Reflection of quasi P and quasi SV waves from the surface of a monoclinic medium

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Dedicated

To the Muslims of Palestine

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Abstract

In this dissertation a review of [1] and [2] is presented. In particular the propagation of plane waves in an anisotropic elastic medium possessing monoclinic and transversly isotropic symmetry are discussed, respectively. The phase velocities for the quasai longitudinal and transverse waves propagation in the plane of elastic symmetry are given [1]. It is studied that in this case that there do not exist any purely longitudinal and transverse waves. Pure longitudinal and transverse waves may be found in some specific and ideal directions and conditions. The precise form of reflection coefficients are obtained that are used to prove the mentioned statement and also to proved that in monoclinic medium, the angle of incidence is not equal to angle of reflection. The problem of propagation for the guided waves in a fluid loaded transversely isotropic cylinder is discussed [2]. Graphs are presented for both the problem for illustration purposes.

Contents

Chapter 1

Introduction

Waves are disturbance in a medium which carry information from one place to another place. Being carriers of information, waves find primary importance in our daily life. Sound, heat and light are all the type of waves and no one can ignore the importance of the waves for all of us. All the wireless communication uses electromagnetic waves for transmitting and receiving information. Electromagnetic waves are not only fastest but also the quick and reliable source of sending the information from one place to other. Heat comes from sun to earth in the form of electromagnetic radiation, thus enabling life to sustain on earth. Radars and ultrasonics use sound waves for communications. Even we can listen to a person when the sound waves from his/her mouth reaches to our ears. Waves are indeed very important for nature to sustain its periodicity.
The study of guided waves propagation can be tracked back to early in 1920 mainly inspired by the field of seismology. Since then there has been an increase effort on the analytical and critical study of guided waves propagation in cylindrical structures. It was early 1990s that guided waves were considered as a practical method for the non-destructive testing for engineering structure. Today guided waves are working and functioning in the field of oil gas and chemical industries [3]. For compressible, underground system and navigation in the sea or testing the situations, Guided waves was first developed in Joint Industry Project (JIP) at Russia in 1920 with funding from many oil and power companies is US, Japan and Korea. The wave property in the pipe is the same once the ultrasonics waves is generating in the pipe. Guided waves testing has many capabilities and limitations in the field applications. Its success de-

pends highly on the support from the manufacturer or selling company and capability of guided waves system. The underground construction companies took a huge contribution in advertising the guided waves system which also has many characteristics [3]. Using the previous other many experiments and method the guided waves are proved best method for solving and testing many hidden facts like underground situation and laying the pipes that are several kilometres long. With the help of guided waves, one can detect the internal and external metal loss. Likewise on the other hand the guided waves system has some drawbacks that in this system, the interpretation of data is totally operator dependent and difficult to find small pitting defects [4].

There are many types of waves like longitudinal, transverse, Rayleigh and Love waves which travel in a medium and their propagation in mediums like monoclinic, hexagonal and many others, depends on the material properties. Monoclinic medium is one of all the crystallographic structure in which the waves propagate with a certain behaviour due to the symmtry of the material. In the paper by Singh and Khurana [1] discussed the reflection and the propagation of waves along the surface of a monoclinic medium and derive the expression that is the correction of the previous result done by the author Chattopadhyay and Choudhury [1] in (1995). Basic concept of continuum mechanics are used to derive explicit expression of the reflection coefficients for quasi P and SV waves.

Chapter 2 is concerned with the basic definitions and concepts of the theory of elasticity. This chapter also contains the basic concept related to crystall systems and the details of mathematics required throughout the thesis.

Chapter 3 contains the review of citeSingh propagation of waves in a monoclinic medium. The mathematical expressions and various approaches are used in this setting to study waves propagating behaviour, that signifies and explain useful properties of waves. After review the previous research work of authors and by studying the new research work on that problem we obtain a reflection coefficients for the waves. This will help to examine the true relation of the waves with the monoclinic medium.

Chapter 4 gives the review of [2]. Various algebric expression for both the cases of fluid loaded cylinder and empty cylinder are presented in [2].

Chapter 2

Preliminaries

2.1 Introduction

In this chapter we make the review of the very basics of the theory of elasticity. The first portion is a simple review of the early development of the theory and also its importance today. The next portion consists of the basics and compelling definition the yet to come description in the rest of dissertation. The last part consists of a brief introduction to the Guided and the concept of longitudinal waves.

2.2 The basics of elasticity

The dynamic theory of elasticity contain the study of wave propagation in elastic solids. This field has gained enormous attention in the mid of 19th century. The early efforts involved the study of a concept that light could be considered as the propagation of a disturbance in a elastic aether. Cauchy and Poisson did a lot of work on the same idea and their efforts hence resulted in the theory of elasticity. Further, Ostrogadsky, Green, Lame, Stocks, Clebsch and Christoffel added their contribution in the extension of the theory [7]. In the later year of 19th century, the theory gained huge popularity through the work in the field by Rayleigh, Lamb and Love. Wave propagation in elastic solid has been an active area for investigation of the earthquake phenomena and recording the nuclear explosion. It is a significant science in the engineering applications, ultrasonics, applied mathematics, electromagnetic theory and acoustics.

2.3 The concept of continuum

Although the matter comprises of the atoms, molecules etc in such a way that there exist many gaps between them, but we assume the matter is uniformly and continuously distributed i.e density is a continuous function of position and therefore it is meaningful to take derivative.

2.4 Representation of physical properties by tensor

By definition, a tensor of rank r is a set of 3^r components denoted by r indices which transform as follow

$$A'_{...ijk....} = a_i^l a_j^m a_k^n ....A_{..lmn..}, \tag{2.1}$$

where a_α^β denotes the transformation matrix from the unprimed to primed frame of reference and the indices i,j,k,l varries from 1–4. Formally, a matrix is used for the tensor representation. The tensor analysis is a mathematical tool to deal with the physical properties of materials. A physicist W. Voigt introduce the idea in the 19 century with a view to describe the strain field of the solid. When such a change of axes is symmetry element of the crystal, the identity of physical properties in each set if co-ordinates provides relation between the components of the tenor that describe those physical properties, and finally reduce the number of independent components.

2.5 Strain, stress and their relationship

This section contains the idea of stress, strain and the constants that characterize the properties of an elastic material. By definition a medium is called a flexible or elastic medium if it go back to its preliminary state when the external forces are eliminated. This return to the initial state is due to internal forces. When force is applied to a material, it undergoes a change of shape or a deformation. These measures of deformation is referred to as strain. Let the field defining the displacement of particles be denoted by $u(x,t)$. As a direct implication of a notion of a continuum, the deformation of the medium can be expressed as the gradient of the displacement vector *u(x,t)*. The restriction of the linearized theory the deformation is simply defined through the strain tensor **S**, with components,

$$S_{ij} = \frac{1}{2}(e_{i,j} + e_{j,i}). \tag{2.2}$$

It is clear that $S_{ij} = S_{ji}$ i.e **S** is a symmetric tensor of rank 2. External forces are essential to produced deformation in a solid. Those forces may be exerted on the surface, by using the mechanical contact, or inside the solid by force field. The effect of this force field in or at the medium can be measured by a force per unit volume (gravitational field) or by torque per unit volume (electric field in a polar crystal). Mechanical tension, or stresses appear in the distorted solid tending to restore it to rest state and therefore enduring the equilibrium of the medium. This stress propagates via the bending force among atoms, the range of which few inter-atomic distances is a totally small microscopic scale. Therefore, the medium surrounding any given volume acts on it via boundary surface, we intend to define at a given point of a surface perpendicular to a coordinate axis. The stress tensor T_{ik}, in an orthonormal frame, is mathematically given by

$$T_{ik} = \lim_{\Delta_{e_k}} \to 0 \quad \lim(\frac{\Delta F_i}{\Delta e_k}), \tag{2.3}$$

Where ΔF_i is the i-th part of force ΔF applied on the surface element Δe_k (perpendicular to k axis) by the medium in the positive direction. T_{ik} is the i-th component of the force applied on a unit surface perpendicular to the k-axis. It can be easily proved that $T_{ik} = T_{ki}$. In an elastic material, there is one to one correspondence between stress and pressure in elastic solid. It's far recognised that the elastic behaviour of most substances is adequately (for small deformation) by using first order term in a Taylor expansion of the equation,

$$T_{ik}(e_{kl}) = T_{ik}(0) + (\frac{\partial T_{ij}}{\partial e_{kl}})e_{kl=0}\ e_{kl} + \frac{1}{2}(\frac{\partial^2 T_{ij}}{\partial e_{kl}\partial e_{mn}})e_{kl=0}\ e_{kl}e_{mn} + \cdots, \tag{2.4}$$

or , since $T_{ij}(0) = 0$, (2.4) gives

$$T_{ij} = c_{ijkl}e_{kl}, \tag{2.5}$$

Where

$$c_{ijkl} = (\frac{\partial T_{ij}}{\partial e_{kl}})e_{kl=\ 0}. \tag{2.6}$$

The coefficient c_{ijkl}, that mention the most common linear relationship between the second rank tensor T_{ij} and e_{kl}, are the components of the fourth rank tensor called the elastic stiffness tensor, and is therefore called Hook's law. A fourth rank tensor has $3^4 = 81$ components. But since T_{ij} *and* S_{kl} are symmetric tensor, the elastic constants defined by Eq (2.5) does not change under a permutation of i and j or k and l which therefore gives,

$$c_{ijkl} = c_{jikl}; c_{ijkl} = c_{ijlk} \tag{2.7}$$

Hooks law (2.5) can be written in terms of displacements, after using Eq (2.2)

$$T_{ij} = \frac{1}{2} c_{ijkl} \frac{\partial u_k}{\partial x_l} + \frac{1}{2} c_{ijkl} \frac{\partial u_l}{\partial x_k}, \tag{2.8}$$

and since $c_{ijkl} = c_{jikl}$, (2.8) gives

$$T_{ij} = c_{ijkl} \frac{\partial u_l}{\partial x_k}. \tag{2.9}$$

From the above equations, we are therefore left with 36 independent elastic constants instead of 81. An ordered pair of indices (i, j) takes only six distinct values, which are numbered from 1 to 6 in the following way

$$(11) \leftrightarrow 1, (22) \leftrightarrow 2, (33) \leftrightarrow 3,$$

$$(23) = (32) \leftrightarrow 4, (13) = (31) \leftrightarrow 5, (12) = (21) \leftrightarrow 6. \tag{2.10}$$

Equation (2.10) defines the famous voigt notations and the independent elastic modulli are thus labelled by only two indices α and β, ranging from 1 to 6 can be written as

$$c_{\alpha\beta} = c_{ijkl}, \tag{2.11}$$

where $\alpha \leftrightarrow (ij), \beta \leftrightarrow (kl)$. The medium is elastically homogenous if the coefficient C_{ijkl} are constant. The material is elastically isotropic wherein there aren't any preferred direction in the material, and the elastic constant must be the same whatever the orientation of the cartesian coordinate system be. The elastic isotropy means that the steady c_{ijkl} may be expressed as

$$c_{ijkl} = \lambda \delta_{ij} \delta_{ij} + \mu(\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}). \tag{2.12}$$

Hooks law then take the form,

$$T_{ij} = \lambda S_{kk} \delta_{ij} + 2\mu S_{ij}. \tag{2.13}$$

(2.12) and (2.13) are two elastic constant λ *and* μ, which are Lame elastic constants.

2.6 Transversely isotropic material

The elements of the matrix $c_{\alpha\beta},\ \alpha\beta = 1, 2, \cdots 6$ for *transversely isotropic* material (hexagonal crystal) can be written as

$$c_{ijkl} = a_i^m a_j^n a_k^o a_l^p c_{mnop}. \tag{2.14}$$

Now the hexagonal crystal has at least one diad axis and one A_6 axis. If Ox_3-axis is put along the diad axis, the matrix a for the frame change is diagonal:

$$[a] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.15}$$

using (2.14) and (2.15) becomes

$$c_{ijkl} = a_i^m a_j^n a_k^o a_l^p c_{ijkl}. \tag{2.16}$$

This means that all coefficient with a odd index (for which $a_i^m a_j^n a_k^o a_l^p = -1$) vanish. Therefore $c_{\alpha\beta}$ have thirteen elastic constants, for which the set of indices includes zero, two or four times appearence of the index 3. The crystals of trigonal, tetragonal, hexagonal classes have only one direct or inverse axis of order greater then two. The rotation matrix a is no more diagonal (axis points in the x_3 direction).

$$a = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{with } \ \phi = \tfrac{2\pi}{n} \neq \pi \ , \tag{2.17}$$

where ϕ is an angle so the invariant relation (2.14) is now more tough to judge for it involves many components at the same time. For this purpose, we have to diagonalize matrix in (2.15), finally we $c_{\alpha\beta}$ take the form,

$$c_{ijkl} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & (c_{11} - c_{11})/2 \end{bmatrix}. \tag{2.18}$$

2.7 Monoclinic system

The monoclinic system is one of the structural classes in which crystal solids can be assigned. Crystals that belong to this class are referred to have three axes of unequal lengths say, a, b, and c of which a is perpendicular to b and c, however b and c are not perpendicular to each other.

