. So, the oscillatory behavior of a function means analyzing the number and locations of the zeros of that function.

The second order ordinary differential equation

$$u'' + u = 0, \tag{2.1.2}$$

has two solutions $u_1(t) = \sin(t)$ satisfying $u(0) = 0, u'(0) = 1$, and $u_2(t) = \cos(t)$ satisfying $u(0) = 1, u'(0) = 0$, which are linearly independent. The positive zeros of $u_1(t)$ and $u_2(t)$ are $\pi, 2\pi, 3\pi, \cdots$, and $\pi/2, 3\pi/2, 5\pi/2, \cdots$, respectively. Here note that between two consecutive zeros of $u_1(t)$, there is a zero of $u_2(t)$, and vice versa.

From the practical point of view the following result is fundamental.

Theorem 2.1.1. [9] *(Sturm separation theorem) If $u_1(t)$ and $u_2(t)$ are two linearly independent solutions of*

$$u'' + P(t)u' + Q(t)u = 0,$$

then the zeros of these functions are separate and come alternatively in the sense that $u_1(t)$ vanishes exactly once between any two successive zeros of $u_2(t)$, and conversely.

Proof. Let t_1 and t_2 be two successive zeros of u_1 with $t_1 < t_2$. Since u_1 and u_2 are linearly independent, their Wronskian $W(t) = u_1(t)u_2'(t) - u_1'(t)u_2(t)$ never vanishes. It means that $u_2(t_1) \neq 0$ and $u_2(t_2) \neq 0$, or else their Wronskian is zero at these points.
Suppose that the conclusion is not true, that is $u_2(t) \neq 0$ for all $t \in [t_1, t_2]$. Then the function $g(t) = \frac{u_1(t)}{u_2(t)}$ is well-defined and continuous on $[t_1, t_2]$, and continuously differentiable on (t_1, t_2). Since $u_1(t_1) = u_1(t_2) = 0$, so $g(t_1) = g(t_2) = 0$. By Rolle's theorem, there exists $z \in (t_1, t_2)$ so that $g'(z) = 0$. Computing $g'(z)$, we have that

$$0 = g'(z) = \frac{u_1'(z)u_2(z) - u_1(z)u_2'(z)}{[u_2(z)]^2} = \frac{-W(z)}{[u_2(z)]^2}. \tag{2.1.3}$$

But this means $W(z) = 0$, which contradicts the fact that u_1 and u_2 are linearly independent. Therefore u_2 must have a zero in (t_1, t_2).
By interchanging the roles of u_1 and u_2, we note that between any two successive zeros of u_2, there is a zero of u_1. As a result, there cannot be more than one zero of u_2 between two successive zeros of u_1. □

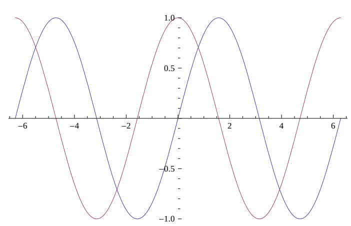


Figure 2.1: Plot of zeros of two linearly independent solutions.

Corollary 2.1.2. *Let one nontrivial solution to $u'' + P(t)u' + Q(t)u = 0$ is oscillatory on $[a, \infty)$. Then its all solutions are oscillatory.*

Proof. Suppose u_1 be a nontrivial solution having infinite number of zeros on $[a, \infty)$. Then by the Sturm separation theorem, if there is any other solution u_2 such that u_1 and u_2 are not linearly dependent, then between each successive pair of zeros of u_1 there must be a zero of u_2, and so u_2 must also possess an infinite number of zeros on $[a, \infty)$. And if u_1 and u_2 are linearly dependent, then

$$u_2 = \lambda u_1, \tag{2.1.4}$$

for some $\lambda =$ constant. Since u_1 is nontrivial. Thus u_2 also has an infinite number of zeros on $[a, \infty)$. □

Theorem 2.1.3. [9] *Suppose u be a nontrivial solution of $u'' + P(t)u' + Q(t)u = 0$ on $[a, b]$. Then on the interval $[a, b]$, u has at most a finite number of zeros.*

Proof. Suppose on the contrary that u has an infinite number of zeros on the interval $[a,b]$. By the Bolzano-Weierstrass theorem "there exist in $[a,b]$ a point t_0, and a sequence of zeros $t_n \neq t_0$ such that $\lim_{n\to\infty} t_n = t_0$." Since u is continuous and differentiable at t_0, we have

$$u(t_0) = \lim_{n\to\infty} u(t_n) = 0$$

and

$$u'(t_0) = \lim_{n\to\infty} \frac{(u(t_n) - u(t_0))}{(t_n - t_0)} = 0.$$

Theorem 2.1.4. ***(Existence and uniqueness theorem)*** *Consider the initial value problem:*

$$\begin{aligned}
&u^{(n)} + a_1(t)u^{(n-1)} + \cdots + a_n(t)u = h(t), \\
&u(t_0) = u_0, \\
&u'(t_0) = u_1, \\
&\vdots \\
&u^{(n-1)}(t_0) = u_{n-1}.
\end{aligned} \tag{2.1.5}$$

If $a_j(t), s$ and $h(t)$ are continuous on $\mathbb{R}$ then for any value of t_0 and $u_0, \ldots, u_{n-1}$, the problem (2.1.5) *possess a unique solution defined on* $\mathbb{R}$.

u must be the trivial solution which is a contradiction. Therefore, u has at most a finite number of zeros in $[a,b]$.

2.2 The Sturm comparison theorem

The equation

$$u'' + P(t)u' + Q(t)u = 0, \tag{2.2.1}$$

can be written as

$$w'' + q(t)w = 0. \tag{2.2.2}$$

By putting $u = wv$, where $w = e^{\frac{-1}{2}\int P dt}$ and $q(t) = Q(t) - \frac{1}{4}P(t)^2 - \frac{1}{2}P(t)$. Here (2.2.1) is the standard form and (2.2.2) is the normal form of a homogeneous second order linear equation. Since $v(t) > 0$ for all t, so the above transformation of (2.2.1) into (2.2.2) has not any effect on the zeros of the solutions, and consequently the oscillation behavior remains unchanged. ☐

The Sturm Separation Theorem compares the zeros of two solutions to the same equation. For solutions of two different equations, it may still be possible to relate their zeros. For example, consider the equations $u'' + m^2u = 0$ and $u'' + n^2u = 0$. The first has a general solution of the form $u_1(t) = A_1 \sin(m(t - \theta_1))$ and the second $u_2(t) = A_2 \sin(n(t - \theta_2))$.

The distance between successive zeros of u_1 is π/m and of u_2 is π/n. Consequently, for $n > m$, the distance between two zeros of u_1 is more than the distance between two zeros of u_2. Thus for $n_2 > m_2$, between two successive zeros of u_1, there is a zero of u_2. Similar result holds when the constants m_2 and n_2 are replaced by functions of t.

Theorem 2.2.1. *[Sturm comparison theorem] Suppose u_1 be a nontrivial solution to*

$$u'' + q_1(t)u = 0, \quad a < t < b, \tag{2.2.3}$$

and u_2 be a nontrivial solution to

$$u'' + q_2(t)u = 0, \quad a < t < b. \tag{2.2.4}$$

Let $q_2(t) \geq q_1(t)$ for all $t \in (a, b)$. If t_1 and t_2 are two successive zeros of u_1 on (a, b) with $t_1 < t_2$, then there exists a zero of u_2 in (t_1, t_2), unless $q_2(t) = q_1(t)$ on $[t_1, t_2]$ in which case u_1 and u_2 are linearly dependent on $[t_1, t_2]$.

Proof. Suppose $u_2(t) \neq 0$ in (t_1, t_2). We want to show that $q_1(t) = q_2(t)$ and u_1, u_2 are linearly dependent on $[t_1, t_2]$.We may suppose that $u_1(t) > 0$ and $u_2(t) > 0$ in (t_1, t_2). We have

$$\begin{aligned}
\frac{d}{dt}(W(u_2, u_1)) &= \frac{d}{dt}(u_2 u_1' - u_2' u_1), \\
&= u_2' u_1' + u_2 u_1'' - u_2'' u_1 - u_2' u_1', \\
&= u_2 u_1'' - u_2'' u_1, \\
&= u_2(-q_1 u_1) - (-q_2 u_2) u_1, \\
&= u_1 u_2 (q_2 - q_1) \geq 0,
\end{aligned}$$

for all t in (t_1, t_2). Therefore, $W(u_2, u_1)$ is non-decreasing on (t_1, t_2). However, since $u_1(t_1) = u_1(t_2) = 0$ and u_1 is positive on (t_1, t_2), we must have $u_1'(t_1) \geq 0$ and $u_1'(t_2) \leq 0$. Hence, $W(u_2, u_1)(t_1) = u_2(t_1)u_1'(t_1) \geq 0$ and $W(u_2, u_1)(t_2) = u_2(t_2)u_1'(t_2) \leq 0$.
Since $W(u_2, u_1)(t)$ is non-decreasing, the only way for it to be nonnegative at t_1 and non-positive at t_2 is $W(u_2, u_1)(t) = 0$ for all t in $[t_1, t_2]$. This implies $\frac{d}{dt}(W(u_2, u_1)(t)) = 0$ in $[t_1, t_2]$. That means $q_1(t) = q_2(t)$ in $[t_1, t_2]$. Then u_1 and u_2 satisfy the same equation on $[t_1, t_2]$, and their Wronskian vanishes on this interval. Hence, u_1 and u_2 are linearly dependent. ☐

Remark 2.2.2. The Sturm comparison theorem asserts that either u_2 has a zero between t_1 and t_2 or $u_2(t_1) = u_2(t_2) = 0$ (since u_1 and u_2 are linearly dependent in the later case).

Corollary 2.2.3. *Let $q(t) \leq 0$ for all $t \in [a, b]$. If u is a nontrivial solution of $u'' + q(t)u = 0$ on $[a, b]$, then u has at most one zero on $[a, b]$.*

Proof. Since $u(t) \equiv 1$ is a solution to the equation $u'' + 0.u = 0$ and $q(t) \leq 0$, it follows from (1.2.4) that if u has two or more zeros in $[a, b]$, then w must have a zero between them. Since w is never zero, u can have at most one zero in $[a, b]$. ☐

Example 2.2.4. The equation

$$t^2 u'' + t u' + (t^2 - p^2) u = 0, t > 0, \tag{2.2.5}$$

is called Bessel's equation. For $a > 0$, we discuss the number of zeros in the interval $[a, a + \pi)$. The substitution $u = wt^{\frac{-1}{2}}$ transforms equation (2.2.5) into the form,

$$\frac{d^2w}{dt^2} + \left(1 - \frac{p^2 - \frac{1}{4}}{t^2}\right) w = 0, t > 0. \tag{2.2.6}$$

Since $u = wt^{\frac{-1}{2}}$, the distribution of zeros of a solution u to (2.2.6) is the same as the corresponding solution w to (2.2.6). Let's compare the solutions of (2.2.6) with those of $w'' + w = 0$. Observe that $w(t) = A\sin(t - a)$ is a solution of $w'' + w = 0$ and has zeros at a and $a + \pi$.

Case 1.($p > 1/2$). In this case, $4p^2 - 1 > 0$ such that $1 - (\frac{p_2 - \frac{1}{4}}{t_2}) < 1$ for t in $[a, a + \pi)$. By the Sturm comparison theorem 2.2.1, a solution to (2.2.6) cannot have more than one zero in $[a, a + \pi)$ because $w(t) = A\sin(t - a)$ does not have a zero in $(a, a + \pi)$.

Case 2. ($0 \leq p < 1/2$). In this case, $4p_2 - 1 < 0$ such that $1 - (\frac{p_2 - \frac{1}{4}}{t_2}) > 1$ for t in $[a, a + \pi)$. By the Sturm comparison theorem 2.2.1, a solution to (2.2.6) must have a zero in $(a, a + \pi)$, since a and $a + \pi$ are successive zeros of $w(t) = A\sin(t - a)$.

Case 3. ($p = 1/2$). In this case, (2.2.6) reduces to $w'' + w = 0$, which has the general solution $w(t) = A\sin(t - a)$. As a result, it has exactly one zero in $[a, a + \pi)$.

Kneser's theorem

In mathematics, in the field of ordinary differential equations, the Kneser theorem, named after Adolf Kneser, provides criteria for a differential equation is oscillating or not.

Theorem 2.2.5. [11] *Consider an ordinary linear homogenous differential equation of the form*

$$u'' + q(t)u = 0,$$

with

$$q : [0, +\infty) \to \mathbb{R}$$

continuous. The equation is non-oscillating if

$$\limsup_{t \to +\infty} t^2 q(t) < \frac{1}{4}$$

and oscillating if

$$\liminf_{t \to +\infty} t^2 q(t) > \frac{1}{4}.$$

Example 2.2.6. To illustrate the theorem consider

$$q(t) = \left(\frac{1}{4} - a\right) t^{-2} \quad \text{for} \quad t > 0,$$

where a is real and non-zero. According to the theorem, solutions will be oscillating or not depending on whether a is positive (non-oscillating) or negative (oscillating) because

$$\limsup_{t \to +\infty} t^2 q(t) = \liminf_{t \to +\infty} t^2 q(t) = \frac{1}{4} - a.$$

To find the solutions for this choice of $q(t)$, and verify the theorem for this example, substitute

$$u(t) = t^n,$$

which gives

$$n(n-1)+\frac{1}{4}-a=\left(n-\frac{1}{2}\right)^2-a=0.$$

This means that (for non-zero a) the general solution is

$$u(t)=At^{\frac{1}{2}+\sqrt{a}}+Bt^{\frac{1}{2}-\sqrt{a}},$$

where A and B are arbitrary constants. It is not hard to see that for positive a the solutions do not oscillate while for negative $a=-\omega^2$ the identity

$$t^{\frac{1}{2}\pm i\omega}=\sqrt{t}\;e^{\pm(i\omega)\ln t}=\sqrt{t}\;(\cos(\omega\ln t)\pm i\sin(\omega\ln t))$$

shows that they do.

Lemma 2.2.7. ***(Picone's identity)*** *Let the functions u, v, pu', p_1v' be differentiable and $v(t)\neq 0$ in J. Then the following identity holds:*

$$[\frac{u}{v}(vpu'-up_1v')]'=u(pu')'-\frac{u^2}{v}(p_1v')'+(p-p_1)u'^2+p_1(v'-\frac{u}{v}v')^2. \tag{2.2.7}$$

Proof. Expanding the left side, we have
$\frac{u}{v}(v(pu')'+v'pu'-u(p_1v')'-u'p_1v')+(\frac{u'}{v}-\frac{u}{v^2}v')(vpu'-vp_1zv')=u(pu')'-\frac{u^2}{v}(p_1v')'+pu'^2-\frac{2uu'p_1v'}{v}+\frac{u^2p_1v'^2}{v^2}=u(pu')'-\frac{u^2}{v}(p_1v')'+(p-p_1)u'^2+p_1(u'-\frac{u}{v}v')^2.$ ☐

Theorem 2.2.8. [12] ***(Sturm Picone's theorem)*** *If $\alpha,\ \beta\in J$ are the consecutive zeros of a nontrivial solution $u(t)$ of*

$$(p(t)u')'+q(t)u=0 \tag{2.2.8}$$

and if $p_1(t),\ q_1(t)$ are continuous and $0<p_1(t)\leq p(t),\ q_1(t)\geq q(t)$ in $[\alpha,\beta]$, then every nontrivial solution $v(t)$ of the differential equation

$$(p_1(t)v')'+q_1(t)v=0 \tag{2.2.9}$$

has a zero in $[\alpha,\beta]$.

Proof. Let $v(t)\neq 0$ in $[\alpha,\beta]$, then Lemma 2.2.7 is applicable and from (2.2.7) and the DE's (2.2.8) and (2.2.9), we find

$$[\frac{u}{v}(vpu'-up_1v')]'=(q_1-q)u^2+(p-p_1)u'^2+p_1(u'-\frac{u}{v}v')^2.$$

Integrating the above identity and using $u(\alpha)=u(\beta)=0$, we obtain

$$\int_\alpha^\beta[(q_1-q)u^2+(p-p_1)u'^2+p_1(u'-\frac{u}{v}v')^2]dt=0,$$

which is a contradiction unless $q_1(t)\equiv q(t),\ p_1(t)\equiv p(t)$ and $u'-(u/v)v'\equiv 0$. The last identity is the same as $\frac{d}{dt}(u/v)\equiv 0$, and hence $u(t)/v(t)\equiv$ constant. However, since $u(\alpha)=0$ this constant must be zero, and so $u(t)/v(t)\equiv 0$, or $u(t)\equiv 0$. This contradiction implies that v must have a zero in $[\alpha,\beta]$. ☐

Theorem 2.2.9. *The only solution of the differential equation* (2.2.8) *which vanishes infinitely often in $J=[\alpha,\beta]$ is the trivial solution.*

Theorem 2.2.10. [12] ***(Leighton's oscillation theorem)*** *If* $\int^{\infty}(\frac{1}{p(t)})dt = \infty$ *and* $\int^{\infty} q(t)dt = \infty$, *then the DE* (2.2.8) *is oscillatory in* $J = (0, \infty)$.

