Numerical solutions for steady flow problems involving nanofluids and regular fluids

by

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In the name of Allah, the most beneficent, the most merciful.

Dedicated to

My loving parents.

Abstract

In this dissertation we discuss numerical solutions for steady flow problems involving nanofluids and regular fluids. The governing partial differential equations (PDEs) are reduced into the system of non-linear ordinary differential equations (ODEs), while considering appropriate similarity transformations. The numerical solution of the resulting non-linear ODEs are obtained by using shooting method with the fifth order Runge-Kutta time integration technique and the results are compared with the built-in solver *bvp4c* of MATLAB. Graphs are drawn for the influence of various parameters on the flow field. The present analysis shows the effects of velocity ratio on the flow of the field and effect of thermophoresis parameter, Brownian motion, Prandtl and Schmidt numbers on the temperature and concentration profile, respectively. We find a good agreement between our results and the results in the literature.

Preface

Nanofluids have become an active field of research due to its applications in technological and industrial processes. Application of nanofluids includes microelectronics, fuel cells, biomedicine, engine cooling, domestic refrigerator, chiller, heat exchanger, nuclear reactor coolant. Research on nanofluids have potential to improve the heat transfer and energy efficiency in industrial and engineering areas including industrial coolants, smart fluids, removal of geothermal power, nanofluids in automobile fuels, brake fluids, car radiator coolant, microelectronics cooling, bio and pharmaceutical industry. Assorted benefits of the applications of nanofluids include improved heat transfer, heat transfer system size reduction, minimal clogging, microchannel cooling, and reduction of systems. It is important to mention here that nanotechnology is widely used in the industry because the materials having nanometer sized particles possess unique chemical and physical properties. The size of these nanoparticles in diameter is less than 100nm. The arrangement of the dissertation is as follows:

Chapter 1 is introductory in nature. It presents some basic definitions. Some details about the shooting method and *bvp4c* method is also part of this chapter.

Chapter 2 is the review work of Meraj et al [1]. It is concerned with the numerical solution for the stagnation-point flow of nanofluid over an exponentially stretching sheet. First, we convert partial differential equations (PDEs) into a system of non-linear ordinary differential equation (ODEs) and than solve this system of non-linear ODEs by using shooting method and *bvp4c*. The numerical analysis of the obtained results is presented at the end of this chapter.

Chapter 3 is the review work of Elbashbeshy et al [2]. It is concerned with the effects of temperature-dependent viscosity on heat transfer over a continuous moving surface. First we convert partial differential equations (PDEs) into system of non-linear ordinary differential equation (ODEs) and than solve this system of non-linear ODEs by using shooting method and *bvp4c*. The numerical analysis of the obtained results is presented at the end of this chapter.

Chapter 4 contains the conclusion of the thesis and future work.

Acknowledgements

In the name of almighty Allah, the merciful, the beneficent, all praises (belong) to Allah alone, the Cherisher and Sustainer of the world. He is the First, he is the Last, He is the hidden, and He knows about everything. He brings the night into the day and brings the day into the night, and He knows the thoughts of hearts (Al-Quran).

I have the pearls of my eyes to admire countless blessings of Allah Almighty because the words are bound, knowledge is limited and time of life is too short to express His dignity. It is the one of His infinite benedictions that He bestowed upon me with the potential and ability to complete the present research program and to make a meek contribution to the deep oceans of knowledge already existing.

My special praise for the Holy Prophet HAZRAT MUHAMMAD (Peace Be Upon Him), the greatest educator, the everlasting source of guidance and knowledge for humanity. He taught the principles of morality and eternal values.

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Contents

Chapter 1

Introduction

This chapter includes some basic definitions used in this research work.

1.1 Nanofluids

The nanofluids are comparatively new class of fluids which contain base fluid with nanometer sized particle (1-100 nm) called nanoparticles, suspended within them. The nanoparticles used in nanofluids are usually made of metals (Cu, Ag, Au, Al, Fe), oxide ceramics (Al_2O_3, CuO, TiO_2), nitride ceramics (AlN, SiN), carbide ceramics (SiC, tiC), semiconductors and carbon nanotubes. In general, base fluids include water, ethylene, glycol and oil. Nanofluids possess better thermophysical properties such as thermal conductivity, thermal diffusivity, viscosity and convective heat transfer coefficients compared with base fluids like oil or water. Nanofluids are valuable in many applications in heat transfer including microelectronics, fuel cells, biomedicine, engine cooling, domestic refrigerator, chiller, heat exchanger and nuclear reactor coolant.

1.2 Dynamic Viscosity

Dynamic viscosity is defined as the proportion of shear stress to the rate of strain. It is denoted by μ.

$$\mu = \frac{\text{shear stress}}{\text{strain rate}} \tag{1.2.1}$$

The unit of dynamic viscosity is $\frac{kg}{ms}$.

1.3 Kinematic Viscosity

The ratio of dynamic viscosity μ to the density ρ of the fluid is called kinematic viscosity. Generally it is denoted by ν.

$$\nu = \frac{\mu}{\rho} \tag{1.3.1}$$

The unit of kinematic viscosity is $\frac{m^2}{s}$.

1.4 Types of Flow

Following are some important types of flows given below.

1.4.1 Steady Flow

If the flow velocity differs from point to point but does not change with time than the flow is called " steady".

1.4.2 Incompressible Flow

If the density ρ does not change with time and space than the flow is called incompressible. Mathematically, it can be written as

$$\frac{d\rho}{dt} = 0. \tag{1.4.1}$$

1.4.3 Laminar Flow

Laminar flow is a kind of fluid flow in which the fluid travels smoothly.

1.5 Law of Conservation of Mass (Continuity Equation)

Let us consider a fixed volume in space (see Figure 1.1). The rate of increase of mass inside the volume is

$$\frac{d}{dt}\int_v \rho dV = \int_v \frac{\partial \rho}{\partial t} dV, \tag{1.5.1}$$

where V is the volume. Since the volume is fixed, so $\frac{d}{dt}$ can be use inside the integral. The rate of mass flow away from the volume is the surface integral

$$\int_A \rho V \cdot dA, \tag{1.5.2}$$

because of the positive sign $\rho V.dA$ is the outward flux. As a result of the law of conservation of mass"rate of increase of mass within a fixed volume must be equal to the rate of inflow throughout the boundaries", that is

$$\int_v \frac{\partial \rho}{\partial t} dV = -\int_A \rho V.dA. \tag{1.5.3}$$

Using the divergence theorem a surface integral can be transformed into the volume integral and so

$$\int_A \rho V.dA = \int_v \nabla.(\rho V) dV. \tag{1.5.4}$$

From Eqs. (1.5.3) and (1.5.4), we get

$$\int_v \Big[\frac{\partial \rho}{\partial t} + \nabla.(\rho V)\Big] dV = 0. \tag{1.5.5}$$

The above equation is satisfied for any volume which is achievable when the integrand vanishes. This requires

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho V) = 0. \tag{1.5.6}$$

The equation (1.5.6) is called Mass conservation equation.

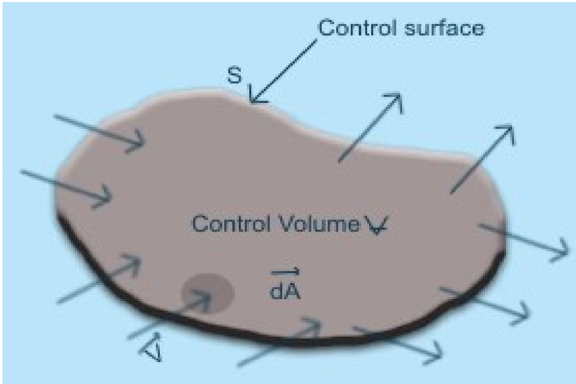

Figure 1.1

1.6 Different Forms of Continuity Equation

ρ is constant for incompressible fluid so

$$\frac{\partial \rho}{\partial t} = 0, \tag{1.6.1}$$

so continuity equation becomes

$$\nabla \cdot (\rho \mathbf{V}) = 0, \tag{1.6.2}$$

which represents a steady flow. Since ρ is constant therefore above equation becomes

$$\rho(\nabla \cdot \mathbf{V}) = 0. \tag{1.6.3}$$

And $\rho \neq 0$, so

$$\nabla \cdot \mathbf{V} = 0. \tag{1.6.4}$$

1.7 Law of Conservation of Momentum

The law of conservation of momentum states that the total linear momentum of an isolated system remains constant in spite of changes occurring inside the system. In vector form, it can be written as

$$\rho \frac{d\mathbf{V}}{dt} = \nabla \cdot \mathbf{T} + \rho \mathbf{b}, \tag{1.7.1}$$

where
$\mathbf{V}$ = velocity field,
$\mathbf{T}$ = stress tensor,
$\rho\mathbf{b}$ = body force.

1.8 Some Common Useful Non-Dimensional Parameters

1.8.1 Reynolds Number

The Reynolds number is the ratio of inertial forces to viscous forces. Mathematically, it can be written as

$$\mathrm{Re} = \frac{\text{inertial force}}{\text{viscous force}} = \frac{\rho v L}{\mu} = \frac{vL}{\nu}, \tag{1.8.1}$$

where
v = velocity of the object relative to the fluid,
L = characteristic length,
ρ = mass density,
μ = dynamic viscosity,
ν = kinematic viscosity.

1.8.2 Prandtl Number

The Prandtl number is defined as the fraction of momentum diffusivity to thermal diffusivity. Mathematically, it can be expressed as

$$Pr = \frac{\text{momentum diffusivity}}{\text{thermal diffusivity}} = \frac{\nu}{\alpha} = \frac{C_p \mu}{k}, \tag{1.8.2}$$

where
ν = kinematic viscosity $=\frac{\mu}{\rho}$,
α = thermal diffusivity$=\frac{k}{\rho C_p}$,
μ = dynamic viscosity,
k = thermal conductivity,

C_p = specific heat,
ρ = density.

1.8.3 Schmidt number

Schmidt number is defined as the ratio of momentum diffusivity (viscosity) to mass diffusivity. Mathematically, it can be written as

$$Sc = \frac{\text{viscous diffusion rate}}{\text{mass diffusion rate}} = \frac{\nu}{D}, \tag{1.8.3}$$

where
ν = kinematic viscosity,
D = mass diffusivity.

