Moving Frame Formalism for Differential Equations

by

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Abstract

Two geometrical objects are said to be equivalent, if they can be mapped to each other using a diffeomorphism. Élie Cartan introduced the method of moving frames, which is a powerful tool to verify the equivalence of two objects, leading to explicit necessary and sufficient conditions for the equivalence of those objects. These conditions are found in the form of differential invariants of the two objects under the class of diffeomorphisms, providing a key to the solution of many equivalence problems. In this thesis the Cartan algorithm is applied to study the equivalence of linear second order ordinary differential equations (ODEs), Lagrangian of second order ODEs, a class of systems of linear constant coefficients ODEs (coupled and un-coupled) under fiber-preserving diffeomorphism. The symmetry algebras are identified in each case using the rank of the invariant coframes. This study contains two new results. First we identify a ten dimensional Lie algebra of fiber-preserving transformations for an un-coupled system of ODEs. Secondly the fiber-preserving algebra for a particular class of coupled system of ODEs is six dimensional. Therefore the Cartan approach also confirms that there is more than one class of systems of linear ODEs in contrast to scalar linear ODEs.

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Asmat Amin

To My Parents

and Beloved Brother

Syed Tahir Amin

CHAPTER 0

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Chapter 1

Introduction

An important problem in differential geometry is the equivalence problem, which is to determine the necessary and sufficient conditions (algebraic or differential) under which two geometric objects can be regarded as equivalent. The two objects could be manifolds, differential equations, Lagrangians, polynomials, differential operators etc. In order to investigate equivalence between objects from a geometric point of view we require basic tools from differential geometry and Lie groups [7, 11, 12]. The first chapter briefly explains the basic concepts of smooth manifolds, tangent and cotangent spaces, differential forms, exterior derivatives, Lie groups [4, 11, 16, 24], Lie algebras and Maurer-Cartan forms.

1.1 Differentiable manifold

A differentiable manifold is the generalization of a Euclidean space, roughly speaking, an m-dimensional manifold M is a space which locally looks like $\mathbb{R}^n$. More specifically, a differentiable manifold is defined as follows.

Definition 1.1.1. An m-dimensional differentiable manifold is a topological space M with a family $(M_i)_{i\in I}$ of subsets, for which the following properties holds,

1. $M = \bigcup_{i\in I} M_i$;
2. For every $i \in I$ there exists a one-to-one mapping, $\phi_i\colon M_i \to \mathbb{R}^n$ such that $\phi_i(M_i)$ is open in $\mathbb{R}^n$;
3. For $M_i \cap M_j \neq \emptyset$, $\phi_i(M_i \cap M_j)$ is open in $\mathbb{R}^n$, and the composition map $\phi_j \circ \phi_i^{-1}\colon \phi_i(M_i \cap M_j) \to \phi_j(M_i \cap M_j)$ is differentiable for arbitrarily i and j,

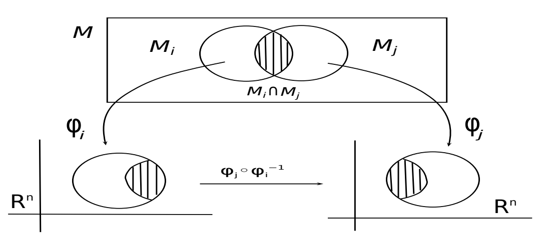

Figure 1.1: Manifold

where each ϕ_i is called a chart and $(M_i, \phi_i)i \in I$, is called an atlas and the map

$$\phi_j \circ (\phi_i)^{-1} \colon \phi_i(M_i \cap M_j) \to \phi_j(M_i \cap M_j), \tag{1.1}$$

defined on the intersection of two charts is called the coordinate transformation or transition function.

Example 1.1.2. A simple example of a manifold is $\mathbb{R}^n$ or any subset of $\mathbb{R}^n$ which is covered by a single coordinate chart.

Example 1.1.3. An n-sphere, $\mathbb{S}^n$ is the locus of all points which are at some fixed distance from the origin in $\mathbb{R}^{n+1}$. It is an n-dimensional manifold. In particular, $\mathbb{S}^1$ represents a unit circle and $\mathbb{S}^2$ corresponds to a sphere in a three dimensional space.

Example 1.1.4. The general linear group $GL(n;\mathbb{R})$, consisting of all $n \times n$ real non-singular matrices is a manifold of dimension n^2.

1.1.1 Maps between manifolds

Definition 1.1.5. A diffeomorphism $\phi : M \to N$ (where M and N are C^∞ manifolds) is a smooth map which also has a smooth inverse.

A function 'f' is said to be smooth, or of class C^∞, if its derivative exists to all orders. For example, all polynomials and the exponential functions are C^∞, because all of their derivatives are continuous. Below we give a counter example of a function which belongs to the space C^0 but fails to be in the space C^1.

Example 1.1.6. The space C^0 consists of all continuous functions, and the space C^1 consist of functions whose first order derivative exists and is of class C^0. For example

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0, \end{cases} \tag{1.2}$$

is continuous but not differentiable at $x = 0$, so $f(x)$ is of class C^0 but not C^1.

We now consider an example of a diffeomorphism in $\mathbb{R}^2$.

Example 1.1.7. Let $f : M(= \mathbb{R}^2) \to N(= \mathbb{R}^2)$ be a mapping between manifolds, given by

$$f(x, y) = (x - 2y, 2x + 3y). \tag{1.3}$$

We can verify that the above mapping is a diffeomorphism as the mapping has all continuous derivatives. Furthermore the smoothness of the inverse map can be assured by the non-vanishing Jacobian

$$J_f = \begin{pmatrix} 1 & -2 \\ 2 & 3 \end{pmatrix} \tag{1.4}$$

which implies that

$$|J_f| = 7 \neq 0. \tag{1.5}$$

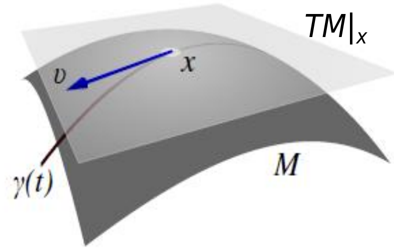

Figure 1.2: Tangent vector at x.

1.1.2 Tangent vector and vector field

A tangent vector at a point $x \in M$, is a vector which is tangent to a smooth curve passing through that point. The collection of all such tangent vectors forms a tangent space $TM|_x$ to M at x. The dimension of the tangent space $TM|_x$ is the same as the dimension of the manifold M.

Example 1.1.8. The tangent space $TM|_x$ to $\mathbb{S}^2$ at $x \in \mathbb{S}^2$, consists of all those vectors which are orthogonal to the radial line through x as shown in the Figure 1.3. The tangent space is a 2-dimensional space associated to $\mathbb{S}^2$ at x, and is equal to the dimension of the manifold $\mathbb{S}^2$.