Proof. Suppose $u(t)$ be a non-oscillatory solution of the differential equation (2.2.8) which we assume to be positive in $[t_0, \infty)$, where $t_0 > 0$. Then the Riccati equation

$$v' + q(t) + \frac{v^2}{p(t)} = 0, \tag{2.2.10}$$

has a solution $v(t)$ in $[t_0, \infty)$. This solution satisfies the equation

$$v(t) = v(t_0) - \int_{t_0}^{t} q(s)ds - \int_{t_0}^{t} \frac{v^2(s)}{p(s)} ds. \tag{2.2.11}$$

Since $\int^{\infty} q(s)ds = \infty$, we can always find an $t_1 > t_0$ such that

$$v(t_0) - \int_{t_0}^{t} q(s)ds < 0,$$

for all t in $[t_1, \infty)$. Thus, from (2.2.11) it follows that

$$v(t) < -\int_{t_0}^{t} \frac{v^2(s)}{p(s)} ds,$$

for all $t \in [t_1, \infty)$. Suppose that

$$r(t) = \int_{t_0}^{t} \frac{v^2(s)}{p(s)} ds, \quad t \in [t_1, \infty).$$

Then $v(t) < -r(t)$ and

$$r'(t) = \frac{v^2(t)}{p(t)} > \frac{r^2(t)}{p(t)}, \tag{2.2.12}$$

for all t in $[t_1, \infty)$. Integrating (2.2.12) from $t_1 > t_0$ to ∞, we get

$$\frac{-1}{r(\infty)} + \frac{1}{r(t_1)} > \int_{t_1}^{\infty} \frac{1}{p(s)} ds,$$

and therefore

$$\int_{t_1}^{\infty} \frac{1}{p(s)} ds < \frac{1}{r(t_1)} < \infty,$$

which is a contradiction. Hence, the solution $u(t)$ is oscillatory. $\square$

Example 2.2.11. We consider the differential equation (2.2.6). For all a, there exists a sufficiently large t_0 such that $1 + [(1 - 4a^2)/4t^2] > 1/2$ for all $t \geq t_0$, and therefore

$$\int^{\infty} (1 + \frac{1 - 4a^2}{4t^2}) dt = \infty.$$

Thus, Theorem 2.2.10 implies that (2.2.6) is oscillatory for all a.

Chapter 3

Delay differential equation

In this chapter, we shall discuss the oscillatory behavior of delay differential equations. The main purpose is to introduce some basic concepts from the theory of differential equations with oscillatory solutions, to sketch some important results from the theory of oscillation of ordinary differential equations.
Delay differential equations are different from ordinary differential equations in this regard that the derivative at current time depends on the solution and possibly its derivative at prior times. The simplest constant delay equations with multiple delays have the form

$$u'(t) = f(t, u(t), u(t-\tau_1), u(t-\tau_2), \ldots, u(t-\tau_k)), \tag{3.0.1}$$

where the time delays (lags) τ_j = positive constants. Broadly speaking, state dependent delays may depend on the solution, that is $\tau_i = \tau_i(t, u(t))$.
Systems of delay differential equations play an important role in all fileds of science and mainly in the biological sciences. Baker, Paul and Wille [13] contains references for many applications for delay differential equations.
It is thought-provoking that traditional point-wise modeling assumptions can be replaced by more realistic distributed assumptions. For example, when the birth rate of predators is affected by prior levels of predators or prey except by only the current levels in a predator-prey model. Because of the considerable difference in the properties of systems of delay differential equations and the systems of ordinary differential equations, it has become an active area of research. Ruan and Martin [14] and Raghothama and Narayanan [15] have given the examples of such applications. Further Shampine, Gladwell, and Thompson [16] have given an explanation for several common models.
To specify a system of delay differential equations additional information is required. As the derivative in (3.0.1) depends on the solution at the previous time $t-\tau_j$ so, to state the value of the solution before time $t = 0$, it is essential to provide an initial history function. Usually in common models the history function is a constant vector, but non-constant history functions come across frequently. Most of the problems have jump derivative discontinuity at the initial time. Additionally, any solution or jump derivative discontinuity in the history function at points former to the initial time need to be handled properly because such discontinuities are propagated to future times.

Definition 3.0.12. A delay differential equation is a differential equation where the time derivatives at the current time depend on the solution and possibly its derivatives at previous times:

$$u'(t) = F(t, u(t), u(t-\tau_1), u(t-\tau_2), \ldots, u(t-\tau_n), u'(t-\sigma_1), u'(t-\sigma_2), \ldots, u'(t-\sigma_m)), \quad t \geq t_0$$

$$u(t) = \phi(t), t \leq t_0.$$

Instead of a simple initial condition, an initial history function $\phi(t)$ needs to be specified. The quantities $\tau_i \geq 0, i = 1, \ldots, n.$ and $\sigma_i \geq 0, i = 1, \ldots, k$ are called the delays or time lags. The delays may be constants functions $\tau(t)$ and $\sigma(t)$ of t (time-dependent delays) or functions $\tau(t, u(t))$ and $\sigma(t, u(t))$ (state-dependent delays). Delay equations with delays σ of the derivatives are referred to as neutral delay differential equations (NDDEs).

Sometimes it is very challenging to solve delay differential equations (DDEs) of second or higher orders, even there are methods to solve (DDEs) but they are laborious. However, we can solve and plot graph of the some simple (DDEs) using Mathematica.
In this Chapter, we give oscillation criteria for (DDEs), so that we can say something about the solution of delay differential equation without solving it.

3.1 Difference between ordinary and delay differential equations

The most clear difference between ordinary differential equations and delay differential equations [16, 17] is the initial data. The solution of an ordinary differential equation can be obtained by its value at the initial point. Though, to solve a delay differential equation, we need an initial history to get a solution.
The oscillatory behavior of a functional differential equation with a deviating argument and of the associated ordinary differential equation are not always the same. Indeed, the retarded differential equation

$$u''(t) - u(t-\pi) = 0$$

has $\sin(t)$ and $\cos(t)$ as oscillatory solutions. On the other hand, the associated ordinary differential equation

$$u''(t) - u(t) = 0$$

has the non-oscillatory solutions e^{-t} and e^{t}. Conversely, we see that the delay equation

$$u''(t) + \frac{1}{2t^2}u(\frac{t}{4}) = 0, t > 0$$

has a non-oscillatory solution $u(t) = \sqrt{t}$, while the associated ordinary equation

$$u''(t) + \frac{1}{2t^2}u(t) = 0$$

has $t\cos(\ln t)$ and $t\sin(\ln t)$ as oscillatory solutions.
Consider the equation

$$u''(t) + \frac{1}{2}u'(t) - \frac{1}{2}u(t-\pi) = 0,$$

for $t \geq 0$, whose solution $u(t) = 1 - \sin(t)$ has an infinite sequence of multiple zeros. This solution also has an oscillatory property.
Consider the equation

$$u''(t) - u(-t) = 0.$$

This equation has an oscillatory solution $u_1(t) = \sin(t)$ and a non-oscillatory solution $u_2(t) = e^t + e^{-t}$. Such a change in the oscillatory behavior of a differential equation is obviously generated or disrupted by the delay and so the study of oscillatory solutions of differential equations with deviating arguments is very important in applications.

3.2 Oscillation behavior of delay differential equations of first order

The simplest delay differential equation is given by [18]

$$u'(t) = -u(t - \tau), \tag{3.2.1}$$

where $\tau > 0$ is called the delay. For $\tau = 0$, we get the simple ODE

$$u'(t) = -u(t), \tag{3.2.2}$$

having general solution as, $u(t) = u(0)e^{-t}$, which decays to zero.
If we prescribe $u(t)$ for $-\tau \leq t \leq 0$, then (3.2.1) should have a unique solution for $t > 0$. Suppose we set

$$u(t) = 1, \quad -\tau \leq t \leq 0, \tag{3.2.3}$$

as "initial history" for (3.2.1). Then, on the interval $0 \leq t \leq \tau$ the argument of u on the right side satisfies $t - \tau \leq 0$ so,

$$u'(t) = -u(t - \tau) = -1,$$

and hence

$$u(t) = u(0) + \int_0^t (-1)ds = 1 - t, \ 0 \leq t \leq \tau. \tag{3.2.4}$$

On $\tau \leq t \leq 2\tau$, we have $0 \leq t - \tau \leq \tau$ so by (3.2.4), we get

$$u'(t) = -u(t - \tau) = -[1 - (t - \tau)],$$

and therefore

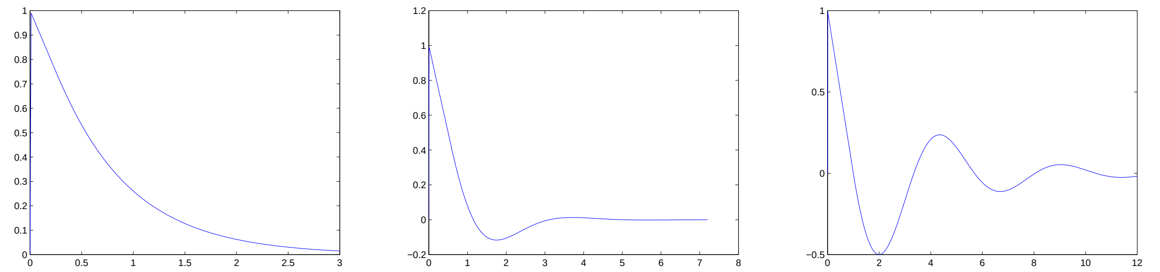
$$\begin{aligned} u(t) &= u(\tau) + \int_\tau^t -[1 - (s - \tau)])ds, \\ &= 1 - \tau + [-s + \frac{1}{2}(s - \tau)^2|_{s=\tau}^{s=t} \\ &= 1 - t + (t - \tau)^2/2, \quad \tau \leq t \leq 2\tau. \end{aligned} \tag{3.2.5}$$

So generalizing, we have

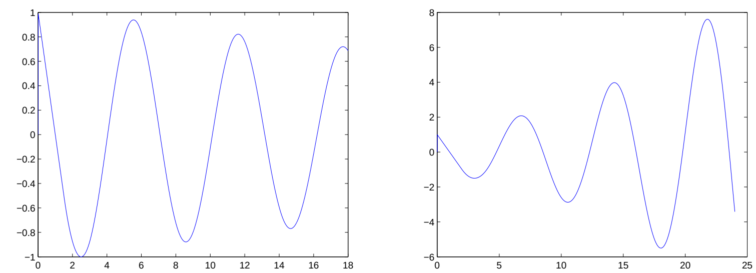
$$u(t) = 1 + \sum_{k=1}^{n} (-1)^k \frac{[t - (k-1)\tau]^k}{k!}, \quad (n-1)\tau \leq t < n\tau, n \geq 1. \tag{3.2.6}$$

Therefore, $u(t)$ is a polynomial of degree n on each subinterval of the form $[(n-1)\tau, n\tau]$. It follows that $u(t)$ is a smooth function, except at each $n\tau,\ n > 0$. The above method that we used to solve the initial-value problem (3.2.1) and (3.2.3) is called the method of steps.

We have explored the numerical solution of this initial value problem by the help of MATLAB. Our purpose is to inspect the behavior of the solution on the interval $t > 0$ for different values of the delay τ. Fig(1.1) and Fig(1.2) shows the graph of $u(t)$ and $u'(t)$.

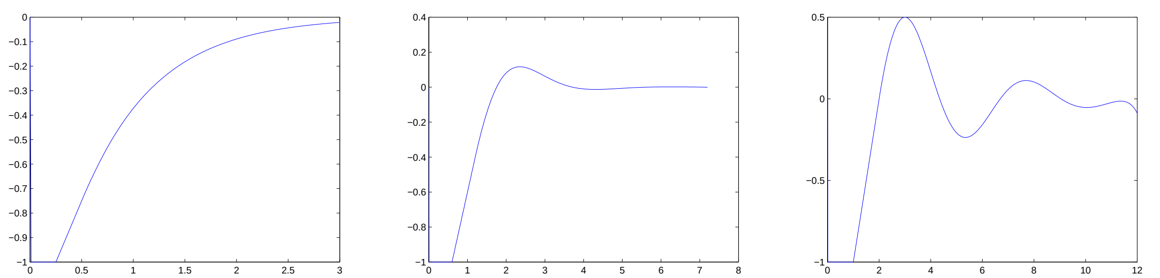


(a) No Oscillation for $\tau = 0.25$. (b) Hint of oscillation for $\tau = 0.6$. (c) Damped oscillation for $\tau = 1.0$.

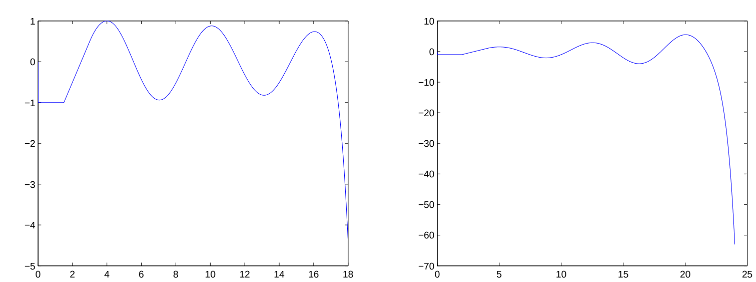


(d) Damped oscillation for $\tau = 1.5$. (e) Undamped oscillation for $\tau = 2$.

Figure 3.1: Solution of equation (3.2.1) for various τ.



(a) No Oscillation for $\tau = 0.25$. (b) Hint of oscillation for $\tau = 0.6$. (c) Damped oscillation for $\tau = 1.0$.



(d) Damped oscillation for $\tau = 1.5$. (e) Undamped oscillation for $\tau = 2$.

Figure 3.2: Derivative of solution of equation (3.2.1) for various τ.

We can easily observe that when we take $\tau = 0.25$ the solution looks very much like the solution of the ordinary differential equation (3.2.2) with initial condition $u(0) = 1$, and it does not oscillate. For $\tau = 0.6$ the solution oscillates. Actually, regardless of appearances, it changes sign repeatedly. We can prove that all solutions oscillate whenever $\tau > e^{-1}$. Why does $\tau > e^{-1} \approx 0.36$ result in oscillations? We answer this later in this chapter. As t increases the oscillations appear to be more prominent but still they are damped. That is, the breadth is decreasing, at least until $\tau = 2$ where now the breadth grows.

Necessary and sufficient conditions for oscillation

We determine necessary and sufficient conditions [19] for the oscillation of all solutions of the delay differential equation.

Theorem 3.2.1. ***(A necessary condition for oscillation)***
Consider a delay differential equation

$$u'(t) + mu(t-\tau) - nu(t-\sigma) = 0, \tag{3.2.7}$$

where $m,\ n,\ \tau,\ \sigma \in \mathbb{R}^+, \tau \geq \sigma$. *Then all solutions of* (3.2.7) *are oscillatory if*

$$m\tau - n\sigma > \frac{1}{e}.$$

Theorem 3.2.2. ***(A sufficient condition for oscillation)***
Suppose

$$m_i, \tau_i \geq 0, \qquad i = 1, 2, \ldots, n.$$

Then

$$\sum_{i=1}^{n} m_i \tau_i > \frac{1}{e},$$

is sufficient for the oscillation of all the solutions of the delay equation

$$u'(t) + \sum_{i=1}^{n} m_i u(t - \tau_i) = 0.$$

Example 3.2.3. Consider a delay differential equation

$$u'(t) + 2u(t - \frac{5}{4}\pi) = 0. \tag{3.2.8}$$

Here $m = 2$ and $\tau = \frac{5}{4}\pi$. So

$$m\tau = 7.85 > \frac{1}{e}.$$

Thus the necessary condition for oscillation is satisfied. Hence the (3.2.8) has an oscillatory solution of the form $u(t) = \cos(2t)$.

Example 3.2.4. Consider a delay differential equation

$$u'(t) + 2u(t - \frac{5}{2}\pi) - u(t - \frac{3}{2}\pi) = 0. \tag{3.2.9}$$

Here $m = 2, n = 1, \tau = \frac{5}{2}\pi$ and $\sigma = \frac{3}{2}\pi$. Also

$$m > n, \tau \geq \sigma.$$

Thus the necessary condition for oscillation

$$m\tau - n\sigma > \frac{1}{e}$$

is satisfied, therefore the (3.2.9) has an oscillatory solution of the form $u(t) = \cos(t)$.

Example 3.2.5. Consider a delay differential equation

$$u'(t) + u(t - \frac{9\pi}{2}) = 0. \tag{3.2.10}$$

Here $m = 1$ and $\tau = \frac{9\pi}{2}$. So

$$m\tau = \frac{9\pi}{2} > \frac{1}{e}.$$

Thus the necessary condition for oscillation is satisfied, hence the (3.2.10) has an oscillatory solution of the form $u(t) = \sin(t)$.