If the atoms or group atom organizations in the solid are shown by points to points and if they are linked, the resulting lattice consist of an orderly stacking of blocks, or unit cells. The monoclinic unit is recognised via a single axis, known as an axis of two fold of symmetry, approximately which the cellular can be rotated through 180 without changing its shape and structure. Many crystals belonging to the monoclinic system than to any other. Beta, Sulphur, Gypsum, Borax lie in the monoclinic system.

2.8 Basics of monoclinic system

We consider the homogenous anisotropic elastic medium of a monoclinic type and as we know that monoclinic material is the sole plane of flexible direction,

the components of the fourth order elasticity tensor for a monoclinic system is

$$[a] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \tag{2.19}$$

when x_1 axis is parallel to the axis of symmtry

$$c'_{ijkl} = q_{pi}q_{qj}q_{rk}q_{sl}c_{pqrs}, \tag{2.20}$$

where q_{pi} is +1 or -1 if p=i, q=j, r=k, s=l. If the index 1 or 2 or 3 appear odd times in monoclinic system then it tend to zero on choosing which plane we are dealing with.

$$c'_{ijkl} = -c_{ijkl} \tag{2.21}$$

For example if we are taking x_2x_3 plane then there is one plane x_1 that is fixed and the index 1 if appear odd times then it tends to zero in the whole monoclinic system.

$$c'_{ijkl} = -(1)^3 1 c_{pqrs}, \tag{2.22}$$

So in this way we will have the elasticity matrix of the monoclinic medium and after solving all the 36 components of the elasticity matrix we have only 13 independent components left in the monoclinic medium matrix. Here the matrix and notation are given as For instance, if we calculate $c_{15'}$, we get

$$c'_{1113} = q_{p1}q_{q1}q_{r1}q_{s3}c_{pqrs}. \tag{2.23}$$

Now apply summation on r and s

$$c_{1113'} = q_{11}q_{11}q_{11}q_{13}c_{1111} + q_{11}q_{11}q_{31}q_{13}c_{1131} + q_{11}q_{11}q_{11}q_{33}c_{1113} + q_{11}q_{11}q_{31}q_{33}c_{1133}, \tag{2.24}$$

Using Eq 2.19 we get

$$c'_{15} = (1)(-1)c_{15} + 0 + 0 + 0,$$
$$c'_{15} = (1)(-1)c_{15},$$
$$2c_{15} = 0,$$

which implies that

$$c_{15} = 0. \tag{2.25}$$

Similarly other components can be found.

$$c'_{ijkl} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & 0 & 0 \\ c_{12} & c_{22} & c_{23} & c_{24} & 0 & 0 \\ c_{13} & c_{23} & c_{33} & c_{34} & 0 & 0 \\ c_{41} & c_{42} & c_{43} & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & c_{56} \\ 0 & 0 & 0 & 0 & c_{56} & c_{66} \end{bmatrix} \tag{2.26}$$

2.9 Propagation equation

The displacement of any point is considered as time independent i.e.

$$u_i = u_i(x_k, t). \tag{2.27}$$

The force density per unit volume of stressed material is given by

$$F_i = \frac{\partial T_{ij}}{\partial x_j}. \tag{2.28}$$

Where (2.28) is the equation of motion and we ignore the effect of gravity and say that this force gives rise to the acceleration along the $i-th$ axis for the unit volume mass ρ,

$$\rho\frac{\partial^2 u_i}{\partial t^2} = \frac{\partial T_{ij}}{\partial x_j}. \tag{2.29}$$

Here ρ is the density so making use of Hooks law the equation of motion becomes

$$\rho\frac{\partial^2 u_i}{\partial t^2} = c_{ijkl}\frac{\partial^2 u_i}{\partial x_j \partial x_k}. \tag{2.30}$$

This is a set of three-second order differential equations, which govern the wave motion in a fluid in three dimensional case. For a homogeneous, isotropic, linearly elastic body, equation of motion (2.30) after using (2.12) and (2.13) and (2.14) can be written in the form,

$$\mu u_{i,jj} + (\lambda + \mu)u_{j,ij} = \rho \ddot{u}_i, \tag{2.31}$$

when there are no internal body forces acting on the body. Similarly for transversely isotropic material (2.30) takes the form

$$\frac{1}{2}(c_{11} - c_{12}) + u_{\alpha,\pi\pi} + \frac{1}{2}(c_{11} + c_{12}) + u_{\pi,\alpha\pi} + c_{44}u_{\alpha,33} + (c_{13} - c_{44})u_{\pi,\alpha\pi} = \rho\ddot{u_\alpha}, \tag{2.32}$$

the above Eq (2.30) has a general solution of the form

$$u_i = f(\mathbf{n}.\mathbf{x} - vt)\mathbf{p}, \tag{2.33}$$

where $\boldsymbol{n}$ and $\boldsymbol{p}$ are unit vector and v has the dimensions of velocity. The unit vector $\mathbf{n}$ is the direction in which wave travels with the rate v. This vector is called propagation vector and the unit vector $\mathbf{p}$ shows the displacement $\mathbf{u}$ and is known as polarization vector.

2.10 Wave propagation in solids

There is a great variety of elastic waves. Certainly, similar to the propagation situations, which will talk over with, amongst Rayleigh waves, lamb waves, love waves and Stoneley waves. Basically there exist different of waves as follows.

The *longitudinal* or *compression* waves are characterized by using a particle displacement parallel to the direction of propagation, i.e a polarization parallel to the vector. A longitudinal wave creates the variation in the distance among parallel plane containing given particles, in order that the volume occupied by a given wide variety of particle is not constant. The *transverse* of *shear* waves are such that particle displacement is perpendicular to the wave vector.

The velocities of the longitudinal and transverse waves are denoted through c_L and c_T respectively. These waves travel in an unbound medium by way of which we suggest in practise that the size of the medium is a greater than the beam size and that surface effects are negligible.

2.11 Longitudinal and transverse wave motion

Mechanical waves are waves which propagate through a material medium (solid, liquid, or gas) at a wave speed which depends on the elastic and inertial properties of that medium. There are fundamental forms of wave movement for mechanical waves: **longitudinal** waves and **transverse** waves [13].

2.12 Longitudinal waves

In a longitudinal wave the particle displacement is parallel to the direction of wave propagation. The particles do not move up and down with the wave but they simply oscillate back and forth about their equilibrium positions. Examples of P Waves (Primary Waves) in the earthquake is longitudinal waves. P waves travel with high speed and travel first as compared to S waves [14, 15].

2.13 Transverse waves

In a transverse wave the displacement of the particles is perpendicular to the direction of wave propagation. The particles do not move along with the wave, they simply oscillate up and down about their individual equilibrium positions as the wave passes by.
The S waves (secondary waves) in an earthquake are examples of transverse waves. S waves propagate with a pace slower than P waves, arriving several seconds later.

2.14 Rayleigh surface waves

A particles in solid, through which a rayleigh surface passes, moves in elliptical path, with the most important axis of ellipses on a solid surface. As depth into the solid

increases, "width" of elliptical direction decreases. Rayleigh waves in a elastic solid are different from surface waves in water in a completely critical way. In the water wave all the particles travel in the counter clockwise direction. But, in rayleigh the surface wave, the particles on the floor shows a clock wise direction, However particles in a depth of more than 1/5th of a wavelength trace out clock watch ellipse [11].
The Rayleigh surface waves make the most harm during the earthquake. They travel with the velocity slower then S waves, and arrive later but with much greater amplitude. These are also waves that are most easily felt during an earthquake and involve both up-down and side to side motion.

2.15 Hook's law in cylindrical coordinates

In this section we re-wright hooks law from equation (2.6). Now the respective component in cylindrical coordinates with the repressntation are computed so applying the summation on the values of i,j,k,l accordingly we have

$$T_{11} = c_{11kl}e_{kl}. \tag{2.34}$$

$$c_{1111}e_{11} + c_{1112}e_{12} + c_{1113}e_{13} + c_{1121}e_{21} + c_{1122}e_{22} + c_{1123}e_{23} + c_{1131}e_{31} + c_{1132}e_{32} + c_{1133}e_{33},$$

but due to symmetry as

$$T_{11} = c_{1111}e_{11} + 2c_{1112}e_{12} + 2c_{1113}e_{13} + 2c_{1132}e_{32} + c_{1122}e_{22} + c_{1133}e_{33}. \tag{2.35}$$

Now the respective component in cylindrical coordinates with the repressntation are and the notation are shown in Eq (2.10). Thus after simplification we have

$$r = 1, \theta = 2 \; and \; z = 3. \tag{2.36}$$

So the equation becomes after expanding them according to (2.34)

$$\begin{aligned} T_{rr} =& c_{11}\frac{\partial u_r}{\partial x_r} + c_{16}\left(\frac{\partial u_r}{\partial x_\theta} + \frac{\partial u_\theta}{\partial x_r}\right) + c_{15}\left(\frac{\partial u_r}{\partial x_z} + \frac{\partial u_z}{\partial x_r}\right) \\ &+ c_{14}\left(\frac{\partial u_\theta}{\partial x_z} + \frac{\partial u_z}{\partial x_\theta}\right) + c_{12}\frac{\partial u_\theta}{\partial x_\theta} + c_{13}\frac{\partial u_z}{\partial x_z}. \end{aligned} \tag{2.37}$$

Like wise we can solve other stress and normal components also and we get

$$\begin{aligned}T_{\theta\theta} =& c_{21}\frac{\partial u_r}{\partial x_r} + c_{26}\left(\frac{\partial u_r}{\partial x_\theta} + \frac{\partial u_\theta}{\partial x_r}\right) + c_{25}\left(\frac{\partial u_r}{\partial x_z} + \frac{\partial u_z}{\partial x_r}\right)\\ &+ c_{24}\left(\frac{\partial u_\theta}{\partial x_z} + \frac{\partial u_z}{\partial x_\theta}\right) + c_{23}\frac{\partial u_z}{\partial x_z} + c_{22}\frac{\partial u_\theta}{\partial x_\theta},\end{aligned} \tag{2.38}$$

$$\begin{aligned}T_{zz} =& c_{12}\frac{\partial u_r}{\partial x_r} + c_{34}\left(\frac{\partial u_\theta}{\partial x_z} + \frac{\partial u_z}{\partial x_\theta}\right) + c_{36}\left(\frac{\partial u_r}{\partial x_\theta} + \frac{\partial u_\theta}{\partial x_r}\right)\\ &+ c_{35}\left(\frac{\partial u_r}{\partial x_z} + \frac{\partial u_z}{\partial x_r}\right) + c_{32}\frac{\partial u_\theta}{\partial x_\theta} + c_{33}\frac{\partial u_z}{\partial x_z},\end{aligned} \tag{2.39}$$

and one of the shear component is

$$\begin{aligned}T_{r\theta} =& c_{16}\frac{\partial u_r}{\partial x_r} + c_{66}\left(\frac{\partial u_r}{\partial x_\theta} + \frac{\partial u_\theta}{\partial x_r}\right) + c_{46}\left(\frac{\partial u_\theta}{\partial x_z} + \frac{\partial u_z}{\partial x_\theta}\right)\\ &+ c_{56}\left(\frac{\partial u_z}{\partial x_r} + \frac{\partial u_r}{\partial x_z}\right) + c_{26}\frac{\partial u_\theta}{\partial x_\theta} + c_{36}\frac{\partial u_z}{\partial x_z}.\end{aligned} \tag{2.40}$$

2.16 Buchwald representation in cylindrical coordinates

As we know the Helmholtz representation which is given by

$$u = \nabla\phi + \nabla \wedge \psi, \tag{2.41}$$

where ψ andϕ are differential operator and is given by

$$\nabla \ = \frac{e_1}{h_1}\frac{\partial}{\partial q_1} + \frac{e_2}{h_2}\frac{\partial}{\partial q_2} + \frac{e_3}{h_3}\frac{\partial}{\partial q_3}, \tag{2.42}$$

