Approximate similarity solutions of nonlinear partial differential equations

by
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This Dissertation is dedicated to my parents and husband

for their love and support.

Contents

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Abstract

In this dissertation we generate a method to find approximate similarity soluion of nonlinear PDEs. The method of solving PDEs via similarity transform mostly results in ODEs whose analytical solution is difficult to find. Such situations require solving reduced ODEs numerically. Since there is no pratical approach to use the numerical solution of reduced ODE to govern solution of original PDE. In this dissertation we have given a systematic procedure to deal with such situations.

In the first chapter we provide a brief review of some basic concepts of Symmetry Analysis then an algorithm of finding symmetries of PDEs is given and at the end we discuss symmetry method to solve boundary value problems of PDEs.

In second chapter we have discussed methods to solve the reduced BVPs of ODEs numerically.

In the third chapter we have provided a method to find approximate similarity solution of nonlinear PDEs and implemented our method to find approximate similarity solution for non linear diffusion equation and for different cases of IBVP of unsteady gas flow through porous semi-infinite medium.

The thesis is concluded in fourth chapter.

Chapter 1

Introduction to Symmetry Analysis

In this chapter, some background of symmetry analysis and algorithm which generates the Lie symmetries of partial differential equations is provided [1–6]. We consider the reduction of partial differential equations (PDEs) to ordinary differential equations (ODEs) using the silmilarity variables. This method is used to find an exact solution of PDE via invariance of differential equation under a particular group. Later in this chapter we highlight the systematic procedure of extending the symmetry method for a PDE to investigate boundary value problems of PDE.

1.1 Definitions

1.1.1 Group:

A group G is defined to be a set of elements with a law of composition ψ between elements which satisfy following axioms [7]:

• Closure law: For any $g_1, g_2 \in G$, $\Psi(g_1, g_2) \in G$.

• Associative Property: For any $g_1, g_2, g_3 \in G$,

$$\Psi(g_1, \Psi(g_2, g_3)) = \Psi(\Psi(g_1, g_2), g_3). \tag{1.1.1}$$

• Identity element: There exists a unique element $e \in G$ such that for all $g \in G$

$$\Psi(g, e) = \Psi(e, g) = g. \tag{1.1.2}$$

• Inverse element: For any $g \in G$ there exists a unique $g^{-1} \in G$ such that

$$\Psi(g, g^{-1}) = \Psi(g^{-1}, g) = e. \tag{1.1.3}$$

A group G is said to be **Abelian** group if for all $g_1, g_2 \in G$,

$$\Psi(g_1, g_2) = \Psi(g_2, g_1).$$

A **Subgroup** of G is a subset of G whose elements form a group with the same law of composition Ψ.

1.1.2 Group of transformations:

Let us consider $\mathbf{x} = (x_1, x_2, ..., x_n)$ which lie in region $A \subset \mathbf{R}^n$. Then set of transformations [7]

$$\mathbf{x}^* = \mathbf{X}(\mathbf{x}; \epsilon), \tag{1.1.4}$$

which is defined for each $\mathbf{x} \in A$ and parameter $\epsilon \in W \subset \mathbf{R}$ and $\psi(\epsilon, \delta)$ defines law of composition of parameters $\epsilon, \delta \in W$, forms a 1-parameter group of transformations in A if following holds:

• For each $\epsilon \in W$, the transformation is one-to-one onto A. (Hence $\mathbf{x}^* \in A$.)

• W forms a group with law of compostion ψ.

• For each $\mathbf{x} \in A$, we have $\mathbf{x}^* = \mathbf{x}$, when $\epsilon = \epsilon_o$ corresponds to identity element e.

• If $\mathbf{x}^* = \mathbf{X}(\mathbf{x}; \epsilon)$ and $\mathbf{x}^{**} = \mathbf{X}(\mathbf{x}^*; \delta)$, then we have

$$\mathbf{x}^{**} = \mathbf{X}(\mathbf{x}; \psi(\epsilon, \delta)). \tag{1.1.5}$$

1.1.3 One-parameter Lie group of transformations:

A 1-parameter Lie group of transformation [7] in addition to the definition 1.1.2 satisfies

• The parameter ϵ is continuous, that is W is an interval in $\mathbf{R}$. Without loss of generality, $\epsilon = 0$ corresponds to identity e.

• $\mathbf{X}$ is infinitely differentiable w.r.t $\mathbf{x} \in A$ and is an analytic function of $\epsilon \in W$.

• $\psi(\epsilon, \delta)$ is an analytic function of δ and ϵ where $\delta, \epsilon \in W$.

1.1.4 Symmetry:

A symmetry is defined to be a one-parameter group of transformations which maps solutions into solutions.

The infinitesimal generator of one-parameter Lie group of transformations is the operator

$$\mathbf{X} = \mathbf{X}(\mathbf{x}) = \zeta(\mathbf{x}) \times \underline{\nabla} = \sum_{i=1}^{n} \zeta_i(\mathbf{x})(\frac{\partial}{\partial x_i}), \tag{1.1.6}$$

where ∇ is the gradient operator.

1.1.5 Lie algebra:

A Lie algebra is defined to be a vector space T over $\mathbf{R}$ or $\mathbf{C}$ with a bilinear bracket operation (the commutator)$[.,.] : T \times T \longrightarrow T$ satisfying the following properties.

• Bilinearity:

$$[c_1\mathbf{X}_1+c_2\mathbf{X}_2, \mathbf{X}_3] = c_1[\mathbf{X}_1, \mathbf{X}_3]+c_2[\mathbf{X}_2, \mathbf{X}_3], \qquad [\mathbf{X}_3, c_1\mathbf{X}_1+c_2\mathbf{X}_2] = c_1[\mathbf{X}_3, \mathbf{X}_1]+c_2[\mathbf{X}_3, \mathbf{X}_2], \tag{1.1.7}$$

for all $c_1, c_2 \in \mathbf{R}$ or $\mathbf{C}$ and $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3 \in T$.

• Antisymmertry:

$$[\mathbf{X}, \mathbf{X}] = 0, \tag{1.1.8}$$

for all $\mathbf{X} \in T$.

• Jacobi's identity:

$$[\mathbf{X}_1, [\mathbf{X}_2, \mathbf{X}_3]] + [\mathbf{X}_2, [\mathbf{X}_3, \mathbf{X}_1]] + [\mathbf{X}_3, [\mathbf{X}_1, \mathbf{X}_2]] = 0, \tag{1.1.9}$$

for all $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3 \in T$.

1.1.6 Invariant functions:

An infinitely differentiable function $G(\mathbf{x})$ is an invariant function [7] of the Lie group of transformation (1.1.3) iff, for any group of transformations,

$$G(\mathbf{x}) \equiv G(\mathbf{x}^*).$$

Theorem 1.1.1. *$G(\boldsymbol{x})$ is invariant under the Lie group of transformation (1.1.3) iff*

$$\boldsymbol{X}G(\boldsymbol{x}) = 0.$$

Proof. First of all we compute $G(\mathbf{x}^*)$. Taylor series expansion gives us

$$\begin{aligned}
G(\mathbf{x}^*) &= \sum_0^\infty \frac{\epsilon^k}{k!}[\frac{d^k}{d\epsilon^k}G(\mathbf{x}^*)]_{\epsilon=0},\\
&= \sum_0^\infty \frac{\epsilon^k}{k!}(\frac{d^{k-1}}{d\epsilon^{k-1}}[\frac{d}{d\epsilon}G(\mathbf{x}^*)])_{\epsilon=0},\\
&= \sum_0^\infty \frac{\epsilon^k}{k!}(\frac{d^{k-1}}{d\epsilon^{k-1}}[\sum_n^{i=1}\frac{\partial}{\partial x_i^*}G(\mathbf{x}^*)\frac{dx_i^*}{d\epsilon}])_{\epsilon=0},\\
&= \sum_0^\infty \frac{\epsilon^k}{k!}(\frac{d^{k-1}}{d\epsilon^{k-1}}[\zeta_i(\mathbf{x}^*)\sum_n^{i=1}\frac{\partial}{\partial x_i^*}G(\mathbf{x}^*)])_{\epsilon=0},\\
&= \sum_0^\infty \frac{\epsilon^k}{k!}(\frac{d^{k-1}}{d\epsilon^{k-1}}[\mathbf{X}(\mathbf{x}^*)G(\mathbf{x}^*)])_{\epsilon=0},\\
&= \sum_0^\infty \frac{\epsilon^k}{k!}(\frac{d^{k-2}}{d\epsilon^{k-2}}[\frac{d}{d\epsilon}(\mathbf{X}(\mathbf{x}^*)G(\mathbf{x}^*))])_{\epsilon=0},\\
&= \sum_0^\infty \frac{\epsilon^k}{k!}(\frac{d^{k-2}}{d\epsilon^{k-2}}[\mathbf{X}(\mathbf{x}^*)(\mathbf{X}(\mathbf{x}^*)G(\mathbf{x}^*))])_{\epsilon=0},\\
&= \sum_0^\infty \frac{\epsilon^k}{k!}(\frac{d^{k-2}}{d\epsilon^{k-2}}[(\mathbf{X}^2(\mathbf{x}^*)G(\mathbf{x}^*))])_{\epsilon=0},\\
&\,.\\
&\,.\\
&\,.\\
G(\mathbf{x}^*) &= \sum_0^\infty \frac{\epsilon^k}{k!}([\mathbf{X}^k(\mathbf{x}^*)G(\mathbf{x}^*)]_{\epsilon=0}).
\end{aligned}$$

Then,

$$G(\mathbf{x}^*) = \sum_0^\infty \frac{\epsilon^k}{k!}\mathbf{X}^k(\mathbf{x})G(\mathbf{x}). \tag{1.1.10}$$

Now suppose that $\mathbf{X}G(\mathbf{x}) = 0$, now we need to prove that $G(\mathbf{x}^*) = G(\mathbf{x})$

$$\mathbf{X}G(\mathbf{x}) = 0 \Rightarrow \mathbf{X}^n G(\mathbf{x}) = 0, \tag{1.1.11}$$

then from Eq. (1.1.10) it follows that $G(\mathbf{x}^*) = G(\mathbf{x})$.

Converesely, suppose that $G(\mathbf{x}^*) = G(\mathbf{x})$ we need to prove that $\mathbf{X}G(\mathbf{x}) = 0$.

$$G(\mathbf{x}^*) = G(\mathbf{x}) \Rightarrow G(\mathbf{x}) = G(\mathbf{x}) + \sum_{1}^{\infty} \frac{\epsilon^k}{k!}\mathbf{X}^k(\mathbf{x})G(\mathbf{x}),$$

$$\Rightarrow \sum_{1}^{\infty} \frac{\epsilon^k}{k!}\mathbf{X}^k(\mathbf{x})G(\mathbf{x}) = 0.$$

Therefore, $\mathbf{X}G(\mathbf{x}) = 0$. □

1.2 Algorithm: Finding point symmetries of PDE:

In Lie point symmetry methods we find all the symmetries of the PDE. Let us consider an n^{th} order PDE,

$$F(t, x, v, v_t, v_x, ...) = 0. \tag{1.2.1}$$

Now to find one-parameter Lie group of transformation for above equation, consider the transformations

$$\begin{aligned}
\bar{t} &= t + \epsilon\tau(t, x, v), \\
\overline{x} &= x + \epsilon\xi(t, x, v), \\
\overline{v} &= v + \epsilon\eta(t, x, v), \\
\overline{v}_i &= v_i + \epsilon\eta_i(t, x, v, v_1), \\
&. \\
&. \\
&. \\
\overline{v}_{i_1 i_2 \dots i_k} &= v_{i_1 i_2 \dots i_k} + \epsilon\eta_{i_1 i_2 \dots i_k}(t, x, v, v_1, ...v_k),
\end{aligned}$$

where ϵ is called group parameter. Corresponding generator is

$$\mathbf{H} = \tau(t, x, v)\partial_t + \xi(t, x, v)\partial_x + \eta(t, x, v)\partial_v, \tag{1.2.2}$$

where,

$$(\frac{\partial\bar{t}}{\partial\epsilon})_{\epsilon=0} = \tau(t, x, v), \qquad (\frac{\partial\overline{x}}{\partial\epsilon})_{\epsilon=0} = \xi(t, x, v), \qquad (\frac{\partial\overline{v}}{\partial\epsilon})_{\epsilon=0} = \eta(t, x, v). \tag{1.2.3}$$

We require an n^{th} extension of $\mathbf{H}$ to apply on an n^{th} order PDE, i.e.

$$\mathbf{H}^{[n]}F|_{F=0} = 0, \tag{1.2.4}$$

where,

$$\mathbf{H}^{[n]} = \mathbf{H} + \eta_i(t,x,v,v_1)\partial_{v_i} + ... + \eta_{i_1,i_2...i_k}(t,x,v,v_1,v_2,...v_k)\partial_{v_{i_1,i_2,...,i_k}}, \tag{1.2.5}$$

is the nth order prolongation of the generator $\mathbf{H}$. $\eta_i, ..., \eta_{i_1,i_2...i_k}$ are called the prolongation coefficients and are defined as

$$\begin{aligned} \eta_i &= D_i\eta - (D_i\tau)v_t - (D_i\xi)v_x, & (1.2.6)\\ \eta_{i_1,i_2...i_k} &= D_{i_k}\eta_{i_1,i_2...i_{k-1}} - (D_{i_k}\tau)v_{i_1,i_2...i_{k-1}t} - (D_{i_k}\xi)v_{i_1,i_2...i_{k-1}x}, & (1.2.7) \end{aligned}$$

and

$$\begin{aligned} D_x &= \partial_x + v_x\partial_v + v_{xx}\partial_{v_x} + v_{xt}\partial_{v_t} + ..., & (1.2.8)\\ D_t &= \partial_t + v_t\partial_v + v_{xt}\partial_{v_x} + v_{tt}\partial_{v_t} + ... & (1.2.9) \end{aligned}$$