We consider the oscillatory behavior of solutions of the following linear differential inequalities and equations with retarded argument

$$u'(t) + p(t)u(\tau(t)) \leq 0, \tag{3.2.11}$$

$$u'(t) + p(t)u(\tau(t)) \geq 0, \tag{3.2.12}$$

$$u'(t) + p(t)u(\tau(t)) = 0. \tag{3.2.13}$$

Here $p, \tau \in C[\mathbb{R}_+, \mathbb{R}_+]$, $\tau(t) < t$, and $\lim_{t\to\infty} \tau(t) = +\infty$.
Let us begin with the following result.

Theorem 3.2.6. [20] *If*

$$\lim_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds > \frac{1}{e}, \tag{3.2.14}$$

then

(a) (3.2.11) *has no eventually positive solutions,*

(b) (3.2.12) *has no eventually negative solutions,*

(c) all solutions of (3.2.13) *are oscillatory.*

Proof. Without loss of generality, we assume that $\tau(t)$ is nondecreasing, otherwise we set

$$\delta(t) = max(\tau(s) = s \in [0, t]).$$

It is easy to prove that (3.2.14) is equivalent with $\lim_{t\to\infty} \int_{\delta(t)}^{t} p(s)ds > 1/e$.
First, we prove the validity of statement (a).
Assume that $u(t)$ is an eventually positive solution of (3.2.11) such that $u(\tau(t)) > 0$ for $t > t_1$.
Because of (3.2.14), there exists a $t_2 \geq t_l$ such that

$$\int_{\tau(t)}^{t} p(s)ds \geq c > e^{-1}. \tag{3.2.15}$$

For $t \geq t_2$. Since $u'(t) < 0$ for $t \geq t_1$, from (3.2.11) we get

$$u'(t) + p(t)u(t) < 0. \tag{3.2.16}$$

Dividing (3.2.16) by $u(t)$ and integrating from $\tau(t)$ to t, we obtain

$$ln\frac{u(t)}{u(\tau(t))}+\int_{\tau(t)}^{t}p(s)ds<0,\ t\geq t_2 \tag{3.2.17}$$

and hence

$$ln\left(\frac{u(\tau(t))}{u(t)}\right)\geq\int_{\tau(t)}^{t}p(s)ds\geq c,\qquad t\geq t_2.$$

Since $e^x\geq ex$, for $x\geq 0$, it follows that

$$\frac{u(\tau(t))}{u(t)}\geq ec,\qquad t\geq t_2.$$

Repeating the above procedure, there exists a sequence t_k such that

$$\frac{u(\tau(t))}{u(t)}\geq (ec)^k,\qquad t\geq t_k. \tag{3.2.18}$$

From (3.2.15), there exists a t^* such that

$$\int_{\tau(t)}^{t^*}p(s)ds\geq c/2$$

and

$$\int_{t^*}^{t}p(s)ds\geq c/2\ \text{ for } t\geq t_k.$$

Integrating (3.2.11) from $\tau(t)$ to t^* yields

$$u(t^*)-u(\tau(t))+\int_{t^*}^{t}p(s)u(\tau(s))ds\leq 0.$$

This implies that

$$u(\tau(t))\geq u(\tau(t^*)c/2. \tag{3.2.19}$$

Similarly, we obtain

$$u(t)-u(t^*)+\int_{t^*}^{t}p(s)u(\tau(s))ds\leq 0,$$

and consequently

$$u(t^*)\geq u(\tau(t))c/2. \tag{3.2.20}$$

Combining (3.2.19) and (3.2.20), there results the inequality

$$u(t^*)\geq u(\tau(t^*))(c/2)^2. \tag{3.2.21}$$

From (3.2.18) and (3.2.21), it follows that

$$\left(\frac{2}{c}\right)^2\geq\frac{u(\tau(t^*))}{u(t^*)}\geq (ec)^k, \tag{3.2.22}$$

for all $t\geq t_k$. Now we choose k sufficiently large such that

$$(ec)^k>(2/c)^2. \tag{3.2.23}$$

Which is possible because $ec>1$. Therefore (3.2.22) is a contradiction. A parallel argument holds for (3.2.12), therefore we obtain the conclusion (c). The proof is complete. □

We shall next discuss the special case with $p(t) \equiv p > 0$ and $\tau(t) \equiv t - \tau,\ \tau > 0$.

Theorem 3.2.7. *Assume that p and τ are positive numbers in* (3.2.13)*. Further, assume that*

$$p\tau e < 1. \tag{3.2.24}$$

Then (3.2.13) *has a non-oscillatory solution.*

Proof. Let us look at a solution of (3.2.13) of the form, $u(t) = exp(\lambda t)$. It follows that

$$F(\lambda) = -\frac{1}{\tau} + p\exp(-\lambda\tau) = 0.$$

Observe that $F(0) = p > 0$ and

$$F(-\frac{1}{\tau}) = -\frac{1}{\tau} + pe = \frac{p\tau e - 1}{\tau} \leq 0.$$

Hence there exists a negative real number $\lambda \in [-1/\tau, 0)$ such that $\exp(\lambda t)$ is a non-oscillatory solution of (3.2.13). □

Corollary 3.2.8. *If p and τ are positive numbers in* (3.2.13)*, then*

$$p\tau e > 1 \tag{3.2.25}$$

is necessary and sufficient for all solutions of (3.2.13) *to oscillate.*

Example 3.2.9. The equation

$$u'(t) + \frac{1}{e}u(t-1) = 0 \tag{3.2.26}$$

has a non-oscillatory solution $u(t) = \exp(-t)$, by 3.3.4, because $p\tau e = 1$.

Example 3.2.10. We consider

$$u'(t) + \frac{1}{(eIn2)t}u(\frac{t}{2}) = 0, \tag{3.2.27}$$

where $p(t) = 1/(e\ln 2)t$. Obviously, we have

$$\int_{t/2}^{t} p(s)ds = e^{-1}. \tag{3.2.28}$$

Hence (3.2.27) does not satisfy condition (3.2.14). In fact, equation (3.2.27) has non-oscillatory solution $u(t) = t^{\alpha}$ where $\alpha = \frac{-1}{\ln 2}$. In view of 3.3.4 and above examples, condition (3.4.4) is the best possible condition for all solutions of (3.2.13) to be oscillatory.

Necessary and sufficient conditions for oscillations

The autonomous case: Consider the linear autonomous delay differential equation

$$\dot{u}(t) + \sum_{i=1}^{n} p_i u(t - \tau_i) = 0. \tag{3.2.29}$$

Where the coefficients p_i are real numbers and the delays τ_i are non-negative real numbers. With (3.2.29) one associates its characteristic equation

$$\lambda + \sum_{i=1}^{n} p_i e^{-\lambda\tau_i} = 0. \tag{3.2.30}$$

Our first aim in this section is to establish the following fundamental result for the oscillation of all solutions of (3.2.29).

Lemma 3.2.11. *(a) Let $u \in C[[-\tau, \infty), \mathbb{R}]$ and let $\sigma_0 < \infty$ be the abscissa of convergence of the Laplace transform $U(s)$ of $u(t)$. Then the Laplace transform of $u(t-\tau)$ has the same abscissa of convergence and*

$$L[u(t-\tau)] = \int_0^\infty e^{-st} u(t-\tau) dt = e^{-s\tau} U(s) + e^{-s\tau} \int_{-\tau}^0 e^{-st} u(t) dt.$$

For all s with $Re(s) > \sigma_0$.

(b) Let $u \in C^1[[0, \infty), \mathbb{R}]$ and let $\sigma_0 < \infty$ be the abscissa of convergence of the Laplace transform $U(s)$ of $u(t)$. Then the Laplace transform of $\dot{u}(t)$ has the same abscissa of convergence and

$$L[\dot{u}(t)] = \int_0^\infty e^{-st} \dot{u}(t) dt = sU(s) - u(0),$$

for all s with $Re(s) > \sigma_0$.

$$\int_0^\infty e^{-st} \dot{u}(t) dt = sU(s) - u(0), \;\; Re(s) > \sigma_0$$

and for $i = 1, 2, \ldots, n$.

$$\int_0^\infty e^{-st} u(t-\tau_i) dt = e^{-s\tau_i} U(s) + e^{-s\tau_i} \int_{-\tau_i}^0 e^{-st} u(t) dt, \;\; Re(s) > \sigma_0.$$

Theorem 3.2.12. [21] *Assume that*

$$p_i \in R$$

and

$$\tau_i \in \mathbb{R}^+$$

for $i = 1, 2, \ldots, n$.
Then the following statements are equivalent.

(a) Every solution of (3.2.29) *oscillates.*

(b) The characteristic equation (3.2.30) *has no real roots.*

Proof. The proof that (a) implies (b) is elementary. This is because if the characteristic equation (3.2.30) has a real root λ_0 then $e^{\lambda_0 t}$ is a non-oscillatory solution of (3.2.29).
The proof that (b) implies (a) makes use of Laplace transforms and 1.5.3. Assume, for the sake of contradiction, that (b) holds and that (3.2.29) has an eventually positive solution $u(t)$. As (3.2.29) is autonomous, we may (and do) assume that $x(t) > 0$ for $t \geq -\tau$ where

$$\tau = \max_{1 \leq i \leq n} \tau_i.$$

Clearly $\tau > 0$, for otherwise equation (3.2.30) has a real root. And by Lemma 1.5.2 for $u \in \mathcal{C}[[0, \infty), \mathbb{R}]$, there exist constants M and μ such that

$$|u(t)| \leq Me^{\mu t}, \;\; t \geq -\tau.$$

Thus the Laplace transform

$$U(s) = \int_0^\infty e^{-st} u(t) dt, \tag{3.2.31}$$

exists for $\mathrm{Re}(s) > \mu$. Let σ_0 be the abscissa of convergence of $U(s)$, $\sigma_0 = \mathrm{Inf}\{\sigma \in \mathbb{R} : U(\sigma) \text{ exists}\}$. Then for any $i = 1, \ldots, n$, the Laplace transform of $u(t - \tau_i)$ exists and has abscissa of convergence σ_0. Furthermore by Lemma 3.2.11

$$\int_0^\infty e^{-st}\dot{u}(t)dt = sU(s) - u(0), \mathrm{Re}(s) > \sigma_0$$

and for $i = 1, 2, \ldots, n$.

$$\int_0^\infty e^{-st}u(t - \tau_i)dt = e^{-s\tau_i}U(s) + e^{-s\tau_i}\int_{-\tau_i}^0 e^{-st}u(t)dt,\ \mathrm{Re}(s) > \sigma_0.$$

Therefore, by taking Laplace transforms of both sides of (3.2.29) we obtain

$$F(s)U(s) = \phi(s), \qquad \mathrm{Re}(s) > \sigma_0 \tag{3.2.32}$$

where

$$F(s) = s + \sum_{i=1}^n p_i e^{-s\tau_i} \tag{3.2.33}$$

and

$$\phi(s) = u(0) - \sum_{i=1}^n p_i e^{-s\tau_i}\int_{-\tau_i}^0 e^{-st}u(t)dt. \tag{3.2.34}$$

Clearly, $F(s)$ and $\phi(s)$ are entire functions. Also by hypothesis, $F(s) \neq 0$ for all real s. It follows from (3.2.32) that

$$U(s) = \frac{\phi(s)}{F(s)},\ \mathrm{Re}(s) > \sigma_0. \tag{3.2.35}$$

We now claim that $\sigma_0 = -\infty$. Otherwise, $\sigma_0 > -\infty$ and by 1.5.3 the point $s = \sigma_0$ must be a singularity of the quotient $\frac{\phi(s)}{F(s)}$. But this quotient has no singularity on the real axis (the numerator and denominator are entire functions and by hypothesis the denominator has no real zeros). Thus $\sigma_0 = -\infty$ and (3.2.35) becomes

$$U(s) = \frac{\phi(s)}{F(s)} \tag{3.2.36}$$

for all $s \in \mathbb{R}$. One can now see that as s approaches to $-\infty$, through real values, (3.2.36) leads to a contradiction because $U(s)$ and $F(s)$ are always positive while $\phi(s)$ becomes eventually negative. The positivity of $U(s)$ follows from (3.2.31) and the fact that $u(t) > 0$ for $t \geq 0$. The positivity of $F(s)$ follows from (3.2.33) and the facts that $F(\infty) = \infty$ and that the characteristic equation has no real roots. Without loss of generality we may assume that the delays in (3.2.29) are distinct and that the coefficients p_i are different from zero. Let τ_{i_0} be the maximum delay in (3.2.29). Then the corresponding coefficient $p_{i_0} > 0$, for otherwise $\lim_{s\to-\infty} F(s) = -\infty$ and the dominant term in (3.2.34), as $s \to -\infty$, is $p_{i_0}e^{-s\tau_{i_0}}$. Clearly $\lim_{s\to-\infty}\phi(s) = -\infty$. The proof is complete. □

3.3 Oscillation behavior of delay differential equations of second order

Example 3.3.1. Consider the second order equation with delay

$$u''(t) + u(\pi - t) = 0. \tag{3.3.1}$$

It has both an oscillatory solution $u_l = \sin(t)$ and a non-oscillatory solution $u_2 = e^t - e^{\pi - t}$.
As we mentioned in chapter one, for second order linear ODE either all solutions oscillate or all solutions are non-oscillatory. Thus we see that second order equations with delay create some new problems in oscillation theory. For example, consider

$$u''(t) + p(t)u(\tau(t)) = 0. \tag{3.3.2}$$

We need to establish various sets of conditions [22] under which either: (a) all solutions are oscillatory, (b) all solutions are non-oscillatory, (c) the equation has a non-oscillatory solution, (d) the equation has an oscillatory solution or (e) the equation has both oscillatory and non-oscillatory solutions.

Example 3.3.2. The equation with delay given by

$$u''(t) - u(t - \pi) = 0, \tag{3.3.3}$$

has the oscillatory solutions $u = \sin(t)$ and $u = \cos(t)$. But

$$u''(t) - u(t) = 0,$$

has no oscillatory solution.

3.3.1 Second order neutral delay differential equation

In this section the oscillatory properties of the solutions of neutral differential equations of the form

$$\frac{d^2}{dt^2}[u(t) + P(t)u(t - \tau)] + Q(t)u(t - \sigma) = 0, \ t \geq t_0 \tag{3.3.4}$$

are investigated, where $P(t), Q(t) \in C([t_0, \infty); \mathbb{R})$ and the delays τ and σ are nonnegative real numbers. Let $\phi(t) \in C([t_0 - \rho, t_0]; \mathbb{R})$, where $\rho = max\,(\tau, \sigma)$, is a given function and let z_1 be a given constant. We have the following conditions:

H_1. $P(t) \in C([t_0, \infty); \mathbb{R}),\ p_1 \leq P(t) \leq p_2$ for $t \in [t_0, \infty)$, where p_1 and p_2 are constants.

H_2. $Q(t) \in C([t_0, \infty); \mathbb{R}),\ Q(t) \geq q = \text{constant.} > 0$ for $t \in [t_0, \infty)$.

Lemma 3.3.3. *Assume conditions H1 and H2 fulfilled. Let $u(t)$ be an eventually positive solution of equation* (3.3.4)*. Set*

$$z(t) = u(t) + P(t)u(t - \tau).$$

Then the following statement is true: The functions $z(t)$ and $z'(t)$ are strictly monotone and either

$$\lim_{t\to\infty} z(t) = \lim_{t\to\infty} z'(t) = -\infty,$$

or

$$\lim_{t\to\infty} z(t) = \lim_{t\to\infty} z'(t) = 0, \quad z(t) < 0$$

and $z'(t) > 0$. In particular, $z(t)$ is always negative.

Theorem 3.3.4. *[23] Consider the neutral delay differential equation* (3.3.4) *and assume conditions* H_1 *and* H_2 *fulfilled. Furthermore, assume that* $P(t)$ *is not eventually negative. Then each solution of equation* (3.3.4) *oscillates.*

Proof. Assume, for the sake of contradiction, that there is an eventually positive solution $u(t)$ of equation (3.3.4). Set

$$z(t) = u(t) + P(t)u(t-\tau).$$

Then, eventually, $z(t)$ takes nonnegative values. However, Lemma 3.3.3 implies that $z(t)$ is eventually negative. This is a contradiction and the proof is complete. □

Example 3.3.5. The neutral delay differential equation

$$\frac{d^2}{dt^2}[u(t) + (1/2 + \sin t)u(t-2\pi)] + (3/2 + \sin t)u(t-4\pi) = 0,\ t \geq 0. \tag{3.3.5}$$

Satisfies the conditions of Theorem 3.3.4. Therefore each solution of equation (3.3.6) oscillates. For example $u(t) = \sin t(3/2 + \sin t)^{-1}$ is an oscillating solution.

Example 3.3.6. For the neutral delay differential equation

$$\frac{d^2}{dt^2}[u(t) + (t-1)^{-1/2}u(t-1)] + \frac{1}{4}t^{-3/2}(t-2)^{-1/2}u(t-2) = 0,\ t > 2. \tag{3.3.6}$$

All conditions of Theorem 3.3.4, except for H_2, hold. Note, however, that $u(t) = \sqrt{t}$ is a non-oscillating solution.