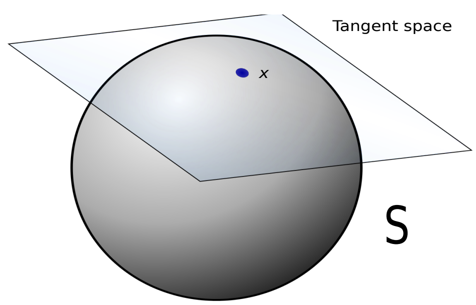

Figure 1.3: Tangent space $TM|_x$ to $\mathbb{S}^2$ at $x \in \mathbb{S}^2$

Since we can define a tangent space at each point on the manifold, therefore these tangent spaces are sewn together to form a tangent bundle TM, $TM = U_{x\in M}TM|x$ of the manifold, which is a $2m$-dimensional manifold.

Definition 1.1.9. A vector field '$\mathbf{v}$' on M is a map that assigns a tangent vector to each point of the manifold,

$$\mathbf{v} : M \to TM, \quad x \to \mathbf{v}_x \in TM|_x. \tag{1.6}$$

In local coordinates the vector field is given as

$$\mathbf{v} = \sum_{i=1}^{m} \xi^i(x) \frac{\partial}{\partial x^i}, \tag{1.7}$$

where the coefficient $\xi^i(x)$ are smooth functions, and $\frac{\partial}{\partial x^i}$ are the tangent vectors to the coordinates axes and form a basis for the tangent space $TM|_x$.

Definition 1.1.10. A parameterized curve $\phi : \mathbb{R} \to M$ is called an integral curve of the vector field '$\mathbf{v}$' if its tangent vector coincides with the vector field '$\mathbf{v}$' at each point.

Thus if '$\mathbf{v}$' is a vector field on M then through each point of M, there is a curve, called an integral curve of '$\mathbf{v}$', whose velocity vector field coincides with '$\mathbf{v}$'. Then the collection of all integral curves through the points of M may be thought as the motion on the manifold, called a local flow of the vector field.

Example 1.1.11. Let us consider two vector fields on $M = \mathbb{R}$, given by

$$\mathbf{v}_1 = \frac{\partial}{\partial x}, \quad \mathbf{v}_2 = x^2 \frac{\partial}{\partial x}. \tag{1.8}$$

The vector field $\mathbf{v}_1$, generates a translational flow on the manifold. On the other hand the vector field $\mathbf{v}_2$, corresponds to an inversional flow.

Example 1.1.12. On $M = \mathbb{R}^2$, the vector fields which generate translational flows in the x and y directions are given by

$$\mathbf{v}_1 = \frac{\partial}{\partial x}, \quad \mathbf{v}_2 = \frac{\partial}{\partial y}. \tag{1.9}$$

Now in the 2-dimensional space $\mathbb{R}^2$, we have a freedom to generate a rotational flow in the clockwise and anti-clockwise direction which are given by the vector fields

$$\mathbf{v}_3 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, \quad \mathbf{v}_4 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, \tag{1.10}$$

respectively.

Example 1.1.13. On $M = \mathbb{R}^3$, the vector fields

$$\mathbf{v}_1 = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, \quad \mathbf{v}_2 = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}, \quad \mathbf{v}_3 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \tag{1.11}$$

generates rotational flow around x, y and z-axes, respectively.

Definition 1.1.14. A Lie bracket is an operator which assigns to any two vector fields $\mathbf{v}_1$ and $\mathbf{v}_2$ on a smooth manifold M, a third vector field denoted by $[\mathbf{v}_1, \mathbf{v}_2]$, and is defined by

$$[\mathbf{v}_1, \mathbf{v}_2] = \mathbf{v}_1\mathbf{v}_2 - \mathbf{v}_2\mathbf{v}_1.$$

Example 1.1.15. Consider two vector fields in $\mathbb{R}^3$,

$$\mathbf{v}_1 = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, \quad \mathbf{v}_2 = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}, \tag{1.12}$$

which are the rotations in yz and $zx-$planes (i.e., rotation around x and y axis respectively) then their Lie-bracket is

$$\begin{aligned} [\mathbf{v}_1, \mathbf{v}_2] &= \mathbf{v}_1\mathbf{v}_2 - \mathbf{v}_2\mathbf{v}_1, \\ &= (y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y})(x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}) - (x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x})(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}), \\ &= -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, \\ &= x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \end{aligned} \tag{1.13}$$

which corresponds to rotational flow in $xy-$plane (i.e., rotation around z-axis).

A Lie bracket satisfies the following properties, which help in simplifying several computational steps:

1. A Lie bracket is bilinear i.e.,
$$[a\mathbf{v}_1 + b\mathbf{v}_2, \mathbf{v}_3] = a[\mathbf{v}_1, \mathbf{v}_3] + b[\mathbf{v}_2, \mathbf{v}_3]; \tag{1.14}$$

2. The Lie bracket is anti-symmetric i.e.,
$$[\mathbf{v}_1, \mathbf{v}_2] = -[\mathbf{v}_2, \mathbf{v}_1]; \tag{1.15}$$

3. The Lie bracket satisfies the Jacobi identity,
$$[\mathbf{v}_1, [\mathbf{v}_2, \mathbf{v}_3]] + [\mathbf{v}_2, [\mathbf{v}_3, \mathbf{v}_1]] + [\mathbf{v}_3, [\mathbf{v}_1, \mathbf{v}_2]] = 0, \tag{1.16}$$
for any triple of vector fields $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

1.1.3 Differential forms

Definition 1.1.16. Let $x \in M$ be a point on M, a differential 1-form is a linear functional on the tangent space at x,
$$\omega : T_xM \to \mathbb{R}. \tag{1.17}$$

Thus at each point x, ω_x is the element of the dual space of T_xM [3, 15, 22, 24, 28]. The space of 1-forms is the dual vector space to the tangent space $TM|_x$, and is called the cotangent space, denoted by $T^*M|_x$. The cotangent spaces are sewn together to form the cotangent bundle,
$$T^*M = U_{x\in M}T^*M|_x. \tag{1.18}$$
The evaluation of ω on the tangent vector '$\mathbf{v}$' will be indicated by the bilinear pairing $< \omega; \mathbf{v} >$. In local coordinates $x = (x^1, ..., x^m)$, the differentials dx^i of the coordinate functions provide a basis for the cotangent space at each point of the coordinate chart, which forms the dual to the coordinate basis $\frac{\partial}{\partial x^i}$ of the tangent space. Thus
$$< dx^i, \frac{\partial}{\partial x^i} >= \delta^i_j, \tag{1.19}$$
where δ^i_j denotes the Kronecker delta, which is 1 if $i = j$ and is zero otherwise. The differential forms of higher degree are defined as alternating multi-linear maps on the tangent spaces. Thus a differential k-form $\wedge^k\Omega$ at a point $x \in M$, is a k-linear map that takes k tangent vectors as input and return a real number
$$\wedge^k\Omega : TM|_x \times ... \times TM|_x \quad \to \mathbb{R}, \tag{1.20}$$
which is anti-symmetric in its arguments, meaning that
$$< \wedge^k\Omega : \mathbf{v}_{\pi_1}, ..., \mathbf{v}_{\pi_k} >= (\mathrm{sign}\pi) < \wedge^k\Omega; \mathbf{v}_1, ..., \mathbf{v}_k >, \tag{1.21}$$
for any permutation π of the indices $\{1, \ldots, k\}$. The space of all k-forms at x is the k-fold exterior power of the cotangent space at x, denoted by $\wedge^k T^*M|_x$ and form a vector space of dimension $\begin{pmatrix} m \\ k \end{pmatrix}$.