The scale factors h_i are in general functions of the coordinates q_j and the unit vector e_i generally vary in direction from point to point in space. After apply the ∇ operator we get

$$\nabla\phi = \frac{e_1}{h_1}\frac{\partial\phi}{\partial q_1} + \frac{e_2}{h_2}\frac{\partial\phi}{\partial q_2} + \frac{e_3}{h_3}\frac{\partial\phi}{\partial q_3}, \tag{2.43}$$

and the

$$\nabla \wedge \psi = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 e_1 & h_2 e_2 & h_3 e_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1\psi_r & h_2\psi_\theta & h_3\psi_z \end{vmatrix} \tag{2.44}$$

The corresponding scale factor and unit bases are $h_1 = 1, h_2 = r, h_3 = 1$ and $e_1 = e_r, e_2 = e_\theta, e_3 = k$. After using the value of h_1 , h_2, h_3 and e_1, e_2, e_3 in (2.44) we get the expression

$$e_r\left(\frac{\partial\psi_z}{\partial\theta} - r\frac{\partial\psi_\theta}{\partial z}\right) - re_\theta\left(\frac{\partial\psi_z}{\partial r} - \frac{\partial}{\psi_r}\partial z\right) + k\left(r\frac{\partial\psi_\theta}{\partial r} - \frac{\partial\psi_r}{\partial\theta}\right). \tag{2.45}$$

By combining (2.44) and (2.37) of helm Holtz together we get the selective term of u, v and w that is

$$u = \frac{\partial\phi}{\partial r} + \frac{1}{r}\frac{\partial\psi_z}{\partial\theta} - \frac{\partial\psi_\theta}{\partial z}, \tag{2.46}$$

similarly for v and w expression

$$v = \frac{1}{r}\frac{\partial\phi}{\partial\theta} + \frac{\partial\psi_r}{\partial z} - \frac{\partial\psi_z}{\partial r}, \tag{2.47}$$

and like wise the w component and now due to the nature few terms are not linked and they vanish so after simplification the terms are

$$u_r = \frac{\partial\phi}{\partial r} + \frac{1}{r}\frac{\partial\psi_\chi}{\partial\theta}, \tag{2.48}$$

$$u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} - \frac{\partial\chi}{\partial r}, \tag{2.49}$$

and

$$u_z = \frac{\partial\psi}{\partial z}. \tag{2.50}$$

2.17 Laplacian operator in cylindrical coordinates

Here we make the conversion of cartesian coordinate to cylindrical coordinate. For this first we define the basic of cartesian and polar coordinates respectively. Organize a point x and y measure a couple of pairs of points from the point of view the durable linear lines in the plane are called a nuclear assistant axis actually meet. Polar coordination system, a two dimensional systematic system using a row grid. The r and θ from the point of viewpoint approaches between the original and angle line and row axis. The Laplacian operator is very important in mathematics. It is nearly ubiquitous. Its form is simple and symmetric in Cartesian coordinates

$$\nabla^2 = \frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2} + \frac{\partial}{\partial z^2}, \tag{2.51}$$

and in the polar co-ordinate it is given by

$$\nabla^2 = \frac{\partial}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \theta^2} + \frac{\partial}{\partial z^2}, \tag{2.52}$$

Converting them to cylindrical coordinates, the details of calculating itsform in cylindrical co-ordinates is follows. It is good to begin with the simpler case.The z component does not change and for the x and y components, the transformations are $V = V(x, y)$ where

$$x = r\cos\theta \; and \; y = r\sin\theta. \tag{2.53}$$

By partial derivative with respect to r we have

$$\frac{\partial V}{\partial r} = \frac{\partial V}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial V}{\partial y}\frac{\partial y}{\partial r}, \tag{2.54}$$

$$\frac{\partial V}{\partial r} = \cos\theta\frac{\partial V}{\partial x} + \sin\theta\frac{\partial V}{\partial y}. \tag{2.55}$$

Same for the case in θ

$$\frac{\partial V}{\partial \theta} = \frac{\partial V}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial V}{\partial y}\frac{\partial y}{\partial \theta}, \tag{2.56}$$

$$\frac{\partial V}{\partial \theta} = -r\sin\theta\frac{\partial V}{\partial x} + r\cos\theta\frac{\partial V}{\partial y}. \tag{2.57}$$

Eq (2.55) and (2.57) when solved for x give

$$\frac{\partial V}{\partial x} = \cos\theta\frac{\partial V}{\partial r} - \frac{\sin\theta}{r}\frac{\partial V}{\partial \theta}, \tag{2.58}$$

$$\frac{\partial}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}. \tag{2.59}$$

Also we get with respect to y

$$\frac{\partial V}{\partial y} = \sin\theta\frac{\partial V}{\partial r} + \frac{\cos\theta}{r}\frac{\partial V}{\partial \theta}, \tag{2.60}$$

$$\frac{\partial}{\partial y} = \cos\theta\frac{\partial}{\partial r} + \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}. \tag{2.61}$$

$$\frac{\partial V^2}{\partial x^2} = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right), \tag{2.62}$$

taking the second derivative we have

$$\begin{aligned}\frac{\partial V^2}{\partial x^2} &= \cos\theta^2 \frac{\partial V^2}{\partial r^2} + 2\frac{\sin\theta\cos\theta}{r^2}\frac{\partial V}{\partial\theta} - \frac{2\sin\theta\cos\theta}{r}\frac{\partial V}{\partial\theta\partial r} \\ &+ 2\frac{\sin^2\theta}{r^2}\frac{\partial V^2}{\partial\theta^2} + \frac{\sin^2\theta}{r^2}\frac{\partial V}{\partial r}.\end{aligned} \tag{2.63}$$

Similarly with respect to y we have

$$\frac{\partial V^2}{\partial y^2} = \left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right), \tag{2.64}$$

which implies

$$\begin{aligned}\frac{\partial V^2}{\partial y^2} &= \sin\theta^2 \frac{\partial V^2}{\partial r^2} - 2\frac{\sin\theta\cos\theta}{r^2}\frac{\partial V}{\partial\theta} + \frac{2\sin\theta\cos\theta}{r}\frac{\partial V}{\partial\theta\partial r} \\ &+ 2\frac{\cos^2\theta}{r^2}\frac{\partial V^2}{\partial\theta^2} + \frac{\cos^2\theta}{r^2}\frac{\partial V}{\partial r},\end{aligned} \tag{2.65}$$

by addition of (2.63) and (2.65) we have

$$\frac{\partial V^2}{\partial x^2} + \frac{\partial V^2}{\partial y^2} = \frac{\partial V^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial V^2}{\partial\theta^2} + \frac{1}{r}\frac{\partial V}{\partial r}. \tag{2.66}$$

and the laplacian in (r, θ) can be written as

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial V}{\partial r}) + \frac{1}{r}\frac{\partial V^2}{\partial\theta^2}, \tag{2.67}$$

cylindrical coordinates (r, θ, z) the same method we can derive the laplace operator in cylindrical coordinates

$$V = V(x, y, z), \quad x = r\cos\theta, \; y = r\sin\theta, \quad z = z. \tag{2.68}$$

By partial derivative selective term of u,v and w are

$$\frac{\partial V}{\partial r} = \frac{\partial V}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial V}{\partial y}\frac{\partial y}{\partial r}, \tag{2.69}$$

$$\frac{\partial V}{\partial r} = \frac{\partial V}{\partial x}\cos\theta + \frac{\partial V}{\partial y}\sin\theta. \tag{2.70}$$

Likewise in θ

$$\frac{\partial V}{\partial\theta} = \frac{\partial V}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial V}{\partial y}\frac{\partial y}{\partial r}, \tag{2.71}$$

$$\frac{\partial V}{\partial \theta} = -\frac{\partial V}{\partial x} r\sin\theta + \frac{\partial V}{\partial y} rcos\theta. \tag{2.72}$$

At the same time in z we have,

$$\frac{\partial V}{\partial z} = \frac{\partial V}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial V}{\partial y}\frac{\partial y}{\partial z} + \frac{\partial V}{\partial z}\frac{\partial z}{\partial z}, \tag{2.73}$$

$$\frac{\partial V}{\partial z} = \frac{\partial V}{\partial z}. \tag{2.74}$$

After simplification we have

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial V}{\partial r}) + \frac{1}{r}\frac{\partial V^2}{\partial \theta^2} + \frac{\partial V^2}{\partial z^2}. \tag{2.75}$$

Chapter 3

Reflection of longitudinal and transverse waves at surface of a monoclinic medium

In an anisotropic elastic solid medium, three types of waves with the equal time orthogonal particles movement may be propagated. In reality, the particle movement is neither simply longitudinal nor simply transverse. Because of this, three types of body waves in a isotropic medium known as as q(P), q(SV) and q(SH), in place of P, SV and SH, the symbols used for propagation in an isotropic medium. Here the symbol q is used for the quasi, means a similar properties and behaviour and P and SV are representing as the longitudinal and transverse wave.

The monoclinic medium contain one plane of elastic symmetry. For wave propagation inside a plane of symmetry, SH movement is decoupled from the P-SV motion. At the same time because the particle movement of SH waves are purely transverse, it is not completely longitudinal nor transverse in the case of P-SV waves.

3.1 Problem formulation

We take the homogenous anisotropic monoclinic elastic medium. Taking plane symmetry as the x_2x_3 plane the hooks law shown in equation (**??**) and (**??**) so in displacement form, the component in the matrix form are

$$T_{11} = c_{11}e_{11} + c_{12}e_{22} + c_{13}e_{33} + 2c_{14}e_{23}, \tag{3.1}$$

$$T_{22} = c_{12}e_{11} + c_{22}e_{22} + c_{23}e_{33} + 2c_{24}e_{23}, \tag{3.2}$$

$$T_{33} = c_{13}e_{11} + c_{23}e_{22} + c_{33}e_{33} + 2c_{34}e_{23}, \tag{3.3}$$

$$T_{23} = c_{14}e_{11} + c_{24}e_{22} + c_{34}e_{33} + 2c_{44}e_{23}, \tag{3.4}$$

$$T_{13} = 2(c_{55}e_{13} + c_{56}e_{12}), \tag{3.5}$$

$$T_{12} = 2(c_{56}e_{13} + c_{66}e_{12}). \tag{3.6}$$

Here T_{ij} is showing stress tensor and e_{ij} showing strain tensor. For plane wave propagating in the x_2x_3 plane we have

$$T_{11} = c_{12}\frac{\partial u_2}{\partial x_2} + c_{13}\frac{\partial u_3}{\partial x_3} + c_{14}(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}), \tag{3.7}$$

since we are taking the x_2x_3 plane, the derivative with respect to x_1 are zero. Now using equation of motion (2.29) without body force. Now the summation is applied on Eq (2.29), so after comparing the equation of motion and all the components with respect to x_1 will cancel and remaining are

$$c_{66}\frac{\partial^2 u_1}{\partial x_2^2} + 2c_{56}\frac{\partial^2 u_1}{\partial x_2 \partial x_3} + c_{55}\frac{\partial^2 u_1}{\partial x_3^2} = \rho\frac{\partial^2 u_1}{\partial t^2}, \tag{3.8}$$

$$c_{22}\frac{\partial^2 u_2}{\partial x_2^2} + c_{44}\frac{\partial^2 u_2}{\partial x_3^2} + c_{24}\frac{\partial^2 u_3}{\partial x_2^2} + c_{34}\frac{\partial^2 u_3}{\partial x_3^2} + 2c_{24}\frac{\partial^2 u_2}{\partial x_2 \partial x_3} + (c_{23}+c_{44})\frac{\partial^2 u_3}{\partial x_2 \partial x_3} = \rho\frac{\partial^2 u_2}{\partial t^2}, \tag{3.9}$$

$$c_{24}\frac{\partial^2 u_2}{\partial x_2^2} + c_{34}\frac{\partial^2 u_2}{\partial x_3^2} + c_{44}\frac{\partial^2 u_3}{\partial x_3^2} + c_{33}\frac{\partial^2 u_3}{\partial x_3^2} + 2c_{34}\frac{\partial^2 u_3}{\partial x_2 \partial x_3} + (c_{23}+c_{44})\frac{\partial^2 u_2}{\partial x_2 \partial x_3} = \rho\frac{\partial^2 u_3}{\partial t^2}. \tag{3.10}$$