Theorem 3.3.7. *Consider the neutral delay differential equation* (3.3.4) *and assume that condition* H_1 *and* H_2 *are satisfied with*

$$-1 \leq p_1 \leq p_2 < 0.$$

Suppose also that there exists a positive constant r *such that*

$$\frac{Q(t)}{P(t+\tau-\sigma)} \leq -r$$

and

$$r^{1/2}(\frac{\sigma-\tau}{2}) > \frac{1}{e}.$$

Then each solution of equation (3.3.4) *oscillates.*

Example 3.3.8. For the neutral delay differential equation

$$\frac{d^2}{dt^2}[u(t) - (e^2 + e^{-t})u(t-1)] + e^2(e-1)u(t-2) = 0,\ t \geq 0,$$

all conditions of Theorem 3.3.7, except for $-1 \leq p_1$, are satisfied. Note that $u(t) = e^t$ is a non-oscillating solution of this equation.

3.4 An application of delay differential equations

Consider the autonomous delay differential equation [24]

$$\frac{dx(t)}{dt} = x(t)\left[a - \sum_{j=1}^{n} b_j x(t-\tau_j)\right], \qquad t \geq 0. \tag{3.4.1}$$

Where $a, b_j, \tau_j \ \ (j = 1, 2, \ldots, n)$ are positive constants. Equation (3.4.1) corresponds to a generalization of an equation of the form

$$\frac{dN(t)}{dt} = rN(t)\left[1 - \frac{N(t-\tau)}{K}\right], \tag{3.4.2}$$

in which r, τ, K are positive numbers. It has been suggested by Hutchinson [1948] that (3.4.2) can be used to model the dynamics of a single species population growing towards a saturation level K with a constant reproduction rate r, the term $\left[1 - \frac{N(t-\tau)}{K}\right]$ in (3.4.2) denotes a density dependent feedback mechanism which takes r units of time to respond to changes in the population density represented in (3.4.2) by N. By a change of variables, (3.4.2) can be brought to an equation of the form

$$\frac{du(s)}{ds} = -\alpha u(s-1)\left[1 + u(s)\right]. \tag{3.4.3}$$

Where α is a positive constant. Eq (3.4.3) has been studied by numerous authors and notably, by Kakutani and Markus [1958], Jones [1962] and Wright [1955].

It is intuitively expected that, if all the delays τ_j in (3.4.1) are sufficiently small (relative to a and b_j), then the asymptotic behavior as $t \to \infty$ of solutions of (3.4.1) will be similar to that of the solutions of

$$\frac{dx(t)}{dt} = x(t)\left[a - \left(\sum_{j=1}^{n} b_j\right) x(t)\right].$$

It has been known that if τ is sufficiently large, then nonconstant positive solutions of (3.4.2) oscillate about its positive equilibrium.

Since fluctuating populations are susceptible to extinction due to sudden and unforeseen environmental disturbances, a knowledge of the conditions under which population densities fluctuate indefinitely will be of some interest in planning and designing control as well as management strategies.

Examples of delay differential equations: The well known logistic equation elaborates the growth of a single population is given by

$$N'(t) = N(t)[b - aN(t)].$$

Here one may assume that population density negatively affects the per capita growth rate according to $\frac{dN}{Ndt} = b - aN(t)$ due to environmental humiliation. Hutchinson [1948] indicated the negative effects that high population densities have on the environment. This directed him towards the delayed logistic equation

$$N'(t) = N(t)[b - aN(t-r)]. \tag{3.4.4}$$

Where $a, b, r > 0$. Here r is the delay. Wright's guess, made in 1955, concerning the solutions of (3.4.4) remains open. Arino, Wang, and Wolkowicz introduced an alternative model that has simpler dynamics.

May it would be more realistic to assume that the density dependence is distributed over an interval in the past except concentrated at a single time instant. It will give:

$$N'(t) = N(t)[b - a\int_0^\infty N(t-s)k(s)ds], \tag{3.4.5}$$

where kernel k is normalized so that $\int_0^\infty k(s)ds = 1$. A drawback of the delayed logistic equation is that the birth and death rates are not clearly distinguished. Nicholson's data on population fluctuations of the sheep blowfly Lucillia cuprina motivated the model now referred to as the Nicholson's blowfly equation:

$$N'(t) = bN(t-r)]\exp(-N(t-r)/N_0) - \delta N(t). \tag{3.4.6}$$

Eggs laid by blowfly develops into larvae that finally becomes adult flies. This model is used only for mature flies, to which food supply is given at a constant rate. It is supposed that r units of time is taken by eggs to develop into mature flies. The first term on the right side describes addition of new adults, it should be taken as a probability of survival from egg to adult: $aN(t-r) \times c\exp(-N(t-r)/N_0)$. With the increase in population size, the probability rate of survival decreases because intra specific food competition among the immature flies increases. The final term on the right accounts for death.

Chapter 4

Oscillation of solutions of impulsive differential equations

Impulsive differential equations, that is, differential equations involving impulse effect, represent a real framework for mathematical modeling to real world problems. Significant progress has been made in the theory of impulsive conditions. There are many real life processes and phenomena that are characterized by rapid changes in their state. The duration of these changes is relatively short compared to the overall duration of the whole process and the changes turn out to be irrelevant to the development of the studied process. The mathematical models in such cases can be adequately created with the help of impulsive equations. Some examples of such processes we can found in Physics, Biology, population dynamics, ecology, pharmacokinetics, and others.

In the general case, the impulsive equations consist of two parts:

(a) Differential equation, that defines the continuous part of the solution.

(b) Impulsive part, that defines the instantaneous changes and the discontinuity of the solution.

The first part of the impulsive equations, that is described by differential equations, could consist of ordinary differential equations, integro-differential equations, functional differential equations, partial differential equations, etc.

The second part of the impulsive equations is called a jump condition. The points, at which the impulses occur, are called moments of impulses. The functions, that define the amount of impulses, are called impulsive functions.

The type of the moments of impulses defines different types of impulsive equations. The two types of impulsive equations are:

(a) Impulsive equations with fixed moments of impulses (the impulses occur at initially given fixed points).

(b) Impulsive equations with variable moments of impulses (the impulses occur on initially given sets, i.e. the impulse occurs when the integral curve of the solution hits a given set).

In this chapter, we shall study the oscillation of solutions of impulsive differential equations of first and second order.

4.1 Oscillatory behavior of first-order linear equations

Let us consider the linear impulsive differential equation

$$u' + b(t)u = 0,\ t \neq t_k, \quad \Delta u|_{t=t_k} + b_k u = 0,\ k \in N, \tag{4.1.1}$$

with the following inequalities:

$$u' + b(t)u \leq 0,\ t \neq t_k, \quad \Delta u|_{t=t_k} + b_k u \leq 0,\ k \in N, \tag{4.1.2}$$

$$u' + b(t)u \geq 0,\ t \neq t_k, \quad \Delta u|_{t=t_k} + b_k u \geq 0,\ k \in N. \tag{4.1.3}$$

Lemma 4.1.1. *If $u \in \mathcal{C}[[0,\infty),\mathbb{R}]$ satisfies the impulse equation*

$$u'(t) + b(t)u = 0,\ t \neq t_k,$$

$$\Delta u(t)|_{t=t_k} + b_k u = 0,\ K \in \mathbb{N},$$

then

$$u(t) = u(t_0)exp(-\int_{t_0}^{t} b(s)ds) \prod_{t_0 \leq t_k < t} (1 - b_k).$$

Proof. Since $u \in \mathcal{C}[[0,\infty),\mathbb{R}]$ satisfies

$$u'(t) + b(t)u = 0,\ t \neq t_k.$$

Integrating from t_0 to t, we have

$$u(t) = u(t_0)e^{-\int_{t_0}^{t} b(s)ds}.$$

For $t \in [t_0, t_1]$

$$u(t_1^-) = u(t_0)e^{-\int_{t_0}^{t_1} b(s)ds} \tag{4.1.4}$$

and by impulse condition

$$u(t_1^+) = u(t_1^-) - b_1 u(t_1^-) = u(t_1^-)(1 - b_1).$$

From (4.1.4) substituting the value of $u(t_1^-)$, we have

$$u(t_1^+) = u(t_0)e^{-\int_{t_0}^{t_1} b(s)ds}(1 - b_1).$$

Integrating from t_1 to t, we have

$$u(t) = u(t_1^+)e^{-\int_{t_1}^{t} b(s)ds},$$

for $t \in [t_1, t_2]$,

$$u(t_2^-) = u(t_1^+)e^{-\int_{t_1}^{t} b(s)ds}. \tag{4.1.5}$$

From (4.3.2) substituting the value of $u(t_1^+)$, we have

$$u(t_2^-) = u(t_0)e^{-\int_{t_0}^{t} b(s)ds}(1 - b_1).$$

Again by impulse condition

$$u(t_2^+) = (1 - b_2)u(t_2^-),$$

substituting the value of $u(t_2^-)$, we get,

$$u(t_2^+) = u(t_0)e^{-\int_{t_0}^{t} b(s)ds}(1-b_1)(1-b_2),$$

$$\vdots$$

$$u(t) = u(t_0)exp(-\int_{t_0}^{t} b(s)ds) \prod_{t_0 \le t_k < t} (1-b_k).$$

□

Theorem 4.1.2. [25] *Suppose that $b \in PLC(I)$ and $1 - b_k \neq 0,\ k \in N$. Then the following statements are equivalent.*

(1) The sequence $\{1 - b_k\}$ has infinite many negative terms.

(2) The inequality (4.1.2) has no eventually positive solution.

(3) The inequality (4.1.3) has no eventually negative solution.

(4) All non-zero solutions of (4.1.1) are oscillatory.

Proof. (1) implies (2). Suppose that the sequence $\{1 - b_k\}$ have infinite many negative terms. And we let that the statement (2) is false, it means that we are assuming that the inequality (4.1.2) has an eventually positive solution $u(t), t \geq T_0$. Let $1 - b_k < 0$. for $t_k \geq T_0$. Then by (4.1.2) we have,

$$u(t_k^+) \leq (1 - b_k)u(t_k) < 0,$$

which contradicts our hypothesis.

(2) implies (3). if $u(t)$ is a solution of the inequality (4.1.2), then $-u(t)$ is a solution of the inequality (4.1.3) and vice versa. This shows the rationality of this relation.

(2) and (3) implies (4). Actually, if (4.1.1) has no eventually positive and no eventually negative solution, then every nonzero solution $u(t)$ of (4.1.1) is oscillatory.

(4) implies (1). If $u(t)$ is an oscillatory solution of (4.1.3), then the equality

$$u(t) = u(t_0)exp(-\int_{t_0}^{t} b(s)ds) \prod_{t_0 \le t_k < t} (1-b_k),$$

follows that the sequence $1 - b_k$ has infinite many negative terms. □

Theorem 4.1.3. [25] *Suppose that $b \in PLC(I)$ and $1 - b_k \neq 0,\ k \in N$. Then the following statements are equivalent.*

(1) The sequence $\{1 - b_k\}$ has finite many negative terms.

(2) The inequality (4.1.2) has an eventually positive solution.

(3) The inequality (4.1.3) has an eventually negative solution.

(4) All non-zero solutions of (4.1.1) are not oscillatory.

We know that the equation (4.1.1) without impulsive effects has non-oscillatory solutions. On the other hand equation (4.1.1) with impulsive effects possess oscillatory solutions. Consequently we have that impulsive effects defines the oscillatory properties of linear differential equations of the first order.

4.2 Sturmian theory for second-order linear equations

In the investigation of qualitative properties of the solutions of linear and non-linear equations, Sturm comparison theory plays an significant role. In 1996, the first paper on the Strumian theory of differential equations with impulsive effect was published .

Consider the second-order linear impulsive differential equations

$$u'' + a(t)u = 0,\ t \neq t_k,\ \ \Delta u(t_k) = 0,\ \Delta u'(t_k) + a_k u(t_k) = 0,\ k \in N, \tag{4.2.1}$$

$$v'' + b(t)v = 0,\ t \neq t_k,\ \ \Delta v(t_k) = 0,\ \Delta v'(t_k) + b_k v(t_k) = 0,\ k \in N, \tag{4.2.2}$$

where a and b are continuous for $t \in I,\ t \neq t_k$, and they have a discontinuity at the points $t_k \in I$, where they are continuous from the left.

The following theorem is an important result, and is also applicable on the differential inequalities.

Theorem 4.2.1. [26] *Assume the following.*
(1) Equation (4.2.2) *possess a solution* $v(t)$ *so that*

$$v(t) > 0,\ t \in (a,b),\ v(a^+) = v(b^-) = 0. \tag{4.2.3}$$

(2) The following inequalities are valid:

$$p(t) \geq q(t),\ p_k \geq q_k,\ t \in (a,b),\ t_k \in (a,b). \tag{4.2.4}$$

(3) $p(t) > q(t)$ *in a subinterval of* (a,b) *or* $p_k > q_k$ *for some* $t_k \in (a,b)$. *Then* (4.2.1) *has no positive solution* $x(t)$ *defined on (a,b).*

Proof. Let (4.2.1) has a solution $u(t)$ so that $u(t) > 0,\ t \in (a,b)$. Then

$$(u'(t)v(t) - u(t)v'(t))' = u''(t)v(t) - u(t)v''(t),\ t \in (a,b),\ t \neq t_k. \tag{4.2.5}$$

Taking integral, we have

$$u'(b^-)v(b^-) - u(b^-)v'(b^-) - u'(a^+)v(a^+) + u(a^+)v'(a^+) =$$
$$\int_a^b [u''(t)v(t) - u(t)v''(t)]dt + \sum_{a<t_k<b} [\Delta u'(t_k)v(t_k) - u(t_k)v'(t_k)].$$

From (4.2.1) and (4.2.2), condition (1), and the above inequality, we have that

$$0 \leq \int_a^b (q(t) - p(t))u(t)v(t)dt + \sum_{a<t_k<b} (q_k - p_k)u(t_k)v(t_k). \tag{4.2.6}$$

However, conditions (2) and (3) shows that the right side of the inequality above is negative, which is a contradiction. Hence the proof is complete. □

Corollary 4.2.2. *Assume the following.*
(1) Equation (4.2.2) *has a solution* $v(t)$ *such that*

$$v(t) \neq 0,\ t \in (a,b),\ v(a^+) = v(b^-) = 0. \tag{4.2.7}$$

(2) The following inequalities are valid:

$$p(t) \geq q(t),\ p_k > q_k,\ t \in (a,b),\ t_k \in (a,b). \tag{4.2.8}$$

(3) $p(t) > q(t)$ in some subinterval of (a,b) or $p_k > q_k$ for some $t_k \in (a,b)$. Then, every solution $u(t)$ of (4.2.1) *has at least one zero in (a, b).*

Corollary 4.2.3. *If conditions (1) and (2) of Corollary 4.2.2 are satisfied, then we can conclude the following.*
(1) Every solution $u(t)$ of (4.2.1) *for which $|u(a^+)| + |u(b^-)| > 0$ has at least one zero in (a,b).*
(2) Every solution $u(t)$ of (4.2.1) *has at least one zero in (a,b).*

Corollary 4.2.4. *Assume the following.*
(1) There exists a solution $v(t)$ of (4.2.2) *and a sequence of disjoint intervals $(a_n,b_n) \subset I$ such that*

$$\lim_{n\to\infty} a_n = \infty,\ v(a_n^+) = v(b_n^-) = 0,\ v(t) \neq 0, \tag{4.2.9}$$

for $t \in (a_n,b_n),\ n \in N$.
(2) The following inequalities are valid for $t \in (a_n,b_n),\ t_k \in (a_n,b_n)$, and $n \in N$,

$$p(t) \geq q(t), \quad p_k \geq q_k. \tag{4.2.10}$$

Then (4.2.1) *has all the oscillatory solutions, and additionally, they change sign in each interval $[a_n,b_n]$.*

Corollary 4.2.5. *(Comparison theorem) Let the inequalities $p(t) \geq q(t),\ p_k \geq q_k$ hold for $t > T \geq \alpha$ and $t_k > T$. Then, all solutions of* (4.2.2) *are non-oscillatory if* (4.2.1) *has a non-oscillatory solution.*

4.3 Oscillation theory for the second-order linear equations

Bainovetal provided a Sturmian type comparison theorem and zeros separation theorem for linear impulsive second-order differential equations. In recent times, this theory has been prolonged in different directions, special attention has been paid on PiconeŠs formulas, and Leighton type comparison theorems.
Consider the linear impulsive second-order differential equation

$$u'' + p(t)u = 0,\ t \neq t_k, \Delta u(t_k) = 0, \quad \Delta u'(t_k) + p_k u(t_k = 0,\ k \in \mathbb{N}, \tag{4.3.1}$$

$$v'' + p(t)v = 0,\ t \neq t_k, \Delta v(t_k) = 0, \quad \Delta v'(t_k) + q_k v(t_k = 0,\ k \in \mathbb{N}, \tag{4.3.2}$$

where p and q are continuous for $t \in I,\ t \neq t_k$, and they have a discontinuity at the points $t_k \in I$, where they are continuous from the left. The following theorem is important and is also applicable for differential equations.

