Approximate Solution of System of Nonlinear Partial Differential Equations and New Transformations

By

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Dedication

This thesis is dedicated to my Parents and Siblings for their endless devotion, support and encouragement.

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All praises are for Almighty Allah, the most gracious and the most merciful, Who created this entire universe. I am highly grateful to Him for showering His countless blessings upon me and giving me the ability and strength to complete this thesis successfully and blessing me more than I deserve.

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Abstract

In this dissertation, a systematic approach is used to find an approximate solution of a system of nonlinear PDEs. The system of nonlinear PDEs is reduced into a system of ODEs by using similarity transformations, which are solved numerically. We approximated numerical solutions by suitable functions. These simialrity transformations are again used to obtain approximate solution in the form of functions for the sytem of nonlinear PDEs. Residues for the system of ODEs and PDEs are calculated in accordance with the approximate solutions. Then, Lie symmetry method is used to find new transformations for the system of nonlinear PDEs.

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Chapter 1

Introduction

Mathematics is the science of relations, structures and order that has developed gradually from elemental practices of describing the shapes of objects, counting and measuring. It deals with quantitative calculation and logical reasoning. As a theoretical discipline, it opens ways for the feasible correspondence among abstractions without any concern whether such abstractions have counterparts in the real world.

Differential equations play a very important role not only in mathematics but also in various fields like physics, engineering, chemistry, computer sciences, economics and biology. The study of differential equations is important as they have been developed through the mathematical formulation of many problems of real life. For example, problem of growth and decay, problem of motion of a rocket, finding current in electric circuit, carbon dating problem and temperature problem. Such problems can be described through a single equation or a set of equations that relate various quantities and their rates of change.

In the language of mathematics, variable is a term which is liable to vary and they can be dependent or independent. The term derivative is defined as the rate of change of one variable with respect to another. An equation containing derivative(s) of one or more dependent variable(s) with respect to one or more independent variable(s) is called differential equation. Following are some examples of differential equations:

$$\frac{dy}{dx} + 2xy = e^x.$$

$$\frac{du}{dx} - \frac{dv}{dx} = x.$$

$$\frac{\partial u}{\partial y} + \frac{\partial u}{\partial x} = 0.$$

Differential equations first came into existence with the development of calculus by Newton [1]. Two Swiss mathematicians Jacob Bernoulli and Johann Bernoulli [2] worked on differential equations and their hard work ends up in making differential equation as a segregated branch of mathematics.
Newton's second law of motion is defined as:

$$\boldsymbol{F} = m\boldsymbol{a},$$

where $\boldsymbol{F}$ is the force, m is mass of an object and $\boldsymbol{a}$ is acceleration which is derivative of velocity $\boldsymbol{v}$, with respect to time t. Therefore, the above equation can be written in the form of differential equation as

$$\boldsymbol{F} = m\frac{dv}{dt}. \tag{1.1}$$

That is $\boldsymbol{F}$ can be defined as a function of velocity and time.
The order of differential equation is define as the highest derivative appearing in a differential equation. e.g.

$$\frac{d^3y}{dx^3} + \frac{dy}{dx} = 0,$$

is a third order differential equation.
The degree of a differential equation is the power of the highest derivative appearing in the equation. e.g.

$$\left(\frac{d^2y}{dx^2}\right)^4 + 5\left(\frac{dy}{dx}\right)^2 - 4y = x,$$

has degree 4.

Differential equations can be classified as Ordinary Differential Equation (ODE) and Partial Differential Equation (PDE). An equation which contains only ordinary derivative of one or more dependent variable(s) with respect to a single independent variable is known as ordinary differential equation. On the other hand, a partial differential equation is the one which contains the derivatives of dependent variable with respect to two or more independent variables.

Differential equations are also classified as linear, nonlinear, homogeneous and non-homogeneous. For linearity, an equation is characterized by the following properties:

- The dependent variable and all its derivatives are of first degree that is the power of each term involving dependent variable or its derivative is one.

- There is no product of dependent variable and its derivative.
- No transcendental functions (sine, cosine, exponential, etc.) in terms of dependent variable.

Otherwise, they are known as nonlinear differential equations.

Some examples of linear and nonlinear ordinary differential equations and partial differential equations are given below:

$$x\frac{d^3y}{dx^3} + e^x\frac{d^2y}{dx^2} - xy = 2$$

$$y^3\left(\frac{dy}{dx}\right)^2 - 3(x^2-1)y = 2,$$

and

$$\frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} = 0$$

$$u^2\left(\frac{\partial u}{\partial x_1}\right)^3 + \frac{\partial u}{\partial x_2} + u^4 = 0.$$

A linear ordinary differential equation of n^{th} order given as

$$a_n(x)\frac{d^ny}{dx^n} + ... + a_1(x)\frac{dy}{dx} + a_0(x)y = F(x),$$

where $a_n(x) \neq 0$, is said to be homogeneous differential equation if $F(x) = 0$. Otherwise, it is said to be non-homogeneous differential equation.

In eighteenth century, the study of partial differential equations appeared as an important tool in the explanation of mechanics of continuum and more generally, as the principal mode of analytical study of models in the physical science because of the work of Euler, d'Alembert, Lagrange and Laplace [3].

In nineteenth century, Riemann's work also shed light on the importance of PDEs in different branches of mathematics [3]. In the field of physics, engineering and many other applied disciplines, PDEs are always used to model different systems. In 20th century, they were not only like a bridge between important issues of applied mathematics and physical sciences but also helped a lot in the evolution of new ideas in pure mathematics.

Solution of a partial differential equation satisfies the given equation. Generally, system of nonlinear PDEs is more difficult to handle than a single PDE for solution. If we are not able to find the analytical solution of a partial differential equation then we may use a numerical methods for obtaining approximate solution.

The steadily revolving flow of a viscous fluid above a planar surface has a system of equations which is a system of nonlinear PDEs [4]. This system of PDEs is converted into a system of ODEs by using specific similarity transformations. As the reduced system of ODEs is nonlinear and it is difficult to find its exact solution. Therefore, we use a systematic approach to obtain an approximate closed form solution.

1.1 History

Researchers have been working on the steadily revolving flow of a viscous fluid above a solid surface since long time back. Karman was the first person who studied such kind of flows where axial velocity component was considered in the direction of rotating disk in 1921 [5]. In 1940, Bodewadt was the one who worked on the steadily revolving flow of a viscous fluid above solid surface considering reverse Karman flow [6]. His research in this aspect opens ways for many other researchers to work in this area. The concept of heat transfer from a steadily rotating disk by keeping the temperature constant for varying values of Prandtl number was disscused by Millsaps et al. in 1952 [7]. The steady flow and heat transfer arising due to the rotation of a viscous fluid at a larger distance from a stationary disk is extended to the case where the disk surface admits partial slip was discussed by Sahoo et al. in 2014 [8].

In 2013 [9], Sahoo et al. tried to find out solution for the system of nonlinear coupled equations by a finite difference and Keller-box method for the Bodewadt flow with partial slip. Shevchuk discussed the case when disk temperature was varying radially for the heat transfer in a fluid which is rotating with a rotating disk and found out self similar solutions for the flow in 2009 [10]. In 2011, Sahoo also discussed the analysis of heat transfer with electrically conducting non-Newtonian fluids and the effects of slip on steady Bodewadt flow in [11].

Bodewadt boundary layer flow was also extended to the case when stationary disk is allowed to stretch in the radial direction by Turkyilmazoglu in 2015 [12]. MacKerrell in his research discusses the stability of bodewadt flow which was published in 2005 [13]. The case when the flow is steady and partial slip is under consideration then its effect on Bodewadt flow of a non-Newtonian fluid was discussed by Sahoo et al. in 2013 [14].

In 2017, Rahman and Andersson explained the heat transfer in steadily revolving Bodewadt flow [4]. In this thesis, coupled nonlinear PDEs are obtained by modeling the above mentioned problem discussed in [4].

1.2 MATLAB

MATLAB is the basic tool to find the solutions of those equations for which we are not able to find analytical solution. MATLAB is defined as multi-paradigm numerical computing environment and proprietary programming language developed by moler [15]. Numerical solutions for nonlinear PDEs are obtained by using MATLAB. We have many algorithms for mathematical models which are built-in in MATLAB.

Basic utilization of MATLAB includes:

- Computation of mathematical problems.
- Development of various algorithm.
- Visualization, data analysis and exploration.
- Simulation, modeling and prototyping.
- Application development including graphical user interface building.
- Scientific and engineering graphics.

bvp4c is one of the built-in tool used to solve boundary value problems in MATLAB. It is difficult to get the exact guess for the BVPs. *bvp4c* uses a right approach by restricting the error which helps in dealing with poor guess.

bvp4c needs three main arguments which are:

- A function *BC* for evaluating the residual in the boundary conditions.
- A structure *solinit* that provides a guess for a mesh.
- The *solution* on this mesh [16].

We require a proper guess (used to find out the solution the we need) which assists the solver to compute that solution. The guess is supplied to *bvp4c* as a structure formed by the auxiliary function *bvpinit*. The syntax for *bvp4c* is given by:

$$sol = bvp4c(@odefun,\ @bcfun,\ solinit).$$

The first order reduced ODEs are represented by new variables in *bvp4c*. BVPs are evaluated by the function *'odefun'* in the form of first order ODEs and given as:

function dydx=odefun(x,y,parameters).

Here, y and *dydx* are column vectors and x is scalar. So *odefun* provides column vectors expressed by *dydx*.
Residuals in boundary value problems are being evaluated by using function *bcfun* and it has the form:

$$function\ residual = bcfun(y_a, y_b),$$

where y_a and y_b are column vectors along with numerical solution at $x = a$ and $x = b$ respectively. The structure *solinit* provides initial guess in *bvp4c*. *bvpinit* is a helper function because of the difficulty in making guess structure.

$$solinit = bvpinit(x,\ yinit,\ params).$$

In [17] Shampine et al. discussed boundary value problems for ordinary differential equation in MATLAB using *bvp4c*.

1.3 Fluid Mechanics

Fluid mechanics is the branch of physics that deals with the mechanics of fluid and the external forces being exerted on them and how these fluids respond to those forces. In this section, we discuss some basic definitions of fluid mechanics as they are used in later chapters.

1.3.1 Density

Density ρ is the measure of mass, m, per unit volume, v [18].

$$\rho = \frac{m}{v}.$$

1.3.2 Viscosity

Viscosity is defined as the measure of the resistance of a fluid which is being deformed by the shear stress [18] and is denoted by μ. There are many fluids which have higher viscosities than water because it has lowest viscosity. For example, honey, vegetable oil, etc.

1.3.3 Kinematic Viscosity

Kinematic viscosity is the ratio of dynamic viscosity to density and is denoted by ν. Mathematically,

$$\nu = \frac{\mu}{\rho}.$$

1.3.4 Boundary Layer

Boundary layer is a thin region close to a surface of body. It is a layer where viscous forces are more dominant than inertial forces. This layer divides the flow past a surface into two different regions. One of them is a region where influence of viscosity is important while other region shows negligible effect of viscosity.

1.3.5 Bodewadt Flow

Bodewadt flow is basically a reversed Karman flow. In this flow, the axial velocity component is directed away from the planar surface rather than towards the rotating disk [4].

1.3.6 Steady Flow

Steady flow is a flow in which terms like pressure, temperature, velocity etc are time independent [19].

1.3.7 Angular Velocity

Angular velocity is the change of angular displacement with respect to time t. It is also known as rotational velocity.

1.3.8 Navier Stoke's Equation

Momentum is conserved when the fluid is in motion. Navier-Stokes equations represent the conservation of momentum. They are nonlinear partial differential equations. Mathematically,

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}.\boldsymbol{\nabla})\,\boldsymbol{v} + \frac{\boldsymbol{\nabla} P}{\rho} - \frac{\mu \nabla^2 \boldsymbol{v}}{\rho} = 0,$$

where P is the fluid pressure, ρ is the fluid density, μ is the dynamic viscosity and $\nabla^2 = (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$ is the differential operator.

1.3.9 Continuity Equation

Continuity equation states that the rate at which mass enters a system is equal to the rate at which mass leaves the system. Continuity equation is used when the mass of a fluid is conserved. Mathematically,

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}.(\rho \boldsymbol{v}) = 0,$$

Chapter 2

Solution of System of Nonlinear PDEs by using Approximation Method

In this chapter, review work of [20] is presented. The boundary layer flow of a rotating fluid over an exponentially stretching sheet is discussed by Javed et. al. in [21]. The flow is governed by nonlinear partial differential equations.