Consider $\mathbf{p}(0, p_2, p_3)$ represents the unit propagation vector, c the phase velocity and k the wave number of plane waves propagating in x_2x_3-plane. A solution of equation of motion (3.3) representing a plane wave is of the form

$$u_1 = A\ exp[ik(ck - x_2p_2 - x_3p_3)]. \tag{3.11}$$

So inserting equation above equation (3.8) in (3.5) we get

$$c_{66}p_2^2 + 2c_{56}p_2p_3 + c_{55}p_3^2 = \rho c^2. \tag{3.12}$$

This equation gives the phase velocity of an SH wave propagating in a different direction in the plane symmetry of monoclinic medium in an arbitrary direction. We want to solve plane wave solution of Eq (3.9) and (3.10) in the form (3.8)

$$[u_2, u_3] = A[d_2, d_3]\ exp\ [ik(ct - x_2p_2 - x_3p_3)], \tag{3.13}$$

here $\mathbf{d}(0, d_2, d_3)$ the unit displacement vector. Putting above expression for u_2 and u_3 in the equations of motion (3.9) and (3.10) we have

$$(U - \rho c^2)d_2 + Vd_3 = 0, \tag{3.14}$$

$$Vd_2 + (Z - \rho c^2)d_3 = 0, \tag{3.15}$$

where

$$U = c_{22}p_2^2 + c_{44}p_3^2 + 2c_{24}p_2p_3, \tag{3.16}$$

$$V = c_{24}p_2^2 + c_{34}p_3^2 + (c_{23} + c_{44})p_2p_3, \tag{3.17}$$

$$Z = c_{44}p_2^2 + c_{33}p_3^2 + 2c_{34}p_2p_3. \tag{3.18}$$

Eq (3.14) and (3.15) yeild

$$d_2/d_3 = V/(\rho c^2 - U) = (\rho c^2 - Z)/V. \tag{3.19}$$

Therefore, ρc^2 satisfies quadratic equation

$$\rho^2c^4 - (U + Z)\rho c^2 + (UZ - V^2) = 0, \tag{3.20}$$

with solutions

$$2\rho c^2(p_2, p_3) = (U + Z) \pm [(U - Z)^2 + 4V^2]^{1/2}. \tag{3.21}$$

The upper sign in (3.21) is for longitudinal and lower sign is for transverse waves. Eliminating ρc^2 from the two equations in (3.19) we find

$$(d_2^2 - d_3^2)V = d_2d_3(U\text{-}Z), \tag{3.22}$$

inserting the expression for U V and Z from Eq (3.18) we obtain

$$\begin{aligned}[c_{24}(d_3^2 - d_2^2) + (c_{22} - c_{44})d_2d_3]p_2^2 + [c_{34}(d_3^2 - d_2^2) + (c_{44} - c_{33})d_2d_3]p_3^2 \\ + [(c_{23} + c_{44})(d_3^2 - d_2^2) + 2(c_{24} - c_{34})d_2d_3]p_2p_3 = 0,\end{aligned} \tag{3.23}$$

we may write Eq (3.22) in the form

$$\frac{d_2 d_3}{d_3^2 - d_2^2} = V/Z\text{-}U. \tag{3.24}$$

Note that $U = U(p_2, p_3)$ etc., Eq (3.24) can be used to find the direction of the displacement vector **d** for a given direction of a propagation **p**. Substituting tan e= p_2/p_3, tanϕ=d_2/d_3, we have

$$\phi = \frac{1}{2}tan^{-}1(\Omega), \frac{\pi}{2} + \frac{1}{2}tan^{-}1(\Omega), \tag{3.25}$$

where

$$\Omega = 2\frac{c_{24} \quad \tan^2 e + (c_{23} + c_{44})tan e + c_{34}}{[(c_{44} - c_{22})\tan^2 e + 2(c_{34} - c_{24})tan e + c_{33} - c_{44}]}. \tag{3.26}$$

3.2 Reflection of longitudinal and transverse waves

Take a homogenous, elastic half-space that occupying the area x_3=0 (fig 3.1). Elastic plane symmetry can be written as x_2x_3 plane. Plane qP or qSV waves are an incident on traction free boundary $x_3 = 0$ [13, 14]. We consider the plain strain problem for which

$$u_1 = 0, u_2 = u_2(x_2, x_3, t), u_3 = u_3(x_2, x_3, t). \tag{3.27}$$

Incident q(P) and q(SV) waves will create reflected q(P) and q(SV) waves. So total displacement is mention by

$$u_2 = \sum_{j=1}^{4} A_j e^{iP_j} \qquad u_3 = \sum_{j=1}^{4} B_j e^{iP_j}, \tag{3.28}$$

where

$$P_1 = \omega[t - (x_2 \sin e_1 - x_3 \cos e_1)/c_1],$$

$$P_2 = \omega[t - (x_2 \sin e_2 - x_3 \cos e_2)/c_2],$$

$$P_3 = \omega[t - (x_2 \sin e_3 + x_3 \cos e_3)/c_3],$$

$$P_4 = \omega[t - (x_2 \sin e_4 + x_3 \cos e_4)/c_4], \tag{3.29}$$

ω is representing the angular frequency. We mention the elements corresponding to the variety of waves by taking notation.

(1) for incident longitudinal waves and e_1 for incidence angle of longitudinal,

(2) for incident transverse waves and e_2 for incidence angle of transverse,

(3) for reflecting longitudinal waves and e_3 for reflecting angle of longitudinal,

(4) is for reflecting transverse waves and e_4 for reflecting angle of transverse.

Thus for example, for the incident qP waves, c_1 representing the phase velocity, e_1 the incidence angle, $P_1(x_2, x_3, t)$ phase factor, A_1 the amplitude factor of the u_2 displacement and B_1 that of the u_3 component [12].

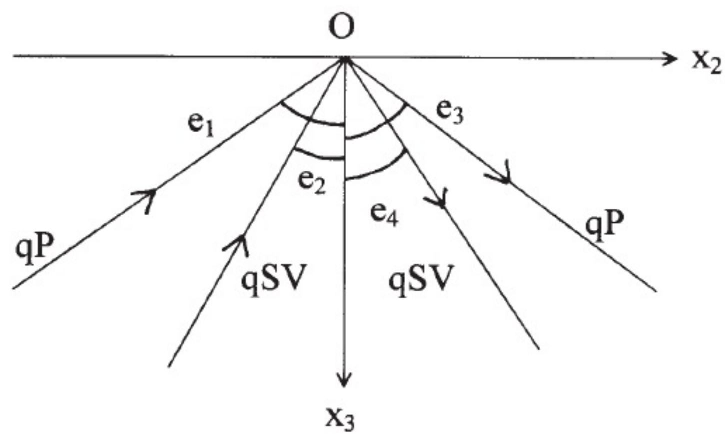

Figure 3.1: Reflect the (qP) and SV waves on the planes free boundary at ($x_3 = 0$) of mono-clinic space.

So qP and qSV for each incidence, it appears that the qP and qSV must satisfy equation of motion,

$$A_i = F_i B_i \quad (i = 1, 2, 3, 4; \text{no summation}), \tag{3.30}$$

where

$$F_i = V_i/(\rho c_i^2 - U_i) = (\rho c_i^2 - Z_i)/V_i, \quad (i = 1, 2, 3, 4), \tag{3.31}$$

$$2\rho c_i^2 = (U_i + Z_i) + [(U_i - Z_i)^2 + 4V_i^2]^{1/2} \quad (i = 1, 3), \tag{3.32}$$

$$2\rho c_i^2 = (U_i + Z_i) - [(U_i - Z_i)^2 + 4V_i^2]^{1/2} \quad (i = 2, 4). \tag{3.33}$$

The expression for $U_i, V_i$and Z_i are collected from the basic algebric expression. For incident longitudinal waves, $p_2 = \sin e_1$, $p_3 = -\cos e_1$; for incidence transverse waves, $p_2 = \sin e_2$, $p_3 = -\cos e_2$; for reflected longitudinal waves, $p_2 = \sin e_3$, $p_3 = \cos e_3$ and for reflected transverse waves, $p_2 = \sin e_4$, $p_3 = \cos e_4$.
Look the figure after putting the values we have,

$$U_1 = c_{22} \sin^2 e_1 + c_{44} \cos^2 e_1 - 2c_{24} \sin e_1 \cos e_1, \tag{3.34}$$

$$V_1 = c_{24} \sin^2 e_1 + c_{34} \cos^2 e_1 - (c_{23} + c_{44}) \sin e_1 \cos e_1, \tag{3.35}$$

$$Z_1 = c_{44} \sin^2 e_1 + c_{33} \cos^2 e_1 - 2c_{34} \sin e_1 \cos e_1, \tag{3.36}$$

$$U_3 = c_{22} \sin^2 e_3 + c_{44} \cos^2 e_3 + 2c_{24} \sin e_3 \cos e_3, \tag{3.37}$$

$$V_3 = c_{24} \sin^2 e_3 + c_{34} \cos^2 e_3 + (c_{23} + c_{44}) \sin e_3 \cos e_3, \tag{3.38}$$

$$Z_3 = c_{44} \sin^2 e_3 + c_{33} \cos^2 e_3 + 2c_{34} \sin e_3 \cos e_3. \tag{3.39}$$

Similarly (U_2, V_2, Z_2) are taken with help of (U_1, V_1, Z_1) by exchanging e_1 by e_2 and (U_4, V_4, Z_4) are collected by (U_3, V_3, Z_3) on exchanging e_3 by e_4. We only solve one of all the equation in order to understand and for getting the relation.
Now the overall displacement given by Eq (3.28) and it must obey the traction free boundary because in monoclinic system we have $\tau_{23} = \tau_{33} = 0$ at $x_3 = 0$. The simple algebric equations thus yields,

$$\begin{aligned}
&\big[(c_{24}A_1 + (c_{44}B_1)) \sin e_1/c_1 - (c_{44}A_1 + (c_{34}B_1)) \cos e_1/c_1\big] e^{ip_1(x_2,0)} \\
&+\big[(c_{24}A_2 + (c_{44}B_2)) \sin e_2/c_2 - (c_{44}A_2 + (c_{34}B_2)) \cos e_2/c_2\big] e^{ip_2(x_2,0)} \\
&+\big[(c_{24}A_3 + (c_{44}B_3)) \sin e_3/c_3 + (c_{44}A_3 + (c_{34}B_3)) \cos e_3/c_3\big] e^{ip_3(x_2,0)} \\
&+\big[(c_{24}A_4 + (c_{44}B_4)) \sin e_4/c_4 + (c_{44}A_4 + (c_{34}B_4)) \cos e_4/c_4\big] e^{ip_4(x_2,0)} = 0
\end{aligned} \tag{3.40}$$

similarly

$$
\begin{aligned}
&\left[(c_{23}A_1+(c_{34}B_1))\sin e_1/c_1-(c_{34}A_1+(c_{33}B_1))cose_1/c_1\right]e^{ip_1(x_2,0)}\\
&+\left[(c_{23}A_2+(c_{34}B_2))\sin e_2/c_2-(c_{34}A_2+(c_{33}B_2))\cos e_2/c_2\right]e^{ip_2(x_2,0)}\\
&+\left[(c_{23}A_3+(c_{34}B_3))\sin e_3/c_3+(c_{34}A_3+(c_{34}B_3))\cos e_3/c_3\right]e^{ip_3(x_2,0)}\\
&+\left[(c_{23}A_4+(c_{34}B_4))\sin e_4/c_4+(c_{34}A_4+(c_{33}B_4))\cos e_4/c_4\right]e^{ip_4(x_2,0)}=0 \quad (3.41)
\end{aligned}
$$

Since above two equation must obeyed for each value of x_2, we have

$$P_1(x_2,0)=P_2(x_2,0)=P_3(x_2,0)=P_4(x_2,0), \quad (3.42)$$

equation (3.29) and (3.42) imply

$$\frac{\sin e_1}{c_1(e_1)}=\frac{\sin e_2}{c_2(e_2)}=\frac{\sin e_3}{c_3(e_3)}=\frac{\sin e_4}{c_4(e_4)}=\frac{1}{c_a}, \quad (3.43)$$

here c_a the apparent phase is the speed or velocity. It's kind of snell's law for a monoclinic medium.

