Generalized Separation of Variables:
Exact Solutions of Nonlinear Partial Differential Equations

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To My Loving Parents

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Abstract

This thesis comprises the discussion on generalized separation of variables method. Some exact solutions have been obtained by applying generalized separation of variables method. In this context, we suggest a method to construct the exact solutions of nonlinear partial differential equations(PDEs). The method involves searching for transformations that *reduce the dimensionality* of the equation. New families of exact solutions of nonlinear second-order partial differential equations that govern processes of heat and mass transfer are illustrated.

Exact solutions of hyperbolic-type equation, Korteweg-De-Vries type, and nonlinear wave equation are constructed by applying this method. Obviously, the generalized separation of variables method can also be effective for constructing exact solutions of many other nonlinear PDEs, as well.

The thesis also discusses about the construction of the exact solutions of nonlinear PDEs, by applying an efficient method, commonly known as $\frac{G'}{G}$ expansion method. This method appears to be effective in seeking exact solutions of nonlinear equations.

Contents

Chapter 1

Introduction

In scientific research, seeking the exact solution of nonlinear equations is a hot topic. We outline generalized separation of variables which is applied to second order partial differential equations(PDEs). To solve linear heat -and mass transfer equations and other linear equations of mathematics, the method of separation of variables is the most widely used. Moreover the method make it possible to construct exact solutions of nonlinear wave equations [7].

Exact solutions of heat and mass transfer equations play a vital role in forming a proper understanding of qualitative features of various thermal and diffusion processes. Four basic approaches are most frequently encountered for searching the exact solution of nonlinear differential equations:

(1) searching the travelling wave solution, (2) searching the self similar solutions, and (3) solution in the form of the sum or product of two functions of different arguments, (4) application of groups to search for symmetries of the equation.

The method of separation of variables that is outlined below includes the first two approaches as its special cases and quite often allows finding exact solutions that cannot be obtained by application of groups.

1.1 Structure of exact solutions for heat and mass transfer equations

1.1.1 Self similar solution

There are various techniques to reduce the PDE into an ODE (or at least a PDE in a smaller number of independent variables), which includes various integral transforms, when we are dealing with linear PDEs. Such techniques are much less prevalent when we deal with nonlinear PDEs. However, there is an approach that identifies equations for which the solution depends on certain groupings of the independent variables rather than depending on each of the independent variables separately. Firstly, we will describe this technique for a linear PDE. For our convenience we consider the one dimensional case. Self similar solutions of one dimensional heat equation are solution of the form

$$U(x,t) = t^{\alpha} g\Big(\frac{x}{t^{\lambda}}\Big), \tag{1.1.1}$$

where α and λ are arbitrary constants. The unknown function $g(\frac{x}{t^{\lambda}})$ is identified by an ODE, which is obtained by substituting the solution of equation (1.1.1) into the original PDE. Generally, self similar solution are said to be the solution of the form [8].

$$U(x,t) = \chi(t) g\Big(\frac{x}{\phi(t)}\Big), \tag{1.1.2}$$

where $\chi(t)$ and $\phi(t)$ are selected for the reason of simplicity in some particular problem.

1.1.2 Separation of variables for linear equations

Most of the linear PDEs can be solved by the separation of variables. We consider the linear second order PDE of the form

$$G\Big(x, t, U, \frac{\partial U}{\partial x}, \frac{\partial U}{\partial t}, \frac{\partial^2 U}{\partial x^2}, \frac{\partial^2 U}{\partial t^2}, \frac{\partial^2 U}{\partial x \partial t}\Big) = 0, \tag{1.1.3}$$

with two independent variables x and t and an unknown function $U = U(x,t)$, that is to be determined. Procedure for solving equation of the type Eq. (1.1.3) involves some stages.

Let us consider the examples of particular solution by using the method described for constructing exact solutions of linear equations .

(1). Firstly, we search for the particular solution of the form

$$U(x,t) = \chi(x)\lambda(t). \tag{1.1.4}$$

Now substituting expression (1.1.4) into Eq. (1.1.3) for the function $\chi(x)$ and $\lambda(t)$, rewriting the equation, if possible, such that the left hand side of the resulting equation depends only on $x, \chi, \chi'_x, \chi''_{xx}$ and its right hand side depends only on $t, \lambda, \lambda'_t, \lambda''_{tt}$. Thus in non degenerate case, both sides are equal to some constant C, which is called as the separation constant. Thus the solution of Eq. (1.1.4) can be determined by ODE obtained for $\chi(x)$ and $\lambda(t)$. This method is called as the separation of variables in linear equations.

(2). Since a linear combination of exact solutions of a linear equation is also an exact solution of this equation, which is called as the principle of linear superposition. Here we restrict our consideration to the equation in two independent variables x and t, and one dependent variable U.

Thus, the separable linear equations have exact solution in the the form of the sum

$$U(x,t) = \psi_1(x)\chi_1(t) + \psi_2(x)\chi_2(t) + ... + \psi_m(x)\chi_m(t). \tag{1.1.5}$$

Some of the linear equation admit the exact solutions of the form [9]

$$U(x,t) = \psi(x) + \chi(t), \tag{1.1.6}$$

where $\psi(x)$ and $\chi(t)$ can be determined by the corresponding ODE obtained by substituting Eq. (1.1.6) into linear equation.

Example. Let us consider a linear equation

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\Big[a(x)\frac{\partial U}{\partial x}\Big] + b(x)\frac{\partial U}{\partial x} + \chi(t). \tag{1.1.7}$$

Eq. (1.1.7) admits more complicated solution of the form

$$U(x,t) = \psi(x)\phi_1(t) + \phi_2(t), \tag{1.1.8}$$

where $\phi_1(t) = e^{bt}$ and $\phi_2(t)$ can be identified by solving the first order ODE

$$\phi_2'(t) = b\phi_2(t) + \chi(t).$$

1.1.3 Separation of variables for nonlinear equations

Exact Solution of a PDE is a solution, which is defined in the whole domain where a PDE is defined, which can be represented as a finite expression. The exact solution of a nonlinear PDE is redundant, if there exist more general solution then such redundant solution can be considered as a particular case of general solutions of nonlinear PDEs [10].

Much effort has been spent to construct the exact solutions of nonlinear PDEs, because of their vital role in understanding the nonlinear problems. Since some of the nonlinear PDEs admit exact solution of the type given in Eq. (1.1.4). In this case unknown function $\chi(x)$ and $\lambda(t)$ can be identified by using ODEs, those are obtained by substituting Eq. (1.1.4) into the original equation and followed by nonlinear separation of variables.

Let us consider particular examples related to the method described for constructing exact solutions of nonlinear equations by separation of variables.

Example 1. The nonlinear heat equation

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\Big(\beta U^k \frac{\partial U}{\partial x}\Big), \tag{1.1.9}$$

admits the exact solutions of the form (1.1.4). Here the term βU^k is defined as the thermal diffusivity, where β and k are constants.

Example 2. The nonlinear heat equation

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\Big(\beta e^{\gamma U} \frac{\partial U}{\partial x}\Big), \tag{1.1.10}$$

admits the ansatz of the form (1.1.6). Here the term $\beta e^{\gamma U}$ is defined as the thermal diffusivity, where β and γ are constants.

Let us consider some examples for constructing exact solutions of nonlinear PDEs by generalized separation of variables.

1.1.4 Generalized separation of variables

We restrict our consideration to the nonlinear PDEs involving two independent variables x and t and one dependent variable u [9]. Most of the nonlinear PDEs involving quadratic and higher order derivatives of the form

$$a_1(x)b_1(t)\Pi_1(u) + a_2(x)b_2(t)\Pi_2(u) + ... + a_m(x)b_m(t)\Pi_m(u) = 0, \tag{1.1.11}$$

(where $\Pi_i(u)$ are the products of powers of functions u and their partial derivatives) have the exact solution of the form

$$u(x,t) = \phi_1(x)\psi_1(t) + \phi_2(x)\psi_2(t) + ... + \phi_m(x)\psi_m(t). \tag{1.1.12}$$

Such solutions are called as *generalized separable solution.*

General form of functional differential equation

Substituting Eq. (1.1.11) into Eq. (1.1.12), we obtain the differential equation of the form

$$\Phi_1(X)\Psi_1(T) + \Phi_2(X)\Psi_2(T) + ... + \Phi_m(X)\Psi_m(T) = 0, \tag{1.1.13}$$

for $\psi_i(t)$ and $\phi_i(x)$, where the functionals $\Phi_i(X)$ and $\Psi_j(T)$ depend on variables x and t respectively such that

$$\begin{aligned}\Phi_i(X) &= \Phi_i(x, \phi, \phi_1', \phi_1'', ..., \phi_m', \phi_m''),\\ \Psi_j(T) &= \Psi_j(t, \psi, \psi_1', \psi_1'', ..., \psi_m', \psi_m'').\end{aligned} \tag{1.1.14}$$

Now we describe two different methods for solving functional differential equations [7].

Generalized separation of variables: Differentiation method

The procedure for solving functional differential equation involves three steps:

(1). We can assume that $\Phi_m(X) \neq 0$. Dividing (1.1.14) by $\Phi_m(X)$ and differentiating the result with respect to x yields an equation which is of the same form as Eq. (1.1.14) but consisting of less number of terms. Thus

$$\tilde{\Phi}_1(X)\tilde{\Psi}_1(T) + \tilde{\Phi}_2(X)\tilde{\Psi}_2(T) + ... + \tilde{\Phi}_{m-1}(X)\tilde{\Psi}_{m-1}(T) = 0, \tag{1.1.15}$$

where

$$\tilde{\Phi}_i(X) = [\Phi_i(X)/\Phi_j(X)]_x',$$

$$\tilde{\Psi}_i(T) = \Psi_i(T),$$

continuing the similar procedure, finally we are left with the separable equation in two-terms only, as given below

$$\widehat{\Phi}_1(X)\widehat{\Psi}_1(T) + \widehat{\Phi}_2(X)\widehat{\Psi}_2(T) = 0. \tag{1.1.16}$$

Now we consider two situations:

Non degenerate case: $\widehat{\Psi}_1(T) + \widehat{\Psi}_2(T) \neq 0$ and $\widehat{\Phi}_1(X) + \widehat{\Phi}_2(X) \neq 0$.

The solution of Eq. (1.1.16) can be identified by the resulting ODEs as given below

$$\widehat{\Psi}_1(T) + C\widehat{\Psi}_2(T) = 0, \qquad \widehat{\Phi}_1(X) - C\widehat{\Phi}_2(X) = 0, \tag{1.1.17}$$

where C is an arbitrary constant.

Degenerate case:

If $\Psi_1(T) = 0, \Psi_2(T) = 0$, when $\Phi_1(X), \Phi_2(X)$ are arbitrary.

If $\Phi_1(X) = 0, \Phi_2(X) = 0$, when $\Psi_1(T), \Psi_2(T)$ are arbitrary.

(2). Thus the solutions of the two term Eq. (1.1.16) should be substituted into the original functional differential Eq. (1.1.13), to eliminate the unwanted integration constants.

(3). The case for $\Phi_m(X) = 0$ must be considered separately.

Generalized separation of variables: Splitting method

To avoid the difficulties while applying the differentiating method, it is convenient to split the initial problem into two simpler subproblems. Below we briefly describe the major steps of splitting method.

Case 1**: Functional differential equation consisting of an even number of terms**: m=2k

(1). In the first step, one can show by differentiation and induction, that the functional Eq. (1.1.13) has the following solution [9].

$$\Phi_i(X) = A_{i1}\Phi_{k+1}(X) + A_{i2}\Phi_{k+2}(X) + ... + A_{ik}\Phi_{2k}(X), \qquad (i = 1, 2, 3.....k),$$
$$\Psi_{k+i}(T) = -A_{1i}\Psi_1(T) - A_{2i}\Psi_2(T) - ... - A_{ki}\Psi_k(T), \qquad (i = 1, 2, 3.....k), \tag{1.1.18}$$

where $\Phi_1(X), ..., \Phi_k(X), \Psi_{k+1}(T), ..., \Psi_{2k}(T)$ are unknown functions, whereas $\Phi_{k+1}(X), ..., \Phi_{2k}(X)$, $\Psi_1(T), ..., \Psi_k(T)$ are known quantities. Expression (1.1.18) is consisting of k^2 arbitrary constants A_{ij}.
Example: Consider the functional equation consisting of an even number of terms of the form

$$\Phi_1(X)\Psi_1(T) + \Phi_2(X)\Psi_2(T) + \Phi_3(X)\Psi_3(T) + \Phi_4(X)\Psi_4(T) = 0. \tag{1.1.19}$$

Thus Eq. (1.1.19) has the solution

$$\Phi_1 = B_1\Phi_3 + B_2\Phi_4, \qquad \Phi_2 = B_3\Phi_3 + B_4\Phi_4,$$
$$\Psi_3 = -B_1\Psi_1 - B_3\Psi_2, \qquad \Psi_4 = -B_2\Psi_1 - B_4\Psi_2. \tag{1.1.20}$$

Comparing Eqs. (1.1.18) and (1.1.20) for $k = 2$, we get $A_{11} = B_1$, $A_{12} = B_2$, $A_{21} = B_3$ and $A_{22} = B_4$ as arbitrary constants, whereas $\Phi_i(X)$ are functions of one arguments and $\Psi_i(T)$ are functions of another argument.
Case 2. Functional differential equation consisting of an odd number of terms: $m = 2k - 1$
Two different solutions exist for Eq. (1.1.13), if it is consisting of odd number of terms which includes $k(k - 1)$ arbitrary constants.
Example: Let us consider the functional equation consisting of an odd number of terms of the form

$$\Phi_1(X)\Psi_1(T) + \Phi_2(X)\Psi_2(T) + \Phi_3(X)\Psi_3(T) = 0. \tag{1.1.21}$$

Eq. (1.1.21) has the solution

$$\begin{gathered} \Phi_1 = B_1\Phi_3, \quad \Phi_2 = B_2\Phi_3, \quad \Psi_3 = -B_1\Psi_1 - B_2\Psi_2, \\ \Psi_1 = B_1\Psi_3, \quad \Psi_2 = B_2\Psi_3, \quad \Phi_3 = -B_1\Phi_1 - B_2\Phi_2. \end{gathered} \tag{1.1.22}$$

Thus, in the first solution, the arbitrary constants are given the names as follows: $B_1 = A_{11}$, and $B_2 = A_{21}$ for $k = 2$, whereas, for finding the second solution, one can use a simple technique, since $B_1 = -1/A_{12}$ and $B_2 = A_{11}/A_{12}$, we obtain

$$\Phi_3 = -B_1\Phi_1 - B_2\Phi_2,$$

substituting above equation in (1.1.21), we have

$$\Phi_1(X)\Psi_1(T) + \Phi_2(X)\Psi_2(T) + (-B_1\Phi_1 - B_2\Phi_2)\Psi_3(T) = 0,$$

or

$$\Phi_1(X)[\Psi_1(T) - B_1\Psi_3(T)] + \Phi_2(X)[\Psi_2(T) - B_2\Psi_3(T)] = 0.$$

Equating the expressions in parentheses with zero, we obtain the following

$$\Psi_1(T) = B_1\Psi_3(T) \quad \text{and} \quad \Psi_2(T) = B_2\Psi_3(T),$$

where B_1 and B_2 are arbitrary constants [9].

1.1.5 Functional separation of variables

Structure of solutions:
It is important to note that, most of the nonlinear PDEs have the exact solution of the form

$$u(x,t) = F(z), \qquad z = \sum_{m=1}^{n} \phi_m(x)\psi_m(t). \tag{1.1.23}$$

Such solution are called as functionally separable solutions, where the functions $\phi_m(x)$, $\psi_m(t)$ and $F(z)$ are unknown in advance, and must be determined to find the solution.
We consider two simplest functional separable solutions

$$U = G(z), \qquad z = \psi(x)\chi(t), \tag{1.1.24}$$

or

$$U = G(z), \qquad z = \psi(x) + \chi(t), \tag{1.1.25}$$

where $G(z)$ is some function.

Functional separation of variables of a special form

For easiness, one can specify some functions in solution (1.1.23) in advance and other unknown functions can be identified. Such solutions are called as *functionally separable solutions* of a *special form*.
A generalized separable solution is a functional separable solution of a special form corresponding to $F(z) = z$. Now we consider functionally separable solutions of Eq. (1.1.23) of special form [10]

$$w = G(z), \qquad z = \phi(t) + x\phi(t), \tag{1.1.26}$$

$$w = G(z), \qquad z = t\psi_1(x) + \psi_2(x). \tag{1.1.27}$$

The arguments x and t in solution (1.1.27) can be eliminated, such solution will be called as *generalized travelling wave solution* of the original equation. Substituting expression (1.1.23) into the original equation and then eliminating t by using the expression for z, thus a functional differential equation will be formed in two variables x and z.
Remark: Functional separation of variables of a special form are given below

$$w = G(z) \quad z = t^2\psi_1(x) + \psi_2(x), \quad (z \text{ is quadratic in } t),$$

$$w = G(z) \quad z = \psi_1(x)e^{\lambda t} + \psi_2(x), \quad (z \text{ contains an exponential of } t).$$

Functional separation of variables: Differentiation method

In general, substituting solution (1.1.23) into the PDE which is under consideration, yields a functional differential equation with three arguments, two simple variables x and t and one composite argument z. In some cases, the resulting equation can be reduced by using differentiation method to a functional differentiation equation containing two variables (either variable x or t is eliminated).

Functional separation of variables: Splitting method

The procedure for constructing exact solutions of functional differential equations consists of some steps, which are outlined below.

Step 1: Substitute solution (1.1.23) into the nonlinear PDE to produce a functional differential equation in three arguments (where the two arguments x and t are simple, whereas the third variable z is composite.

Step 2: With the help of elementary differential substitution (by selecting and renaming terms with derivatives), transforming the functional differential equation into pure functional equation in three arguments x, t and z.

Step 3: Reducing the functional equation containing three arguments into a functional equation in two arguments (the variables x or t is swapped) by the differentiation method.

Step 4: Constructing a solution for the two argument functional differential equation, obtained in step (3) (by using the formula discussed in section (1.1.4)).

Step 5: Constructing a solution by solving the set of ODEs, obtained by the step 4 solutions, and differential substitution made in step 2.

Step 6: Substituting the solution set obtained in step 5 into the original functional differential equation of step 1, to determine the unknown quantities.

Step 7: Analyze separately the degenerate cases (which arise due to violation of assumption made in the solution procedure), which are possible.

Example: Let us consider the unsteady-state heat equation with a nonlinear source term [9]

$$u_t = u_{xx} + f(u). \tag{1.1.28}$$

To find the exact solution of Eq. (1.1.28) by functional separation of variable method, considering the ansatz of the form

$$u = u(\zeta), \quad \zeta = \phi(t)x + \chi(t), \tag{1.1.29}$$

which is consisting of three variables x, t and ζ. Now to determine the unknown functions $u(\zeta), \phi(t)$ and $\chi(t)$, differentiating the Eq. (1.1.29) with respect to x and t

$$\begin{aligned} \frac{\partial u}{\partial t} &= u'(\zeta)[\phi'(t)x + \chi'(t)], \\ \frac{\partial u}{\partial x} &= u'(\zeta)\phi(t), \\ \frac{\partial^2 u}{\partial x^2} &= u''(\zeta)\phi^2(t). \end{aligned} \tag{1.1.30}$$

Substituting Eq. (1.1.30) into the Eq. (1.1.28), we obtain

$$u'(\zeta)[\phi'(t)x + \chi'(t)] = u''(\zeta)\phi^2(t) + f(u). \tag{1.1.31}$$

Dividing the Eq. (1.1.31) by $u'(\zeta) \neq 0$, thus we have

$$\phi'(t)x + \chi' = \frac{u''(\zeta)}{u'(\zeta)}\phi^2 + \frac{f(u)}{u'(\zeta)}. \tag{1.1.32}$$

Now substituting the value of x from Eq. (1.1.29) in Eq. (1.1.32), we get the functional differential equation in two variables t and ζ

$$\frac{\phi'(t)\chi}{\phi(t)} - \chi' - \frac{\phi'(t)\zeta}{\phi(t)} + \frac{u''(\zeta)}{u'(\zeta)}\phi^2(t) + \frac{f(u)}{u'(\zeta)} = 0, \tag{1.1.33}$$

comparing the functional differential Eq. (1.1.33) with Eq. (1.1.19), we obtain

$$\Phi_1 = \frac{\phi'(t)\chi}{\phi(t)} - \chi', \quad \Phi_2 = -\frac{\phi'(t)}{\phi(t)}, \quad \Phi_3 = \phi^2, \qquad \Phi_4 = 1,$$

$$\Psi_1 = 1, \qquad \Psi_2 = \zeta, \qquad \Psi_3 = \frac{u''(\zeta)}{u'(\zeta)}, \qquad \Psi_4 = \frac{f(u)}{u'(\zeta)}.$$

Substituting the above expressions into the Eq. (1.1.19), we get

$$\frac{\phi'(t)\chi}{\phi} - \chi'(t) = B_1\Phi_3 + B_2\Phi_4 = B_1\phi^2 + B_2. \tag{1.1.34}$$

Dividing the Eq. (1.1.34) by ϕ, thus we have

$$\frac{\phi'(t)\chi}{\phi^2} - \frac{\chi'(t)}{\phi} = B_1\phi(t) + \frac{B_2}{\phi(t)},$$

or

$$-\frac{d}{dt}\left(\frac{\chi}{\phi}\right) = B_1\phi(t) + \frac{B_2}{\phi(t)}.$$

Integrating the above equation with respect to t, we get

$$\chi(t) = -\phi(t)\left[B_1 \int \phi(t)dt + B_2 \int \frac{dt}{\phi(t)} + c_2\right]. \tag{1.1.35}$$

Substituting Eq. (1.1.33) into the Eq. (1.1.19) produces

$$-\phi^{-3}\phi'(t) - B_4\phi^{-2} = B_3. \tag{1.1.36}$$

Taking $w = \phi^{-2}(t)$, and $\frac{dw}{dt} = -2\phi^{-3}\phi'(t)$ in Eq. (1.1.36), we obtain

$$\frac{d}{dt}(we^{-2B_4t}) = 2e^{-2B_4t}B_3.$$

Integrating the above equation with respect to t produces $w(t) = c_1e^{2B_4t} - \frac{B_3}{B_4}$.
Substituting $w(t) = \phi^{-2}(t)$ in Eq. (1.1.35), we get

$$\phi(t) = \pm\Big(c_1e^{2B_4t} - \frac{B_3}{B_4}\Big)^{-1/2}. \tag{1.1.37}$$

Now to solve $u(\zeta)$, we have the following equation

$$\frac{u''(\zeta)}{u'(\zeta)} = -B_1 - B_3\zeta. \tag{1.1.38}$$

The solution of Eq. (1.1.38) can be identified by integrating with respect to ζ, thus we have

$$u(\zeta) = c_4\int e^{\left[-B_1\zeta-\frac{1}{2}B_3\zeta^2\right]}d\zeta + c_5. \tag{1.1.39}$$

and

$$\frac{f(u)}{u'(\zeta)} = -B_2 - B_4\zeta,$$

or

$$f(u) = -c_3(B_2 + B_4\zeta)e^{\left(-B_1\zeta-\frac{1}{2}B_3\zeta^2\right)},$$

where c_1, c_2, c_3, c_4 are arbitrary constants. Now substituting Eqs. (1.1.35), (1.1.37) and (1.1.39) in Eq. (1.1.29), ζ in Eq. (1.1.29) takes the form

$$\zeta = \pm\Big(c_1e^{2B_4t} - \frac{B_3}{B_4}\Big)^{-1/2}x - \phi(t)\Big[B_1\int\phi(t)dt + B_2\int\frac{dt}{\phi(t)} + c_2\Big]. \tag{1.1.40}$$

Thus from the above example, it is to be noted that

(i) The splitting method reduces the three argument functional differential equation solution into the pure functional equation solution (by reducing it into a functional equation in two arguments).