For $n = 2$ the form of $\mathbf{H}^{[2]}$ is given by

$$\mathbf{H}^{[2]} = \tau(t,x,v)\partial_t + \xi(t,x,v)\partial_x + \eta(t,x,v)\partial_v + \eta_1\partial_{v_t} + \eta_2\partial_{v_x} + \eta_{11}\partial_{v_{tt}} + \eta_{12}\partial_{v_{tx}} + \eta_{22}\partial_{v_{xx}}, \tag{1.2.10}$$

where,

$$\begin{aligned} \eta_1 &= D_t(\eta) - v_tD_t(\tau) - v_xD_t(\xi),\\ &= \eta_t + (\eta_v - \tau_t)v_t - \xi_tv_x - \tau_v(v_t)^2 - \xi_vv_tv_x, & (1.2.11)\\ \eta_2 &= D_x(\eta) - v_tD_x(\tau) - v_xD_x(\xi),\\ &= \eta_x + (\eta_v - \xi_x)v_x - \tau_xv_t - \xi_v(v_x)^2 - \tau_vv_tv_x, & (1.2.12)\\ \eta_{11} &= D_t(\eta_1) - v_{tt}D_t(\tau) - v_{tx}D_t(\xi),\\ &= \eta_{tt} + [2\eta_{tv} - \tau_{tt}]v_t + [\eta_{vv} - 2\tau_{tv}](v_t)^2 - \tau_{vv}(v_t)^3 - 3\tau_vv_tv_{tt} - \xi_{tt}v_x - 2\xi_{vt}v_tv_x\\ &\quad -2\xi_tv_{tx} - \xi_{vv}v_x(v_t)^2 - \xi_vv_xv_{tt} - 2\xi_vv_tv_{xt}, & (1.2.13) \end{aligned}$$

$$
\begin{aligned}
\eta_{12} &= D_x(\eta_1) - v_{tt}D_x(\tau) - v_{tx}D_x(\xi), \\
&= \eta_{tx} + [2\eta_{tv} - \xi_{tx}]v_x + [\eta_{vx} - \tau_{tx}](v_t) + [\eta_{vv} - \tau tv - \xi_{vx}]v_t v_x + [\eta_v - \tau_t - \xi_x]v_{tx} \\
&\quad -\tau_{vx}(v_t)^2 - \tau_{vv}v_x(v_t)^2 - \xi_v v_t v_{xx} - \xi_{tv}(v_x)^2 - \xi_t v_{xx} - \tau_x v_{tt} - \xi_{vv}v_t(v_x)^2 \\
&\quad -2\xi_v v_x v_{tx} - \tau_v v_x v_{tt} - 2\tau_v v_t v_{xt}, \qquad (1.2.14) \\
\eta_{22} &= D_x(\eta_2) - v_{tx}D_x(\tau) - v_{xx}D_x(\xi), \\
&= \eta_{xx} + [2\eta_{xv} - \xi_{xx}]v_x - \tau_{xx}v_t + [\eta_v - 2\xi_x]v_{xx} - 2\tau_x v_{tx} + [\eta_{vv} - 2\xi_{xv}](v_x)^2 \\
&\quad -2\tau_{xv}v_t v_x - \xi_{vv}(v_x)^3 - \tau_{vv}v_t(v_x)^2 - 3\xi_v v_x v_{xx} - \tau_v v_t v_{xx} - 2\tau_v v_x v_{tx}, \qquad (1.2.15)
\end{aligned}
$$

Condition (1.2.4) gives a system of nonlinear PDEs in τ, ξ and η. Also τ, ξ and η must depend only on t, x and u since we are dealing with point symmetries. After comparing the coefficients of powers of derivatives of v we get system of PDEs. The solution of this system yields the value of τ, ξ and η. The number of constants in $\mathbf{H}$ determines the dimension of the Lie algebra.

1.2.1 Example:

Let us consider heat equation [7, 8]

$$v_t = v_{xx}. \qquad (1.2.16)$$

Applying 2^{nd} extension of $\mathbf{H}$, we get

$$\eta_1 = \eta_{22}. \qquad (1.2.17)$$

Following the algorithm given in Section (1.2), we get following system of equations.

$$
\begin{aligned}
\tau_v &= 0, & (1.2.18) \\
\tau_x &= 0, & (1.2.19) \\
\xi_v &= 0, & (1.2.20) \\
-2\xi_x + \tau_t &= 0, & (1.2.21) \\
\xi_t + 2\eta_{xv} - \xi_{xx} &= 0, & (1.2.22) \\
\eta_{vv} &= 0, & (1.2.23) \\
-\eta_{xx} + \eta_t &= 0. & (1.2.24)
\end{aligned}
$$

Solving above system we get the general solution given below

$$\begin{aligned}\tau &= c_1 + c_3 t + \frac{1}{4}c_4 t^2, &(1.2.25)\\ \xi &= c_2 + \frac{1}{2}(c_3 + c_4 t)x + c_6 t, &(1.2.26)\\ \eta &= (-\frac{1}{8}c_4 x^2 - \frac{1}{4}c_4 t - \frac{1}{2}c_6 x + c_5)v + V(t,x), &(1.2.27)\end{aligned}$$

where

$$V_t = V_{xx}, \tag{1.2.28}$$

is function of integration.

Taking one of the constants to be 1 and remaining constants to be zero we get all the symmetries as follows

$$\begin{aligned}\mathbf{H}_1 &= \frac{\partial}{\partial t}, \qquad \mathbf{H}_2 = \frac{\partial}{\partial x}, \qquad \mathbf{H}_3 = t\frac{\partial}{\partial t} + \frac{1}{2}x\frac{\partial}{\partial x},\\ \mathbf{H}_4 &= \frac{1}{2}tx\frac{\partial}{\partial x} + \frac{1}{2}t^2\frac{\partial}{\partial t} - (\frac{1}{8}x^2 v + \frac{1}{2}tv)\frac{\partial}{\partial v},\\ \mathbf{H}_5 &= v\frac{\partial}{\partial v}, \qquad \mathbf{H}_6 = t\frac{\partial}{\partial x} - \frac{1}{2}xv\frac{\partial}{\partial v}, \qquad \mathbf{H}_7 = V(t,x)\frac{\partial}{\partial v}. \end{aligned} \tag{1.2.29}$$

1.3 Group reductions and point transformations:

Once we obtain the generators, we solve the following invariant surface condition to reduce PDEs to ODEs for each symmetry

$$\frac{dt}{\tau(t,x,v)} = \frac{dx}{\xi(t,x,v)} = \frac{dv}{\eta(t,x,v)}. \tag{1.3.1}$$

When we solve Eq. (1.3.1), we will get two first integrals denoted by u and z. Thus the reduction variables will be

$$u = w(z). \tag{1.3.2}$$

When we substitute Eq. (1.3.2) in Eq. (1.2.1) we obtain an ODE. Its solution gives us the value of w.

1.3.1 Example:

In Example (1.2.1) the invariant surface condition for $\mathbf{H}_3$ gives us

$$\frac{dt}{t} = 2\frac{dx}{x} = \frac{dv}{0}. \tag{1.3.3}$$

Now,

$$2\frac{dx}{x} = \frac{dt}{t} \quad \Rightarrow \quad z = \frac{x^2}{t}, \tag{1.3.4}$$

$$\frac{dv}{0} = 2\frac{dx}{x} \quad \Rightarrow \quad v = u. \tag{1.3.5}$$

Using Eq. (1.3.2) the reduction variables can be written as

$$v = w(\frac{x^2}{t}), \tag{1.3.6}$$

and after substituting in Eq. (1.2.16) we get

$$4zw'' + (2+z)w' = 0. \tag{1.3.7}$$

The solution of ODE given by Eq. (1.3.7) is

$$w(z) = 2\sqrt{\pi}k_1 erf(\frac{1}{2}\sqrt{z}) + k_2, \tag{1.3.8}$$

where k_1 and k_2 are arbitrary constants and erf represents the error function.
Substituting Eq. (1.3.8) in Eq. (1.3.6) we get the exact solution given below

$$v(x,t) = 2\sqrt{\pi}k_1 erf(\frac{1}{2}\sqrt{\frac{x^2}{t}}) + k_2. \tag{1.3.9}$$

Sometimes it is hard to find the exact solution of reduced ODE. In such cases we consider the numerical approach. Once we get the solution of reduced ODE, we get the solution of initial PDE by using the reciprocal bijection of point transformation.
The symmetries obtained above generates the point transformations which leave the differential equation invariant. In order to get these point transformations we solve

$$\frac{\partial \bar{t}}{\partial \epsilon} = \tau(\bar{t},\bar{x},\bar{v}), \qquad \frac{\partial \bar{x}}{\partial \epsilon} = \xi(\bar{t},\bar{x},\bar{v}), \qquad \frac{\partial \bar{v}}{\partial \epsilon} = \eta(\bar{t},\bar{x},\bar{v}), \tag{1.3.10}$$

subject to conditions

$$\bar{t}|_{\epsilon=0} = t, \qquad \bar{x}|_{\epsilon=0} = x, \qquad \bar{v}|_{\epsilon=0} = v. \tag{1.3.11}$$

1.3.2 Example:

Let us find the point transformation for $\mathbf{H}_3$ in Example (1.2.1). Eq. (1.3.10) implies

$$\frac{\partial \bar{t}}{\partial \epsilon} = \bar{t}, \qquad \frac{\partial \overline{x}}{\partial \epsilon} = \frac{1}{2}\overline{x}, \qquad \frac{\partial \overline{v}}{\partial \epsilon} = 0, \tag{1.3.12}$$

which implies

$$\bar{t} = e^{\epsilon} a_1, \qquad \overline{x} = e^{\frac{1}{2}\epsilon} a_2, \qquad \overline{v} = a_3. \tag{1.3.13}$$

where a_1, a_2 and a_3 are arbitrary constants. Using conditions given in Eq. (1.3.11) the point transformation obtained from $\mathbf{H}_3$ gives us the Lie scaling group.

$$\bar{t} = e^{\epsilon} t, \qquad \overline{x} = e^{\frac{\epsilon}{2}} x, \qquad \overline{v} = v. \tag{1.3.14}$$

Thus we have illustrated the algorithm by finding symmetries of heat equation and using one of these symmetries to find exact solution of heat equation. Again we used the same symmetry to find point transformation which leave the differential equation invariant.

1.4 Lie symmetry solution of initial-boundary value problem:

In this section a review of process of finding Lie symmetry solution of Initial boundary value problems (IBVPs) is given [9, 10]. The process of applying Lie symmetry method to IBVPs of PDE requires finding a one parameter group of transformations that leaves the problem invariant and then using these transformations either to obtain similarity reduction or to construct invariant solution.
The process consists of following steps:

Step 1. Symmetry Algebra of PDE:

Determining Lie symmetries of governing PDE.

Step 2. Invariance of boundaries:

Finding the conditions under which most general symmetry operator $\mathbf{H}$ obtained in step 1 leaves the boundaries invariant.

Step 3. Invariance of boundary conditions:

Finding restrictions which are imposed on **H** due to invariance of boundary conditions. Above three steps determine symmetry operator that leaves the IBVP invariant.

Step 4. Reductions or construction of similarity solutions:

Finding the similarity reductions and similarity solution of the IBVP by using the similarity variables of symmetry operator of IBVP.
We consider a test problem to describe above procedure.

1.4.1 Example:

Transient condition in a semi-infinite solid with constant surface temperature.

$$\frac{\partial T}{\partial t} = \beta \frac{\partial^2 T}{\partial x^2}, \tag{1.4.1}$$

subject to initial and boundary conditions,

$$T|_{t=o} = T_i, \qquad T|_{x=0} = T_s, \qquad T|_{x\to\infty} = T_i. \tag{1.4.2}$$

The symmetry algebra of Eq. (1.4.1) is well known and is spanned by the vecor field,

$$\begin{aligned}
\mathbf{H}_1 &= \frac{\partial}{\partial t}, \qquad \mathbf{H}_2 = \frac{\partial}{\partial x}, \qquad \mathbf{H}_3 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}, \qquad \mathbf{H}_4 = 2t\frac{\partial}{\partial x} - \frac{1}{\beta}xT\frac{\partial}{\partial T} \\
\mathbf{H}_5 &= 4t^2\frac{\partial}{\partial t} + 4xt\frac{\partial}{\partial x} - \frac{1}{\beta}(x^2 + 2\beta t)T\frac{\partial}{\partial T}, \qquad \mathbf{H}_6 = T\frac{\partial}{\partial T}, \qquad \mathbf{H}_\infty = g(t,x)\frac{\partial}{\partial T}.
\end{aligned}$$

We consider general symmetry operator of Eq. (1.4.1)

$$\mathbf{H} = c_1\mathbf{H}_1 + c_2\mathbf{H}_2 + c_3\mathbf{H}_3 + c_4\mathbf{H}_4 + c_5\mathbf{H}_5 + c_6\mathbf{H}_6,$$

and look for the operator that will preserve boundary and boundary conditions given by Eq. (1.4.2)
The invariance of the boundaries $x = 0$, $t = 0$ or equivalently we can say

$$\begin{aligned}
[\mathbf{H}(x-0)]_{x=0} &= 0, \\
[\mathbf{H}(t-0)]_{t=0} &= 0,
\end{aligned}$$

which implies

$$c_1 = c_2 = c_4 = 0. \tag{1.4.3}$$

Thus, $\mathbf{H}$ must be

$$\mathbf{H} = c_3\mathbf{H}_3 + c_5\mathbf{H}_5 + c_6\mathbf{H}_6.$$

Also the invariance of initial and boundary conditions i.e

$$\begin{aligned} [\mathbf{H}(T - T_i)]_{t=0} &= 0, \qquad on \qquad T = T_i, \\ [\mathbf{H}(T - T_s)]_{x=0} &= 0, \qquad on \qquad T = T_s, \end{aligned}$$

implies

$$c_5 = 0 = c_6. \tag{1.4.4}$$

Thus under symmetry given below, the IBVP given by Eqs. (1.4.1), (1.4.2) is invariant

$$\mathbf{H} = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}, \tag{1.4.5}$$

where we have chosen $c_3 = 1$.

Now solving the characteristic system for $\mathbf{H}I = 0$, we get $I_1 = \frac{x^2}{t}$ and $I_2 = T$ as differential invariants of Eq. (1.4.5). Thus the similarity variables for $\mathbf{H}$ are

$$V(x,t) = \frac{x^2}{t}, \qquad W(V) = T. \tag{1.4.6}$$

When we substitute similarity variables in Eq. (1.4.1), we get corresponding similarity solution of Eq. (1.4.1) of the form $T = W(V)$ where $W(V)$ satisfies the ODE

$$4V\frac{d^2W}{dV^2} + (2 + \frac{V}{\beta})\frac{dW}{dV} = 0. \tag{1.4.7}$$

We integrate Eq. (1.4.7) by substituting

$$U = \frac{dW}{dV}.$$

Thus we get

$$W(V) = k_1 \int \frac{e^{\frac{-V}{4\beta}}}{\sqrt{V}} dV + k_2. \tag{1.4.8}$$

If we make change of variable

$$y^2 = \frac{V}{4\beta},$$

the above solution becomes

$$\begin{aligned} W &= 4k_1\sqrt{\beta}\int e^{-y^2}dy + k_2, \\ &= 2k_1\sqrt{\beta}\sqrt{\pi}\cdot erf(y) + k_2. \end{aligned} \tag{1.4.9}$$

Where erf represents the error function. Hence from above equations the exact solution of Eq. (1.4.1) that is invariant under Eq. (1.4.5) is given below

$$T(x,t) = 2k_1\sqrt{\beta}\sqrt{\pi}\cdot erf(\frac{\frac{x}{\sqrt{t}}}{2\sqrt{\beta}}) + k_2. \tag{1.4.10}$$

Applying initial and boundary conditions we get

$$k_1 = \frac{T_i - T_s}{2\sqrt{\beta\pi}}, \qquad k_2 = T_s,$$

so the solution of the IBVP given by Eqs. (1.4.1) and (1.4.2) becomes

$$T(x,t) = (T_i - T_s)erf(\frac{x}{2\sqrt{\beta t}}) + T_s. \tag{1.4.11}$$

Remark:

In case of non-linear problems sometimes it happens that the reduced ODE obtained from similarity variables cannot be integrable in terms of known functions. In such cases reduced BVP of ODE can be solved numerically to understand the solution.