Definition 1.1.17. An exterior derivative is a map which maps k-form to $(k+1)$-form, i.e.,

$$d : \wedge^k \Omega \to \wedge^{k+1} \Omega. \tag{1.22}$$

For 0-forms, exterior differentiation is just the operation of taking differential

$$d : f \to df, \quad df := \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i, \tag{1.23}$$

where x^i are local coordinates.

The exterior derivative 'd' satisfies the following properties:

1. $d(\Omega + \phi) = d\Omega + d\phi$;
2. $d(\Omega \wedge \phi) = d\Omega \wedge \phi + (-1)^k \Omega \wedge d\phi$, where Ω is k-form and ϕ are l-forms;
3. d(dΩ)=0.

Example 1.1.18. Consider a basic example ($M = \mathbb{R}^3$) and take the zero form (just a function on $\mathbb{R}^3$)

$$\Omega = \sin(x^3 + y) + \tan(2z), \tag{1.24}$$

computing the exterior derivative, we get

$$d\Omega = 3x^2 \cos(x^3 + y)dx + \cos(x^3 + y)dy + 2\sec^2(2z)dz,$$

which is a 1-form. Note that if we take exterior derivative again

$$\begin{aligned} d(d\Omega) &= -3x^2 \sin(x^3 + y)dy \wedge dx - 3x^2 \sin(x^3 + y)dx \wedge dy + 0, \\ &= 3x^2 \sin(x^3 + y)dx \wedge dy - 3x^2 \sin(x^3 + y)dx \wedge dy, \end{aligned} \tag{1.25}$$

we obtain $d^2\Omega = 0$, which is always true for the exterior derivative and is an important property.

Example 1.1.19. Let Ω be a 1-form

$$\Omega = (x^2 - y)dx + (y + z)dy + (x^2 + z)dz.$$

If we compute the exterior derivative we obtain a 2-form

$$d\Omega = dx \wedge dy - dy \wedge dz + 2xdx \wedge dz. \tag{1.26}$$

1.1.4 Geometrical interpretation of forms

In order to comprehend forms geometrically we consider a three-dimensional manifold $M = \mathbb{R}^3$, and the vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^\mathbf{3}$,

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}. \tag{1.27}$$

Then a two form $dx_1 \wedge dx_2$, on vectors **a** and **b** is

$$dx_1 \wedge dx_2 \left\{ \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \right\} = det \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = a_1 b_2 - a_2 b_1, \tag{1.28}$$

which may be understood geometrically as the projection of **a** and **b** onto (x_1, x_2) plane. The determinant above gives the signed area of the parallelogram spanned by **a** and **b**, thus $(dx_1 \wedge dx_2)$ deserves to be called the (x_1, x_2) component of the signed area. The 3-form $(dx_1 \wedge dx_2 \wedge dx_3)$ tell us that (x_1, x_2, x_3) component of the signed volume of the parallelepiped spanned by the three vectors (x_1, x_2, x_3). In this example a 1-form is just a function on $M = \mathbb{R}^3$, and all the k-forms for $k > 3$ are zero. We now introduce a basic operation on differential forms known as pull-back.

Definition 1.1.20. Let $F : M \to N$ be a smooth map, then there is an induced map on differential forms, called the pull-back of F denoted by F^*, which maps the differential forms on N back to the differential forms on M.

1.2 Lie groups

In this section we give a brief introduction of Lie groups. These are important as they carry the structure of a differentiable manifold besides group properties. More precisely,

Definition 1.2.1. A Lie group G is a C^∞ manifold endowed with a group structure in which group multiplication and inversion are C^∞ operations, i.e., $(g, h) \to g \cdot h$ and $g \to g^{-1},\ g, h \in G$, define smooth maps.

Example 1.2.2. The simplest example of a Lie group is $\mathbb{R}^n$, where the group operation is given by vector addition. The identity element is the zero vector and the inverse of any vector x is $-x$.

Example 1.2.3. $GL(n, \mathbb{R})$ consisting of all invertible $n \times n$ real matrices from $\mathbb{R}^n \to \mathbb{R}^n$. The matrix multiplication defined the group operation. The manifold $GL(n, \mathbb{R})$ is an n^2-dimensional manifold.

Example 1.2.4. The special linear or unimodular group is

$$SL(n) = \{A \in GL(n) |\ \ \text{det} A = 1\}, \tag{1.29}$$

consisting of all volume preserving transformations. $SL(n)$ is a Lie-group of dimension $(n^2 - 1)$.

Example 1.2.5. The orthogonal group

$$O(n) = \{A \in GL(n, \mathbb{R}) |\ \ A^T A = I\}, \tag{1.30}$$

is the group of norm preserving transformations of dimension $\frac{1}{2}n(n- 1)$, and consists of rotation and reflection in $\mathbb{R}^n$.

Example 1.2.6. The special orthogonal group consist of rotations

$$\begin{aligned} SO(n) &= O(n) \cap SL(n), \\ SO(n) &= \{A \in O(n, R);\ \ \text{det} A = 1\}. \end{aligned} \tag{1.31}$$

The dimension of $SO(n)$ is $\frac{1}{2}n(n - 1)$.

In most cases, groups are not given to us in the abstract form, rather concretely as a family of transformations acting on the manifold [1]. Below we use Lie groups as the transformation groups and investigate their group actions.

Definition 1.2.7. A transformation group acting on a smooth manifold M is determined by a Lie group G and smooth map,

$$\Phi : G \times M \to M, \tag{1.32}$$

given by

$$\Phi(g, x) = g \cdot x, \tag{1.33}$$

which satisfies

$$e \cdot x = x, \quad g \cdot (h \cdot x) = (g \cdot h) \cdot x, \tag{1.34}$$

where $x \in M$, $g \in G$.

Example 1.2.8. An obvious example is provided by the usual linear action of general linear group $GL(n, R)$, acting by matrix multiplication on column vector $\mathbf{x} \in \mathbb{R}^{\mathbf{n}}$. In particular $G = SO(2)$, the group of rotations in plane

$$G = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \; ; \; 0 \leq \theta < 2\pi \right\}, \tag{1.35}$$

where θ denotes the angle of rotation.