(ii) Solving the system of ODEs by splitting the original problem into several subproblems.

1.1.6 Exact travelling wave solutions of nonlinear equations

A travelling wave is a wave of permanent form in which the medium is travelling in the direction of propagation of wave. Commonly, solution of nonlinear equation are determined in terms of travelling wave solution, where waves are represented by the form [15]

$$U(x,t) = U(\xi), \qquad \xi = x + \lambda t, \tag{1.1.41}$$

where λ is speed of the wave. Since it is possible to reduce the PDE (in x,t) into an ODE (in ξ), which can be solved by particular methods. A travelling wave solution occurs in different kinds of mathematical problems.

The travelling wave solution of a linear wave equation was first obtained by d'Alembert in 1747.
Methodology: Suppose that a nonlinear equation in two variables x and t is given by

$$F(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, u_{xxx}, u_{ttt}, ...) = 0, \tag{1.1.42}$$

where $u(x,t)$ is an unknown function, F is a polynomial in u and its partial derivatives involving the highest order derivatives and other nonlinear terms. Major steps of this method are given below [16].
Step 1: We use generalized wave transformation of the form

$$u(x,t) = u(\xi), \quad \xi = x - vt. \tag{1.1.43}$$

Eq. (1.1.43) permits us to reduce Eq. (1.1.42) to an ODE

$$F[u, -vu', u', v^2u'', -vu'', u''...] = 0. \tag{1.1.44}$$

Step 2: Suppose that the ODE (1.1.44) has the formal solution [4].

$$u(\xi) = \sum_{i=0}^{k} B_i(x)\Big(\frac{F'}{F}\Big)^i, \tag{1.1.45}$$

where $B_i(x)$ are functions of x, the functions $F(\xi)$ and $B_i(x)$ are unknown, and are to be identified later, and $B_k(x) \neq 0$.
Step 3: The positive integer k can be identified by considering the homogenous balance between the highest degree derivative and nonlinear terms in Eq. (1.1.44).
Step 4: Substitute Eq. (1.1.45) into the Eq. (1.1.44), obtain the function $F'(\xi)$, calculate all the necessary derivatives $u'(\xi)$, $u''(\xi)$,... of the unknown function $u(\xi)$.
As a result, we obtain a polynomial of $\frac{F'}{F}$ and its derivatives, gathering all the terms of same powers of $\frac{F'}{F}$ and its derivatives, and equating with zero all the coefficients of this polynomial, which produces a system of equations, which on solving yields the values of $B_i(x)$ and $F(\xi)$, using the values in Eq. (1.1.45), we can obtain the exact solution of Eq. (1.1.42).
The outline of the thesis is as follows.
Chapter 2 discusses about how we can use generalized separation of variables and structures of ansatzes which are admitted by the nonlinear hyperbolic type equation to determine the exact solutions.
Chapter 3 of this thesis illustrates the approaches to find the exact solution of nonlinear equations namely Korteweg-De-Vries equation and nonlinear wave equation.
Chapter 4 comprises discussion on the exact travelling wave solution method to determine the exact solution of PDEs, particularly Benjamen Bona Mohany equation by applying $\frac{G'}{G}$-expansion approach [15].
A conclusion chapter highlighting the contributions made in this thesis are given at the end.

Chapter 2

Generalized separation of variables for nonlinear partial differential equations

A method of separation of variables is one of the important methods that is widely used in mathematics to seek the exact solutions of linear equations. This method is used to find exact solutions of equations with two independent variables x and t and an unknown function w in the form of product of functions of different variables.

$$w = \delta(x)\beta(t). \tag{2.0.1}$$

Solution structure (2.0.1) can be regarded as an ansatz that transforms the equation under study to an ODE with an unknown function $\delta(x)$ or an unknown function $\beta(t)$. Exact solutions of nonlinear PDEs can be constructed by arranging the k terms in the form of a finite sum

$$w(x,t) = \sum_{i=1}^{k} g_i(x)a_i(t), \tag{2.0.2}$$

to determine the unknown functions $g_i(x)$ and $a_i(t)$. To construct the exact solution of nonlinear PDE, the following generalization of ansatz (2.0.1) is

$$u = \sum_{i=1}^{m} w_i(t)d_i(x) + g(x,t), \qquad m \geq 1. \tag{2.0.3}$$

used. Ansatz (2.0.3) contains an unknown function $g(x,t)$, m unknown function $d_i(x)$ and m unknown functions $w_i(t)$, which are to be determined with the condition that anastz (2.0.3) reduces the given equation to a system of m ODEs with unknown functions $a_i(t)$.
If we consider $m = 1$ in Eq. (2.0.3), this system becomes an ODE with an unknown function $w_1(t)$. We consider an ansatz which is similar to Eq. (2.0.3) and obtained by replacing u by x, x by u, whereas t will remain same in Eq. (2.0.3)

$$x = \sum_{i=1}^{m} w_i(t)d_i(u) + g(u,t). \tag{2.0.4}$$

Solution structure Eq. (2.0.3) or (2.0.4) are called solution with separated variables, and the method used for their construction is called the generalized procedure of separation of variables [2]. We use the ansatzes of the type Eq. (2.0.3) and Eq. (2.0.4) to construct the exact solution of the nonlinear hyperbolic-type equation.

2.1 Exact solutions of the nonlinear hyperbolic-type equation

Consider the general form of second order linear partial differential equation in two variables with constant coefficients:

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + F = 0,$$

which satisfies the condition
$B^2 - AC > 0$ is known as hyperbolic equation.
Let us consider hyperbolic equation of the form [2]

$$\frac{\partial^2 w}{\partial t^2} = aw\frac{\partial^2 w}{\partial x^2} + b\Big(\frac{\partial w}{\partial x}\Big)^2 + c. \tag{2.1.1}$$

We will also consider the following generalizations of equation (2.1.1), when $c = 0$, and $c = \phi(t)$, i.e

$$\frac{\partial^2 w}{\partial t^2} = aw\frac{\partial^2 w}{\partial x^2} + b\Big(\frac{\partial w}{\partial x}\Big)^2 + \phi(t)w. \tag{2.1.2}$$

2.1.1 Exact solutions by separation of variables of nonlinear hyperbolic-type equation when ($c = 0$)

Consider the special case of Eq. (2.1.1), when $c = 0$, i.e

$$\frac{\partial^2 w}{\partial t^2} = aw\frac{\partial^2 w}{\partial x^2} + b\Big(\frac{\partial w}{\partial x}\Big)^2. \tag{2.1.3}$$

Now to construct the exact solution of Eq. (2.1.3), we use the ansatz

$$w = u(t)d(x) + g(x,t), \qquad u(t) \neq 0. \tag{2.1.4}$$

Above solution structure is some times used to construct the exact solution of nonlinear equations.
Now by differentiating Eq. (2.1.4) with respect to t and x

$$\frac{\partial^2 w}{\partial t^2} = u''d + g_{tt}, \tag{2.1.5}$$

$$\frac{\partial w}{\partial x} = ud' + g_x, \tag{2.1.6}$$

$$\frac{\partial^2 w}{\partial x^2} = ud'' + g_{xx}. \tag{2.1.7}$$

Substituting Eq. (2.1.5)-(2.1.7) into Eq. (2.1.1), we obtain the equation

$$u''d + g_{tt} = a(ud + g)(ud'' + g_{xx}) + b(ud' + g_x)^2, \tag{2.1.8}$$

or

$$u''d + g_{tt} = a(u^2dd'') + augd'' + audg_{xx} + agg_{xx} + bu^2(d')^2 + b(g_x)^2 + 2bug_x(d)',$$

or

$$u''d + g_{tt} - u(ag_{xx}d + 2bg_xd' + agd'') - u^2[add'' + b(d')^2] - agg_{xx} - b(g_x)^2 = 0.$$

Above equation must be an ODE with $u = u(t)$ as an unknown function. Thus

$$add'' + b(d')^2 = \alpha d, \qquad \alpha \in \Re, \tag{2.1.9}$$

and

$$adg_{xx} + 2bd'g_x + ad''g = \tilde{\gamma}(t)d. \tag{2.1.10}$$

For $\alpha = 0$, Eq. (2.1.9) becomes

$$ad(d)'' + b(d')^2 = 0,$$

which has a particular solution $d = x^\mu$, when $a = \frac{\mu b}{1-\mu}$, $\alpha = 0$ and $\mu \neq 1$.
Now substituting $d = x^\mu$ and $b = a\frac{1-\mu}{\mu}$ into Eq. (2.1.10), we obtain

$$ax^\mu g_{xx} + 2a\mu\frac{1-\mu}{\mu}x^{\mu-1}g_x + a\mu(\mu-1)x^{\mu-2}g = \tilde{\gamma}(t)x^\mu,$$

$$g_{xx} + 2(1-\mu)\frac{g_x}{x} + \mu(\mu-1)\frac{g}{x^2} = \frac{\tilde{\gamma}(t)}{a},$$

$$x^2g_{xx} + 2(1-\mu)xg_x + \mu(\mu-1)g = \gamma(t)x^2, \quad \text{where} \quad \frac{\tilde{\gamma}(t)}{a} = \gamma(t). \tag{2.1.11}$$

Let us consider following three cases:
Case 1: For $\mu = 2$, substituting $d = x^2$ and $a = -2b$ in Eq. (2.1.11), it reduces to the following form

$$x^2g_{xx} - 2xg_x + 2g = \gamma(t)x^2. \tag{2.1.12}$$

Since Eq. (2.1.12) is Cauchy Euler equation, let $x = e^\xi$, then Eq. (2.1.12) becomes

$$(\Delta^2 - 3\Delta + 2)g = \gamma(t)e^{2\xi},$$

where $xD = \Delta$, and $x^2D^2 = \Delta(\Delta - 1)$.
The solution for homogenous equation: $(\Delta^2 - 3\Delta + 2) = 0$, is

$$g_c = \tilde{\beta}(t)x^2 + \eta(t)x.$$

The particular solution of Eq. (2.1.12): $(\Delta^2 - 3\Delta + 2)g_p = \gamma(t)e^{2\xi}$, is given by

$$(\Delta - 2)g_p = \gamma(t)e^{2\xi},$$

or

$$\int \frac{d}{d\xi}(e^{-2\xi}g_p) = \gamma(t)\xi,$$

or

$$g_p = \gamma(t)\xi e^{2\xi} = \gamma(t)\ln(x)x^2.$$

Thus the general solution of Eq. (2.1.12) is

$$g(x,t) = \gamma(t)x^2 \ln \mid x \mid + \tilde{\beta}(t)x^2 + \eta(t)x. \tag{2.1.13}$$

Substituting Eq. (2.1.13) in Eq. (2.1.4) by using $d(x) = x^2$, we obtain

$$w(x,t) = \gamma(t)x^2 \ln \mid x \mid + \beta(t)x^2 + \eta(t)x, \quad \text{where} \quad \beta(t) = \tilde{\beta}(t) + u(t). \tag{2.1.14}$$

Establishing $\eta(t) = 0$ in Eq. (2.1.14), we obtain

$$w = \gamma(t)x^2 \ln \mid x \mid + \beta(t)x^2. \tag{2.1.15}$$

Now differentiating Eq. (2.1.15) with respect to t and x

$$\frac{\partial^2 w}{\partial t^2} = \beta''(t)x^2 + \gamma''(t)x^2 \ln \mid x \mid,$$

$$\frac{\partial w}{\partial x} = \gamma(t)(x + 2x \ln \mid x \mid) + \beta(t)2x,$$

$$\frac{\partial^2 w}{\partial x^2} = \gamma(t)(3 + 2\ln \mid x \mid) + 2\beta.$$

Substituting above equations in Eq. (2.1.3) produces

$$\gamma''(t)x^2 \ln \mid x \mid + \beta''(t)x^2 = -2b(\gamma x^2 ln \mid x \mid + \beta(t)x^2)(3\gamma + 2\gamma ln \mid x \mid + 2\beta)$$
$$+ b(\gamma(t)x + 2x\gamma ln \mid x \mid + 2\beta x)^2). \tag{2.1.16}$$

Using Eq. (2.1.16), we obtain the following system of equations for $\gamma(t)$ and $\beta(t)$

$$\gamma''(t) = -2b\gamma^2(t), \qquad \beta''(t) = -2b\beta\gamma + b\gamma^2(t). \tag{2.1.17}$$

In order to get a particular solution of $\gamma''(t) = -2b\gamma^2(t)$, let us consider $\gamma(t) = At^n$ and $\gamma''(t) = An(n-1)t^{n-2}$. Now substituting these values in Eq. (2.1.17), we obtain

$$n(n-1)t^{n-2} = -2bAt^{2n}, \quad \text{which is satisfied by} \quad n = 3.$$

Thus using $A = \frac{-3}{bt^5}$, $\gamma(t)$ is given by

$$\gamma(t) = \frac{-3t^{-2}}{b}. \tag{2.1.18}$$

Substituting Eq. (2.1.18) in $\beta''(t) = -2b\beta\gamma + b\gamma^2(t)$, we obtain

$$\beta''(t) = 6\beta t^{-2} + 9\frac{t^{-4}}{b}, \tag{2.1.19}$$

or

$$t^2\beta''(t) - 6\beta = 9\frac{t^{-2}}{b},$$

which is a nonhomogenous Cauchy Euler equation. Thus the solution of corresponding homogenous equation is:

$$y_c = c_1 t^{-2} + c_2 t^3,$$

and particular solution of Eq. (2.1.19) can be obtained by letting $y_1 = t^{-2}$, and $y_2 = t^3$ in above equation, and taking their Wronskian, since $W[y_1, y_2] = y_1 y_2' - y_2 y_1' = 5$, thus we have

$$y_p = y_1(t) \int \frac{y_2(t)g(t)dt}{w(y_1, y_2)} + y_2 \int \frac{y_1(t)g(t)dt}{w(y_1, y_2)} = t^{-2} \int \frac{t^3(9t^{-4})dt}{5b} + t^3 \int \frac{t^{-2}(9t^{-4})dt}{5b},$$

$$y_p = \frac{-9t^{-2}\ln t}{5b} + \frac{9t^{-2}}{25b}.$$

General solution of Eq. (2.1.17) is

$$\beta(t) = c_1t^3 + c_2t^{-2} + \frac{9t^{-2}}{25b} - \frac{9t^{-2}\ln t}{5b} = c_1t^3 + \Big(c_2 + \frac{9}{25b}\Big)t^{-2} - \frac{9t^{-2}\ln t}{5b}$$
$$= c_1t^3 + c_3t^{-2} - \frac{9\ln t t^{-2}}{5b} \quad \text{where} \quad c_3 = \Big(c_2 + \frac{9}{25b}\Big). \tag{2.1.20}$$

Now using Eqs. (2.1.20) and (2.1.18), exact solution of Eq. (2.1.15) is given by

$$w(x,t) = \frac{-3t^{-2}x^2\ln \mid x \mid}{b} + \Big(c_1t^3 + c_3t^{-2} - \frac{9t^{-2}\ln \mid t \mid}{5b}\Big)x^2, \tag{2.1.21}$$

where c_1 and c_3 are arbitrary constants.

Case 2: For $\mu = 3$, Eq. (2.1.11) reduces to a following form by using $d = x^3$

$$x^2g_{xx} - 4xg_x + 6g = \gamma(t)x^2, \tag{2.1.22}$$

which is Cauchy Euler equation, which reduces to the form

$$(\Delta^2 - 5\Delta + 6)g = \gamma(t)e^{2\xi}.$$

The solution of homogenous equation: $(\Delta^2 - 5\Delta + 6) = 0$, is given by

$$g_c = \beta(t)x^2 + \tilde{\eta}(t)x^3.$$

For particular solution of Eq. (2.1.22), we proceed as follows

$$(\Delta - 2)g_p = \frac{\gamma(t)e^{2\xi}}{(\Delta - 3)} = -\gamma(t)e^{2\xi},$$

thus from above equation, we obtain

$$\int \frac{d}{d\xi}(e^{-2\xi}g_p) = -\int \gamma(t)d\xi,$$

or

$$g_p = -\gamma(t)\xi e^{2\xi} = -\gamma(t)x^2\ln x.$$

Finally general solution of Eq. (2.1.22) is given by

$$g(x,t) = -\gamma(t)x^2\ln x + \beta(t)x^2 + \tilde{\eta}(t)x^3. \tag{2.1.23}$$

Now substituting Eq. (2.1.23) in Eq. (2.1.4), we have

$$w(x,t) = \beta(t)x^2 + \eta(t)x^3 - \gamma(t)\ln x\; x^2, \qquad \text{where} \qquad \eta(t) = u(t) + \tilde{\eta}(t). \tag{2.1.24}$$

Substituting Eq. (2.1.24) into Eq. (2.1.3), and establishing $\gamma(t) = 0$, we get

$$w(x,t) = \beta(t)x^2 + \eta(t)x^3. \tag{2.1.25}$$

Differentiating the Eq. (2.1.25) with respect to t and x

$$\frac{\partial^2 w}{\partial t^2} = \beta''(t)x^2 + \eta''(t)x^3,$$

$$\frac{\partial w}{\partial x} = 2x\beta(t) + 3x^2\eta(t),$$

$$\frac{\partial^2 w}{\partial x^2} = 2\beta(t) + 6x\eta(t).$$

Now substituting the above equations in Eq. (2.1.3), we obtain

$$\beta''(t)x^2 + \eta''(t)x^3 = a\Big(2\beta^2x^2 + 8\beta\eta x^3 + 6\eta^2x^4\Big) + b\Big(4\beta^2x^2 + 9\eta^2x^4 + 12\beta\eta x^3\Big).$$

Thus from above equation, we have following system of equations for $\beta(t)$ and $\eta(t)$, by taking, $a = \frac{-3b}{2}$

$$\beta'' = b\beta^2, \qquad \eta'' = 0. \tag{2.1.26}$$

The equation $\eta'' = 0$, has the solution: $\eta(t) = c_1t + c_2$.
Now we search for the particular solution of $\beta'' = b\beta^2$, by substituting $\beta = At^{n+2}$ and $\beta'' = A(n+2)(n+1)t^n$ in $\beta'' = b\beta^2$, of Eq. (2.1.26), it reduces to

$$A(n+2)(n+1)t^n = bA^2t^{2n+4}, \qquad \text{for} \qquad n = 1,$$

thus for $A = \frac{6}{bt^5}$, Eq. (2.1.26) becomes

$$\beta(t) = \frac{6}{b}t^{-2}.$$

Substituting $\beta(t)$ and $\eta(t)$ in Eq. (2.1.25), thus we have

$$w(x,t) = \frac{6t^{-2}}{b}x^2 + [c_1t + c_2]x^3, \tag{2.1.27}$$

where c_1 and c_2 are arbitrary constants. The ansatz $w = \beta(t)x^2 + \eta(t)x^3$ is a special case of more general ansatz [14]

$$w(x,t) = \delta_3(t)x^3 + \delta_2(t)x^2 + \delta_1(t)x + \delta_0(t),$$

where $\delta_1(t) = 0$ and $\delta_0(t) = 0$.
Case 3: Now consider $\mu \neq 1, 2, 3$ in Eq. (2.1.11), thus we have

$$x^2g_{xx} + 2(1-\mu)xg_x + \mu(\mu-1)g = \gamma(t)x^2.$$

To find the general solution of Eq. (2.1.11), since the homogenous part of the above equation is Cauchy Euler equation:

$$x^2g_{xx} + 2(1-\mu)xg_x + \mu(\mu-1)g = 0,$$

or

$$[\Delta(\Delta-1) + 2(1-\mu)\Delta + \mu(\mu-1))]g = 0,$$

or

$$\Delta = \frac{-(1-2\mu) \pm \sqrt{(1-2\mu)^2 - 4(\mu^2-\mu)}}{2},$$

$$g_c = \tilde{\delta_2}(t)\ x^{\mu} + \delta_0(t)x^{\mu-1}.$$

Now particular solution of Eq. (2.1.11) is given by: $g_p = \frac{\gamma(t)e^{2\xi}}{[\Delta^2+\Delta(1-2\mu)+\mu^2-\mu]}$, or

$$g_p = \frac{\gamma(t)x^2}{(\mu-3)(\mu-2)}. \tag{2.1.28}$$

Consequently the general solution of Eq. (2.1.11) is

$$g(x,t) = \frac{\gamma(t)x^2}{(\mu-3)(\mu-2)} + \tilde{\delta_2}(t)\ x^{\mu} + \delta_0(t)x^{\mu-1}. \tag{2.1.29}$$

Now substituting (2.1.29) in Eq. (2.1.4), we obtain

$$w = \delta_2(t)x^{\mu} + \frac{\gamma(t)x^2}{(\mu-3)(\mu-2)} + \delta_0(t)x^{\mu-1}, \quad \text{where} \quad u(t) + \tilde{\delta_2}(t) = \delta_2(t).$$

Using the Eq. (2.1.4), we get

$$w = \delta_2(t)x^{\mu} + \delta_1(t)x^2 + \delta_0(t)x^{\mu-1}, \qquad \delta_1(t) = \frac{\gamma(t)}{(\mu-3)(\mu-2)}. \tag{2.1.30}$$

Now to find the exact solution of Eq. (2.1.30), differentiating the Eq. (2.1.30) with respect to x and t, thus we have the following equations

$$\begin{aligned} \frac{\partial^2 w}{\partial t^2} &= \delta_1''(t)x^2 + \delta_2''(t)x^{\mu} + \delta_0''(t)x^{\mu-1}, \\ \frac{\partial w}{\partial x} &= \delta_1(t)2x + \delta_2(t)\mu x^{\mu-1} + \delta_0(t)(\mu-1)x^{\mu-2}, \\ \frac{\partial^2 w}{\partial x^2} &= 2\delta_1(t) + \delta_2(t)\mu(\mu-1)x^{\mu-2} + \delta_0(t)(\mu-1)(\mu-2)x^{\mu-3}. \end{aligned} \tag{2.1.31}$$

Substituting Eq. (2.1.31) in Eq. (2.1.3)

$$\begin{aligned} \delta_1''(t)x^2 + \delta_2''(t)x^{\mu} + \delta_0''(t)x^{\mu-1} = a\Big[\delta_2(t)x^{\mu} + \delta_1(t)x^2 + \delta_0(t)x^{\mu-1}\Big]\Big[2\delta_1(t) + \delta_2(t)\mu(\mu-1)x^{\mu-2} + \\ \delta_0(t)(\mu-1)(\mu-2)x^{\mu-3}\Big] + b\Big[\delta_1(t)2x + \delta_2(t)\mu x^{\mu-1} + \delta_0(t)(\mu-1)x^{\mu-2}\Big]^2. \end{aligned} \tag{2.1.32}$$

Using Eq. (2.1.32), we obtain the following system of equation for $\delta_1(t)$ and $\delta_2(t)$

$$\delta_1''(t) = (2a+4b)\delta_1^2. \tag{2.1.33}$$

.

$$\delta_2''(t) = (2a + a\mu(\mu-1) + 4b\mu)\delta_1\delta_2, \qquad \text{where} \quad a = \frac{\mu b}{1-\mu}, \ \mu = \frac{a}{a+b}.$$

Substituting $\mu = \frac{a}{a+b}$ in above equation, we get

$$\delta_2''(t) = \Big(\frac{-a^2 b}{(a+b)^2} + \frac{2a^2+6ab}{a+b}\Big)\delta_1\delta_2. \tag{2.1.34}$$

Now to find the particular solution of Eq. (2.1.33) by considering $\delta_1 = At^{n+3}$ and $\delta_1''(t) = A(n+3)(n+2)t^{n+1}$, thus we have

$$(n+3)(n+2)At^{n+1} = (2a+4b)A^2t^{2n+6}, \quad \text{satisfied by} \quad n = 0.$$

Thus by using $A = \frac{3}{(a+2b)t^5}$, solution of Eq. (2.1.33) is

$$\delta_1 = \frac{3}{a+2b} t^{-2}. \tag{2.1.35}$$

Using Eq. (2.1.35), Eq. (2.1.34) reduces to the following form

$$\delta_2''(t) = (6 - 5\mu + \mu^2) a \delta_1 \delta_2.$$

Thus we have

$$\delta_2''(t) = (6 - 5\mu + \mu^2) \frac{3at^{-2}}{a+2b} \delta_2, \qquad b = \frac{a(1-\mu)}{\mu},$$

or

$$t^2 \delta_2''(t) = (9\mu - 3\mu^2)\delta_2. \tag{2.1.36}$$

For the solution of Eq. (2.1.36), let $e^x = \xi$, $x = \ln \xi$ and $\xi^2 \delta_2''(t) = (\Delta^2 - \Delta)\delta_2$ and substituting these values in Eq. (2.1.36), thus we have

$$(\Delta^2 - \Delta - 9\mu + 3\mu^2)\delta_2 = 0. \tag{2.1.37}$$

Solving Eq. (2.1.37), we consider following two cases;
Case $3a$: When $3\mu^2 - 9\mu \geq \frac{1}{4}$;

$$(\Delta - \frac{1}{2})^2 + 3\mu^2 - 9\mu - \frac{1}{4} = 0 \quad \text{implies} \quad \Delta = \frac{1}{2} + \iota\sigma,$$

where $\sigma = \sqrt{3\mu^2 - 9\mu - \frac{1}{4}}$.
Thus in this case, Eq. (2.1.37) has the following solution

$$\delta_2 = e^{\frac{x}{2}}(c_1 \cos \sigma x + c_2 \sin \sigma x).$$

or

$$\delta_2 = \xi^{\frac{1}{2}}[c_1 \cos(\sigma \ln \xi) + c_2 \sin(\sigma \ln \xi)], \tag{2.1.38}$$

where $\quad \sigma^2 = (3\mu^2 - 9\mu - \frac{1}{4}) > 0$.
Case $3b$: When $3\mu^2 - 9\mu \leq \frac{1}{4}$;
$(\Delta - \frac{1}{2})^2 - (9\mu + \frac{1}{4} - 3\mu^2) = 0$ implies $\Delta = \frac{1}{2} \pm \sigma, \qquad \sigma = \sqrt{9\mu - 3\mu^2 + \frac{1}{4}} > 0$,

$$\delta_2 = \xi^{\frac{1}{2}}(c_1 \xi^{\sigma} + c_2 \xi^{-\sigma}), \tag{2.1.39}$$

where $\sigma^2 = 9\mu - 3\mu^2 + \frac{1}{4} > 0$. Substituting $\mu = \frac{9 \pm 2\sqrt{21}}{6}$ in Eq. (2.1.36) produces

$$t^2 \frac{\partial^2 \delta_2}{\partial t^2} = \Big[9\Big(\frac{9 \pm 2\sqrt{21}}{6}\Big) - 3\Big(\frac{9 \pm 2\sqrt{21}}{6}\Big)^2 \delta_2\Big].$$

Substituting $t^2 \delta_2''(t) = (\Delta^2 - \Delta)$ in above equation, we have $(4\Delta^2 - 4\Delta + 1)\delta_2 = 0$, or $\Delta = \frac{1}{2}, \frac{1}{2}$, thus we obtain

$$\delta_2(t) = e^{\frac{x}{2}}(c_1 + c_2 x),$$

or

$$\delta_2(t) = \xi^{\frac{1}{2}}(c_1 + c_2 \ln \xi), \tag{2.1.40}$$

where c_1 and c_2 are arbitrary constants.