2.1 System of Nonlinear PDEs

Consider a boundary layer flow for a viscous incompressible fluid. It is a three dimensional flow. This flow is considered on a stretching surface in a rotating frame. Here, flow is steady. Assumption is applied on the flow that it is generated when elastic boundary surface is stretched in such a way that its velocity is directed towards *x-axis* exponentially. The fluid is moving in such a way that its mass and momentum are conserved, so the governing continuity equation and Navier Stoke's equations are [21]

$$\frac{\partial}{\partial x}(u_1) + \frac{\partial}{\partial z}(w_1) = 0, \tag{2.1}$$

$$u_1 \frac{\partial}{\partial x}(u_1) + w \frac{\partial}{\partial z}(u_1) - 2\overline{\Omega} v_1 = \nu \frac{\partial^2 u_1}{\partial z^2}, \tag{2.2}$$

$$u_1 \frac{\partial}{\partial x}(v_1) + w \frac{\partial}{\partial z}(v_1) + 2\overline{\Omega} u_1 = \nu \frac{\partial^2 v_1}{\partial z^2}, \tag{2.3}$$

where u_1, v_1 and w_1 are the velocity components in the x, y and z directions respectively, ν is the kinematic viscosity and $\overline{\Omega}$ is the angular velocity which is constant.

For the given problem the suitable boundary conditions are:

$$\begin{aligned} &u_1 = Ue^{(\frac{x}{L})}, \qquad v_1 = 0, \qquad w_1 = 0 \qquad at \qquad z = 0, \\ &u_1 \to 0, \qquad v_1 \to 0 \qquad as \qquad z \to \infty, \end{aligned} \tag{2.4}$$

where U is a constant which has dimensions of velocity and L is the reference length.

2.2 Reduction to a Nonlinear System of ODEs

Similarity transformations are used to get required reduced system of ODEs. Following are the similarity transformations used for the conversion of PDEs (2.1)-(2.3) into ODEs [21]:

$$\eta = \sqrt{\frac{U}{2\nu L}} e^{(\frac{x}{2L})} z, \tag{2.5}$$

$$u_1 = Ue^{(\frac{x}{L})} f'(\eta), \tag{2.6}$$

$$v_1 = Ue^{(\frac{x}{L})} g(\eta), \tag{2.7}$$

$$w_1 = -\sqrt{\frac{vU}{2L}} e^{(\frac{x}{2L})} [f(\eta) + \eta f'(\eta)], \tag{2.8}$$

where prime shows differentiation with respect to η.

Now next step would be taking partial derivatives of the similarity transformations for the system of PDEs (2.1)-(2.3). By putting the results in the system, we get the simplified version of ODEs.

Taking partial derivatives of Eqs. (2.6) and (2.7) with respect to x and z respectively, we get

$$\frac{\partial}{\partial x}(u_1) = e^{(\frac{x}{L})} \frac{U}{L} f'(\eta) + e^{(\frac{x}{L})} \frac{U}{2L} \sqrt{\frac{U}{2\nu L}} e^{(\frac{x}{2L})z} f''(\eta), \tag{2.9}$$

$$\frac{\partial}{\partial z}(u_1) = e^{(\frac{x}{L})} U f''(\eta) \sqrt{\frac{U}{2\nu L}} e^{(\frac{x}{2L})}, \tag{2.10}$$

$$\frac{\partial}{\partial x}(v_1) = e^{(\frac{x}{L})}\frac{U}{L}g'(\eta) + e^{(\frac{x}{L})}\frac{U}{2L}\sqrt{\frac{U}{2\nu L}}e^{(\frac{x}{2L})}g'(\eta), \tag{2.11}$$

$$\frac{\partial}{\partial z}(v_1) = e^{(\frac{x}{L})}U\sqrt{\frac{U}{2\nu L}}e^{(\frac{x}{2L})}g'(\eta). \tag{2.12}$$

Now, by taking partial derivative of Eq. (2.8) with respect to z

$$\frac{\partial}{\partial z}(w_1) = -e^{(\frac{x}{L})}\frac{U}{2L}\sqrt{\frac{U}{2\nu L}}e^{(\frac{x}{2L})z}f''(\eta) - e^{(\frac{x}{L})}\frac{U}{L}f'(\eta). \tag{2.13}$$

Differentiating Eq. (2.10) and Eq. (2.12) with respect to z, we get

$$\frac{\partial^2}{\partial z^2}(u_1) = e^{(\frac{2x}{L})}\frac{U^2}{2\nu L}f'''(\eta), \tag{2.14}$$

$$\frac{\partial^2}{\partial z^2}(v_1) = e^{(\frac{2x}{L})}\frac{U^2}{2\nu L}g''(\eta). \tag{2.15}$$

Using Eqs. (2.9) and (2.13), the PDE (2.1) vanishes. Eq. (2.2) reduces to the following nonlinear ODE by using Eqs. (2.9), (2.10) and (2.14)

$$f''' - 2f'^2 + ff'' + 4\Omega g = 0. \tag{2.16}$$

Similarly, Eq. (2.3) reduces to the following nonlinear ODE by using Eqs. (2.11), (2.12) and (2.15)

$$g'' - 2gf' + fg' - 4\Omega f' = 0. \tag{2.17}$$

The corresponding boundary conditions are transformed as

$$\begin{aligned} &f = 0, \quad f' = 0, \quad g = 0 \quad at \quad \eta = 0, \\ &f' = 0, \quad g = 0 \quad as \quad \eta \to \infty, \end{aligned} \tag{2.18}$$

where $\Omega = e^{(\frac{x}{L})}\frac{L}{U}\overline{\Omega}$ is a dimensionless local rotation.

2.3 Numerical Solution of the Reduced Nonlinear System of ODE

The resulting ODEs given by Eqs. (2.16) and (2.17) are nonlinear. It is difficult to solve this reduced nonlinear system of ODEs analytically, so a MATLAB built in function *bvp4c* has been used to get numerical solution of f' and g.

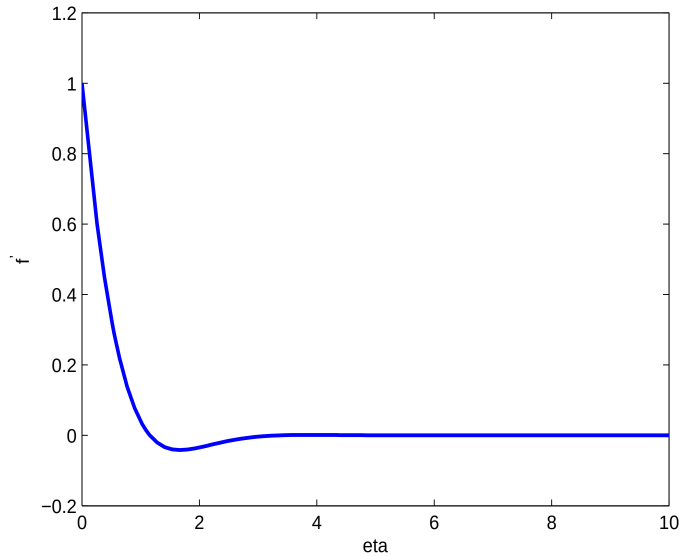

Figure 2.1: Numerical solution of the function $f^{'}$

Graph of numerical solution of $f^{'}$, denoted as $f^{'}_{num}$ is shown in Figure 2.1 and graph of numerical solution of g, denoted as g_{num} is shown in Figure 2.2.

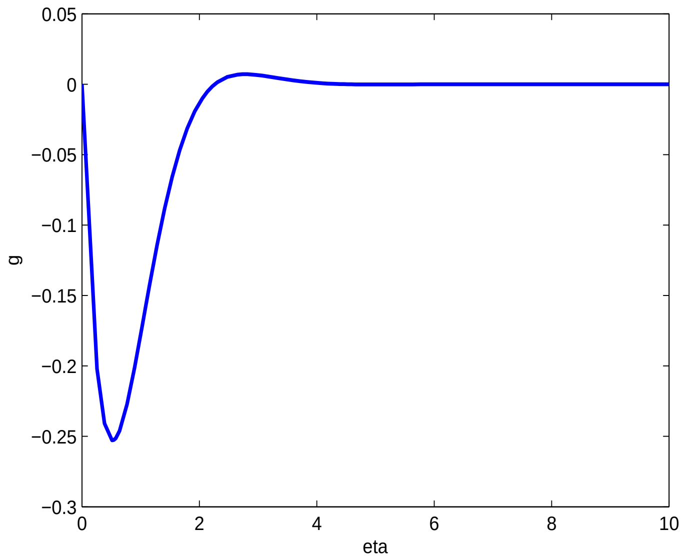

Figure 2.2: Numerical solution of the function g

2.3.1 Approximation for f'

The graph of numerical solution of the function f', in Figure 2.1 resembles the graph of function

$$f_1 = \frac{\cos(1.35297\eta)}{(0.8\eta + 1)^3}. \tag{2.19}$$

Figure 2.3 shows the correspondence between f'_{num} and f_1 . The difference between f'_{num} and f_1 is shown in Figure 2.4.

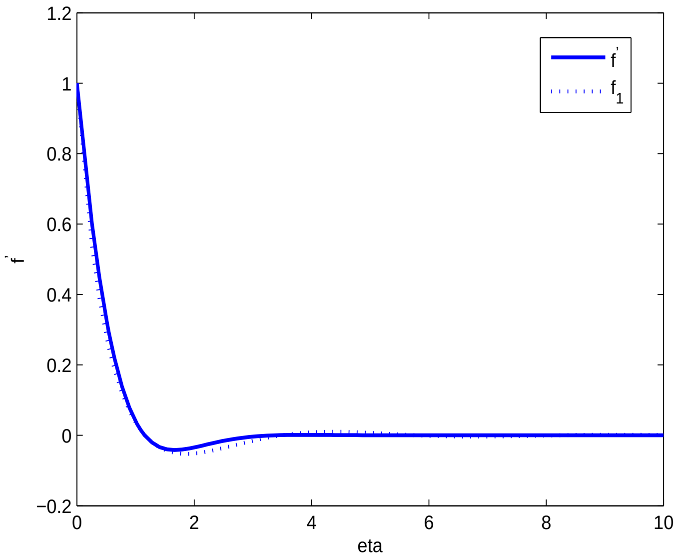

Figure 2.3: Graph of f'_{num} and f_1

The graph of difference shows resemblance to the graph of function

$$f_2 = \alpha\eta(\eta - x_0)(\eta - x_1)(\eta - x_2)(\eta - x_3)(\eta - x_4)e^{-\beta\eta}. \tag{2.20}$$

We have use different data points in f_2 to get the values of x_0, x_1, x_2, x_3 and x_4 which are 1.1055, 1.3568, 3.4673, 5.8291 and 8.0905 respectively. We get the following function that approximates the difference curve:

$$f_2 = -0.0024248\eta(\eta - 1.1055)(\eta - 1.3568)(\eta - 3.4673)(\eta - 5.8291)(\eta - 8.0905)e^{-4.4221\eta}. \tag{2.21}$$

The approximation, f_2, of the difference between f'_{num} and f_1 is illustrated in Figure 2.5. The approximate solution, f'_{approx} is