1.8.4 Nusselt Number

The fraction of convective to conductive heat transfer is called Nusselt number. It is denoted by Nu. Mathematically, it can be expressed as

$$Nu = \frac{\text{convective heat transfer}}{\text{conductive heat transfer}} = \frac{hL}{k}, \tag{1.8.4}$$

where
L = characteristic length,
h = convective heat transfer coefficient of the fluid,
k = thermal conductivity of the fluid.

1.8.5 Sherwood Number

The proportion of convective to diffusive mass transport is called Sherwood number. It is denoted by Sh. Mathematically, it can be written as

$$Sh = \frac{\text{convective mass transfer coefficient}}{\text{diffusive mass transfer coefficient}} = \frac{kL}{D}, \tag{1.8.5}$$

where
L = characteristic length,
D = mass diffusivity,
k = mass transfer coefficient.

1.9 Boundary Layer

A boundary layer is the layer of fluid in the immediate vicinity of a bounding surface where the effects of viscosity are significant.

1.10 Laminar Boundary Layer

A laminar boundary layer is one where the flow takes place in layers, i.e., each layer slides past the closest layers. When the Reynolds numbers are small than laminar boundary layers are formed.

1.11 Thermal Boundary Layer

The layer of a liquid or gaseous heat-transfer agents between the free stream and a heat-exchange surface. In this layer the temperature of the heat- transfer agent changes from that of the wall to that of the free stream.

1.12 Numerical Techniques

1.12.1 Shooting Method for Non-Linear Differential Equation

We explain shooting method for second order ODE. We consider the following ODE

$$y'' = f(x, y, y'), \qquad a \leq x \leq b, \tag{1.12.1}$$

with boundary conditions,

$$y(a) = B_1, \quad y(b) = B_2. \tag{1.12.2}$$

In shooting method first we reduce the boundary value problem (BVP) into an initial value problem (IVP), i.e.

$$y'' = f(x, y, y') \qquad for \qquad a \leq x \leq b,$$

through

$$y(a) = B_1, \qquad y'(a) = \alpha(unknown). \tag{1.12.3}$$

In Eq (1.12.3) α is unknown which has to be found.

$$lim_{k\to\infty} \quad y(b, \alpha) = y(b) = B_2. \tag{1.12.4}$$

We produce a sequence of α_1 , α_2 ,α_3 ,..... through α_0 as the initial guess. The iteration has to be prevented when

$$y(b, \alpha) - B_2 = 0. \tag{1.12.5}$$

Eq. (1.12.5) is a nonlinear equation in variable α. To produce the sequence α_k we use the Newton-Raphson method. Only initial guess α_0 is required in Newton's method and produces the left behind terms by

$$\alpha_k = \alpha_{k-1} - \frac{y(b, \alpha_{k-1}) - B_2}{\frac{dy}{ds}(b, \alpha_{k-1})}. \tag{1.12.6}$$

For two or more variable the Newton-Raphson formula is

$$\alpha_k = \alpha_{k-1} - \frac{(y(b, \alpha_{k-1}) - B_2)}{\mid J \mid}. \tag{1.12.7}$$

where $\mid J \mid$ is the Jacobian matrix.

1.12.2 bvp4c

Any nonlinear boundary value problem (BVP) ordinary differential equation (ODE) can be solved by using MATLAB *bvp4c* solver. The solver apply *collocation* method. It initiates solution with an initial guess supplied at initial mesh points and changes step-size (hence changes mesh) to obtain the particular accuracy. For more detail see reference [3].

Chapter 2

Numerical Solution for Stagnation-Point Flow of Nanofluid over an Exponentially Stretching Sheet

The structure of this chapter is prepared as follows. In section 2.1, the introduction is given. In sections 2.2 and 2.3, we study the problem and present the governing equations. In section 2.4, we examine the numerical results with the help of tables and graphs.

2.1 Introduction

This chapter is the review work of Mustafa et al [1]. The steady flow past a flat plate with a uniform free stream was investigated by Blasius [4]. In distinguishing the Blasius problem, the boundary layer flow over a continuously moving plate in a ambient fluid was discovered by Sakiadis [5]. Crane [6] completed this concept for a sheet which is stretched with a velocity linearly proportional to the distance from the origin. The flow analysis over an exponentially stretching sheet has been scarcely presented. Enhancement of heat transfer is important in improving performance of electronic devices. Heat transfer and thermal energies of usual cooling agents is relatively smaller. Thus it was tried to suspend nanoparticles into fluids to procedure high effective heat transfer fluids. The term nanofluid was first used by Choi [10] to refer to the fluids with suspended nanoparticles. Recent investigations on nanofluids have revealed that the thermal conductivity increases with decreasing grain size. In recent years, nanofluids have attracted attention as a new generation of heat transfer fluids in building heating, heat exchangers, plants, and automotive cooling applications because of their excellent thermal performance. This chapter explains the numerical solution for stagnation-point flow of nanofluid over an exponentially stretching sheet. By using similarity transformation we reduce the governing partial differential equations (PDEs) into ordinary differential equations(ODEs). Then these equations are solved by shooting method with fifth order Runge-Kutta integration method. The solution is confirmed with the built-in solver *bvp4c* of MATLAB. Graphs are integrated for the influence of singular parameters on the stream field.

2.2 Problem Formulation

In this section, we study the laminar boundary layer flow of a nanofluid in the given region of stagnation-point flowing towards an exponentially stretching sheet placed at $y = 0$. The x-axis is taken along the sheet and y-axis is taken perpendicular to the sheet and the flow is restricted to $y \geq 0$. We also incorporate the effects of Brownian motion and thermophoresis. We denote $U_w(x) = ae^{\frac{x}{L}}$ and $U_\infty(x) = be^{\frac{x}{L}}$ as the velocities of the sheet and external flow respectively. Let the temperature and the nanoparticles concentration are to be taken as $T_w = T_\infty + ce^{\frac{x}{L}}$ and $C_w = C_\infty + de^{\frac{x}{L}}$ where T_∞ and C_∞ represents the ambient temperature and concentration respectively. The conservation of mass and momentum are governed by the following boundary layer equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2.2.1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U_\infty \frac{dU_\infty}{dx} + \nu_f \frac{\partial^2 u}{\partial y^2}, \tag{2.2.2}$$

where u is the velocity component along x-direction and v is the velocity component along y-direction, ν_f is the kinematic viscosity. The boundary conditions for the given problem are

$$u = U_w(x) = ae^{\frac{x}{L}}, v = 0, \quad \text{at} \quad y = 0, \tag{2.2.3}$$

$$u \to U_e(x) = be^{\frac{x}{L}}, \quad \text{as} \quad y \to \infty. \tag{2.2.4}$$

We introduce the following dimensionless variables [1]

$$\eta = \sqrt{\frac{a}{2\nu_f L}}e^{\frac{x}{2L}}y, \quad u = ae^{\frac{x}{L}}f'(\eta), v = -\sqrt{\frac{\nu_f a}{2L}}e^{\frac{x}{2L}}[f(\eta) + (\eta)f'(\eta)], \tag{2.2.5}$$

$$\frac{\partial u}{\partial x} = \frac{a}{L}e^{\frac{x}{L}}f' + \frac{a}{2L}e^{\frac{x}{L}}y\sqrt{\frac{a}{2\nu_f L}}e^{\frac{x}{2L}}f'', \quad u\frac{\partial u}{\partial x} = \frac{a^2}{L}e^{\frac{2x}{L}}f'^2 + \frac{a^2}{2L}e^{\frac{2x}{L}}y\sqrt{\frac{a}{2\nu_f L}}e^{\frac{x}{2L}}f'f'', \tag{2.2.6}$$

$$\frac{\partial u}{\partial y} = ae^{\frac{x}{L}}f''\sqrt{\frac{a}{2\nu_f L}}e^{\frac{x}{2L}}, \quad v\frac{\partial u}{\partial y} = -\frac{a^2}{2L}e^{\frac{2x}{L}}ff'' - \frac{a^2}{2L}e^{\frac{2x}{L}}y\sqrt{\frac{a}{2\nu_f L}}e^{\frac{x}{2L}}f'f'', \tag{2.2.7}$$

where

$$\frac{dU_\infty}{dx} = be^{\frac{x}{L}}\frac{1}{L}, \quad U_\infty\frac{dU_\infty}{dx} = \frac{b^2}{L}e^{\frac{2x}{L}}, \tag{2.2.8}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{a^2}{2L\nu_f}e^{\frac{2x}{L}}f''', \quad \nu_f\frac{\partial^2 u}{\partial y^2} = \frac{a^2}{2L}e^{\frac{2x}{L}}f''', \tag{2.2.9}$$

$$\frac{\partial v}{\partial y} = -\frac{a}{L}e^{\frac{x}{L}}f' - \frac{a}{2L}e^{\frac{x}{L}}y\sqrt{\frac{a}{2\nu_f L}}e^{\frac{x}{2L}}f'', \tag{2.2.10}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{a^2}{2L}e^{\frac{2x}{L}}[2f'^2 - ff''], \tag{2.2.11}$$

$$U_\infty\frac{dU_\infty}{dx} + v_f\frac{\partial^2 u}{\partial y^2} = \frac{b^2}{L}e^{\frac{2x}{L}} + \frac{a^2}{2L}e^{\frac{2x}{L}}f'''. \tag{2.2.12}$$

By substituting the Eqs. (2.2.6) and (2.2.10) in Eq. (2.2.1), we obtain

$$\frac{a}{L}e^{\frac{x}{L}}f' + \frac{a}{2L}e^{\frac{x}{L}}y\sqrt{\frac{a}{2\nu_f L}}e^{\frac{x}{2L}}f'' - \frac{a}{L}e^{\frac{x}{L}}f' - \frac{a}{2L}e^{\frac{x}{L}}y\sqrt{\frac{a}{2\nu_f L}}e^{\frac{x}{2L}}f'' = 0.$$

Therefore, Eq. (2.2.1) is identically satisfied. Now, by substituting the Eqs. (2.2.11) and (2.2.12) in Eq. (2.2.2), we obtain

$$\frac{a^2}{2L}e^{\frac{2x}{L}}[2f'^2 - ff''] = \frac{b^2}{L}e^{\frac{2x}{L}} + \frac{a^2}{2L}e^{\frac{2x}{L}}f''',$$

$$[2f'^2 - ff''] = \frac{2Le^{-\frac{2x}{L}}}{a^2}\Big[\frac{b^2}{L}e^{\frac{2x}{L}} + \frac{a^2}{2L}e^{\frac{2x}{L}}f'''\Big],$$

$$[2f'^2 - ff''] = \frac{2b^2}{a^2} + f'''.$$

$$\Rightarrow f''' + ff'' - 2f'^2 + 2\lambda^2 = 0. \quad \Big(\text{where } \lambda = \frac{b}{a}\Big) \tag{2.2.13}$$

In the last step we convert the boundary conditions into the new variable.
When $\eta = 0$, then we get

$$u = ae^{\frac{x}{L}}f'(\eta). \tag{2.2.14}$$

By equating Eqs. (2.2.3) and (2.2.14), we get

$$ae^{\frac{x}{L}} = ae^{\frac{x}{L}}f'(0),$$

$$f'(0) = 1. \tag{2.2.15}$$

When $\eta = 0$ then

$$v = -\sqrt{\frac{\nu_f a}{2L}}e^{\frac{x}{2L}}f(0). \tag{2.2.16}$$

By comparing Eqs. (2.2.3) and (2.2.16) we obtain

$$f(0) = 0. \tag{2.2.17}$$

Both values of u compared when $y \to \infty$ and $\eta \to \infty$.

$$be^{\frac{x}{L}} = ae^{\frac{x}{L}}f'(\infty),$$

$$f'(\infty) = \frac{b}{a},$$

$$f'(\infty) = \lambda, \tag{2.2.18}$$

where $\lambda = \frac{b}{a}$ is the ratio of the velocity of external flow to the velocity of sheet.