Now substituting $\delta_1(t)$ and $\delta_2(t)$ in Eq. (2.1.11), we obtain the following exact solutions:

(a). Since

$$w = \delta_1 x^2 + \delta_2 x^\mu.$$

Therefore exact soluton of Eq. (2.1.11) using Eq. (2.1.35) and Eq. (2.1.38) is given by

$$w(x,t) = \Big(\frac{3}{a+2b}t^{-2}\Big)x^2 + x^\mu\Big[\xi^{\frac{1}{2}}(c_1\cos(\sigma\ln\xi) + c_2(\sin\sigma\ln\xi)\Big]. \tag{2.1.41}$$

(b). The exact solution of Eq. (2.1.11) by substituting Eqs. (2.1.35) and (2.1.39) is given by

$$w = \Big(\frac{3}{a+2b}t^{-2}\Big)x^2 + \xi^{\frac{1}{2}}(c_1\xi^\sigma + c_2\xi^{-\sigma})x^\mu, \quad \text{where} \quad \mu = \frac{9 \pm 2\sqrt{21}}{6} > 0, \tag{2.1.42}$$

and $\sigma^2 = 9\mu - 3\mu^2 + \frac{1}{4} > 0$.

(c). The exact solution of Eq. (2.1.11) by using Eqs. (2.1.35) and (2.1.40) is given by

$$w = \Big(\frac{3t^{-2}}{a+2b}\Big)x^2 + x^\mu\xi^{\frac{1}{2}}(c_1 + c_2\ln\xi), \qquad \text{where} \qquad \sigma^2 = \Big(3\mu^2 - 9\mu - \frac{1}{4}\Big), \tag{2.1.43}$$

where c_1 and c_2 are arbitrary constants.

2.1.2 Exact solution by separation of variables of nonlinear hyperbolic-type equation ($c \neq 0$)

Consider the hyperbolic equation for $c \neq 0$

$$\frac{\partial^2 w}{\partial t^2} = aw\frac{\partial^2 w}{\partial x^2} + b\Big(\frac{\partial w}{\partial x}\Big)^2 + c. \tag{2.1.44}$$

Exact solution of Eq. (2.1.44) can be constructed by using the ansatz of the form

$$w = u(t) + g(x,t). \tag{2.1.45}$$

The above ansatz is obtained by taking a special case $i = 1$ in general ansatz (2.0.3), where $g(x,t)$ is generalization of $\phi(x)$ in ansatz of the form

$$w(x,t) = \phi(x) + \chi(t),$$

differentiating Eq. (2.1.45) with respect to x and t, we have

$$\begin{aligned} \frac{\partial w}{\partial x} &= g_x, \\ \frac{\partial^2 w}{\partial x^2} &= g_{xx}, \\ \frac{\partial^2 w}{\partial t^2} &= u''(t) + g_{tt}. \end{aligned} \tag{2.1.46}$$

Now substituting Eq. (2.1.46) into the Eq. (2.1.44), we have

$$u''(t) + g_{tt} = a(u+g)g_{xx} + b(g_x)^2 + c, \tag{2.1.47}$$

which must be an ODE with $u = u(t)$ as an unknown function, that is to be determined.
Let us consider the function $g(x,t)$ is of the form

$$g(x,t) = \delta_2(t)x^2 + \delta_1(t)x + \tilde{\delta_0}(t). \tag{2.1.48}$$

Substituting Eq. (2.1.48) in Eq. (2.1.46), we have

$$w(x,t) = u(t) + \delta_2(t)x^2 + \delta_1(t)x + \tilde{\delta_0}(t).$$

Thus we obtain the following ansatz

$$w(x,t) = \delta_2(t)x^2 + \delta_1(t)x + \delta_0(t), \qquad \text{where} \qquad \delta_0 = u(t) + \tilde{\delta_0}(t). \tag{2.1.49}$$

Differentiating Eq. (2.1.49) with respect to t and x, we have

$$\begin{aligned} \frac{\partial w}{\partial x} &= 2\delta_2(t)x + \delta_1(t), \\ \frac{\partial^2 w}{\partial x^2} &= 2\delta_2(t), \\ \frac{\partial^2 w}{\partial t^2} &= \delta_2''(t)x^2 + \delta_1''(t)x + \delta_0''(t). \end{aligned} \tag{2.1.50}$$

Now substituting Eq. (2.1.50) in Eq. (2.1.44), it reduces to the following form thus we have

$$\delta_2''x^2 + \delta_1''x + \delta_0'' = 2a(\delta_2^2x^2 + \delta_1\delta_2 x + \delta_0\delta_2) + 4b\delta_2^2x^2 + 4b\delta_1^2 + 4b\delta_1\delta_2 x + c.$$

Thus from above equation, we have the following system of equation for the functions $\delta_i(t)$

$$\delta_2'' = (2a + 4b)\delta_2^2, \tag{2.1.51}$$

$$\delta_1'' = (2a + 4b)\delta_1\delta_2, \tag{2.1.52}$$

$$\delta_0'' = 2a\delta_0\delta_2 + 4b\delta_1^2 + c. \tag{2.1.53}$$

Now solving Eq. (2.1.51) by substituting $\delta_2 = At^{n+3}$ and $\delta_2''(t) = A(n+3)(n+2)t^{n+1}$. Using $A = \frac{3}{(a+2b)t^5}$, Eq. (2.1.51) has the following solution

$$\delta_2(t) = \frac{3}{a+2b}t^{-2}. \tag{2.1.54}$$

Now substituting Eq. (2.1.54) in Eq. (2.1.52), we obtain Cauchy Euler equation in the form as given below

$$t^2\delta_1'' - 6\delta_1 = 0.$$

Thus we have the following solution

$$\delta_1(t) = c_1e^{3t} + c_2e^{-2t}. \tag{2.1.55}$$

Now solving Eq. (2.1.53) by substituting Eq. (2.1.54) and Eq. (2.1.55), we obtain

$$t^2\delta_0'' - \frac{6a}{a+2b}\delta_0 = 4bt^2(c_1e^{3t} + c_2e^{-2t})^2 + t^2c,$$

which is nonhomogenous Cauchy Euler equation, thus we have

$$\left(\Delta^2 - \Delta - \frac{6a}{a+2b}\right)\delta_0 = bt^2(c_1e^{3t} + c_2e^{-2t})^2 + t^2c,$$

The solution of homogenous equation: $(\Delta^2 - \Delta - \frac{6a}{a+2b}) = 0$ is

$$g_c = \frac{k_1}{2} e^{t\left[1+\sqrt{(\frac{25a+2b}{a+2b})}\right]} + \frac{k_2}{2} e^{t\left[1-\sqrt{(\frac{25a+2b}{a+2b})}\right]},$$

where a and b are constants. Now let us consider a special case for $a = b = 1$ in above equation, which gives $\Delta = 2, -1$. Thus

$$g_c = k_1 e^{2t} + k_2 e^{-t}.$$

$(\Delta^2 - \Delta - \frac{6a}{a+2b}) g_p = bt^2(c_1 e^{3t} + c_2 e^{-2t})^2 + t^2 c, \qquad$ where $a = b = 1$
$(\Delta - 2)(\Delta + 1) f_p = t^2 c_1^2 e^{6t} + t^2 c_2^2 e^{-4t} + 2t^2 c_3 e^t + t^2 c$
$g_p = \frac{t^2 c_1^2 e^{6t}}{(\Delta-2)(\Delta+1)} + \frac{t^2 c_2^2 e^{-4t}}{(\Delta-2)(\Delta+1)} + \frac{2t^2 c_3 e^t}{(\Delta-2)(\Delta+1)} + \frac{t^2 c}{(\Delta-2)(\Delta+1)},$
$g_p = \frac{t^2 c_1^2 e^{6t}}{28} + \frac{t^2 c_2^2 e^{-4t}}{18} - t^2 c_3 e^t - \frac{t^2 c}{2}.$
Thus the general solution of Eq. (2.1.53) is given by

$$\delta_0(t) = k_1 e^{2t} + k_2 e^{-t} + \frac{t^2 c_1^2 e^{6t}}{28} + \frac{t^2 c_2^2 e^{-4t}}{18} - t^2 c_3 e^t - \frac{t^2 c}{2}. \tag{2.1.56}$$

Using Eqs. (2.1.54)-(2.1.56) in Eq. (2.1.49), we have exact solution of the form

$$w(x,t) = \Big(\frac{3}{a+2b} t^{-2}\Big) x^2 + (c_1 e^{3t} + c_2 e^{-2t}) x + k_1 e^{2t} + k_2 e^{-t} + \frac{t^2 c_1^2 e^{6t}}{28} + \frac{t^2 c_2^2 e^{-4t}}{18} - t^2 c_3 e^t - c_4 t^2. \tag{2.1.57}$$

2.1.3 Exact solution by separation of variables of nonlinear hyperbolic-type equation $[c = \phi(t)]$

Now considering a special case for [2]

$$\frac{\partial^2 w}{\partial t^2} = aw \frac{\partial^2 w}{\partial x^2} + b\Big(\frac{\partial w}{\partial x}\Big)^2 + \phi(t) w. \tag{2.1.58}$$

We consider following cases,

Case 1: For $\mu = 2$ and $a = -2b$, the ansatz Eq. (2.1.14) for $\eta(t) = 0$ takes the form

$$w = \gamma(t) x^2 \ln \mid x \mid + \beta(t) x^2, \tag{2.1.59}$$

$$\begin{aligned} \frac{\partial^2 w}{\partial t^2} &= \beta''(t) x^2 + \gamma''(t) x^2 \ln \mid x \mid, \\ \frac{\partial w}{\partial x} &= \gamma(t)(x + 2x \ln \mid x \mid) + \beta(t) 2x, \\ \frac{\partial^2 w}{\partial x^2} &= \gamma(t)(3 + 2 \ln \mid x \mid) + 2\beta. \end{aligned} \tag{2.1.60}$$

Using Eq. (2.1.60) in Eq. (2.1.58), we get

$$\begin{aligned} \gamma''(t) x^2 \ln \mid x \mid + \beta'' x^2 = -2b[\gamma x^2 ln \mid x \mid + \beta x^2][3\gamma + 2\gamma ln \mid x \mid + 2\beta] + \\ b[\gamma x + 2x\gamma ln \mid x \mid + 2\beta x]^2 + \phi[\gamma x^2 \ln \mid x \mid + \beta x^2], \end{aligned} \tag{2.1.61}$$

comparing the coefficients in Eq. (2.1.61), we obtain

$$\begin{aligned} \gamma''(t) &= -2b\gamma^2(t) + \phi\gamma. \\ \beta''(t) &= -2b\beta\gamma + b\gamma^2(t) + \phi\beta. \end{aligned} \tag{2.1.62}$$

Case 2: For $\mu = 3$ and $a = \frac{-3b}{2}$, considering a special case of Eq. (2.1.24) when $\gamma(t) = 0$.

$$w = \beta(t)x^2 + \eta(t)x^3. \tag{2.1.63}$$

Differentiating the Eq. (2.1.63) with respect to t and x, thus we have

$$\begin{aligned} \frac{\partial^2 w}{\partial t^2} &= \beta''(t)x^2 + \eta''(t)x^3, \\ \frac{\partial w}{\partial x} &= 2x\beta(t) + 3x^2\eta(t), \\ \frac{\partial^2 w}{\partial x^2} &= 2\beta(t) + 6x\eta(t). \end{aligned} \tag{2.1.64}$$

Substituting Eq. (2.1.64) in Eq. (2.1.58), we obtain

$$\beta''(t)x^2 + \eta''(t)x^3 = a\big[\beta(t)x^2 + \eta(t)x^3\big]\big[2\beta + 6x\eta\big] + b\big[2x\beta + 3\eta x^2\big]^2 + \phi(t)\big[\beta x^2 + \eta x^3\big]. \tag{2.1.65}$$

Comparing coefficients of x^2 and x^3 in Eq. (2.1.65), thus we have

$$\begin{aligned} \beta'' &= b\beta^2 + \phi\beta, \\ \eta'' &= \phi\eta. \end{aligned} \tag{2.1.66}$$

Case 3: Establishing $\delta_0(t) = 0$, thus Eq. (2.1.28) reduces to the following form

$$w = \delta_2(t)x^\mu + \delta_1(t)x^2. \tag{2.1.67}$$

Now differentiating Eq. (2.1.67) with respect to t and x , we have

$$\begin{aligned} \frac{\partial^2 w}{\partial t^2} &= \delta_1''(t)x^2 + \delta_2''(t)x^\mu, \\ \frac{\partial w}{\partial x} &= \delta_1(t)2x + \delta_2(t)\mu x^{\mu-1}, \\ \frac{\partial^2 w}{\partial x^2} &= 2\delta_1(t) + \delta_2(t)\mu(\mu-1)x^{\mu-2}. \end{aligned} \tag{2.1.68}$$

Substituting Eq. (2.1.68) in Eq. (2.1.58) produces

$$\begin{aligned} \delta_1''(t)x^2 + \delta_2''(t)x^\mu = a[\delta_2(t)x^\mu + \delta_1(t)x^2][2\delta_1(t) + \delta_2(t)\mu(\mu-1)x^{\mu-2}]+ \\ b[\delta_1(t)2x + \delta_2(t)\mu x^{\mu-1}]^2 + \phi(t)[\delta_2 x^\mu + \delta_1 x^2]. \end{aligned} \tag{2.1.69}$$

Comparing coefficients of x^2 and x^μ in Eq. (2.1.69), we obtain the following equations

$$\delta_1'' = (2a + 4b)\delta_1^2 + \phi\delta_1. \tag{2.1.70}$$

.

$$\delta_2'' = (2a + a\mu(\mu-1) + 4\mu b)\delta_1\delta_2 + \phi\delta_2,$$

where $a = \frac{b\mu}{1-\mu}$. Now substituting the value of $\mu = \frac{a}{a+b}$ in above equation, we get

$$\delta_2''(t) = \Big(\frac{-a^2 b}{(a+b)^2} + \frac{2a^2 + 6ab}{a+b}\Big)\delta_1\delta_2 + \phi\delta_2. \tag{2.1.71}$$

Since $\phi(t)$ is some general function. By knowing $\phi(t)$ and by solving Eqs. (2.1.70) and (2.1.71) and substituting the result in Eq. (2.1.58) we can obtain the exact solution of Eq. (2.1.58).

Chapter 3

Exact solutions of nonlinear partial differential equations

In this chapter, we continue the investigation that was started in previous chapter to construct the exact solution of nonlinear equation. By replacing $w \longmapsto x, \quad x \longmapsto w, \ t \longmapsto t$ in Eq. (2.0.3), we obtain an ansatz which is similar to Eq. (2.0.3) and is of the form

$$x = \sum_{i=1}^{m} u_i(t) d_i(w) + g(w, t).$$

We use the above ansatz to construct the exact solution of the nonlinear Korteweg-De-Vries equation and nonlinear wave equation.

3.1 Generalized separation of variables for Korteweg-De-Vries equation

Korteweg-De-Vries equation was first introduced by Boussinesq (1877), and rediscovered by Diederik Korteweg and Gustav de Vries (1895) for a mathematical explanation of the solitary wave phenomenon discovered by S. Russell in 1844. They take the general form [12]

$$\partial_t w + \partial_x^3 w + \partial_x G(w) = 0,$$

where $w(x, t)$ is a function of one space and one time variable, and $G(w)$ is some polynomial of w.
Korteweg -De-Vries equation is [2]

$$\frac{\partial w}{\partial t} + G(w)\Big(\frac{\partial w}{\partial x}\Big)^k + \frac{\partial^3 w}{\partial x^3} = 0, \tag{3.1.1}$$

where k is a real parameter. In Eq. (3.1.1), $G(w)$ is an unknown function that is to be determined. To determine $G(w)$, we use ansatz of the form

$$x = u_1(t) d(w) + u_2(t). \tag{3.1.2}$$

In above ansatz the unknown terms are $u_1(t), u_2(t)$ and $d(w)$. Differentiating the Eq. (3.1.2) with respect to x, we have

$$\frac{\partial w}{\partial x} = \frac{1}{u_1(t) d'}. \tag{3.1.3}$$

Differentiatin the Eq. (3.1.2) with respect to x

$$u_1(t)d''(w_x)^2 + u_1(t)d'w_{xx} = 0,$$

or

$$w_{xx} = -\frac{d''}{u_1^2(t)(d')^3}.$$

Now differentiating the above equation with respect to x

$$u_1(t)d'''(w_x)^3 + 3u_1(t)d''w_xw_{xx} + u_1(t)d'w_{xxx} = 0,$$

or

$$\frac{\partial^3 w}{\partial x^3} = -\frac{d'''}{u_1^3(t)(d')^4} + 3\frac{(d'')^2}{u_1^3(t)(d')^5}. \tag{3.1.4}$$

Differentiating the Eq. (3.1.2) with respect to t

$$-u_1(t)d'w_t = u_1'(t)d(w) + u_2'(t),$$

or

$$\frac{\partial w}{\partial t} = -\frac{u_1'(t)d(w) + u_2'(t)}{u_1(t)d'}. \tag{3.1.5}$$

Now substituting Eqs. (3.1.3)-(3.1.5) in Eq. (3.1.1), we have

$$-\frac{u_1'd}{u_1d'} - \frac{u_2'}{u_1d'} + G(w)\frac{1}{u_1^k(d')^k} - \frac{d'''}{u_1^3(t)(d')^4} + 3\frac{(d'')^2}{u_1^3(d')^5} = 0. \tag{3.1.6}$$

To check the linear independence of functions $y_1 = \frac{d}{d'}$, and $y_2 = \frac{1}{d'}$, which are coefficients of $\frac{-u_1'}{u_1}$ and $\frac{-u_2'}{u_1}$. We observe that Wronskian $= W[y_1, y_2] \neq 0$, so $y_1,\ y_2$ are linearly independent.
Assume that $k \neq 3$, we require that the coefficients of the functions $\frac{1}{u_1^3}$ can be represented as a linear combination, over the field of the real numbers of the functions $\frac{d}{d'}$ and $\frac{1}{d'}$. Thus we have

$$-\frac{d'''}{(d')^4} + 3\frac{(d'')^2}{(d')^5} = \lambda\frac{d}{d'} + \mu\frac{1}{d'}, \tag{3.1.7}$$

or

$$-d'''d' + 3(d'')^2 = \lambda d(d')^4 + \mu(d')^4. \tag{3.1.8}$$

Now inserting Eq. (3.1.7) in Eq. (3.1.6), we obtain

$$-\frac{u_1'd}{u_1d'} - \frac{u_2'}{u_1d'} + \frac{G(w)}{u_1^k(d')^k} + \frac{\lambda}{u_1^3}\frac{d}{d'} + \frac{\mu}{u_1^3}\frac{1}{d'} = 0. \tag{3.1.9}$$

In viewing the Eq. (3.1.9), $G(w)$ is

$$G(w) = [u_1'u_1^{k-1} - \lambda u_1^{k-3}]d(d')^{k-1} + [u_2'u_1^{k-1} - \mu u_1^{k-3}](d')^{k-1}. \tag{3.1.10}$$

Therefore

$$u_1'u_1^{k-1} - \lambda u_1^{k-3} = \delta_1, \tag{3.1.11}$$

$$u_2'u_1^{k-1} - \mu u_1^{k-3} = \delta_2, \tag{3.1.12}$$

where δ_1 and δ_2 are constants.

For $k \neq 3$, Korteweg-de-vries equation admits ansatz (3.1.2) for $G(w)$ of the form

$$G(w) = \delta_1 d(d')^{k-1} + \delta_2 (d')^{k-1}, \tag{3.1.13}$$

where $d = d(w)$ is an arbitrary solution of the Eq. (3.1.8).