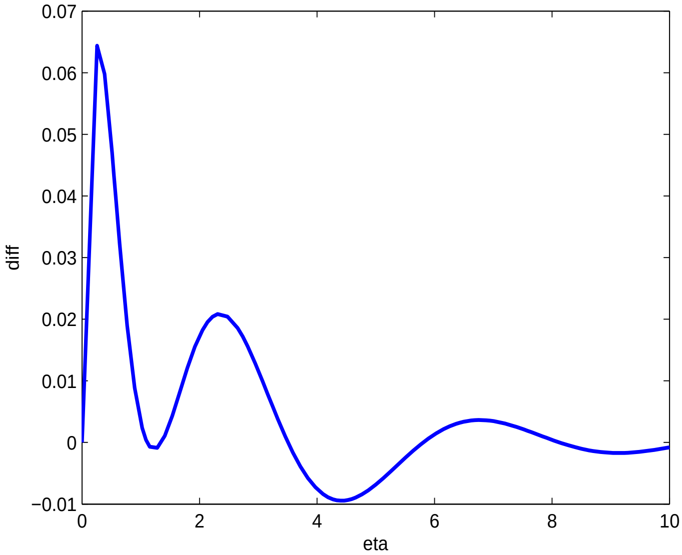

Figure 2.4: Graph of difference between f'_{num} and f_1

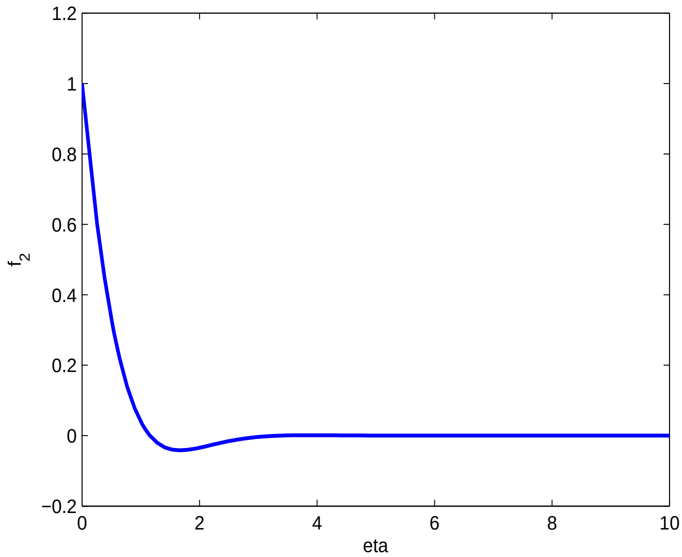

Figure 2.5: Graph of f_2

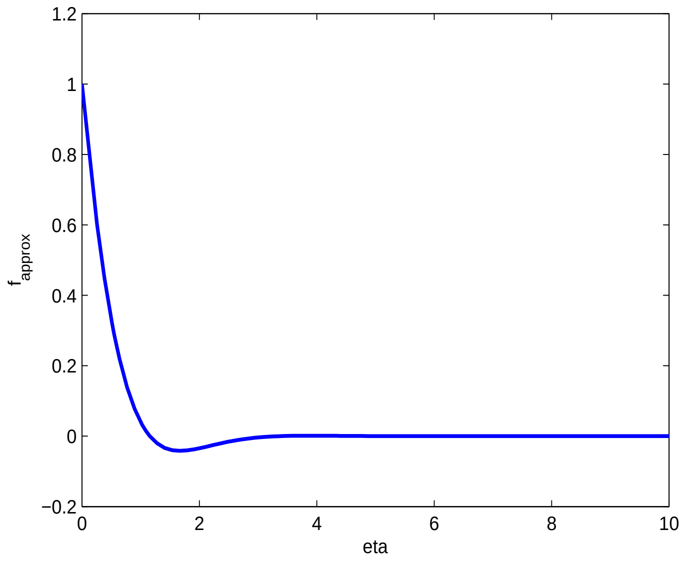

Figure 2.6: Graph of f'_{approx}

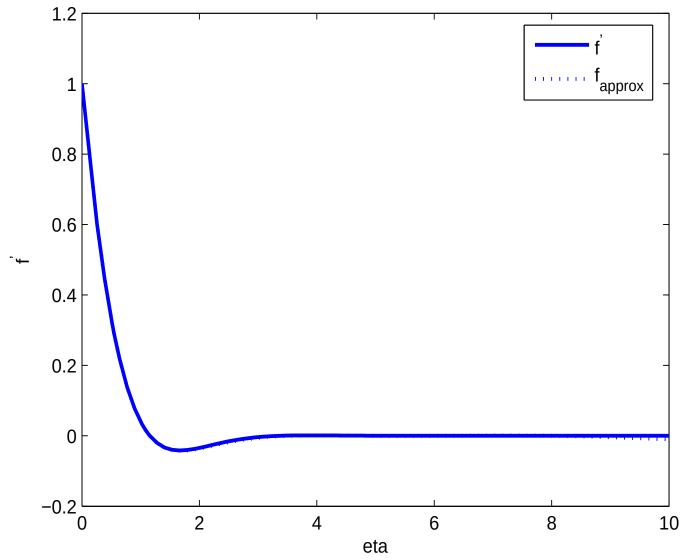

Figure 2.7: Approximate solution of the function f'_{num} and f'_{approx}

$$f'_{approx} = f_1 + f_2,$$

or

$$\begin{aligned} f'_{approx} =& \frac{\cos(1.35297\eta)}{(0.8\eta+1)^3} - 0.0024248\eta(\eta-1.1055) \\ & (\eta-1.3568)(\eta-3.4673)(\eta-5.8291)(\eta-8.0905)e^{-0.9124\eta}. \end{aligned} \tag{2.22}$$

Figure 2.6 shows the graph of f'_{approx}. Figure 2.7 shows a graph of numerical solution along with its approximate solution.

2.3.2 Approximation for g

The graph of numerical solution of the function g, in Figure 2.2 resembles the graph of function

$$g_1 = -\frac{\sin(1.3699\eta)}{(0.54\eta+1)^4}. \tag{2.23}$$

Figure 2.9 shows the correspondence between g_{num} and g_1. The difference between g_{num} and g_1 and is shown in Figure 2.10.

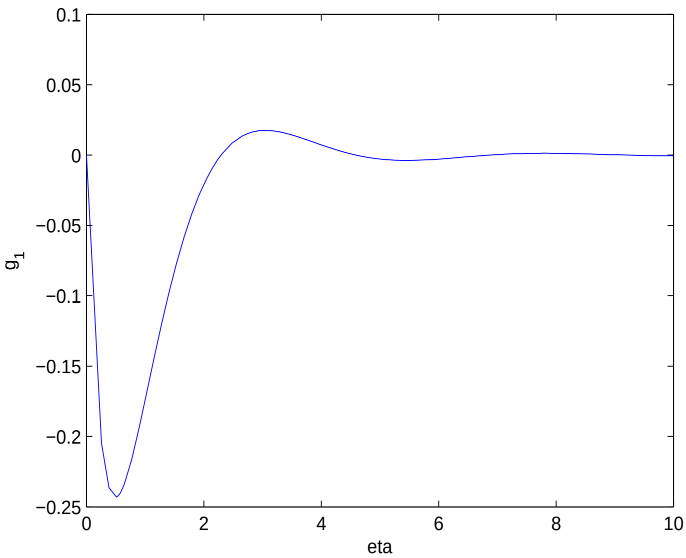

Figure 2.8: Graph of g_1

The graph of difference shows resemblance to the graph of function

$$g_2 = \sin(0.05\eta)(\eta-x_0)(\eta-x_1)(\eta-x_2)e^{-1.45\eta}. \tag{2.24}$$

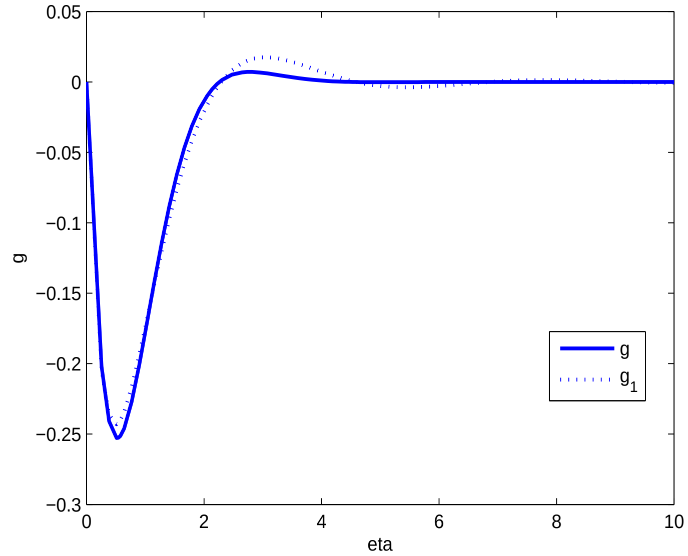

Figure 2.9: Graph of g_{num} and g_1

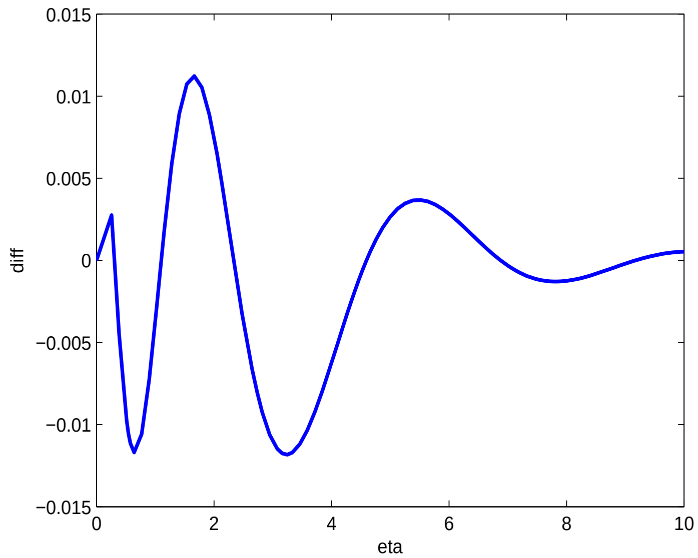

Figure 2.10: Graph of difference between g_{num} and g_1

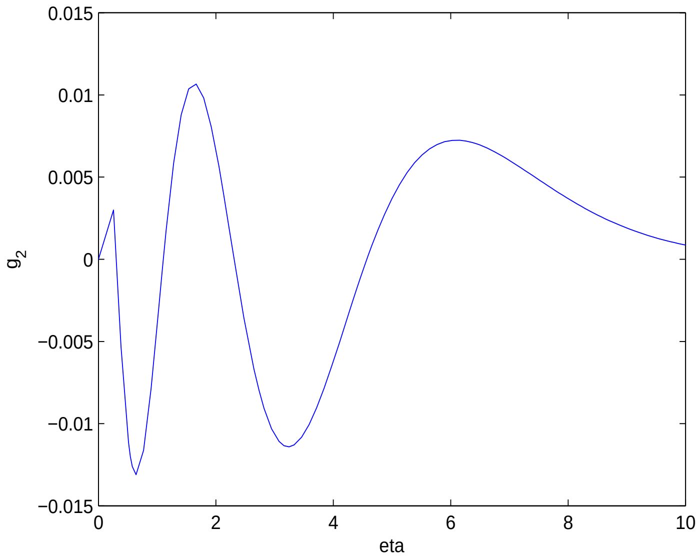

Figure 2.11: Graph of g_2

Using different data points in g_2 we get the values of x_0, x_1, x_2, x_3 and x_4 which are 0.3015, 1.1055, 2.3116 and 4.5729 respectively. We get the following function that approximates the difference curve:

$$g_2 = \sin(0.05\eta)(\eta - 0.3015)(\eta - 1.1055)(\eta - 2.3116)(\eta - 4.5729)e^{-1.45\eta}. \tag{2.25}$$

The approximation, g_2, denoted by data 2, of the difference between g_{num} and g_1 is illustrated in Figure 2.12.

The approximate solution, g_{approx} is obtained as

$$g_{approx} = g_1 + g_2,$$

or

$$\begin{aligned} g_{approx} = & -\frac{\sin(1.3699\eta)}{(0.54\eta + 1)^4} + sin(0.05\eta) \\ & (\eta - 0.3015)(\eta - 1.1055)(\eta - 2.3116)(\eta - 4.5729)e^{-1.45\eta}. \end{aligned} \tag{2.26}$$

Figure 2.13 shows the graph of g_{approx}. Figure 2.14 shows a graph of numerical solution g_{num} along with its approximate solution g_{approx}.