2.3 Transport Equation

The governing equations of energy and nanoparticles volume fraction are given by

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2}+\tau\left[D_B\frac{\partial C}{\partial y}\frac{\partial T}{\partial y}+\frac{D_T}{T_\infty}\left(\frac{\partial T}{\partial y}\right)^2\right], \tag{2.3.1}$$

$$u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}=D_B\frac{\partial^2 C}{\partial y^2}+\frac{D_T}{T_\infty}\frac{\partial^2 T}{\partial y^2}, \tag{2.3.2}$$

where
T = temperature,
C = nanoparticles concentration,
α = thermal diffusivity,
D_B = Brownian motion coefficient,
D_T = thermophoretic diffusion coefficient,
$\tau=\frac{(\rho C)_p}{(\rho C)_f}$ is the ratio of heat capacity of the nanoparticle material to heat capacity of the fluid.

2.3.1 Prescribed Surface Temperature (PST)

In this case the boundary conditions are

$$T=T_w=T_\infty+ce^{\frac{x}{L}},\quad C=C_w=C_\infty+de^{\frac{x}{L}},\quad \text{at}\ \ y=0, \tag{2.3.3}$$

$$T\to T_\infty,\quad C\to C_\infty,\ \text{as}\ \ y\to\infty. \tag{2.3.4}$$

Here $c,d>0$ are constants.We introduce the dimensionless temperature $\theta(\eta)=\frac{T\ -\ T_\infty}{T_w\ -\ T_\infty}$ and nanoparticles concentration $\phi(\eta)=\frac{C\ -\ C_\infty}{C_w\ -\ C_\infty}$.
Now we convert the Eq. (2.3.1) into ODE. As given that

$$T=T_w=T_\infty+ce^{\frac{x}{L}},$$

$$T=T_w-T_\infty=ce^{\frac{x}{L}},$$

$$T=ce^{\frac{x}{L}}\theta(\eta)+T_\infty, \tag{2.3.5}$$

$$\frac{\partial T}{\partial x}=\frac{c}{L}e^{\frac{x}{L}}\theta+\frac{c}{2L}e^{\frac{x}{L}}y\sqrt{\frac{a}{2\nu_f L}}e^{\frac{x}{2L}}\theta', \tag{2.3.6}$$

use $T=T_w-T_\infty=ce^{\frac{x}{L}}$ in Eq. (2.3.6) we obtain

$$\frac{\partial T}{\partial x}=\frac{(T_w-T_\infty)}{L}\theta+\frac{(T_w-T_\infty)}{2L}y\sqrt{\frac{a}{2\nu_f L}}e^{\frac{x}{2L}}\theta',$$

$$u\frac{\partial T}{\partial x}=\frac{a(T_w-T_\infty)}{L}e^{\frac{x}{L}}f'\theta+\frac{a(T_w-T_\infty)}{2L}e^{\frac{x}{L}}y\sqrt{\frac{a}{2\nu_f L}}e^{\frac{x}{2L}}\theta' f', \tag{2.3.7}$$

$$\frac{\partial T}{\partial y}=(T_w-T_\infty)\sqrt{\frac{a}{2\nu_f L}}e^{\frac{x}{2L}}\theta',$$

$$v\frac{\partial T}{\partial y}=-\frac{a(T_w-T_\infty)}{2L}e^{\frac{x}{L}}f\theta'-\frac{a(T_w-T_\infty)}{2L}e^{\frac{x}{L}}y\sqrt{\frac{a}{2\nu_f L}}e^{\frac{x}{2L}}\theta' f' \tag{2.3.8}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{a(T_w-T_\infty)}{L}e^{\frac{x}{L}}f'\theta-\frac{a(T_w-T_\infty)}{2L}e^{\frac{x}{L}}f\theta'. \tag{2.3.9}$$

$$\frac{\partial^2 T}{\partial y^2}=\frac{a}{2v_f L}(T_w-T_\infty)e^{\frac{x}{L}}\theta'', \tag{2.3.10}$$

$$\frac{\partial C}{\partial y}\frac{\partial T}{\partial y}=\frac{a}{2v_f L}(T_w-T_\infty)(C_w-C_\infty)e^{\frac{x}{L}}\theta'\phi', \tag{2.3.11}$$

$$\left(\frac{\partial T}{\partial y}\right)^2=\frac{a}{2v_f L}(T_w-T_\infty)^2e^{\frac{x}{L}}\theta'^2. \tag{2.3.12}$$

By substituting the Eqs. (2.3.9) to (2.3.12) in Eq. (2.3.1), we obtain

$$\begin{aligned}\frac{a(T_w-T_\infty)}{L}e^{\frac{x}{L}}f'\theta-\frac{a(T_w-T_\infty)}{2L}e^{\frac{x}{L}}f\theta' =&\alpha\frac{a}{2v_f L}(T_w-T_\infty)e^{\frac{x}{L}}\theta''+\tau D_B\frac{a}{2\nu_f L}(T_w-T_\infty)\\ &(C_w-C_\infty)e^{\frac{x}{L}}\theta'\phi'+\tau\frac{D_T}{T_\infty}\frac{a}{2\nu_f L}(T_w-T_\infty)^2e^{\frac{x}{L}}\theta'^2\end{aligned}$$

$$\frac{a(T_w-T_\infty)}{2L}e^{\frac{x}{L}}[2f'\theta-f\theta']=\frac{a(T_w-T_\infty)}{2L}e^{\frac{x}{L}}\left[\frac{\alpha}{\nu_f}\theta''+\frac{\tau D_B(C_w-C_\infty)}{\nu_f}\theta'\phi'+\frac{\tau D_T(T_w-T_\infty)}{\nu_f T_\infty}\theta'^2\right]$$

$$2f'\theta - f\theta' = \frac{\alpha}{\nu_f}\theta'' + \frac{(\rho C)_p D_B (C_w - C_\infty)}{(\rho C)_f \nu_f}\theta'\phi' + \frac{(\rho C)_p D_T (T_w - T_\infty)}{(\rho C)_f \nu_f T_\infty}\theta'^2 \qquad (2.3.13)$$

Here $Pr = \frac{\nu_f}{\alpha}$ is the Prandtl number, $N_b = \frac{(\rho C)_p D_B (C_w - C_\infty)}{(\rho C)_f \nu_f}$ is the Brownian motion parameter and $N_t = \frac{(\rho C)_p D_T (T_w - T_\infty)}{(\rho C)_f \nu_f T_\infty}$ is the thermophoresis parameter. After substituting all these values in Eq. (2.3.13), it becomes

$$\frac{1}{Pr}\theta'' + f\theta' - 2f'\theta + N_b\theta'\phi' + N_t\theta'^2 = 0. \qquad (2.3.14)$$

Now we convert the Eq. (2.3.2) into ODE.
As given that

$$C = C_w = C_\infty + de^{\frac{x}{L}},$$

$$C = C_w - C_\infty = de^{\frac{x}{L}},$$

$$C = de^{\frac{x}{L}}\phi(\eta) + C_\infty, \qquad (2.3.15)$$

$$\frac{\partial C}{\partial x} = \frac{d}{L}e^{\frac{x}{L}}\phi + \frac{d}{2L}e^{\frac{x}{L}}y\sqrt{\frac{a}{2\nu_f L}}e^{\frac{x}{2L}}\phi' \qquad (2.3.16)$$

Use $C = C_w - C_\infty = de^{\frac{x}{L}}$ in Eq. (2.3.16), we obtain

$$\frac{\partial C}{\partial x} = \frac{(C_w - C_\infty)}{L}\phi + \frac{(C_w - C_\infty)}{2L}y\sqrt{\frac{a}{2\nu_f L}}e^{\frac{x}{2L}}\phi',$$

$$u\frac{\partial C}{\partial x} = \frac{a(C_w - C_\infty)}{L}e^{\frac{x}{L}}f'\phi + \frac{a(C_w - C_\infty)}{2L}e^{\frac{x}{L}}y\sqrt{\frac{a}{2\nu_f L}}e^{\frac{x}{2L}}\phi' f'. \qquad (2.3.17)$$

$$\frac{\partial C}{\partial y} = (C_w - C_\infty)\sqrt{\frac{a}{2\nu_f L}}e^{\frac{x}{2L}}\phi',$$

$$v\frac{\partial C}{\partial y} = -\frac{a(C_w - C_\infty)}{2L}e^{\frac{x}{L}}f\phi' - \frac{a(C_w - C_\infty)}{2L}e^{\frac{x}{L}}y\sqrt{\frac{a}{2\nu_f L}}e^{\frac{x}{2L}}\phi' f'. \qquad (2.3.18)$$

$$u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = \frac{a(C_w - C_\infty)}{L}e^{\frac{x}{L}}f'\phi - \frac{a(C_w - C_\infty)}{2L}e^{\frac{x}{L}}f\phi'. \qquad (2.3.19)$$

$$\frac{\partial^2 C}{\partial y^2} = \frac{a}{2v_f L}(C_w - C_\infty)e^{\frac{x}{L}}\phi''. \qquad (2.3.20)$$

$$\frac{D_T}{T_\infty}\frac{\partial^2 T}{\partial y^2} = \frac{D_T a}{T_\infty 2\nu_f L}(T_w - T_\infty)e^{\frac{x}{L}}\theta''. \qquad (2.3.21)$$