Chapter 2

Numerical solution of boundary value problems for ordinary differential equations

There are many methods to find the numerical solutions of BVPs for ODEs. In this chapter we will illustrate "Shooting method" for solving BVPs numerically. Later we will discuss solution of BVPs in $MATLAB$ with *bvp4c*.

2.1 Existence and uniqueness of solution of BVP:

Consider a BVP for the 2^{nd} order ODE of the form [11],

$$y'' = g(x, y, y'), \qquad a \leq b, \qquad y(a) = \alpha, y(b) = \beta. \tag{2.1.1}$$

Theorem 2.1.1. *Suppose g is continuous on set*

$$H = (x, y, y'); \qquad a \leq x \leq b, -\infty < y < \infty, -\infty < y' < \infty.$$

and partial derivatives g_y and $g_{y'}$ are continuous on H.
If $g_y(x, y, y') > 0$, for all $(x, y, y') \in H$, and there exist a constant m such that $|g_{y'}(x, y, y')| \leq m$, for all $(x, y, y') \in H$ then the BVP given by Eq. (2.1.1) has unique solution.

Example:

Determine if the following BVP has a unique solution

$$y'' + \sin(y') + e^{-xy} = 0, \qquad 1 \leq x \leq 2, \qquad y(1) = y(2) = 0.$$

First of all we rewrite above BVP, i.e.

$$y'' = -\sin(y') - e^{-xy}, \qquad so \qquad g(x, y, y') = -\sin(y') - e^{-xy}.$$

Now we check conditions of uniqueness.

$$g(x, y, y') = -\sin(y') - e^{-xy}, \qquad g_y(x, y, y') = xe^{-xy}, \qquad g_{y'}(x, y, y') = -\cos(y'),$$

are continuous on

$$H = (x, y, y'); \qquad 1 \leq x \leq 2, -\infty < y < \infty, -\infty < y' < \infty.$$

Also

1. $g_y(x, y, y') = xe^{-xy} > 0$ on H.
2. $|g_{y'}(x, y, y')| = |-\cos(y')| \leq 1 = m$.

So the above BVP has a unique solution in H.

Example:

In this example we will check conditions under which a linear BVP has a unique solution. Consider a linear BVP of the form

$$y'' = p(x)y' + q(x)y + r(x), \qquad a \leq x \leq b, \qquad y(a) = \alpha, y(b) = \beta.$$

Observe that

$$g(x, y, y') = p(x)y' + q(x)y + r(x), \qquad g_y(x, y, y') = q(x), \qquad g_{y'}(x, y, y') = p(x),$$

are continuous on H iff $p(x)$, $q(x)$ and $r(x)$ are continuous for $a \leq x \leq b$. Next we check conditions (1) and (2).

1. $g_y(x, y, y') = q(x) > 0$ for $a \le x \le b$.

2. Since $g_{y'}$ is continuous on $[a, b]$, so $g_{y'}$ is bounded.

Thus the linear BVP has a unique solution if $p(x)$, $q(x)$ and $r(x)$ are continuous and $q(x) > 0$ for $a \le x \le b$.

2.2 Shooting Method:

Shooting method is a method used to solve BVPs of ODEs by reducing them to solutions of IVPs. It is used to solve both linear and nonlinear BVPs. This method is called Shooting, by analogy to the procedure of firing objects at a stationary target.

2.2.1 The Linear Shooting method:

Consider a linear BVP of the form [11]

$$y'' = p_1(x)y' + p_2(x)y + p_3(x), \qquad a \le x \le b, \qquad y(a) = \alpha_1, y(b) = \alpha_2. \tag{2.2.1}$$

where $p_1(x), p_2(x)$ and $p_3(x)$ are continuous and $p_2(x) > 0$ for $a \le x \le b$. Now find the solution of following two initial value problems (IVPs)

$$\begin{aligned} y'' &= p_1(x)y' + p_2(x)y + p_3(x), \qquad a \le x \le b, \qquad y(a) = \alpha_1, y'(a) = 0, && (2.2.2)\\ y'' &= p_1(x)y' + p_2(x)y, \qquad a \le x \le b, \qquad y(a) = 0, y'(a) = 1. && (2.2.3)\end{aligned}$$

Say the solutions are $y_1(x)$ and $y_2(x)$ respectively. Consider the following linear combination of $y_1(x)$ and $y_2(x)$ and call it $y(x)$.

$$y(x) = y_1(x) + \frac{\alpha_2 - y_1(b)}{y_2(b)} y_2(x). \tag{2.2.4}$$

Then $y(x)$ is the solution of the BVP given by Eq. (2.2.1).

Verification:

In order to verify whether $y(x)$ is solution of BVP given by Eq. (2.2.1), we consider

$$y'' = y_1''(x)+\frac{\alpha_2 - y_1(b)}{y_2(b)}y_2''(x) = p_1(x)y_1'+p_2(x)y_1+p_3(x)+\frac{\alpha_2 - y_1(b)}{y_2(b)}(p_1(x)y_2'+p_2(x)y_2+p_3(x)),$$

$$= p_1(x)(y_1'(x) + \frac{\alpha_2 - y_1(b)}{y_2(b)}y_2'(x)) + p_2(x)(y_1(x) + \frac{\alpha_2 - y_1(b)}{y_2(b)}y_2(x)) + p_3(x).$$

Thus $y(x)$ is solution of $y'' = p_1(x)y' + p_2(x)y + p_3(x)$.

Now we will check the boundary conditions of BVP.

$$\begin{aligned} y(a) &= y_1(a) + \frac{\alpha_2 - y_1(b)}{y_2(b)}y_2(a) = \alpha_1 + \frac{\alpha_2 - y_1(b)}{y_2(b)}(0) = \alpha_1, \\ y(b) &= y_1(b) + \frac{\alpha_2 - y_1(b)}{y_2(b)}y_2(b) = \alpha_2. \end{aligned}$$

Thus Linear shooting method requires solution of two independent IVPs given by Eqs. (2.2.2) and (2.2.3) in order to solve BVP given by Eq. (2.2.1).

Example:

Consider the BVP given by

$$y'' = \frac{2}{x^2}\ln x - \frac{2}{x^2}y - \frac{4}{x}y', \qquad 1 \le x \le 2, \qquad y(1) = \frac{1}{2}, y(2) = \ln 2. \tag{2.2.5}$$

Exact solution is given by

$$y(x) = -\frac{1}{x^2}(-x^2 \ln x + \frac{3}{2}x^2 - 4x + 2). \tag{2.2.6}$$

Here in this BVP, $p_1(x) = -\frac{4}{x}$, $p_2(x) = -\frac{2}{x^2}$ and $p_3(x) = \frac{2}{x^2}\ln x$ are continuous on $[1, 2]$. Since we have $p_2(x) \ngtr 0$ so we cannot say about uniqueness of above BVP.

Now solving following two IVPs,

$$\begin{aligned} y_1'' &= \frac{2}{x^2}\ln x - \frac{2}{x^2}y_1 - \frac{4}{x}y_1', \qquad 1 \le x \le 2, \qquad y_1(1) = \frac{1}{2}, y_1'(1) = 0, \\ y_2'' &= -\frac{2}{x^2}y_2 - \frac{4}{x}y_2', \qquad 1 \le x \le 2, \qquad y_2(1) = 0, y_2'(1) = 1. \end{aligned}$$

Next we setup these initial value problems into a system of four 1^{st} order differential equations given by

$$\begin{aligned}
v_1' &= v_2, \\
v_2' &= \frac{2}{x^2}\ln x - \frac{2}{x^2}v_1 - \frac{4}{x}v_2, \qquad v_1(1) = \frac{1}{2}, v_2(1) = 0, \\
v_3' &= v_4, \\
v_4' &= -\frac{2}{x^2}v_3 - \frac{4}{x}v_4, \qquad v_3(1) = 0, v_4(1) = 1.
\end{aligned}$$

Above system can be easily solved by $MATLAB$ build in function $ode45$ which solves by using variable step RKF45-method. The solution to above BVP is shown by following graphs.

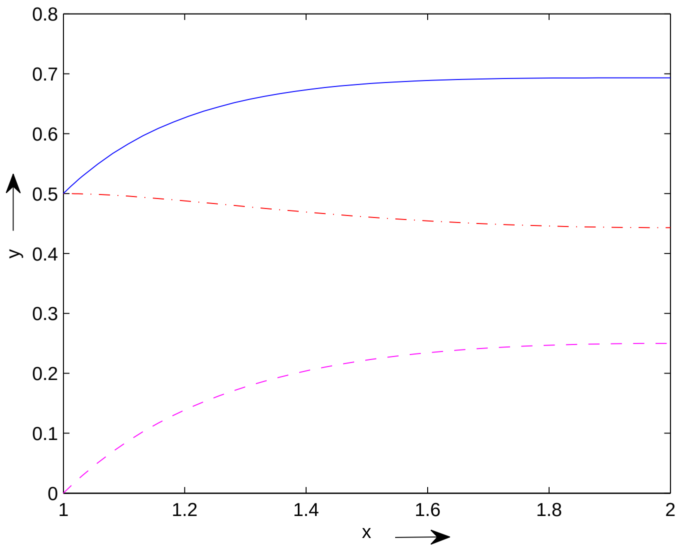

Figure 2.1: Solid line represent numerical solution of BVP and dotted lines represent solution of two IVPs

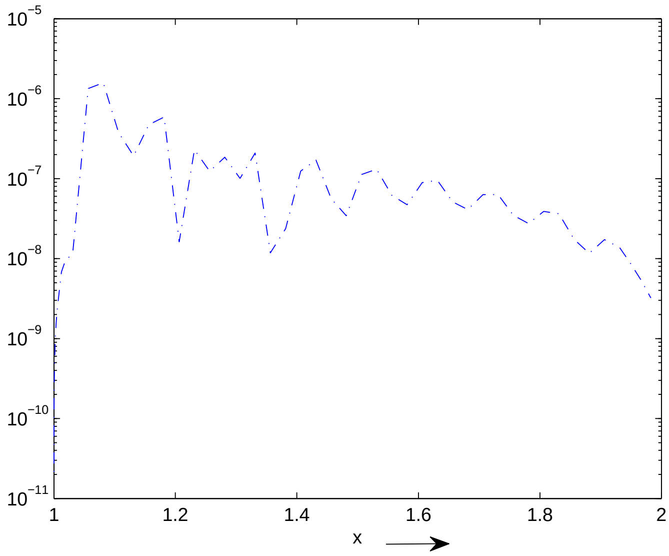

Figure 2.2: Difference between numerical and exact solution

2.2.2 Shooting method for non-linear problems:

Let us consider BVP of 2^{nd} order ODE of the form [11]

$$y'' = g(x, y, y'), \qquad a \le x \le b, \qquad y(a) = \alpha_1, y(b) = \alpha_2, \tag{2.2.7}$$

where $g(x, y, y')$ is not linear in y and y'. Assume $y(x)$ to be the unique solution of above BVP then it can be found out by solving sequence of following IVPs

$$y'' = g(x, y, y'), \qquad a \le x \le b, \qquad y(a) = \alpha_1, y'(a) = w_k, \tag{2.2.8}$$

where w_k are real numbers. Let $y(x, w_k)$ be the solution of IVP given by Eq. (2.2.8). We need to find a sequence w_k such that

$$\lim_{k\to\infty} y(b, w_k) = y(b).$$

We can choose w_0 to be

$$w_0 = y'(a) \approx \frac{y(b) - y(a)}{b - a} = \frac{\alpha_2 - \alpha_1}{b - a}.$$

If $y(b, w_o)$ is not sufficiently close to α_2 we correct our approximation by finding w_1, w_2 and so on until we reach a w from a sequence w_k such that

$$y(b, w) - \alpha_2 = 0, \tag{2.2.9}$$

Our aim is to find this w that is the solution of the equation. Observe that $y(b, w) - \alpha_2 = 0$ is a one variable non-linear equation. Using Newton's method to approximate solution of Eq. (2.2.9)

$$w_k = w_{k-1} - \frac{y(b, w_{k-1}) - \alpha_2}{\frac{dy}{dw}(b, w_{k-1})},$$

Since we do not have $y(x)$ explicitly so in order to determine $\frac{dy}{dw}(b, w_{k-2})$ let $y(x, w)$ be the solution of IVP given by Eq. (2.2.8). Then we have

$$y''(x, w) = g(x, y(x, w), y'(x, w)), \qquad a \leq x \leq b, \qquad y(a, w) = \alpha_1, y'(a, w) = w.$$

Differentiating above equation w.r.t w we have,

$$\begin{aligned} \frac{\partial y''(x, w)}{\partial w} &= \frac{\partial g(x, y(x, w), y'(x, w))}{\partial w}, \\ &= g_x x_w + g_y \frac{\partial y(x, w)}{\partial w} + g_{y'} \frac{\partial y'(x, w)}{\partial w}. \end{aligned}$$

Since x and w are independent, so $x_w = 0$. Hence we have

$$\frac{\partial y''(x, w)}{\partial w} = g_y \frac{\partial y(x, w)}{\partial w} + g_{y'} \frac{\partial y'(x, w)}{\partial w}, \tag{2.2.10}$$

for $a \leq x \leq b$. The initial conditions are

$$\frac{\partial y(a, w)}{\partial w} = \frac{d}{dw}[\alpha_1] = 0, \qquad \frac{\partial y'(a, w)}{\partial w} = \frac{d}{dw}[w] = 1.$$

Define $z(x, w) = \frac{\partial y(x,w)}{\partial w}$. Since

$$\frac{\partial^3 y(x, w)}{\partial x^2 \partial w} = \frac{\partial}{\partial w}[\frac{\partial^2 y(x, w)}{\partial x^2}] = \frac{\partial}{\partial w}[y''(x, w)].$$

Let us denote

$$z''(x, w) = \frac{\partial}{\partial w}[y''(x, w)],$$

Thus IVP given by Eq. (2.2.10) becomes

$$z''(x,w) = g_y z(x,w) + g_{y'} z'(x,w), \qquad a \le x \le b, \qquad z(a,w) = 0, z'(a,w) = 1. \qquad (2.2.11)$$

Using the information from $z(x,w)$ we can update w_k as follows

$$w_k = w_{k-1} - \frac{y(b,w_{k-1}) - \alpha_2}{z(b,w_{k-1})}.$$

Algorithm:

Thus the shooting method for non-linear BVP consists of following steps

$$y'' = g(x,y,y'), \qquad a \le x \le b, \qquad y(a) = \alpha_1, y(b) = \alpha_2.$$

Step 1:

Compute

$$w_0 = \frac{\alpha_2 - \alpha_1}{b - a}.$$

Step 2:

For $k \ge 1$, Solve the system of two IVPs for $y(x)$ and $z(x)$

$$\begin{aligned} y'' &= g(x,y,y'), \qquad a \le x \le b, \qquad y(a) = \alpha_1, y'(a) = w_{k-1}, \\ z''(x,w) &= g_y z(x,w) + g_{y'} z'(x,w), \qquad a \le x \le b, \qquad z(a,w_{k-1}) = 0, z'(a,w_{k-1}) = 1. \end{aligned}$$

Step 3:

Update

$$w_k = w_{k-1} - \frac{y(b,w_{k-1}) - \alpha_2}{z(b,w_{k-1})}.$$

For a given ϵ , terminate the algorithm if $|y(b,w_{k-1}) - \alpha_2| < \epsilon$.