Theorem 4.3.1. [26] *Assume the following.*
(1) Equation (4.3.2) *has a solution $v(t)$ so that*

$$v(t) > 0,\ t \in (a,b),\ v(a^+) = v(b_-) = 0.$$

(2) The following inequalities hold:

$$p(t) \geq q(t),\ p_k \geq q_k,\ t \in (a,b),\ t_k \in (a,b).$$

(3) $p(t) > q(t)$ in a subinterval of (a,b) or $p_k > q_k$ for some $t_k \in (a,b)$.
Then (4.3.1) has no positive solution $u(t)$ defined on (a,b).

Proof. Let (4.3.1) has a solution $u(t)$ so that $u(t) > 0,\ t \in (a,b)$. Then

$$(u'(t)v(t) - u(t)v'(t))' = u''(t)v(t) - u(t)v''(t),\ t \in (a,b),\ t \neq t_k.$$

Taking integration, we have

$$u'(b^-)v(b^-) - u(b^-)v'(b^-) - u'(a^+)v(a^+) + u(a^+)v'(a^+)$$
$$= \int_a^b [u''(t)v(t) - u(t)v''(t)dt + \sum_{a<t_k<b} [\Delta u'(t_k)v(t_k) - u(t_k)\Delta v'(t_k).$$

From (4.3.1) and (4.3.2), condition (1), and the above inequality, we get

$$0 \leq \int_a^b (q(t) - p(t))u(t)v(t)dt + \sum_{a<t_k<b} (q_k - p_k)u(t_k)v(t_k.$$

However, from conditions (2) and (3), it follows that the right side of the above inequality is negative, which is a contradiction. Hence the proof is complete. □

Corollary 4.3.2. *Assume the following.*
(1) Equation (4.3.2) has a solution $v(t)$ so that

$$v(t) \neq 0,\ t \in (a,b),\ v(a^+)v(b^-) = 0.$$

(2) The following inequalities hold:

$$p(t) \geq q(t),\ p_k \geq q_k,\ t \in (a,b),\ t_k \in (a,b).$$

(3) $p(t) > q(t)$ in some subinterval of (a,b) or $p_k > q_k$ for some $t_k \in (a,b)$. Then, every solution $u(t)$ of (4.3.1) has at least one zero in (a,b).

Corollary 4.3.3. *If conditions (1)and (2) of Corollary 4.3.2 are fulfilled, then we have the following.*
(1) Every solution $u(t)$ of (4.3.1) for which $|u(a^+)| + |u(b^-)| > 0$ has at least one zero in (a,b).
(2) Every solution $u(t)$ of (4.3.1) has at least one zero in (a,b).

Corollary 4.3.4. *Assume the following.*
(1) There exists a solution $v(t)$ of (4.3.2) and a sequence of disjoint intervals $(a_n, b_n) \subset J$ so that

$$\lim_{n\to\infty} a_n = \infty,\ v(a_n^+) = v(b_n^-) = 0,$$

and $v(t) \neq 0$ for $t \in (a_n, b_n),\ n \in N$.
(2) The following inequalities hold for $t \in (a_n, b_n),\ t_k \in (a_n, b_n)$, and

$$n \in N,\ p(t) \geq q(t),\ p_k \geq q_k.$$

Then (4.3.1) has all the oscillatory solutions, and furthermore, they change sign in each interval $[a_n, b_n]$.

Corollary 4.3.5. *Suppose the inequalities $p(t) \geq q(t),\ p_k \geq q_k$ valid for $t > T \geq \alpha$ and $t_k > T$. Then, if (4.3.1) has a non-oscillatory solution then all solutions of (4.3.2) are so.*

Chapter 5

The oscillation of fractional differential equations

Fractional differential equations are of great interest because of their applications in real life problems. Caputo derivatives of fractional order $0 < \alpha < 1$ are widely used in modeling several physical phenomena, and therefore are significant to study. Impulsive fractional differential equations represent a real framework for mathematical modeling to real world problems.

In first section of this chapter we shall study oscillation of fractional differential equations with impulse effect. Second section is about the oscillation of delay fractional differential equations.

5.1 On the oscillation of fractional differential equations with impulse effect

In this section, we study oscillation theory for fractional differential equations with impulse effect. Oscillation criteria are obtained for a class of nonlinear Fractional differential equations of the form

$$\begin{aligned} &{}^cD_a^{\alpha} u + f_1(t,u) = v(t) + f_2(t,u),\ t \neq t_k,\ 0 < \alpha < 1. \\ &u^k(a) = b_k, \\ &\Delta(u(t_k) = I_k(u(t_k)),\ t = t_k, \end{aligned} \tag{5.1.1}$$

and $k = 0, 1, 2, \ldots, m-1$. Where ${}^cD_a^{\alpha}$ is the Caputo fractional derivative.

We suppose that f_1, f_2 and v are continuous. The (5.1.1) is equivalent to the Volterra integral equation

$$u(t) = \sum_{k=0}^{m-1} \frac{b_k(t-a)^k}{k!} + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1}[v(s)+f_2(s,u(s))-f_1(s,u(s))]ds + \sum_{k=0}^{m-1} I_k(u(t_k)),\ t \in (t_{m-1}, T],\ T > a. \tag{5.1.2}$$

In other words every solution of (5.1.1) is a solution of (5.1.2) and conversely. We consider only those solutions which are continuous on $[a, \infty]$, and not identically zero on any half line (b, ∞) for some $b \geq a$.

S. R. Grace, R. P. Agarwal, P. J. Y. Wong and A. Zafer [27] have developed an oscillating criteria for fractional differential equations with initial condition. The aim of the section is to apply the criteria of [27] to fractional differential equation with impulsive effect.

5.1.1 Main results

Lemma 5.1.1. [28] *A function $u \in \mathcal{C}[[0,\infty),\mathbb{R}]$ is solution of* (5.1.1) *if and only if u satisfies* (5.1.2).

Proof. we consider (5.1.1)

$$\begin{aligned}
&{}^cD_a^\alpha u + f_1(t,u) = v(t) + f_2(t,u),\ t \neq t_k,\ 0 < \alpha < 1.\\
&u^k(a) = b_k,\\
&\Delta(u(t_k) = I_k(u(t_k)),\ t = t_k,
\end{aligned}$$

and $k = 0, 1, 2, \ldots, m-1$. where, ${}^cD_a^\alpha$ is the Caputo fractional derivative.
Applying fractional integral operator on both sides of (5.1.1), we have

$$^cI_a^{\alpha c}D_a^\alpha u = {}^cI_a^\alpha[v(t) + f_2(t,u) - f_1(t,u)].$$

Using Theorem 1.3.7, we have

$$u(t) - \sum_{k=0}^{m-1} \frac{u^k(a)(t-a)^k}{k!} = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1}[v(s) + f_2(s,u(s)) - f_1(s,u(s))]ds,$$

Applying initial and impulsive conditions , we have

$$u(t) = \sum_{k=0}^{m-1} \frac{b_k(t-a)^k}{k!} + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1}[v(s) + f_2(s,u(s)) - f_1(s,u(s))]ds + \sum_{k=0}^{m-1} I_k(u(t_k)),\ t \in (t_{m-1},T],\ T > a.$$

Conversely suppose that u Satisfies (5.1.2), then by applying fractional differential operator on both sides of (5.1.2), we have (5.1.1). □

Next we will make use of the conditions

$$\begin{aligned}
&(C_1)\ uf_i(t,u) > 0\ (i = 1,2), u \neq 0, t \geq a,\\
&(C_2)\ |f_1(t,u)| \geq p_1(t)|u|^\beta,\ |f_2(t,u)| \leq p_2(t)|u|^\gamma,\\
&\textit{and}\\
&(C_3)\ \sum_{k=0}^{m-1} |I_k(u(t_k))| \leq (m-1)l_1, u \neq 0, t \geq a.
\end{aligned}$$

Where $p_1, p_2 \in \mathcal{C}([a,\infty),\mathbb{R}^+)$ and $\beta, \gamma > 0$ are real numbers.

Lemma 5.1.2. *For $U \geq 0$ and $V > 0$, we have*

$$U^\lambda + (\lambda-1)V^\lambda - \lambda UV^{\lambda-1} \geq 0, \qquad \lambda > 1 \tag{5.1.3}$$

and

$$U^\lambda - (1-\lambda)V^\lambda - \lambda UV^{\lambda-1} \leq 0, \qquad \lambda < 1. \tag{5.1.4}$$

Where equality holds if and only if $U = V$.

Now we may present our first theorem when $f_2 = 0$.

Theorem 5.1.3. *Let $f_2 = 0$ and condition (C_1) hold. If*

$$lim_{t\to\infty}inf\ t^{1-m}\int_a^t (t-s)^{\alpha-1}v(s)ds = -\infty, \tag{5.1.5}$$

and

$$lim_{t\to\infty}sup\ t^{1-m}\int_a^t (t-s)^{\alpha-1}v(s)ds = \infty, \tag{5.1.6}$$

then every solution of (5.1.1) *is oscillatory.*

Proof. Let $u(t)$ be a non oscillatory solution of (5.1.1) with $f_2 = 0$. Let $T > a$ is large enough such that $u(t) > 0$ for $t \geq T$.
Suppose that $F(t) = v(t) - f_1(t, u(t))$, then we see from (5.1.2) that

$$u(t) = \sum_{k=0}^{m-1} \frac{b_k(t-a)^k}{k!} + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}[v(s) - f_1(s,u(s))]ds + \sum_{k=0}^{m-1} I_k(u(t_k)),\ t \in (t_{m-1}, T], \quad T > a,$$

substituting (c_3), we have

$$u(t) \leq \sum_{k=0}^{m-1} \frac{|b_k|(t-a)^k}{k!} + \frac{1}{\Gamma(\alpha)}\int_a^T (t-s)^{\alpha-1}|F(s)|ds + \frac{1}{\Gamma(\alpha)}\int_T^t (t-s)^{\alpha-1}v(s)ds + (m-1)l_1, t \geq T.$$

$$\Gamma(\alpha)t^{1-m}u(t) \leq c(T) + t^{1-m}\int_T^t (t-s)^{\alpha-1}v(s)ds, t \geq T, \tag{5.1.7}$$

where

$$C(T) = \sum_{k=0}^{m-1} \frac{|b_k|(T-a)^k}{T^{m-1}k!}\Gamma(\alpha) + T^{1-m}\int_a^T (T-s)^{\alpha-1}|F(s)|ds + \Gamma(\alpha)(m-1)l_1T^{1-m}, t \geq T.$$

The improper integral on the right hand side is convergent. Applying the limit inferior of both sides of inequality (5.1.7) as $t \to \infty$. We have a contradiction to condition (5.1.5). A similar argument gives a contradiction with (5.1.6), when $u(t)$ is eventually negative. $\square$

Next we have the following results.
The proof of these results are the similar as in [27] with $C(T)$ as above

Theorem 5.1.4. *Let conditions (C_1) and (C_2) hold with $\beta > 1$ and $\gamma = 1$. If*

$$lim_{t\to\infty}inf\ t^{1-m}\int_a^t (t-s)^{\alpha-1}[v(s) + H_\beta(s)]ds = -\infty$$

and

$$lim_{t\to\infty}sup\ t^{1-m}\int_a^t (t-s)^{\alpha-1}[v(s) + H_\beta(s)]ds = \infty,$$

where

$$H_\beta(s) = (\beta-1)\beta^{\frac{\beta}{1-\beta}}p_1^{\frac{1}{1-\beta}}p_2^{\frac{\beta}{\beta-1}}(s),$$

then every solution of equation (5.1.2) *is oscillatory.*

Theorem 5.1.5. *Let conditions* (C_1) *and* (C_2) *hold with* $\beta = 1$ *and* $\gamma < 1$. *If*

$$lim_{t\to\infty}inf\ t^{1-m}\int_a^t (t-s)^{\alpha-1}[v(s)+H_\gamma(s)]ds = -\infty$$

and

$$lim_{t\to\infty}sup\ t^{1-m}\int_a^t (t-s)^{\alpha-1}[v(s)+H_\gamma(s)]ds = \infty,$$

where

$$H_\gamma(s) = (1-\gamma)\gamma^{\frac{\gamma}{\gamma-1}} p_1^{\frac{\gamma}{\gamma-1}} p_2^{\frac{1}{1-\gamma}}(s),$$

then every solution of equation (5.1.2) *is oscillatory.*

Theorem 5.1.6. *Let conditions* (C_1) *and* (C_2) *hold with* $\beta > 1$ *and* $\gamma < 1$. *If*

$$lim_{t\to\infty}inf\ t^{1-m}\int_a^t (t-s)^{\alpha-1}[v(s)+H_{\beta,\gamma}(s)]ds = -\infty$$

and

$$lim_{t\to\infty}sup\ t^{1-m}\int_a^t (t-s)^{\alpha-1}[v(s)+H_{\beta,\gamma}(s)]ds = \infty,$$

where

$$H_{\beta,\gamma}(s) = (\beta-1)\beta^{\frac{\beta}{1-\beta}} p_1^{\frac{1}{1-\beta}} \xi^{\frac{\beta}{\beta-1}}(s) + (1-\gamma)\gamma^{\frac{\gamma}{\gamma-1}} \xi^{\frac{\gamma}{\gamma-1}} p_2^{\frac{1}{1-\gamma}}(s),$$

with $\xi \in \mathcal{C}([a,\infty),\mathbb{R}^+)$, *then every solution of equation* (5.1.2) *is oscillatory.*

5.2 On the oscillation of fractional-order delay differential equations with constant coefficients

In this section, we review some oscillation results from [29] including sufficient conditions or necessary and sufficient conditions for the oscillation of fractional-order delay differential equations with constant coefficients. For this, α-exponential function which is a kind of functions that play the same role of the classical exponential functions and Laplace transformation formulations of fractional-order derivatives are used.

In this section, we will focus on first order delay differential equation including fractional-order derivative of the form

$$\dot{u}(t) + pD^\alpha u(t-\tau) + qu(t-\sigma) = 0, \tag{5.2.1}$$

where $p, q, \tau, \sigma \in \mathbb{R}, 0 < \alpha < 1$, and its general form

$$\dot{u}(t) + pD^\alpha u(t-\tau) + \sum_{i=1}^{m} q_i u(t-\sigma_i) = 0, \tag{5.2.2}$$

where $p, q_i, \tau, \sigma_i \in \mathbb{R},\quad i = 1,2,\dots,m,\ \alpha = \frac{\text{odd integer}}{\text{odd integer}}$ such that $0 < \alpha < 1$, some oscillation results are given.

In classical calculus, the function $e^{\lambda t}$ plays an important role in solving ordinary differential equations with constant coefficients, and it satisfies

$$\frac{d}{dt}e^{\lambda t} = \lambda e^{\lambda t}.$$

In fractional calculus, the following α-exponential function:

$$e_{\alpha}^{\lambda t} = t^{\alpha-1}\sum_{k=0}^{\infty}\frac{\lambda^k t^{\alpha k}}{\Gamma((k+1)\alpha)}, \quad (t>0)$$

plays an important role as $e^{\lambda t}$ in solving ordinary differential equations with constant coefficients, and $e_{\alpha}^{\lambda t}$ satisfies the following differential equations:

$$D_0^{\alpha}u(t) = \lambda u(t)$$

and

$$D_0^{\alpha}u(t-\tau) = \lambda u(t-\tau), \quad (t>0),$$

where derivative $D_0^{\alpha}u(t-\tau)$ has the property of

$$D_0^{n\alpha}u(t-\tau) = (D_0^{(n-1)\alpha}(D_0^{\alpha})u(t-\tau)).$$

As α is a positive rational number, getting $\alpha = k/n = k\beta$ satisfying $0<\alpha<1$, where $1 \leq k < n, k \neq n$, we can rewrite equation (5.2.1) in the form

$$D_0^{n\beta}u(t) + pD_0^{k\beta}u(t-\tau) + qu(t-\sigma) = 0.$$

Moreover, for the equation (5.2.1) with constant coefficients, substituting $u(t) = e_{k\beta}^{\lambda t}$, we have

$$(D_0^{n\beta} + pD_0^{k\beta}e_{k\beta}^{-\lambda\tau} + qe_{k\beta}^{-\lambda\sigma})e_{k\beta}^{\lambda t} = f_1(\lambda)e_{k\beta}^{\lambda t},$$

where

$$f_1(\lambda) = \lambda^n + p\lambda e_{k\beta}^{-\lambda\tau} + qe_{k\beta}^{-\lambda\sigma}, \tag{5.2.3}$$

is the characteristic polynomial of equation (5.2.1), and for the equation (5.2.2) with constant coefficients

$$(D_0^{n\beta} + pD_0^{k\beta}e_{k\beta}^{-\lambda\tau} + \sum_{i=1}^{n}q_i e_{k\beta}^{-\lambda\sigma_i} = f_2(\lambda)e_{k\beta}^{\lambda t},$$

where

$$f_2(\lambda) = \lambda^n + p\lambda e_{k\beta}^{-\lambda\tau} + \sum_{i=1}^{n}q_i e_{k\beta}^{-\lambda\sigma_i}, \tag{5.2.4}$$

is the characteristic polynomial of equation (5.2.2).
Assume that equations (5.2.1) and (5.2.2) have a solution of the form $ce_{\alpha}^{\lambda t}$ in the sense of Riemann-Liouville derivative, then λ must be a root of $f_i(\lambda),\ i=1,2$ in (5.2.3) and (5.2.4). Thus, equations (5.2.1) and (5.2.2) have a solution $ce_{\alpha}^{\lambda t}$ if and only if λ is a root of $f_i(\lambda),\ i=1,2$.