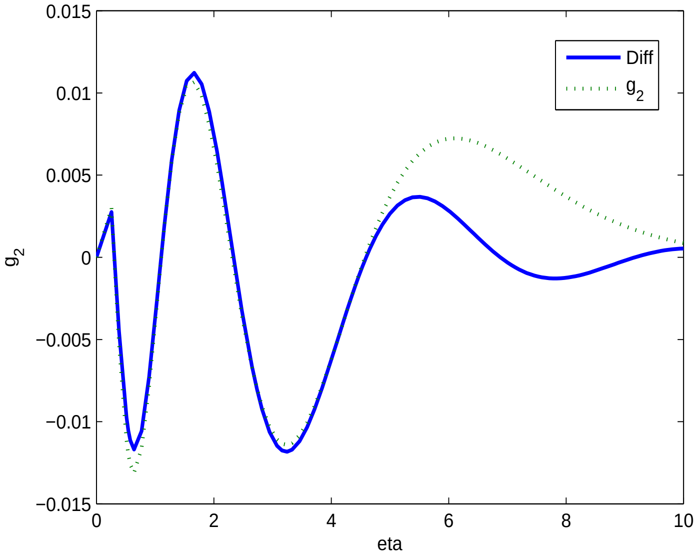

Figure 2.12: Graph of difference alongwith its approximation g_2

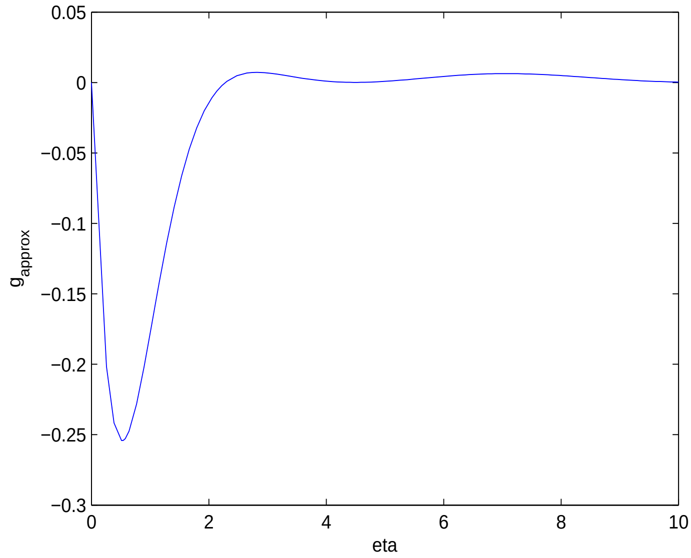

Figure 2.13: Graph of g_{approx}

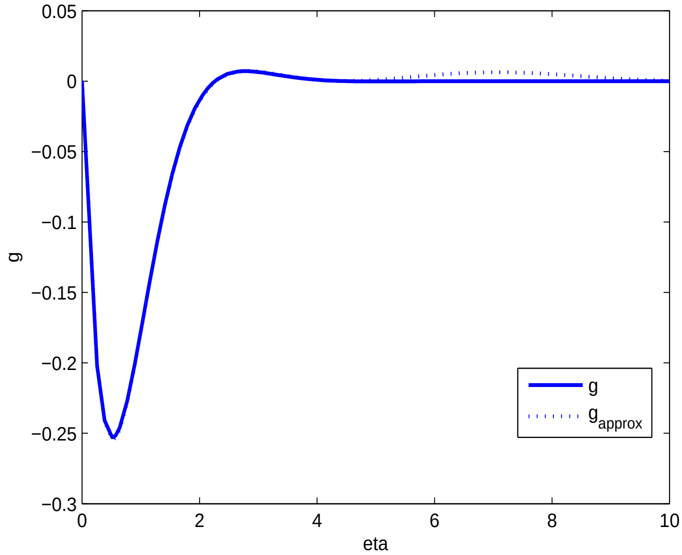

Figure 2.14: Graph of g_{num} and g_{approx}

Chapter 3

Approximate Solutions of System of Nonlinear Partial Differential Equations

Modeling of mathematical problems always play an important role in natural sciences. In physics, many theories have been described by the mathematical modeling like wave mechanics and fluid dynamics etc. Nonlinear partial differential equations are obtained from these processes for which exact solutions are not easy to find. Numerical approximation method is being used to find the solutions of these nonlinear partial differential equations. Therefore, nonlinear PDEs are being transformed into ODEs. For their reduction in ODEs, similarity transformations are used most oftenly.

Most of the times, analytical solutions for these reduced ODEs do not exist. In order to cope up with such situations, a method is used to approximate the numerical solution of the reduced ordinary differential equations by a function. This chapter explains the approach which is used to find the solutions of nonlinear partial differential equations. Residues of approximate solutions have been found just to check the authenticity of approximate solutions.

3.1 System of Nonlinear Partial Differential Equations

Many physical processes in fluid dynamics can also be described through PDEs like Navier-Stokes equations. Consider a steadily revolving flow of a viscous fluid in cylindrical polar coordinates (r, θ, z). The fluid is moving in such a way that its mass and momentum are conserved, so the governing continuity equation and Navier Stoke's equations are [4],

$$\frac{\partial}{\partial r}(ru) + \frac{\partial}{\partial z}(rw) = 0, \tag{3.1}$$

$$u\frac{\partial u}{\partial r} - \frac{v^2}{r} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left[\frac{\partial^2 u}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{u}{r}\right) + \frac{\partial^2 u}{\partial z^2}\right], \tag{3.2}$$

$$u\frac{\partial v}{\partial r} + \frac{uv}{r} + w\frac{\partial v}{\partial z} = \nu\left[\frac{\partial^2 v}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{v}{r}\right) + \frac{\partial^2 v}{\partial z^2}\right], \tag{3.3}$$

$$u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left[\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\left(\frac{\partial w}{\partial r}\right) + \frac{\partial^2 w}{\partial z^2}\right], \tag{3.4}$$

where the velocity components are *(u, v, w)* and they are in the radial, circumferential and axial directions respectively. The kinematic viscosity of the fluid is ν. There is an assumption of rotational symmetry about the vertical $z - axis$, i.e. $\frac{\partial}{\partial\theta} = 0$.
For such kind of flows the favorable boundary conditions are:

$$\begin{aligned} &u = 0,\ v = 0,\ w = 0 \qquad at \qquad z = 0,\\ &u = 0,\ v = r\Omega,\ p = \frac{r^2\Omega^2}{2} \qquad as \qquad z \to \infty, \end{aligned} \tag{3.5}$$

in which Ω is the angular velocity of revolving fluid at some distance from the surface.

3.2 Reduction to a Nonlinear System of ODEs

Conversion of system of nonlinear partial differential equations into a system of ordinary differential equations requires similarity transformations. Following are the similarity transformations for the above mentioned system of nonlinear partial differential equations [4]

$$u(r, z) = r\Omega F(\eta), \tag{3.6}$$

$$v(r, z) = r\Omega G(\eta), \tag{3.7}$$

$$w(r, z) = \sqrt{\nu\Omega}H(\eta), \tag{3.8}$$

$$p(r, z) = -\rho\nu\Omega P(\eta) + \frac{\rho r^2\Omega^2}{2}, \tag{3.9}$$

where η is a dimensionless variable defined by

$$\eta = z\sqrt{\frac{\Omega}{\nu}}. \tag{3.10}$$

To get the simplified version of the above nonlinear PDEs in the form of reduced ODEs, the next step would be taking the partial derivatives of the similarity transformations (3.6)-(3.9) in accordance with the system of PDEs (3.1)-(3.4). By taking partial derivatives of Eqs. (3.6)-(3.9) with respect to r and z respectively, we get

$$\frac{\partial u}{\partial r} = \Omega F(\eta), \tag{3.11}$$

$$\frac{\partial u}{\partial z} = r\Omega F'(\eta)\sqrt{\frac{\Omega}{\nu}}, \tag{3.12}$$

$$\frac{\partial v}{\partial r} = \Omega G(\eta), \tag{3.13}$$

$$\frac{\partial v}{\partial z} = r\Omega G'(\eta)\sqrt{\frac{\Omega}{\nu}}, \tag{3.14}$$

$$\frac{\partial w}{\partial r} = 0, \tag{3.15}$$

$$\frac{\partial w}{\partial z} = \Omega H'(\eta), \tag{3.16}$$

$$\frac{\partial p}{\partial r} = r\rho\Omega^2, \tag{3.17}$$

$$\frac{\partial p}{\partial z} = \nu\rho\Omega\sqrt{\frac{\Omega}{\nu}}P'(\eta). \tag{3.18}$$

Differentiating (3.11)-(3.16) with respect to r and z, we get

$$\begin{aligned} &\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\right) = \frac{\partial}{\partial r}\Bigg(\Omega F(\eta)\Bigg), \\ &\frac{\partial^2 u}{\partial r^2} = 0. \end{aligned} \tag{3.19}$$

$$\begin{aligned}\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right) &= \frac{\partial}{\partial z}\left(r\Omega F'(\eta)\sqrt{\frac{\Omega}{\nu}}\right),\\ \frac{\partial^2 u}{\partial z^2} &= r\frac{\Omega^2}{\nu^2}F''(\eta).\end{aligned} \tag{3.20}$$

$$\begin{aligned}\frac{\partial}{\partial r}\left(\frac{\partial v}{\partial r}\right) &= \frac{\partial}{\partial r}\left(\Omega G(\eta)\right),\\ \frac{\partial^2 v}{\partial r^2} &= 0.\end{aligned} \tag{3.21}$$

$$\begin{aligned}\frac{\partial}{\partial z}\left(\frac{\partial v}{\partial z}\right) &= \frac{\partial}{\partial z}\left(r\Omega G'(\eta)\sqrt{\frac{\Omega}{\nu}}\right),\\ \frac{\partial^2 v}{\partial z^2} &= r\frac{\Omega^2}{\nu^2}G''(\eta).\end{aligned} \tag{3.22}$$

$$\begin{aligned}\frac{\partial}{\partial z}\left(\frac{\partial w}{\partial z}\right) &= \frac{\partial}{\partial z}\left(\Omega H'(\eta)\right),\\ \frac{\partial^2 w}{\partial z^2} &= \Omega\sqrt{\frac{\Omega}{\nu}}H''(\eta).\end{aligned} \tag{3.23}$$

Now, using Eqs. (3.11) and (3.15), the PDE (3.1) reduces to the following non linear ODE

$$2F + H' = 0. \tag{3.24}$$

Then, using Eqs. (3.11), (3.12), (3.17) and (3.18), the PDE (3.2) reduces to the following non linear ODE

$$F'' - HF' - F^2 + G^2 = 1. \tag{3.25}$$

Again, using Eqs. (3.13), (3.14), (3.19) and (3.20), the PDE (3.3) reduces to the following non linear ODE

$$G'' - HG' - 2F' = 0. \tag{3.26}$$

Finally, using Eqs. (3.16), (3.18) and (3.23), the PDE (3.4) reduces to the following non linear ODE

$$P' + 2FH - 2F' = 0. \tag{3.27}$$

The corresponding boundary conditions specified in (3.4) are transformed as

$$\begin{aligned} &F = 0, \quad G = 0, \quad H = 0 \quad at \quad \eta = 0, \\ &F = 0, \quad G = 1, \quad P = 0 \quad as \quad \eta \to \infty. \end{aligned} \tag{3.28}$$

Now, from Eq. (3.24), we obtained value of

$$F(\eta) = -\frac{H'(\eta)}{2},$$

and by substituting the value of $F(\eta)$ in the Eqs. (3.25), (3.26) and (3.27), we get more simplified system of ODEs with new boundary conditions given below:

$$2H''' - 2HH'' + H'^2 - 4G^2 = -4, \tag{3.29}$$

$$G'' - HG' + H'G = 0, \tag{3.30}$$

$$P' + H'' - HH' = 0, \tag{3.31}$$

where prime shows differentiation with respect to η and the boundary conditions are:

$$\begin{aligned} &H = 0, \quad H' = 0, \quad G = 0 \quad at \quad \eta = 0, \\ &H' = 0, \quad G = 1 \quad as \quad \eta \to \infty. \end{aligned} \tag{3.32}$$

Furthermore, the pressure P from Eq. (3.31) can be calculated, once H is determined.

3.3 Numerical Solution of the Reduced Nonlinear System of Ordinary Differential Equations

The above ODEs are nonlinear. It is difficult to solve reduced nonlinear system of ODEs analytically, so a MATLAB built in function *bvp4c* has been used to get numerical solutions.

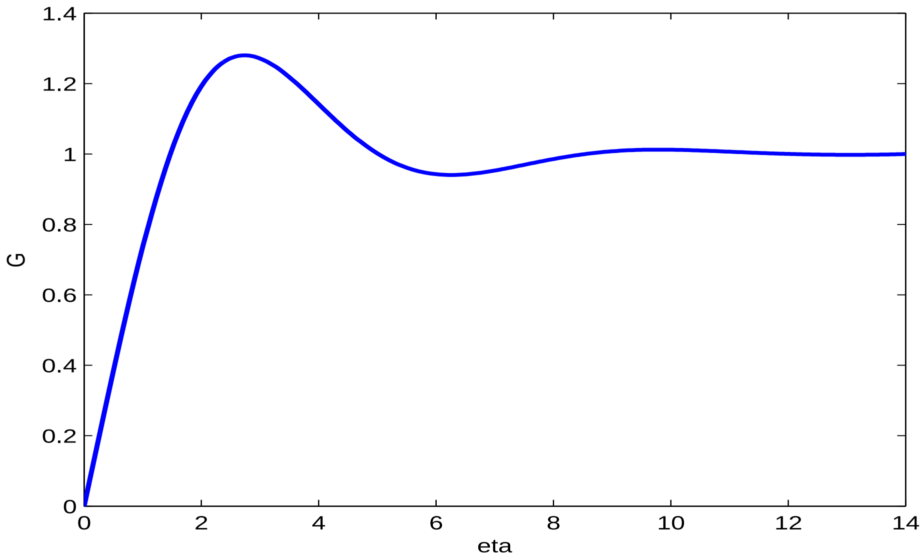

Figure 3.1: Numerical solution of the function G

Graph of numerical solution of G, denoted as G_{num} is shown in Figure 3.1 and graph of numerical solution of H, denoted as H_{num} is shown in Figure 3.2.