Considering some special cases for $d = d(w), \lambda$ and μ in Eq. (3.1.8):

Case a: For $d = \ln w$ if $\lambda = 0$ and $\mu = 1$;

Case $1a$: When $k \neq 2$. Taking $d = \ln w$ in Eq. (3.1.13), $G(w)$ takes the form

$$G(w) = (\delta_1 \ln w + \delta_2) w^{1-k}. \tag{3.1.14}$$

Now $d(w)$ can be determined by using Eq. (3.1.2) as given below

$$d(w) = \frac{x}{u_1(t)} - \frac{u_2(t)}{u_1(t)}, \tag{3.1.15}$$

where $u_1(t)$ and $u_2(t)$ are unknowns, and can be determined by using Eq. (3.1.11), thus by substituting $\lambda = 0$ in Eq. (3.1.11), we obtain

$$u_1' u_1^{k-1} = \delta_1,$$

or

$$u_1(t) = k(k\delta_1 t + c_1)^{\frac{1}{k}}. \tag{3.1.16}$$

Now we find $u_2(t)$ by substituting $\mu = 1$ in Eq. (3.1.12)

$$u_2' u_1^{k-1} - u_1^{k-3} = \delta_2.$$

Dividing above equation by Eq. (3.1.11)

$$\frac{u_2' u_1^{k-1} - u_1^{k-3}}{u_1' u_1^{k-1}} = \frac{\delta_2}{\delta_1},$$

or

$$u_2'(t) - \frac{\delta_2}{\delta_1} u_1'(t) = u_1^{-2}(t). \tag{3.1.17}$$

Using a special case $c_1 = 0$, Eq. (3.1.16) reduces to the following form

$$u_1(t) = k(k\delta_1 t)^{\frac{1}{k}}. \tag{3.1.18}$$

Substituting $u_1(t) = k(k\delta_1 t)^{\frac{1}{k}}$ in Eq. (3.1.17) and integrating the Eq. (3.1.17) with respect to t

$$u_2(t) - \frac{\delta_2}{\delta_1} u_1(t) = \int k(k\delta_1 t)^{\frac{-2}{k}} dt, \tag{3.1.19}$$

$$u_2(t) - \frac{\delta_2}{\delta_1} u_1(t) = \frac{k(k\delta_1)^{\frac{-2}{k}}}{k-2} t^{\frac{k-2}{k}} + c_2. \tag{3.1.20}$$

Dividing Eq. (3.1.20) by $u_1(t)$

$$\frac{u_2(t)}{u_1(t)} = \frac{\delta_2}{\delta_1} + \frac{k(k\delta_1)^{\frac{-2}{k}}}{k-2} \frac{t^{\frac{k-2}{k}}}{u_1(t)} + \frac{c_2}{u_1(t)}. \tag{3.1.21}$$

Thus by rearranging, Eq. (3.1.21) takes the form

$$\frac{u_2(t)}{u_1(t)} = \frac{\delta_2}{\delta_1} + \frac{k(k\delta_1)^{\frac{-3}{k}}}{k-2} t^{\frac{k-3}{k}} + c(t)^{\frac{-1}{k}}, \qquad c = c_2 k(k\delta_1)^{\frac{-1}{k}}. \tag{3.1.22}$$

Substituting Eq. (3.1.22) in Eq. (3.1.15), we obtain

$$d(w) = xk(k\delta_1 t)^{\frac{-1}{k}} - \frac{k(k\delta_1)^{\frac{-3}{k}}}{k-2} t^{\frac{k-3}{k}} + c(t)^{\frac{-1}{k}} - \frac{\delta_2}{\delta_1}. \tag{3.1.23}$$

As we know that in this case

$$d(w) = \ln w, \qquad \text{implies} \qquad w = e^{[d(w)]},$$

thus after substituting Eq. (3.1.23) in above equation, $w(x,t)$ takes the form

$$w(x,t) = e^{\left[xk(k\delta_1 t)^{\frac{-1}{k}} - \frac{k(k\delta_1)^{\frac{-3}{k}}}{k-2} t^{\frac{k-3}{k}} + c(t)^{\frac{-1}{k}} - \frac{\delta_2}{\delta_1}\right]}, \quad k \neq 2. \tag{3.1.24}$$

Thus (3.1.24) is the exact solution of Eq. (3.1.1), where c is an arbitrary constant.
Case $2a$**:** When $k = 2$. Taking $\lambda = 0$ in Eq. (3.1.11), it reduces to the following form

$$u_1' u_1^{k-1} = \delta_1. \tag{3.1.25}$$

Integrating with respet to t, Eq. (3.1.25) reduces to the following form

$$u_1(t) = (2\delta_1 t + c_3)^{\frac{1}{2}}.$$

Taking $c_3 = 0$, in above equation, we have

$$u_1(t) = (2\delta_1 t)^{\frac{1}{2}}. \tag{3.1.26}$$

Now substituting Eq. (3.1.26) in Eq. (3.1.17), we get

$$u_2'(t) - \frac{\delta_2}{\delta_1} u_1'(t) = (2\delta_1 t)^{-1}. \tag{3.1.27}$$

Integrating Eq. (3.1.27) with respect to t

$$u_2(t) - \frac{\delta_2}{\delta_1} u_1(t) = (2\delta_1)^{-1} \ln t + c_4. \tag{3.1.28}$$

$$\frac{u_2(t)}{u_1(t)} = \frac{\delta_2}{\delta_1} + (2\delta_1)^{\frac{-3}{2}} t^{\frac{-1}{2}} \ln t + ct^{\frac{-1}{2}}, \qquad c = c_4(2\delta_1)^{\frac{-1}{2}}. \tag{3.1.29}$$

Substituting Eq. (3.1.29) in Eq. (3.1.15) , we have

$$d(w) = x(2\delta_1 t)^{\frac{-1}{2}} - (2\delta_1)^{\frac{-3}{2}} t^{\frac{-1}{2}} \ln t + ct^{\frac{-1}{2}} - \frac{\delta_2}{\delta_1}. \tag{3.1.30}$$

As a result, we obtain the following exact solution of Eq. (3.1.1).

$$w(x,t) = e^{\left[x(2\delta_1 t)^{\frac{-1}{2}} - (2\delta_1)^{\frac{-3}{2}} t^{\frac{-1}{2}} \ln t + ct^{\frac{-1}{2}} - \frac{\delta_2}{\delta_1}\right]}, \quad k = 2. \tag{3.1.31}$$

Case b: For $d = w^\alpha$ if $\lambda = 0$ and $\mu = -1$, Eq. (3.1.13) implies

$$G(w) = \delta_1 w^\alpha (\alpha w^{\alpha-1})^{k-1} + \delta_2 (\alpha w^{\alpha-1})^{k-1},$$

or

$$G(w) = [\delta_1' w^\alpha + \delta_2'] w^{(\alpha-1)(k-1)}, \tag{3.1.32}$$

where $\delta_1' = \alpha^{k-1}\delta_1$ and $\delta_2' = \alpha^{k-1}\delta_2$. Using $\mu = -1$ in Eq. (3.1.12), we have

$$u_2' u_1^{k-1} + u_1^{k-3} = \delta_2.$$

Dividing above equation by Eq. (3.1.25), we get

$$u_2'(t) - \frac{\delta_2}{\delta_1} u_1'(t) = -u_1^{-2}(t). \tag{3.1.33}$$

Integrating the Eq. (3.1.33) with respect to t, we obtain

$$u_2(t) - \frac{\delta_2}{\delta_1} u_1(t) = -\int k(k\delta_1 t)^{\frac{-2}{k}} dt, \tag{3.1.34}$$

or

$$u_2(t) = \frac{\delta_2}{\delta_1} u_1(t) + \frac{k(k\delta_1)^{\frac{-2}{k}}}{2-k} t^{\frac{k-2}{k}} + c_2. \tag{3.1.35}$$

Dividing Eq. (3.1.35) by $u_1(t)$, produces

$$\frac{u_2(t)}{u_1(t)} = \frac{\delta_2}{\delta_1} + \frac{k(k\delta_1)^{\frac{-2}{k}}}{2-k} \frac{t^{\frac{k-2}{k}}}{u_1(t)} + \frac{c_2}{u_1(t)}. \tag{3.1.36}$$

Using Eq. (3.1.36) in Eq. (3.1.15), d(w) takes the form

$$d(w) = xk(k\delta_1 t)^{\frac{-1}{k}} - \frac{k(k\delta_1)^{\frac{-3}{k}}}{2-k} t^{\frac{k-3}{k}} + c(t)^{\frac{-1}{k}} - \frac{\delta_2}{\delta_1}. \tag{3.1.37}$$

As we know that in this case

$$d = w^\alpha, \qquad \text{and} \qquad w(x,t) = [d(w)]^{\frac{1}{\alpha}}.$$

Thus substituting Eq. (3.1.37) in above equation, Eq. (3.1.1) has following exact solution

$$w(x,t) = \Big[xk(k\delta_1 t)^{\frac{-1}{k}} - \frac{k(k\delta_1)^{\frac{-3}{k}}}{2-k} t^{\frac{k-3}{k}} + c(t)^{\frac{-1}{k}} - \frac{\delta_2}{\delta_1}\Big]^{\frac{1}{\alpha}}, \tag{3.1.38}$$

where α is some integer.

Case c: For $\alpha = \frac{1}{2}$ in $d = w^\alpha$ if $\lambda = \mu = 0$. Using $d = w^{\frac{1}{2}}$ in Eq. (3.1.13), $G(w)$ takes the following form

$$G(w) = \delta_1 w^{\frac{1}{2}} (\frac{1}{2} w^{\frac{-1}{2}})^{k-1} + \delta_2 (\frac{1}{2} w^{\frac{-1}{2}})^{k-1},$$

or

$$G(w) = \delta_1' w^{\frac{2-k}{2}} + \delta_2' w^{\frac{1-k}{2}}, \tag{3.1.39}$$

where $\delta_1' = 2^{1-k}\delta_1$ and $\delta_2' = 2^{1-k}\delta_2$. For $\mu = 0$, Eq. (3.1.12) has the following form

$$u_2' u_1^{k-1} = \delta_2. \tag{3.1.40}$$

Dividing Eq. (3.1.40) by Eq. (3.1.25), we obtain

$$\frac{u_2' u_1^{k-1}}{u_1' u_1^{k-1}} = \frac{\delta_2}{\delta_1}. \tag{3.1.41}$$

Integrating the Eq. (3.1.41) with respect to t, we get

$$u_2(t) = \frac{\delta_2}{\delta_1} u_1(t) + c_2. \tag{3.1.42}$$

Dividing Eq. (3.1.42) by $u_1(t)$, we have

$$\frac{u_2(t)}{u_1(t)} = \frac{\delta_2}{\delta_1} + c_2 k (k\delta_1 t)^{\frac{-1}{k}}. \tag{3.1.43}$$

Using Eq. (3.1.43) in Eq. (3.1.15), $d(w)$ takes the following form

$$d(w) = xk(k\delta_1 t)^{\frac{-1}{k}} - \frac{\delta_2}{\delta_1} - ct^{\frac{-1}{k}} \qquad c = c_2 k (k\delta_1)^{\frac{-1}{k}}. \tag{3.1.44}$$

Since, in this case

$$d = w^{\frac{1}{2}} \quad \text{and} \quad w(x,t) = [d(w)]^2,$$

Using above equation, Eq. (3.1.1) has the following exact solution

$$\begin{aligned} w(x,t) &= \Big[xk(k\delta_1 t)^{\frac{-1}{k}} - ct^{\frac{-1}{k}} - \frac{\delta_2}{\delta_1}\Big]^2, \qquad k \neq 2, \\ &= \Big[x(2\delta_1 t)^{\frac{-1}{2}} - ct^{\frac{-1}{2}} - \frac{\delta_2}{\delta_1}\Big]^2, \qquad k = 2, \end{aligned} \tag{3.1.45}$$

where c is an arbitrary constant.

Case d: For $d = \sin^{-1} w$ and $d' = \frac{1}{\sqrt{1-w^2}}$ if $\lambda = 0$ and $\mu = -1$, Eq. (3.1.13) reduces to the following form

$$G(w) = \delta_1 \sin^{-1} w \Big(\frac{1}{1-w^2}\Big)^{\frac{k-1}{2}} + \delta_2 \Big(\frac{1}{1-w^2}\Big)^{\frac{k-1}{2}} = [\delta_1 \sin^{-1} w + \delta_2](1-w^2)^{\frac{1-k}{2}}. \tag{3.1.46}$$

Using $\mu = -1$, Eq. (3.1.12) takes the form

$$u_2' u_1^{k-1} + u_1^{k-3} = \delta_2. \tag{3.1.47}$$

Dividing the Eq. (3.1.47) by Eq. (3.1.25), we obtain

$$\frac{u_2' u_1^{k-1} + u_1^{k-3}}{u_1' u_1^{k-1}} = \frac{\delta_2}{\delta_1}. \tag{3.1.48}$$

$$u_2'(t) - \frac{\delta_2}{\delta_1} u_1'(t) = -u_1^{-2}(t). \tag{3.1.49}$$

By using Eq. (3.1.18), integrating the Eq. (3.1.49) with respect to t

$$u_2(t) - \frac{\delta_2}{\delta_1} u_1(t) = -\int k(k\delta_1 t)^{\frac{-2}{k}} dt, \tag{3.1.50}$$

Rearranging Eq. (3.1.50), it takes the following form

$$u_2(t) = \frac{\delta_2}{\delta_1} u_1(t) + \frac{k(k\delta_1)^{\frac{-2}{k}}}{2-k} t^{\frac{k-2}{k}} + c_2. \tag{3.1.51}$$

Now dividing Eq. (3.1.51) by $u_1(t)$, we obtain

$$\frac{u_2(t)}{u_1(t)} = \frac{\delta_2}{\delta_1} + \frac{k(k\delta_1)^{\frac{-2}{k}}}{2-k}\frac{t^{\frac{k-2}{k}}}{u_1(t)} + \frac{c_2}{u_1(t)}, \tag{3.1.52}$$

or

$$\frac{u_2(t)}{u_1(t)} = \frac{\delta_2}{\delta_1} + \frac{k(k\delta_1)^{\frac{-3}{k}}}{2-k}t^{\frac{k-3}{k}} + c(t)^{\frac{-1}{k}}, \qquad c = c_2k(k\delta_1)^{\frac{-1}{k}}. \tag{3.1.53}$$

Substituting Eq. (3.1.53) into Eq. (3.1.15), $d(w)$ has the following form

$$d(w) = xk(k\delta_1)^{\frac{-1}{k}} - \frac{k(k\delta_1)^{\frac{-3}{k}}}{2-k}t^{\frac{k-3}{k}} + c(t)^{\frac{-1}{k}} - \frac{\delta_2}{\delta_1}. \tag{3.1.54}$$

Since in this case

$$d = \sin^{-1} w, \quad \text{implies} \quad w(x,t) = \sin d(w).$$

Thus using above equation, Eq. (3.1.1) has the following exact solution

$$w(x,t) = \sin\Big[xk(k\delta_1)^{\frac{-1}{k}} - \frac{k(k\delta_1)^{\frac{-3}{k}}}{2-k}t^{\frac{k-3}{k}} + c(t)^{\frac{-1}{k}} - \frac{\delta_2}{\delta_1}\Big]. \tag{3.1.55}$$

Case e: For $d = \operatorname{arccos} hw/2$ and $d' = \frac{1}{2\sqrt{(w/2)^2-1}}$ if $\lambda = 0$ and $\mu = 0$, Eq. (3.1.13) takes the form

$$G(w) = [\delta_1 \operatorname{arccos} hw/2 + \delta_2](w^2-4)^{\frac{1-k}{2}}. \tag{3.1.56}$$

Case f: For $d = \frac{e^w}{e^w-1}$ and $d' = \frac{-e^w}{(e^w-1)^2}$, if $\lambda = 0$ and $\mu = 0$, Eq. (3.1.13) takes the following form

$$G(w) = \Big[\delta_1\frac{e^w}{e^w-1} + \delta_2\Big]\Big(\frac{-e^w}{(e^w-1)^2}\Big)^{\frac{1-k}{2}}. \tag{3.1.57}$$

For $\mu = 0$, Eq. (3.1.12) reduces to the form as given below

$$u_2'u_1^{k-1} = \delta_2. \tag{3.1.58}$$

Dividing Eq. (3.1.58) by Eq. (3.1.25), we get

$$\frac{u_2'u_1^{k-1}}{u_1'u_1^{k-1}} = \frac{\delta_2}{\delta_1}. \tag{3.1.59}$$

$$u_2' = \frac{\delta_2}{\delta_1}u_1'. \tag{3.1.60}$$

Integrating the Eq. (3.1.60) with respect to t, we obtain

$$u_2(t) = \frac{\delta_2}{\delta_1}u_1(t) + c_2. \tag{3.1.61}$$

Dividing Eq. (3.1.61) by $u_1(t)$, we have

$$\frac{u_2(t)}{u_1(t)} = \frac{\delta_2}{\delta_1} + c_2k(k\delta_1t)^{\frac{-1}{k}}. \tag{3.1.62}$$

Substituting Eq. (3.1.62) in Eq. (3.1.15), $d(w)$ has the following form

$$d(w) = xk(k\delta_1t)^{\frac{-1}{k}} - \frac{\delta_2}{\delta_1} - ct^{\frac{-1}{k}} \qquad c = c_2k(k\delta_1)^{\frac{-1}{k}}. \tag{3.1.63}$$

Since in this case

$$d = \frac{e^w}{e^w - 1}, \qquad \text{and} \qquad w(x,t) = \ln\Big[\frac{d(w)}{d(w)-1}\Big].$$

Thus using above equation, Eq. (3.1.1) has following solution

$$w(x,t) = \ln\Big[\frac{xk(k\delta_1 t)^{\frac{-1}{k}} - ct^{\frac{-1}{k}} - \frac{\delta_2}{\delta_1}}{xk(k\delta_1 t)^{\frac{-1}{k}} - \frac{\delta_2}{\delta_1} - ct^{\frac{-1}{k}} - 1}\Big], \qquad k \neq 2. \tag{3.1.64}$$

$$w(x,t) = \ln\Big[\frac{x(2\delta_1 t)^{\frac{-1}{2}} - ct^{\frac{-1}{2}} - \frac{\delta_2}{\delta_1}}{x(2\delta_1 t)^{\frac{-1}{2}} - ct^{\frac{-1}{2}} - \frac{\delta_2}{\delta_1} - 1}\Big], \qquad k = 2, \tag{3.1.65}$$

where c is an arbitrary constant.

Special Case:

For $k = 3$, Eq. (3.1.1) takes the following form

$$\frac{\partial w}{\partial t} + G(w)\Big(\frac{\partial w}{\partial x}\Big)^3 + \frac{\partial^3 w}{\partial x^3} = 0. \tag{3.1.66}$$

Substituting Eqs. (3.1.3)-(3.1.5) in Eq. (3.1.66), we get

$$-\frac{u_1' d}{u_1 d'} - \frac{u_2'}{u_1 d'} + \frac{G(w)}{u_1^3 (d')^3} - \frac{d'''}{u_1^3(t)(d')^4} + 3\frac{(d'')^2}{u_1^3(t)(d')^5} = 0. \tag{3.1.67}$$

Using Eq. (3.1.67), $G(w)$ has the following form

$$G(w) = u_1' u_1^2 d(d')^2 + u_2' u_1^2 (d')^2 + \frac{d'''}{d'} - 3\frac{(d'')^2}{(d')^2}. \tag{3.1.68}$$

Considering $y_1 = d(d')^2$ and $y_2 = (d')^2$ in Eq. (3.1.68), we observe that Wronskian $W(y_1, y_2) \neq 0$, so using the linear independence of the functions $d(d')^2$ and $(d')^2$, we obtain

$$u_1' u_1^2 = \delta_1, \qquad u_2' u_1^2 = \delta_2, \tag{3.1.69}$$

where δ_1 and δ_2 are arbitrary constants. Using Eq. (3.1.69), Eq. (3.1.68) takes the form

$$G(w) = \delta_1 d(d')^2 + \delta_2 (d')^2 + \frac{d'''}{d'} - 3\frac{(d'')^2}{(d')^2}, \tag{3.1.70}$$

where $d(w)$ is an arbitrary smooth function. Considering the following special cases for Eq. (3.1.70):

Case 1: For $d = e^w$, $d' = e^w$, $d'' = e^w$, $d''' = e^w$ in Eq. (3.1.70), we obtain

$$G(w) = \delta_1 e^{3w} + \delta_2 e^{2w} - 2. \tag{3.1.71}$$

Using Eq. (3.1.71) in Eq. (3.1.66), we get

$$\frac{\partial w}{\partial t} + (\delta_1 e^{3w} + \delta_2 e^{2w} - 2)\Big(\frac{\partial w}{\partial x}\Big)^3 + \frac{\partial^3 w}{\partial x^3} = 0. \tag{3.1.72}$$

Now to find $d(w)$, we use the ansatz (3.1.2). Integrating the Eq. (3.1.69) with respect to t, we get

$$u_1(t) = (3\delta_1 t + c_2)^{\frac{1}{3}}. \tag{3.1.73}$$

$$u_2(t) = \frac{\delta_2}{\delta_1} u_1(t) + c_1. \tag{3.1.74}$$

Using (3.1.73) and (3.1.74) in Eq. (3.1.15), $d(w)$ takes the following form

$$d(w) = \frac{x + c_1}{(3\delta_1 t + c_2)^{\frac{1}{3}}} - \frac{\delta_2}{\delta_1}. \tag{3.1.75}$$

Since, in this case

$$d = e^w, \qquad \text{implies} \qquad w = \ln d(w).$$

Thus using above equation, exact solution of Eq. (3.1.66) is given by

$$w(x,t) = \ln\Big[\frac{x + c_1}{(3\delta_1 t + c_2)^{\frac{1}{3}}} - \frac{\delta_2}{\delta_1}\Big], \tag{3.1.76}$$

where c_1 and c_2 are arbitrary constants.

Case 2: For $d = w^{-1}$, $d' = -w^{-2}$, $d'' = 2w^{-3}$, and $d''' = -6w^{-4}$, Eq. (3.1.70) takes the form

$$G(w) = \delta_1 w^{-5} + \delta_2 w^{-4} + 6w^{-1} - 12w^{-2}. \tag{3.1.77}$$

inserting Eq. (3.1.77) into Eq. (3.1.66), we have [2]

$$\frac{\partial w}{\partial t} + (\delta_1 w^{-5} + \delta_2 w^{-4} + 6w^{-1} - 12w^{-2})\Big(\frac{\partial w}{\partial x}\Big)^3 + \frac{\partial^3 w}{\partial x^3} = 0. \tag{3.1.78}$$

Exact solution of Eq. (3.1.78) is given by using $w = \frac{1}{d(w)}$, i.e

$$w(x,t) = \frac{\delta_1}{\delta_1(x + c_1)(3\delta_1 t + c_2)^{\frac{-1}{3}} - \delta_2}. \tag{3.1.79}$$

Theorem 1 [13].

$$\frac{\partial w}{\partial t} + G(w,t)\Big(\frac{\partial w}{\partial x}\Big)^k + \frac{\partial^3 w}{\partial x^3} = 0. \tag{3.1.80}$$

The solution of Eq. (3.1.80) can be found by using the ansatz (3.1.1), if the following conditions are satisfied:

(1). In viewing the Eq. (3.1.80), the function $G(w,t)$ has the form

$$G(w,t) = [u_1' u_1^{k-1} - \lambda u_1^{k-3}] d(d')^{k-1} + [u_2' u_1^{k-1} - \mu u_1^{k-3}](d')^{k-1} = a(t) d(d')^{k-1} + b(t)(d')^{k-1}, \tag{3.1.81}$$

where $d = d(w)$ is an arbitrary solution of Eq. (3.1.8), $a(t)$ and $b(t)$ are functions of t, whereas $u_1 = u_1(t)$ and $u_2 = u_2(t)$ satisfy the system of equations

$$u_1' u_1^{k-1} - \lambda u_1^{k-3} = a(t), \tag{3.1.82}$$

$$u_2' u_1^{k-1} - \mu u_1^{k-3} = b(t). \tag{3.1.83}$$

(2). For $k = 3$, function $G(w,t)$ has the following form

$$G(w,t) = u_1' u_1^2 d(d')^2 + u_2' u_1^2 (d')^2 + \frac{d'''}{d'} - 3\frac{(d'')^2}{(d')^2} = a(t) d(d')^2 + b(t)(d')^2 + \frac{d'''}{d'} - 3\frac{(d'')^2}{(d')^2}, \tag{3.1.84}$$

where $d = d(w)$ is an arbitrary smooth function, $a(t)$ and $b(t)$ are functions of t, with $u_1(t)$ and $u_2(t)$ satisfy the system of equations

$$u_1' u_1^2 = a(t), \quad u_2' u_1^2 = b(t). \tag{3.1.85}$$

Consider the following special cases of Eq. (3.1.8):

Case 1a: For $d = e^w$ if $\lambda = 0$ and $\mu = 1$, Eq. (3.1.84) has the following form

$$G(w,t) = [a(t)e^w + b(t)](e^w)^{k-1}.$$

Using above equation, Eq. (3.1.80) takes the form [2]

$$\frac{\partial w}{\partial t} + [a(t)e^w + b(t)](e^w)^{k-1}\Big(\frac{\partial w}{\partial x}\Big)^k + \frac{\partial^3 w}{\partial x^3} = 0. \tag{3.1.86}$$

For $\lambda = 0$, Eq. (3.1.82) takes the form

$$u_1' u_1^{k-1} = a(t). \tag{3.1.87}$$

Integrating the Eq. (3.1.87) with respect to t, we get

$$u_1(t) = \Big[k \int a(t)dt + c_1\Big]^{\frac{1}{k}}. \tag{3.1.88}$$

For $\mu = 1$, Eq. (3.1.83) reduces to the form

$$u_2' u_1^{k-1} - u_1^{k-3} = b(t), \tag{3.1.89}$$

or

$$u_2' = b(t)u_1^{1-k} + u_1^{-2}.$$

Integrating the above equation with respect to t produces

$$u_2(t) = \int b(t)u_1^{1-k}dt + \int u_1^{-2}(t)dt + c_2.$$

Substituting Eq. (3.1.88) in above equation, we obtain

$$u_2(t) = \int b(t)\Big[k\int a(t)dt + c_1\Big]^{\frac{1-k}{k}} dt + \int \Big[k\int a(t)dt + c_1\Big]^{\frac{-2}{k}} dt + c_2, \tag{3.1.90}$$

where c_1 and c_2 are arbitrary constants. Since $d(w) = e^w$, thus Eq. (3.1.80) has the following exact solution

$$w(x,t) = \ln\Big[\frac{x}{u_1(t)} - \frac{u_1(t)}{u_2(t)}\Big],$$

where $u_1(t)$ and $u_2(t)$ can be substituted by using Eqs. (3.1.87) and (3.1.89).