By substituting the Eqs. (2.3.19) to (2.3.21) in Eq. (2.3.2), we obtain

$$\frac{a(C_w - C_\infty)}{L} e^{\frac{x}{L}} f'\phi - \frac{a(C_w - C_\infty)}{2L} e^{\frac{x}{L}} f\phi' = D_B \frac{a}{2\nu_f L}(C_w - C_\infty) e^{\frac{x}{L}} \phi'' + \frac{D_T a}{T_\infty 2\nu_f L}(T_w - T_\infty) e^{\frac{x}{L}} \theta''$$

$$\frac{a(C_w - C_\infty)}{2L} e^{\frac{x}{L}} [2f'\phi - f\phi'] = \frac{a(C_w - C_\infty)}{2L} e^{\frac{x}{L}} \left[\frac{D_B}{\nu_f}\phi'' + \frac{D_T(T_w - T_\infty)}{T\infty(C_w - C_\infty)\nu_f} \right]$$

$$2f'\phi - f\phi' = \frac{D_B}{\nu_f}\phi'' + \frac{D_T(T_w - T_\infty)}{T\infty(C_w - C_\infty)\nu_f}$$

Dividing both sides of above Eq. by $\frac{\nu_f}{D_B}$ it becomes

$$\frac{\nu_f}{D_B}[2f'\phi - f\phi'] = \phi'' + \frac{D_T(T_w - T_\infty)}{T_\infty(C_w - C_\infty)D_B}\theta''. \tag{2.3.22}$$

Here $Sc = \frac{\nu_f}{D_B}$ is the Schmidt number and $\frac{N_t}{N_b} = \frac{D_T(T_w - T_\infty)}{T_\infty(C_w - C_\infty)D_B}$ is the ratio of thermophoresis parameter to the Brownian motion parameter. After substituting all these values in Eq. (2.3.22), it becomes

$$\phi'' + Sc(f\phi' - 2f'\phi) + \frac{N_t}{N_b}\theta'' = 0. \tag{2.3.23}$$

Now we convert the boundary conditions into the new form. From Eqs. (2.3.3) and (2.3.4), we know that

$T = T_w = T_\infty + ce^{\frac{x}{L}}, \quad C = C_w = C_\infty + de^{\frac{x}{L}}, \quad \text{at} \;\; y = 0,$

$T \to T_\infty, \quad C \to C_\infty, \;\; \text{as} \;\; y \to \infty.$

As given that

$\eta = \sqrt{\frac{a}{2\nu_f L}} e^{\frac{x}{2L}} y$, when $y = 0$ then $\eta = 0$.

$$T = (T_w - T_\infty)\theta(\eta) + T_\infty, \tag{2.3.24}$$

$$C = (C_w - C_\infty)\phi(\eta) + C_\infty. \tag{2.3.25}$$

Substituting $\eta = 0$ in Eq. (2.3.24) and both values of T compared when $y = 0$ and $\eta = 0$

$$ce^{\frac{x}{L}} + T_\infty = (T_w - T_\infty)\theta(0) + T_\infty,$$

we know that $ce^{\frac{x}{L}} = T_w - T_\infty$. So above equation becomes

$$\theta(0) = 1. \tag{2.3.26}$$

Substituting as $\eta = 0$ in Eq. (2.3.25) and both values of C compared when $y = 0$ and $\eta = 0$

$$de^{\frac{x}{L}} + C_\infty = (C_w - C_\infty)\phi(0) + C_\infty,$$

we know that $de^{\frac{x}{L}} = C_w - C_\infty$. So above equation becomes

$$\phi(0) = 1. \tag{2.3.27}$$

Substituting $\eta = \infty$ in Eq. (2.3.24) and both values of T compared when $y \to \infty$ and $\eta \to \infty$

$$T_\infty = (T_w - T_\infty)\theta(\infty) + T_\infty,$$

after simplifying the above equation we obtain

$$\theta(\infty) = 0. \tag{2.3.28}$$

Substituting as $\eta \to \infty$ in Eq. (2.3.25) and both values of C compared when $y \to \infty$ and $\eta \to \infty$

$$C_\infty = (C_w - C_\infty)\phi(\infty) + C_\infty,$$

after simplifying the above equation we obtain

$$\phi(\infty) = 0. \tag{2.3.29}$$

The skin friction coefficient $C_f = \frac{\mu(\frac{\partial u}{\partial y})_{y=0}}{\rho U_w^2}$

$$\rho U_w^2 C_f = \mu(\frac{\partial u}{\partial y}) \tag{2.3.30}$$

$$\frac{\partial u}{\partial y} = ae^{\frac{x}{L}}\sqrt{\frac{a}{2\nu_f L}}e^{\frac{x}{2L}}f''(\eta) \tag{2.3.31}$$

$$U_w^2 = a^2 e^{\frac{2x}{L}}$$

Put Eq. (2.3.31) into Eq. (2.3.30) we get

$$\rho a^2 e^{\frac{2x}{L}} C_f = \mu a e^{\frac{x}{L}}\sqrt{\frac{a}{2\nu_f L}}e^{\frac{x}{2L}}f''(\eta)$$

Dividing both sides of above Eq. by $a^2 e^{\frac{x}{2L}}$ after simplifying it becomes

$$\rho C_f = \frac{\mu}{ae^{\frac{x}{L}}}\sqrt{\frac{a}{2\nu_f L}}e^{\frac{x}{2L}}f''(\eta)$$

Multiplying both sides of above Eq. by $\sqrt{\frac{a}{2\nu_f L}}e^{\frac{x}{2L}}$ and after simplification it becomes

$$\sqrt{\frac{a}{2\nu_f L}}e^{\frac{x}{2L}}\rho C_f = \frac{\mu}{2\nu_f L}f''(\eta)$$

Inserting $\rho = \frac{\mu}{\nu_f}$ in above equation and after simplifying it becomes

$$\sqrt{\frac{a}{2\nu_f L}}e^{\frac{x}{2L}}C_f = \frac{1}{2L}f''(\eta) \tag{2.3.32}$$

Multiplying both sides of Eq. (2.3.32) by $2L$ we get

$$\sqrt{\frac{2aL}{\nu_f}}e^{\frac{x}{2L}}C_f = f''(\eta)$$

$$\sqrt{2R_e}C_f = f''(\eta), \tag{2.3.33}$$

at $y = 0 \Rightarrow \eta = 0$
Put $\eta = 0$ in Eq. (2.3.33), we get

$$\sqrt{2R_e}C_f = f''(0), \Big(Re_x = \frac{U_w x}{\nu_f}\text{is the local Reynolds number}\Big). \tag{2.3.34}$$

The local Nusselt number

$$Nu = -\frac{x(\frac{\partial T}{\partial y})_{y=0}}{(T_w - T_\infty)} \tag{2.3.35}$$

$$\frac{\partial T}{\partial y} = (T_w - T_\infty)\sqrt{\frac{a}{2\nu_f L}}e^{\frac{x}{2L}}\theta'(\eta), \tag{2.3.36}$$

Put Eq. (2.3.36) in Eq. (2.3.35), after simplifying we get

$$Nu = -x\sqrt{\frac{a}{2\nu_f L}}e^{\frac{x}{2L}}\theta'(\eta) \tag{2.3.37}$$

Multiplying both sides of Eq. (2.3.37) by $\frac{1}{\sqrt{x}}$ we get

$$\frac{1}{\sqrt{x}}Nu = -\sqrt{\frac{ax}{2\nu_f L}}e^{\frac{x}{2L}}\theta'(\eta)$$

$$\sqrt{\frac{2L}{x}}Nu\sqrt{\frac{\nu_f}{ax}}e^{-\frac{x}{2L}} = -\theta'(\eta) \tag{2.3.38}$$

at $y = 0$ then $\eta = 0$

Put $\eta = 0$ in Eq. (2.3.38) we get

$$\sqrt{\frac{2L}{x}}\frac{Nu}{Re_x^{1/2}} = -\theta'(0) = Nur, \left(Re_x = \frac{U_w x}{\nu_f}\text{is the local Reynolds number}\right). \tag{2.3.39}$$

The local Sherwood number $Sh = -\frac{x(\frac{\partial C}{\partial y})_{y=0}}{(C_w - C_\infty)}$,

$$Sh = -\frac{x(\frac{\partial C}{\partial y})_{y=0}}{(C_w - C_\infty)}, \tag{2.3.40}$$

$$\frac{\partial C}{\partial y} = (C_w - C_\infty)\sqrt{\frac{a}{2\nu_f L}}e^{\frac{x}{2L}}\phi'(\eta), \tag{2.3.41}$$

Put Eq. (2.3.41) in Eq. (2.3.40) after simplifying we get

$$Sh = -x\sqrt{\frac{a}{2\nu_f L}}e^{\frac{x}{2L}}\phi'(\eta),$$

Multiplying both sides of above Eq.by $\frac{1}{\sqrt{x}}$ we get

$$\frac{1}{\sqrt{x}}Sh = -\sqrt{\frac{ax}{2\nu_f L}}e^{\frac{x}{2L}}\phi'(\eta),$$

$$\sqrt{\frac{2L}{x}}Sh\sqrt{\frac{\nu_f}{ax}}e^{-\frac{x}{2L}} = -\phi'(\eta), \tag{2.3.42}$$

at $y = 0$ then $\eta = 0$

Putting $\eta = 0$ in Eq. (2.3.42) we get

$$\sqrt{\frac{2L}{x}}\frac{Sh}{Re_x^{1/2}} = -\phi'(0) = Shr, \left(Re_x = \frac{U_w x}{\nu_f}\text{is the local Reynolds number}\right). \tag{2.3.43}$$

2.4 Numerical Results and Discussion

Numerical solution of the governing ordinary differential equations (ODEs) for different values of thermophoresis parameter Nt, Brownian motion parameter Nb, Prandtl number Pr and Schmidt number Sc is obtained by using the shooting method with fifth order Runge-Kutta integration method and MATLAB built-in solver *bvp4c*. To examine the effects of different parameters we have constructed Tables 2.1 and 2.2. We have also drawn Figures 2.1 - 2.8 for different parameters.