Example:

Solve the BVP given by

$$y'' = \frac{1}{8}(2x^3 + 32 - yy'), \qquad 1 \leq x \leq 3, \qquad y(1) = 17, y(3) = \frac{43}{3}. \tag{2.2.12}$$

Here

$$g(x, y, y') = \frac{1}{8}(2x^3 + 32 - yy'),$$

$$g_y = -y', \qquad g_{y'} = -y.$$

Solve system of two 2^{nd} order IVPs given by

$$\begin{aligned} y'' &= \frac{1}{8}(2x^3 + 32 - yy'), \qquad 1 \leq x \leq 3, \qquad y(1) = 17, y'(1) = w_k, \\ z'' &= g_y z + g_{y'} z' = -\frac{1}{8}(y'z + yz'), \qquad 1 \leq x \leq 3, \qquad z(1) = 0, z'(1) = 1. \end{aligned}$$

Let $v_1 = y$, $v_2 = y'$, $v_3 = z$ and $v_4 = z'$. Now solve system of four 1^{st} order IVPs

$$\begin{aligned} v_1' &= v_2, \\ v_2' &= \frac{1}{8}(2x^3 + 32 - v_1 v_2), \\ v_3' &= v_4, \\ v_4' &= -\frac{1}{8}(v_2 v_3 + v_1 v_4). \end{aligned}$$

with initial conditions

$$v_1(1) = 17, \qquad v_2(1) = w_k, \qquad v_3(1) = 0, \qquad v_4(1) = 1.$$

We use $MATLAB$ function $ode45$ to solve above system of IVPs. The graph of numerical solution and exact solution of BVP is shown in following Fig.

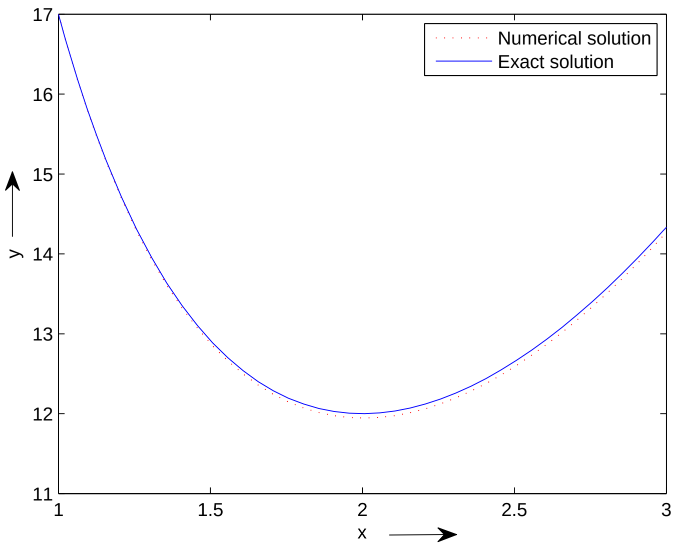

Figure 2.3: Numerical solution and exact solution of BVP

2.3 Solving boundary value problems in MATLAB with *bvp4c*

The MATLAB function *bvp4c* is a BVP solver used to solve two point boundary value problems [12]. It requires introduction of new variables to convert differential equation into a system of first order ODEs of the form

$$w' = g(x, w), \qquad a \leq x \leq b,$$

subject to boundary conditions

$$h(w(a), w(b)) = 0,$$

It also solves the problems involving an unknown parameter q.

$$w' = g(x, w, q), \qquad a \leq x \leq b,$$

subject to boundary conditions

$$h(w(a), w(b), q) = 0,$$

bvp4c implements collocation method to solve BVPs. The approximate solution $W(x)$ is continuous cubic polynomial on each subinterval of a mesh $a = x_0 < x_1 < x_2 < ..... < x_N = b$ that satisfies the boundary conditions and (collocates) satisfies the differential equation at mid point and both ends of each subinterval

$$\begin{aligned} W'(x_n) &= g(x_n, W(x_n)), \\ W'(\frac{(x_n + x_{n+1})}{2}) &= g(\frac{(x_n + x_{n+1})}{2}, W(\frac{(x_n + x_{n+1})}{2})), \\ W'(x_{n+1}) &= g(x_{n+1}, W(x_{n+1})). \end{aligned}$$

This results in a nonlinear system of algebraic equations for coefficients defining $W(x)$. This system is solved iteratively by linearization. The basic method of *bvp4c* is Simpsons method. After we compute $W(x)$ on a mesh then it can be evaluated at any point of the mesh by using *bvpval* function. The collocation at end points of each subinterval and continuity of $W(x)$ implies that $W(x)$ has continuous derivative also on $[a, b]$. The residual $r(x)$, for such an approximation is defined by

$$r(x) = W'(x) - g(x, W(x)),$$

It says that $W(x)$ is exact solution of ODEs

$$W'(x) = g(x, W(x))) + r(x).$$

Also $h(W(a), W(b))$ is residual in boundary conditions. *bvp4c* also controls size of residuals. For uniformly small residuals, $W(x)$ is good solution rather exact solution close to the one supplied to BVP solver.

Chapter 3

Approximate similarity solutions of nonlinear PDEs

A method to solve PDEs via similarity method mostly leads to reduction to ODEs which are not easily integrable in terms of known functions. In such situations we solve reduced ODEs numerically. However there is no systematic approach to use this numerical solution to find solution of the original PDE. In this chapter we describe a practical procedure to deal with such situations and apply it to different cases of IBVP of unsteady gas flow through semi-infinite porous medium and nonlinear diffusion equation.

The modeling of many physical processes like chemical kinetics, diffusion, wave mechanics, fluid mechanics and general transport problems leads to such nonlinear PDEs whose analytic solution is hard to find. Thus the approach of reduction of PDEs to ODEs is quite important and helps in understanding many physical processes. A powerful technique to analyze such nonlinear PDEs and their reduction to ODEs is given by Lie symmetry method also called similarity method. The similarity method makes use of natural symmetries in a PDE to determine similarity variables which give rise to reducion to ODE. A detailed review of this technique is given in chapter one.

There are many notable contributions in application of similarity method to IBVPs. Success of this method relies upon success of solving the reduced ODEs. In many cases such ODEs are not easily integrable thus it demands solving them numerically. There is lack of practical approaches to utilize these numerical solutions to obtain solution of original PDE. Due to this restriction the method has not been utilized in great deal. The aim of this work is to provide a practical way of dealing with such situations.

3.1 Description of the method:

The main idea of our method is to find an approximation of the solution of the reduced ODE so that it can be used to generate approximate solution of the original PDE via inverse similarity transformation. This approximate solution of PDE has more advantages over numerical solution as it displays variables and parameters of the problem, thus it requires less processing time and it can be more useful in real time applications.
The process consists of following steps:

Step 1:

Reduce IBVP of PDE to BVP of ODE by applying similarity transformations.

Step 2:

Find numerical solution of reduced BVP.

Step 3:

Obtain an initial guess for approximate solution of BVP of ODE.

Step 4:

Improve initial guess to get approximate solution upto desired level of accuracy.

Step 5:

Use inverse similarity transform to get approximate solution of IBVP of PDE.
In following subsections we will demonstrate our method by giving some examples.

3.2 Unsteady gas flow through semi-infinite porous medium using transformation $z = \frac{x}{\sqrt{t}}(\frac{B}{4})^{\frac{1}{2}}$, $u(z) = \alpha^{-1}(1-(q(x,t))^2)$

Consider a problem of unsteady gas flow through semi-infinite porous medium [13–15], initially that was filled with gas at uniform pressure $q_0 > 0$ at time $t = 0$. At the outflow face pressure is suddenly reduced from q_0 to $q_1 > 0$ and is thereafter maintained at q_1. The nonlinear partial differntial equation that describe above phenomenon is

$$\nabla^2(q^2) = 2B\frac{\partial q}{\partial t},$$

where B is contant given by properties of the medium. In case of one dimensional medium extending from $x = 0$ to $x = \infty$ the above equation reduces to

$$\frac{\partial}{\partial x}(q\frac{\partial q}{\partial x}) = B\frac{\partial q}{\partial t},$$

with

$$q(x,0) = q_0, \qquad q(0,t) = q_1 < q_0, \qquad q(\infty,t) = q_0.$$

Without loss of generality we assume $q_0 = 1$. Thus IBVP under study is

$$\frac{\partial}{\partial x}(q\frac{\partial q}{\partial x}) = B\frac{\partial q}{\partial t}, \tag{3.2.1}$$

with initial and boundary conditions given below

$$q(x,0) = 1, \qquad 0 < x < \infty \tag{3.2.2}$$

$$q(0,t) = q_1(<1), \qquad 0 \le t < \infty \tag{3.2.3}$$

$$q(\infty,t) = 1. \qquad 0 < x < \infty \tag{3.2.4}$$

First of all we apply Lie symmetry method to above IBVP to obtain a one-parameter group of transformations that leaves the problem invariant and then we use these transformations to obtain similarity reductions.

Applying 2^{nd} extension of generator $\mathbf{H}$ and Eq. (1.2.4) to above PDE (3.2.1), we get

$$-B\eta_1 + \eta q_{xx} + \eta_{22}q + 2\eta_2 q_x = 0.$$

According to algorithm we put values of η_1, η_2 and η_{22}, then we equate coefficients of derivatives of q to zero. Thus we get following system of equations

$$-B\eta_t = 0, \quad (3.2.5)$$

$$-B\eta_q + B\tau_t + B\eta_q - 2B\xi_x = 0, \quad (3.2.6)$$

$$2\eta_x + B\xi_t = 0, \quad (3.2.7)$$

$$B\tau_q = 0, \quad (3.2.8)$$

$$B\xi_q - 2\tau_x = 0, \quad (3.2.9)$$

$$\eta = 0, \quad (3.2.10)$$

$$\eta_{xx} = 0, \quad (3.2.11)$$

$$2\eta_{xq} - \xi_{xx} = 0, \quad (3.2.12)$$

$$-\tau_{xx} = 0, \quad (3.2.13)$$

$$-2\tau_x = 0, \quad (3.2.14)$$

$$\eta_{qq} - 2\xi_{xq} = 0, \quad (3.2.15)$$

$$-2\tau_{xq} = 0, \quad (3.2.16)$$

$$-\xi_{qq} = 0, \quad (3.2.17)$$

$$-\tau_{qq} = 0, \quad (3.2.18)$$

$$-3\xi_q = 0, \quad (3.2.19)$$

$$-\tau_q = 0, \quad (3.2.20)$$

$$-2\tau_q = 0, \quad (3.2.21)$$

$$2(\eta_q - \xi_x) - \eta_q + 2\xi_x = 0, \quad (3.2.22)$$

$$-2\xi_q = 0, \quad (3.2.23)$$

$$-2\tau_q = 0. \quad (3.2.24)$$

Solving above system we get

$$\xi = ax + b, \qquad \tau = 2at + c, \qquad \eta = 0.$$

Taking one of the constants to be one and remaining to be zero we get all the symmetries as follows.

$$\mathbf{H}_1 = \frac{\partial}{\partial t}, \qquad \mathbf{H}_2 = \frac{\partial}{\partial x}, \qquad \mathbf{H}_3 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}. \tag{3.2.25}$$

Now we consider general symmetry operator as follows

$$\mathbf{H} = c_1\mathbf{H}_1 + c_2\mathbf{H}_2 + c_3\mathbf{H}_3,$$

and look for the operator which preserve boundary and boundary conditions. The invariance of boundaries $x = 0$, $t = 0$ or equivalently we say

$$[\mathbf{H}(x-0)]_{x=0} = 0,$$

or

$$(c_2 \cdot 1 + x \cdot c_3)|_{x=0} = 0, \qquad \Rightarrow \qquad c_2 = 0.$$

Also

$$[\mathbf{H}(t-0)]_{t=0} = 0,$$

or

$$(c_1 \cdot 1 + 2t \cdot c_3)|_{t=0} = 0, \qquad \Rightarrow \qquad c_1 = 0.$$

Also the invariance of initial and boundary conditions i.e.

$$\begin{aligned} [\mathbf{H}(q-1)]_{t=0} &= 0, \\ [\mathbf{H}(q-q_1)]_{x=0} &= 0, \end{aligned}$$

implies that under symmetry given below, the IBVP is invariant

$$\mathbf{H} = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}. \tag{3.2.26}$$

Here we choose $c_3 = 1$.

Now solving

$$\frac{dt}{2t} = \frac{dx}{x} = \frac{dq}{0}.$$

Thus

$$\frac{dx}{x} = \frac{dt}{2t}, \qquad \Rightarrow \qquad z = \frac{x^2}{t}.$$

and

$$\frac{dq}{0} = \frac{dx}{x}, \qquad \Rightarrow \qquad q = v.$$

Thus reduction variables will be $v = u(z)$. So we have

$$z(x,t) = \frac{x^2}{t}, \qquad u(z) = q. \tag{3.2.27}$$

Applying following similarity transformation [13, 14] to IBVP (3.2.1)

$$z = \frac{x}{\sqrt{t}}(\frac{B}{4})^{\frac{1}{2}}, \qquad u(z) = \alpha^{-1}(1 - (q(x,t))^2), \tag{3.2.28}$$

with $\alpha = 1 - q_1^2$.