Example 1.2.9. The real affine group $A(n)$ is define as the group of affine transformations $\mathbf{x} \to \mathbf{Ax} + \mathbf{b}$ in $\mathbb{R}^n$

$$A(n) = \left\{ \begin{pmatrix} A & \mathbf{b} \\ 0 & 1 \end{pmatrix} \; ; \; A \in GL(n) \text{ and } \mathbf{b} \in \mathbb{R}^{\mathbf{n}} \right\}. \tag{1.36}$$

The affine group has dimension $n(n + 1)$. Its subgroups is special affine group

$$SA(n) = \left\{ \begin{pmatrix} A & \mathbf{b} \\ 0 & 1 \end{pmatrix} \; ; \; A \in SL(n) \text{ and } \mathbf{b} \in \mathbb{R}^{\mathbf{n}} \right\}. \tag{1.37}$$

Example 1.2.10. Another important example is provided by the Euclidean group of transformation $E(n) = O(n) \times \mathbb{R}^n$

$$E(n) = \left\{ \begin{pmatrix} A & \mathbf{b} \\ 0 & 1 \end{pmatrix} \; ; \; A \in O(n) \text{ and } \mathbf{b} \in \mathbb{R}^{\mathbf{n}} \right\}, \tag{1.38}$$

which is generated by the group of orthogonal transformations and translations. The group operation of $E(n)$ is as follow, if we represent a vector $\mathbf{x} \in \mathbb{R}^{\mathbf{n}}$, by the $(n + 1)$-dimensional vector $(\mathbf{x}, 1)^T$, the element of $E(n)$ acts on $\mathbf{x}$ by the matrix multiplication

$$\begin{pmatrix} A & \mathbf{b} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} = \begin{pmatrix} A\mathbf{x} + \mathbf{b} \\ 1 \end{pmatrix}. \tag{1.39}$$

1.2.1 Lie algebra

A Lie algebra is a tangent space associated to a Lie group, but before we give a formal definition of Lie algebra we introduce some basic concepts related to a Lie algebra.

Definition 1.2.11. For a fixed element $g \in G$ in a Lie group we can define a pair of smooth maps, the left translation $L_g : h \to g \cdot h$, and right translation $R_g : h \to h \cdot g$.

Definition 1.2.12. A vector field '$\mathbf{v}$' on G is called left invariant if it is unaffected by the left multiplication of any group element, i.e., $dL_g(\mathbf{v}) = \mathbf{v}$ and right invariant if $dR_g(\mathbf{v}) = \mathbf{v}$, for all $g \in G$.

Definition 1.2.13. The left (right) Lie algebra of Lie group is the space of all left (right) invariant vector field on G. Thus associated to any Lie group there are two different Lie algebras, $\mathfrak{g}_L$ and $\mathfrak{g}_R$ respectively.

Remark 1.2.14. Whenever we take a Lie algebra corresponding to any Lie group we mean $\mathfrak{g} = \mathfrak{g}_R$. Every right invariant vector field is uniquely determined by its value at the identity e, because $\mathbf{v}|_g = dR_g(\mathbf{v}|_e)$. Thus we can identify the right Lie algebra with the tangent space to G at the identity element $\mathfrak{g}_R \simeq TG|_e$, a finite dimensional vector space having the same dimension as G.

Definition 1.2.15. A Lie algebra $\mathfrak{g}$ is a vector space equipped with the bracket operation $[.,.]$:$\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which is bilinear, antisymmetric $[\mathbf{v}, \mathbf{w}] = -[\mathbf{w}, \mathbf{v}]$ and satisfies the Jacobi identity,

$$[\mathbf{u}, [\mathbf{v}, \mathbf{w}] + [\mathbf{v}, [\mathbf{w}, \mathbf{u}] + [\mathbf{w}, [\mathbf{u}, \mathbf{v}] = 0. \tag{1.40}$$

Example 1.2.16. Consider the two parameters group $A(1)$ of the affine transformation $x \to ax + b$ on the line $x \in \mathbb{R}$, the group multiplication law is given by

$$(a, b) \cdot (c, d) = (ac, ad + b), \tag{1.41}$$

and the identity element is $e = (1, 0)$. The right translation is,

$$R_{(a,b)}(c, d) = (c, d) \cdot (a, b) = (ac, bc + d). \tag{1.42}$$

A basis for the right Lie algebra $\mathfrak{a}(1)_R$ corresponding to the coordinate basis $\frac{\partial}{\partial a}|_e, \frac{\partial}{\partial b}|_e$ of $TA(1)|_e$ is therefore,

$$\mathbf{v}_1 = dR_{(a,b)}[\frac{\partial}{\partial a}|_e] = a\frac{\partial}{\partial a} + b\frac{\partial}{\partial b}, \quad \mathbf{v}_2 = dR_{(a,b)}[\frac{\partial}{\partial b}|_e] = \frac{\partial}{\partial b}, \tag{1.43}$$

and their commutation relation is $[\mathbf{v}_1, \mathbf{v}_2] = -\mathbf{v}_2$.

Definition 1.2.17. Let $\mathbf{v}_1, ..., \mathbf{v}_r$ be a basis of Lie algebra $\mathfrak{g}$. We define the associated structure constants C_{ij}^k by the bracket relations

$$[\mathbf{v}_i, \mathbf{v}_j] = \sum_{k=1}^{r} C_{ij}^k \mathbf{v}_k. \tag{1.44}$$

By using anti-symmetric property and Jacobi identity the following relations can be obtained

$$C_{ij}^k = -C_{ji}^k, \quad \sum_{l=1}^{r} \{C_{ij}^l C_{lk}^m + C_{ki}^l C_{lj}^m + C_{jk}^l C_{li}^m\} = 0. \tag{1.45}$$

1.2.2 Maurer-Cartan forms

In the theory of Lie groups the most important are the invariant differential forms associated with the right (left) action of the Lie group on itself. They are also known as the Maurer-Cartan forms [24]. Below we define it.

Definition 1.2.18. A differential form Ω on a Lie group G is right invariant if it is unaffected by the right multiplication of the Lie group element i.e., $dR_g^*\Omega = \Omega$ for all $g \in G$. The right invariant 1-form on G is known as (right invariant Maurer-Cartan forms).