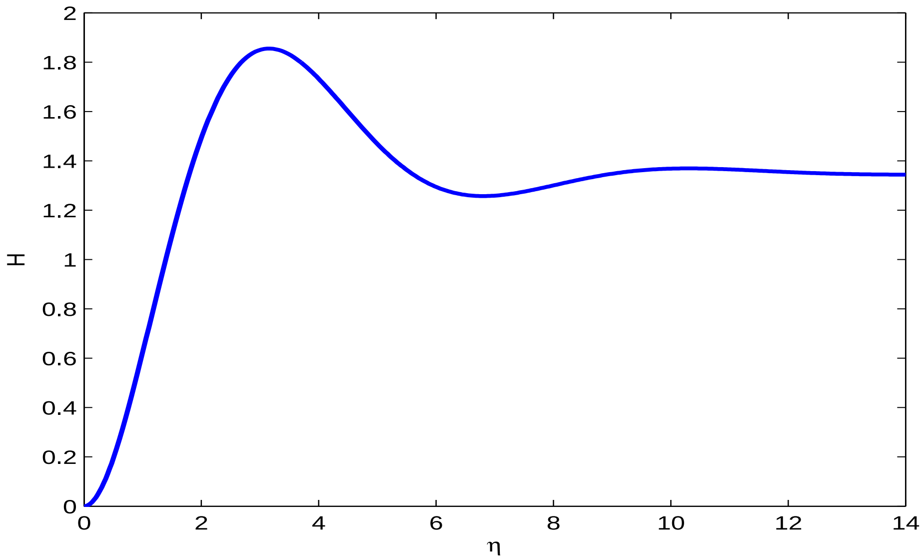

Figure 3.2: Numerical solution of the function H

3.3.1 Approximation of Numerical Solution of the Function G

The graph of numerical solution of the function G, in Figure 3.1 resembles the graph of function

$$G_1 = 1 - e^{-a\eta} + b\eta e^{-a\eta}. \tag{3.33}$$

By putting different data points in the function G_1, we get values of a and b as 1.3333 and 0.99668 respectively which gives a reduction in the difference of graphs of numerical solution G and G_1. So

$$G_1 = 1 - e^{-1.3333\eta} + 0.99668\eta e^{-1.3333\eta}. \tag{3.34}$$

Figure 3.3 shows graph of G_1, whereas, Figure 3.4 shows the correspondence between G_{num} and G_1. The difference between G_{num} and G_1 and is shown in Figure 3.5.

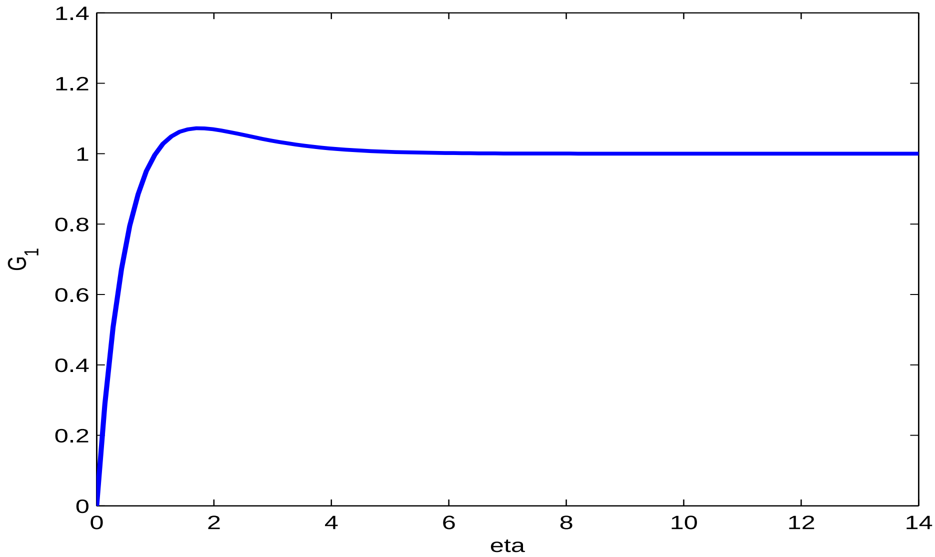

Figure 3.3: Graph of G_1

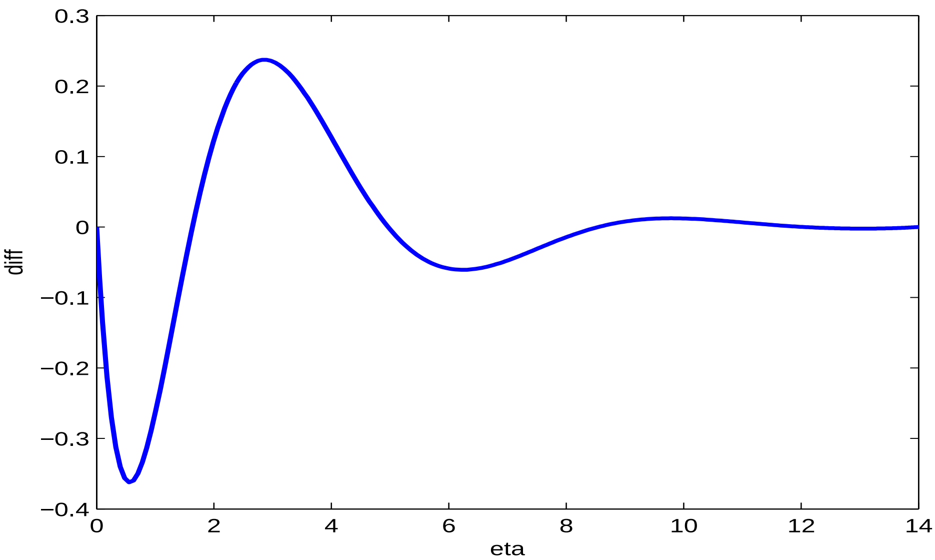

Figure 3.4: Graph of difference between G_{num} and G_1

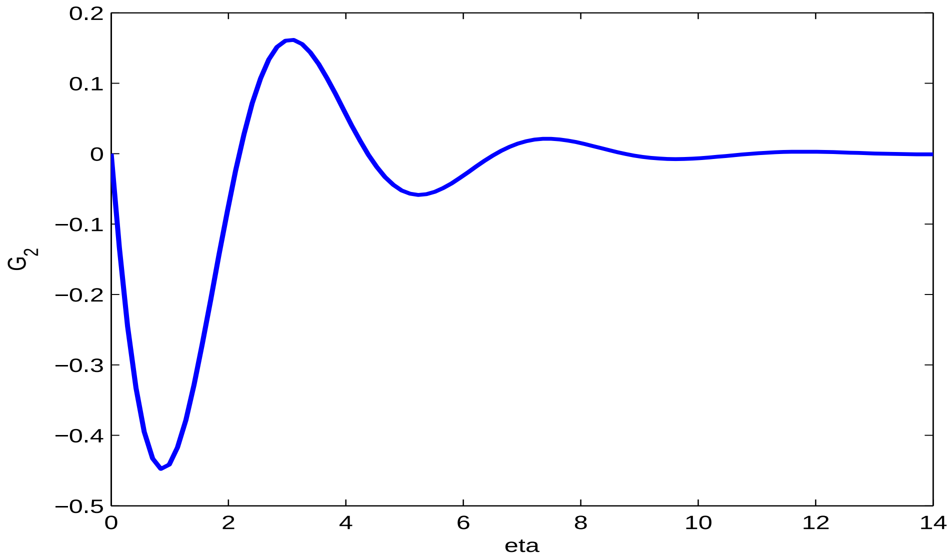

Figure 3.5: Graph of G_2

The graph of difference shows resemblance to the graph of function

$$G_2 = a\sin(b\eta - \Pi)e^{-c\eta}. \tag{3.35}$$

We have use different data points in G_2 to get the values of a, b and c which are 0.70753, 1.43753 and 0.46539 respectively. We get the following function that approximates the differencce curve:

$$G_2 = 0.70753\sin(1.43753\eta - \Pi)e^{-0.46539\eta}. \tag{3.36}$$

The approximation, G_2, of the difference between G_{num} and G_1 is illustrated in Figure 3.6.

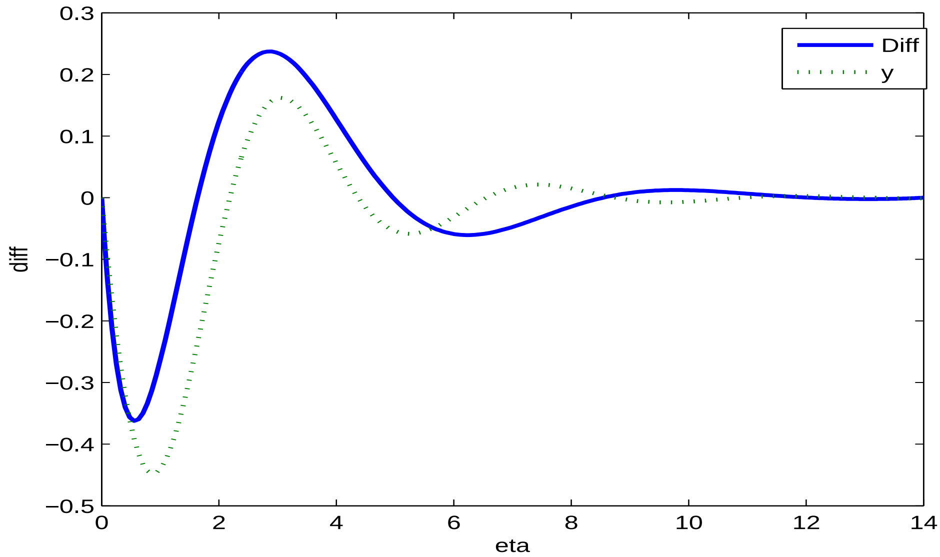

Figure 3.6: Graph of difference alongwith its approximation G_2

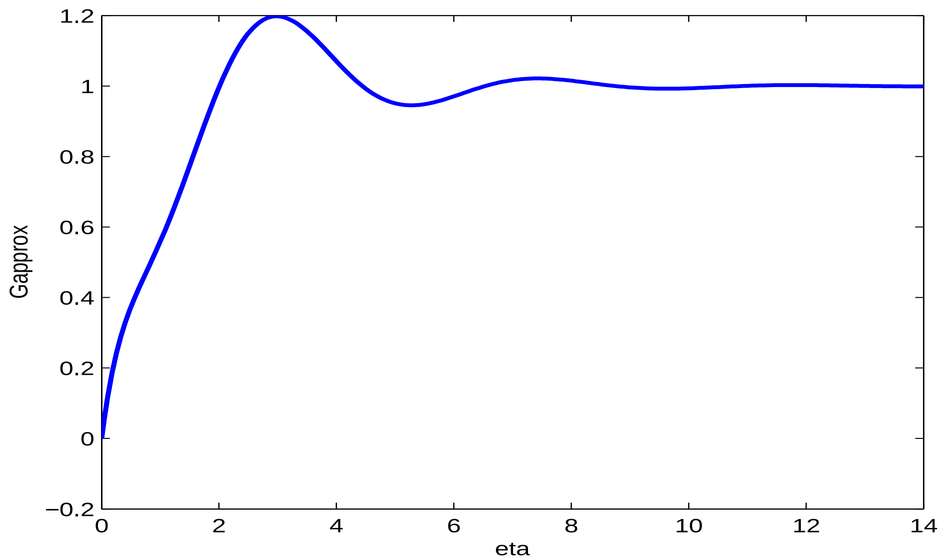

Figure 3.7: Graph of G_{approx}

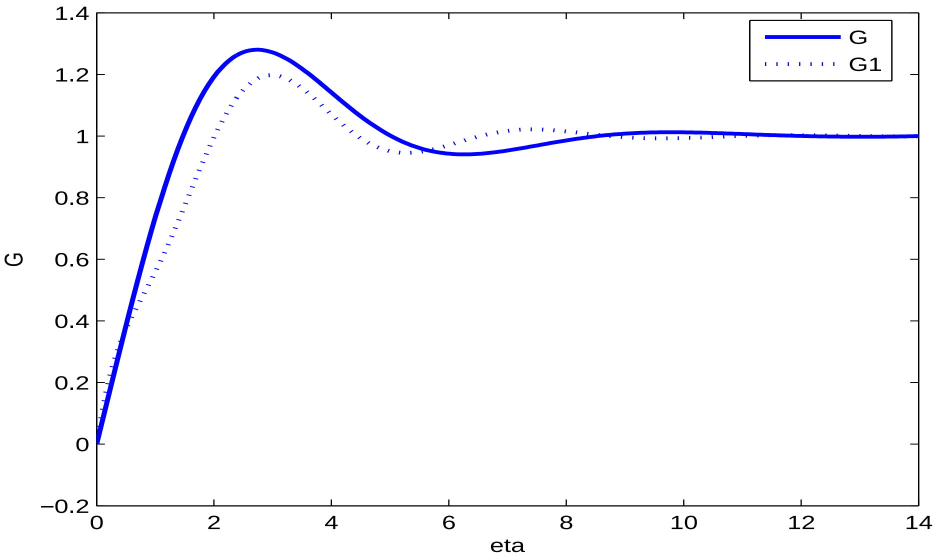

Figure 3.8: Graph of G_{num} and G_{approx}

The approximate solution, G_{approx} is

$$G_{approx} = G_1 + G_2,$$

or

$$G_{approx} = 1 - e^{-1.3333\eta} + 0.99668\eta e^{-1.3333\eta} + 0.70753\sin(1.43753\eta - \Pi)e^{-0.46539\eta}. \quad (3.37)$$

Figure 3.7 shows the graph of G_{approx}. Figure 3.8 shows the graph of G_{num} and G_{approx}. The maximum error 0.3623 reduces to 0.2395.