2.4.1 Velocity, Heat and Mass Transfer Rates

In Table 2.1 the dimensionless velocity gradient on the sheet is estimated for different values of λ. Hence skin friction coefficient is compared by assuming suitably large values of λ. In Table 2.2 we have given the values of Nur and Shr consequent to singular values of Pr and Sc. We observed that increase in Pr and Sc reduce the thermal boundary layer thickness and curves become steeper. The reduced Nusselt and Sherwood numbers, being proportional to the corresponding initial slopes, increase with an increase in Pr and Sc respectively.

λ	$\sqrt{2ReC_f} = f''(0)$	
	shooting method	bvp4c
0	-1.2844	-1.2818
0.1	-1.2540	-1.2536
0.2	-1.1952	-1.1951
0.5	-0.8798	-0.8798
0.8	-0.3979	-0.3978
1.2	0.4521	0.4515

Table 2.1: Numerical values of skin friction coefficient $f''(0)$ for different values of velocity ratio parameter λ.

Pr	Sc	$Nur = -\theta'(0)$		$Shr = -\phi'(0)$	
		shooting method	bvp4c	shooting method	bvp4c
0.4	1	0.7577	0.7499	0.9801	0.9740
0.7		1.0356	1.0343	0.7823	0.7787
1.0		1.2604	1.2602	0.6149	0.6127
1.2		1.3907	1.3907	0.5151	0.5133
1.0	0.4	1.2800	1.2810	-0.0946	0.1173
	0.7	1.2686	1.2686	0.2983	0.2922
	1.2	1.2561	1.2559	0.7983	0.7969
	1.5	1.2508	1.2505	1.0439	1.0432

Table 2.2: Numerical values of Nur and Shr for different values of Pr and Sc when $\lambda = 0.2$,$N_b = N_t = 0.1$.

2.4.2 Velocity Profile

The effects of velocity profile for different values of velocity ratio λ are shown in Figure 2.1. It is clear from Figure 2.1 that when $\lambda > 1$ the velocity increases and boundary layer thickness decreases. Figure 2.1 also shows that when $\lambda < 1$ the flow of boundary layer structure is reversed. Here the sheet velocity $U_w(x)$ exceeds the velocity of external stream $U_\infty(x)$. It is also noticed that when $\lambda = 1$ boundary layer is not formed.

2.4.3 Temperature Profiles

The effects of Brownian motion and thermophoresis parameters on the temperature is shown in Figure. 2.2. It is noticed that the temperature and the thermal boundary layer thickness increases by increasing N_b and N_t. The behavior of Prandtl number Pr on the temperature θ is shown in Figure. 2.3. A larger Prandtl number result in a comparatively lower thermal diffusivity. As expected, the variation in the temperature is more obvious for smaller values of Pr than its larger values. Figure.

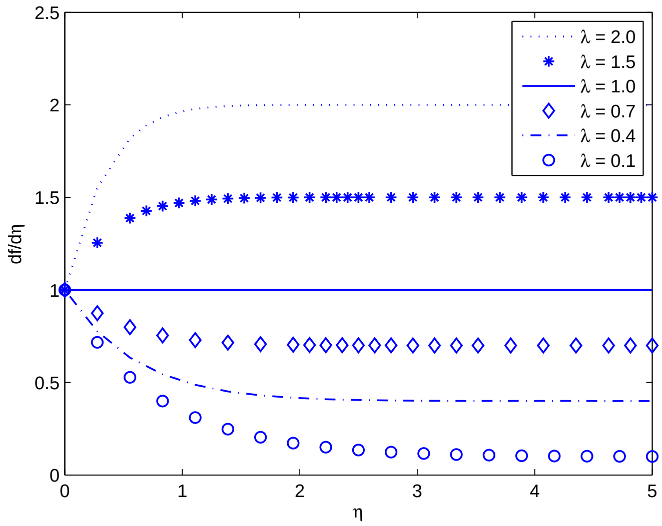

Figure 2.1: Influence of λ on $f'(\eta)$.

2.4 illustrates the effect of velocity ratio λ on the temperature θ. The temperature and the thermal boundary layer thickness decrease with an increase in λ .

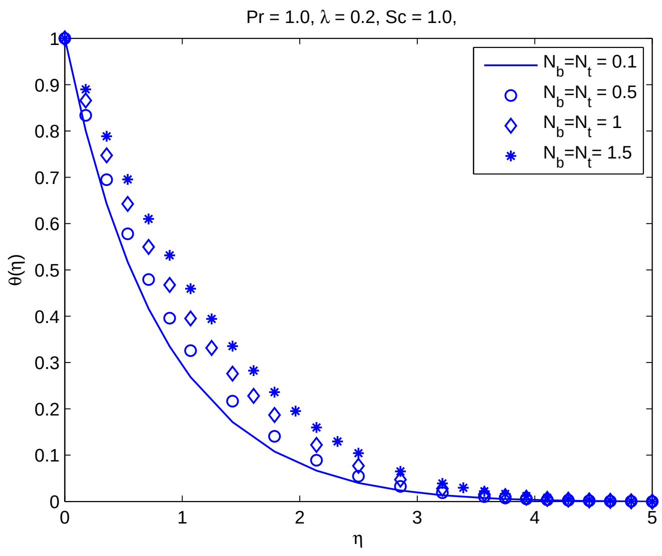

Figure 2.2: Influence of N_b and N_t on $\theta(\eta)$.

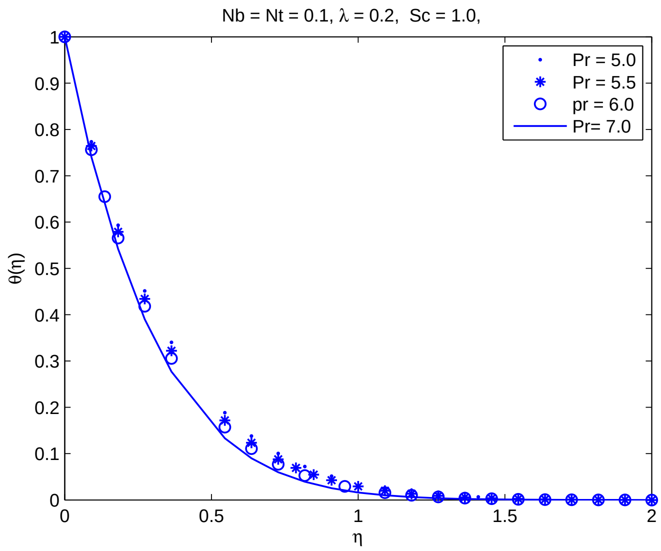

Figure 2.3: Influence of Pr on $\theta(\eta)$.

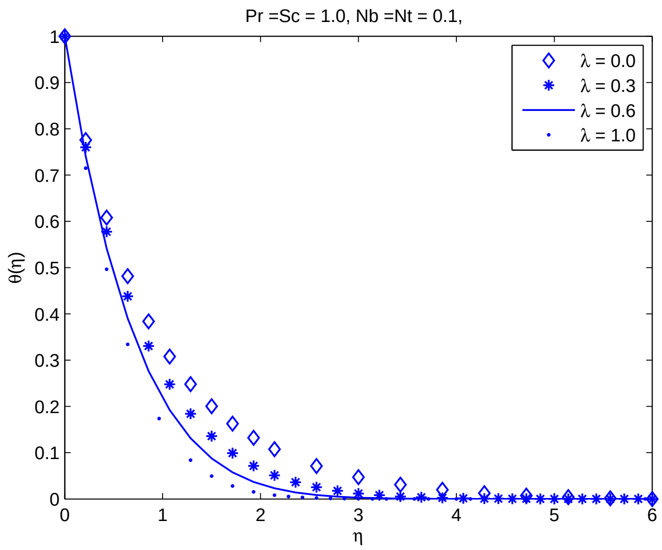

Figure 2.4: Influence of λ on $\theta(\eta)$.

2.4.4 Nanoparticles Concentration Profiles

Plotting of the concentration function against η for different values of the Brownian motion parameter Nb is shown in Figure. 2.5. We observed that concentration ϕ is only affected for the values of Nb in the range $0 < Nb \leq 2$. Figure. 2.6 illustrate the influence of thermophoresis parameter Nt on the concentration boundary layer. It is initiate that concentration ϕ is increase by increasing Nt. This conclusion is recognized to the fact that an increase in N_t considerably increase the mass flux due to temperature gradient. Figure. 2.7 shows the behaviour of Schmidt number Sc on the concentration field ϕ. As Sc steadily increases, this corresponds to a weaker molecular diffusivity and thinner concentration boundary layer. Figure. 2.8 illustrates that the influence of λ on the nanoparticles concentration ϕ is almost similar to that accounted for the temperature θ.

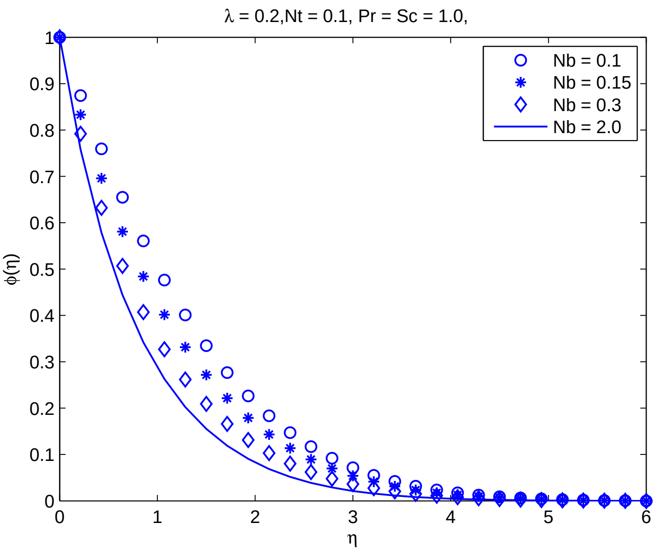

Figure 2.5: Influence of N_b on $\phi(\eta)$.