Case 1b: For $d(w) = w^{\frac{1}{\alpha}}$ if $\lambda = \mu = 0$, Eq. (3.1.82) has the following form

$$G(w,t) = [a_1(t)w^{\frac{1}{\alpha}} + b_1(t)](w^{\frac{1-\alpha}{\alpha}})^{k-1}, \tag{3.1.91}$$

where $a_1(t) = \alpha^{1-k}a(t)$ and $b_1(t) = \alpha^{1-k}b(t)$. Using Eq. (3.1.91), Eq. (3.1.80) take the form

$$\frac{\partial w}{\partial t} + [a_1(t)w^{\frac{1}{\alpha}} + b_1(t)](w^{\frac{1-\alpha}{\alpha}})^{k-1}\Big(\frac{\partial w}{\partial x}\Big)^k + \frac{\partial^3 w}{\partial x^3} = 0. \tag{3.1.92}$$

Thus the exact solution of Eq. (3.1.80) is given by

$$w(x,t)=\Big[\frac{x}{u_1(t)}-\frac{u_1(t)}{u_2(t)}\Big]^{\alpha}, \tag{3.1.93}$$

where $u_2(t)$ is an unknown function. Now to determine $u_2(t)$ by considering $\mu=-1$ in Eq. (3.1.83), we have

$$u_2'u_1^{k-1}+u_1^{k-3}=b(t). \tag{3.1.94}$$

Integrating the Eq. (3.1.94) with respect to t, we get

$$u_2(t)=\int b(t)u_1^{1-k}dt-\int u_1^{-2}(t)dt+c_2.$$

Substituting Eq. (3.1.88) in above equation, we obtain

$$u_2(t)=\int b(t)\Big[k\int a(t)dt+c_1\Big]^{\frac{1-k}{k}}dt-\int\Big[k\int a(t)dt+c_1\Big]^{\frac{-2}{k}}dt+c_2, \tag{3.1.95}$$

where c_1 and c_2 are arbitrary constant. Thus by substituting Eqs. (3.1.88) and (3.1.95) into Eq. (3.1.93), we can find the exact solution of Eq. (3.1.80).

Case 1*c*: For $d(w)=\arctan 2w$ and $d'=\frac{2}{1+4w^2}$ if $\lambda=0$ and $\mu=0$, Eq. (3.1.82) takes the form

$$G(w,t)=[a_1(t)\arctan 2w+b_1(t)]\Big(\frac{1}{1+4w^2}\Big)^{k-1}, \tag{3.1.96}$$

where $a_1(t)=2a(t)$ and $b_1(t)=2b(t)$. Using Eq. (3.1.96), Eq. (3.1.80) takes the form

$$\frac{\partial w}{\partial t}+[a_1(t)\arctan 2w+b_1(t)]\Big(\frac{1}{1+4w^2}\Big)^{k-1}\Big(\frac{\partial w}{\partial x}\Big)^{k}+\frac{\partial^3 w}{\partial x^3}=0. \tag{3.1.97}$$

Solution of Eq. (3.1.80) is obtained by using the function $w(x,t)=\frac{1}{2}[\tan d(w)]$, where $d(w)$ is given by Eq. (3.1.15)

$$w(x,t)=\frac{1}{2}\tan\Big[\frac{x}{u_1(t)}-\frac{u_1(t)}{u_2(t)}\Big], \tag{3.1.98}$$

where $u_1(t)$ can be substituted by using Eq. (3.1.87), and $u_2(t)$ is given by

$$u_2'u_1^{k-1}=b(t),\qquad \mu=0. \tag{3.1.99}$$

Integrating Eq. (3.1.99) with respect to t

$$u_2(t)=\int b(t)u_1^{1-k}dt+c_2.$$

Substituting Eq. (3.1.88) in above equation, we have the following result

$$u_2(t)=\int b(t)\Big[k\int a(t)dt+c_1\Big]^{\frac{1-k}{k}}dt+c_2, \tag{3.1.100}$$

where c_1 and c_2 are arbitrary constants.

Case 1*d*: For $d(w)=\arcsin hw$ and $d'(w)=\frac{1}{\sqrt{1+w^2}}$ if $\lambda=0$ and $\mu=1$, Eq. (3.1.82) takes the form

$$G(w,t)=a(t)\arcsin hw\Big(\frac{1}{\sqrt{1+w^2}}\Big)^{k-1}+b(t)\Big(\frac{1}{\sqrt{1+w^2}}\Big)^{k-1}=\Big[a(t)\arcsin hw+b(t)\Big](1+w^2)^{\frac{1-k}{2}}. \tag{3.1.101}$$

Using Eq. (3.1.101), Eq. (3.1.80) takes the form

$$\frac{\partial w}{\partial t}+\Big[a(t)\arcsin hw+b(t)\Big](1+w^2)^{\frac{1-k}{2}}\Big(\frac{\partial w}{\partial x}\Big)^k+\frac{\partial^3 w}{\partial x^3}=0.$$

Thus the solution of Eq. (3.1.80) is obtained by using $d(w)=\arcsin hw$, where $d(w)$ is obtained by using Eq. (3.1.15)

$$w(x,t)=\sinh\Big[\frac{x}{u_1(t)}-\frac{u_2(t)}{u_1(t)}\Big], \tag{3.1.102}$$

where $u_1(t)$ and $u_2(t)$ are obtained by using Eqs. (3.1.88) and (3.1.90).

3.2 Generalized separation of variables for nonlinear wave equation

In this section, we want to construct the exact solution of nonlinear equation with the generalized separation of variables [2]. By replacing $w\longmapsto t,\ x\longmapsto w,\ t\longmapsto x$ in Eq. (2.0.3), we obtain an ansatz which is similar to Eq. (2.0.3), and is of the form

$$t=\sum_{i=1}^{m}u_i(x)d_i(w)+f(w,x).$$

We use the above ansatz to construct the exact solutions of the nonlinear equations.
Consider the nonlinear equation [3]

$$\frac{\partial^2 w}{\partial t^2}=bG'(w)\Big(\frac{\partial w}{\partial x}\Big)^2+G(w)\frac{\partial^2 w}{\partial x^2}, \tag{3.2.1}$$

which has many applications especially in the field of liquid crystal theory, wave and gas dynamics.
The purpose of this section is to searching for the set of functions $G(w)$, for which Eq. (3.2.1) admits the following ansatz.

1. Exact solution of the form $t=u_1(x)d(w)+u_2(x)$

Let us consider an ansatz for Eq. (3.2.1)

$$t=u_1(x)d(w)+u_2(x). \tag{3.2.2}$$

Differentiating the Eq. (3.2.2) with respect to t, we obtain

$$\frac{\partial w}{\partial t}=\frac{1}{u_1(x)d'(w)}. \tag{3.2.3}$$

Again differentiating Eq. (3.2.2) with respect to t, thus we have

$$\frac{\partial^2 w}{\partial t^2}=-\frac{d''}{u_1^2(d')^3}. \tag{3.2.4}$$

Now differentiating the Eq. (3.2.1) with respect to x produces

$$u_1'(x)d(w)+u_1(x)d'(w)\frac{\partial w}{\partial x}+u_2'(x)=0.$$

$$\frac{\partial w}{\partial x}=-\frac{u_1'(x)d(w)+u_2'(x)}{u_1(x)d'(w)}. \tag{3.2.5}$$

Again differentiating the Eq. (3.2.5) with respect to x, we obtain

$$u_1''(x)d(w) + 2u_1'(x)d'(w)\frac{\partial w}{\partial x} + u_1(x)d''(w)\Big(\frac{\partial w}{\partial x}\Big)^2 + u_1(x)d'(w)\frac{\partial^2 w}{\partial x^2} + u_2''(x) = 0,$$

or

$$\begin{aligned}\frac{\partial^2 w}{\partial x^2} = -\frac{u_2''}{u_1 d'} - \frac{u_1'' d}{u_1 d'} + \frac{2d(u_1')^2}{d'(u_1)^2} + \frac{2u_2' u_1'}{(u_1)^2 d'} - \frac{(u_1')^2 d^2 d''}{(u_1)^2 (d')^3} - \\ \frac{(u_2')^2 d''}{(u_1)^2 (d')^3} - \frac{2u_2' u_1' d d''}{(u_1)^2 (d')^3}.\end{aligned} \tag{3.2.6}$$

Inserting (3.2.3), (3.2.5) in Eq. (3.2.1)

$$\begin{aligned}-\frac{d''}{u_1^2 (d')^3} = aG'\Big[\frac{u_1' d + u_2'}{u_1 d'}\Big]^2 + G\Big[-\frac{u_2''}{u_1 d'} - \frac{u_1'' d}{u_1 d'} + \frac{2d(u_1')^2}{d'(u_1)^2} + \frac{2u_2' u_1'}{(u_1)^2 d'} - \\ \frac{(u_1')^2 d^2 d''}{(u_1)^2 (d')^3} - \frac{(u_2')^2 d''}{(u_1)^2 (d')^3} - \frac{2u_2' u_1' d d''}{(u_1)^2 (d')^3}\Big],\end{aligned} \tag{3.2.7}$$

or

$$\begin{aligned}&Gu_1 u_2''(d')^2 + Gu_1 u_1'' d(d')^2 - [2Gd(d')^2 + bG'd^2 d' - Gd^2 d''](u_1')^2 - \\ &[2G(d')^2 + 2bdd'G' - 2Gdd'']u_1' u_2' - [-Gd'' + bG'd'](u_2')^2 - d'' = 0.\end{aligned} \tag{3.2.8}$$

The coefficient functions $Gd(d')^2$ and $G(d')^2$, which multiply $u_1 u_1'$ and $u_2' u_1$ are linearly independent over $\Re$. To check their linear independence, let $Y_1 = Gd(d')^2$ and $Y_2 = G(d')^2$.
We observe that $W[Y_1, Y_2] \neq 0$, which implies, $Y_1 = Gd(d')^2$ and $Y_2 = G(d')^2$ are linearly independent.
We restrict the coefficients of $(u_1')^2, u_1' u_2', (u_2')^2$ and function d'', that they can be represented as a linear combination of functions $Gd(d')^2$ and $G(d')^2$ over $\Re$, then we get the following form

$$\begin{aligned}&bG'd^2 d' - Gd^2 d'' = \alpha_1 Gd(d')^2 + \lambda_1 G(d')^2 \\ &bG'dd' - Gdd'' = \alpha_2 Gd(d')^2 + \lambda_2 G(d')^2, \\ &bG'd' - Gd'' = \alpha_3 Gd(d')^2 + \lambda_3 G(d')^2, \\ &d'' = \alpha Gd(d')^2 + \lambda G(d')^2,\end{aligned} \tag{3.2.9}$$

where $\alpha, \alpha_i, \lambda, \lambda_i \in \Re, \quad i = 1, 2, 3$. Now using system of equations of Eq. (3.2.9), we obtain

$$(\alpha_3 d - \alpha_2)Gd(d')^2 + (\lambda_3 d - \lambda_2)G(d')^2 = 0,$$

$$(\alpha_2 d - \alpha_1)Gd(d')^2 + (\lambda_2 d - \lambda_1)G(d')^2 = 0,$$

$$(d^2\alpha_3 - \alpha_1)Gd(d')^2 + (d^2\lambda_3 - \lambda_1)G(d')^2 = 0.$$

As we know from above system of equations that, $Gd(d')^2$ and $G(d')^2$ are linearly independent, thus we have the following system of six equations

$$\alpha_3 d - \alpha_2 = 0, \quad \text{and} \quad \lambda_3 d - \lambda_2 = 0, \tag{3.2.10}$$

$$\alpha_2 d - \alpha_1 = 0, \quad \text{and} \quad \lambda_2 d - \lambda_1 = 0, \tag{3.2.11}$$

$$d^2\alpha_3 - \alpha_1 = 0, \quad \text{and} \quad d^2\lambda_3 - \lambda_1 = 0. \tag{3.2.12}$$

Since determinant of the coefficient matrix of system of Eqs. (3.2.10)-(3.2.12) is non zero, it follows that, $\alpha_i, \lambda_i = 0$ for $i = 1, 2, 3$.

Solving first three equations of Eq. (3.2.9), we obtain

$$bG'd' - Gd'' = 0. \tag{3.2.13}$$

Integrating Eq. (3.2.13) with respect to w, we have

$$G(w) = \beta(d')^{\frac{1}{b}}, \tag{3.2.14}$$

where $\beta \neq 0$, is a constant.

If (3.2.1) admits the ansatz (3.2.2), then the function $G(w)$ can be obtained by using (3.2.12). Now inserting Eq. (3.2.14) in

$$d'' = \alpha G d(d')^2 + \lambda G(d')^2,$$

thus we obtain

$$d'' = \beta(\alpha d + \lambda)(d')^{\frac{1+2b}{b}}, \tag{3.2.15}$$

where $d(w)$ is an arbitrary solution of ODE (3.2.15). $d(w)$ can be determined by integrating the Eq. (3.2.13) with respect to w. Let us consider the case for $\alpha = 0$.

Case 1: If $b \neq -1$, let us consider a transformation in Eq. (3.2.15)

$$\beta\lambda = \epsilon\frac{b}{b+1}, \qquad \epsilon = \pm 1.$$

Thus Eq. (3.2.15) reduces to the form

$$d'' = \epsilon\frac{b}{b+1}(d')^{(1+2b)/b}, \qquad b \neq -1, \tag{3.2.16}$$

or

$$(d')^{-(1+2b)/b} d'' = \epsilon\frac{b}{b+1}.$$

Now integrating the Eq. (3.2.16) with respect to w, we obtain

$$d' = (-\epsilon w + a)^{-b/(1+b)}. \tag{3.2.17}$$

Integrating the Eq. (3.2.17) with respect to w, we have

$$d(w) = -\epsilon(1+b)(-\epsilon w + a)^{1/(1+b)} + c, \tag{3.2.18}$$

where a and c are arbitrary constants. Substituting (3.2.17) in Eq. (3.2.14), we have

$$G(w) = \beta(-\epsilon w + a)^{-1/(1+b)}, \qquad b \neq -1.$$

Now substituting $-\epsilon w + a = W$ in above equation, we get

$$G = \beta W^{-1/(1+b)}, \qquad b \neq -1. \tag{3.2.19}$$

Eq. (3.2.1) after transforming from the function $w \longmapsto W = W(x,t)$ takes the form

$$\frac{\partial^2 W}{\partial t^2} = \beta W^{-1/(1+b)}\frac{\partial^2 W}{\partial x^2} - \frac{\beta b}{1+b}W^{-(2+b)/(1+b)}\Big(\frac{\partial W}{\partial x}\Big)^2. \tag{3.2.20}$$

Since from Eq. (3.2.17), $d' = (W)^{-b/(1+b)}$, which gives on substituting in Eq. (3.2.16)

$$d'' = \epsilon b/(b+1)W^{-1/(1+b)}(d')^2.$$

Now to find the $u_1(x)$ and $u_2(x)$, inserting above equation and Eq. (3.2.12) into the (3.2.8), thus we have the following form

$$\begin{aligned}
&\beta W^{-1/(1+b)}u_1u_2''(d')^2 + \beta W^{-1/(1+b)}u_1u_1''d(d')^2 - \\
&\Big[2\beta W^{-1/(1+b)}d(d')^2 - b/(b+1)W^{-(2+b)/(1+b)}d^2d' - \beta W^{-1/(1+b)}d^2d''\Big](u_1')^2 - \\
&\Big[2\beta W^{-1/(1+b)}(d')^2 - 2dd'b/(b+1)W^{-(2+b)/(1+b)} - 2\beta W^{-1/(1+b)}dd''\Big]u_1'u_2' - \\
&\Big[-\beta W^{-1/(1+b)}d'' - b/(b+1)W^{-(2+b)/(1+b)}d'\Big](u_2')^2 - \epsilon b/(b+1)W^{-1/(1+b)}(d')^2 = 0.
\end{aligned} \tag{3.2.21}$$

From Eq. (3.2.21), we obtain the equation for the functions u_1 and u_2

$$d(d')^2\Big[u_1u_1'' - 2(u_1')^2\Big] + (d')^2\Big[u_1u_2'' - 2u_1'u_2' - \frac{\epsilon b}{\beta(b+1)}\Big] = 0.$$

Since $d(d')^2$ and $(d')^2$ are linearly independent.

$$u_1u_1'' - 2(u_1')^2 = 0, \qquad u_1u_2'' - 2u_1'u_2' - \frac{\epsilon b}{\beta(b+1)} = 0. \tag{3.2.22}$$

The solution of $u_1u_1'' = 2(u_1')^2$ is given by

$$u_1 = \frac{c_1}{x+c_2}, \quad \text{and} \quad u_1' = \frac{-c_1}{(x+c_2)^2}.$$

Now we will find the solution of

$$u_2'' - 2\frac{u_1'}{u_1}u_2' - \frac{\epsilon b}{u_1\beta(b+1)} = 0,$$

or

$$u_2'' + \frac{2}{(x+c_2)}u_2' = \frac{\epsilon b(x+c_2)}{c_1\beta(b+1)}.$$

Multiplying the above equation with $I.F = e^{\int \frac{2}{(x+c_2)}dx} = (x+c_2)^2$, we obtain

$$(x+c_2)^2u_2'' + 2(x+c_2)u_2' = \frac{\epsilon b(x+c_2)^3}{c_1\beta(b+1)}.$$

Thus we have

$$u_2(x) = \frac{\epsilon b(x+c_2)^3}{12c_1\beta(b+1)} + c_3.$$

Now dividing above equation by $u_1(x)$, we obtain

$$\frac{u_2(x)}{u_1(x)} = \frac{\epsilon b(x+c_2)^4}{12c_1^2\beta(b+1)} + \frac{c_3(x+c_2)}{c_1}.$$

And $d(w)$ can be determined by using Eq. (3.2.2)

$$d(w) = \frac{t}{u_1} - \frac{u_2(x)}{u_1(x)},$$

or by using above results, $d(w)$ takes the following form

$$d(w) = \frac{(x+c_2)}{c_1}t - \frac{\epsilon b(x+c_2)^4}{c_3\beta(b+1)} - \frac{c_3(x+c_2)}{c_1}.$$

Using Eq. (3.2.18), we have $W = \frac{-1}{\epsilon(1+b)}[d(w)]^{1+b}$. Thus the exact solution of Eq. (3.2.20) is given by

$$W = \frac{-1}{\epsilon(1+b)}\Big[\frac{(x+c_2)}{c_1}t - \frac{\epsilon b(x+c_2)^4}{c_3\beta(b+1)} - \frac{c_3(x+c_2)}{c_1}\Big]^{1+b}, \tag{3.2.23}$$

where c_1, c_2, c_3, c_4 are arbitrary constants.

Now taking a transformation $x \longrightarrow \eta$, $t \longrightarrow \nu$, and $W = U^{1+b}$ in Eq. (3.2.20). Differentiating $W = U^{1+b}$ with respect to t and x, thus we have

$$\frac{\partial W}{\partial x} = (1+b)U^b\Big(\frac{\partial U}{\partial \eta}\Big),$$

$$\frac{\partial^2 W}{\partial x^2} = b(1+b)U^{b-1}\Big(\frac{\partial U}{\partial \eta}\Big)^2 + (1+b)U^b\Big(\frac{\partial^2 U}{\partial \eta^2}\Big),$$

$$\frac{\partial^2 W}{\partial t^2} = b(1+b)U^{b-1}\Big(\frac{\partial U}{\partial \nu}\Big)^2 + (1+b)U^b\Big(\frac{\partial^2 U}{\partial \nu^2}\Big).$$

After substituting the above equations, Eq. (3.2.20) takes the form

$$\frac{\partial^2 U}{\partial \eta^2} = \frac{1}{\beta}U\frac{\partial^2 U}{\partial \nu^2} + \frac{b}{\beta}\Big(\frac{\partial U}{\partial \nu}\Big)^2. \tag{3.2.24}$$

The Eq. (3.2.24) is a hyperbolic-type equation [2].