From (3.32) (3.36) and (3.39) we notice that if $e_1=e_3$, $c_1\neq c_3$. Therefore,the reflection angle of qP waves does not identical just like incidence angle of (qP) waves. Likewise, reflection angle of qSV waves does not identical to the incidence angle of qSV waves. Assuming the reflection angle of qP (qSV) waves is similar to incidence angle of qP (qSV) waves, this is a reason the reflected coefficient collected in these studies are wrong. Consequently, $c_1=c_3$ if $e_1=e_3$. Above (3.43) highlights that angle of reflection of qP (qSV) waves is identical to angle of incidence qP(qSV) waves. By taking help from the relation (3.30), (3.42) and (3.43) in Eq (3.40) and (3.41) we have

$$a_1B_1+a_2B_2+a_3B_3+a_4B_4=0 \quad (3.44)$$

$$b_1B_1+b_2B_2+b_3B_3+b_4B_4=0 \quad (3.45)$$

here

$$a_1=c_{24}F_1+c_{44}-(c_{44}F_1+c_{34})\cot\ e_1, \quad (3.46)$$

$$a_2 = c_{24}F_2 + c_{44} - (c_{44}F_2 + c_{34}) \cot\ e_2, \tag{3.47}$$

$$a_3 = c_{24}F_3 + c_{44} + (c_{44}F_3 + c_{34}) \cot\ e_3, \tag{3.48}$$

$$a_4 = c_{24}F_4 + c_{44} + (c_{44}F_4 + c_{34}) \cot\ e_4, \tag{3.49}$$

$$b_1 = c_{23}F_1 + c_{34} - (c_{34}F_1 + c_{33}) \cot\ e_1, \tag{3.50}$$

$$b_2 = c_{23}F_2 + c_{34} - (c_{34}F_2 + c_{33}) \cot\ e_2, \tag{3.51}$$

$$b_3 = c_{23}F_3 + c_{34} + (c_{34}F_3 + c_{33}) \cot\ e_3, \tag{3.52}$$

$$b_4 = c_{23}F_4 + c_{34} - (c_{34}F_4 + c_{33}) \cot e_4. \tag{3.53}$$

In the case of qP waves $A_2 = B_2 = 0$ and Eq (3.44) and (3.45) becomes

$$a_1B_1 + a_3B_3 + a_4B_4 = 0 \tag{3.54}$$

$$b_1B_1 + b_3B_3 + b_4B_4 = 0 \tag{3.55}$$

After doing, we get the amplitude ratio in the form of

$$\frac{B_3}{B_1} = a_4B_1 - a_1B_4/\Delta, \tag{3.56}$$

$$\frac{B_4}{B_1} = a_1B_3 - a_3B_1/\Delta, \tag{3.57}$$

where

$$\Delta = a_3B_4 - a_4B_3, \tag{3.58}$$

referring Eq (3.30) we have

$$\frac{A_3}{A_1} = \frac{F_3}{F_1}\left(\frac{B_3}{B_1}\right), \frac{A_4}{A_1} = \frac{F_4}{F_1}\left(\frac{B_4}{B_1}\right). \tag{3.59}$$

In sense of incident qSV waves $A_1 = B_1 = 0$ so that we have

$$a_2B_2 + a_3B_3 + a_4B_4 = 0 \tag{3.60}$$

$$b_2B_2 + b_3B_3 + b_4B_4 = 0 \tag{3.61}$$

and the same process as above gives us the amplitude ratios of qSV waves which is

$$\frac{A_3}{A_2} = \frac{F_3}{F_2}\left(\frac{B_3}{B_2}\right), \frac{A_4}{A_2} = \frac{F_4}{F_2}\left(\frac{B_4}{B_2}\right). \tag{3.62}$$

As we know that for isotropic medium

$$c_1 = c_3 = [\lambda + 2\mu/\rho]^{1/2} = \alpha, \tag{3.63}$$

$$c_2 = c_4 = (\mu/\rho)^{1/2} = \beta, \tag{3.64}$$

$$e_1 = e_3, e_2 = e_4 = f, \tag{3.65}$$

$$\frac{\sin e}{\alpha} = \frac{\sin f}{\beta}, \tag{3.66}$$

$$F_1 = -F_3 = -\tan e, F_2 = -F_4 = \cot f, \tag{3.67}$$

$$a_1 = a_3 = 2\mu,\ \ a_2 = a_4 = -\mu \cos 2f/\sin^2 f, \tag{3.68}$$

$$b_1 = -b_3 = -2\mu(\alpha/\beta)^2 \cos 2f/\sin 2e, \tag{3.69}$$

$$b_2 = -b_4 = -2\ \mu\ \cot f. \tag{3.70}$$

Substituting those values in the equation in (3.56), (3.59) and (3.62) we get the amplitude ratio for isotropic half space in the shape of

$$\frac{A_3}{A_1} = -\frac{B_3}{B_1} = \frac{\sin 2e \sin 2f - (\alpha/\beta^2)\cos^2 2f}{\sin 2e \sin 2f + (\alpha/\beta^2)\cos^2 2f}, \tag{3.71}$$

$$\frac{A_3}{A_1} = \frac{\cot f}{\tan e}\left(\frac{B_4}{B_1}\right) = \frac{(\alpha/\beta^2)\cos^2 4f}{\sin 2e \sin 2f + (\alpha/\beta^2)\cos^2 2f}, \tag{3.72}$$

$$\frac{A_3}{A_2} = \frac{\tan e}{\cot f}\left(\frac{B_3}{B_2}\right) = \frac{4\sin^2 e \cos 2f}{\sin 2e \sin 2f + (\alpha/\beta^2)\cos^2 2f}, \tag{3.73}$$

$$\frac{A_4}{A_2} = -\frac{B_4}{B_2} = -\frac{A_3}{A_1} = \frac{B_3}{B_1}. \tag{3.74}$$

The above equations or expression signifies the ratios for amplitude for an isotropic half space matched by the accompanying results of Ben-Menahem and Singh [14].

3.3 Results and discussion

The reflection coefficients given via Chattopadhyay and Choudhary (1995) for the reflection of longitudinal and transverse waves at the plane free boundary of a monoclinic elastic half-space are wrong, because of two assumptions done via those authors, that is qP are longitudinal and (qSV) are transverse waves and the angle of reflection of the waves of qP (qSV) is the same as an angle of incidance of qP (qSV). In current study, we got the right reflection coefficients by evaluating the problem.

Eqs (3.56) and (3.59) are showing the amplitude ratios when qP waves are incident on a monoclinic elastic half space. From these equation, A_3/A_1 and A_4/A_1 are dimensions for components that are horizontal of the displacement and B_3/B_1 and B_4/B_1 are the ratios for components that are vertical of the displacement. Likewise, (3.59) gives the amplitude ratios for incident qSV waves. Equally by these (3.28) and (3.30), we notice for instance, entire displacement of the incident qP waves is

$$(A_1^2 + B_1^2)^{1/2} e^{iP_1}, \tag{3.75}$$

using equation (3.30) we have after applying summation i=(1,2,3,4) for i=1,

$$A_1 = F_1 B_1, \tag{3.76}$$

using this in above equation we found

$$(F_1^2 B_1^2 + B_1^2)^{1/2} e^{iP_1}, \tag{3.77}$$

$$B_1^2 (1 + F_1^2) e^{iP_1}, \tag{3.78}$$

similarly using index for 2, 3 ,4 we have

$$B_2^2 (1 + F_2^2) e^{iP_2}, \tag{3.79}$$

$$B_3^2 (1 + F_3^2) e^{iP_3}, \tag{3.80}$$

$$B_4^2 (1 + F_4^2) e^{iP_4}, \tag{3.81}$$

so on comparison and cancellation of above mentioned coefficients and upon division of (3.80) and (3.78) and (3.81) and (3.78) the reflective classes can be expressed individually

$$R_{PP} = \left(\frac{1 + F_3^2}{1 + F_1^2} \right)^{1/2} * \frac{B_3}{B_1}, R_{PS} = \left(\frac{1 + F_4^2}{1 + F_1^2} \right)^{1/2} * \frac{B_4}{B_1}, \tag{3.82}$$

for incident qP waves and

$$R_{SP} = \left(\frac{1+F_3^2}{1+F_2^2}\right)^{1/2} * \frac{B_3}{B_2}, R_{SS} = \left(\frac{1+F_4^2}{1+F_2^2}\right)^{1/2} * \frac{B_4}{B_2}, \tag{3.83}$$

for incident qSV waves.

The reflections coefficients present here are in means of four angles and velocities e_i and $c_i(e_i), i = 1, 2, 3, 4$. If we have incident qP wave, e_1 and, thats why, $c_1(e_1)$ is meant to find. We have to find e_3 and e_4 for provided e_1. The velocities $c_3(e_3)$ and $c_4(e_4)$ may then be calculated from precise algebric formula. We provide underneath the technique for solving e_3 and e_4 for given e_1 in the case of incident (qP) waves and for given e_2 in the case of incident qSV waves.

There is a Snell's law for monoclinic medium provided by (3.14) in which apparent velocity, may be taken as $c_a = c/p_2$, where $p(0, p_2, p_3)$ is the propagation vector. We specify the dimensionless apparent velocity through the relationship,

$$\bar{c} = c_a/\beta = c/(p_2\beta), \tag{3.84}$$

so the equation becomes

$$\bar{c}^4 - (\bar{U} + \bar{Z})\bar{c}^2 + (\bar{U}\bar{Z} - \bar{V}^2) = 0 \tag{3.85}$$

here

$$\bar{U} = p^2 + 2\bar{c_{24}} + \bar{c_{22}}, \tag{3.86}$$

$$\bar{V} = \bar{c_{34}}p^2 + (1 + \bar{c_{23}})p + \bar{c_{24}}, \tag{3.87}$$

$$\bar{Z} = \bar{c_{33}}p^2 + 2\bar{c_{34}} + 1. \tag{3.88}$$

for qP incidents, p= -cot e_1; for qSV incident, p =-cot e_2, for qP reflected, p= -cot e_3; for qSV reflected , p= -cot e_4, equation (3.85) done for two $\bar{c^2}$, two solutions for (qP) and (qSV) waves. After putting the above expression we have

$$g_0p^4 + g_1p^3 + g_2p^2 + g_3p + g_4 = 0, \tag{3.89}$$

where

$$g_0 = \bar{c_{33}} - \bar{c_{34}}^2, \tag{3.90}$$

$$g_1 = 2(\bar{c_{24}} c_{33} - \bar{c_{34}} c_{23}), \tag{3.91}$$

$$g_2 = 1 + \bar{c_{22}} c_{33} + 2\bar{c_{24}} c_{34} - (1 + \bar{c_{23}}^2) - (1 + \bar{c_{33}}^2)\bar{c}^2, \tag{3.92}$$

$$g_3 = 2[\bar{c_{22}} c_{34} - \bar{c_{23}} c_{24} - (\bar{c_{24}} + \bar{c_{34}})\bar{c}^2], \tag{3.93}$$

$$g_4 = \bar{c}^4 - (1 + \bar{c_{22}})\bar{c}^2 + \bar{c_{22}} - \bar{c_{24}}^2. \tag{3.94}$$

If we define $q = 1/p$ then (3.89) transform into

$$g_0 q^4 + g_3 q^3 + g_2 q^2 + g_1 q + g_0 = 0, \tag{3.95}$$

so above equation consist of two positive roots say q_3 and q_4. According to reflective SV and reflective P waves.

$$e_3 = \tan^{-1}(q_3), \quad e_4 = \tan^{-1}(q_4). \tag{3.96}$$

In case of isotropic medium we have

$$g_0 = \gamma, g_1 = 0, \tag{3.97}$$

$$g_2 = 2\gamma - (1 + \gamma)\bar{c}^2, g_3 = 0, \tag{3.98}$$

$$g_4 = (\bar{c}^2 - 1)(\bar{c}^2 - \gamma), \tag{3.99}$$

so equation (3.89) reduces to

$$\gamma p^4 + [2\gamma - (1 + \gamma)\bar{c}^2]p^2 + (\bar{c}^2 - 1)(\bar{c}^2 - \gamma) = 0, \tag{3.100}$$

i.e,

$$\gamma(p^2 - \bar{c}^2 + 1)(p^2 - \bar{c}^2/\gamma + 1) = 0. \tag{3.101}$$

however in the current scenario, the snell law take the form

$$\frac{\sin e}{\alpha} = \frac{\sin f}{\beta} = \frac{1}{c_a}, \tag{3.102}$$

so from this Eq (3.84) shows that

$$\bar{c} = c_a/\beta = \text{cosec} f = \sqrt{\gamma}\text{cosec } e. \tag{3.103}$$

Therefore the root are given by

$$p^2 = \bar{c}^2 - 1 = \cot^2 f, \tag{3.104}$$

according to SV waves, and more

$$p^2 = \bar{c}^2/\gamma - 1 = \cot^2 e, \tag{3.105}$$

instead to P waves. So, w e can select $(q = 1/p)$,
$q_1 = -\tan\ e$, $q_2 = -\tan\ f$, $q_3 = \tan\ e$, $q_4 = \tan\ f$.
This factor helps us in finding the reflection angle of longitudinal and transverse waves. In case of monoclinic medium (3.95) makes the form of quadratic in q^2 so $q_1 = -q_3$, $q_2 = -q_4$, so for monoclinic medium the angle of qP (qSV) waves is not identical to the incidence angle of qP (qSV) waves but it may be right for other classes but not for monoclinic. For numerical calculation, we supposed that $c_{22}/c_{44} = 19.8/6.67$, $c_{33}/c_{44} = 24.9/6.67$, $c_{23}/c_{44} = 7.8/6.67$, $c_{24}/c_{44} = c_{34}/c_{44} =$ C.