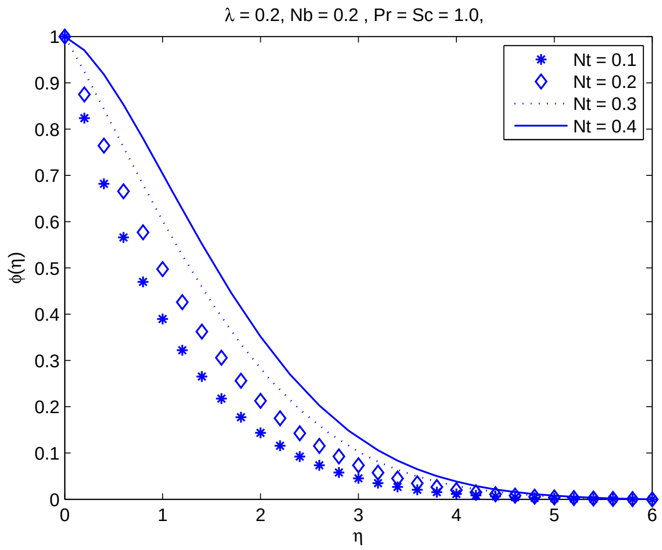

Figure 2.6: Influence of N_t on $\phi(\eta)$.

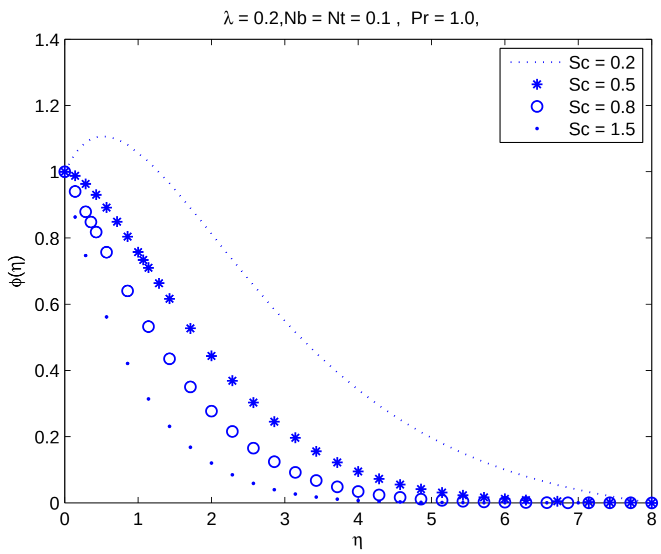

Figure 2.7: Influence of Sc on $\phi(\eta)$.

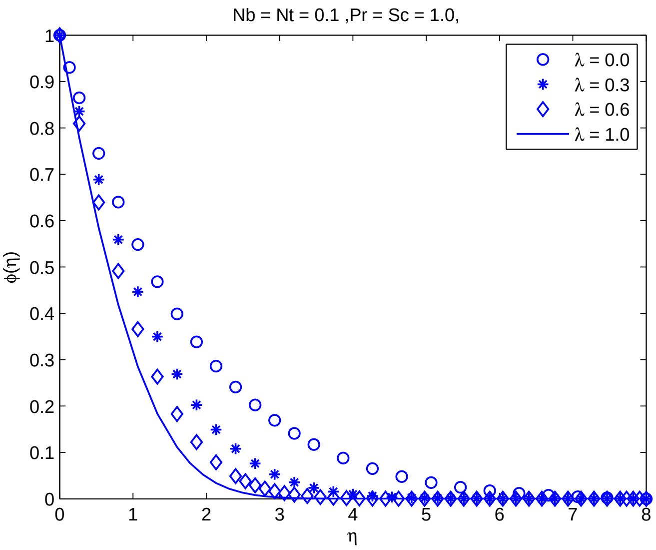

Figure 2.8: Influence of λ on $\phi(\eta)$.

Chapter 3

The effects of temperature-dependent viscosity on heat transfer over a continuous moving surface

Structured of this chapter is prepared as follows. In section 3.1, the introduction is given. In section 3.2, we study the problem and present the governing equations. In section 3.3, we examine the numerical results with the help of graphs.

3.1 Introduction

This chapter is the review of Elbashbeshy et al [21]. This chapter explain the effects of temperature-dependent viscosity on heat transfer over a continuous moving surface. By using similarity transformation, we reduce the governing partial differential equations(PDEs) into ordinary differential equations(ODEs). Then these equations have been solved by shooting method with fifth order Runge-Kutta integration method. The solution have been confirmed with the built-in solver *bvp4c* of MATLAB.

3.2 Problem Formulation

In this section we consider the steady two dimensional laminar flow on a continuous stretching surface with uniform surface temperature T_w and velocity U_w moving axially through a stationary fluid. The conservation equations of the laminar boundary layer are given in [22] and [23] as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{3.2.1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{\rho_\infty}\frac{\partial}{\partial y}(\mu\frac{\partial u}{\partial y}), \tag{3.2.2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{k}{\rho_\infty C_p}\frac{\partial^2 T}{\partial y^2}, \tag{3.2.3}$$

where u is the velocity component along x-direction and v is the velocity component along y-direction, T is the temperature inside the boundary layer, ρ_∞ is the density away from the hot plate, μ is the dynamic viscosity, k is the thermal conductivity, C_p is the specific heat at constant pressure and T_∞ is the free stream temperature. The boundary conditions for the given problem are

$$u = U_w, v = 0, \quad T = T_w \;\text{ at }\; y = 0, \qquad u = 0, \quad T = T_\infty \;\text{ as }\; y \to \infty. \tag{3.2.4}$$

For a viscous fluid, Ling and Dybbs [22] suggest a viscosity dependence on temperature T of the form

$$\mu = \frac{\mu_\infty}{[1 + \gamma(T - T_\infty)]}, \tag{3.2.5}$$

so that viscosity is an inverse linear function of the temperature T. Equation (3.2.5) can be written as

$$\frac{1}{\mu} = \alpha(T - T_r), \tag{3.2.6}$$

where

$$\alpha = \frac{\gamma}{\mu_\infty} \qquad \textit{and} \qquad T_r = \frac{T_\infty - 1}{\gamma} \tag{3.2.7}$$

In the above Eq. (3.2.7), both α and T_r are constant and their values depend on the reference state and γ is a thermal property of the fluid.

We introduce the following dimensionless variables

$$\eta = y\sqrt{\frac{U_w}{2\nu_\infty x}}, \quad \nu_\infty = \frac{\mu_\infty}{\rho_\infty}, \quad \psi(x,y) = \sqrt{2\nu_\infty U_w x} f(\eta), \tag{3.2.8}$$

$$\theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty} \tag{3.2.9}$$

We choose a stream function $\psi(x, y)$ as

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}. \tag{3.2.10}$$

Now we use Eqs. (3.2.8) and (3.2.10) to find u and v.

$$u = \sqrt{2\nu_\infty U_w x} f'(\eta) \frac{\partial \eta}{\partial y}$$

$$u = \sqrt{2\nu_\infty U_w x} f'(\eta) \frac{\partial}{\partial y}\left(y\sqrt{\frac{U_w}{2\nu_\infty x}}\right)$$

$$u = U_w f'(\eta) \tag{3.2.11}$$

$$v = -\frac{\partial}{\partial x}(\sqrt{2\nu_\infty U_w x} f(\eta))$$

$$v = -\sqrt{2\nu_\infty U_w x}\frac{\partial}{\partial x}(f(\eta)) - f(\eta)\frac{\partial}{\partial x}(\sqrt{2\nu_\infty U_w x})$$

$$v = -\sqrt{2\nu_\infty U_w x} f'(\eta)\frac{\partial \eta}{\partial x} - f(\eta)\frac{1}{2\sqrt{2\nu_\infty U_w x}} 2\nu_\infty U_w$$

$$v = -\sqrt{2\nu_\infty U_w x} f'(\eta)\frac{\partial}{\partial x}\left(y\sqrt{\frac{U_w}{2\nu_\infty x}}\right) - \frac{\nu_\infty U_w}{\sqrt{2\nu_\infty U_w x}} f(\eta)$$

$$v = \frac{U_w y}{2x} f'(\eta) - \frac{\nu_\infty U_w}{\sqrt{2\nu_\infty U_w x}} f(\eta) \tag{3.2.12}$$

$$\frac{\partial u}{\partial x} = -\frac{U_w^{\frac{3}{2}} y}{2^{\frac{3}{2}}\sqrt{\nu_\infty} x^{\frac{3}{2}}} f''(\eta), \quad u\frac{\partial u}{\partial x} = -\frac{U_w^{\frac{5}{2}} y}{2^{\frac{3}{2}}\sqrt{\nu_\infty} x^{\frac{3}{2}}} f'(\eta) f''(\eta), \tag{3.2.13}$$

$$\frac{\partial u}{\partial y}=\frac{U_w^{\frac{3}{2}}}{\sqrt{\nu_\infty}x}f''(\eta),\qquad v\frac{\partial u}{\partial y}=\frac{U_w^{\frac{5}{2}}y}{2^{\frac{3}{2}}\sqrt{\nu_\infty}x^{\frac{3}{2}}}f'(\eta)f''(\eta)-\frac{U_w^2}{2x}f(\eta)f''(\eta),\tag{3.2.14}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{U_w^2}{2x}f(\eta)f''(\eta)\tag{3.2.15}$$

$$\frac{\partial}{\partial y}\Big(\mu\frac{\partial u}{\partial y}\Big)=\frac{\partial}{\partial y}\Big(\mu\frac{U_w^{\frac{3}{2}}}{\sqrt{\nu_\infty}x}f''(\eta)\Big)$$

$$\frac{\partial}{\partial y}\Big(\mu\frac{\partial u}{\partial y}\Big)=\frac{\mu U_w^2}{2\nu_\infty x}f'''(\eta)+\frac{\mu U_w^2\theta'}{2\nu_\infty x(\theta_r-\theta)}$$

$$\frac{1}{\rho_\infty}\frac{\partial}{\partial y}\Big(\mu\frac{\partial u}{\partial y}\Big)=\frac{\mu U_w^2}{2\nu_\infty x\rho_\infty}\Big(f'''+\frac{\theta'}{\theta_r-\theta}f''\Big)\tag{3.2.16}$$

Therefore, Eq. (3.2.1) is identically satisfied. Now, by substituting the Eqs. (3.2.15) and (3.2.16) in Eq. (3.2.2) we obtain

$$-\frac{U_w^2}{2x}f(\eta)f''(\eta)=\frac{\mu U_w^2}{2\nu_\infty x\rho_\infty}\Big(f'''+\frac{\theta'}{\theta_r-\theta}f''\Big)$$

$$-\frac{1}{\mu}\rho_\infty\nu_\infty ff''=f'''+\frac{\theta'}{\theta_r-\theta}f''.$$