2. Exact solution of the form $U = w(\eta)d(\nu) + g(\eta,\nu)$

To find the exact solution of Eq. (3.2.24), we use ansatz of the form

$$U = w(\eta)d(\nu) + g(\eta,\nu). \tag{3.2.25}$$

Now by differentiating Eq. (3.2.25) with respect to η and ν, we have

$$\beta w''d - w(g_{\nu\nu}d + 2bg_\nu d' + gd'') - w^2(dd'' + b(d')^2) + \beta g_{\eta\eta} - agg_{\nu\nu} - b(g_\nu)^2 = 0. \tag{3.2.26}$$

Eq. (3.2.26) must be an ODE with unknown function $w(\eta)$, thus we have

$$dd'' + b(d')^2 = \delta d, \qquad \delta \in \Re. \tag{3.2.27}$$

and

$$(g_{\nu\nu}d + 2bg_\nu d' + gd'') = \alpha(\eta)d. \tag{3.2.28}$$

Eq. (3.2.27) has the following particular solution:

$$d = \nu^\lambda, \quad b = \frac{1-\lambda}{\lambda}, \quad \delta = 0, \quad \lambda \neq 1.$$

Substituting $d = \nu^\lambda$ and $b = \frac{1-\lambda}{\lambda}$ into Eq. (3.2.28), we get

$$\nu^2 g_{\nu\nu} + 2(1-\lambda)\nu g_\nu + \lambda(\lambda-1)g = \alpha(\eta)\nu^2. \tag{3.2.29}$$

Let us consider following three cases:

Case a: For $\lambda = 4$, $d = \nu^4$ and $b = \frac{-3}{4}$, Eq. (3.2.28) takes the form

$$\nu^2 g_{\nu\nu} - 6\nu g_\nu + 12g = \alpha(\eta)\nu^2, \tag{3.2.30}$$

Thus the general solution of Eq. (3.2.30) is given by

$$g(\eta,\nu) = \frac{\alpha(\eta)}{2} + \gamma(\eta)\nu^3 + \tilde{\phi}(\eta)\nu^4. \tag{3.2.31}$$

Substituting Eq. (3.2.31) in Eq. (3.2.25), we have

$$U = \frac{\alpha(\eta)}{2}\nu^2 + \gamma(\eta)\nu^3 + \phi(\eta)\nu^4, \qquad \phi(\eta) = w(\eta) + \tilde{\phi}(\eta). \tag{3.2.32}$$

We consider that $\phi(\eta) = 0$ in Eq. (3.2.32), we obtain

$$U = \frac{\alpha(\eta)}{2}\nu^2 + \gamma(\eta)\nu^3,$$

substituting the above equation into Eq. (3.2.24), we get

$$\beta\frac{\alpha''}{2}\nu^2 + \beta\gamma''\nu^3 = \Big[\frac{\alpha^2}{2}\nu^2 + 4\alpha\gamma\nu^3 + 6\gamma^2\nu^4\Big] - \frac{3}{4}\Big[\alpha^2\nu^2 + 6\alpha\gamma\nu^3 + 9\gamma^2\nu^4\Big].$$

Thus we obtain the following system of equations for the functions $\alpha(\eta)$ and $\gamma(\eta)$

$$\alpha'' = \frac{-\alpha^2}{2\beta}, \qquad \beta\gamma'' = \frac{-1}{2}\alpha\gamma.$$

let $\alpha(\eta) = A\eta^n$, solutions of above equations are given by

$$\alpha(\eta) = \frac{-4\beta}{\eta^2}, \quad \text{and} \quad \gamma(\eta) = c_1\eta^2 + c_2\eta^{-1}.$$

Thus the exact solution of Eq. (3.2.24) is given by

$$U = -2\beta\eta^{-2}\nu^2 + [c_1\eta^2 + c_2\eta^{-1}]\nu^3, \tag{3.2.33}$$

where c_1 and c_2 are arbitrary constants.

Case b: For $\lambda = 3$, $d = \nu^3$ and $b = \frac{-2}{3}$, Eq. (3.2.28) has the following form

$$\nu^2 g_{\nu\nu} - 4\nu g_\nu + 6g = \alpha(\eta)\nu^2. \tag{3.2.34}$$

Thus the general solution of Eq. (3.2.34) is

$$g(\eta,\nu) = -\alpha(\eta)\nu^2 \ln\nu + \tilde{\gamma}(\eta)\nu^3 + \phi(\eta)\nu^2.$$

Now using above equation in Eq. (3.2.25), it takes the form

$$U(\nu,\eta) = \gamma(\eta)\nu^3 + \phi(\eta)\nu^2 - \alpha(\eta)\nu^2 \ln \nu, \qquad \gamma(\eta) = w(\eta) + \tilde{\gamma(\eta)}. \tag{3.2.35}$$

Establishing $\alpha(\eta) = 0$, in Eq. (3.2.35), we get

$$U(\nu,\eta) = \gamma(\eta)\nu^3 + \phi(\eta)\nu^2.$$

Substituting above equation in Eq. (3.2.24), we obtain the following system of equations for $\gamma(\eta)$ and $\phi(\eta)$

$$\phi''(\eta) = \frac{-2}{3\beta}\phi^2, \qquad \gamma''(\eta) = 0.$$

The above equations have the following particular solutions

$$\gamma(\eta) = c_1\eta + c_2, \qquad \phi(\eta) = -\frac{3\beta}{\eta^2}.$$

Thus the exact solution of the Eq. (3.2.24) has the following form

$$U(\nu,\eta) = [c_1\eta + c_2]\nu^3 - 3\beta\eta^{-2}\nu^2, \tag{3.2.36}$$

where c_1 and c_2 are arbitrary constants.
Case c: For $\lambda = 2$, $d = \nu^2$ and $b = \frac{-1}{2}$, Eq. (3.2.28) has the form

$$\nu^2 g_{\nu\nu} - 2\nu g_\nu + 2g = \alpha(\eta)\nu^2, \tag{3.2.37}$$

which has the general solution of the form

$$g(\nu,\eta) = \alpha(\eta)\nu^2 \ln \mid \nu \mid + \phi(\eta)\nu + \tilde{\gamma}(\eta)\nu^2. \tag{3.2.38}$$

By inserting Eq. (3.2.38), Eq. (3.2.25) takes the form

$$U = \gamma(\eta)\nu^2 + \phi(\eta)\nu + \alpha(\eta)\nu^2 \ln \mid \nu \mid, \qquad \gamma(\eta) = w(\eta) + \tilde{\gamma}(\eta). \tag{3.2.39}$$

Establishing $\phi(\eta) = 0$ in Eq. (3.2.39), we obtain the following system of equations for $\alpha(\eta)$ and $\gamma(\eta)$

$$\beta\alpha'' = \alpha^2, \qquad \beta\gamma'' = \alpha\gamma - \frac{1}{2}\alpha^2.$$

The above equation have the following particular solutions

$$\alpha(\eta) = \frac{6\beta}{\eta^2}, \qquad \gamma(\eta) = c_1\eta^3 + c_3\eta^{-2} + \frac{18\beta}{5}\eta^{-2}\ln\eta.$$

Now the exact solution of Eq. (3.2.24) is given by

$$U = \left[c_1\eta^3 + c_3\eta^{-2} + \frac{18\beta}{5}\eta^{-2}\ln \mid \eta \mid\right]\nu^2 + \frac{6\beta}{\eta^2}\nu^2 ln \mid \nu \mid. \tag{3.2.40}$$

Case d: Now considering $\lambda \neq 1, 2, 3$ in Eq. (3.2.25), thus we have

$$\nu^2 g_{\nu\nu} + 2(1-\lambda)\nu g_\nu + \lambda(\lambda - 1)g = \alpha(\eta)\nu^2,$$

which has the following general solution

$$g(\eta,\nu) = \frac{\alpha(\eta)\nu^2}{(\lambda-3)(\lambda-2)} + \tilde{\delta}_2(\eta)\nu^{\lambda} + \delta_0(\eta)\nu^{\lambda-1}. \tag{3.2.41}$$

Now inserting (3.2.41) into Eq. (3.2.25), it takes the form

$$U(\eta,\nu) = \frac{\alpha(\eta)}{(\lambda-3)(\lambda-2)}\nu^2 + \delta_2(\eta)\nu^{\lambda} + \delta_0(\eta)\nu^{\lambda-1}, \qquad w(\eta) + \tilde{\delta}_2(\eta) = \delta_2(\eta),$$

or

$$U = \delta_1(\eta)\nu^2 + \delta_2(\eta)\nu^{\lambda} + \delta_0(\eta)\nu^{\lambda-1}, \quad \text{where} \quad \delta_1(\eta) = \frac{\alpha(\eta)}{(\lambda-3)(\lambda-2)}. \tag{3.2.42}$$

Establishing $\delta_0(\eta) = 0$, Eq. (3.2.42) takes the form

$$U = \delta_1(\eta)\nu^2 + \delta_2(\eta)\nu^{\lambda}.$$

Substituting above equation in Eq. (3.2.24), we have following system of equations for $\delta_1(\eta)$ and $\delta_2(\eta)$

$$\delta_1''(\eta) = \frac{(2+4b)}{\beta}\delta_1^2, \qquad \lambda = \frac{1}{1+b}. \tag{3.2.43}$$

$$\delta_2'' = [6 - 5\lambda + \lambda^2]\delta_1\delta_2. \tag{3.2.44}$$

Now the particular solution of Eq. (3.2.43) is

$$\delta_1(\eta) = \frac{3\beta}{(1+2b)}\eta^{-2}. \tag{3.2.45}$$

And the particular solution of Eq. (3.2.44) is obtained by considering two cases:
Case $1d$: When $\xi^2 = 3\lambda^2 - 9\lambda > \frac{1}{4}$;

$$\delta_2(\eta) = \eta^{\frac{1}{2}}[c_1\cos\xi\ln\eta + c_2\sin\xi\ln\eta]. \tag{3.2.46}$$

Thus in this case, the exact solution of Eq. (3.2.24) is given by

$$U(\eta,\nu) = \frac{3\beta}{(1+2b)}\eta^{-2}\nu^2 + \eta^{\frac{1}{2}}[c_1\cos\xi\ln\eta + c_2\sin\xi\ln\eta]\nu^{\lambda}. \tag{3.2.47}$$

Case $2d$: When $\xi^2 = 3\lambda^2 - 9\lambda < \frac{1}{4}$;

$$\delta_2(\eta) = \eta^{\frac{1}{2}}(c_1\eta^{\xi} + c_2\eta^{-\xi}), \quad \text{where} \quad \lambda = \frac{9 \pm 2\sqrt{21}}{6}. \tag{3.2.48}$$

Thus in this case, the exact solution of Eq. (3.2.24) is given by

$$U(\eta,\nu) = \frac{3\beta}{(1+2b)}\eta^{-2}\nu^2 + \eta^{\frac{1}{2}}(c_1\eta^{\xi} + c_2\eta^{-\xi})\nu^{\lambda}. \tag{3.2.49}$$

Now substituting $\lambda = \frac{9\pm2\sqrt{21}}{6}$ in the Eq. (3.2.44), we obtain

$$\delta_2(\eta) = \eta^{\frac{1}{2}}(c_1 + c_2\ln\eta). \tag{3.2.50}$$

Thus in this case, we obtain the following exact solutions

$$U(\eta,\nu) = \frac{3\beta}{1+2b}\eta^{-2}\nu^2 + \nu^{\lambda}\eta^{\frac{1}{2}}(c_1 + c_2\ln\eta),$$

where c_1 and c_2 are arbitrary constants.

Now returning to the variables from $U \longrightarrow W$, $\nu \longrightarrow x$ and $\eta \longrightarrow t$, Eq. (3.2.22) has the following exact solution

$$W^{1/1+b} = \frac{3\beta}{(1+2b)} x^{-2} t^2 + x^{\frac{1}{2}} [c_1 \cos(\xi \ln x) + c_2 \sin(\xi \ln x)] t^{\lambda}.$$

Case 2: For $b = -1$, and $\alpha = 0$, Eq. (3.2.15) reduces to the form

$$d'' = kd', \qquad k = \beta\lambda. \tag{3.2.51}$$

The solution of Eq. (3.2.51) is the function

$$d = \frac{1}{k} e^{kw} + b, \tag{3.2.52}$$

where b is a constant. Taking $b = 0$ in Eq. (3.2.52) and for $d' = e^{kw}$, Eq. (3.2.14) takes the form

$$G(w) = \beta e^{-kw}. \tag{3.2.53}$$

Inserting (3.2.53) into Eq. (3.2.1), we have

$$\frac{\partial^2 w}{\partial t^2} = \beta e^{-kw} \frac{\partial^2 w}{\partial x^2} - bk\beta e^{-ku} \Big(\frac{\partial w}{\partial x}\Big)^2. \tag{3.2.54}$$

Substistuting $G(w) = \beta e^{-kw}$, $G'(w) = -\beta k e^{-kw}$ and $d'' = ke^{-kw}(d')^2$ in Eq. (3.2.8), we obtain

$$\begin{aligned} &\beta u_1 u_2''(d')^2 + \beta u_1 u_1'' d(d')^2 - [2\beta d(d')^2 - b\beta k d^2 d' - \beta d^2 d''](u_1')^2 - \\ &\quad [2\beta(d')^2 - 2bdd'\beta k - 2\beta dd'']u_1'u_2' - [-\beta d'' - b\beta k d'](u_2')^2 - k(d')^2 = 0. \end{aligned} \tag{3.2.55}$$

Thus we have

$$d(d')^2[u_1 u_1'' - 2(u_1')^2] + (d')^2 \Big[u_1 u_2'' - 2u_1'u_2' - \frac{k}{\beta}\Big] = 0.$$

Since $d(d')^2$ and $(d')^2$ are linearly independent, so their cofficient are

$$u_1 u_1'' - 2(u_1')^2 = 0, \quad u_1 u_2'' - 2u_1'u_2' - \frac{k}{\beta} = 0.$$

Integrating the above system of equation with respect to x, we have

$$u_1 = \frac{c_1}{x + c_2} \quad \text{and} \quad u_1' = \frac{-c_1}{(x+c_2)^2}. \tag{3.2.56}$$

Now we will find the solution of

$$u_2'' - 2\frac{u_1'}{u_1} u_2' = \frac{k}{\beta u_1},$$

or

$$u_2'' + \frac{2}{(x+c_2)} u_2' = \frac{kc_1(x+c_2)}{\beta}.$$

Multiplying the above equation with $I.F = e^{\int \frac{2}{(x+c_2)} dx} = (x+c_2)^2$, we obtain

$$(x+c_2)^2 u_2'' + 2(x+c_2)u_2' = \frac{k(x+c_2)^3}{\beta},$$

and $\int[\frac{d}{dx}(x+c_2)^2u_2']dx = \int \frac{k(x+c_2)^3}{\beta}dx$, or $u_2' = \frac{k(x+c_2)^2}{4\beta}$, thus we have

$$u_2 = \frac{k(x+c_2)^3}{12\beta} + c_3. \tag{3.2.57}$$

Now dividing Eq. (3.2.57) by Eq. (3.2.56) produces

$$\frac{u_2}{u_1} = \frac{kc_1^2(x+c_2)^4}{12\beta} + c_3(x+c_2). \tag{3.2.58}$$

Since for $b=0$, Eq. (3.2.52) has the form

$$d = \frac{1}{k}e^{kw}, \qquad \text{and} \qquad w(x,t) = \frac{1}{k}\ln[kd(w)]. \tag{3.2.59}$$

Now substituting Eqs. (3.2.56) and (3.2.58) in Eq. (3.2.2), $d(w)$ takes the following form

$$d(w) = kc_1(x+c_2)t - \Big[\frac{k^2c_1^2(x+c_2)^4}{12\beta} + c_3(x+c_2) + c_4\Big]. \tag{3.2.60}$$

Inserting Eq. (3.2.60) into (3.2.59), the exact solution of (3.2.54) has the following form

$$w(x,t) = \frac{1}{k}\ln\Big[kc_1(x+c_2)t - \frac{k^2c_1^2(x+c_2)^4}{12\beta} + c_3(x+c_2) + c_4\Big], \tag{3.2.61}$$

where c_1, c_2, c_3, c_4 are arbitrary constants.

Case 3: For $\alpha \neq 0$, $\lambda = 0$ and $b = -1$, Eq. (3.2.15) reduces to the integration of an ODE

$$d'' = \beta\alpha dd'.$$

Assuming $\beta\alpha = 2$, the above equation has the following particular solution:

$$d(w) = a\tan(aw+\delta_0), \qquad \text{or} \qquad d(w) = -a\tanh(aw+\delta_0).$$

Now consider that function $d = d(w)$ in above equation is defined to a transformation $d \longrightarrow \mu_1 d + \mu_2$, such that $\mu_1 = a^{-1}$ and $\mu_2 = 0$

$$d(w) = a^{-1}\tan(aw+\delta_0), \qquad \text{or} \qquad d(w) = -a^{-1}\tanh(aw+\delta_0).$$

Thus differentiating the above equation with respect to w, we have

$$d'(w) = \sec^2(aw+\delta_0), \qquad \text{or} \qquad d'(w) = -\sec^2 h(aw+\delta_0). \tag{3.2.62}$$

Substituting Eq. (3.2.62) in (3.2.14), $G(w)$ takes the form

$$G(w) = \beta\cos^2(aw+\delta_0), \quad \text{or} \quad G(w) = -\beta\cosh^2(aw+\delta_0). \tag{3.2.63}$$

Replacing the variable $aw + \delta_0 = W$ in Eq. (3.2.63)

$$G = \beta\cos^2 W, \quad \text{or} \quad G = -\beta\cosh^2(W).$$

Now differentiating the above equations, thus we have

$$G' = \beta\cos^2 W\tan W, \quad \text{or} \quad G' = \beta\cosh^2 W\tanh W.$$

After transformation from the function w to $W = W(x,t)$, (3.2.1) takes the form

$$\frac{\partial^2 W}{\partial t^2} = \beta\cos^2 W\frac{\partial^2 W}{\partial x^2} + 2\beta\cos^2 W\tan W\Big(\frac{\partial W}{\partial x}\Big)^2. \tag{3.2.64}$$

$$\frac{\partial^2 W}{\partial t^2} = -\beta\cosh^2 W\frac{\partial^2 W}{\partial x^2} + 2\beta\cosh^2 W\tanh W\Big(\frac{\partial W}{\partial x}\Big)^2. \tag{3.2.65}$$

Since $d'' = 2\frac{1}{d'}d(d')^2$, now substituting $d'' = 2b^2\cos^2(W)d(d')^2$ and (3.2.63) in Eq. (3.2.8), thus we have

$$u_1u_2''(d')^2 + u_1u_1''d(d')^2 - [d(d')^2 + 2b\tan W d^2d' - d^2d''](u_1')^2 -$$
$$[2(d')^2 + 4bdd'\tan W - 2dd'']u_1'u_2' + [d'' - 2b\tan W d'](u_2')^2 - \frac{2b^2}{\beta}d(d')^2 = 0. \tag{3.2.66}$$

Substituting $d'' = -2b^2\cosh^2(W)d(d')^2$ and Eq. (3.2.63) in Eq. (3.2.8), thus we have

$$u_1u_2''(d')^2 + u_1u_1''d(d')^2 + [-d(d')^2 + 2b\tanh W d^2d' + d^2d''](u_1')^2 +$$
$$[-2(d')^2 + 4bdd'\tanh W + 2dd'']u_1'u_2' + [d'' + 2b\tanh W d'](u_2')^2 - \frac{2b^2}{\beta}d(d')^2 = 0. \tag{3.2.67}$$

So we obtain the the equation for the functions u_1 and u_2

$$d(d')^2\Big[u_1u_1'' - 2(u_1')^2 - \frac{2b^2}{\beta}\Big] + (d')^2[u_1u_2'' - 2u_1'u_2'] = 0.$$

Since $d(d')^2$ and $(d')^2$ are linearly independent, so their coefficient are

$$u_1u_1'' - 2(u_1')^2 - \frac{2b^2}{\beta} = 0, \qquad u_1u_2'' - 2u_1'u_2' = 0. \tag{3.2.68}$$

Now solving first equation of Eq. (3.2.68), we have

$$u_1 = \frac{\beta[cx + c_2]^2 + 128b^2}{16\beta c_1}. \tag{3.2.69}$$

Therefore, solution of the second equation of Eq. (3.2.68) can be identified by using the solution of the first equation

$$u_2 = \frac{cu_1^3}{3} + c_4,$$

or

$$u_2 = \frac{c}{3}\Big[\frac{\beta[cx + c_2]^2 + 128b^2}{16\beta c_1}\Big]^3 + c_4. \tag{3.2.70}$$

$$\frac{u_2}{u_1} = \frac{c(\beta[cx + c_2]^2 + 128b^2)^2}{768\beta^2c_1^2} + \frac{c_4 16\beta c_1}{\beta[cx + c_2]^2 + 128b^2}. \tag{3.2.71}$$

Since

$$d(w) = a\tan(W), \qquad \text{implies} \qquad W = \arctan[d(w)/a].$$

$d(w)$ can be determined by using ansatz (3.2.2)

$$d(w) = -\frac{c(\beta[cx + c_2]^2 + 128b^2)^2}{768\beta^2c_1^2} + \frac{(t - c_4)16\beta c_1}{\beta[cx + c_2]^2 + 128b^2}. \tag{3.2.72}$$

Now using Eq. (3.2.72), exact solution of Eq. (3.2.64) has the following form

$$W = \arctan\Big[-\frac{c(\beta[cx+c_2]^2+128b^2)^2}{768a\beta^2c_1^2}+\frac{(t-c_4)16\beta c_1}{a\beta[cx+c_2]^2+128b^2}\Big]. \tag{3.2.73}$$

As we know that

$$d(w) = -a\tanh W, \qquad \text{or} \qquad W = \arctan h[-d(w)/a].$$

Thus the exact solution of Eq. (3.2.65) using above equation has the form

$$W = \arctan h\Big[\frac{c(\beta[cx+c_2]^2+128b^2)^2}{768a\beta^2c_1^2}-\frac{(t-c_4)16\beta c_1}{a\beta[cx+c_2]^2+128b^2}\Big], \tag{3.2.74}$$

where c_1, c_2, c_4, c are arbitrary constants.