Figure (3.2) shows the reflection angle of longitudinal waves for the various values of incidence angle of qP waves after putting three values of C. In figure (3.2), for $C > 0$, the reflection angle is larger then incidence angle. On other hand, if the value of $C < 0$, the reflection angle is smaller then incidence angle. Figure (3.3) is representing the reflection angle of qSV (transverse) waves for the different values of angle of incidence of longitudinal waves. Similarly figure (3.4) and (3.5) are representing incident qSV waves. The changing of the reflection coefficients R_{pp} for the incident qP-reflected qP waves along with the incidence angle of qP waves is representing in graph (3.6). The variation of the reflection coefficients R_{ps}, R_{sp} and R_{ss} is being representing in the figure (3.7-3.9). From figure (3.6-3.9) we notice that the anisotropy has a magnificent and clear influence on reflection coefficients [18].

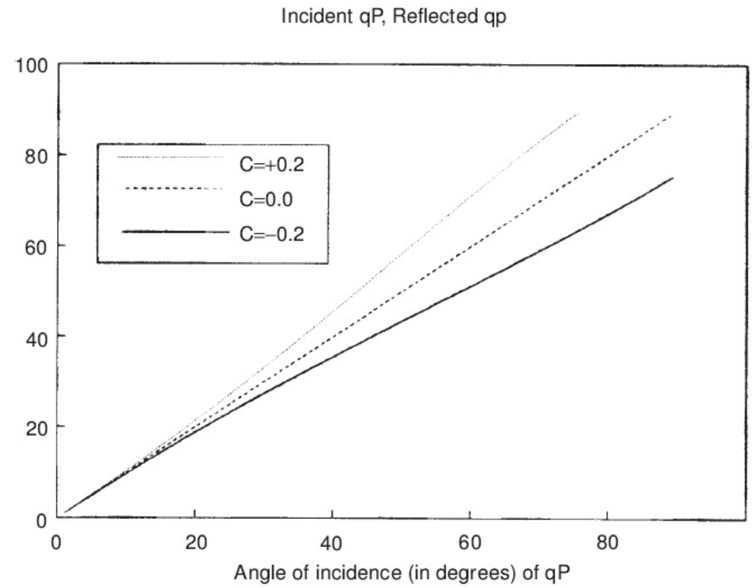

Figure 3.2: Change in reflection angle (e_3) of qP together with angle of incidence (e_1) of qP waves.

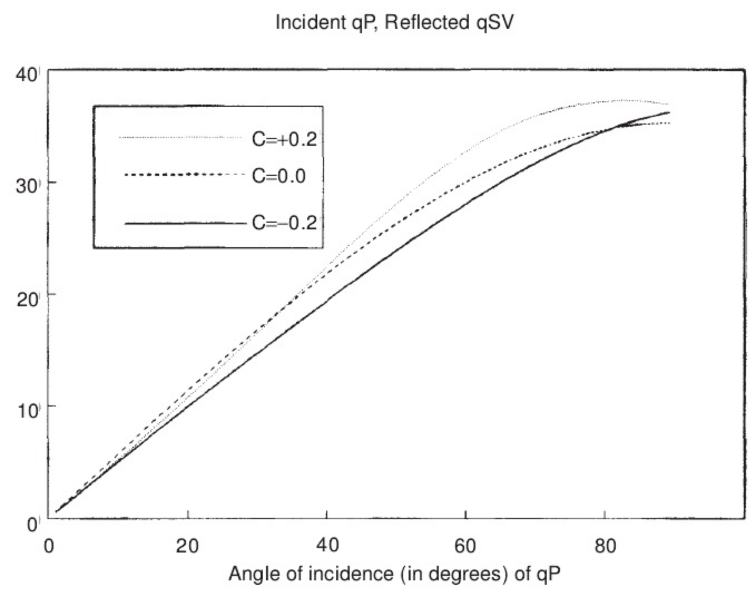

Figure 3.3: Change in reflection angle (e_4) of qSV waves together with incidence angle (e_1) of qP waves.

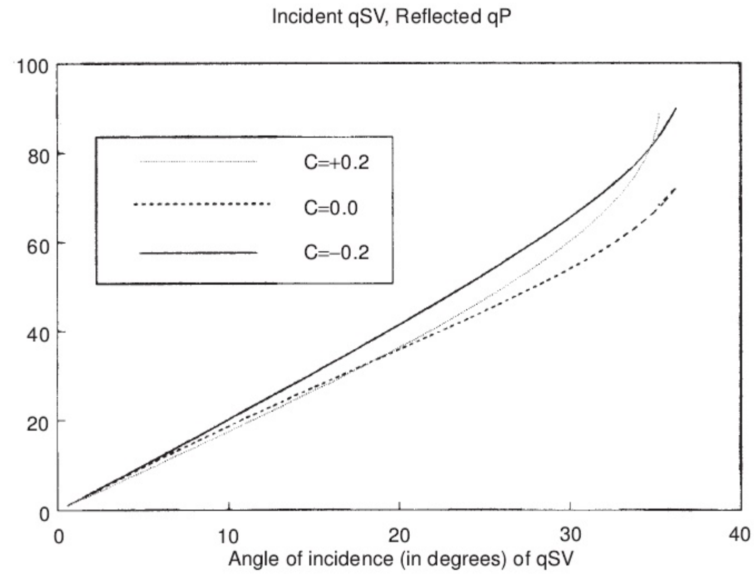

Figure 3.4: Change in reflection angle (e_3) of qP waves together with incidence angle (e_2) of qSV waves.

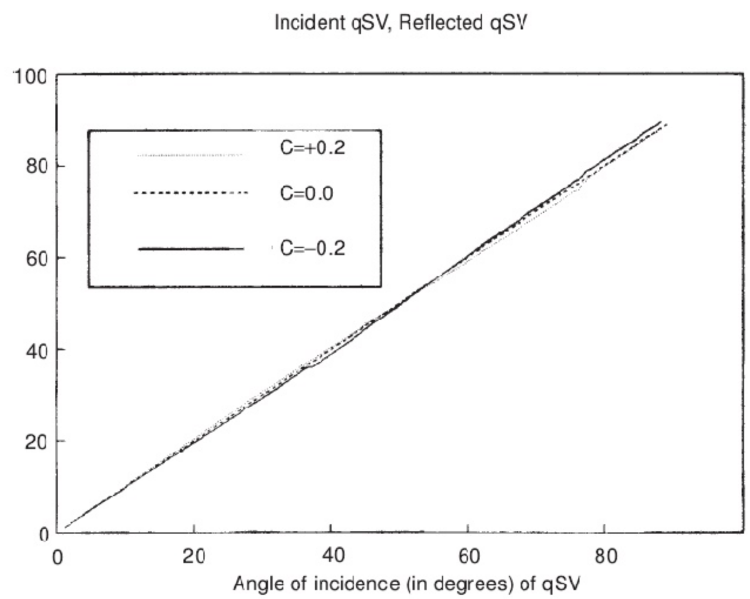

Figure 3.5: Change in reflection angle (e_4) of qSV waves together with incidence angle (e_2) of qSV waves.

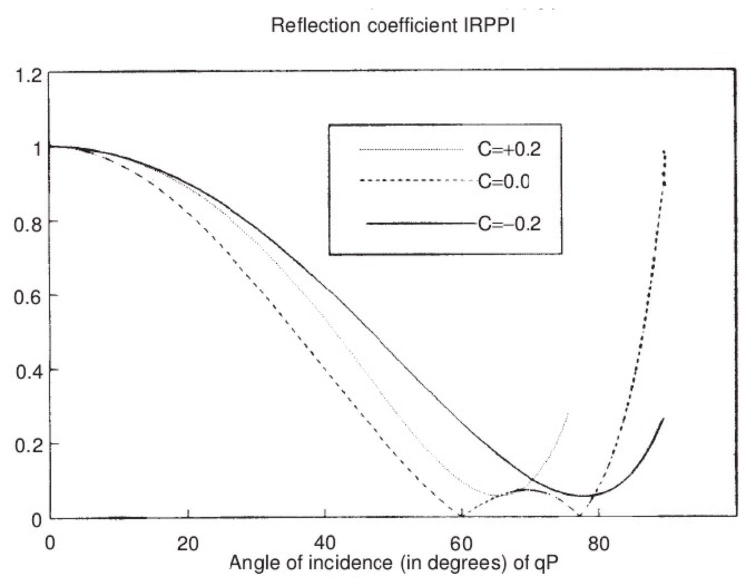

Figure 3.6: Change in reflection angle Coefficient $|R_{pp}|$ together with incidence (e_1)angle of qP waves.

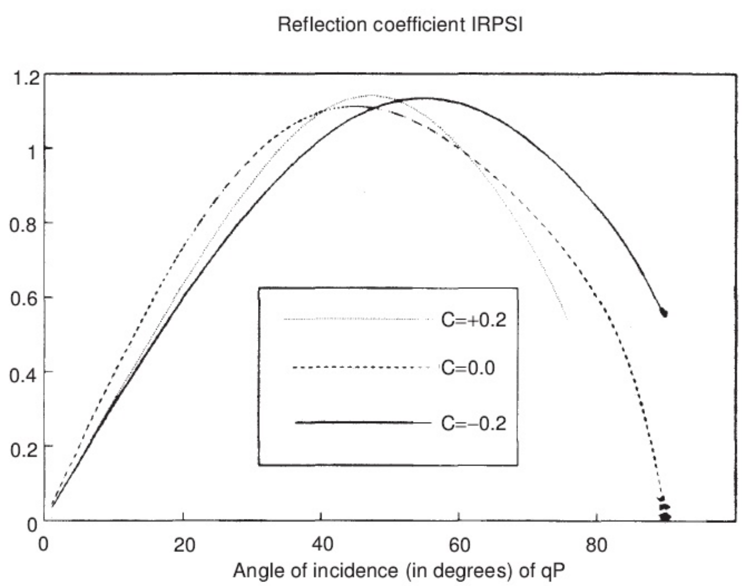

Figure 3.7: Change in reflection angle Coefficient $|R_{ps}|$ together with incidence (e_1)angle of qP waves.

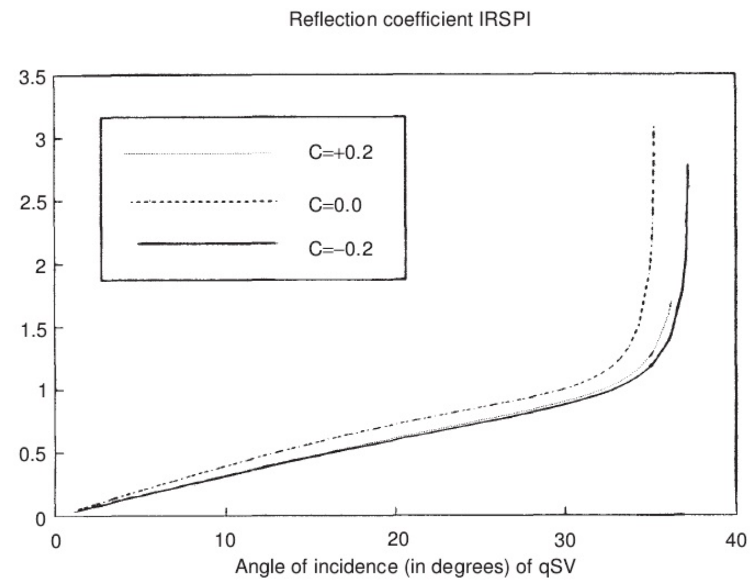

Figure 3.8: Change in reflection angle Coefficient $|R_{sp}|$ together with incidence (e_2)angle of qSV waves.

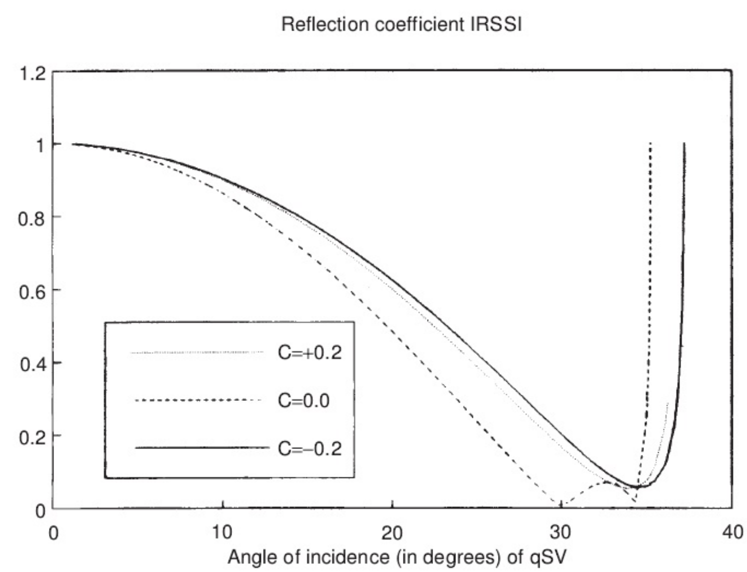

Figure 3.9: Change in reflection angle Coefficient $|R_{ss}|$ together with incidence (e_2)angle of qSV waves.