From Eq. (3.2.28) we can write

$$q = \sqrt{1 - \alpha u}. \tag{3.2.29}$$

Again from Eq. (3.2.28)

$$z = \frac{x}{\sqrt{t}}(\frac{B}{4})^{\frac{1}{2}} \tag{3.2.30}$$

Differentiating Eq. (3.2.30) partially w.r.t x and t respectively, we have

$$\frac{\partial z}{\partial x} = \frac{1}{\sqrt{t}}(\frac{B}{4})^{\frac{1}{2}}, \tag{3.2.31}$$

$$\frac{\partial z}{\partial t} = \frac{-x}{2(t)^{\frac{3}{2}}}(\frac{B}{4})^{\frac{1}{2}}. \tag{3.2.32}$$

Now differentiating Eq. (3.2.29) partially w.r.t x we have

$$\frac{\partial q}{\partial x} = \frac{1}{2\sqrt{1-\alpha u}}\frac{\partial u}{\partial x},$$

or

$$\frac{\partial q}{\partial x} = \frac{1}{2\sqrt{1-\alpha u}}\frac{du}{dz}\frac{\partial z}{\partial x}. \tag{3.2.33}$$

Multiplying Eq. (3.2.29) with Eq. (3.2.33) we get

$$q\frac{\partial q}{\partial x} = \frac{\sqrt{1-\alpha u}}{2\sqrt{1-\alpha u}}(\frac{1}{\sqrt{t}}(\frac{B}{4})^{\frac{1}{2}})\frac{du}{dz}, \tag{3.2.34}$$

Now differentiating Eq. (3.2.34) partially w.r.t x we get

$$\frac{\partial}{\partial x}(q\frac{\partial q}{\partial x}) = \frac{1}{2\sqrt{t}}\sqrt{\frac{B}{4}}\frac{d^2u}{dz^2}\frac{\partial z}{\partial x}. \tag{3.2.35}$$

Using Eq. (3.2.31) in Eq. (3.2.35) we get

$$\frac{\partial}{\partial x}(q\frac{\partial q}{\partial x}) = \frac{1}{2\sqrt{t}}\sqrt{\frac{B}{4}}\frac{d^2u}{dz^2}(\frac{1}{\sqrt{t}}(\frac{B}{4})^{\frac{1}{2}}),$$

or

$$\frac{\partial}{\partial x}(q\frac{\partial q}{\partial x}) = \frac{B}{8t}\frac{d^2u}{dz^2}. \tag{3.2.36}$$

Again differentiating Eq. (3.2.29) partially w.r.t t we get

$$\frac{\partial q}{\partial t} = \frac{1}{2\sqrt{1-\alpha u}}\frac{\partial u}{\partial t},$$

or

$$\frac{\partial q}{\partial t} = \frac{1}{2\sqrt{1-\alpha u}}\frac{du}{dz}\frac{\partial z}{\partial t},$$

or

$$\frac{\partial q}{\partial t} = \frac{1}{2\sqrt{1-\alpha u}}(\frac{-x}{2(t)^{\frac{3}{2}}}(\frac{B}{4})^{\frac{1}{2}})\frac{du}{dz}. \tag{3.2.37}$$

Substitute (3.2.36) and (3.2.37) in (3.2.1) we get

$$\frac{d^2u}{dz^2} = (\frac{-8xt}{4(t)^{\frac{3}{2}}\sqrt{1-\alpha u}}\sqrt{\frac{B}{4}})\frac{du}{dz},$$

or

$$\frac{d^2u}{dz^2} = \frac{-2}{\sqrt{1-\alpha u}}(\frac{x}{\sqrt{t}}\sqrt{\frac{B}{4}})\frac{du}{dz},$$

or

$$u'' + \frac{2z}{\sqrt{1-\alpha u}}u' = 0.$$

And the boundary conditions are

When $t = 0$, $q(x, 0) = 1$ implies $z \to \infty$, $u(z \to \infty) = \alpha^{-1}(1-1) = 0$.

When $x \to \infty$, $q(\infty, t) = 1$ implies $z \to \infty$, $u(z \to \infty) = \alpha^{-1}(1-1) = 0$.

When $x = 0$, $q(0, t) = q_1$ implies $z = 0$, $u(z = 0) = \alpha^{-1}(1-q_1^2) = 1$.

Thus the similarity transformation reduces IBVP given by Eq. (3.2.1) to Eq. (3.2.4) to BVP of ODE given below

$$u'' + \frac{2z}{\sqrt{1-\alpha u}}u' = 0, \tag{3.2.38}$$

$$u(z=0) = 1, \qquad u(z \to \infty) = 0. \tag{3.2.39}$$

Next step is to find the numerical solution of reduced BVP (3.2.38) and (3.2.39). In this example we have taken lower solution of BVP (3.2.38) and (3.2.39) as our initial guess. As shown in calculations below, lower solution in just few iterations leads to an approximate solution. Thus it is a good initial guess. Thus the initial guess for all cases discussed below is taken to be

$$\text{initial approximation} = u_{lower} = 1 - erf(\tfrac{z}{\sqrt{q_1}}),$$

where erf denotes error function.
We adopt following procedure to improve our initial approximation. The lower solution has form

$$1 - erf(cz), \tag{3.2.40}$$

with $c = c_0 = \frac{1}{\sqrt{q_1}}$ gives us initial approximation u_{c_0} i.e the lower solution. Numerical simulations tells us that as we decrease value of c from c_0 by a small decrement, lower solution moves towards the numerical solution.
Given a function $W(z)$ and $\epsilon > 0$, we say that $g(z)$ lies within $\epsilon-$band of $W(z)$ on interval J if

$$|g(z) - W(z)| < \epsilon. \qquad \forall z \in J$$

For numbers $\epsilon > 0$, $\delta_i > 0$ and a suitable value n, use sequence of values

$$c = c_i = c_0 - \delta_i, \qquad (i = 1, 2, ..., n)$$

in equation (3.2.40) which generates a sequence of curves u_{c_i} which approach uniformly towards the numerical solution, finally results in the curve

$$u_{approx} = u_{c_n},$$

lying in an $\epsilon-$band of numerical solution u_{num}. We choose ϵ according to our desired accuracy and value of c_n is approximated by numerical simulations.
In following subsection we implement above procedure and find approximate similarity solution for some cases of values of q_1.

3.2.1 Approximate similarity solution of IBVP for $q_1 = 0.9$

Here in this case $c = \frac{1}{\sqrt{0.9}}$ and the lower solution of BVP of ODE becomes

$$u_{lower} = 1 - erf(cz) \approx 1 - erf(1.05409z),$$

which is our initial approximation u_{c_0}. To approximate solution of BVP of ODE, we first solve BVP (3.2.38), (3.2.39) numerically by applying nonlinear Shooting method defined in Section (2.2.2). We call this solution u_{num}. The problem is also solved numerically with MATLAB function bvp4c. The graph is not shown here. We get same approximate solution in both cases. Therefore all the rest of problems in this dissertation are solved only by using bvp4c. The graph of numerical solution and initial guess i.e the lower solution is shown in Fig. (3.1)

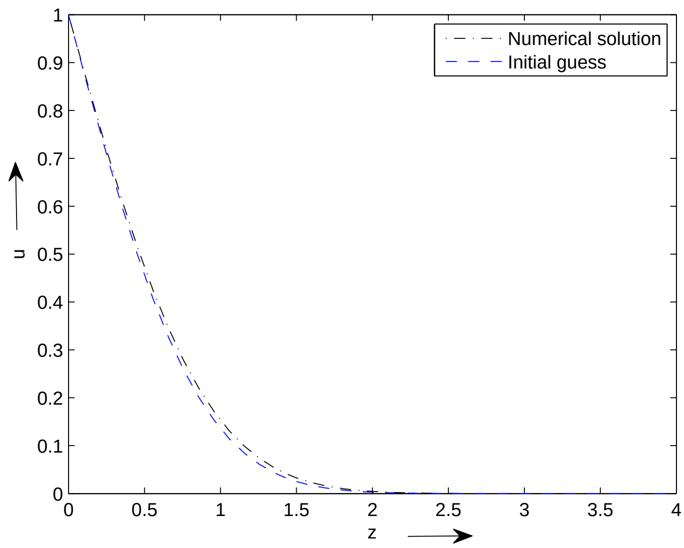

Figure 3.1: Graphs of u_{num} and u_{lower} for case $q_1 = 0.9$.

$$Max|u_{num} - u_{lower}| = 0.0196.$$

$$Error(z) = u_{num}(z) - u_{lower}(z),$$

is shown in following Fig (3.2).

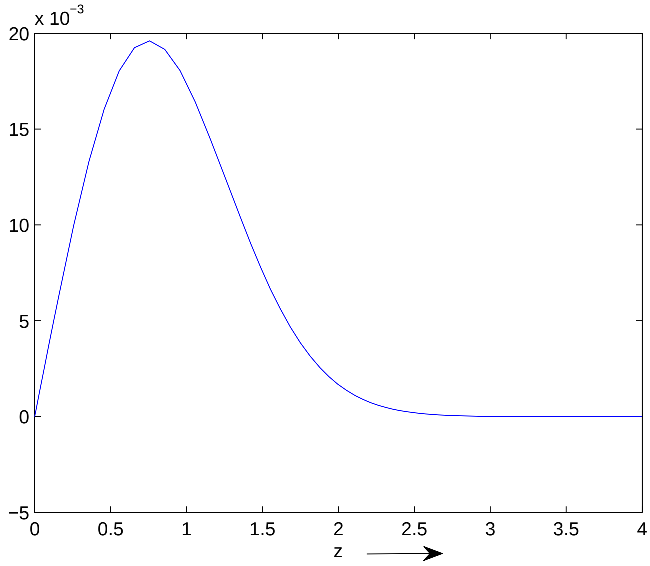

Figure 3.2: Error between u_{num} and u_{lower}.

By applying the procedure explained in previous subsection we improve the approximation u_{c_i} uniformly to get an approximate solution of ODE given below

$$u_{approx}(z) = 1 - erf(1.0122z),$$

The graph of numerical solution $u_{num}(z)$, initial approximation $u_{lower}(z)$ and the approximate solution $u_{approx}(z)$ are shown in Fig. (3.3). The dotted curve is taken from a sequence of curves u_{c_i} approaching uniformly towards the numerical solution.

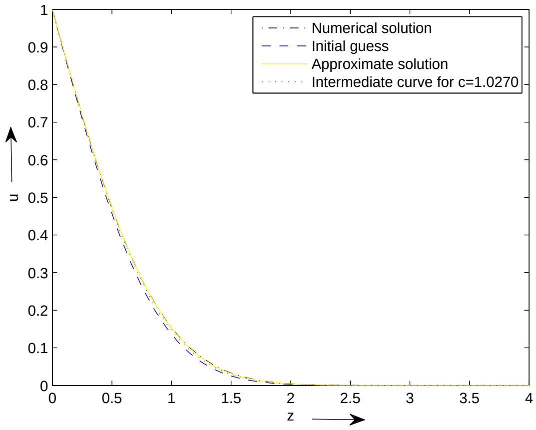

Figure 3.3: Graph of $u_{num}(z)$, $u_{lower}(z)$ and $u_{approx}(z)$.

$$Max|u_{num} - u_{approx}| = 0.0014.$$

$$Error(z) = u_{num}(z) - u_{approx}(z),$$

is given in Fig. (3.4).

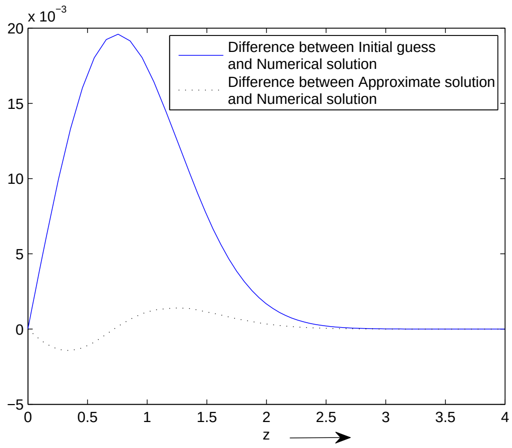

Figure 3.4: The solid line represents error between lower solution and numerical solution whereas dotted line gives error between approximate solution and numerical solution.

When we repeat above procedure by using upper solution

$$u_{upper} = 1 - erf(z),$$

we get same approximate solution given by

$$u_{approx}(z) = 1 - erf(1.0122z).$$

In this case the graph of numerical solution $u_{num}(z)$, initial approximation $u_{upper}(z)$ and the approximate solution $u_{approx}(z)$ are shown in Fig. (3.5).

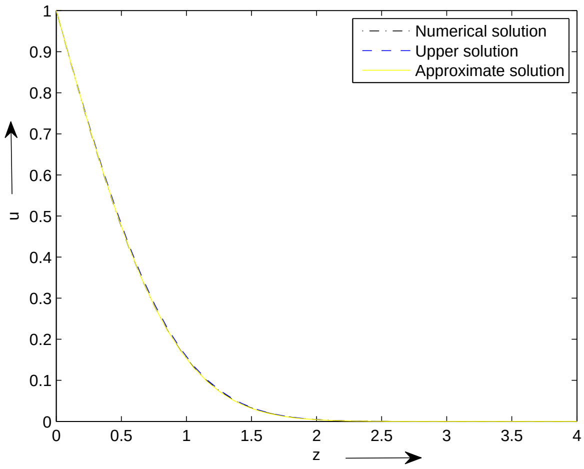

Figure 3.5: Graphs of u_{num}, u_{upper} and u_{approx} for case $q_1 = 0.9$.

Finally by using similarity variables given by Eq. (3.2.28) we find the approximate solution of IBVP (3.2.1)-(3.2.4) given as

$$q(x,t) = \sqrt{0.81 + 0.19erf(1.0120\frac{x}{\sqrt{t}}(\frac{B}{4})^{\frac{1}{2}}}).$$

3.2.2 Approximate similarity solution of IBVP for $q_1 = 0.3$

Here we have $c = \frac{1}{\sqrt{0.3}}$ and the lower solution becomes

$$u_{lower} = 1 - erf(cz) \approx 1 - erf(1.852574z),$$

which is taken as initial approximation u_{c_0}. The graph of numerical solution and initial guess i.e the lower solution is shown in Fig. (3.6)

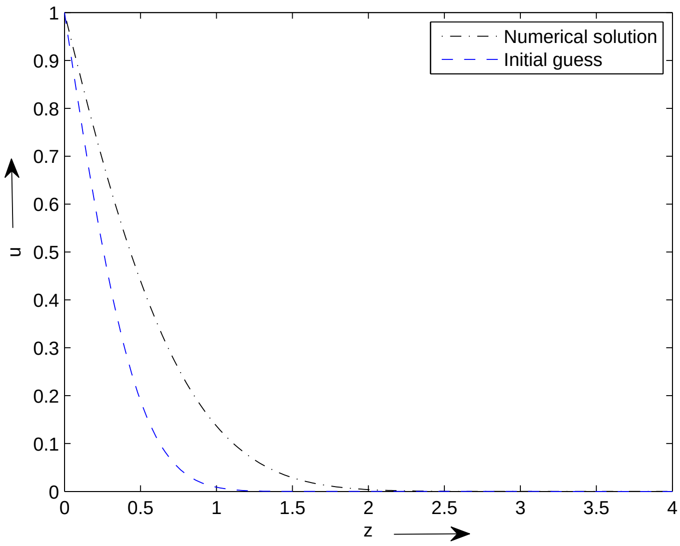

Figure 3.6: Graphs of u_{num} and u_{lower} for case $q_1 = 0.3$.

$$Max|u_{num} - u_{lower}| = 0.2499.$$

$$Error(z) = u_{num}(z) - u_{lower}(z),$$

is shown in following Fig.