5.2.1 First-order delay differential equations with fractional-order derivative

In this subsection we give some oscillation results for the first-order delay differential equation with fractional-order derivative and constant coefficients of the form (5.2.1) and (5.2.2).

Theorem 5.2.1. *Assume that* $p, q, \tau, \sigma \in \mathbb{R}^+$ *and* $\alpha = \frac{odd\ integer}{odd\ integer}$ *such that* $0 < \alpha < 1$.
Then the following statements are equivalent.

(a) Every solution of equation (5.2.1) *oscillates.*

(b) The characteristic equation (5.2.3) *has no real roots.*

Proof. (a) implies (b). The proof is obvious. As the statement (a) holds, if the characteristic equation (5.2.3) has a real root λ_0 then $e^{\lambda_0 t}$ is a non-oscillatory solution of equation (5.2.1). Which is a contradiction to the statement (b).
(b) implies (a). We will make the proof using Caputo's Laplace transform formulation of the fractional derivative. Assume, on the contrary, that (b) holds and that equation (5.2.1) has an eventually positive solution $u(t)$. Since equation (5.2.1) is autonomous, we may assume that $u(t) > 0$ for $t \geq -\tau$. As α is rational number, we can write $\alpha = k/n = k\beta$ satisfying $0 < \alpha < 1$, where $1 \leq k < n,\ k \neq n$. In this case, equation (5.2.1) can be recast into the following delay differential equation with fractional-order derivatives

$$D_0^{n\beta}u(t) + pD_0^{k\beta}u(t-\tau) + qu(t-\sigma) = 0. \tag{5.2.5}$$

One can show that there exist M and μ such that $|u(t)| \leq Me_\alpha^{\mu t},\ t \geq -\tau$. Thus the Laplace transform

$$U(s) = \int_0^\infty e_\alpha^{-st}u(t)dt \tag{5.2.6}$$

exists for $\mathbb{R}(s) > \mu$. Let ϕ_0 be the abscissa of convergence of $U(s)$, that is $\phi_0 = inf\Big\{\phi \in \mathbb{R} : u(\phi) \text{ exists}\Big\}$.
By taking the Laplace transforms of both sides of the equation (5.2.5) we obtain the characteristic equation

$$s^nU(s) - u(0) + qe^{-s\sigma}U(s) + qh(s\sigma) + p\lambda[e^{-s\tau}U(s) + h(s\tau)] = 0,$$

where

$$h(s\sigma) = e^{-s\sigma}\int_{-\sigma}^0 e^{-st}u(t)dt,$$

or

$$F(s)U(s) = Q(s),\ \mathbb{R}(s) > \phi_0, \tag{5.2.7}$$

where

$$F(s) = s^n + qe^{-s\sigma} + p\lambda e^{-s\tau}, \tag{5.2.8}$$

$$Q(s) = u(0) + q\lambda(s\sigma) + p\lambda h(s\tau). \tag{5.2.9}$$

Clearly, $F(s)$ and $Q(s)$ are entire functions. Also by hypothesis, $F(s) \neq 0$ for all real s. Therefore we can write from (5.2.7)

$$U(s) = \frac{Q(s)}{F(s)},\ \mathbb{R}(s) > \phi_0. \tag{5.2.10}$$

We now claim that $\phi_0 = -\infty$. Otherwise $\phi_0 > -\infty$ and the point $s = \phi_0$ must be a singularity of the quotient $Q(s)/F(s)$. But this quotient has no singularity on the real axis. Because of the numerator and dominator are entire functions and by hypothesis the dominator has no real zeros. Thus $\phi_0 = -\infty$ and (5.2.10) becomes

$$U(s) = \frac{Q(s)}{F(s)}, \tag{5.2.11}$$

for all $s \in \mathbb{R}$. Now we can see that as $s \to -\infty$, trough real values, (5.2.11) leads to a contraction because $U(s)$ and $F(s)$ are always positive while $Q(s)$ becomes eventually negative. $\square$

Example 5.2.2.

$$\dot{u}(t) + u^{1/3}(t - \pi/2) + \sqrt{3}u(t - 2\pi/3) = 0, \tag{5.2.12}$$

where $\alpha = 1/3, p = 1, q = \sqrt{3}$ and $\tau = \pi/2, \sigma = 3\pi/2$. All the conditions of Theorem 5.2.1 are satisfied. Hence all the solutions of the equation (5.2.12) are oscillatory.

Theorem 5.2.3. *Assume thatp,* $q_i, \tau, \sigma_i \in \mathbb{R}^+,\ i = 1, 2, \ldots, m,$ *and* $\alpha = \frac{\textit{odd integer}}{\textit{odd integer}}$ *such that* $0 < \alpha < 1$. *Then the following statements are equivalent.*

(a) Every solution of equation (5.2.2) *oscillates,*

(b) The characteristic equation (5.2.4) *has no real roots.*

The proof is same as the proof of Theorem 5.2.1.

Example 5.2.4. We consider first-order delays differential equation with fractional- order derivatives and constant coefficients of the form

$$\dot{u}(t) + 2D^{1/3}u(t - \pi/4) + (3\sqrt{3} - 5)u(t - \pi/6) + \frac{(9 - 3\sqrt{3})}{\sqrt{2}}u(t - 3\pi/4) = 0, \tag{5.2.13}$$

where $\alpha = 1/3, p = 2, q_1 = 3\sqrt{3} - 5, q_2 = \frac{9-3\sqrt{3}}{\sqrt{2}}$ and $\tau = \pi/4, \sigma_1 = \pi/6, \sigma_2 = 3\pi/4$. All the conditions of Theorem 5.2.3 are satisfied. Hence all the solutions of the equation (5.2.13) are oscillatory.

5.2.2 Second-order delay differential equations with fractional-order derivative

We give some oscillation results for the second-order delay differential equation with fractional-order derivative of the general form

$$\ddot{u}(t) + pD^{\alpha}u(t - \tau) + \sum_{i=1}^{m} q_i u(t - \sigma_i) = 0, \tag{5.2.14}$$

where $p, q_i, \tau, \sigma_i \in \mathbb{R}, 0 < \alpha < 1$.

As we stated above that α is a positive rational number, getting $\alpha = k/n = k\beta$ satisfying $0 < \alpha < 1$, where $1 \leq k < n, k \neq n$, we can write equation (5.2.14) as

$$(D_0^{2n\beta} + pD_0^{k\beta}e_{k\beta}^{-\lambda\tau} + \sum_{i=1}^{m} q_i e_{k\beta}^{-\lambda\sigma_i})e_{k\beta}^{\lambda t} = f_3(\lambda)e_{k\beta}^{\lambda t},$$

where

$$f_3(\lambda) = \lambda^{2n} + p\lambda e_{k\beta}^{-\lambda\tau} + \sum_{i=1}^{m} q_i e_{k\beta}^{-\lambda\sigma_i}, \tag{5.2.15}$$

is called the characteristic polynomial of equation (5.2.14). Assume that (5.2.14) has a solution of the form $ce_{\alpha}^{\lambda t}$ in the sense of Riemann Liouville derivative, then λ must be a root of $f_3(\lambda)$. Thus (5.2.14) has a solution $ce_{\alpha}^{\lambda t}$ if and only if λ is a root of $f_3(\lambda)$.

Theorem 5.2.5. *Assume that* $p, q_i, \tau, \sigma_i \in \mathbb{R}^+$ *and* $\alpha = \frac{\textit{odd integer}}{\textit{odd integer}}$ *such that* $0 < \alpha < 1$. *Then the following statements are equivalent.*

(a) Every solution of equation (5.2.14) *oscillates.*

(b) The characteristic equation (5.2.15) *has no real roots.*

Proof. (a) implies (b). The proof is obvious.

(b) implies (a). As α is rational number, we can write $\alpha = k/n = k\beta$ satisfying $0 < \alpha < 1$, where $1 \leq k < n,\ k \neq n$. In this case, equation (5.2.14) can be recast into the following delay differential equation with fractional-order derivatives

$$D_0^{2n\beta}u(t) + pD_0^{k\beta}u(t-\tau) + \sum_{i=1}^{m} q_i u(t-\sigma_i) = 0. \tag{5.2.16}$$

By taking the Laplace transforms of the equation (5.2.16), we obtain the characteristic equation

$$s^{2n}U(s) - u(0) + \sum_{i=1}^{m} q_i e^{-s\sigma_i}U(s) + \sum_{i=1}^{m} q_i h(s\sigma_i) + p\lambda[e^{-s\tau}U(s) + h(s\tau)] = 0,$$

where

$$h(s\sigma_i) = e^{-s\tau}\int_{-\sigma_i}^{0} e^{-st}u(t)dt,$$

or

$$F(s)U(s) = Q(s),\ \mathbb{R}(s) > \phi_0, \tag{5.2.17}$$

where

$$F(s) = s^n + \Sigma_{i=1}^{m} q_i e^{-s\sigma_i} + p\lambda e^{-s\tau}$$

$$Q(s) = u(0) + \Sigma_{i=1}^{m} q_i \lambda(s\sigma_i) + p\lambda h(s\tau).$$

Clearly, $F(s)$ and $Q(s)$ are entire functions. Also by hypothesis, $F(s) \neq 0$ for all real s. Therefore we can write from (5.2.17)

$$U(s) = \frac{Q(s)}{F(s)},\ \mathbb{R}(s) > \phi_0. \tag{5.2.18}$$

We now claim that $\phi_0 = -\infty$. Otherwise $\phi_0 > -\infty$ and the point $s = \phi_0$ must be a singularity of the quotient $\frac{Q(s)}{F(s)}$. But this quotient has no singularity on the real axis. Because of the numerator and dominator are entire functions and by hypothesis the dominator has no real zeros. Thus $\phi_0 = -\infty$ and (5.2.18) becomes

$$U(s) = \frac{Q(s)}{F(s)}, \tag{5.2.19}$$

for all $s \in \mathbb{R}$. Now we can see that as $s \to -\infty$, trough real values, (5.2.19) leads to a contraction because $U(s)$ and $F(s)$ are always positive while $Q(s)$ becomes eventually negative. The proof is complete. $\square$

Example 5.2.6. We consider the fractional-order delay differential equation of the form

$$\ddot{u}(t) + pD^{3/5}u(t - 13\pi/10) + qu(t - 17\pi/10) = 0, \tag{5.2.20}$$

where $\alpha = 3/5, p = \frac{1}{4}\sqrt{2}\sqrt{5-\sqrt{5}}, q = \frac{1}{4}\sqrt{5} + \frac{1}{4}$ and $\tau = 13\pi/10, \sigma = 17\pi/10$. All the conditions of Theorem 5.2.5 are satisfied. Hence all the solutions of the equation (5.2.20) are oscillatory.

Chapter 6

Iterative methods for solving fractional differential equations

The Daftardar-Jafari method (DJ-method) developed in [30], has been used by many researchers for solving linear and nonlinear ordinary and partial differential equations of integer and fractional order. The method converges to the exact solution, if it exists through successive approximations. For concrete problems, a few number of approximations can be used for numerical purposes with high degree of accuracy. The DJ-method does not require any restrictive assumptions for nonlinear terms as required by some existing methods. The aim of this chapter is to effectively employ DJ-method to obtain solutions for different type of equations.

The variational iteration method requires the determination of Lagrange multiplier [31–33] in its computational algorithm, DJ-method is independent of any such requirements. Moreover, unlike the Adomian decomposition method [34–42], where the calculation of the tedious Adomian polynomials is needed to deal with nonlinear terms, DJ-method handles linear and nonlinear terms in a simple and straightforward way without any additional requirements.

Many problems in chemistry, physics and biology have their mathematical setting as fractional differential equations. Therefore, methods to solve fractional differential equations, are receiving significant attention in recent years. In this chapter we utilize the DJ-method to obtain solutions of nonlinear fractional equations. The method when unite with algebraic computing software (Mathematica, e.g.) turns out to be powerful.

6.1 Method for solving fractional differential equations

In this section we use the Daftardar and Jafari method for solving fractional differential equations.

A variety of problems in physics, chemistry and biology have their mathematical setting as integral equations. Therefore, developing methods to solve integral equations, is receiving increasing attention in recent years. In this chapter we describe an iterative method which can be utilized to obtain solutions of nonlinear functional equations. The method when combined with algebraic computing software (Mathematica, e.g.) turns out to be powerful.

6.1.1 DJ-method

Considering the following general functional equation [43]:

$$u = f + N(u), \tag{6.1.1}$$

where N is a nonlinear operator from a Banach space $B \to B$ and f is a known function. We are looking for a solution u of (6.1.1) having the series form

$$u = \sum_{i=0}^{\infty} u_i. \tag{6.1.2}$$

The nonlinear operator N can be decomposed as

$$N(\sum_{i=0}^{\infty} u_i) = N(u_0) + \sum_{i=1}^{\infty} \left\{ N(\sum_{j=0}^{i} u_j) - N(\sum_{j=0}^{i-1} u_j) \right\}. \tag{6.1.3}$$

From (6.1.2) and (6.1.3), the eq (6.1.1) is equivalent to

$$\sum_{i=0}^{\infty} u_i = f + N(u_0) + \sum_{i=1}^{\infty} \left\{ N(\sum_{j=0}^{i} u_j) - N(\sum_{j=0}^{i-1} u_j) \right\}.$$

We define the recurrence relation:

$$\begin{aligned}
u_0 &= f, \\
u_1 &= N(u_0), \\
u_2 &= N(u_0 + u_1) - N(u_0), \\
u_3 &= N(u_0 + u_1 + u_2) - N(u_0 + u_1), \\
&\vdots \\
u_{m+1} &= N(u_0 + \cdots + u_m) - N(u_0 + \cdots + u_{m-1}),
\end{aligned}$$

where

$$m = 1, 2, \ldots$$

and

$$u = f + \sum_{i=0}^{\infty} u_i$$

If N is a contraction, i.e. $||N(x) - N(u)|| \leq K||x - u||, 0 < k < 1$, then

$$||u_{m+1}|| = ||N(u_0 + \cdots + u_m) - N(u_0 + \cdots + u_{m-1})|| \leq K||u_m|| \leq K^m||u_0||, \quad m = 0, 1, 2, \ldots,$$

and the series $\sum_0^{\infty} u_i$ absolutely and uniformly converges to a solution of (6.1.1), which is unique, in view of Banach fixed point theorem.

6.2 Nonlinear fractional differential equations

Consider the fractional differential equation

$$
{}^cD^{\alpha}u - f(t,u(t)) = g(t), \tag{6.2.1}
$$

$$
u(a) = c_1, u'(a) = c_2.
$$

For $1 < \alpha \leq 2$, $|g(t)| \leq M$. Applying Caputo integral, we have

$$
u(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s))ds + \int_a^t \frac{(t-s)^{\alpha-1}g(s)}{\Gamma(\alpha)}ds + c_1 + c_2 t,
$$

where $|t-a| \leq h$, f is a continuous function of its arguments and satisfies Lipschitz condition,

$$
|f(t,\phi(t)) - f(t,\psi(t))| < K|\phi - \psi|. \; Let \; |f(t,\phi(t))| < M'.
$$

Define

$$
\begin{aligned}
u_0(t) &= \int_a^t \frac{(t-s)^{\alpha-1}g(s)}{\Gamma(\alpha)}ds + c_1 + c_2 t,\\
u_1(t) &= \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u_0(s))ds,\\
u_{m+1} &= \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s,u_0+\cdots+u_m) - f(s,u_0+\cdots+u_{m-1})|ds, \; m = 1,2,\ldots
\end{aligned}
$$

We prove $\sum_{i=1}^{\infty} u_i(x)$ is uniformly convergent.

$$
\begin{aligned}
&|u_1(t)| \leq \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,y_0(s))|ds \leq \frac{M'}{\Gamma(\alpha)}\frac{(t-a)^{\alpha}}{\alpha} \leq M'\frac{h^{\alpha}}{\Gamma(\alpha+1)},\\
&|u_2(t)| \leq \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,u_0+u_1) - f(s,u_0)|ds \leq K\int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|u_1|ds \leq \frac{M'Kh^{\alpha+1}}{\Gamma(\alpha+2)}\\
&\vdots\\
&|u_{m+1}(t)| \leq \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,u_0+u_1+\cdots+u_m) - f(s,u_0+\cdots+u_{m-1})|ds \leq\\
&K\int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|u_{m-1}|ds \leq \frac{M'K^m h^{\alpha+m}}{\Gamma(\alpha+m+1)}.
\end{aligned}
$$

Hence $\sum_{i=0}^{\infty} u_i(t)$ is absolutely and uniformly convergent and $u(t)$ satisfies the differential equation (6.2.1). If equation (6.2.1) does not possess unique solution, then this iterative method will give a solution among many other solutions.