Chapter 4

Guided wave in an isotropic elastic solid with fluid interaction

4.1 Guided waves

In this chapter we cover the basic definition of guided wave: Guided waves are those that travel a long distance with little loss in energy. Guided wave method is a non-destructive evaluation method, from which the hidden information can be find out like under ground or in the depth of the oceans and also in the underground pipes. Guided waves are fitted best in the detection of internal or external metal loss. Guided waves are basically very different from conventional ultrasonic testing. The frequency used in the inspection depends on the thickness of the structure, but guided wave testing typically uses ultrasonic frequencies in the range of 10 kHz to several MHz Below the figure is representing the guided wave [3, 16].

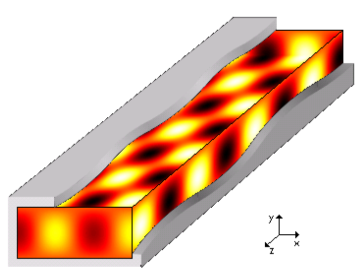

Figure 4.1: Guided wave

4.2 Crystal structure

A crystal shape is a different and regular arrangements of atoms in a crystal. A crystal shape consists of a unit cellular, and atoms are organized in a selected way; that's periodically repeated in 3 dimensions on a lattice. We can also say that crystals structure has been ordered manage nuclear, ions or invasions in a crystalline cloth. This system is ordered by it internal nature of the constituent rearrangement to shape symmetric styles that repeat along the major instructions of three-dimensional area. The smallest group of the inside of the cloth is the unit which reproduce the unit Shape cellular. The unit cell defines exactly the crystalline formula and shape flexibility, which is created by translating unit cell repetition with its most important axis. Repeated patterns are kept on the perspective Bravis flexible. The most important axis, or edges of unit seals and angle lengths and there are elastic obstacles between them, it is also called as an lattice parameters [8, 17].

4.3 Transverse isotropic materials

The transverse transmission is one with isotropic material bodily homes that are symmetric about an axis that is regular to a plane of isotropy. This transverse plane has infinite planes of symmetry and for that reason, inside this plane, the material residences are the same in all instructions. A transversely isotropic fabric has countless rotational symmetry about a completely unique route. In different words, there's a route about which you can rotate it with out changing its appearance [7, 9].

4.4 Derivation through the displacement equation

For the derivation further we take the displacement equation and as we know that

$$T_{ij,j} = \rho u_i, \tag{4.1}$$

where

$$T_{ij} = c_{ijkl} \frac{\partial u_k}{\partial x_l}. \tag{4.2}$$

The expression of u is

$$u = \frac{\partial \phi}{\partial r} + \frac{1}{r}\frac{\partial \psi_z}{\partial \theta} - \frac{\partial \psi_\theta}{\partial z}, \tag{4.3}$$

similarly for v and w are

$$v = \frac{1}{r}\frac{\partial \phi}{\partial \theta} + \frac{\partial \psi_r}{\partial z} - \frac{\partial \psi_z}{\partial r}. \tag{4.4}$$

Now due to the nature few terms are not linked and they vanish so after simplification the terms are

$$u_r = \frac{\partial \phi}{\partial r} + \frac{1}{r}\frac{\partial \psi_\chi}{\partial \theta}, \tag{4.5}$$

$$u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} - \frac{\partial \chi}{\partial r}, \tag{4.6}$$

and

$$u_z = \frac{\partial \psi}{\partial z}. \tag{4.7}$$

So now expanding the Eq (4.1)

$$T_{11,1} + T_{21,1} + T_{31,1} = \rho \ddot{u}_1, \tag{4.8}$$

$$T_{12,2} + T_{22,2} + T_{32,2} = \rho \ddot{u}_2, \tag{4.9}$$

$$T_{13,3} + T_{23,3} + T_{33,3} = \rho \ddot{u}_3. \tag{4.10}$$

We solve (4.8) according to the r θ and z

$$T_{11,1} = \frac{\partial}{\partial x_l} = \left(c_{11kl} \frac{\partial u_k}{\partial x_l} \right), \tag{4.11}$$

so collecting all terms w.r.t "r"

$$\frac{\partial}{\partial r}\left[c_{11kl}\frac{\partial u_k}{\partial x_l} + c_{21kl}\frac{\partial u_k}{\partial x_l} + c_{31kl}\frac{\partial u_k}{\partial x_l} \right] = \rho \ddot{u}_1, \tag{4.12}$$

like wise with respect to θ and z we have,

$$\frac{\partial}{\partial \theta}\left[c_{12kl}\frac{\partial u_k}{\partial x_l} + c_{22kl}\frac{\partial u_k}{\partial x_l} + c_{32kl}\frac{\partial u_k}{\partial x_l} \right] = \rho \ddot{u}_2, \tag{4.13}$$

$$\frac{\partial}{\partial z}\left[c_{13kl}\frac{\partial u_k}{\partial x_l} + c_{23kl}\frac{\partial u_k}{\partial x_l} + c_{33kl}\frac{\partial u_k}{\partial x_l} \right] = \rho \ddot{u}_3. \tag{4.14}$$

In order to prevent from long calculation we would solve them one by one and then compile in the end for the sake of convineance, so apply summation on k and l we get

$$\frac{\partial}{\partial x_l}.\frac{\partial u_k}{\partial x_l}\Big(c_{11kl}+c_{21kl}+c_{31kl}\Big),$$

$$\begin{aligned} T_{11,1} &= \frac{\partial}{\partial x_r}\left[\frac{\partial u_1}{\partial x_l}+\Big(c_{111l}+c_{211l}+c_{311l}\Big)+\frac{\partial u_2}{\partial x_l}+\Big(c_{112l}+c_{212l}+c_{312l}\Big)\right. \\ &+ \left.\frac{\partial u_3}{\partial x_l}\Big(c_{113l}+c_{213l}+c_{313l}\Big)\right]. \end{aligned} \tag{4.15}$$

Like wise we apply the summation on l and we get a lots of terms in which we pre-define the terms that are going to zero. According to the property of hexagonal any indices of $1, 2, 3$ if coming odd times tends to be zero and hence this would reduce our number of components. The only components that exist after cancellation of others are

$$c_{11}, c_{12}, c_{44}, c_{66}, c_{13}. \tag{4.16}$$

These are only five components that exists, so the Ist equation of r is

$$\frac{\partial}{\partial r}\left[\frac{\partial u_r}{\partial r}c_{11}+\frac{\partial u_\theta}{\partial r}c_{66}+\frac{\partial u_z}{\partial r}c_{44}+\frac{\partial u_r}{\partial \theta}c_{66}+\frac{\partial u_\theta}{\partial \theta}c_{12}+0+\frac{\partial u_r}{\partial z}c_{44}+0+\frac{\partial u_z}{\partial z}c_{13}\right]. \tag{4.17}$$

Here $1, 2$ *and* 3 are representing r θ and z now the next term according to θ

$$\frac{\partial}{\partial x_2}.\frac{\partial u_k}{\partial x_l}\Big(c_{12kl}+c_{22kl}+c_{32kl}\Big),$$

apply the same method for θ and z as above gives,

$$\frac{\partial}{\partial \theta}\left[\frac{\partial u_r}{\partial r}c_{21}+\frac{\partial u_\theta}{\partial r}c_{66}+\frac{\partial u_r}{\partial \theta}c_{66}+\frac{\partial u_\theta}{\partial \theta}c_{22}+\frac{\partial u_z}{\partial \theta}c_{44}+0+\frac{\partial u_\theta}{\partial z}c_{44}+0+\frac{\partial u_z}{\partial z}c_{23}\right]. \tag{4.18}$$

$$\frac{\partial}{\partial z}\left[\frac{\partial u_r}{\partial \theta}c_{31}+\frac{\partial u_z}{\partial r}c_{55}+\frac{\partial u_\theta}{\partial \theta}c_{23}+\frac{\partial u_z}{\partial \theta}c_{44}+\frac{\partial u_r}{\partial z}c_{55}+0+\frac{\partial u_\theta}{\partial z}c_{44}+0+\frac{\partial u_z}{\partial z}c_{33}\right]. \tag{4.19}$$

Now substitution of the values in (4.17) to (4.19)

$$u_r = \frac{\partial \phi}{\partial r}+\frac{1}{r}\frac{\partial \psi_\chi}{\partial \theta}, \tag{4.20}$$

$$u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta}-\frac{\partial \chi}{\partial r}, \tag{4.21}$$

and

$$u_z = \frac{\partial \psi}{\partial z}. \tag{4.22}$$

Picking up the first term of r we have

$$\frac{\partial}{\partial r}\left[\frac{\partial u_r}{\partial r}c_{11} + \frac{\partial u_\theta}{\partial r}c_{66} + \frac{\partial u_z}{\partial r}c_{44} + \frac{\partial u_r}{\partial \theta}c_{66} + \frac{\partial u_\theta}{\partial \theta}c_{12} + 0 + \frac{\partial u_r}{\partial z}c_{44} + 0 + \frac{\partial u_z}{\partial z}c_{13}\right] = \rho\frac{\partial^2 u_r}{\partial t^2}. \tag{4.23}$$

Now using the values of Eq (4.20) to (4.22) in (4.23).

$$\begin{aligned} T_{rr,r} &= \frac{\partial}{\partial r}\left[\frac{\partial}{\partial r}\left(\frac{\partial \phi}{\partial r} + \frac{1}{r}\frac{\partial \psi_\chi}{\partial \theta}\right)c_{11} + \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta} - \frac{\partial \chi}{\partial r}\right)c_{66} + \frac{\partial}{\partial r}\left(\frac{\partial \psi}{\partial z}\right)c_{44}\right. \\ &+ \left.\frac{\partial}{\partial \theta}\left(\frac{\partial \phi}{\partial r} + \frac{1}{r}\frac{\partial \chi}{\partial \theta}\right)c_{66}\right] + \left[\frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta} - \frac{\partial \chi}{\partial r}\right)c_{12} + \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial r} + \frac{1}{r}\frac{\partial \psi_\chi}{\partial \theta}\right)c_{44}\right. \\ &+ \left.\frac{\partial}{\partial z}\left(\frac{\partial \psi}{\partial z}\right)c_{13}\right]. \end{aligned} \tag{4.24}$$

Also we know that

$$c_{66} = (c_{11} - c_{12})/2, \tag{4.25}$$

and we replace the value ahead and now expanding the derivative of r accordingly we get

$$\begin{aligned} T_{rr,r} &= \frac{\partial}{\partial r}\left(\frac{\partial^2 \phi}{\partial r} + \frac{1}{r}\frac{\partial^2 \chi}{\partial \theta \partial r} - \frac{1}{r^2}\frac{\partial \chi}{\partial \theta}\right)c_{11} + \left(\frac{1}{r}\frac{\partial^2 \phi}{\partial \theta \partial r} - \frac{1}{r^2}\frac{\partial \phi}{\partial \theta} - \frac{\partial^2 \chi}{\partial r^2}\right)c_{66} \\ &+ \left(\frac{\partial^2 \phi}{\partial r \partial z}\right)c_{44} + c_{66}\left(\frac{\partial^2 \phi}{\partial r \partial \theta} + \frac{1}{r}\frac{\partial^2 \chi}{\partial \theta^2}\right) + c_{12}\left(\frac{1}{r}\frac{\partial^2 \phi}{\partial \theta^2} - \frac{\partial^2 \chi}{\partial \theta \partial r}\right) \\ &+ c_{44}\left(\frac{\partial^2 \phi}{\partial r \partial z} + \frac{1}{r}\frac{\partial^2 \chi}{\partial \theta \partial z}\right) + c_{13}\left(\frac{\partial^2 \psi}{\partial z^2}\right) = \rho\frac{\partial^2 u_r}{\partial t^2}. \end{aligned} \tag{4.26}$$

Similarly in the same way we do for the next two equations so after comparing the external derivative which cancel with the right hand side derivative and re-arranged will gives us

$$c_{11}\nabla^2\phi + c_{44}(\frac{\partial^2 \phi}{\partial z^2}) + (c_{13} + c_{44})\frac{\partial^2 \psi}{\partial z^2} = \rho_1\frac{\partial^2 \phi}{\partial t^2}. \tag{4.27}$$