3.3.2 Approximation of Numerical Solution of the Function H

The graph of numerical solution of the function H, in Figure 3.2 resembles the graph of function

$$H_1 = (c\eta - 1.344)e^{-d\eta} + 1.344. \quad (3.38)$$

Using different data points in the function H_1, we get values of c and d as 0.91 and 0.657 respectively which gives a reduction in the difference of graphs of H_{num} and H_1. So

$$H_1 = (0.91\eta - 1.344)e^{-0.657\eta} + 1.344. \quad (3.39)$$

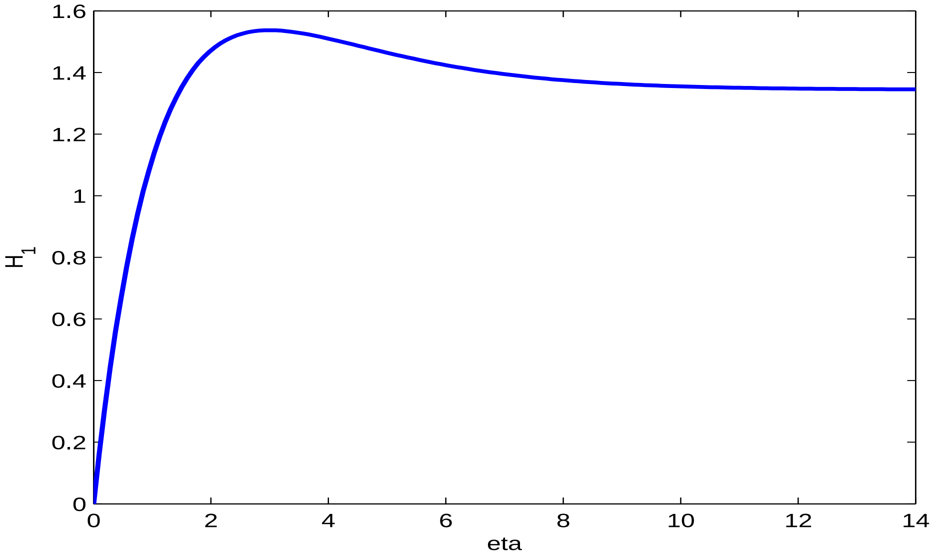

Figure 3.9: Graph of H_1

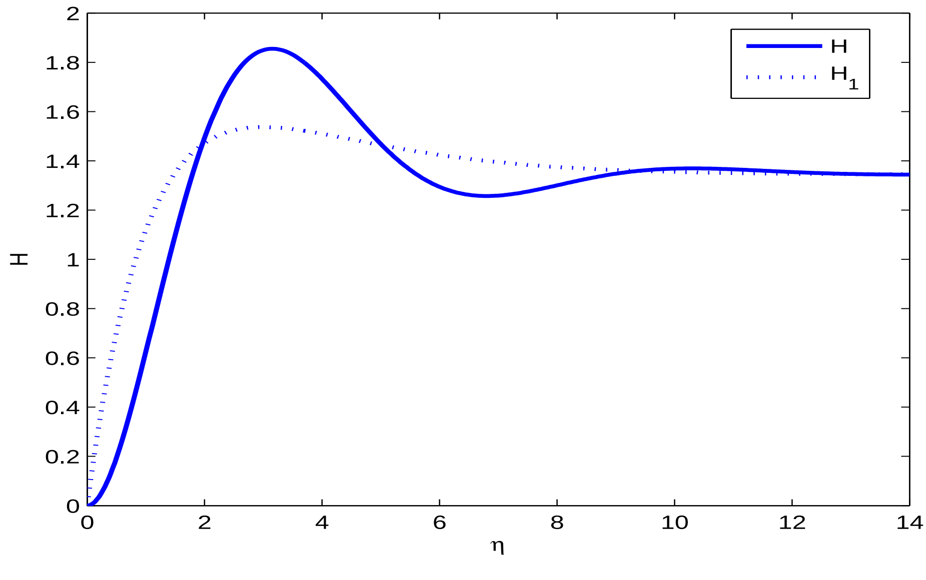

Figure 3.10: Graph of H_{num} and H_1

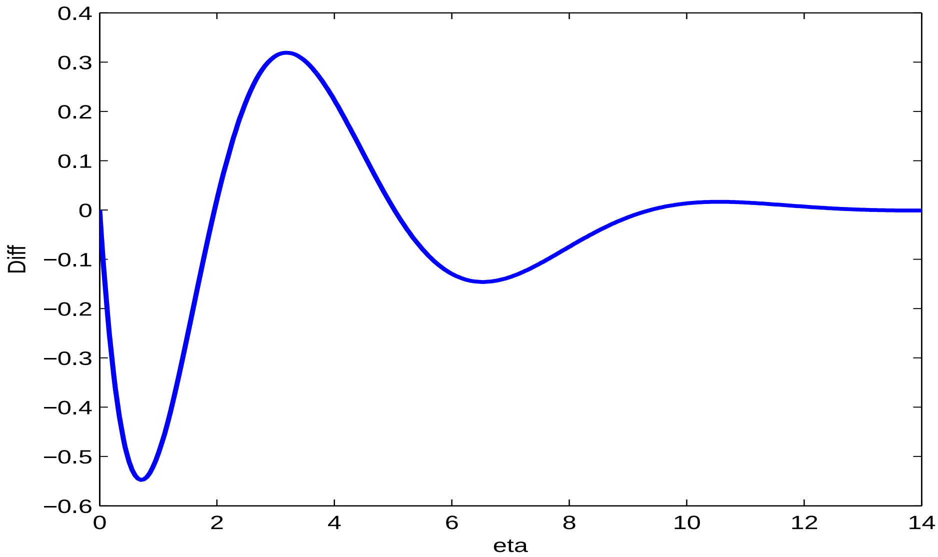

Figure 3.11: Graph of difference between H_{num} and H_1

Figure 3.9 shows graph of H_1, whereas, Figure 3.10 shows the correspondence between H_{num} and H_1. The difference between H_{num} and H_1 and is shown in Figure 3.11. The graph of difference shows resemblance to the graph of function

$$H_2 = m \sin(k\eta - \Pi)e^{-l\eta}. \tag{3.40}$$

We have use different data points in H_2 to get the values of m, k and l which are 0.955, 1.29576 and 0.36862 respectively. Finally, we get the following function that approximates the difference curve. The approximation, H_2, of the difference between H_{num} and H_1 is illustrated in Figure 3.12.

$$H_2 = 0.955 \sin(1.29576\eta - \Pi)e^{-0.36862\eta}. \tag{3.41}$$

The approximate solution, H_{approx} is

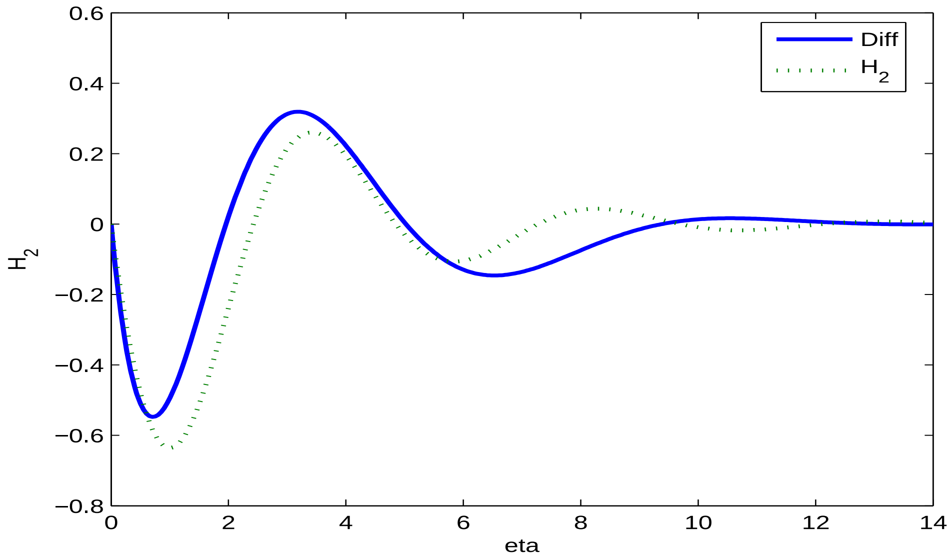

Figure 3.12: Graph of difference along with its approximation H_2

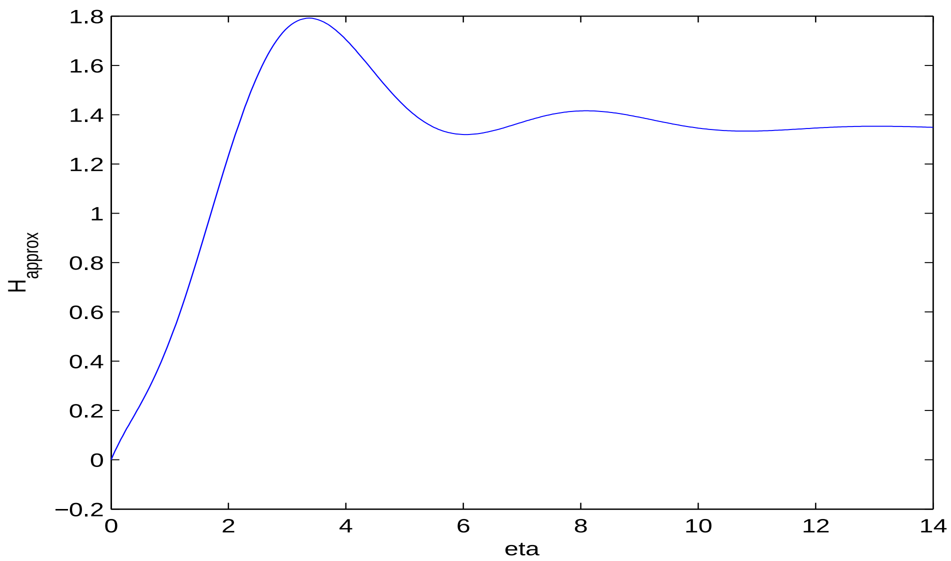

Figure 3.13: Graph of H_{approx}

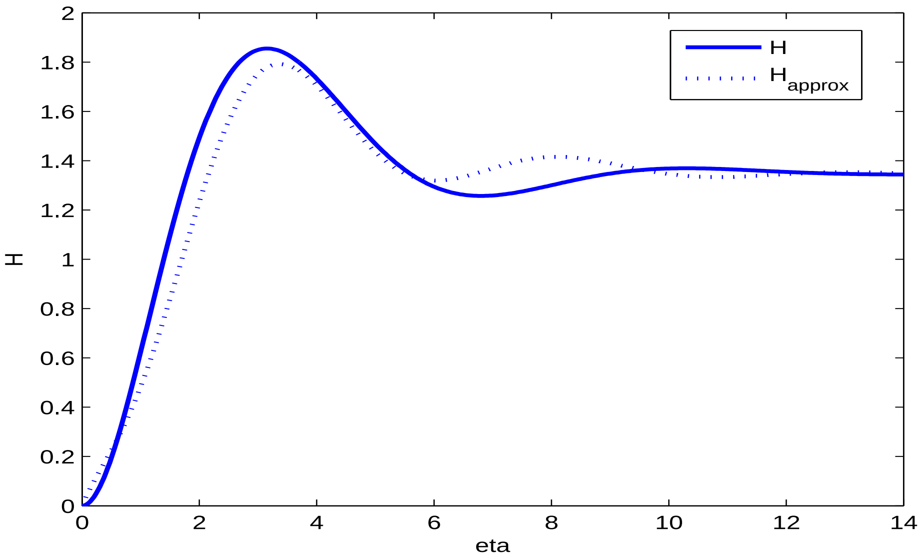

Figure 3.14: Graph of H_{num} and H_{approx}

The approximation is

$$H_{approx} = H_1 + H_2,$$

or

$$H_{approx} = (0.91\eta - 1.344)e^{-0.657\eta} + 1.344 + 0.955\sin(1.29576\eta - \Pi)e^{-0.36862\eta}. \quad (3.42)$$

Figure 3.13 shows the graph of H_{approx}. Figure 3.14 shows the graph of H_{num} and H_{approx}. The maximum error 0.5477 reduces to 0.2726.

3.3.3 Residual analysis for the Approximate Solution of Reduced ODEs

The residues of the ODEs (3.29) and (3.30) with respect to approximate solutions given by Eqs. (3.37) and (3.42) are shown in Figures (3.15) and (3.16).