The space of Maurer-Cartan forms is naturally dual to the Lie algebra of G and hence form a vector space of the same dimension as the Lie group. If we choose a basis $\mathbf{v}_1, ..., \mathbf{v}_r$ of the Lie algebra $\mathfrak{g}$, then there is a dual basis (or co-frame) $\alpha^1, ..., \alpha^r$ consisting of the Maurer-Cartan forms satisfying $< \alpha^i; \mathbf{v}_j >= \delta^i_j$ and satisfy the fundamental structure equation

$$d\alpha^k = -\frac{1}{2}\sum_{i,j=1}^{r} C^k_{ij}\alpha^i \wedge \alpha^j = -\sum_{i,j=1}^{r} C^k_{ij}\alpha^i \wedge \alpha^j, \quad i < j \text{ and } k = 1, \ldots, r. \tag{1.46}$$

The coefficients C^k_{ij} are the same as the structure constants corresponding to our choices of basis of the Lie algebra $\mathfrak{g}$. If the group G is given as parameterized matrix Lie group, then a basis for the space of the Maurer-Cartan forms can be found among the entries of the matrix of the 1-form

$$\gamma = dg \cdot g^{-1}, \quad or \quad \gamma^i_j = \sum_{k=1}^{r} dg^i_k (g^{-1})^k_j, \tag{1.47}$$

each entry γ^i_j is clearly right invariant 1-form because if h is any fixed group element then

$$(R_h)^*\gamma = d(g \cdot h) \cdot (g \cdot h)^{-1} = dg \cdot g^{-1} = \gamma. \tag{1.48}$$

Example 1.2.19. Consider the special affine group of transformation $SA(2)$ in two dimensions. The matrix g has the form,

$$g = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & 0 & 1 \end{pmatrix}, \tag{1.49}$$

and

$$g^{-1} = \begin{pmatrix} a_5 & -a_2 & a_2a_6 - a_5a_3 \\ -a_4 & a_1 & a_4a_3 - a_1a_6 \\ 0 & 0 & 1 \end{pmatrix}, \tag{1.50}$$

with $a_1a_5 - a_2a_4 = 1$ and $(a_3, a_6) = \mathbf{b} \in \mathbb{R}^2$, the exterior derivative matrix is,

$$dg = \begin{pmatrix} da_1 & da_2 & da_3 \\ da_4 & da_5 & da_6 \\ 0 & 0 & 0 \end{pmatrix}. \tag{1.51}$$

Thus

$$dg \cdot g^{-1} = \begin{pmatrix} a_5da_1 - a_4da_2 & a_1da_2 - a_2da_1 & (a_2a_6 - a_5a_3)da_1 + (a_4a_3 - a_1a_6)da_2 + da_3 \\ a_5da_4 - a_4da_5 & a_1da_5 - a_2da_4 & (a_2a_6 - a_5a_3)da_4 + (a_4a_3 - a_1a_6)da_5 + da_6 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.52}$$

Since

$$
\begin{aligned}
&a_5 a_1 - a_4 a_2 = 1, \\
&a_1 da_5 + a_5 da_1 - a_2 da_4 - a_4 da_2 = 0,
\end{aligned}
\tag{1.53}
$$

or

$$
a_5 da_1 - a_4 da_2 = -(a_1 da_5 - a_2 da_4). \tag{1.54}
$$

Therefore the Maurer-Cartan forms are

$$
\begin{aligned}
\alpha^1 &= a_5 da_1 - a_4 da_2, \\
\alpha^2 &= a_1 da_2 - a_2 da_1, \\
\alpha^3 &= (a_2 a_6 - a_5 a_3) da_1 + (a_4 a_3 - a_1 a_6) da_2 + da_3, \\
\alpha^4 &= a_5 da_4 - a_4 da_5, \\
\alpha^5 &= (a_2 a_6 - a_5 a_3) da_4 + (a_4 a_3 - a_1 a_6) da_5 + da_6,
\end{aligned}
\tag{1.55}
$$

and for the special case when $\mathbf{b} = (a_3, 0) \in \mathbb{R}^2$, the Maurer-Cartan forms are found to be,

$$
\begin{aligned}
\alpha^1 &= a_5 da_1 - a_4 da_2, \\
\alpha^2 &= a_1 da_2 - a_2 da_1, \\
\alpha^3 &= a_3 a_4 da_2 - a_3 a_5 da_1 + da_3, \\
\alpha^4 &= a_5 da_4 - a_4 da_5, \\
\alpha^5 &= a_3 a_4 da_5 - a_3 a_5 da_4.
\end{aligned}
\tag{1.56}
$$

Chapter 2

Cartan equivalence method

We now define frames and coframes on a manifold, equivalence criteria and structure equations followed by key steps involved in Cartan algorithm which includes normalization, absorption and prolongation. [24].

2.1 Frames and coframes

Let M be a smooth m-dimensional manifold. A frame on M is a set of vector fields $\mathbf{v} = \{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_m}\}$, with the property that it forms a basis for the tangent space $TM|_x$ at each point $x \in M$. Dually a coframe on M is a set of one-forms $\theta = \{\theta^1, ..., \theta^m\}$, which form the basis for the cotangent space $T^*M|_x$, at each point $x \in M$.

Remark 2.1.1. A frame and coframe are dual to each other if and only if they form dual bases for the tangent and cotangent spaces to M at each point. Given a coframe $\{\theta^1, ..., \theta^m\}$ we shall denote the dual frame vector fields by $\{\frac{\partial}{\partial\theta^1}, ..., \frac{\partial}{\partial\theta^m}\}$ such that

$$\langle\theta^i, \frac{\partial}{\partial\theta^j}\rangle = \delta^i_j, \quad i, j = 1, \ldots, m. \tag{2.1}$$

If we introduce local coordinates $x = (x^1, ..., x^m)$ on M, then the coframe can be written in term of coordinates coframe

$$\theta^i = \sum_j a^i_j(x) dx^j, \tag{2.2}$$

where $a^i_j(x)$ is a non singular $m \times m$ matrix of functions and the dual frame is given by

$$\frac{\partial}{\partial\theta^j} = \sum_i b^i_j(x) \frac{\partial}{\partial x^i}, \quad \text{where} \quad b^i_j(x) = (a^i_j(x))^{-1}, \tag{2.3}$$

i.e., $b^i_j(x)$ is the inverse of the matrix $a^i_j(x)$.

2.2 Equivalence of coframes

In this section we will find the necessary and sufficient conditions under which the two coframes are equivalent. Suppose we have two manifolds M and $\bar{M}$ of the same dimensions m, and let

$\theta = \{\theta^1, ..., \theta^m\}$ and $\bar{\theta} = \{\bar{\theta^1}, ..., \bar{\theta^m}\}$, be the given coframes on M and $\bar{M}$, respectively. The basic equivalence problem for the coframes is to determine whether the two coframes can be mapped to each other by a diffeomorphism,

$$\Phi : M \to \bar{M}, \quad \text{such that} \quad \Phi^* \bar{\theta}^i = \theta^i, \quad i = 1, \dots, m, \tag{2.4}$$

i.e., whether there exists a diffeomorphism which pull-back the coframe on $\bar{M}$ to the coframe on M. Cartan made the fundamental observation that the invariance of the exterior derivative 'd ' under smooth maps is the key to the solution of coframe equivalence problem. Thus if (2.4) holds, we must have

$$\Phi^* d\bar{\theta}^i = d\theta^i. \tag{2.5}$$

Since θ is a 1-form, then its exterior derivative is a two-form and thus we can write $d\theta^i$ in terms of wedge product of θ's, which gives the fundamental structure equation

$$d\theta^i = \sum_{j,k=1}^{m} T^i_{jk} \theta^j \wedge \theta^k, \quad 1, \dots, m, \tag{2.6}$$

associated with the given coframe, where T^i_{jk} are the structure functions and are $\frac{1}{2}m^2(m-1)$ in numbers. The structure functions may be constant or even zero. Now any equivalent coframe $\bar{\theta}$ on $\bar{M}$ will have analogue structure equations

$$d\bar{\theta}^i = \sum_{j,k=1}^{m} \bar{T}^i_{jk} \bar{\theta}^j \wedge \bar{\theta}^k, \quad 1, \dots, m. \tag{2.7}$$