Case 4: For $\alpha \neq 0$ and $b \neq \frac{-1}{2}$ in Eq. (3.2.15), assuming that

$$\beta\alpha = \frac{2b}{\epsilon}, \qquad \epsilon = \pm 1,$$

considering $\lambda = 0$, Eq. (3.2.15) takes the form

$$d'' = \frac{2b}{\epsilon}d(d')^{1+2b/b}. \tag{3.2.75}$$

Integrating Eq. (3.2.75) with respect to w, we obtain $(d')^{-1/b} = \frac{1}{\epsilon}d^2$, integrating again the above equation with respect to w, we get

$$d = \epsilon(1+2b)(\epsilon w+b)^{1/1+2b}. \tag{3.2.76}$$

Differentiating Eq. (3.2.76) with respect to w, we obtain $d' = (\epsilon w+b)^{\frac{-2b}{1+2b}}$, substituting this in Eq. (3.2.14) produces

$$G(w) = \beta(\epsilon w+b)^{-2/1+2b}, \qquad G'(w) = \frac{-2\beta}{1+2b}(\epsilon w+b)^{-(3+2b)/1+2b}. \tag{3.2.77}$$

Now by taking a transformation $\epsilon w + b = W$, Eq. (3.2.21) takes the form [3]

$$\frac{\partial^2 W}{\partial t^2} = \beta(W)^{-2/1+2b}\frac{\partial^2 W}{\partial x^2} - \frac{2b\beta}{1+2b}(W)^{-(3+2b)/1+2b}\Big(\frac{\partial W}{\partial x}\Big)^2. \tag{3.2.78}$$

Thus the problem is now reduced to integrate Eq. (3.2.78). Now substituting Eq. (3.2.77) and $d'' = -\frac{2a}{\beta}d(d')^2G(w)$ in Eq. (3.2.8)

$$\begin{aligned}&\beta(W)^{-2/1+2b}u_1u_2''(d')^2+\beta(W)^{-2/1+2b}u_1u_1''d(d')^2-\\&\quad\Big[2\beta(W)^{-2/1+2b}d(d')^2+b\frac{-2\beta}{1+2b}(W)^{-3-2b/1+2b}d^2d'-\beta(W)^{-2/1+2b}d''\Big](u_1')^2-\\&\quad\Big[2\beta(W)^{-2/1+2b}(d')^2+2bdd'\frac{-2\beta}{1+2b}(W)^{\frac{-3-2b}{1+2b}}-2\beta(W)^{-2/1+2b}dd''\Big]u_1'u_2'-\\&\quad\Big[-\beta(W)^{-2/1+2b}d''+b\frac{-2\beta}{1+2b}(W)^{\frac{-3-2b}{1+2b}}d'\Big](u_2')^2+2bd(d')^2(W)^{-2/1+2b}=0.\end{aligned} \tag{3.2.79}$$

Therefore, we obtain the following equations for $u_1(x)$ and $u_2(x)$

$$d(d')^2\Big[u_1u_1''-2(u_1')^2+\frac{2b}{\beta}\Big]+(d')^2\Big[u_1u_2''-2u_1'u_2'\Big]=0. \tag{3.2.80}$$

Since $d(d')^2$ and $(d')^2$ are linearly independent, so we have

$$u_1 u_1'' - 2(u_1')^2 + \frac{2b}{\beta} = 0, \qquad \text{and} \qquad u_1 u_2'' - 2u_1' u_2' = 0. \tag{3.2.81}$$

Now solving first equation of Eq. (3.2.81), let us consider $p = \frac{du_1}{dx}$ and $\frac{d^2u_1}{dx^2} = p\frac{dp}{du_1}$

$$u_1 p \frac{dp}{du_1} - 2(p)^2 + \frac{2b}{\beta} = 0.$$

Since $I.F = \frac{1}{u_1{}^4}$, so

$$\frac{d}{du_1}\Big(\frac{z}{u_1^4}\Big) = -\frac{4b}{u_1^5\beta}.$$

Integrating with respect to u_1 produces

$$u_1 = \frac{\sqrt{b}x}{\sqrt{\beta}} + c_1 \qquad \text{where} \qquad c_1 = 0. \tag{3.2.82}$$

Now solving second equation of Eq. (3.2.81), $u_2(x)$ has the following form

$$u_2 = c_3\Big(\frac{\sqrt{b}x}{\sqrt{\beta}}\Big)^3 + c_4. \tag{3.2.83}$$

$$\frac{u_2}{u_1} = \frac{c_3 b}{\beta}x^2 + \frac{c_4\sqrt{\beta}}{\sqrt{b}x}. \tag{3.2.84}$$

Inserting Eqs. (3.2.82) and (3.2.83) into Eq. (3.2.2), $d(w)$ has the form

$$d(w) = \Big[\frac{t\sqrt{\beta}}{\sqrt{b}x} - \frac{c_3 b}{\beta}x^2 - \frac{c_4\sqrt{\beta}}{\sqrt{b}x}\Big]. \tag{3.2.85}$$

Substituting (3.2.85) in Eq. (3.2.76), we have

$$W = \Big(\frac{1}{2\epsilon(1+2b)}\Big[\frac{t\sqrt{\beta}}{\sqrt{b}x} - \frac{c_3 b}{\beta}x^2 - \frac{c_4\sqrt{\beta}}{\sqrt{b}x}\Big]\Big)^{1+2b}. \tag{3.2.86}$$

which is the exact solution of Eq. (3.2.78), where c_3 and c_4 are arbitrary constants.
Now considering a transformation by changing the variable $W = U^{1+2b}$ in Eq. (3.2.78)

$$\begin{aligned}
\frac{\partial W}{\partial x} &= (1+2b)U^{2b}\frac{\partial U}{\partial x},\\
\frac{\partial^2 W}{\partial x^2} &= 2b(1+2b)U^{2b-1}\Big(\frac{\partial U}{\partial x}\Big)^2 + (1+2b)U^{2b}\frac{\partial^2 U}{\partial x^2},\\
\frac{\partial W}{\partial t} &= (1+2b)U^{2b}\frac{\partial U}{\partial t},\\
\frac{\partial^2 W}{\partial t^2} &= 2b(1+2b)U^{2b-1}\Big(\frac{\partial U}{\partial t}\Big)^2 + (1+2b)U^{2b}\frac{\partial^2 U}{\partial t^2}.
\end{aligned} \tag{3.2.87}$$

Now substituting (3.2.87) in Eq. (3.2.78), we obtain following equation [3]

$$\frac{\partial^2 U}{\partial x^2} = \frac{2b}{\beta}U\Big(\frac{\partial U}{\partial t}\Big)^2 + \frac{1}{\beta}U^2\frac{\partial^2 U}{\partial t^2}. \tag{3.2.88}$$

Now to construct the exact solution of Eq. (3.2.88), we use ansatz of the form

$$U = u(x)d(t) + g(x,t), \qquad u(x) \neq 0. \tag{3.2.89}$$

Now by differentiating Eq. (3.2.89) with respect to t and x

$$\frac{\partial^2 U}{\partial x^2} = u''d + g_{xx},$$
$$\frac{\partial U}{\partial t} = ud' + g_t, \tag{3.2.90}$$
$$\frac{\partial^2 U}{\partial t^2} = ud'' + g_{tt}.$$

Substituting Eq. (3.2.90) into Eq. (3.2.88), we obtain the equation

$$u''d + g_{xx} = \frac{1}{\beta}(ud+g)^2(ud''+g_{tt}) + \frac{2b}{\beta}(ud+g)(ud'+g_t)^2,$$

or

$$(u''d + g_{xx})\beta = (u^2d^2 + g^2 + 2udg)(ud''+g_{tt}) + 2b(ud+g)(u^2(d')^2 + g_t^2 + 2ud'g_t),$$

or

$$(u''d + g_{xx})\beta = u^3d^2d'' + g^2ud'' + 2u^2dgd'' + u^2d^2g_{tt} + g^2g_{tt} + 2udgg_{tt} +$$
$$2b[(u^3d(d')^2 + udg_t^2 + 2u^2dd'g_t + u^2g(d')^2 + gg_t^2 + 2ud'gg_t)], \tag{3.2.91}$$

or

$$(u''d)\beta - u^3[d^2d'' + 2bd(d')^2] - u^2[2dgd'' + d^2g_{tt} + 4bdd'g_t + 2bg(d')^2] -$$
$$u[g^2d'' + 2dgg_{tt} + 2bdg_t^2 + 4bd'gg_t] - g^2g_{tt} - 2bgg_t^2 + g_{xx}\beta = 0. \tag{3.2.92}$$

The above equation must be an ODE with $u = u(x)$ as an unknown function. Hence

$$d^2d'' + 2bd(d')^2 = \gamma d, \qquad \gamma \in \Re, \tag{3.2.93}$$

$$2dgd'' + d^2g_{tt} + 4bdd'g_t + 2bg(d')^2 = \alpha(x)d. \tag{3.2.94}$$

For $\gamma = 0$, Eq. (3.2.93) becomes

$$dd'' + 2b(d')^2 = 0,$$

which has a particular solution: $d = t^\mu$, $\quad b = \frac{1-\mu}{2\mu}$, $\quad \gamma = 0$, $\quad \mu \neq 0, 1$.
Now substituting $d = t^\mu$, and $b = \frac{1-\mu}{2\mu}$ into Eq. (3.2.94), we obtain

$$t^2g_{tt} + 2(1-\mu)tg_t + \mu(\mu-1)g = \alpha(x)t^{2-\mu}. \tag{3.2.95}$$

To find general solution of Eq. (3.2.95), let us consider three cases for $d = t^\mu$:
Case 1: For $\mu = 2$, we get

$$t^2g_{tt} - 2tg_t + 2g = \alpha(x). \tag{3.2.96}$$

The solution for homogenous equation:

$$g_c = \tilde{\lambda}(x)t^2 + \eta(x)t.$$

Thus the general solution of Eq. (3.2.95) is

$$g(x,t) = \tilde{\lambda}(x)t^2 + \eta(x)t + \alpha(x). \tag{3.2.97}$$

By substituting Eq. (3.2.97) in Eq. (3.2.89), we obtain

$$U = \lambda(x)t^2 + \eta(x)t + \alpha(x), \qquad \lambda(x) = \tilde{\lambda}(x) + u(x). \tag{3.2.98}$$

Taking $\alpha(x) = 0$ in Eq. (3.2.98) and differentiating with respect to t and x, we have

$$\lambda''(x)t^2 + \eta''(x)t = \frac{-1}{2\beta}[4\lambda^3(x)t^4 + 4\lambda^2\eta(x)t^3 + 5\lambda\eta^2t^2 + \eta^3t + 4t^3\lambda^2\eta] + \frac{2}{\beta}[\lambda^3(x)t^4 + \lambda\eta^2(x)t^2 + 2\lambda^2\eta t^3]. \tag{3.2.99}$$

As a result, we obtain the following system of equations for $\lambda(x)$ and $\eta(x)$

$$\lambda''(x) = \frac{-1}{2\beta}\lambda\eta^2(x), \quad \eta''(x) = \frac{-1}{2\beta}\eta^3(x). \tag{3.2.100}$$

System (3.2.100) can be solved completely in the implicit form. The second equation of Eq. (3.2.100) can be determined by applying Jacobi elliptic function method [6], and using the solution of first equation, we can obtain the solution of first equation in system (3.2.100).

Case 2: Now consider $\mu \neq 1, 2, 3$ in Eq. (3.2.95)

$$t^2g_{tt} + 2(1-\mu)tg_t + \mu(\mu-1)g = \alpha(x)t^{2-\mu}.$$

Homogenous part of the above equation has the solution:

$$g_c = \tilde{\delta_2}(x)t^{\mu} + \delta_0(x)t^{\mu-1}.$$

Now particular solution of Eq. (3.2.95) is given by

$$g_p = \frac{\alpha(x)e^{(2-\mu)\xi}}{2(2\mu-3)(\mu-1)}.$$

Consequently the general solution of Eq. (3.2.95) has the following form

$$g(x,t) = \frac{\alpha(x)t^{2-\mu}}{2(2\mu-3)(\mu-1)} + \tilde{\delta_2}(x)t^{\mu} + \delta_0(x)t^{\mu-1}. \tag{3.2.101}$$

Using Eq. (3.2.101), Eq. (3.2.89) reduces to the following form

$$U = \delta_2(x)t^{\mu} + \frac{\alpha(x)t^{2-\mu}}{2(2\mu-3)(\mu-1)} + \delta_0(x)t^{\mu-1}, \quad \text{where} \quad u(x) + \tilde{\delta_2}(x) = \delta_2(x),$$

or

$$U(x,t) = \delta_2(x)t^{\mu} + \delta_1(x)t^{2-\mu} + \delta_0(x)t^{\mu-1}, \qquad \delta_1(x) = \frac{\alpha(x)}{2(2\mu-3)(\mu-1)}. \tag{3.2.102}$$

Establishing $\delta_0(x) = 0$, in Eq. (3.2.102) and differentiating the Eq. (3.2.102) with respect to x and t, thus we have the following equations

$$\begin{aligned} \frac{\partial^2 U}{\partial x^2} &= \delta_1''(x)t^{2-\mu} + \delta_2''(x)t^{\mu}, \\ \frac{\partial U}{\partial t} &= (2-\mu)\delta_1(x)t^{1-\mu} + \delta_2(x)\mu t^{\mu-1}, \\ \frac{\partial^2 U}{\partial t^2} &= (2-\mu)(1-\mu)\delta_1(x)t^{-\mu} + \delta_2(x)\mu(\mu-1)t^{\mu-2}. \end{aligned} \tag{3.2.103}$$

Substituting (3.2.103) in Eq. (3.2.88), we obtain the following system of equations for $\delta_1(x)$ and $\delta_2(x)$

$$\delta_2''(x) = \frac{2}{\beta}\Big((2\mu^2 - 5\mu + 3\Big)\delta_1\delta_2^2,$$

$$\delta_1''(x) = \frac{4}{\beta\mu}\Big(\mu^3 - 2\mu^2 + 1\Big)\delta_1^2\delta_2.$$

Inserting $\mu = \frac{1}{2b+1}$ into above equations, we have

$$\begin{aligned}\delta_2''(x) &= \frac{4b(6b+1)}{\beta(2b+1)^2}\delta_1\delta_2^2,\\ \delta_1''(x) &= \frac{4}{\beta}\Big(\frac{(2b+1)^3 - 4b - 1}{(2b+1)^2}\Big)\delta_1^2\delta_2.\end{aligned} \tag{3.2.104}$$

Using $b = \frac{-1}{4}$, in above equation, Eq. (3.2.104) reduces to the form

$$\delta_1(x) = \frac{\alpha(x)}{2}, \qquad \delta_2''(x) = \frac{\alpha(x)}{\beta}\delta_2^2.$$

System (3.2.104) has the particular solution

$$\delta_1(x) = c_1x^{-1} + c_2x^2, \qquad \text{and} \qquad \delta_2(x) = \frac{\beta}{c_1x + c_2x^4}.$$

Thus, Eq. (3.2.102) takes the form

$$U(x,t) = \Big(\frac{\beta}{c_1x + c_2x^4}\Big)t^{\mu} + \Big(c_1x^{-1} + c_2x^2\Big)t^{2-\mu},$$

which is the exact solution of Eq. (3.2.88).

Now returning to the variables from $U \longrightarrow W$, Eq. (3.2.78) has the following exact solution

$$W^{1/1+2b} = \Big(\frac{\beta}{c_1x + c_2x^4}\Big)t^{\mu} + \Big(c_1x^{-1} + c_2x^2\Big)t^{2-\mu}.$$

For $b = \frac{-1}{4}$, the exact solution of Eq. (3.2.78) takes the form

$$W(x,t) = \Big[\Big(\frac{\beta}{c_1x + c_2x^4}\Big)t^{\mu} + (c_1x^{-1} + c_2x^2)t^{2-\mu}\Big]^2.$$

3. **Exact solution of the form:**

$$U(x,t) = u(x) + g(x,t). \tag{3.2.105}$$

The above ansatz is obtained by taking a special case of solution structure (2.0.3). Let us consider the function $g(x,t)$ is of the form

$$g(x,t) = \phi_2(x)t^2 + \phi_1(x)t + \tilde{\phi_0}(x). \tag{3.2.106}$$

Substituting (3.2.106) in (3.2.105), we obtain the following ansatz

$$U(x,t) = \phi_2(x)t^2 + \phi_1(x)t + \phi_0(x), \qquad \phi_0(x) = u(x) + \tilde{\phi_0}(x). \tag{3.2.107}$$

Establishing $\phi_0(x) = 0$, and differentiating the resulting form of Eq. (3.2.107) with respect to t and x, thus we have

$$\phi_2''(x)t^2 + \phi_1''(x)t = \frac{2b}{\beta}\Big(4\phi_2^3t^4 + 4\phi_1^2\phi_2t^2 + 8\phi_1\phi_2^2t^3 + \phi_1^3t\Big) + \frac{1}{\beta}\Big(2\phi_2^3t^4 + 2\phi_2\phi_1^2t^2 + 4t^3\phi_1\phi_2^2\Big).$$

From above equation, we obtain the following system of equations for $\phi_1(x)$ and $\phi_2(x)$

$$\phi_2'' = \frac{2+8b}{\beta}\phi_2\phi_1^2, \qquad \phi_1'' = \frac{2b}{\beta}\phi_1^3. \tag{3.2.108}$$

For $b = \frac{-1}{4}$, the first equation of Eq. (3.2.108) has the solution

$$\phi_2(x) = c_1 x + c_2.$$

Now to find the solution of second equation of Eq. (3.2.108), we apply Jacobi elliptic function method [11], homogeneous balance between the highest order derivative and the nonlinear terms in (3.2.108) is $m + 2 = 3m$, which gives $m = 1$. Then the elliptic function $\mathrm{sn}\,(x)$ is given by $\mathrm{sn}\,(x) = \sin x$,
and $\mathrm{cn}\,(x)$ is given by $\mathrm{cn}\,(x) = \cos x$,
and $\mathrm{dn}\,(x) = \sqrt{1 - k^2\mathrm{sn}^2\,(x)}$.

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{cn}\,(x) = -\mathrm{sn}\,(x)\,\mathrm{dn}\,(x),$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{dn}\,(x) = -k^2\mathrm{sn}\,(x)\,\mathrm{cn}\,(x).$$

Now using $\phi_1(x) = \sum_{i=0}^{m} a_i\mathrm{cn}^i\,(x)$, thus $\phi_1(x)$ takes the form [6]

$$\phi_1(x) = a_0 + a_1\mathrm{cn}\,(x). \tag{3.2.109}$$

$$\frac{d\phi_1}{dx} = -a_1\mathrm{sn}\,(x)\,\mathrm{dn}\,(x).$$

$$\frac{d^2\phi_1}{dx^2} = a_1\Big[(2k^2-1)\mathrm{cn}\,(x) - 2k^2\mathrm{cn}^3\,(x)\Big]. \tag{3.2.110}$$

Substituting Eqs. (3.2.109) and (3.2.110), second equation of system (3.2.108) reduces to the following form

$$a_1[(2k^2-1)\mathrm{cn}\,(x) - 2k^2\mathrm{cn}^3\,(x)] = \frac{-1}{2\beta}[a_0 + a_1\mathrm{cn}\,(x)]^3. \tag{3.2.111}$$

$$2a_1\beta(2k^2-1)\mathrm{cn}\,(x) - 4\beta a_1k^2\mathrm{cn}^3\,(x) + a_0^3 + 3a_0^2a_1\mathrm{cn}\,(x) + 3a_1^2a_0\mathrm{cn}^2\,(x) + a_1^3\mathrm{cn}^3\,(x) = 0,$$

or

$$[2a_1(2k^2-1)\beta + 3a_0^2a_1]\mathrm{cn}\,(x) + a_0^3 + 3a_1^2a_0\mathrm{cn}^2\,(x) + (a_1^3 - 4\beta a_1k^2)\mathrm{cn}^3\,(x) = 0.$$

Equating the coefficients with zero in above equation, we get $a_0 = 0$ and $a_1 = 2k\sqrt{\beta}$. Thus, using a_0 and a_1, Eq. (3.2.109) takes the form

$$\phi_1(x) = 2k\sqrt{\beta}\mathrm{cn}\,(x).$$

Thus using $\phi_1(x)$ and $\phi_2(x)$, Eq. (3.2.88) has the exact solution of the form

$$U = (c_1x + c_2)t^2 + (2k\sqrt{\beta}\cos x)t.$$

Thus for $b = \frac{-1}{4}$, Eq. (3.2.78) has the solution of the form

$$W = U^{1/2} = [\phi_2(x)t^2 + \phi_1(x)t + \phi_0(x)]^{1/2}. \tag{3.2.112}$$

Generalizing the Eq. (3.2.78) by changing the variable [3], we get

$$\frac{\partial^2 W}{\partial t^2} = \beta(W)^{-k/1+kb}\frac{\partial^2 W}{\partial x^2} - \frac{kb\beta}{1+kb}(W)^{-(3+kb)/1+kb}\Big(\frac{\partial W}{\partial x}\Big)^2. \tag{3.2.113}$$

By using $W = U^{1+kb}$, Eq. (3.2.78) reduces to the form

$$\frac{\partial^2 U}{\partial x^2} = \frac{2b}{\beta} U^{k-1} \left(\frac{\partial U}{\partial t}\right)^2 + \frac{1}{\beta} U^k \frac{\partial^2 U}{\partial t^2}. \tag{3.2.114}$$

$$W = U^{1/k} = [\phi_2(x)t^2 + \phi_1(x)t + \phi_0(x)]^{1/k}, \tag{3.2.115}$$

which is the exact solution obtained by generalizing the Eq. (3.2.78). Thus ansatz (2.0.4) is very important in constructing the exact solutions of nonlinear PDEs.

Chapter 4

Exact travelling wave solutions of nonlinear partial differential equations

Suppose that a nonlinear equation in two variables x and t is given by

$$F(u, u_x, u_t, u_{xx}, u_{tt}, u_{xt}, u_{xxx}, u_{ttt}, ...) = 0, \tag{4.0.1}$$

where $u(x,t)$ is an unknown function, F is a polynomial in u and its partial derivatives involving the highest order derivatives and other nonlinear terms.

Main steps of this method are given below:

Step 1: We use generalized wave transformation of the form

$$u(x,t) = u(z), \quad z = \phi(x)t + \psi(x), \tag{4.0.2}$$

where $\phi(x)$ and $\psi(x)$ are differentiable function of x. By using the Eq. (4.0.2), Eq. (4.0.1) has the form [4, 15, 16]

$$F[u, (\phi'(x)t + \psi'(x)u'(z), ...] = 0. \tag{4.0.3}$$

Step 2: Suppose that the ODE (4.0.3) has the formal solution

$$u(z) = \sum_{i=0}^{k} B_i(x)\Big(\frac{F'}{F}\Big)^i, \tag{4.0.4}$$

where $B_i(x)$ are function of x, the functions $F(z)$ and $B_i(x)$ are unknown, and are to be determined later and $B_k(x) \neq 0$.

Step 3: The positive integer k can be determined by considering the homogenous balance between the highest degree derivative and nonlinear terms in Eq. (4.0.3).

Step 4: Substitute Eq. (4.0.4) into Eq. (4.0.3), and obtain the function $F'(z)$, and calculate all the necessary derivatives $u'(z)$, $u''(z)$,... of the unknown function $u(z)$. As a result, we obtain a polynomial of $\frac{F'}{F}$ and its derivatives, gathering all the terms of same powers of $\frac{F'}{F}$ and its derivatives, and equating with zero all the coefficients of this polynomial, which yields a system of equations, which on solving gives the values of $B_i(x)$ and $F(z)$, using these values in Eq. (4.0.4), we can get the exact solution of Eq. (4.0.1).