6.3 Illustrative examples

Initial and boundary value problem:

(1) Consider the following non linear differential equation [43]

$$u' = -u^2, \tag{6.3.1}$$

$$u(1) = 1.$$

With exact solution $u = 1/x$.

Applying Riemann Liouville integral, the corresponding Volterra integral equation is

$$u = 1 - \int_1^x u^2 dt.$$

By DJ-method:

$$\begin{aligned}
&u_0 = 1,\\
&u_1 = N(u_0) = -\int_1^x u_0^2 dt = 1 - x,\\
&u_2 = N(u_0 + u_1) - N(u_0) = \frac{4}{3} - 3x + 2x^2 - \frac{x^3}{3},\\
&u_3 = N(u_0 + u_1 + u_2) - N(u_0 + u_1) = \frac{113}{63} - \frac{64x}{9} + \frac{34x^2}{3} - \frac{85x^3}{9} + \frac{41x^4}{9} - \frac{4x^5}{3} + \frac{2x^6}{9} - \frac{x^7}{63},\\
&\vdots\\
&u = u_0 + u_1 + u_2 + \dots\\
&\ = 2 - x + \frac{4}{3} - 3x + 2x^2 - \frac{x^3}{3} + \dots
\end{aligned}$$

We have solved this Volterra integral equation using DJ-method and results are compared with exact solution of (6.3.1) in the following figures. These figures shows the graphs of numerical solution and the exact solution of (6.3.1). In the table 6.1, after five iterations by DJ-method, we compare the result with exact solution of (6.3.1) for different values of x.

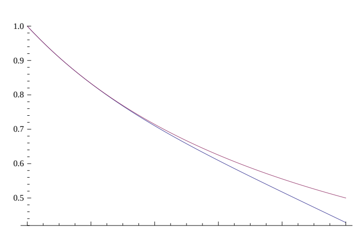


Figure 6.1: Exact and approximate solution of (6.3.1) after three iterations

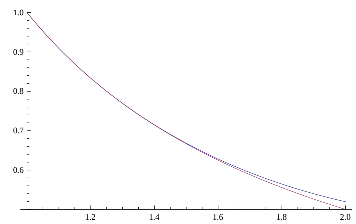


Figure 6.2: Exact and approximate solution of (6.3.1) after four iterations

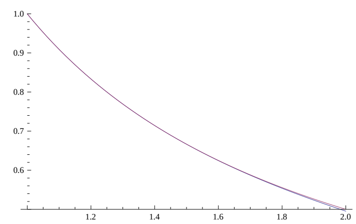


Figure 6.3: Exact and approximate solution of (6.3.1) after five iterations

x	$Exact.Sol$	$App.Sol$	$Error\%$
1.0	1.0	1.0	0.0
1.1	0.909090	0.909091	0.00011
1.2	0.833333	0.833332	0.00012
1.3	0.769231	0.769216	0.00195
1.4	0.714286	0.714222	0.0089608
1.5	0.66667	0.666472	0.0297087
1.6	0.625	0.624530	0.0752
1.7	0.588235	0.587268	0.164661
1.8	0.55556	0.55378	0.321427
1.9	0.526316	0.523328	0.570923
2.0	0.5	0.4953	0.94

Table 6.1: Percentage error of equation (6.3.1) for different values of x after five iterations.

(2) Consider the following differential equation

$$u'' + u = 0, \tag{6.3.2}$$

$$u(0) = 1, \ u'(0) = 2.$$

With exact solution $u = 2\sin t + \cos t$.

The initial value problem in (6.3.2) is equivalent to the integral equation

$$u(t) = -\int_0^t \frac{(t-s)}{\Gamma(2)} u(s)ds + 1 + 2t.$$

Applying DJ-method, we have,

$$
\begin{aligned}
&u_0 = 1 + 2t,\\
&u_1 = N(u_0) = -\int_0^t (t-s)u_0(s)ds\\
&= -\int_0^t (t-s)(1+2s)ds\\
&= \frac{-t^2}{2} - \frac{t^3}{3},\\
&u_2 = N(u_0+u_1) - N(u_0)\\
&= -\int_0^t (t-s)[u_0(s)+u_1(s)]ds + \int_0^t (t-s)u_0(s)ds\\
&= \frac{t^4}{4!} + \frac{t^5}{60}.\\
&\vdots
\end{aligned}
$$

so

$$
\begin{aligned}
u &= u_0 + u_1 + u_2 + \dots\\
&= 1 + 2t - \frac{t^2}{2!} - \frac{t^3}{3} + \frac{t^4}{4!} + \frac{t^5}{60} + \dots\\
&= 2t - 2\frac{t^3}{3!} + 2\frac{t^5}{5!} + 1 - \frac{t^2}{2!} + \frac{t^4}{4!} + \dots\\
&= 2(t - \frac{t^3}{3!} + \frac{t^5}{5!} + \dots) + (1 - \frac{t^2}{2!} + \frac{t^4}{4!} + \dots)\\
&= 2\sin t + \cos t.
\end{aligned}
$$

(3) Consider the following fractional order differential equation

$$
{}^cD^{\alpha}u(x) + u(x) = 0, \tag{6.3.3}
$$

$$
u(0) = 1,\ u'(0) = 0.
$$

With exact solution

$$
u(x) = \sum_{k=0}^{\infty} \frac{(-x^{\alpha})^k}{\Gamma(\alpha k + 1)}.
$$

Let $\alpha = 1.2$. Then we have,

$$
{}^cD^{1.2}u(x) = -u(x),
$$

$$
u(0) = 1,\ u'(0) = 0.
$$

Applying Riemann Liouville integral, we have

$$
u(x) = -\int_0^x \frac{(x-s)^{0.2}}{\Gamma(1.2)}u(s)ds + 1.
$$

By DJ-method, we have

$$f(x) = 1,\ N(u) = -\int_0^x \frac{(x-s)^{0.2}}{\Gamma(1.2)} u(s)ds$$

so,

$$\begin{aligned}
&u_0 = 1, \\
&u_1 = N(u_0) = -\int_0^x \frac{(x-s)^{0.2}}{\Gamma(1.2)} u_0(s)ds \\
&= -0.90760x^{1.2}, \\
&u_2 = N(u_0 + u_1) - N(u_0) \\
&= 0.335435x^{2.4}, \\
&u_3 = N(u_0 + u_1 + u_2) - N(u_0 + u_1) \\
&= -0.0747313x^{3.6} \\
&\vdots \\
&u = u_0 + u_1 + u_2 + \dots \\
&u = \sum_{k=0}^{\infty} \frac{(-x^\alpha)^k}{\Gamma(\alpha k + 1)}.
\end{aligned}$$

(4) Consider the following boundary value problem

$$u'' + u = 0, \tag{6.3.4}$$

$$u(0) = 1,\ u(\pi/2) = 2,$$

with exact solution $u = 2\sin t + \cos t$.

Applying Riemann Liouville integral the corresponding integral equation is:

$$u(t) = 1 + \frac{2}{\pi}t + \frac{2t}{\pi}\int_0^{\frac{\pi}{2}} (\frac{\pi}{2} - s)u(s)ds - \int_0^t (t-s)u(s)ds.$$

By DJ-method,

$$\begin{aligned}
&u_0 = 1 + \frac{2t}{\pi}, \\
&u_1 = N(u_0) = \frac{\pi t}{3} - \frac{t^2}{2} - \frac{t^3}{3\pi}, \\
&u_2 = -\frac{\pi t}{3} + \frac{2(\frac{\pi^2}{6} + \frac{11\pi^4}{2880})t}{\pi} - \frac{\pi t^3}{18} + \frac{t^4}{24} + \frac{t^5}{60\pi}, \\
&u_3 = -\frac{2(\frac{\pi^2}{6} + \frac{11\pi^4}{2880})t}{\pi} + \frac{2(\frac{\pi^2}{6} + \frac{11\pi^4}{2880} + \frac{47\pi^6}{483840})t}{\pi} - \frac{11\pi^3 t^3}{8640} + \frac{\pi t^5}{360} - \frac{t^6}{720} - \frac{t^7}{2520\pi}, \\
&\vdots \\
&u = u_0 + u_1 + u_2 + \dots \\
&= 1 + \frac{2t}{\pi} + \frac{\pi t}{3} - \frac{t^2}{2} - \frac{t^3}{3\pi} - \frac{\pi t}{3} + \frac{2(\frac{\pi^2}{6} + \frac{11\pi^4}{2880})t}{\pi} - \frac{\pi t^3}{18} + \frac{t^4}{24} + \frac{t^5}{60\pi} + \dots
\end{aligned}$$

Using DJ-method, following figures shows the graphs of numerical solution and exact solution after three and four iterations.

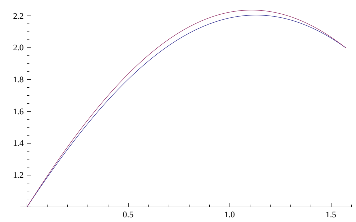


Figure 6.4: Exact and approximate solution of (6.3.4) for $m = 3$.

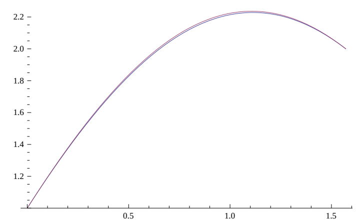


Figure 6.5: Exact and approximate solution of (6.3.4) for $m = 4$.

6.3.1 Boundary value problem with variable coefficients

(5) Consider the boundary value problem [44]

$$^cD^\alpha u(x) - \frac{(1-x)}{(1+x)^2}u(x) = \frac{1}{(1+x)^2},\ 1 < \alpha \leq 2$$

$$u(0) = 1,\ u(1) = \frac{1}{2}.$$

With exact solution $1/1 + x$.

For $\alpha = 2$ corresponding integral equation is:

$$u(x) = 1 - \frac{x}{2} + x\ln 2 + x - \ln(1+x) + \int_0^x \frac{(x-s)(1-s)}{(1+s)^2}u(s)ds - x\int_0^1 \frac{(1-s)^2}{(1+s)}u(s)ds.$$

By DJ-method we have the following iterations. Here *Log* stands for ln.

```
u0 = 1 - x/2 + x Log[2] - Log[1 + x],

u1 = -(1/4) x (19 + (-27 + 20 Log[2]) Log[4]) + 1/4 (x (18 + x - 20
Log[2] - x Log[4]) + 2 Log[1 + x] (-9 + x (-7 + Log[64]) + Log[1024]
+ (3 + x) Log[1 + x])),

u2 = 1/4 x (19 + (-27 + 20 Log[2]) Log[4]) - 1/4 x (19 - 56 Log[2] +
8 Log[2]^2 + Log[4] + x (-3 - 2 Log[2] (-53 + 54 Log[4]) + 40
```

```
Log[2]^2 (-1 + Log[16])) + Log[16] Log[256] + x^2 (3 + Log[4]
Log[16] - Log[1024]) - 2 (-3 + Log[16]) (-9 + x (-7 + Log[64]) +
Log[1024]) Log[1 + x] - 2 (3 + x) (-3 + Log[16]) Log[1 + x]^2) + 1/4
(x^2 (-1 +34 Log[2] - 40 Log[2]^2) + x (17 - 50 Log[2] + 40
Log[2]^2) - x (-1 - 30 Log[2] + 40 Log[2]^2) - x^2 (1 - 66 Log[2] +
80 Log[2]^2) - x^3 (-1 + Log[4]) - x^3 (-2 + Log[16]) + 2 (1 + 2 x)
(-9 + x (-7 + Log[64]) + Log[1024]) Log[1 + x] + x (-15 - 102 Log[2]
+ 120 Log[2]^2 + x^2 (-1 + Log[4]) + 2 x (-8 + 20 Log[2]^2 -
Log[256]) + Log[1048576]) Log[1 + x] + 2 (3 + 7 x + 2 x^2) Log[1 +
x]^2 - 2 (3 + x) (-9 + x (-8 + Log[64]) + Log[1024]) Log[1 + x]^2 -
2 (3 + x)^2 Log[1 + x]^3) + 1/4 (-x (18 + x - 20 Log[2] - x Log[4])
- 2 Log[1 + x] (-9 + x (-7 + Log[64]) + Log[1024] + (3 + x) Log[1 +
x])),

u = u0+u1+u2+...
  =1 - x/2 + x Log[2] - Log[1 + x] - 1/4 x (19 - 56
Log[2] + 8 Log[2]^2 + Log[4] + x (-3 - 2 Log[2] (-53 + 54 Log[4]) +
40 Log[2]^2 (-1 + Log[16])) + Log[16] Log[256] + x^2 (3 + Log[4]
Log[16] - Log[1024]) - 2 (-3 + Log[16]) (-9 + x (-7 + Log[64]) +
Log[1024]) Log[1 + x] - 2 (3 + x) (-3 + Log[16]) Log[1 + x]^2) + 1/4
(x^2 (-1 + 34 Log[2] - 40 Log[2]^2) + x (17 - 50 Log[2] + 40
Log[2]^2) - x (-1 - 30 Log[2] + 40 Log[2]^2) - x^2 (1 - 66 Log[2] +
80 Log[2]^2) - x^3 (-1 + Log[4]) - x^3 (-2 + Log[16]) + 2 (1 + 2 x)
(-9 + x (-7 + Log[64]) + Log[1024]) Log[1 + x] + x (-15 - 102 Log[2]
+ 120 Log[2]^2 + x^2 (-1 + Log[4]) + 2 x (-8 + 20 Log[2]^2 -
Log[256]) + Log[1048576]) Log[1 + x] + 2 (3 + 7 x + 2 x^2) Log[1 +
x]^2 - 2 (3 + x) (-9 + x (-8 + Log[64]) + Log[1024]) Log[1 + x]^2 -
2 (3 + x)^2 Log[1 + x]^3) + 1/4 (-x (18 + x - 20 Log[2] - x Log[4])
- 2 Log[1 + x] (-9 + x (-7 + Log[64]) + Log[1024] + (3 + x) Log[1 +
x])) +
 1/4 (x (18 + x - 20 Log[2] - x Log[4]) +2 Log[1 + x] (-9 + x (-7 + Log[64]) +
 Log[1024] + (3 + x) Log[1 + x])),
       .
       .
       .
```

Graph of the solution by DJ-method on Mathematica is given below.

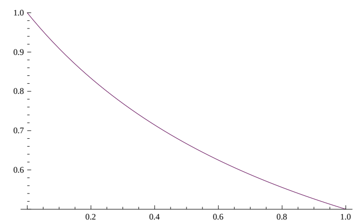


Figure 6.6: Exact and approximate solution of (4) after three iterations.

6.4 Numerical solutions for system of non linear fractional differential equations by the DJ-method

Example 6.4.1. Consider the following non linear system of fractional differential equations [45]:

$$\begin{cases} {}^cD^\alpha u_1 &= & -u_1 + u_2u_3,\ u_1(0) = 1 \\ {}^cD^\alpha u_2 &= & u_1 - u_2u_3 - 2u_2^2,\ u_2(0) = 2 \\ {}^cD^\alpha u_3 &= & u_2^2,\ u_3(0) = 0. \end{cases} \tag{6.4.1}$$

For $\alpha = 1$ system becomes,

$$\begin{aligned} {}^cDu_1 &= -u_1 + u_2u_3,\ u_1(0) = 1 \\ {}^cDu_2 &= u_1 - u_2u_3 - 2u_2^2,\ u_2(0) = 2 \\ {}^cDu_3 &= u_2^2,\ u_3(0) = 0. \end{aligned}$$

The corresponding system of integral equations is:

$$\begin{aligned} u_1 &= 1 + \int_0^t [u_2(s)u_3(s) - u_1(s)]ds \\ u_2 &= 2 + \int_0^t [u_1(s) - u_2(s)u_3(s) - 2u_2^2(s)]ds \\ u_3 &= \int_0^t u_2^2(s)ds. \end{aligned}$$

Applying DJ-method with the help of Mathematica we have,

$$\begin{aligned} &N_1 = \int_0^t [u_2(s)u_3(s) - u_1(s)]ds, \\ &N_2 = \int_0^t [u_1(s) - u_2(s)u_3(s) - 2u_2^2(s)]ds, \\ &N_3 = \int_0^t u_2^2(s)ds. \\ &u_{10} = 1,\ u_{20} = 2,\ u_{30} = 0 \ and \\ &u_{11} = N_1(u_{10}, u_{20}, u_{30}) = \int_0^t [u_{20}(s)u_{30}(s) - u_{10}(s)]ds = -t, \\ &u_{12} = N_1(u_{10} + u_{11}, u_{20} + u_{21}, u_{30} + u_{31}) - N_1(u_{10}, u_{20}, u_{30}) \\ &= \int_0^t [(u_{20} + u_{21})(u_{30} + u_{31})(s) - (u_{10} + u_{11})(s)]ds - \int_0^t [u_{20}(s)u_{30}(s) - u_{10}(s)]ds \\ &= \frac{9t^2}{2} - \frac{28t^3}{3}. \end{aligned}$$

As $\quad u_{21} = N_2(u_{10}, u_{20}, u_{30}) = \int_0^t [u_{10}(s) - u_{20}(s)u_{30}(s) - 2u_{20}^2(s)]ds = -7t$

and $\quad u_{31} = N_3(u_{10}, u_{20}, u_{30}) = \int_0^t u_{20}^2(s)ds = 4t.$

$\vdots$

so,

$u_1 = u_{10} + u_{11} + u_{12} + \cdots = 1 - t + \frac{9t^2}{2} - \frac{28t^3}{3} + \ldots$

$u_2 = u_{20} + u_{21} + u_{22} + \cdots = 2 - 7t + \frac{47t^2}{2} - \frac{70t^3}{3} + \ldots$

$u_3 = u_{30} + u_{31} + u_{32} + \cdots = 4t - 14t^2 + \frac{49t^3}{3} + \ldots$

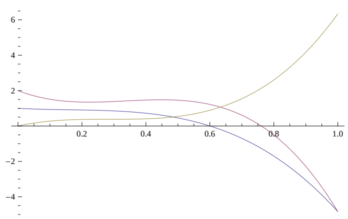


Figure 6.7: Approximate solution of system (6.4.1) after three iterations.