Substituting values from equations (2.6) and (2.7) in equation (2.5) and using the invariance of coframe elements; equation (2.4) we get

$$\sum_{j,k=1}^{m} T^i_{jk}(x) \theta^j \wedge \theta^k = d\theta^i = \Phi^* d\bar{\theta}^i = \sum_{j,k=1}^{m} \bar{T}^i_{jk}(\Phi(x)) \theta^j \wedge \theta^k. \tag{2.8}$$

Since θ^i are linearly independent one-forms thus the coefficient of each $\theta^j \wedge \theta^k$ must agree. This immediately implies the invariance of structure functions

$$\bar{T}^i_{jk}(\bar{x}) = T^i_{jk}(x), \quad \text{when} \quad \bar{x} = \Phi(x), \quad i, j, k = 1, \dots, m. \tag{2.9}$$

The above invariance condition provide an important set of necessary condition for the two coframes to be equivalent.

Remark 2.2.1. The structure functions measure the 'degree of non-commutativity' of the corresponding coframes derivatives. Since we have the dual Lie bracket formulae

$$[\frac{\partial}{\partial \theta^j}, \frac{\partial}{\partial \theta^k}] \quad = -\textstyle\sum_{i=1}^{m} T^i_{jk} \frac{\partial}{\partial \theta^i}. \tag{2.10}$$

Thus $T^i_{jk} = -C^i_{jk}$ (the torsion coefficient in the structure equation are negative time the structure constants as in the case of Lie algebra $\mathfrak{g}$). We now consider a simple example and explain the basic equivalence criteria.

Example 2.2.2. Consider a nontrivial case $M \subset \mathbb{R}^2$, with a coframe on M

$$\theta^1 = xdx + ydy, \quad \theta^2 = ydx + dy, \tag{2.11}$$

we calculate the exterior derivative for structure equations

$$\begin{aligned} d\theta^1 &= 0, \quad d\theta^2 = (1)\ dy \wedge dx, \\ d\theta^1 &= (0)\ \theta^1 \wedge \theta^2, \\ d\theta^2 &= -\frac{1}{x - y^2}\ \theta^1 \wedge \theta^2. \end{aligned}$$

Therefore, the structure functions are

$$J = 0, \quad K = -\frac{1}{x - y^2}. \tag{2.12}$$

Since the structure functions remain invariant therefore we apply a change of variables in above coframe and verify that in the new structure equations we obtain same invariants. Now consider a transformation (diffeomorphism i.e., rotation in $\mathbb{R}^2$ with some fixed angle θ) and calculate the structure functions in the new coordinates.

$$\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{2.13}$$

writing x, y in terms of $\bar{x}$ and $\bar{y}$, we obtain

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}, \tag{2.14}$$

and their differential,

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} d\bar{x} \\ d\bar{y} \end{pmatrix}, \tag{2.15}$$

thus we have,

$$dx = \cos\theta d\bar{x} + \sin\theta d\bar{y}, \tag{2.16}$$

$$dy = -\sin\theta d\bar{x} + \cos\theta d\bar{y}. \tag{2.17}$$

To find the structure equations we calculate the exterior derivative of $\bar{\theta}^1$, therefore

$$\begin{aligned} \bar{\theta}^1 &= (\cos\theta\bar{x} + \sin\theta\bar{y})(\cos\theta d\bar{x} + \sin\theta d\bar{y}) + (-\sin\theta\bar{x} + \cos\theta\bar{y})(-\sin\theta d\bar{x} + \cos\theta d\bar{y}), \\ &= \cos^2\theta\bar{x}d\bar{x} + \sin^2\theta\bar{y}d\bar{y} + \cos^2\theta\bar{y}d\bar{y} + \sin^2\theta\bar{x}d\bar{x}, \\ \bar{\theta}^1 &= \bar{x}d\bar{x} + \bar{y}d\bar{y}, \end{aligned} \tag{2.18}$$

implies that

$$d\bar{\theta}^1 = 0, \quad \bar{J} = 0. \tag{2.19}$$

Similarly, for $\bar{\theta}^2$ we obtain

$$\begin{aligned}\bar{\theta}^2 &= ydx + dy, \\ &= (-\sin\theta\bar{x} + \cos\theta\bar{y})(\cos\theta d\bar{x} + \sin\theta d\bar{y}) + (-\sin\theta d\bar{x} + \cos\theta d\bar{y}),\end{aligned} \tag{2.20}$$

taking the exterior derivative, (in simplified form in the new coordinates as same we calculated for $\bar{\theta}^1$) we get

$$\begin{aligned}d\bar{\theta}^2 &= -\sin^2\theta d\bar{x} \wedge d\bar{y} + \cos^2\theta d\bar{y} \wedge d\bar{x}, \\ d\bar{\theta}^2 &= -d\bar{x} \wedge d\bar{y}, \\ &= -\frac{1}{x-y^2}\,\bar{\theta}^1 \wedge \bar{\theta}^2,\end{aligned}$$

implies that

$$\bar{K} = -\frac{1}{x-y^2}. \tag{2.21}$$

Therefore, we have verified that in the new structure equations the invariants are J and K, as was expected from the Cartan equivalence criteria.

2.3 Formulation of the G-equivalence problem

Among other equivalence problems the most important one is the group-equivalence which we study now. The purpose of G-equivalence between two manifolds is to study the invariance of geometric objects under the Lie group of transformations. This is mainly done on the product manifold $G \times M$ [24].

Definition 2.3.1. Let $G \subset GL(m)$ be a Lie group (structure group), and let ω and $\bar{\omega}$ be coframes defined respectively on m-dimensional manifold M and $\bar{M}$. The G-valued equivalence problem for the coframe is to determine whether there exists a diffeomorphism $\Phi : M \to \bar{M}$ and a G-valued function $g : M \to G$ with the property

$$\Phi^*\bar{\omega} = g(x) \cdot \omega, \tag{2.22}$$

or in full detail

$$\Phi^*\bar{\omega} = \sum_{j=1}^{m} g^i_j(x) \cdot \omega^j, \quad i = 1, \ldots, m, \tag{2.23}$$

where the function $g^i_j(x)$ are the entries of the matrix $g(x)$, which is constrained to belong the structure group G at each point $x \in M$.