After simplifying we get,

$$f'''+\frac{\theta_r-\theta}{\theta_r}ff''+\frac{\theta'}{\theta_r-\theta}f''=0.\tag{3.2.17}$$

Where

$$\theta(\eta)=\frac{T-T_\infty}{T_w-T_\infty}$$
$$T=T_w-T_\infty\theta(\eta)+T_\infty$$

$$\frac{\partial T}{\partial x}=-\frac{U_wy}{2^{\frac{3}{2}}\nu_\infty\sqrt{\frac{U_w}{\nu_\infty x}}x^2}(T_w-T_\infty)\theta'(\eta),\qquad u\frac{\partial T}{\partial x}=-\frac{U_w^{\frac{3}{2}}y}{2^{\frac{3}{2}}\sqrt{\nu_\infty}x^{\frac{3}{2}}}(T_w-T_\infty)f'(\eta)\theta'(\eta),\tag{3.2.18}$$

$$\frac{\partial T}{\partial y}=\sqrt{\frac{U_w}{2\nu_\infty x}}(T_w-T_\infty)\theta'(\eta),\qquad v\frac{\partial T}{\partial y}=\frac{U_w^{\frac{3}{2}}y}{2^{\frac{3}{2}}\sqrt{\nu_\infty}x^{\frac{3}{2}}}(T_w-T_\infty)f'(\eta)\theta'(\eta)-\frac{U_w}{2x}(T_w-T_\infty)f(\eta)\theta'(\eta),\tag{3.2.19}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=-\frac{U_w}{2x}(T_w-T_\infty)f(\eta)\theta'(\eta)\tag{3.2.20}$$

$$\frac{\partial^2 T}{\partial y^2} = \frac{U_w}{2\nu_\infty x}(T_w - T_\infty)\theta''(\eta), \tag{3.2.21}$$

$$\frac{k}{\rho_\infty C_p}\frac{\partial^2 T}{\partial y^2} = \frac{k}{\rho_\infty C_p}\frac{U_w}{2\nu_\infty x}(T_w - T_\infty)\theta''(\eta). \tag{3.2.22}$$

Now, by substituting the Eqs. (3.2.20) and (3.2.22) in Eq. (3.2.3) we obtain

$$-\frac{U_w}{2x}(T_w - T_\infty)f(\eta)\theta'(\eta) = \frac{k}{\rho_\infty C_p}\frac{U_w}{2\nu_\infty x}(T_w - T_\infty)\theta''(\eta)$$

After simplifying we get,

$$\theta'' + Prf\theta' = 0. \tag{3.2.23}$$

where $Pr = \frac{\rho_\infty C_p \nu_\infty}{k}$.

Now we convert boundary condition from Eq. (3.2.4). From Eq. (3.2.11)

$$u = U_w f'(\eta)$$

then compare both values when $y = 0$ and $\eta = 0$, which is

$$U_w f'(\eta) = u_w$$

we get

$$f'(0) = 1. \tag{3.2.24}$$

Putting $\eta = 0$ we obtain

$$v = \frac{U_w y}{2x}f'(0) - \frac{\nu_\infty U_w}{\sqrt{2\nu_\infty U_w x}}f(0) \tag{3.2.25}$$

By comparing Eqs. (3.2.4) and (3.2.25) we obtain

$$f(0) = 0. \tag{3.2.26}$$

Again from Eq. (3.2.4)

$$T = T_w, \qquad when \qquad y = 0$$

$$T = (T_w - T_\infty)\theta(\eta) + T_\infty \tag{3.2.27}$$

after comparing both values of T we get,

$$\theta(0) = 1. \tag{3.2.28}$$

$u = 0 \quad$ when $\quad y \to \infty$ and from Eq. (3.2.11) when $\eta \to \infty$, then

$$u = U_w f'(\infty),$$

by comparing both values we get,

$$f'(\infty) = 0. \tag{3.2.29}$$

Again from Eq. (3.2.4)

$$T = T_w, \quad when \quad y \to \infty$$

and from Eq. (3.2.27) when $\eta \to \infty$ then

$$T = (T_w - T_\infty)\theta(\eta) + T_\infty$$

by comparing both values we get,

$$\theta(\infty) = 0 \tag{3.2.30}$$

Eq. (3.2.24), Eq. (3.2.26), Eq. (3.2.28) to Eq. (3.2.30) are new boundary conditions subjected to Eq. (3.2.4)

$$f(0) = 0, \qquad f'(0) = 1, \qquad \theta(0) = 1, \qquad \eta = 0 \tag{3.2.31}$$

$$f'(\infty) = 0, \qquad \theta(\infty) = 0, \qquad as \qquad \eta \to \infty \tag{3.2.32}$$

The skin friction coefficient is defined by

$$C_f = \frac{2\tau_w}{\rho_\infty U_w^2},$$

where the shearing stress on the plate is defined by

$$\tau_w = \mu(\frac{\partial u}{\partial y}) \mid_{y=0}$$

$$\rho_\infty U_w^2 C_f = 2\mu U_w \sqrt{\frac{U_w}{2\nu_\infty x}} f''(\eta)$$

after simplifying we get,

$$C_f R_e^{\frac{1}{2}} = \frac{\sqrt{2}\theta_r}{\theta_r - 1} f''(0) \tag{3.2.33}$$

The local Nusselt number for heat transfer in the present case is defined by

$$Nu = \frac{-x(\frac{\partial T}{\partial y}) \mid_{y=0}}{T_w - T_\infty}$$

$$Nu = \frac{-x}{(T_w - T_\infty)}(T_w - T_\infty)\sqrt{\frac{U_w}{2\nu_\infty x}}\theta'(\eta)$$

$$Nu = -\sqrt{\frac{U_w x}{2\nu_\infty}}\theta'(\eta)$$

after simplifying we get

$$NuR_e^{-\frac{1}{2}} = -\theta'(0). \tag{3.2.34}$$

3.3 Numerical Results and Discussion

The nonlinear ordinary differential equation (ODEs) given in Eqs. (3.2.17) and (3.2.23) subject to the boundary condition in Eqs. (3.2.31) and (3.2.32) are solved numerically by using shooting method and MATLAB built-in solver *bvp4c*. Figures 3.1 and 3.2 show the variation of the dimensionless velocity $f'(\eta)$ for singular values of θ_r for both air and water. It is found that the $f'(\eta)$ of air and water increases as the viscosity/temperature parameter θ_r decreases, whereas decreasing the value of θ_r to four and two, in fact of increasing the viscosity within the boundary layer, tends to increase the velocity value there instead of reducing the velocities within the layer due to moving the surface. Figures 3.3 and 3.4 show the variation of the dimensionless temperature parameter $\theta(\eta)$ for different values of θ_r for both air and water. It is found that for air, the temperature increases greatly as the viscosity/temperature parameter θ_r increases, whereas for water the temperature increases slightly as θ_r increases, then the temperature gradient $\theta'(0)$ at the stretched surface is decreased.

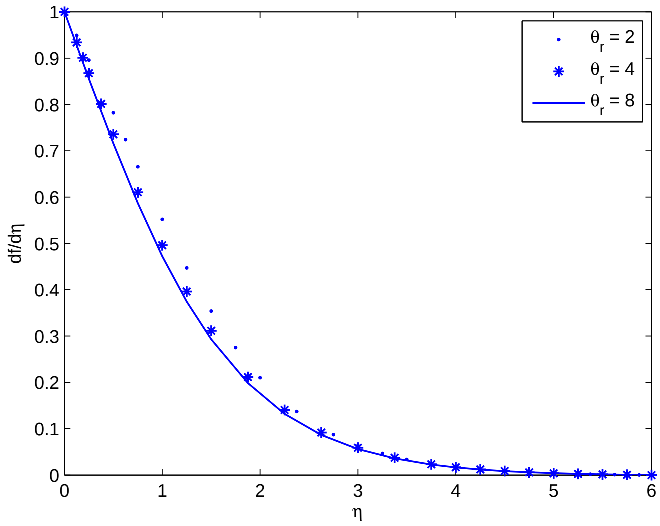

Figure 3.1: Velocity distribution as a function of η for various value of θ_r at $Pr = 0.7$.

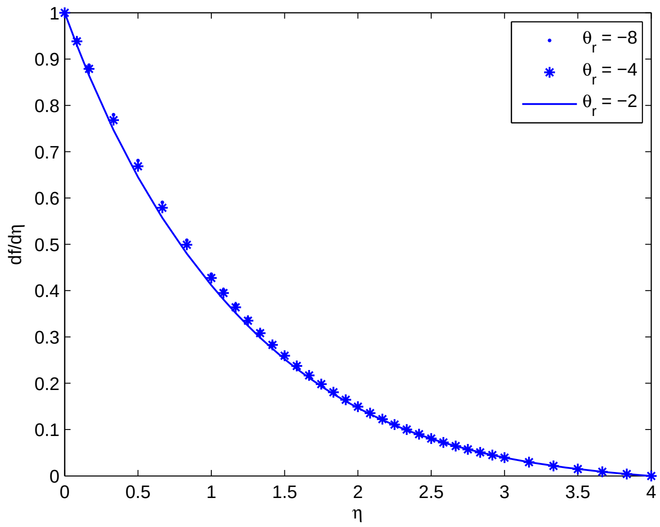

Figure 3.2: Velocity distribution as a function of η for various value of θ_r at $Pr = 7$.

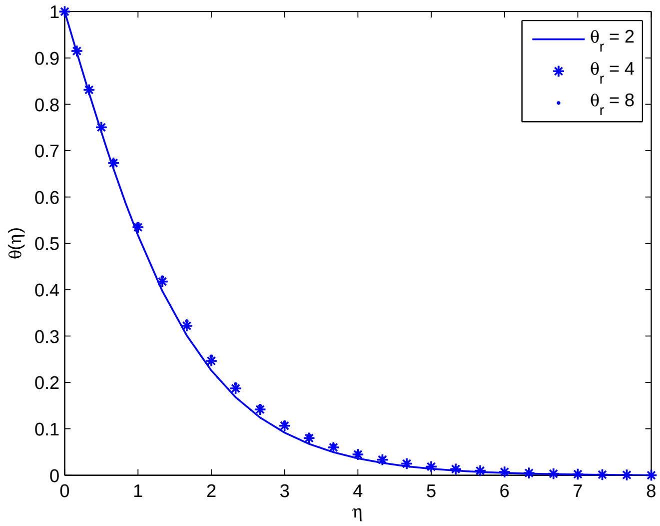

Figure 3.3: Temperature distribution as a function of η for various value of θ_r at $Pr = 0.7$.