Table 3.1: The Residuals for $2H''' - 2HH'' + (H')^2 - 4G^2 = -4$

η	Residues
0.000	10.94
0.1414	9.628
0.5657	5.6
0.848	3.43
1.13	1.783
1.838	-0.1594
1.98	-0.23
2.12	-0.2227
2.54	0.1641
2.97	0.7864
3.11	0.9796
3.677	1.411
3.96	1.369
4.10	1.288
4.667	0.6954
4.94	0.331

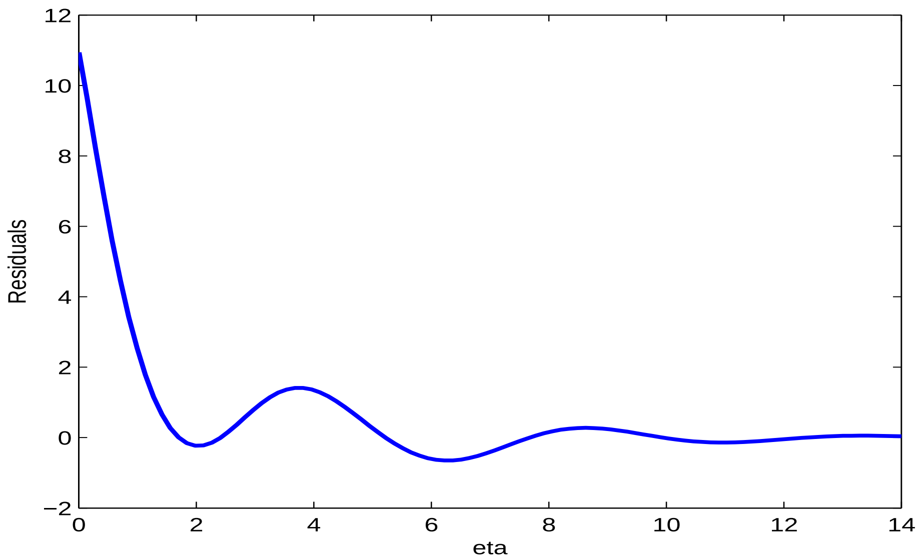

Figure 3.15: Residuals for $2H''' - 2HH'' + (H')^2 - 4G^2 = -4$

Table 3.2: The Residuals for $2H''' - 2HH'' + (H')^2 - 4G^2 = -4$

η	Residues
6.222	-0.6497
6.92	-0.4499
7.07	-0.3682
7.63	-0.0197
7.77	0.0564
8.061	-0.178
9.758	0.0497
10.04	-0.024
10.89	-0.1395
11.03	-0.1398
11.88	-0.0639
12.02	-0.0452
12.44	0.0067
13.15	0.0538
13.43	0.0556
14	0.0372

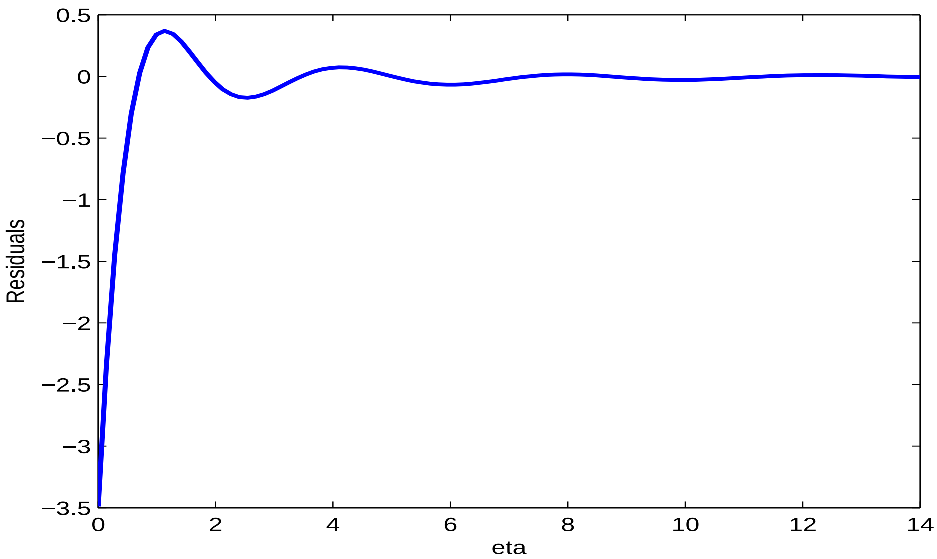

Figure 3.16: Residuals for $G'' - HG' + H'G = 0$

Table 3.3: The Residuals for $G'' - HG' + H'G = 0$

η	Residues
0.0000	-3.489
0.1414	-2.342
0.4242	-0.7854
0.5657	-0.3016
0.9899	0.3396
1.131	0.3695
1.697	0.1141
1.98	-0.04514
2.121	-0.104
2.687	-0.1645
2.97	-0.1154
3.111	-0.08226
3.535	0.01505
4.101	0.07362
4.525	0.05478
4.949	0.00758

Tables (3.1)-(3.4) show the values of residuals for Eqs. (3.29) and (3.30), respectively for different values of η.

3.4 Approximation for the System of Nonlinear PDEs

By putting the expressions of G from Eq. (3.37) and H from Eq. (3.42) in the transformations given by Eqs. (3.7) and (3.8), the approximate solution for the system of PDEs is given by

$$v = r\Omega\left(1 - e^{-1.3333\eta} + 0.99668\eta e^{-1.3333\eta} + 0.70753\sin(1.43753\eta - \Pi)e^{-0.46539\eta}\right), \tag{3.43}$$

$$w = \sqrt{\nu\Omega}\left((0.91\eta - 1.344)e^{-0.657\eta} + 1.344 + 0.955\sin(1.29576\eta - \Pi)e^{-0.36862\eta}\right). \tag{3.44}$$

Table 3.4: The Residuals for $G^{''} - HG^{'} + H^{'}G = 0$

η	Residues
5.091	-0.00923
5.515	-0.05029
5.939	-0.06713
6.081	-0.00667
6.505	-0.05133
6.929	-0.02413
7.071	-0.01486
7.495	0.007468
7.919	0.01656
8.485	0.00798
8.909	-0.00681
9.616	-0.02664
10.04	-0.02864
11.03	-0.00874
12.59	0.00949
13.43	0.00015
13.86	-0.00418
14	-0.00521

The velocity components v and w satisfy the boundary conditions given in Eq. (3.4).

$$\lim_{z\to 0} v(r,z) = r\Omega\Bigg(1 - e^{-1.3333z\sqrt{\frac{\Omega}{\nu}}} + 0.99668z\sqrt{\frac{\Omega}{\nu}}e^{-1.3333z\sqrt{\frac{\Omega}{\nu}}}$$
$$+ 0.70753\sin(1.43753z\sqrt{\frac{\Omega}{\nu}} - \Pi)e^{-0.46539z\sqrt{\frac{\Omega}{\nu}}}\Bigg).$$

After applying limit, the resulting equation is

$$\lim_{z\to 0} v(r,z) = 0. \tag{3.45}$$

Also,

$$\lim_{z\to \infty} v(r,z) = r\Omega\Bigg(1 - e^{-1.3333z\sqrt{\frac{\Omega}{\nu}}} + 0.99668z\sqrt{\frac{\Omega}{\nu}}e^{-1.3333z\sqrt{\frac{\Omega}{\nu}}}$$
$$+ 0.70753\sin(1.43753z\sqrt{\frac{\Omega}{\nu}} - \Pi)e^{-0.46539z\sqrt{\frac{\Omega}{\nu}}}\Bigg).$$

Again by applying the limit, we get

$$\lim_{z\to \infty} v(r,z) = r\Omega. \tag{3.46}$$

The boundary condition for w is also satisfied

$$\lim_{z\to 0} w(r,z) = \sqrt{\nu\Omega}\Bigg((0.91z\sqrt{\frac{\Omega}{\nu}} - 1.344)e^{-0.657z\sqrt{\frac{\Omega}{\nu}}} + 1.344$$
$$+ 0.955\sin(1.29576z\sqrt{\frac{\Omega}{\nu}} - \Pi)e^{-0.36862z\sqrt{\frac{\Omega}{\nu}}}\Bigg).$$

After applying limit, the result is

$$\lim_{z\to 0} w(r,z) = 0. \tag{3.47}$$

Thus we have substantiate that approximate solutions given by Eqs. (3.44) and (3.45) satisfy the boundary conditions mentioned in Eq. (3.5).

3.4.1 Residual Analysis of Approximate Solutions of PDEs

Using approximate solutions given in Eqs. (3.43) and (3.44), we get the residuals of the original system of nonlinear PDEs which is given by Eqs. (3.29) and (3.30). The residual surfaces for Eqs. (3.44) and (3.45) are shown in Figures 3.18 and 3.19 respectively.

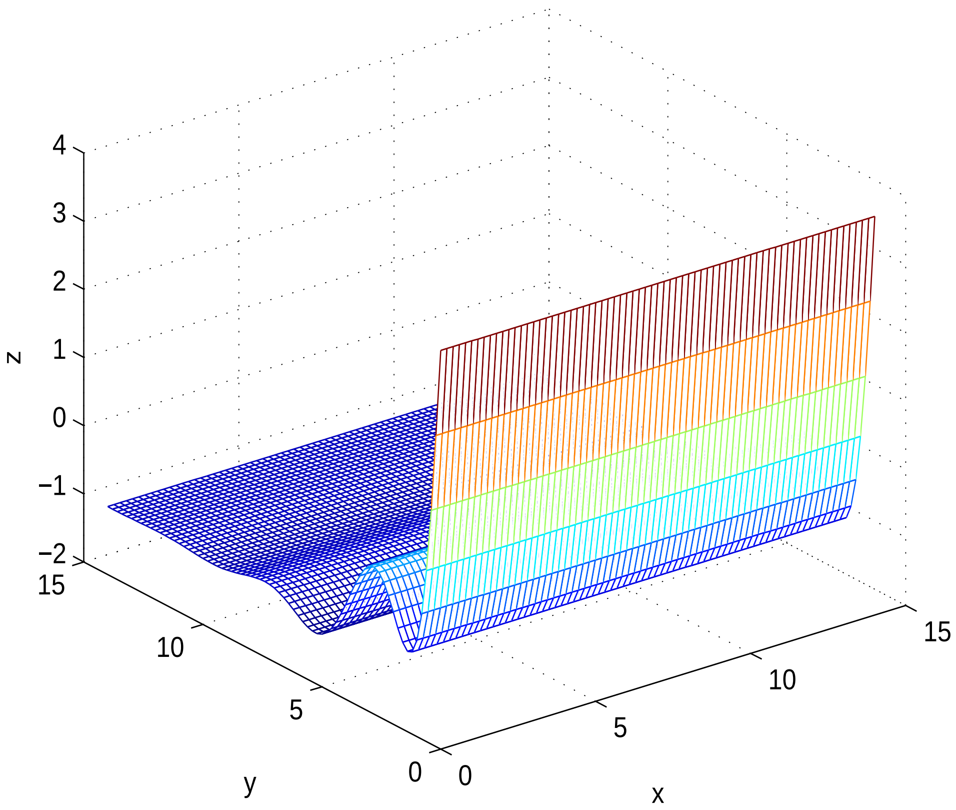

Figure 3.17: Residual for system of second PDE (3.2)

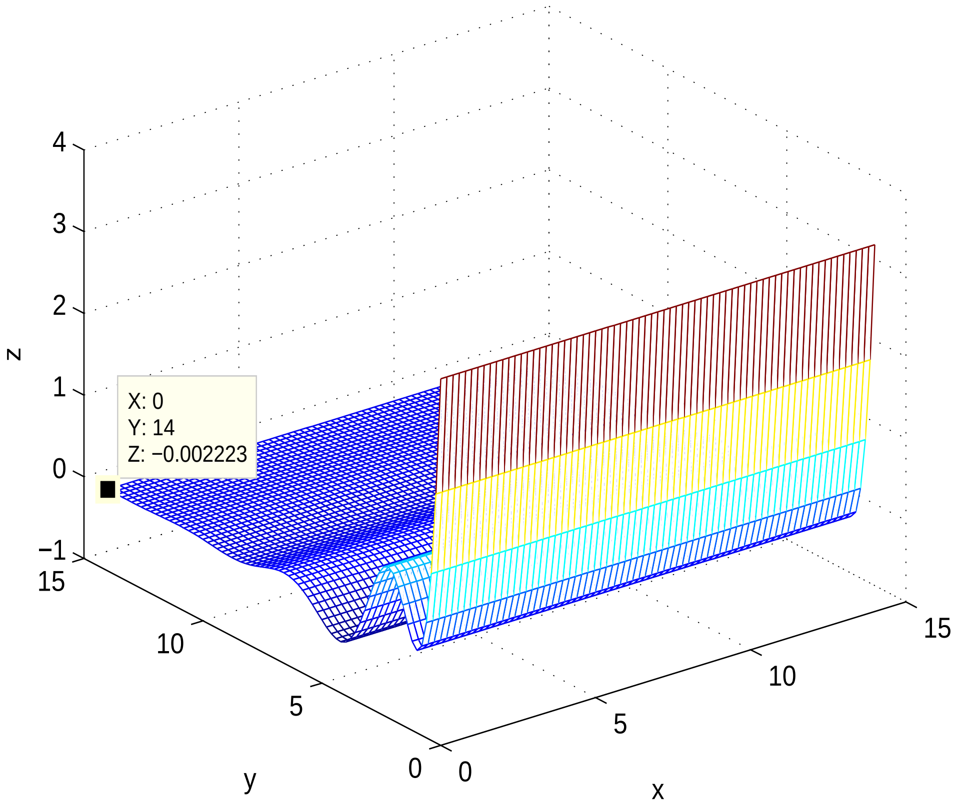

Figure 3.18: Residual for system of third PDE (3.3)

Chapter 4

Similarity Transformations by Using Lie Symmetry Method for System of Nonlinear PDEs

A symmetry is transformation which leaves the geometry of an object apparently unchanged. The transformation which maps each point to itself is known as trivial symmetry of an object and the transformation which maps solution of the system onto a solution of same system is known as continuous symmetry. The symmetry group of a system of a differential equation is a group of transformations of dependent and independent variables leaving the set of all solution invariant [22].