The other two terms are

$$(c_{13} + c_{44})\nabla^2\phi + c_{44}\nabla^2\psi + c_{33}\frac{\partial^2 \phi}{\partial z^2} = \rho_1\frac{\partial^2 \psi}{\partial t^2}, \tag{4.28}$$

and

$$c_{44}\frac{\partial^2\chi}{\partial z^2}+(c_{11}-c_{12})/2\nabla^2\chi=\rho_1\frac{\partial^2\chi}{\partial t^2}. \tag{4.29}$$

Where c_{11} and others are elastic constants for the material. Now assume that any point in the cylinder,

$$\varphi=\sum B_nJ_n(\beta r)\cos(n\theta)\exp i(kz-\omega t), \tag{4.30}$$

likewise (4.31)

$$\psi=\sum c_nJ_n(\beta r)\cos(n\theta)\exp i(kz-\omega t), \tag{4.32}$$

$$\chi=\sum D_nJ_n(\gamma r)\sin(n\theta)\exp i(kz-\omega t). \tag{4.33}$$

where J is showing the bessel function of the first kind and βr and γr are to be determine. The k and ω are the wave number and the frequency of the waves propagating in the cylinder. We have assume these equation for the sake of continuous functions and their continuous derivatives, because the equations we found earlier actually demands double derivatives in order to found the ratio like amplitude and other values. So this is a type of bessel function from which we can get a continuous functions. Now if we put all the above three assumed functions in the earlier found equations we can get the solution in the form of square matrix after simplification.

$$\begin{bmatrix} c_{11}\beta^2-(\rho_1\omega^2-c_{44}k^2) & (c_{13}+c_{44})k^2 & 0 \\ (c_{13}+c_{44})\beta^2 & c_{44}\beta^2-(\rho_1\omega^2-c_{33}k^2) & 0 \\ 0 & 0 & \frac{1}{2}(c_{11}-c_{12})\gamma^2- \\ & & (\rho_1\omega^2-c_{44}k^2) \end{bmatrix}\begin{bmatrix} B_n \\ c_n \\ D_n \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{4.34}$$

This is the matrix formed after solving and re-arranging all the terms of the solved three equations. The next task is to find the values of the unknown from the above matrix. For extraordinary or non-trivial solutions β must be determined from equation,

$$c_{11}c_{44}\beta^4-E\beta^2+F=0. \tag{4.35}$$

This equation is found by solving the above matrix after taking its determinant and re-arranging the terms for the sake of convenance, we named the equation like

$$E=(c_{13}+c_{44})K^2+c_{11}(\rho_1\omega^2-c_{33}k^2)+c_{44}(\rho_1\omega^2-c_{44}k^2), \tag{4.36}$$

and

$$F = (\rho_1\omega^2 - c_{44}k^2)(\rho_1\omega^2 - c_{33}k^2). \tag{4.37}$$

Also the parameter γ is found to be

$$\gamma^2 = \frac{2(\rho_1\omega^2 - c_{44}k^2)}{c_{11} - c_{12}}, \tag{4.38}$$

similarly

$$\beta_1^2 = (E - \Delta)/(2c_{11}c_{44}), \quad \beta_2^2 = (E + \Delta)/(2c_{11}c_{44}). \tag{4.39}$$

Where

$$\Delta = \sqrt{E^2 - 4c_{11}c_{44}F}, \tag{4.40}$$

and the amplitude ratios q_1 and q_2 are found by

$$q_1 = \frac{c_{11}\beta_1^2 + (c_{44}k_2 - \rho_1\omega^2)}{(c_{13} + c_{44})k_2}, \tag{4.41}$$

$$q_2 = -\frac{(c_{13} + c_{44})k_2}{c_{11}\beta_1^2 + (c_{44}k_2 - \rho_1\omega^2)}. \tag{4.42}$$

This is the whole method in order to find the amplitude and the values of other given parameters from the above (4.34) square matrix.

4.5 Dispersion relation

The dispersion relation for the extensional wave propagation for fully saturated, homogeneous, isotropic, porous, round cylinders, subjected to stress-loose open-pore boundary situation was first studied by way of Gartner (1962). Under the attention that the related axial wavelengths are large than the radius of the cylinder in bodily sciences and electric engineering. Dispersion members of the family describes the impact of dispersion in a medium on the residences of a wave journeying inside that medium. A dispersion relation relates the wavelength or wave quantity of a wave to its frequency, from this relation the segment pace and institution pace of the wave have handy expressions which then determine the refractive index of the medium.

4.5.1 Dispersion relation of the free cylinder

In order to find the dispersion relation for the free cylinder we again take the help from previous solved Hooks law (**??**) Also for the transverse isotropic material the components are founded by the hooks law as given below

$$\begin{aligned} T_{rr} &= c_{1111}e_{11} + c_{1112}e_{12} + c_{1113}e_{13} + c_{1121}e_{21} \\ &+ c_{1122}e_{22} + c_{1123}e_{23} + c_{1131}e_{31} + c_{1132}e_{32} + c_{1133}e_{33}. \end{aligned} \tag{4.43}$$

Now we contract (4.70) and get

$$T_{rr} = c_{11}e_{11} + c_{16}e_{12} + c_{15}e_{13} + c_{16}e_{21} + c_{12}e_{22} + c_{14}e_{23} + c_{15}e_{31} + c_{14}e_{32} + c_{13}e_{33}, \tag{4.44}$$

and these are already discussed above and as per the value of the hexagonal, any index of 1 2 and 3 if comes odd times tends to zero and finally we obtain

$$\delta_{rr} = c_{11}e_{11} + c_{12}e_{22} + c_{13}e_{33}. \tag{4.45}$$

So if we give r=1 θ=2 and z=3 we have

$$\delta_{rr} = c_{11}e_{rr} + c_{12}e_{\theta\theta} + c_{13}e_{zz}, \tag{4.46}$$

like wise we also get the other two equations

$$\delta_{r\theta} = (c_{11-c_{12}})e_{r\theta}, \tag{4.47}$$

$$\delta_{rz} = 2c_{44}e_{rz}. \tag{4.48}$$

The next step is to put the potential functions that we discussed previously and before apply we replace "r" that is radius of the cylinder with "a" because this is operating on the free rod and there is no change in the radius so this is the reason of replacing for the sake of convenance and our potential function becomes

$$\varphi = \sum B_n J_n(\beta a)\cos(n\theta)expi(kz - \omega t), \tag{4.49}$$

Like wise

$$\psi = \sum c_n J_n(\beta a)\cos(n\theta)expi(kz - \omega t), \tag{4.50}$$

$$\chi = \sum D_n J_n(\gamma a)\sin(n\theta)expi(kz - \omega t). \tag{4.51}$$

So there are total nine components for which we have solved three of them and we put the potential functions (4.49) to (4.51) in all nine components and among all the components one of them is given below

$$\delta_{rr} = c_{11}e_{11} + c_{12}e_{22} + c_{13}e_{33}, \tag{4.52}$$

basically it would be solved by the hooks law as

$$\delta_{rr} = \frac{\partial}{\partial a}\left[\frac{\partial}{\partial a}\left(\frac{\partial \phi}{\partial a} + \frac{1}{a}\frac{\partial \chi}{\partial \theta}\right)c_{11} + \frac{\partial}{\partial \theta}\left(\frac{1}{a}\frac{\partial \phi}{\partial \theta} - \frac{\partial \chi}{\partial a}\right)c_{12} + \left(\frac{\partial \psi}{\partial z}\right)c_{13}\right], \tag{4.53}$$

now put the potential functions in (4.53)

$$\begin{aligned}
\delta_{rr} &= \frac{\partial}{\partial a}c_{11}\left[\frac{\partial}{\partial a}\left(\frac{\partial}{\partial a}\left(\sum B_n J_n(\beta a)\cos(n\theta) exp\iota(kz - \omega t)\right)\right)\right] \\
&+ \frac{1}{a}\frac{\partial}{\partial \theta}\left(\sum D_n J_n(\gamma a)\sin(n\theta) exp\iota(kz - \omega t)\right) \\
&+ c_{12}\left[\frac{\partial}{\partial \theta}\frac{1}{a}\left(\frac{\partial}{\partial \theta}\left(\sum B_n J_n(\beta a)\cos(n\theta) exp\iota(kz - \omega t)\right)\right)\right] \\
&- \frac{\partial}{\partial a}(\sum D_n J_n(\gamma a)\sin(n\theta) exp\iota(kz - \omega t)) \\
&+ c_{13}\frac{\partial}{\partial z}(\sum c_n J_n(\gamma a)\cos(n\theta) exp\iota(kz - \omega t)) = \rho\frac{\partial^2 u_a}{\partial t^2}.
\end{aligned} \tag{4.54}$$

After operating all the derivatives and cancellation of the repeated expression on both sides of the equations and going through the same methods for all the nine components for which one of them is solved above, we would separate and make the matrix showing all the values of nine components. The matrix is given below

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} B_n \\ c_n \\ D_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{4.55}$$

where the elements a_{ij} of the matrix are founded and values are given below.

$$\begin{aligned}
a_{11} =& c_{11}J_n^{''}(\beta_1 a)(\beta_1 a)^2 + c_{12}J_n^{'}(\beta_1 a)\beta_1 a - n^2 J_n(\beta_1 a) - c_{13}(ka)^2 q_1 J_n(\beta_1 a),\\
a_{12} =& c_{11}q_2 J_n^{''}(\beta_2 a)(\beta_2 a)^2 + c_{12}q_2 J_n^{'}(\beta_2 a)\beta_2 a - q_2 n^2 J_n(\beta_2 a) - c_{13}(ka)^2 J_n(\beta_2 a),\\
a_{13} =& -n(c_{11} - c_{12})[J_n^{'}(\gamma a) - (\gamma a)J_n^{'}(\gamma a)],\\
a_{21} =& 2nJ_n[(\beta_1 a) - (\beta_1 a)J_n^{'}(\beta_1 a)],\\
a_{22} =& 2nq_2[J_n(\beta_2 a) - (\beta_2 a)J_n^{'}(\beta_2 a)],\\
a_{23} =& -J_n^{''}(\gamma a)(\gamma a)^2 + J_n^{'}(\gamma a)\gamma a - n^2 J_n(\gamma a),\\
a_{31} =& (1+q_1)\beta_1 a J_n^{'}(\beta_1 a),\\
a_{32} =& (1+q_2)\beta_2 a J_n^{'}(\beta_2 a),\\
a_{33} =& nJ_n(\gamma a).
\end{aligned} \tag{4.56}$$

4.6 Secular equation

Secular equation is used in matrices and is every other call of the feature equation. The cause is secular because it become first utilized in calculation referring to the planetary motion. Here secular method give one direction of change with time as opposed to periodic. If u have

$$f(t) = at + b \sin t, \tag{4.57}$$

then the "a" is the secular part and "b sint" is the periodic part.

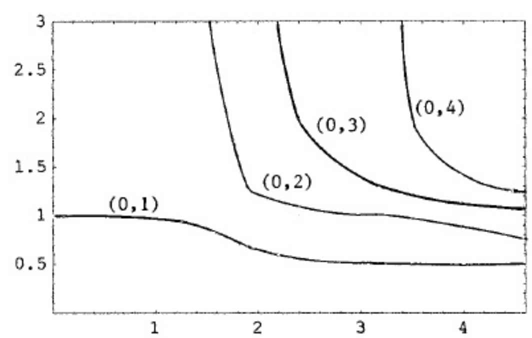

Figure 4.2: The display is curved for the first four long-term modes of cobalt cylinder.

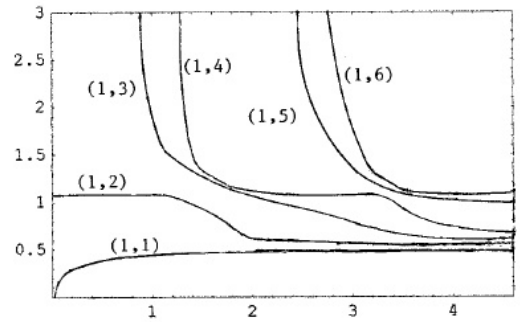

Figure 4.3: The display is curved for the first six long-term modes of cobalt cylinder. $(i.e, n = 1)$ of flexural modes

These are the figure that demonstrate the behaviour of the waves at different modes. Figure 4.2 representing the dispersion curves for the longitudinal waves for a cobalt cylinder. Similarly Fig 4.3 is represented the behaviour of six curves of flexural mode.

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