4.1 Exact travelling wave solutions of Benjamen Bona Mohany equation

T. B. Benjamen, J. L. Bona,and J. J. Mohany introduced *Benjamen-Bona-Mohany equation*(BBM) in 1972 for studying the propagation of long waves. BBM equation is applicable for studying shallow water waves, long wavelength surface waves in liquids [12].
We will consider the BBM equation of the form [15]

$$u_t + au_x + buu_x - cu_{xxt} = 0, \tag{4.1.1}$$

where a, b and c are arbitrary constants.
Using the generalized wave transformation of the form

$$u(x,t) = u(z), \quad z = \phi(x)t + \psi(x),$$

where $\phi(x)$ and $\psi(x)$ are differentiable function of x.
Differentiating Eq. (4.0.2) with respect to t and x, we get

$$\begin{aligned}
\frac{\partial u}{\partial t} &= u'(z)\phi(x),\\
\frac{\partial u}{\partial x} &= u'(z)[\phi'(x)t + \psi'(x)],\\
\frac{\partial^2 u}{\partial x^2} &= u''(z)[\phi'(x)t + \psi'(x)]^2 + u'(z)[\psi''(x) + t\phi''(x)],\\
\frac{\partial}{\partial t}\Big(\frac{\partial^2 u}{\partial x^2}\Big) &= u'''(z)\phi[\phi'(x)t + \psi'(x)]^2 + u''(z)[2(\phi'(x))^2 t + 2\psi'(x)\phi'(x) + t\phi''(x)\phi + \phi\psi''(x)] + u'(z)\phi''(x).
\end{aligned} \tag{4.1.2}$$

Now substituting the Eq. (4.1.2) in Eq. (4.1.1), we obtain

$$\begin{aligned}
&u'(z)\phi(x) + au'(z)[\phi'(x)t + \psi'(x)] + buu'(z)[\phi'(x)t + \psi'(x)] - cu'''(z)\phi[\phi'(x)t + \psi'(x)]^2\\
&\quad -cu''(z)[2(\phi'(x))^2 t + 2\psi'(x)\phi'(x) + t\phi''(x)\phi + \phi\psi''(x)] - cu'(z)\phi''(x) = 0,
\end{aligned} \tag{4.1.3}$$

or

$$\begin{aligned}
&u'(z)[\phi + a\phi'(x)t + a\psi'(x) - c\phi''(x)] + buu'(z)[\phi'(x)t + \psi'(x)] - cu'''(z)\phi[\phi'(x)t + \psi'(x)]^2\\
&\quad -cu''(z)[2(\phi'(x))^2 t + 2\psi'(x)\phi'(x) + t\phi''(x)\phi + \phi\psi''(x)] = 0.
\end{aligned} \tag{4.1.4}$$

Integration the Eq. (4.1.4) with respect to z, thus we obtain the following ODE

$$\begin{aligned}
&2u(z)[\phi + a\phi'(x)t + a\psi'(x) - c\phi''(x)] + bu^2(z)[\phi'(x)t + \psi'(x)] - 2u''(z)[\phi(\phi'(x)t +\\
&\quad \psi'(x))^2] - 2cu'(z)[2(\phi'(x))^2 t + 2\psi'(x)\phi'(x) + t\phi''(x)\phi + \phi\psi''(x)] = 0.
\end{aligned} \tag{4.1.5}$$

Balancing the order of highest degree terms of Eq. (4.1.5) , where order of $u''(z)$ is $m+2$ and u^2 is $2m$, thus on balancing we have $2m = m+2$, which implies $m = 2$. Thus Eq. (4.0.4) reduces to the following form [1]

$$u(z) = B_0 + B_1(x)\Big(\frac{F'}{F}\Big) + B_2(x)\Big(\frac{F'}{F}\Big)^2. \tag{4.1.6}$$

Squaring both sides of the Eq. (4.1.6), we obtain

$$u^2(z) = B_2^2\Big(\frac{F'}{F}\Big)^4 + 2B_1B_2\Big(\frac{F'}{F}\Big)^3 + (B_1^2 + 2B_0B_2)\Big(\frac{F'}{F}\Big)^2 + 2B_0B_1\Big(\frac{F'}{F}\Big) + B_0^2. \tag{4.1.7}$$

Differentiating Eq. (4.1.6) with respect to z, we have

$$u'(z) = B_1(x)\Big(\frac{F''}{F} - \frac{(F')^2}{F^2}\Big) + 2B_2(x)\Big(\frac{F'}{F}\Big)\Big(\frac{F''}{F} - \frac{(F')^2}{F^2}\Big),$$

or

$$u'(z) = \Big(B_1 + 2\frac{F'}{F}B_2\Big)\Big(\frac{F''}{F} - \frac{(F')^2}{F^2}\Big), \tag{4.1.8}$$

$$u''(z) = 2B_2\Big(\frac{F''}{F} - \frac{(F')^2}{F^2}\Big)^2 + \Big(B_1 + 2\frac{F'}{F}B_2\Big)\Big(\frac{F'''}{F} + 2\frac{(F')^3}{F^3} - 3\frac{F''F'}{F^2}\Big). \tag{4.1.9}$$

Substituting Eq. (4.1.6)-(4.1.9) in Eq. (4.1.5), thus we obtain

$$\begin{aligned}
&2\Big[B_0 + B_1\frac{F'}{F} + B_2(\frac{F'}{F})^2\Big]\Big(\phi + a\phi'(x)t + a\psi'(x) - c\phi''(x)\Big) + \\
&\quad b\Big[B_2^2\Big(\frac{F'}{F}\Big)^4 + 2B_1B_2\Big(\frac{F'}{F}\Big)^3 + (B_1^2 + 2B_0B_2)\Big(\frac{F'}{F}\Big)^2 + 2B_0B_1\Big(\frac{F'}{F}\Big) + B_0^2\Big]\Big(\phi'(x)t + \psi'(x)\Big) - \\
&\quad 2c\phi\Big[2B_2\Big(\frac{F''}{F} - \frac{(F')^2}{F^2}\Big)^2 + \Big(B_1 + 2\frac{F'}{F}B_2\Big)\Big(\frac{F'''}{F} + 2\frac{(F')^3}{F^3} - 3\frac{F''F'}{F^2}\Big)\Big] \\
&\quad \Big((\phi'(x))^2t^2 + (\psi'(x))^2 + 2t\phi'(x)\psi'(x)\Big) - \\
&\quad \Big[\Big(2cB_1 + 4c\frac{F'}{F}B_2\Big)\Big(\frac{F''}{F} - \frac{(F')^2}{F^2}\Big)\Big]\Big(2(\phi'(x))^2t + 2\psi'(x)\phi'(x) + t\phi''(x)\phi + \phi\psi''(x)\Big) = 0.
\end{aligned} \tag{4.1.10}$$

Thus we have

$$
\begin{aligned}
&2\phi B_0+2\phi B_1F'F^{-1}+2\phi B_2(F')^2F^{-2}+2aB_0\phi'(x)t+2a\phi'(x)B_1F'(tF^{-1})+\\
&\quad 2B_2a\phi'(x)(F')^2(tF^{-2})+2B_0a\psi'(x)+2a\psi'(x)B_1F'F^{-1}+\\
&\quad 2aB_2(F')^2F^{-2}\psi'(x)-c\phi''(x)2B_0-2c\phi''(x)B_1F'F^{-1}-\\
&\quad 2c\phi''(x)B_2(x)(F')^2F^{-2}+B_2^2b\phi'(x)(F')^4(tF^{-4})+2bB_1B_2\phi'(x)(F')^3(tF^{-3})+\\
&\quad bB_1^2\phi'(x)(F')^2(tF^{-2})+2bB_0B_2\phi'(x)(F')^2(tF^{-2})+2bB_0B_1\phi'(x)(F')(tF^{-1})+\\
&\quad B_0^2b\phi'(x)t+\psi'(x)B_2^2(F')^4F^{-4}+2bB_1B_2\psi'(x)(F')^3F^{-3}+bB_1^2\psi'(x)(F')^2F^{-2}+\\
&\quad 2bB_0B_2\psi'(x)(F')^2F^{-2}+2bB_0B_1\psi'(x)(F')F^{-1}+B_0^2b\psi'(x)-4cB_1(\phi'(x))^2F''(tF^{-1})-\\
&\quad 8c(\phi'(x))^2F'F''(tF^{-2})B_2+4cB_1(\phi'(x))^2(F')^2(tF^{-2})+8c(\phi'(x))^2(F')^3(tF^{-3})B_2-\\
&\quad 4cB_1F''\psi'(x)\phi'(x)(F^{-1})-8c\psi'(x)\phi'(x)F'F''(F^{-2})B_2+4cB_1\psi'(x)\phi'(x)(F')^2(F^{-2})+\\
&\quad 8c\psi'(x)\phi'(x)(F')^3(F^{-3})B_2-2cB_1\phi''(x)\phi F''(tF^{-1})-4c\phi''(x)\phi F'F''(tF^{-2})B_2+\\
&\quad 2cB_1\phi''(x)\phi(F')^2(tF^{-2}+4c\phi''(x)\phi(F')^3(tF^{-3})B_2-2cB_1\phi\psi''(x)F''(tF^{-1})-\\
&\quad 4c\phi\psi''(x)F'F''(tF^{-2})B_2+2cB_1\phi\psi''(x)(F')^2(tF^{-2})+4c\phi\psi''(x)(F')^3(tF^{-3})B_2-\\
&\quad 2(\phi')^2t^2c\phi B_1F'''F^{-1}-2c\phi B_1F'''F^{-1}(\psi'(x))^2-4c\phi B_1F'''F^{-1}\psi'(x)t\phi'(x)-\\
&\quad 4c\phi B_2F'''F^{-2}(\phi'(x))^2F't^2-4cB_2F'''F^{-2}\phi F'(\psi'(x))^2-8tcB_2F'''F^{-2}\phi F'\psi'(x)\phi'(x)-\\
&\quad 4c\phi B_1(F')^3F^{-3}(\phi'(x))^2t^2-4c\phi B_1(F')^3F^{-3}(\psi'(x))^2-8ct\phi B_1(F')^3F^{-3}\psi'(x)\phi'(x)-\\
&\quad 8c\phi B_2(F')^4F^{-4}(\phi'(x))^2t^2-16c\phi B_2(F')^4F^{-4}\phi'(x)\psi'(x)t-\\
&\quad 8(F')^4F^{-4}(\psi'(x))^2c\phi B_2+6c\phi B_1F''F^{-2}(\phi'(x))^2F't^2+6c\phi B_1F''F^{-2}(\psi'(x))^2F'+\\
&\quad 12c\phi B_1F''F^{-2}\phi'(x)\psi'(x)F't+12F''F^{-3}(\phi'(x))^2(F')^2t^2c\phi B_2+12F''F^{-3}(\psi'(x))^2(F')^2c\phi B_2+\\
&\quad 24tF''F^{-3}\psi'(x)(F')^2c\phi B_2\phi'(x)-4c\phi B_2(F'')^2F^{-2}(\phi'(x))^2t^2-\\
&\quad 4c\phi B_2(F'')^2F^{-2}(\psi'(x))^2-8tc\phi B_2(F'')^2F^{-2}\psi'(x)\phi'(x)-8tc\phi B_2(F')^4F^{-4}\psi'(x)\phi'(x)-\\
&\quad 4c\phi B_2(F')^4F^{-4}(\phi'(x))^2t^2-4c\phi B_2(F')^4F^{-4}(\psi'(x))^2+16tc\phi B_2F''(F')^2F^{-3}\psi'(x)\phi'(x)+\\
&\quad 8c\phi B_2F''(F')^2F^{-3}(\phi'(x))^2t^2+8c\phi B_2F''(F')^2F^{-3}(\psi'(x))^2=0,
\end{aligned}
\tag{4.1.11}
$$

equating all the coefficients of $F^0, tF^0, t^2F^0, F^{-1}, tF^{-1}, t^2F^{-1}, F^{-2}, tF^{-2}, t^2F^{-2}, F^{-3}, tF^{-3}, t^2F^{-3}, F^{-4}, tF^{-4}, t^2F^{-4}$ with zero in Eq. (4.1.11), we obtain the following equations respectively [4]

$$2aB_0(x)\phi'(x)+bB_0^2(x)\phi'(x)=0. \tag{4.1.12}$$

Using Eq. (4.1.12), we obtain $\phi'(x)=0$ or $\phi(x)=k$, and $B_0(x)=0$ or $B_0(x)=\frac{-2a}{b}$

$$2\phi B_0+2B_0a\psi'(x)-c\phi''(x)2B_0+2B_0b\psi'(x)=0. \tag{4.1.13}$$

Now solving Eq. (4.1.13) by substituting $\phi(x)=k$ where k is some constant, we obtain

$$\psi'(x)=\frac{-k}{a+b},$$

which gives on integrating with respect to x

$$\psi(x)=\frac{-k}{a+b}x+A_0(x).$$

Equating the coefficients of "F^{-1}" with zero in Eq. (4.1.11), we have

$$2\phi B_1F' + 2a\psi'(x)B_1F' - 2c\phi''(x)B_1F' + 2b\psi'(x)B_0B_1F' - 4c\psi'(x)\phi'(x)B_1F'' - 2c(\psi'(x))^2\phi B_1F''' = 0. \tag{4.1.14}$$

Now solving the Eq. (4.1.14) for $\psi'(x) = \frac{-k}{a+b}$ $\phi(x) = k$ and $\phi'(x) = 0$:

Case 1: Using $B_0 = \frac{-2a}{b}$ in Eq. (4.1.14), we have

$$\frac{(2a+b)(a+b)}{ck^2} = \frac{F'''(z)}{F'(z)}. \tag{4.1.15}$$

Case 2: Using $B_0 = 0$ in Eq. (4.1.14), we obtain

$$\frac{b(a+b)}{ck^2} = \frac{F'''(z)}{F'(z)}. \tag{4.1.16}$$

Now equating the coefficients of "tF^{-1}" with zero in Eq. (4.1.11), we have

$$2a\phi'(x)B_1F' + 2b\phi'(x)B_0B_1F' - 4c(\phi'(x))^2B_1F'' - 2c\phi''(x)B_1F''\phi - 2c\psi''(x)B_1F''\phi - 4c\phi\phi'(x)\psi'(x)B_1F''' = 0. \tag{4.1.17}$$

Equating the coefficients of "t^2F^{-1}" with zero in Eq. (4.1.11), we obtain [16]

$$2c\phi(\phi'(x))^2B_1F'''(z) = 0. \tag{4.1.18}$$

Now equating the coefficients of "F^{-2}" with zero in Eq. (4.1.11)

$$\begin{aligned}&2\phi B_2(F')^2 + 2a\psi'(x)B_2(F')^2 + 2c\phi''(x)B_2(F')^2 + b\psi'(x)B_1^2(F')^2 + 2b\psi'(x)B_0B_2(F')^2 - 8c\psi'(x)B_2F'\phi'(x)F'' - \\ &4c\psi'(x)\phi'(x)B_1(F')^2 - 4c(\psi'(x))^2B_2F'\phi F''' + 6c\phi F''F'(\psi'(x))^2B_1 - 4c(\psi'(x))^2B_2\phi(F'')^2 = 0.\end{aligned} \tag{4.1.19}$$

Equating the coefficients of "tF^{-2}" with zero in Eq. (4.1.11), we have

$$\begin{aligned}&2B_2a\phi'(x)(F')^2 + b(B_1)^2\phi'(x)(F')^2 + 2bB_0B_2\phi'(x)(F')^2 - 8cB_2(\phi'(x))^2F'F'' + \\ &4cB_1(\phi'(x))^2(F')^2 - 4cB_2\phi''(x)F'F''\phi + (F')^2 2cB_1\phi''(x)\phi - 4cB_2\psi''(x)\phi F'F'' + \\ &2cB_1(F')^2\psi''(x)\phi - 8c\phi(F'')^2B_2\phi'(x)\psi'(x) + 12c\psi'(x)\phi\phi'(x)F'F''B_1 - 8c(F'')^2c\psi'(x)\phi\phi'(x)B_2 = 0.\end{aligned} \tag{4.1.20}$$

Now equating the coefficients of "F^{-3}" with zero in Eq. (4.1.11), we obtain

$$\begin{aligned}&2bB_1B_2\psi'(x)(F')^3 + 8c\psi'(x)(F')^3\phi'(x)B_2 + 4c\phi(F')^3B_1(\psi'(x))^2 + \\ &12c\phi B_2(F')^2(\psi'(x))^2F'' + 8c\phi B_2(F')^2(\psi'(x))^2F'' = 0.\end{aligned} \tag{4.1.21}$$

Equating the coefficients of "tF^{-3}" with zero in Eq. (4.1.11)

$$\begin{aligned}&2bB_1B_2(F')^3\phi'(x) + 8cB_2(F')^3(\phi'(x))^2 + 4c\phi B_2(F')^3\phi''(x) + 4c\phi B_2(F')^3\psi''(x) - \\ &8c\phi B_1(F')^3\phi'(x)\psi'(x) + 24c\phi B_2(F')^2\phi'(x)\psi'(x)F'' + 16c\phi B_2(F')^2\phi'(x)\psi'(x)F'' = 0.\end{aligned} \tag{4.1.22}$$

Now equating the coefficients of "t^2F^{-3}" with zero in Eq. (4.1.11), we obtain

$$8c\phi B_2(F')^2F''(\phi'(x))^2 + 12c\phi B_2(\phi'(x))^2(F')^2F'' - 4c\phi B_1(F')^3(\phi'(x))^2 = 0. \tag{4.1.23}$$

Equating the coefficients of "F^{-4}" with zero in Eq. (4.1.11), we have

$$4c\phi B_2(F')^4(\psi'(x))^2 + 8c\phi B_2(F')^4(\psi'(x))^2 - B_2^2(F')^4\psi'(x) = 0, \tag{4.1.24}$$

or

$$B_2^2(F')^4\psi'(x) - 12c\phi B_2(F')^4(\psi'(x))^2 = 0,$$

or

$$B_2(x) = 12c\phi\psi'(x). \tag{4.1.25}$$

Solving Eq. (4.1.25) by substituting $\psi'(x) = \frac{-k}{a+b}$ and $\phi(x) = k$, we get

$$B_2(x) = -\frac{12ck^2}{a+b}. \tag{4.1.26}$$

Equating the coefficients of "tF^{-4}" with zero in Eq. (4.1.11), we obtain

$$8c\phi B_2(F')^4\psi'(x)\phi'(x) + 16c\phi B_2(F')^4\psi'(x)\phi'(x) - b(B_2)^2(F')^4\phi'(x) = 0. \tag{4.1.27}$$

Solving Eq. (4.1.27) by substituting $\psi'(x) = \frac{-k}{a+b}$ and $\phi(x) = k$,

$$B_2(x) = -\frac{24ck^2}{b(a+b)}. \tag{4.1.28}$$

Now equating the coefficients of "t^2F^{-4}" with zero in Eq. (4.1.11)

$$8c\phi B_2(F')^4(\phi'(x))^2 - 4c\phi B_2(F')^4(\phi'(x))^2 = 0. \tag{4.1.29}$$

Equating the coefficients of "t^2F^{-2}" with zero in Eq. (4.1.11), we have

$$4c\phi(\phi'(x))^2F'F'''B_2 - 6c\phi(\phi'(x))^2F'F''B_1 + 4c\phi(\phi'(x))^2(F'')^2B_2 = 0, \tag{4.1.30}$$

or

$$4F'F'''B_2 = 6F'F''B_1 + 4(F'')^2B_2. \tag{4.1.31}$$

Dividing the Eq. (4.1.31) by $(F')^2$, we obtain

$$2\frac{F'''}{F'}B_2 = 3B_1\frac{F''}{F'} + 2\Big(\frac{F''}{F'}\Big)^2B_2. \tag{4.1.32}$$

Substituting $\phi'(x) = 0$ in Eq. (4.1.13) , thus we have

$$10c\phi B_2\psi'(x)F'' + bB_1B_2F' + 2c\phi F'B_1\psi'(x) = 0. \tag{4.1.33}$$

Now substituting $\phi = k, \psi'(x) = \frac{-k}{a+b}$ and $B_2(x) = -\frac{12ck^2}{a+b}$ in Eq. (4.1.33), we obtain

$$B_1F'[bB_2 + 2ck\psi'(x)] = -10c\phi B_2\psi'(x)F'',$$

or

$$B_1F'\Big(b\frac{12ck^2}{a+b} + \frac{2ck^2}{a+b}\Big) = \frac{120c^2k^4}{(a+b)^2}F'',$$

or

$$B_1 = \frac{60ck^2}{(6b+1)(a+b)} \frac{F''}{F'},$$

$$\frac{F''}{F'} = B_1 \frac{(6b+1)(a+b)}{60ck^2}. \tag{4.1.34}$$

By substituting Eq. (4.1.34) and (4.1.16) in Eq. (4.1.32), we have

$$8b = -B_1^2 \frac{(6b+1)(a+b)}{60ck^2} + \frac{8ck^2}{a+b} B_1^2 \Big(\frac{(6b+1)(a+b)}{60ck^2} \Big)^2,$$

or

$$8b = B_1^2 \frac{(6b+1)(a+b)}{60ck^2} \Big(\frac{12b-13}{15} \Big).$$

Thus we obtain

$$B_1(x) = \sqrt{\frac{7200bck^2}{(12b-13)(6b+1)(a+b)}}. \tag{4.1.35}$$

Substituting Eq. (4.1.35) in Eq. (4.1.34), we get

$$\frac{F''}{F'} = \sqrt{\frac{7200bck^2}{(12b-13)(6b+1)(a+b)}} \frac{(6b+1)(a+b)}{60ck^2},$$

or

$$\frac{F''(z)}{F'(z)} = \sqrt{\frac{2b(6b+1)(a+b)}{ck^2(12b-13)}}. \tag{4.1.36}$$

Integrating Eq. (4.1.36) with respect to z, we obtain

$$\ln F'(z) = G_1(x)z + G_2(x),$$

or

$$F'(z) = H_1(x)[e^{zG_1(x)}], \tag{4.1.37}$$

where $G_1(x) = \sqrt{\frac{2b(6b+1)(a+b)}{ck^2(12b-13)}}$ and $G_2(x)$ are constants of integration. Integrating again the Eq. (4.1.37) with respect to z, we have

$$F(z) = \frac{H_1(x)}{G_1(x)} e^{G_1(x)z} + G_3(x). \tag{4.1.38}$$

Using $\phi(x) = k$ and $\psi(x) = \frac{-k}{a+b}x + A_0(x)$ in $z = \phi(x)t + \psi(x)$,

$$z = kt - \frac{kx}{a+b} + A_0(x). \tag{4.1.39}$$

Thus by using Eq. (4.1.39), Eq. (4.1.38) has the form

$$F(z) = G_3(x) + \frac{H_1(x)}{G_1(x)} e^{\left[G_1(x)\left(kt - \frac{kx}{a+b} + A_0(x)\right)\right]}. \tag{4.1.40}$$

And substituting Eq. (4.1.39) in Eq. (4.1.37), we obtain

$$F'(z) = H_1(x) e^{\left[G_1(x)\left(kt - \frac{kx}{a+b} + A_0(x)\right)\right]}. \tag{4.1.41}$$

Now substituting Eq. (4.1.26), (4.1.35), (4.1.40), (4.1.41) and $B_0(x) = 0$, in Eq. (4.1.6), we can obtain the travelling wave solution for the BBM equation, thus Eq. (4.1.6) take the form

$$u_1(x,t) = \sqrt{\frac{7200bck^2}{(12b-13)(6b+1)(a+b)}}\left[\frac{H_1(x)e^{\left[G_1(x)\left(kt-\frac{kx}{a+b}+A_0(x)\right)\right]}}{G_3(x)+\frac{H_1(x)}{G_1(x)}e^{\left[G_1(x)\left(kt-\frac{kx}{a+b}+A_0(x)\right)\right]}}\right]- \frac{12ck^2}{a+b}\left[\frac{H_1(x)e^{\left[G_1(x)\left(kt-\frac{kx}{a+b}+A_0(x)\right)\right]}}{G_3(x)+\frac{H_1(x)}{G_1(x)}e^{\left[G_1(x)\left(kt-\frac{kx}{a+b}+A_0(x)\right)\right]}}\right]^2. \tag{4.1.42}$$

For $B_0(x) = \frac{-2a}{b}$, Eq. (4.1.6) take the form

$$u_2(x,t) = \frac{-2a}{b} + \sqrt{\frac{7200bck^2}{(12b-13)(6b+1)(a+b)}}\left[\frac{H_1(x)e^{\left[G_1(x)\left(kt-\frac{kx}{a+b}+A_0(x)\right)\right]}}{G_3(x)+\frac{H_1(x)}{G_1(x)}e^{\left[G_1(x)\left(kt-\frac{kx}{a+b}+A_0(x)\right)\right]}}\right] - \frac{12ck^2}{a+b}\left[\frac{H_1(x)e^{\left[G_1(x)\left(kt-\frac{kx}{a+b}+A_0(x)\right)\right]}}{G_3(x)+\frac{H_1(x)}{G_1(x)}e^{\left[G_1(x)\left(kt-\frac{kx}{a+b}+A_0(x)\right)\right]}}\right]^2. \tag{4.1.43}$$

Thus Eqs. (4.1.42) and (4.1.43) are exact travelling wave solutions of BBM equation by using ansatz of the the form (4.0.2), when $B_0(x) = 0$ or $\frac{-2a}{b}$ respectively.

Chapter 5

Conclusion

In this thesis, we investigated that the generalized separation of variables can be used to determine the exact solutions of a wide variety of nonlinear PDEs, by using different ansatzes, which when substituted into PDEs can transform them into ODEs. Particularly, the ansatz (2.0.4) has its significance over other ansatzes, because by replacement of variables u, x and t, we can construct an ansatz of a new form, to determine an exact solution of the given PDE.

In this thesis while applying the generalized separation of variables on PDEs, we also used ansatz (2.0.3) which can be applied, as a rule, to the equations with polynomial nonlinearity. Sometimes it may happen, that the construction of solution by applying ansatz (2.0.3) is impossible, then firstly we seek a transformation, which can reduce the given equation to a new equation with polynomial nonlinearity, and then we can construct its solution by applying the ansatz (2.0.3).

The exact travelling wave solution is another efficient method applied to find the exact solution of nonlinear PDEs, by applying the method, commonly known as $\frac{G'}{G}$ expansion method.

The main idea of this method is that the travelling wave solutions of nonlinear equations can be expressed by a polynomial in $\frac{G'}{G}$, where $G = G(\xi)$ satisfies the second order linear ODE $G'' + G' + G = 0$, where $\xi = x - vt$ and then integration can be expressed by an m-th degree polynomial in $\frac{G'}{G}$, where $G = G(\xi)$ is the general solutions of a second order LODE.

The positive integer m is determined by the homogeneous balance between the highest order derivatives and nonlinear terms appearing in the reduced ODE, and the coefficients of the polynomial can be determined by solving a set of simultaneous algebraic equations obtained by the process of using the method. Furthermore we can use this method to many other nonlinear equations.

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