6.5 An approach for solving a system of fractional partial differential equations

In this section we use the Iterative Laplace Transform method [ILTM] [46,47] for solving systems of linear and nonlinear fractional partial differential equations. The Iterative Laplace transform method is a mixture of the Laplace transform method and the DJ-method. Iterative laplace transform method has no round-off errors and therefore, the numerical computations are reduced. The fractional derivative is described in the Caputo sense. Illustrative examples are given to understand the effectiveness of this method.

Definition 6.5.1. The Laplace transform $L[f(t)]$, of the Riemann-Liouville fractional integral is defined as

$$L[I^{\mu} f(t)] = s^{-\mu} F(s).$$

Definition 6.5.2. The Laplace transform $L[f(t)]$, of the Caputo fractional derivative is defined as

$$L[D^{\mu} f(t)] = s^{\mu} F(s) - \sum_{k=0}^{n-1} s^{(\mu-k-1)} f^{(k)}(0),\ n-1 < \mu \leq n.$$

6.5.1 ILTM and system of fractional partial differential equations

Consider the following system of fractional partial differential equations (FDEs) with the initial conditions of the form:

$$D_t^{\alpha_i} u_i(\bar{x}, t) = A_i(u_1(\bar{x}, t), \ldots, u_n(\bar{x}, t)),\ m_i - 1 < \alpha_i \leq m_i,\ i = 1, 2, \ldots, n, \tag{6.5.1}$$

$$\frac{\partial^{(k_i)} u_i(\bar{x}, 0)}{\partial t^{(k_i)}} = h_{ik_i}(\bar{x}),\ k_i = 0, 1, \ldots, m_i - 1,\ m_i \in \mathbb{N}, \tag{6.5.2}$$

where A_i are nonlinear operators and $u_i(\bar{x},t)$ are unknown functions. Taking the laplace transforms L on both sides of (6.5.1), we obtain

$$L\left[D_t^{\alpha_i} u_i(\bar{x},t)\right] = L[A_i(u_1(\bar{x},t),\ldots,u_n(\bar{x},t))],\ i=1,2,\ldots,n.$$

Using initial condition (6.5.2) and by the Laplace transform of the Caputo fractional derivative, we obtain

$$s^{\alpha_i} L[u_i(\bar{x},t)] - \sum_{k=0}^{m_i-1} s^{\alpha_i-k-1} u_i^{(k)}(\bar{x},0) = L[A_i(u_1(\bar{x},t),\ldots,u_n(\bar{x},t))],\ i=1,2,\ldots,n. \tag{6.5.3}$$

Operating with the Laplace inverse on both sides of (6.5.3) we get

$$u_i(\bar{x},t)] = L^{-1}\left[\sum_{k=0}^{m_i-1} s^{-k-1} u_i^{(k)}(\bar{x},0)\right] + L^{-1}[s^{-\alpha_i} L[A_i(u_1(\bar{x},t),\ldots,u_n(\bar{x},t))]], \tag{6.5.4}$$

$$u_i(\bar{x},t)] = f_i + N_i(u_1(\bar{x},t),\ldots,u_n(\bar{x},t))\ i=1,2,\ldots,n.$$

where

$$f_i = L^{-1}\left[\sum_{k=0}^{m_i-1} s^{-k-1} u_i^{(k)}(\bar{x},0)\right],\ i=1,2,\ldots,n,$$

$$N_i(u_1(\bar{x},t),\ldots,u_n(\bar{x},t)) = L^{-1}[s^{-\alpha_i} L[A_i(u_1(\bar{x},t),\ldots,u_n(\bar{x},t))]].$$

We now look for a solution u of (6.5.4) having the series form

$$u_i(\bar{x},t) = \sum_{j=0}^{\infty} u_{ij}(\bar{x},t)\ i=1,2,\ldots,n.$$

The nonlinear operator N_i can be decomposed as
$N_i\left[\sum_{j=0}^{\infty} u_{1j}(\bar{x},t),\ldots,\sum_{j=0}^{\infty} u_{nj}(\bar{x},t)\right] = N_i(u_{10}(\bar{x},t),\ldots,u_{n0}(\bar{x},t))$
$+\sum_{j=1}^{\infty}[N_i(\sum_{k=0}^{j} u_{1k}(\bar{x},t),\ldots,\sum_{k=0}^{j} u_{nk}(\bar{x},t)) - N_i(\sum_{k=0}^{j-1} u_{1k}(\bar{x},t),\ldots,\sum_{k=0}^{j-1} u_{nk}(\bar{x},t))]$.
We define the recurrence relation

$u_{i0}(\bar{x},t) = L^{-1}\left[\sum_{k=0}^{m_i-1} s^{-k-1} u_i^{(k)}(\bar{x},0)\right],$

$u_{i1}(\bar{x},t) = L^{-1}[[s^{-\alpha_i} L[A_i(u_{10}(\bar{x},t),\ldots,u_{n0}(\bar{x},t))]],$

$u_{i(m+1)}(\bar{x},t) = L^{-1}[[s^{-\alpha_i} L[A_i(u_{10}(\bar{x},t),\ldots,+u_{1m}(\bar{x},t)),\ldots,(u_{n0}(\bar{x},t)+\cdots+u_{nm}(\bar{x},t))]]-$

$L^{-1}[[s^{-\alpha_i} L[A_i(u_{10}(\bar{x},t)+\cdots+u_{1(m-1)}(\bar{x},t)),\ldots,\ldots,\ldots,(u_{n0}(\bar{x},t)+\cdots+u_{n(m-1)}(\bar{x},t))]].$

Then

$u_{i1}(\bar{x},t)+\cdots+u_{i(m+1)}(\bar{x},t) = L^{-1}[[s^{-\alpha_i} L[A_i(u_{10}(\bar{x},t),\ldots,+u_{1m}(\bar{x},t)),\ldots,(u_{n0}(\bar{x},t)+\cdots+u_{nm}(\bar{x},t))]].$

The n-term approximate solution of (6.5.1)-(6.5.2) is given by

$$u_i(\bar{x},t) \cong u_{i1}(\bar{x},t)+\cdots+u_{in}(\bar{x},t),\ i=1,2,\ldots,n.$$

The above series solution generally converges very rapidly.

Example 6.5.3. Consider the following FDEs:

$$
\begin{aligned}
{}^cD_t^{\alpha}u - v_x + v + u &= 0, \\
{}^cD_t^{\beta}v - u_x + v + u &= 0, \ (0 < \alpha, \beta \leq 1),
\end{aligned}
$$

with initial conditions

$$u(x,0) = \sinh x, \ v(x,0) = \cosh x.$$

The exact solution, when $\alpha = \beta = 1$ is

$$u(x,t) = \sinh(x-t), \ v(x,t) = \cosh(x-t).$$

The system of linear FDEs corresponds to the following Laplace equations:

$$
\begin{aligned}
u(x,t) &= L^{-1}[s^{-1}u(x,0)] + L^{-1}[s^{-\alpha}L[v_x(x,t) - v(x,t) - u(x,t)]], \\
v(x,t) &= L^{-1}[s^{-1}v(x,0)] + L^{-1}[s^{-\beta}L[u_x(x,t) - v(x,t) - u(x,t)]].
\end{aligned}
$$

First few terms of $u(x,t)$ and $v(x,t)$ are

$$
\begin{aligned}
&u_0(x,t) = \sinh x, \quad v_0(x,t) = \cosh x \\
&u_1(x,t) = N_1(u_0, v_0) = L^{-1}[s^{-\alpha}L[v_{0x}(x,t) - v_0(x,t) - u_0(x,t)]] \\
&= -I^{\alpha} \cosh x t^0 = -\frac{\cosh x t^{\alpha}}{\Gamma(\alpha+1)}, \\
&v_1(x,t) = N_2(u_0, v_0) = -\frac{\sinh x t^{\beta}}{\Gamma(\beta+1)}, \\
&u_2(x,t) = N_1(u_0 + u_1, v_0 + v_1) - N_1(u_0, v_0) \\
&= L^{-1}[s^{-\alpha}L[\frac{-\cosh x t^{\beta}}{\Gamma(\beta+1)} + \frac{\sinh x t^{\beta}}{\Gamma(\beta+1)} + \frac{\cosh x t^{\alpha}}{\Gamma(\alpha+1)}]] \\
&= I^{\alpha}[\frac{-\cosh x t^{\beta}}{\Gamma(\beta+1)} + \frac{\sinh x t^{\beta}}{\Gamma(\beta+1)} + \frac{\cosh x t^{\alpha}}{\Gamma(\alpha+1)}] \\
&\frac{-\cosh x t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{\sinh x t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{\cosh x t^{2\alpha}}{\Gamma(2\alpha+1)}.
\end{aligned}
$$

Similarly,

$$v_2(x,t) = \frac{-\cosh(x) t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{\sinh x t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{\cosh x t^{2\alpha}}{\Gamma(2\alpha+1)}.$$

The series solution is then given by

$$
\begin{aligned}
&u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \ldots \\
&= \cosh(x)[1 + \frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \ldots] - \sinh x[\frac{t^{\beta}}{\Gamma(\beta+1)} + \frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} - \frac{t^{2\beta}}{\Gamma(2\beta+1)} + \ldots].
\end{aligned}
$$

Setting $\alpha = \beta = 1$, we reproduce the solution as follows:

$$u(x,t) = \sinh(x-t), \quad v(x,t) = \cosh(x-t).$$

Example 6.5.4. Consider the following nonlinear system for $0 < \alpha,\ \beta < 1$.

$$\begin{aligned}
&{}^{c}D_t^{\alpha}u - u_{xx} - 2uu_x + (uv)_x = 0,\\
&{}^{c}D_t^{\beta}v - v_{xx} - 2vv_x + (uv)_x = 0.\\
&u(x,0) = \sin(x),\ v(x,0) = \sin(x).
\end{aligned}$$

For $\alpha = \beta = 1$; exact solution of system is:

$$u(x,t) = v(x,t) = \sin(x)e^{-t}.$$

Since

$$\begin{aligned}
&u_i(x,t) = L^{-1}\left[\sum_{k=0}^{m_i-1} s^{-k-1}u_i^{(k)}(x,0)\right] + L^{-1}[s^{-\alpha}L[A_i(u_1(x,t),\dots,u_n(x,t))]],\ i = 1,2,\dots,n.\\
&u(x,t) = L^{-1}\left[s^{-1}u(x,0)\right] + L^{-1}\left[s^{-\alpha}L\left[u_{xx} + 2uu_x - (uv)_x\right]\right],\\
&u_0(x,t) = \sin(x),\ u_{0x} = \cos(x),\ u_{0xx} = -\sin(x),\\
&u_1(x,t) = N_1(u_0,v_0)\\
&= L^{-1}[s^{-\alpha}L[u_{0xx} + 2u_0u_{0x} - (u_0v_0)_x]]\\
&= -I^{\alpha}(\sin(x))t^0 = -\sin(x)\frac{\Gamma(1)}{\Gamma(\alpha+1)}t^{\alpha}.\\
&u_1 = -\frac{\sin(x)t^{\alpha}}{\Gamma(\alpha+1)},\ u_{1x} = -\frac{\cos(x)t^{\alpha}}{\Gamma(\alpha+1)},\ u_{1xx}\\
&= \frac{\sin(x)t^{\alpha}}{\Gamma(\alpha+1)}.
\end{aligned}$$

For $\alpha = 1$

$$\begin{aligned}
&u_1 = -\sin(x)t\\
&u_2(x,t) = N_1(u_0+u_1, v_0+v_1) - N_1(u_0,v_0)
\end{aligned}$$

For $\alpha = \beta = 1$

$$\begin{aligned}
&u_2(x,t) = L^{-1}[s^{-1}L[\sin(x)t]] = \sin(x)I(t^1) = \frac{t^2}{2!}\sin(x)\\
&\vdots\\
&u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \dots\\
&= \sin(x)[1 - t + \frac{t^2}{2!} - \dots] = \sin(x)e^{-t}.
\end{aligned}$$

6.6 Convergence of the iterative method

An iterative method [48] can also be stated as:

$$u = f + N(u),$$

where N is a nonlinear operator from a Banach space $B \to B$, and f is a given element of the Banach space B. u is assumed to be a solution of $u = f + N(u)$. having the series form

$$u = \sum_{0}^{\infty} u_i.$$

The operator N is decomposed as

$$N(u) = N(u_0) + [N(u_0 + u_1) - N(u_0)] + [N(u_0 + u_1 + u_2) - N(u_0 + u_1)] + \ldots$$

Let $G_0 = N(u_0)$ and

$$G_n = N\left(\sum_{i=0}^{n} u_i) - N(\sum_{i=0}^{n-1} u_i\right), \ n = 1, 2, \ldots.$$

Then

$$N(u) = \sum_{i=0}^{\infty} G_i.$$

$$u_0 = f$$

$$u_n = G_{n-1}, \ n = 1, 2, \ldots,$$

then

$$u = \sum_{i=0}^{\infty} u_i$$

is a solution of $u = f + N(u)$.

6.6.1 Convergence of DJ-method

We state some conditions [48] for DJ-method to be convergent.

Theorem 6.6.1. *If N is C^∞ in a neighborhood of u_0 and*

$$||N^n(u_0)|| = Sup\{N^n(u_0)(h_1, \ldots, h_n) : ||h_i|| \le 1, \ 1 \le i \le n\} \le L,$$

for any n and for some real $L > 0$ and $||u_i|| \le M < 1/e, i = 1, 2, \ldots,$ then the series $\sum_{n=0}^{\infty} G_n$ is absolutely convergent, and moreover,

$$||G_n|| \le LM^n e^{n-1}(e-1), \ n = 1, 2, \ldots$$

Theorem 6.6.2. *If N is C^∞ and $||N^n u_0|| \le M \le e^{-1}$, for all n, then the series $\sum_{n=0}^{\infty} G_n$ is absolutely convergent.*

Example 6.6.3. Consider the nonlinear initial value problem,

$$u'(t) = \frac{1}{2} + \frac{1}{8}u^2(t), \quad u(0) = \frac{1}{2}, \quad t \in [0,1]. \tag{6.6.1}$$

By taking integration of both sides of (6.6.1), we have

$$u(t) = \frac{1}{2} + \frac{t}{2} + \frac{1}{8}\int_0^t u^2(s)ds.$$

Here we have $u_0 = \frac{(1+t)}{2}$ and $N(u)(t) = (\frac{1}{8})\int_0^t u^2(s)ds$ then [49]

$$\begin{aligned}(N'(u)y)(t) &= \frac{1}{8}\int_0^t \frac{\partial}{\partial u}(u^2 y)ds \\ &= \frac{1}{8}\int_0^t 2u(s)y(s)ds.\end{aligned}$$

Similarly, $(N''(u)(y_1, y_2))(t) = (\frac{t}{4})y_1(t), y_2(t)$ and $N^{(k)}(u) = 0$ for $k \geq 3$. Since $t \in [0,1]$,

$$\begin{aligned}&||N(u_0)(t)|| = ||\frac{1}{8}\int_0^t \frac{(1+s)^2}{4}ds|| \leq \frac{7}{96} < \frac{1}{e}, \\ &||N'(u_0)(t)|| = ||\frac{1}{4}\int_0^t \frac{(1+s)}{2}ds|| \leq \frac{3}{16} < \frac{1}{e}, \\ &||N''(u_0)(t)|| = ||\frac{t}{4}|| \leq \frac{1}{4} < \frac{1}{e}, \\ &||N^{(k)}(u_0)(t)|| = 0, \quad k \geq 3.\end{aligned}$$

As the conditions of Theorem 6.6.2 are fulfilled, the series solution $u = \sum_{i=1}^{\infty} u_i$ obtained by DJ-method is convergent for $t \in [0,1]$.

Conclusion

An iterative method for solving fractional equations has been discussed. The result for the convergence of solution for nonlinear fractional differential equations is presented. Illustrative examples dealing with initial value problems, boundary value problems, Volterra integral equations, nonlinear fractional differential equations and partial differential equations have been given. The method proves to be simple in its principles and convenient for computer algorithms.

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