Remark 2.3.2. The simplest case when $G = \{e\}$ is the trivial subgroup consisting of just the identity matrix for which equation (2.23) becomes

$$\Phi^*\bar{\omega}^i = \omega^i, \quad i = 1, \ldots, m. \tag{2.24}$$

and so the $\{e\}$-valued equivalence problem is exactly the same as the equivalence problem for coframes.

The preliminary step in Cartan's algorithm is to modify the basic equivalence condition (2.23). In the group property of G, (2.23) will be satisfied if we find a pair of G-valued function $\bar{g}(\bar{x})$ and $g(x)$ such that

$$\bar{g}(\bar{x}) \cdot \bar{\omega} = g(x) \cdot \omega, \tag{2.25}$$

(omitting the pull-back for simplicity), this new form of the G-equivalence problem places the barred coordinates on the same footing as the original coordinates making equation (2.23) more useful. In more appropriate notation equation (2.25) can be written

$$\bar{\theta} = \bar{g}(\bar{x}) \cdot \bar{\omega} \quad \& \quad \theta = g(x) \cdot \omega, \tag{2.26}$$

where θ are the lifted coframe defined on $M \times G$. Thus equation (2.25) becomes

$$\Phi^* \bar{\theta} = \theta, \tag{2.27}$$

and thus the general equivalence problem has been successfully reduced to the equivalence of coframes.

2.4 Cartan's equivalence method

Once the equivalence problem is reformulated in terms of ω on an m-dimensional manifold M along with the structure group $G \subset GL(m)$ we are in a position to apply the Cartan equivalence method [24]. The goal is to normalize the structure group coefficients in a suitably invariant manner which results in a sufficient number of invariant combination. The Cartan method provides an algorithmic approach for finding such invariant combinations. Each group-dependent invariant combination allow us to normalize one group parameter. In favorable case, all the group parameters are normalized and the structure group reduced to the trivial group $\{e\}$. The problem is thus reduced for the equivalence of coframes, which we already know how to deal.

2.5 The structure equations

Consider a coframe $\omega = (\omega^1, ..., \omega^m)$ on m-dimensional manifold M along with the structure group $G \subset GL(m)$. Then the lifted coframe discussed previously has the form

$$\theta = g \cdot \omega, \tag{2.28}$$

or

$$\theta^i = \sum_{j=1}^{m} g^i_j \cdot \omega^j. \tag{2.29}$$

As in the case of the equivalence problem for the coframes the invariance of the exterior derivative is the key. Thus computing the differential of the lifted coframe elements (2.29),

$$\begin{aligned} d\theta^i &= d(\sum_{j=1}^{m} g^i_j \cdot \omega^j), \\ &= \sum_{j=1}^{m} \{dg^i_j \wedge \omega^j + g^i_j d\omega^j\}. \end{aligned} \tag{2.30}$$

Since ω^i form a coframe on M, then the two form $d\omega^i$ can be written in terms of the sum of wedge produce of ω^i and using (2.29), the coframe elements ω^i's can be written in terms of the wedge product of θ^k, so that

$$d\theta^i = \sum_{j=1}^{m} \gamma^i_j \wedge \theta^j + \sum_{j,k=1}^{m} T^i_{jk}(x,g)\theta^j \wedge \theta^k, \quad i = 1, ..., m \text{ and } j < k. \tag{2.31}$$

The torsion coefficient T^i_{jk} are functions which may be constant or depend upon base variable x or group parameter g. The γ^i_j in equation (2.31) are 1-forms

$$\gamma^i_j = \sum_{k=1}^{m} dg^i_k (g^{-1})^k_j, \tag{2.32}$$

or in matrix notation

$$\gamma = dg \cdot g^{-1}, \tag{2.33}$$

and form the matrix of Maurer-Cartan forms on the structure group G. If $(\alpha^1, ..., \alpha^r)$ are the basis for the space of Maurer-Cartan forms corresponding to the local coordinate system of $a = (a^1, ..., a^r)$ in the neighbourhood of the identity of G, then each γ^i_j

$$\gamma^i_j = \sum_{k=1}^{r} A^i_{jk}\alpha^k, \quad i, j = 1, ..., m. \tag{2.34}$$

Thus the final structure equation in terms of the Maurer-Cartan forms for our lifted coframe have the general form

$$d\theta^i = \sum_{k=1}^{r}\sum_{j=1}^{m} A^i_{jk}\alpha^k \wedge \theta^j + \sum_{j,k=1}^{m} T^i_{jk}(x,g)\theta^j \wedge \theta^k, \quad i = 1, ..., m, \quad j < k. \tag{2.35}$$

Similarly the structure equation for the barred coframe $\bar{\theta} = \bar{g} \cdot \bar{\omega}$ on $\bar{M}$

$$d\bar{\theta}^i = \sum_{k=1}^{r}\sum_{j=1}^{m} A^i_{jk}\bar{\alpha}^k \wedge \bar{\theta}^j + \sum_{j,k=1}^{m} \bar{T}^i_{jk}(\bar{x},\bar{g})\bar{\theta}^j \wedge \bar{\theta}^k, \quad i = 1, ..., m, \quad j < k. \tag{2.36}$$

The Cartan algorithm for a G-valued equivalence problem consist of series of important steps. The next step is absorption and prolongation.

2.6 Absorption and normalization

At this stage we replace each Maurer-Cartan form by

$$\alpha^k = \pi^k + \sum_{i}^{p} T^i_{jk}\theta^k, \quad k = 1, ..., r \tag{2.37}$$

in (2.35), and the maximum number of possible torsion coefficients are absorbed. This process is called absorption. The remaining torsion coefficients are called essential torsion, which will explicitly depends on the group parameters. They are normalized by setting them to a fixed number ($\mathbb{R}$), which can be chosen to have convenient values (0 or ± 1) by solving for some group parameter. This completes one loop in the Cartan algorithm. Next we substitute these parameter values back into the coframe and repeat the same process as above. This process will continue until we are left with two possibilities. The most favorable case is when all group parameters are normalized and the process ends with an explicit invariant coframe and thus reducing the equivalence problem to the equivalence problem for coframes. The second possibility is that we are left with one or more unspecified group parameters, but there is no essential torsion coefficient that depends on the group parameters. In the second case we must figure out whether the symmetry group of the equivalence problem is infinite dimensional (or in involution) or finite dimensional. If the system (2.35) is not in involution then we must prolong. We apply the Cartan test after calculating the degree of indeterminacy of the lifted coframe and reduced Cartan characters associated with the structure equations (2.35).