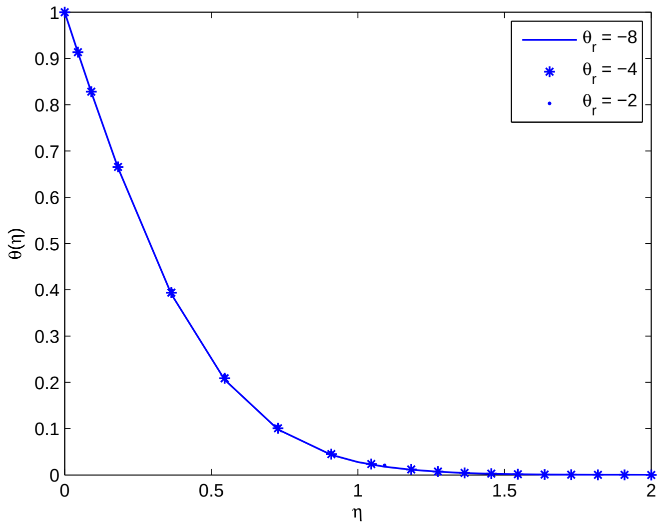

Figure 3.4: Temperature distribution as a function of η for various value of θ_r at $Pr = 7$.

Chapter 4

Conclusion and Outlook

Numerical solution for stagnation-point flow of nanofluid over an exponentially stretching sheet and the effects of temperature-dependent viscosity on heat transfer over a continuous moving surface are studied. The developed mathematical model is solved for the numerical solution by fifth order Runge-Kutta method using a shooting technique and verify the results with *bvp4c*. The key point of this work are as below.

- Temperature and thermal boundary layer thickness increase with an increase in Nb and Nt.
- Temperature and thermal boundary layer thickness decrease with an increase in λ.
- Nanoparticle volume fraction decreases with an increase in Sc.
- The velocity $f'(\eta)$ increases with a decrease of θ_r for both air and water.
- The temperature $\theta(\eta)$ increases with an increase of θ_r for both air and water.

We have only considered the steady flow cases, but unsteady flow can be done in future. The same is true for compressible boundary layer flow. Moreover, the Keller-Box method can be developed to solve the incompressible and compressible flows. In future one can also use other numerical open source software chebfun.

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Appendix

MATLAB code to solve non-linear ODE

function $shooting - Asma - numeric$

global XSTART XSTOP H Pr lambda Nt Nb Sc

$XSTART = 0; \quad XSTOP = 4.4;$

$Pr = 1.0;$

$\lambda = 0.2;$

$Nt = 0.5;$

$Nb = 0.5;$

$Sc = 1.0;$

$H = 0.1;$

$freq = 1;$

$u = [-1 \quad -1 \quad 1];$

$x = XSTART;$

$u = newtonRaphson2(@residual, u);$

$[xSol, ySol] = runKut5(@dEqs, x, inCond(u), XSTOP, H);$

printSol(xSol,ySol,freq)

plot(xSol, ySol(:,4))

$functionF = dEqs(x, y)$

global lambda Pr Sc Nt Nb

$yy1 = 2 * y(2)^2 - y(1) * y(3) - 2 * lambda^2;$

$yy2 = Pr * (2 * y(2) * y(4) - y(1) * y(5) - Nb * y(5) * y(7) - Nt * y(5)^2);$

$yy3 = Sc * (2 * y(2) * y(6) - y(1) * y(7)) - (Nt/Nb) * yy2;$

$F = zeros(1, 7);$

$F(1) = y(2);$

$F(2) = y(3);$

$F(3) = yy1;$

$F(4) = y(5);$

$F(5) = yy2;$

$F(6) = y(7);$

$F(7) = yy3;$

$functiony = inCond(u)$

$y = [0 \quad 1 \quad u(1) \quad 1 \quad u(2) \quad 1 \quad u(3)];$

$functionr = residual(u)$

global XSTART XSTOP H lambda

$r = zeros(length(u), 1);$

$x = XSTART;$

$[xSol, ySol] = runKut5(@dEqs, x, inCond(u), XSTOP, H);$

$lastRow = size(ySol, 1);$

$r(1) = ySol(lastRow, 2) - lambda;$

$r(2) = ySol(lastRow, 4);$

$r(3) = ySol(lastRow, 6);$

Subroutines for the Shooting Method

Subroutine of Newton-Raphson method

$functionroot = newtonRaphson2(func, x, tol)$

$ifnargin == 2; tol = 1.0e4 * eps;$ end

$ifsize(x, 1) == 1; x = x';$ end

$fori = 1 : 10$

$[jac, f0] = jacobian(func, x);$

$ifsqrt(dot(f0, f0)/length(x)) < tol$

$root = x;$return

end

$dx = (jac)/(-f0);$

$x = x + dx;$

$ifsqrt(dot(dx, dx)/length(x)) < tol * max(abs(x), 1.0)$

$root = x;$ return

end

disp(i)

end

error('Too many iterations')

$function[jac, f0] = jacobian(func, x)$

$h = 1.0e - 4;$

$n = length(x);$

$jac = zeros(n);$

$f0 = feval(func, x);$

$fori = 1 : n$

$temp = x(i);$

$x(i) = temp + h;$

$f1 = feval(func, x);$

$x(i) = temp;$

$jac(:, i) = (f1 - f0)/h;$

end

Subroutine of Runge-Kutta method.

$function[xSol, ySol] = runKut5(dEqs, x, y, xStop, h, eTol)$

if size(y,1) ¿ 1 ;$y = y'$; end $ifnargin < 6; eTol = 1.0e - 6$; end

$n = length(y);$

$A = [01/53/103/517/8];$

$$B = [0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 1/5\ \ 0\ \ 0\ \ 0\ \ 0\ \ 3/40\ \ 9/40\ \ 0\ \ 0\ \ 0\ \ 3/10\ \ -9/10\ \ 6/5\ \ 0\ \ 0\ \ -11/54\ \ 5/2\ \ -70/27\ \ 35/27\ \ 0\ \ 1631/55296\ \ 175/51\ \ 575/13824\ \ 44275/110592\ \ 253/4096];$$

$C = [37/3780250/621125/5940512/1771];$

$D = [2825/27648018575/4838413525/55296277/143361/4];$

$xSol = zeros(2, 1); ySol = zeros(2, n);$

$xSol(1) = x; ySol(1, :) = y;$

$stopper = 0; k = 1;$

$for p = 2 : 5000$

$K = zeros(6, n);$

$K(1, :) = h * feval(dEqs, x, y);$

$for i = 2 : 6$

$BK = zeros(1, n);$

$for j = 1 : i - 1$

$BK = BK + B(i, j) * K(j, :);$

end

$K(i, :) = h * feval(dEqs, x + A(i) * h, y + BK);$

end

$dy = zeros(1, n); E = zeros(1, n);$

$for i = 1 : 6$

$dy = dy + C(i) * K(i, :);$

$E = E + (C(i) - D(i)) * K(i, :);$

end

$e = sqrt(sum(E. * E)/n);$

$if e <= eTol$

$y = y + dy; x = x + h;$

$k = k + 1;$

$xSol(k) = x; ySol(k, :) = y;$

$if stopper == 1;$

break

end

end

$if e \ \ = 0; hNext = 0.9 * h * (eTol/e)^0.2;$

else;

$hNext = h;$ end

$if(h > 0) == (x + hNext >= xStop)$

$hNext = xStop - x; stopper = 1;$

end

$h = hNext;$

end

Chapter 2 (bvp4c Codes)

This MATLAB program of chapter 2 to find the solution of the numerical and series solutions for stagnation-point flow of nanofluid over an exponentially stretching sheet using bvp4c method.

$function\ \ power - law - problem$

clear all

close all

$Pr = 1.0;$

$lambda = 0.2;$

$Nt = 0.1;$

$Nb = 0.1;$

$Sc = 1.0;$

$functionysol = bvpex1(x, y)$

$yy1 = 2 * y(2)^2 - y(1) * y(3) - 2 * lambda(i)^2;$

$yy2 = Pr(i) * (2 * y(2) * y(4) - y(1) * y(5) - Nb(i) * y(5) * y(7) - Nt(i) * y(5)^2);$

$yy3 = Sc(i) * (2 * y(2) * y(6) - y(1) * y(7)) - (Nt(i)/Nb(i)) * yy2;$

$ysol = [y(2); y(3); yy1; y(5); yy2; y(7); yy3];$

end $functionres = bcex1(y0, yinf)$

$res = [y0(1); y0(2)-1; yinf(2)-lambda(i); y0(4)-1; yinf(4); y0(6)-1; yinf(6)];$

end $sol1 = bvpinit(linspace(0, 5, 15), [1000000]);$

$sol = bvp4c(@bvpex1, @bcex1, sol1);$

$x = sol.x;$

$y = sol.y;$

$value = deval(sol, o)$

end

Chapter 3 (bvp4c Codes)

This MATLAB program of chapter 3 to find the effects of temperature-dependent viscosity on heat transfer over a continuous moving surface using bvp4c method.

$function \ \ power - law - problem$

clear all

close all

$Pr = 0.7;$

$thetar = 2;$

$functionysol = bvpex1(x, y)$

$yy1 = (y(4) - thetar)/(thetar) * y(1) * y(3) - (y(5))/(thetar - y(4)) * y(3);$

$yy2 = -Pr * y(1) * y(5);$

$ysol = [y(2); y(3); yy1; y(5); yy2];$

end

$functionres = bcex1(y0, yinf)$

$res = [y0(1); y0(2) - 1; yinf(2); y0(4) - 1; yinf(4)];$

end

$sol1 = bvpinit(linspace(0, 6, 25), [1 \quad 0 \quad 0 \quad 0 \quad 0]);$

$sol = bvp4c(@bvpex1, @bcex1, sol1);$

$x = sol.x;$

$y = sol.y;$

$figure(1)$

$plot(x, y(2, :)$

$xlabel('\eta')$

$ylabel('df/d\eta')$

end

end