Differential equations are also solved by using symmetry method. Lie symmetry technique is a powerful way of reducing PDEs into ODEs. Lie symmetry method takes advantage of inherit symmetries in PDEs from the physical system and as a result, we get similarity variables which lead to the reduction of PDEs to the ODEs. Both linear and nonlinear differential equations can be solved by using similarity method which is also recognized as symmetry method. In this method, according to the symmetries available for differential equations one can

- Reduced the number of independent variables involved in differential equation.
- Find the exact solution.
- Decrease the order of differential equation.

4.1 Algorithm for Obtaining Similarity Transformations of PDEs by Lie Symmetry Method

Consider a system of partial differential equations of k^{th} order in l and m independent variables and s dependent variables defined as $\boldsymbol{x} = (x^1, x^2, ..., x^l)$, $\boldsymbol{t} = (t^1, t^2, ..., t^m)$ and $\boldsymbol{v} = (v^1, v^2, ..., v^s)$.

$$G(\boldsymbol{x}, \boldsymbol{t}, \boldsymbol{v}, \boldsymbol{v_1}, ..., \boldsymbol{v_n}) = 0, \tag{4.1}$$

$$\bar{x} = x + \epsilon\xi(\boldsymbol{x}, \boldsymbol{t}, \boldsymbol{v}), \tag{4.2}$$

$$\bar{t} = t + \epsilon\delta(\boldsymbol{x}, \boldsymbol{t}, \boldsymbol{v}), \tag{4.3}$$

$$\bar{v} = v + \epsilon\zeta(\boldsymbol{x}, \boldsymbol{t}, \boldsymbol{v}), \tag{4.4}$$

$$\bar{v_i} = v_i + \epsilon\zeta_i(\boldsymbol{x}, \boldsymbol{t}, \boldsymbol{v}, \boldsymbol{v_1}), \tag{4.5}$$

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$$\bar{v}_{i_1 i_2 ... i_n} = v_{i_1 i_2 ... i_n} + \epsilon\zeta_{i_1 i_2 ... i_n}(\boldsymbol{x}, \boldsymbol{t}, \boldsymbol{v}, \boldsymbol{v_1}, ..., \boldsymbol{v_n}), \qquad n = 1, 2, ... \tag{4.6}$$

where ϵ is the group parameter and v_i represents the derivatives of v with respect to x and t. The k^{th} order prolonged infinitesimal generator is

$$\begin{aligned} X^{(k)} &= \xi\frac{\partial}{\partial x} + \delta\frac{\partial}{\partial t} + \zeta\frac{\partial}{\partial v} + \zeta_i(\boldsymbol{x}, \boldsymbol{t}, \boldsymbol{v}, \boldsymbol{v_1})\frac{\partial}{\partial v_i} + ... \\ &+ \zeta_{i_1 i_2 ... i_n}(\boldsymbol{x}, \boldsymbol{t}, \boldsymbol{v}, \boldsymbol{v_1}, ..., \boldsymbol{v_n})\frac{\partial}{\partial v_{i_1, i_2, ..., i_n}}, \end{aligned} \tag{4.7}$$

where prolongation coefficients are ζ_i,...ζ_{i_1}... $\zeta_{i_1 i_2 ... i_n}$ and given as

$$\zeta_i = D_i\zeta - (D_i\xi)\frac{\partial}{\partial v_x} - (D_i\delta)\frac{\partial}{\partial v_t}, \tag{4.8}$$

and

$$\zeta_{i_1 i_2 \ldots i_n} = D_{i_n}\zeta_{i_1 i_2 \ldots i_{n-1}} - (D_{i_n}\xi)v_{i_1 i_2 \ldots i_{n-1}x} - (D_{i_k}\delta)v_{i_1 i_2 \ldots i_{n-1}t} \tag{4.9}$$

The condition given below must satisfy

$$X^{(k)}G(\boldsymbol{x}, \boldsymbol{t}, \boldsymbol{v}, \boldsymbol{v_1}, \ldots, \boldsymbol{v_n}) = 0, \tag{4.10}$$

by using the following theorem

Theorem 1. *Let*

$$\mathbf{X} = \xi(\mathbf{x}, \mathbf{t}, \mathbf{v})\frac{\partial}{\partial x} + \delta(\mathbf{x}, \mathbf{t}, \mathbf{v})\frac{\partial}{\partial t} + \zeta(\mathbf{x}, \mathbf{t}, \mathbf{v})\frac{\partial}{\partial v},$$

be the infinitesimal generator, and $\mathbf{X}^{(k)}$ *be the* k^{th} *extended infinitesimal generator*

$$\begin{aligned}\mathbf{X}^{(k)} &= \xi(\mathbf{x}, \mathbf{t}, \mathbf{v})\frac{\partial}{\partial x} + \delta(\mathbf{x}, \mathbf{t}, \mathbf{v})\frac{\partial}{\partial t} + \zeta(\mathbf{x}, \mathbf{t}, \mathbf{v})\frac{\partial}{\partial v} + \zeta_i(\mathbf{x}, \mathbf{t}, \mathbf{v}, \mathbf{v_1})\frac{\partial}{\partial v_i} \\ &\quad + \ldots + \zeta_{i_1 i_2 \ldots i_n}(\mathbf{x}, \mathbf{t}, \mathbf{v}, \mathbf{v_1}, \ldots, \mathbf{v_n})\frac{\partial}{\partial v_{i_1, i_2, \ldots, i_n}},\end{aligned}$$

then the one-parameter Lie group of transformations is admitted by the system of PDEs if and only if

$$\boldsymbol{X}^{(k)}G(\boldsymbol{x}, \boldsymbol{t}, \boldsymbol{v}, \boldsymbol{v_1}, \ldots, \boldsymbol{v_n}) = 0,$$

when

$$G(\boldsymbol{x}, \boldsymbol{t}, \boldsymbol{v}, \boldsymbol{v_1}, \ldots, \boldsymbol{v_n}) = 0,$$

which is an infinitesimal criterion for invariance of a system of PDEs [23].

We find the determining equations by comparing coefficients of linearly independent variables on both sides. The solution of determining equations provides the unknown coefficients. Generators of transformations are obtained by the solutions of these determining equations. By using method of characteristics, we solve it to get the similarity variables.

In this chapter, we find similarity variables using Lie symmetry technique of the Navier-Stokes equations [4] given by Eqs. (3.1)-(3.4). We obtained the following symmetries for Eqs. (3.1)-(3.4) by using Maple *PDEtools*

$$\boldsymbol{X}_1 = \frac{\partial}{\partial z}, \tag{4.11}$$

$$\boldsymbol{X}_2 = \frac{\partial}{\partial p}, \tag{4.12}$$

$$\boldsymbol{X}_3 = r\frac{\partial}{\partial r} + z\frac{\partial}{\partial z} - u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v} - 2p\frac{\partial}{\partial p} - w\frac{\partial}{\partial w}. \tag{4.13}$$

4.2 Solution of the System of Nonlinear PDEs by using $\boldsymbol{X}_3$

Using the $\boldsymbol{X_3}$ symmetry, the characteristic equation is

$$\frac{dr}{r} = \frac{dz}{z} = \frac{du}{-u} = \frac{dv}{-v} = \frac{dp}{-2p} = \frac{dw}{-w}. \tag{4.14}$$

The following transformations are obtained from the solution of characteristic equation given as

$$x = \frac{z}{r}, \tag{4.15}$$

then

$$u = \frac{g(x)}{r}, \tag{4.16}$$

$$v = \frac{h(x)}{r}, \tag{4.17}$$

$$w = \frac{f(x)}{r}, \tag{4.18}$$

$$p = \frac{l(x)}{r^2}. \tag{4.19}$$

4.3 Reduction of PDEs to ODEs

Using the similarity transformations given by Eqs. (4.16)-(4.19), the system of nonlinear PDEs (3.1)-(3.4) is reduced to the system of ODEs. By taking partial derivatives of Eqs. (4.16)-(4.19) with respect to r and z, we get

$$\frac{\partial u}{\partial r} = \frac{-xg' - g}{r^2}, \tag{4.20}$$

$$\frac{\partial u}{\partial z} = \frac{g^{'}}{r^2}. \tag{4.21}$$

$$\frac{\partial v}{\partial r} = \frac{-xh^{'} - h}{r^2}, \tag{4.22}$$

$$\frac{\partial v}{\partial z} = \frac{h^{'}}{r^2}. \tag{4.23}$$

$$\frac{\partial w}{\partial r} = \frac{-xf^{'} - f}{r^2}, \tag{4.24}$$

$$\frac{\partial w}{\partial z} = \frac{f^{'}}{r^2}. \tag{4.25}$$

$$\frac{\partial p}{\partial r} = \frac{-xl^{'} - 2l}{r^3}, \tag{4.26}$$

$$\frac{\partial p}{\partial z} = \frac{l^{'}}{r^3}. \tag{4.27}$$

Differentiating Eqs. (4.20)-(4.25) with respect to r and z, we get

$$\frac{\partial^2 u}{\partial r^2} = \frac{4xg' + x^2 g'' + 2g}{r^3}. \tag{4.28}$$

$$\frac{\partial^2 u}{\partial z^2} = \frac{g''}{r^3}. \tag{4.29}$$

$$\frac{\partial^2 v}{\partial r^2} = \frac{4xh' + x^2 h'' + 2h}{r^3}. \tag{4.30}$$

$$\frac{\partial^2 v}{\partial z^2} = \frac{h''}{r^3}. \tag{4.31}$$

$$\frac{\partial^2 w}{\partial r^2} = \frac{4xf' + x^2 f'' + 2f}{r^3}. \tag{4.32}$$

$$\frac{\partial^2 w}{\partial z^2} = \frac{f''}{r^3}. \tag{4.33}$$

By putting Eqs. (4.20)-(4.33) in system of nonlinear PDEs, we get the required system of ODEs given as

$$f' - xg' = 0. \tag{4.34}$$

$$(x^2 + 1)g'' + (3x + xg - f)g' + g^2 + h^2 + xl' + 2l = 0. \tag{4.35}$$

$$(x^2 + 1)h'' + (3x + xg - f)h' = 0. \tag{4.36}$$

$$(x^2 + 1)f'' + (3x + xg - f)f' + f - l' + fg = 0. \tag{4.37}$$

Thus, the system of nonlinear PDEs is reduced to the system of ODEs (4.34)-(4.37) by using new transformation variables obtained from symmetry method. For future work it will be better to find similarity transformations which can satisfy boundary conditions and obtain required results.

Conclusion

In this thesis, an approximate closed form solution of a system of nonlinear PDEs to a boundary layer problem is obtained by a systematic procedure. Firstly, for such kind of systems exact solutions are difficult to find. For this reason, similarity transformations are used to get reduced ODEs from the given system of PDEs. Even it is not easy to get exact solution for the reduced system of ODEs. So, we have used *bvp4c* which is a MATLAB built-in function to get the numerical solution. After that, we approximated our numerical solution by the suitable functions. Validation of approximate solution for the system of ODEs is obtained by taking the difference of exact and approximate solution which is analyzed by finding the residues with the help of approximate solution for this system. Then, transformations have been considered to write approximate solution of the original system of PDEs. We found the residuals using approximate closed-form solution for this system which shows that our approximate solutions are closed to exact solution near the boundary. Finally, we have discussed the algorithm for finding similarity transformations of PDEs by Lie symmetry method and worked out for new similarity transformations for the system of PDEs. Using $\boldsymbol{X}_3$, we get transformations which converts system of nonlinear PDEs into ODEs but unable to get transformed boundary conditions for ODEs.

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