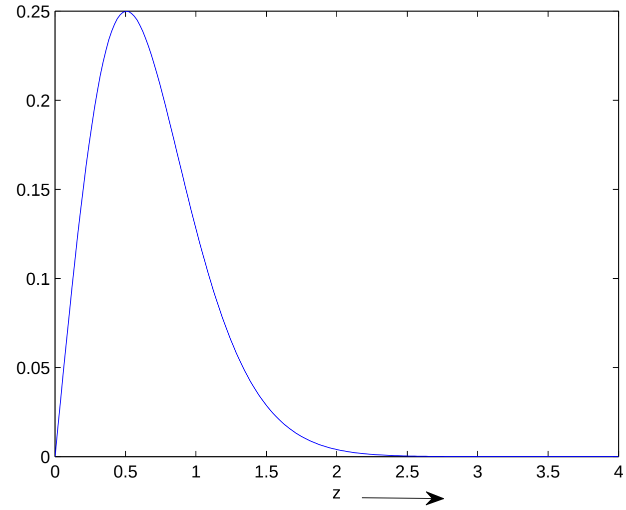

Figure 3.7: Error between u_{num} and u_{lower}.

Approximate solution of BVP of ODE is given below

$$u_{approx}(z) = 1 - erf(1.0800z),$$

The graph of $u_{num}(z)$, $u_{lower}(z)$ and $u_{approx}(z)$ for case $q_1 = 0.3$ are shown in Fig. (3.8)

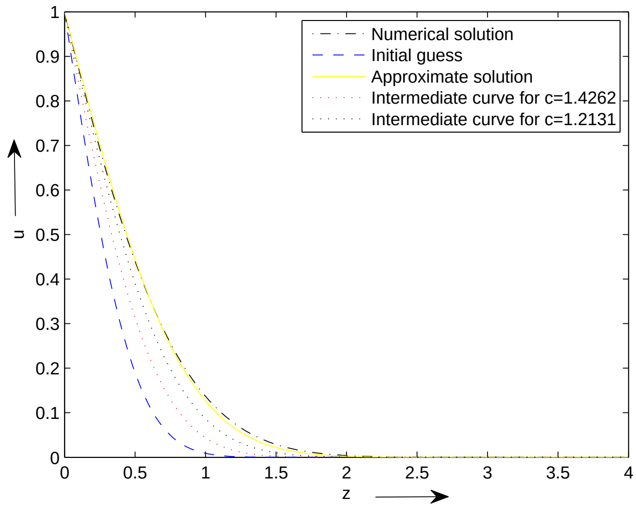

Figure 3.8: Graphs of u_{num}, u_{lower} and u_{approx} for case $q_1 = 0.3$.

$$Max|u_{num} - u_{approx}| = 0.0105.$$

$$Error(z) = u_{num}(z) - u_{approx}(z),$$

is given in Fig. (3.9).

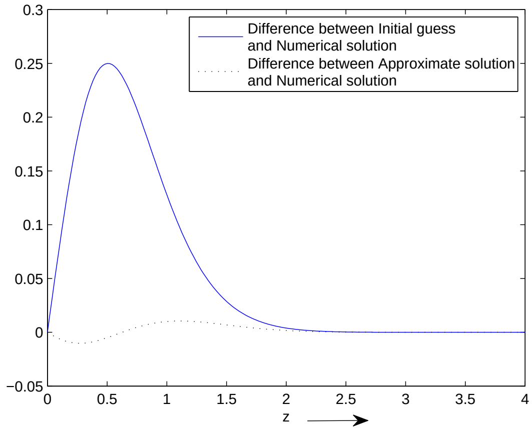

Figure 3.9: The solid line represents error between lower solution and numerical solution whereas dotted line gives error between approximate solution and numerical solution.

When we use upper solution

$$u_{upper} = 1 - erf(z),$$

we get same approximate solution given by

$$u_{approx}(z) = 1 - erf(1.0800z).$$

In this case the graph of $u_{upper}(z)$, $u_{num}(z)$ and $u_{approx}(z)$ are shown in Fig. (3.10).

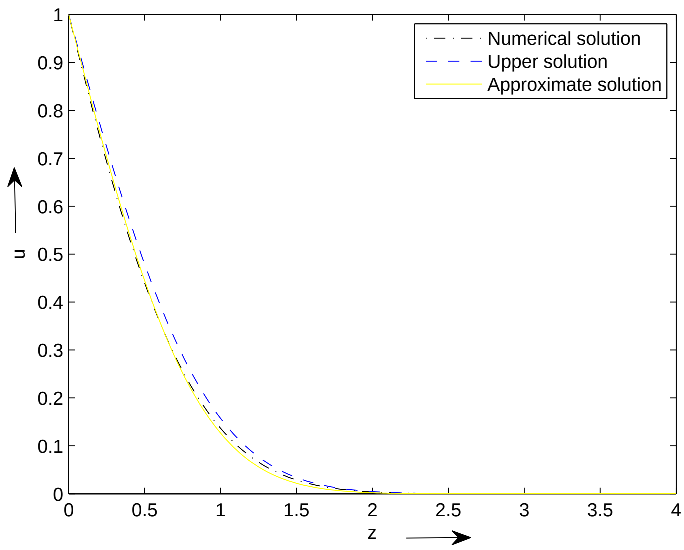

Figure 3.10: Graphs of u_{num}, u_{upper} and u_{approx} for case $q_1 = 0.3$.

The approximate solution of IBVP (3.2.1)-(3.2.4) is given below

$$q(x,t) = \sqrt{0.09 + 0.91erf(1.0800\frac{x}{\sqrt{t}}(\frac{B}{4})^{\frac{1}{2}})}.$$

3.2.3 Approximate similarity solution of IBVP for $q_1 = 0.1$

Here $c = \frac{1}{\sqrt{0.1}}$. Thus the lower solution becomes

$$u_{lower} = 1 - erf(cz) \approx 1 - erf(3.162277z),$$

which is taken as initial approximation u_{c_0}. The graph of numerical solution and initial guess i.e the lower solution is shown in Fig. (3.11)

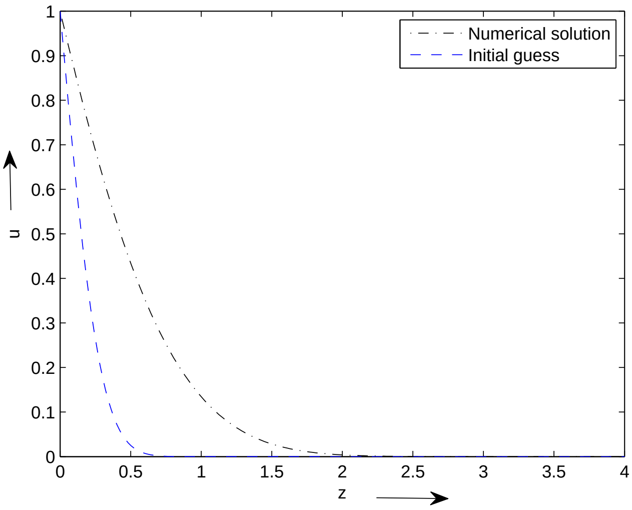

Figure 3.11: Graphs of u_{num} and u_{lower} for case $q_1 = 0.1$.

$$Max|u_{num} - u_{lower}| = 0.4601.$$

$$Error(z) = u_{num}(z) - u_{lower}(z),$$

is shown in following Fig.

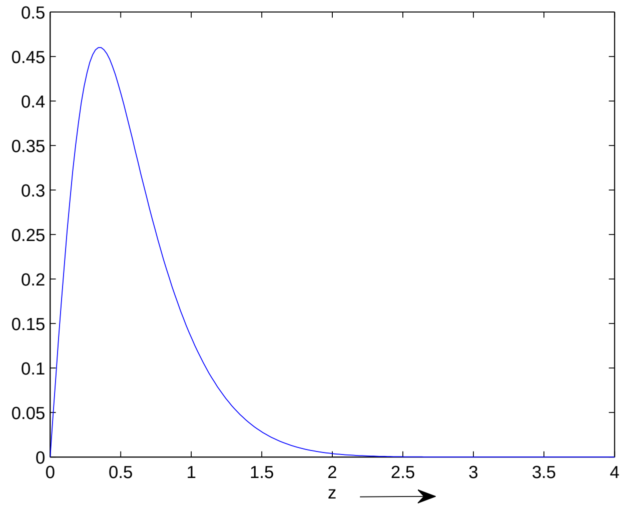

Figure 3.12: Error between u_{num} and u_{lower}.

The approximate solution is given below

$$u_{approx}(z) = 1 - erf(1.0920z),$$

The graph of $u_{num}(z)$, $u_{lower}(z)$ and $u_{approx}(z)$ are shown in Fig. (3.13). The dotted curves represents some intermediate curves involved in simulations of the uniform approximation process from u_{lower} to u_{approx}.

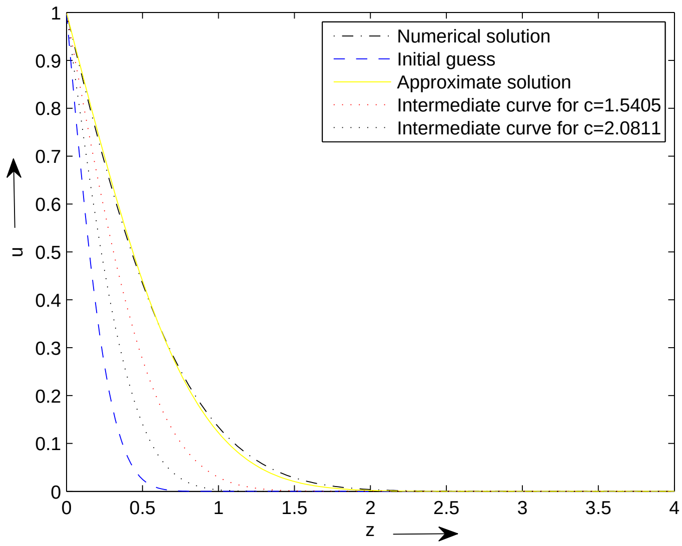

Figure 3.13: Graphs of u_{num}, u_{lower} and u_{approx} for case $q_1 = 0.1$.

$$Max|u_{num} - u_{approx}| = 0.0123.$$

$$Error(z) = u_{num}(z) - u_{approx}(z),$$

is shown in Fig. (3.14).

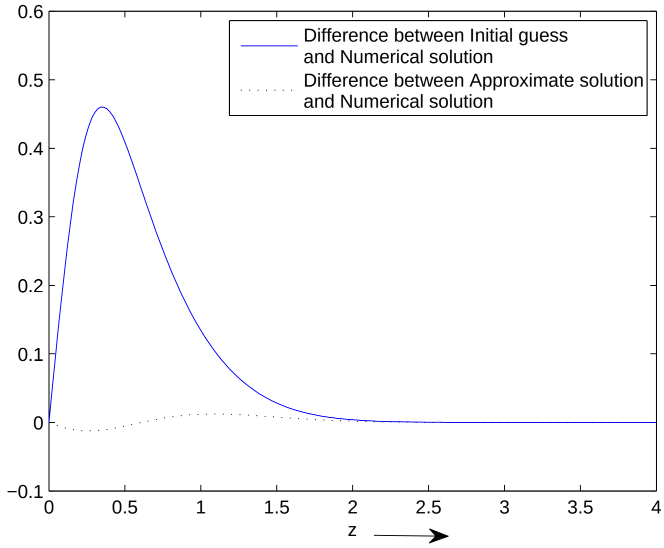

Figure 3.14: The solid line represents error between lower solution and numerical solution whereas dotted line gives error between approximate solution and numerical solution.

Repeating above procedure by using upper solution

$$u_{upper} = 1 - erf(z),$$

gives us same approximate solution i.e

$$u_{approx}(z) = 1 - erf(1.0920z).$$

In this case $u_{num}(z)$, $u_{upper}(z)$ and $u_{approx}(z)$ are shown in Fig. (3.15).

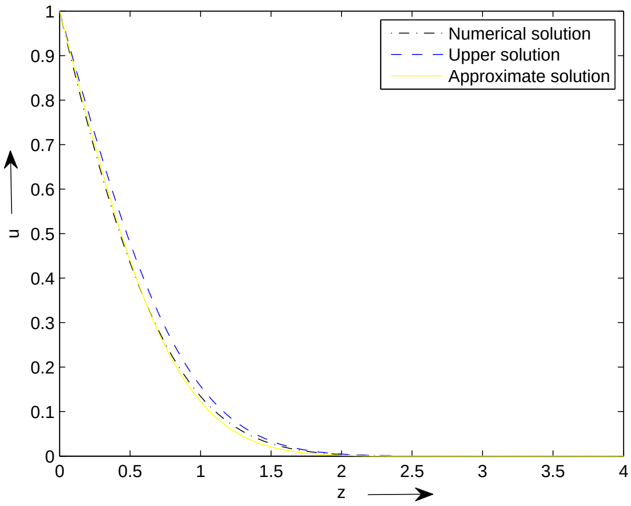

Figure 3.15: Graphs of u_{num}, u_{upper} and u_{approx} for case $q_1 = 0.1$.

Finally for $q_1 = 0.1$ by using similarity variables we get the approximate solution of IBVP (3.2.1)-(3.2.4) given below

$$q(x,t) = \sqrt{0.01 + 0.99erf(1.0920\frac{x}{\sqrt{t}}(\frac{B}{4})^{\frac{1}{2}})}.$$

3.3 Nonlinear diffusion equation

Consider nonlinear diffusion equation [16]

$$u\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \tag{3.3.1}$$

which occurs in problem of thermal expulsion of fluid from slender, long heated tube. The quantity u gives us the flow velocity which is induced in fluid by heating the tube wall. The initial and boundary conditions are

$$u_x(0,t) = -b, \qquad t > 0 \tag{3.3.2}$$

$$u(x,0) = 0, \qquad x > 0 \tag{3.3.3}$$

$$u(\infty,t) = 0, \qquad t > 0 \tag{3.3.4}$$

where constant b is related to heating of tube started at $t = 0$. We assume the PDE is invariant to one-parameter family of groups given by

$$\begin{aligned} u' &= \gamma^a u, \\ t' &= \gamma^b t, \qquad 0 < \gamma < \infty \\ x' &= \gamma x, \end{aligned} \tag{3.3.5}$$

subject to linear constraint

$$a - b = -2.$$

The invariant relation which connects u, x and t is given by

$$u = t^{\frac{a}{b}} w(\frac{x}{t^{\frac{1}{b}}}).$$

The boundary condition (3.3.2) requires that $a = 1$, so $b = 3$. Thus we have

$$u = t^{\frac{1}{3}} w(\frac{x}{\sqrt{3}t^{\frac{1}{3}}}). \tag{3.3.6}$$

The factor $\sqrt{3}$ is introduced for our convenience. Apply following tranformation to IBVP (3.3.1)-(3.3.4)

$$z = \frac{x}{\sqrt{3}t^{\frac{1}{3}}}, \qquad w(z) = \frac{u}{t^{\frac{1}{3}}}. \tag{3.3.7}$$

Now consider

$$u = t^{\frac{1}{3}} w(z). \tag{3.3.8}$$

Differentiating Eq. (3.3.8) partially w.r.t t we get

$$\frac{\partial u}{\partial t} = \frac{1}{3} t^{\frac{-2}{3}} w(z) - \frac{x}{3\sqrt{3}t} w'(z). \tag{3.3.9}$$

Again differentiating Eq. (3.3.8) twice w.r.t x we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{3t^{\frac{1}{3}}} w''(z). \tag{3.3.10}$$

Now substituting Eq. (3.3.9) and Eq. (3.3.10) in Eq. (3.3.1) we get an ODE given below

$$w''(z) = w(z)(w(z) - zw'(z)),$$

with boundary conditions

$$w'(0) = -\sqrt{3}b, \qquad w(\infty) = 0.$$

Taking $b = 1$ in above boundary condition. Thus we finally have boundary value problem given by

$$w'' = w(w - zw'), \tag{3.3.11}$$

with boundary conditions

$$w'(0) = -\sqrt{3}, \qquad w(\infty) = 0. \tag{3.3.12}$$

We need to find the approximate solution w_{approx}. Following the steps mentioned in section (3.1) first of all we will find numerical solution w_{num} of above BVP. By using $MATLAB$ function *bvp4c* we get the numerical solution shown in Fig. (3.16).

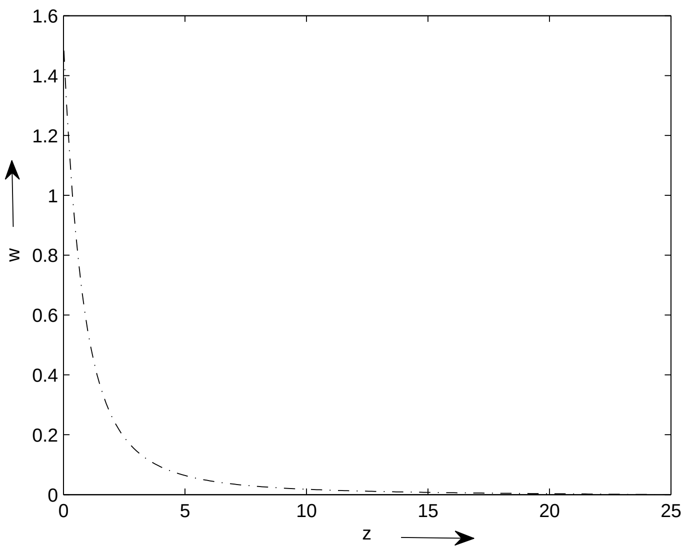

Figure 3.16: Graph of w_{num}.

When we look at the curve of the numerical solution, the highest value it takes on w-axis is 1.5112 and the curve is similar to the graph of an exponential function so our approximate solution can be of the form of $1.5112e^{-az}$. The solution must also satisfy the boundary conditions as well so if we take

$$w(z) = 1.5112e^{-az}.$$

Differentiating w.r.t z we get

$$w'(z) = -1.5112ae^{-az},$$

and

$$w'(0) = -1.5112a = -\sqrt{3}.$$

This gives us the value of a given below

$$a = \frac{\sqrt{3}}{1.5112}.$$

Thus we take approximate function as

$$f(z) = 1.5112e^{\frac{-\sqrt{3}z}{1.5112}}. \tag{3.3.13}$$

Fig. (3.17) shows graph of w_{num} and $f(z)$.

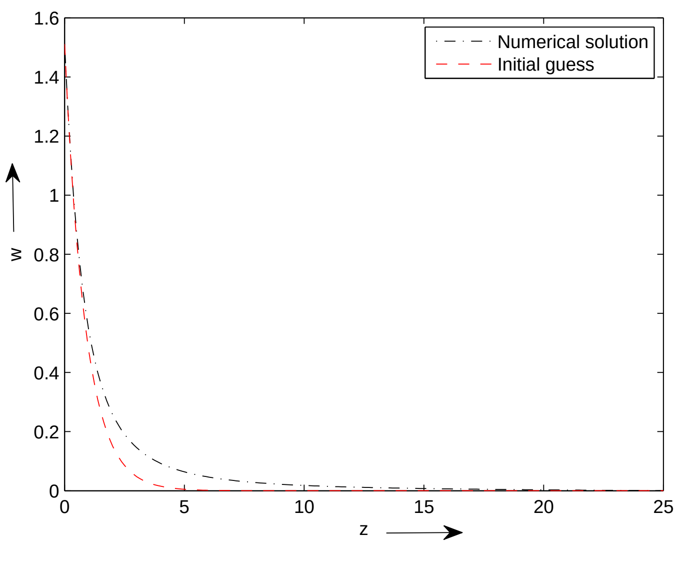

Figure 3.17: Graph of w_{num} and $f(z)$.

Next we calculate maximum error which is calculated to be 0.1073. Fig. (3.18) gives the error graph.

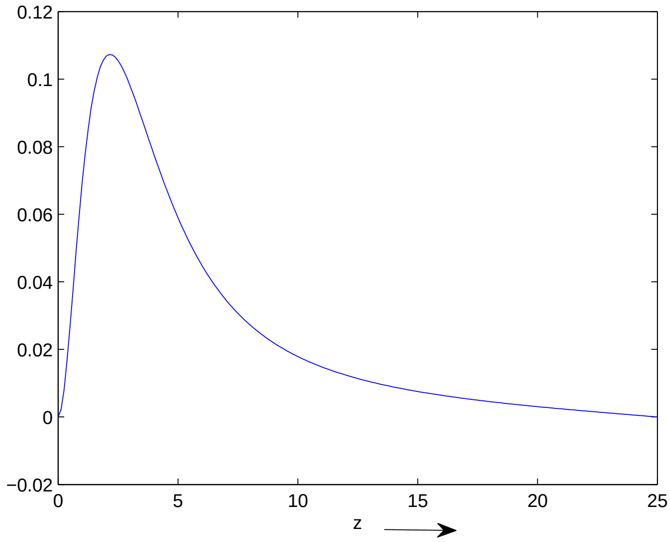

Figure 3.18: Error between w_{num} and $f(z)$.

In order to reduce the maximum error we find the function $g(z)$ which approximate the error graph and then add up this function to $f(z)$ to get approximate solution which will reduce the error. When we look at the error curve it is similar to the graph of function of the form $g(z) = az^b e^{cz}$. In order to find values of a, b and c we take three points on curve i.e (z_1, w_1), (z_2, w_2) and (z_3, w_3). Thus we get following three equations

$$w_1 = az_1^b e^{cz_1}, \tag{3.3.14}$$

$$w_2 = az_2^b e^{cz_2}, \tag{3.3.15}$$

$$w_3 = az_3^b e^{cz_3}. \tag{3.3.16}$$

Dividing Eq. (3.3.14) by Eq. (3.3.15) and then taking log

$$\ln(\frac{w_1}{w_2}) = b\ln(\frac{z_1}{z_2}) + c(z_1 - z_2). \tag{3.3.17}$$

Similarly dividing Eq. (3.3.14) by Eq. (3.3.16) and then taking log we get

$$\ln(\frac{w_1}{w_3}) = b\ln(\frac{z_1}{z_3}) + c(z_1 - z_3). \tag{3.3.18}$$

Multiplying Eq. (3.3.17) by $(z_1 - z_3)$ and Eq. (3.3.18) by $(z_1 - z_3)$ and subtracting the two equations we get

$$b((z_1 - z_3)\ln(\frac{z_1}{z_2}) - (z_1 - z_2)\ln(\frac{z_1}{z_3})) = (z_1 - z_3)\ln(\frac{w_1}{w_2}) - (z_1 - z_2)\ln(\frac{w_1}{w_3}),$$

or

$$b = \frac{(z_1 - z_3)\ln(\frac{w_1}{w_2}) - (z_1 - z_2)\ln(\frac{w_1}{w_3})}{(z_1 - z_3)\ln(\frac{z_1}{z_2}) - (z_1 - z_2)\ln(\frac{z_1}{z_3})}.$$

Now from Eq. (3.3.17)

$$c = \frac{\ln(\frac{w_1}{w_2}) - b\ln(\frac{z_1}{z_2})}{z_1 - z_2}.$$

From Eq. (3.3.14)

$$a = \frac{w_1}{z_1^b e^{cz_1}}.$$

Now taking points as follows

$$z_1 = 0.8794, \qquad z_2 = 2.1357, \qquad z_3 = 7.5377,$$

$$w_1 = 0.0596, \qquad w_2 = 0.1073, \qquad w_3 = 0.0304.$$

Substituting values we get

$$a = 0.12009, \qquad b = 1.4836, \qquad c = -0.57982.$$

Thus

$$g(z) = 0.12009 z^{1.4836} e^{-0.57982z}.$$

The approximate solution $= w_{approx} = f(z) + g(z)$.

Now the maximum error is reduced to 0.0148. Fig. (3.19) shows error graph and $g(z)$. Fig. (3.20) represents graph of w_{num}, $f(z)$ and w_{approx}.

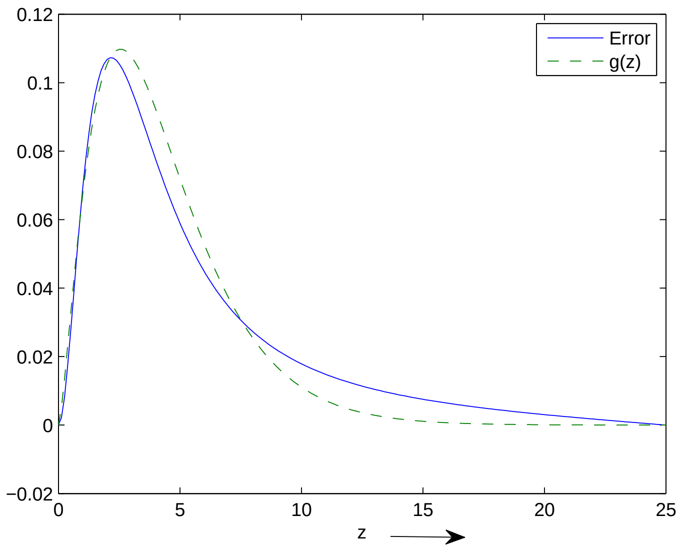

Figure 3.19: Graph of error and $g(z)$.

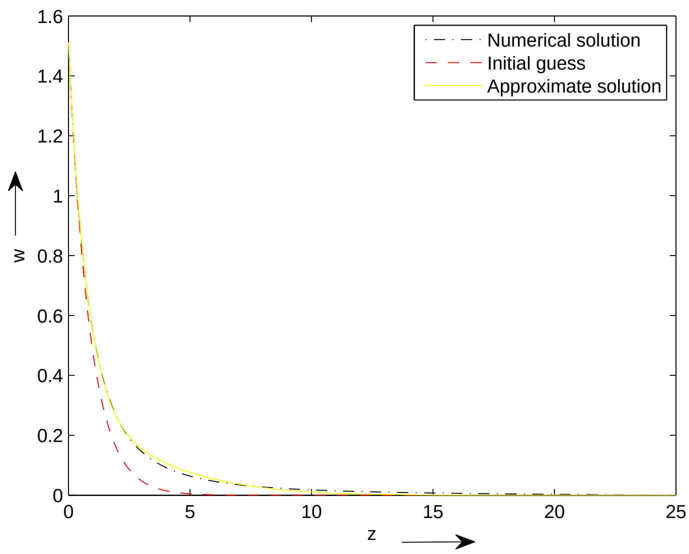

Figure 3.20: Graph of w_{num}, $f(z)$ and w_{approx}.

Fig. (3.21) shows graph of error between numerical solution and initial guess and error between numerical solution and approximate solution.

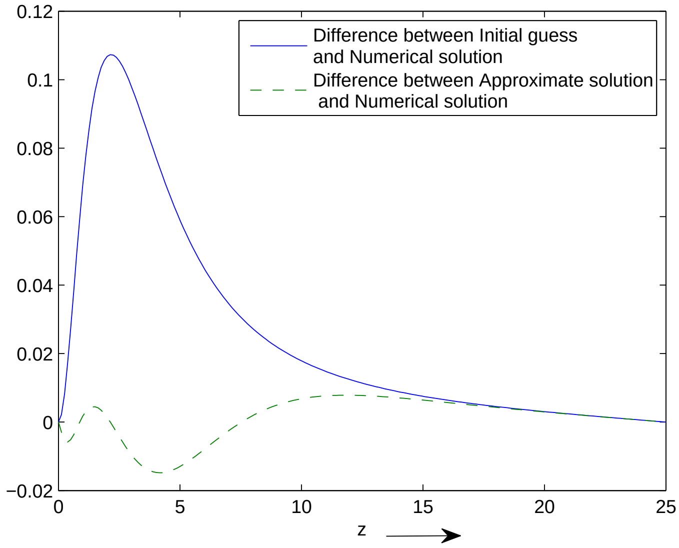

Figure 3.21: Solid line shows graph of error between numerical solution and initial guess and dotted line represents error between numerical solution and approximate solution.

Finally by using similarity variables given by Eq. (3.3.7) we find the approximate solution of IBVP (3.3.1)-(3.3.4) given as

$$u(x,t) = t^{\frac{1}{3}}(1.5112e^{-\frac{x}{1.5112t^{\frac{1}{3}}}} + 0.12009(\frac{x}{\sqrt{3}t^{\frac{1}{3}}})^{1.4836}e^{-0.57982\frac{x}{\sqrt{3}t^{\frac{1}{3}}}}).$$

3.4 Unsteady gas flow through semi-infinite porous medium using transformation $z = \frac{x}{\sqrt{t}}$, $w(z) = q$

Consider again IBVP (3.2.1)-(3.2.4) of unsteady gas flow through semi-infinite porous medium. Now we apply a different similarity transformation and observe how the result varies. Applying following transformation

$$z = \frac{x}{\sqrt{t}}, \tag{3.4.1}$$

$$w(z) = q. \tag{3.4.2}$$

Differentiating Eq. (3.4.2) partially w.r.t x and simplify to get

$$\frac{\partial q}{\partial x} = w'(\frac{1}{\sqrt{t}}). \tag{3.4.3}$$

Multiplying Eq. (3.4.3) with (3.4.3) we get

$$q\frac{\partial q}{\partial x} = \frac{ww'}{\sqrt{t}}. \tag{3.4.4}$$

Differentiating Eq. (3.4.4) partially w.r.t x

$$\frac{\partial}{\partial x}(q\frac{\partial q}{\partial x}) = \frac{1}{t}((w')^2 + ww''). \tag{3.4.5}$$

Again differentiating Eq. (3.4.2) partially w.r.t t and simplifying we get

$$\frac{\partial q}{\partial t} = \frac{-x}{2t^{\frac{3}{2}}}\frac{dw}{dz}. \tag{3.4.6}$$

Substituting Eq. (3.4.5) and Eq. (3.4.6) in Eq. (3.2.1) and solving we get an ODE given below

$$ww'' + (w')^2 + \frac{B}{2}zw' = 0, \tag{3.4.7}$$

With boundary conditions

$$w(0) = q_1, \qquad w(\infty) = 1.$$

Taking $q_1 = 0.9$ and $B = 1$. The boundary condition becomes

$$w(0) = 0.9, \qquad w(\infty) = 1. \tag{3.4.8}$$

Our aim is to find the approximate solution w_{approx}. First of all we will find numerical solution w_{num} by using *bvp4c* shown in Fig. (3.22)

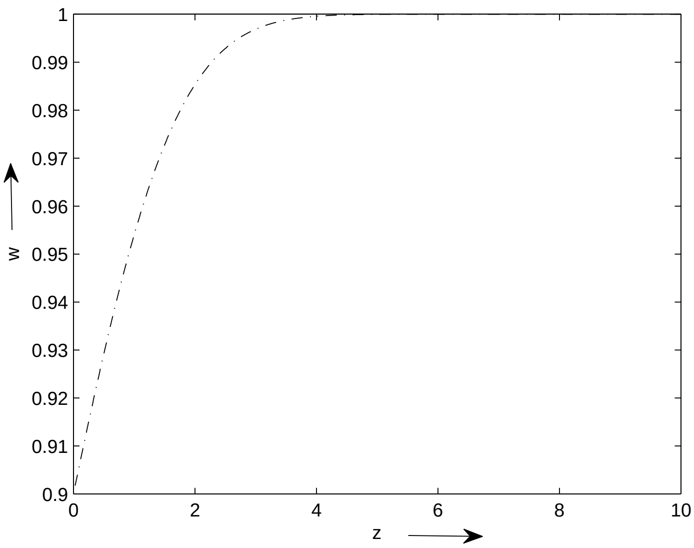

Figure 3.22: Graph of w_{num}.

The curve of the numerical solution is similar to that of an exponential function so we can take an exponential function as our approximation. Since the approximate function must also satisfy the boundary conditions so we take

$$f(z) = 1 - (1 - 0.9)\exp(-bz).$$

We take a point on curve of numerical solution in order to find b. Taking $z = 1.5015$ and $w = 0.9728$ gives

$$f(z) = 1 - (1 - 0.9)\exp(-0.8671z).$$

Fig. (3.23) shows graph of w_{num} and $f(z)$. The maximum error is calculated to be 0.0060. Fig. (3.24) gives the error curve.

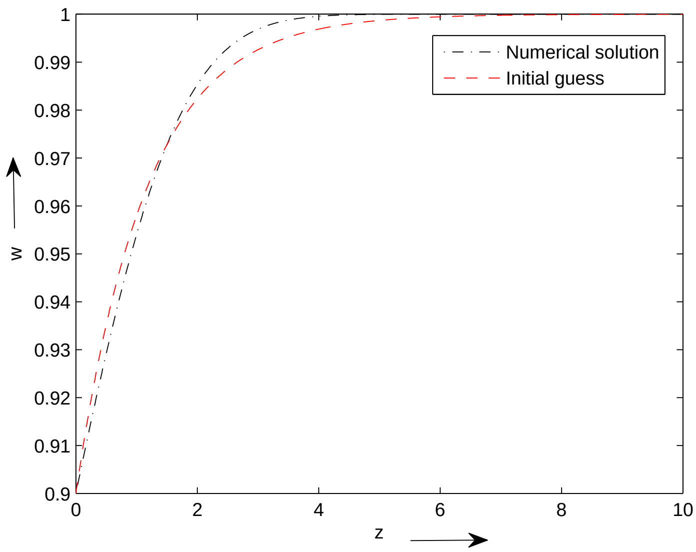

Figure 3.23: Graph of w_{num} and $f(z)$.

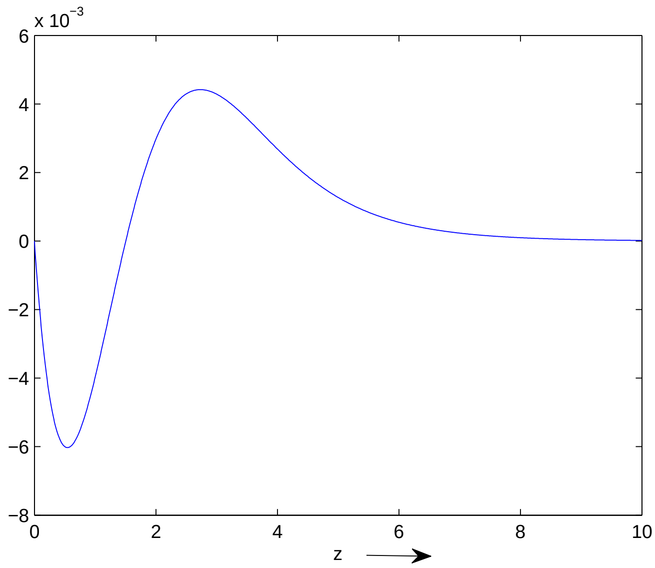

Figure 3.24: Error between w_{num} and $f(z)$.

Our next step is to find a function $g(z)$ which approximates the error curve and then add up this function to $f(z)$ to get the approximate solution which will reduce error and give us better approximation. The error curve is similar to graph of function of the form

$$g(z) = az(z - 1.5015)\exp(bz).$$

In order to find values of a and b, we take two points (z_1, w_1) and (z_2, w_2) on the error curve and solve following system of two equations.

$$\begin{aligned} w_1 &= az_1(z_1 - 1.5015)\exp(bz_1), \\ w_2 &= az_2(z_2 - 1.5015)\exp(bz_2). \end{aligned}$$

Solving above system we get

$$b = \frac{\ln(\frac{w_1}{w_2}) - \ln(\frac{z_1(z_1-1.5015)}{z_2(z_2-1.5015)})}{z_1 - z_2}.$$

$$a = \frac{w_1}{z_1(z_1 - 1.5015)\exp(bz_1)}.$$

Now take $z_1 = 0.4805$, $z_2 = 2.7528$, $w_1 = -0.0060$ and $w_2 = 0.0044$.
Substituting values we get $a = 0.0197$ and $b = -0.9942$. Thus

$$g(z) = 0.0197z(z - 1.5015)\exp(-0.9942z)$$

The approximate solution $= w_{approx} = f(z) + g(z)$
Now the maximum error is reduced to 0.0012. Fig. (3.25) shows graph of error and $g(z)$. Fig. (3.26) shows curves of numerical solution, f(z) and the approximate solution.

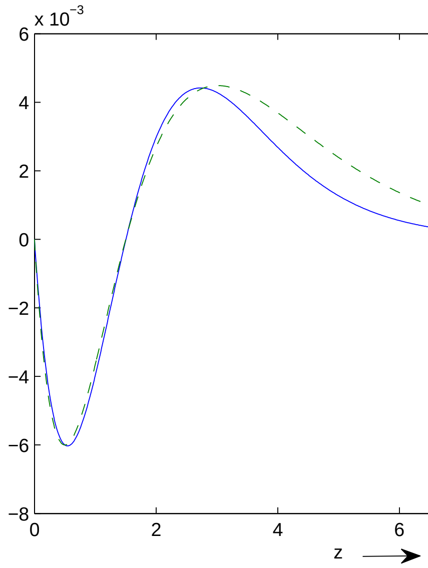

Figure 3.25: Graph of error and $g(z)$.

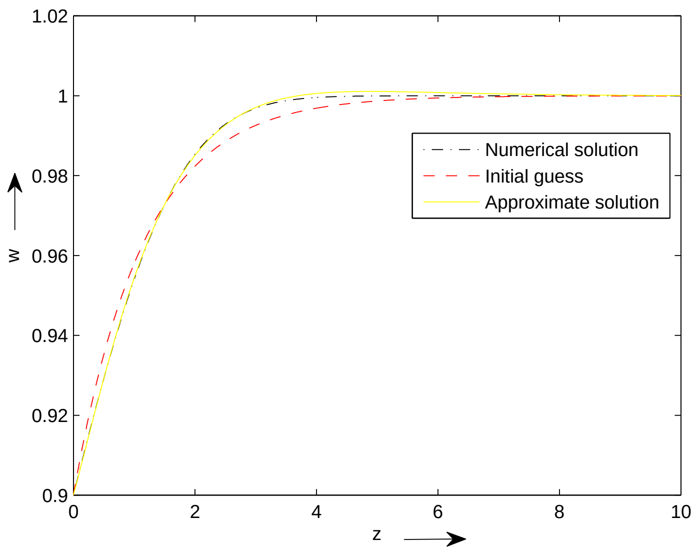

Figure 3.26: Graph of w_{num}, $f(z)$ and w_{approx}.

Fig. (3.27) shows graph of error between numerical solution and initial guess and error between numerical solution and approximate solution.

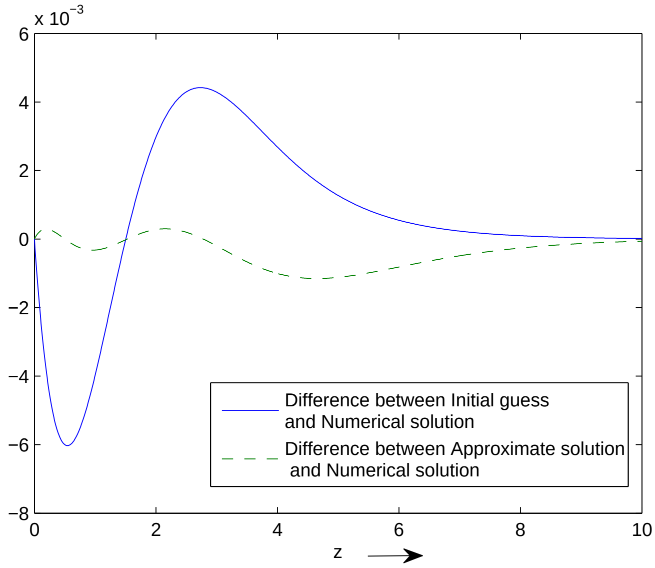

Figure 3.27: Solid line shows graph of error between numerical solution and initial guess and dotted line represents error between numerical solution and approximate solution.

Finally by using similarity variables given by Eq. (3.4.1) we find the approximate solution of IBVP (3.2.1)-(3.2.4) given as

$$q(x,t) = 1-(1-0.9)\exp(\frac{-0.8671x}{\sqrt{t}}) + \frac{0.0197x}{\sqrt{t}}(\frac{x}{\sqrt{t}} - 1.5015)\exp(\frac{-0.9942x}{\sqrt{t}}).$$

Chapter 4

Conclusion

In this dissertation a practical method is presented to obtain approximate solution, in form of function, for a class of PDEs which can be reduced through similarity variables to an ODE but that reduced ODE cannot be easily integrable in terms of known functions.

In chapter one, we have given a brief review of Symmetry Analysis. In the beginning of chapter we have given some basic definitions, later an algorithm of finding Lie symmetries of a PDE is provided. At the end of the chapter we have studied the symmetry analysis of BVP of PDE.

Chapter two is devoted to numerical techniques of BVPs of ODEs. There are many methods to solve BVPs numerically. We have used Shooting Method and MATLAB in built function *bvp4c* in this dissertation. In chapter two we have provided a brief review of these two methods.

In chapter three, we have given a practical way to find approximate solution of nonlinear PDEs. The method involves five steps. First step is to reduce IBVP of PDE to BVP of ODE via similarity variables. Second and third step involves finding numerical solution of reduced ODE and use this to obtain an initial guess for approximate solution. In step four the initial guess is improved upto desired level of accuracy to obtain approximate solution of BVP of ODE. Finally in fifth step with the help of this approximate solution we find the approximate solution of original PDE by using the similarity variables.

Later in this chapter we have given three examples to demonstrate our method. In example one we consider the different cases of IBVP of unsteady flow of gas through porous, semi infinite medium. In this example we take lower and upper solutions as our initial. The combination of numerical solution, simulations and initial approximations are utilized to get approximate solution of the curve $u(z)$ which generated readily, via similarity transform, the approximate solution of surface $q(x,t)$. The idea we present here is to obtain approximate solution of PDE, i.e. approximation of the surface $q(x,t)$ practically involve approximation of the curve $u(z)$ which is easy to deal with as compared to approximation of surface $q(x,t)$ itself.

In second example we consider nonlinear diffusion equation. Here we do not have lower or upper solution so we take a different approach to obtain the initial guess. We first calculate the numerical solution and then by looking at the curve of the numerical solution we guessed a function, considering in mind the boundary conditions, which perfectly fits the curve of the numerical solution. We take this as our initial guess, then we calculate error. In order to improve our initial approximation we guessed the function which fits our error curve and then add up this function to our initial guess to obtain approximate solution. The error in this process reduces significantly from 0.1073 to 0.0148. In the end we obtain the approximate solution of PDE in function form by applying similarity transform.

In third example we consider the problem one again and this time we apply different transformation to see how the things change. Again in this problem we follow the same steps as we already done in example two and reduced the error significantly. It is also suggested that the error may be reduced by fitting an approximate function in the Least Square sense.

The approximate solution of PDE obtained through this method has more advantage over numerical solution. Since it displays parameters and variables of problem, so it requires less time for processing and is more useful in real time problems.

For further work it will be quite interesting to find the numerical solution of PDE and compare this with the approximate solution obtained through this method to see how better our approximation is.

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