The degree of indeterminacy $r^{(1)}$ of the lifted coframe is the numbers of free variable in the solution to the associated linear absorption system and can be found by replacing each Maurer-Cartan form π^i by a linear combination

$$\pi^i = \sum_{j=1}^{r} z^i_j \theta^j, \tag{2.38}$$

we find that some z^i_j will be specified, while some will left arbitrary, called free variables. Next we introduce the reduced Cartan characters associated with the final structure equation (2.35), for given vector $\mathbf{v} = (v^1, ..., v^m) \in \mathbb{R}^m$, define $L[\mathbf{v}]$ to be given be $m \times r$ matrix with entries

$$L^i_k[\mathbf{v}] = \sum_{j=1}^{m} A^i_{jk} v^j, \quad i = 1, .., m, \quad k = 1, ..., r. \tag{2.39}$$

Thus $L[\mathbf{v}]$, is just the coefficient matrix for the Maurer-Cartan form α^k in the final structure equation (2.35) upon replacing each coframe element θ^i by the corresponding entry v^i of the vector $\mathbf{v}$. The maximal rank of the matrix $L[\mathbf{v}]$ over all possible vector $\mathbf{v} \in \mathbb{R}^m$, is called the first reduced character for the coframe denoted by

$$s^{'}_1 = \max\{\text{rank} \quad L[\mathbf{v}], \quad |\mathbf{v} \in \mathbb{R}^m\}. \tag{2.40}$$

The second reduced character $s^{'}_2$ is obtained by computing the maximal rank of $2m \times r$ matrix which is obtained by stacking two copies of the pervious matrix corresponding to two different vectors on top of each other

$$s^{'}_1 + s^{'}_2 \quad = \max\left\{\text{rank} \begin{pmatrix} L[\mathbf{v}_1] \\ L[\mathbf{v}_2] \end{pmatrix} \quad | \, \mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^m \right\}. \tag{2.41}$$

Thus the $(m-1)$ reduced characters $s^{'}_1, ..., s^{'}_{m-1}$ of the lifted coframe

$$s^{'}_1 + s^{'}_2 + ... + s^{'}_k \quad = \max\left\{\text{rank} \begin{pmatrix} L[\mathbf{v}_1] \\ L[\mathbf{v}_2] \\ \vdots \\ L[\mathbf{v}_k] \end{pmatrix} \quad | \, \mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k \in \mathbb{R}^m \;\; , \quad k = 1, ..., m-1. \right\},$$

where $L[\mathbf{v}]$ is $m \times r$ matrix in (2.39). The final reduced character is

$$s^{'}_1 + s^{'}_2 + ... + s^{'}_k = r, \quad \text{each} \;\; s^{'}_i \geq 0. \tag{2.42}$$

Definition 2.6.1. Let θ be a lifted coframe with reduced characters $s_1^{'}, s_2^{'}, ..., s_k^{'}$ and the degree of indeterminacy $r^{(1)}$, then θ is in involution iff it satisfy the Cartan test

$$s_1^{'} + 2s_2^{'} + ... + ks_k^{'} = r^{(1)}. \tag{2.43}$$

Remark 2.6.2. In all cases L.H.S of equation.(2.43) provide an upper bound for the degree of indeterminacy of the coframe, so $r^{(1)} \leq s_1^{'} + 2s_2^{'} + ... + ms_m^{'}$, where the equality holds iff the coframe in involutive.

2.7 Prolongation

Finally, we address the problem in which the Cartan equivalence procedure does not lead to a complete reduction of the structure group and the structure equations are not involutive. In the prolongation process every free parameter in the absorption equation (2.35) provides a new group parameter which must eventually be normalized if we are to solve the prolonged problem. The modified Maurer-Cartan forms are defined by

$$\pi^k = \alpha^k - \sum_{j=1}^{m} z_j^k \theta^j, \quad k = 1, ..., r, \tag{2.44}$$

where z_j^k define the general solution to the absorption equation (2.35) some of the z_j^k's (more specifically $r^{(1)}$ of them) are free variables and can not be invariantly prescribed by the absorption equation alone. Let us denote the free variables by $w^\nu, \quad \nu = 1, .., r^{(1)}$, so that the general solution to the absorption equation (2.35) $z = k[w] + S$, where the linear maps K and S in general depend on x and g. Thus the most general collection of one-forms which solve the absorption equation is of the form

$$\pi = \alpha - (K[w] + S)\theta. \tag{2.45}$$

Consequently our prolongation procedure leads us to introduce a lifted coframe on the prolonged space $M^{(1)} = M \times G$. The base coframe on $M^{(1)}$ will contain first m 1-forms in the original lifted coframe $\theta = g \cdot w$ and second the r modified Maurer-Cartan forms

$$\varpi = \alpha + S\theta, \tag{2.46}$$

or explicitly

$$\varpi^i = \alpha^i + \sum_{j=1}^{m} S_j^i \theta^j, \tag{2.47}$$

which are obtained from the general solution equation (2.45) to the absorption equation by setting all the free variables to zero, $w = 0$. Now θ and ϖ form a coframe on $M^{(1)}$. The free variables w will parameterize a new structure group $G^{(1)}$ which is a subgroup of $GL(m \times r)$ consisting of the blocks lower triangular matrices

$$G^{(1)} \quad = \left\{ \begin{pmatrix} I & 0 \\ K[w] & I \end{pmatrix} \quad | \quad w \in \mathbb{R}^{r^{(1)}} \right\}. \tag{2.48}$$

The structure group $G^{(1)}$ for the prolonged problem has dimension $r^{(1)}$. The lifted coframe will live on the prolonged space

$$M^{(1)} \times G^{(1)} = M \times G \times G^{(1)}, \tag{2.49}$$

and consist of the original lifted coframe θ along with the modified Maurer-Cartan forms $\pi = \varpi + k[w]\theta$. The goal now is to use absorption and normalization procedures to eliminate the new group parameters $w \in G^{(1)}$.

Example 2.7.1. (Equivalence of surfaces) To explain basic steps of the Cartan algorithm, consider the problem of classifying all (local) isometries of the Euclidean plane. These will be (smooth) maps $(\bar{x}, \bar{y}) = \Phi(x, y)$, that preserves the Euclidean metric [24],

$$\Phi^*(d\bar{x}^2 + d\bar{y}^2) = dx^2 + dy^2. \tag{2.50}$$

Introducing the base coframe $\omega^1 = dx,\ \omega^2 = dy$, then the condition equation (2.50) is the same as the condition

$$\Phi^* \begin{pmatrix} \bar{\omega}^1 \\ \bar{\omega}^2 \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix}. \tag{2.51}$$

Thus the structure group is $G = SO(2)$. The next step in the Cartan algorithm is to introduce the lifted coframe, i.e., shifting the coframe to $G \times M$, thus

$$\theta^1 = (\cos t)\omega^1 - (\sin t)\omega^2, \quad \theta^2 = (\sin t)\omega^1 + (\cos t)\omega^2. \tag{2.52}$$

Since

$$\omega^1 = dx, \quad \omega^2 = dy,$$

therefore,

$$d\omega^1 = 0, \quad d\omega^2 = 0. \tag{2.53}$$

Calculating the exterior derivative of θ^1 and θ^2, we obtain

$$\begin{aligned} d\theta^1 &= -\sin t dt \wedge \omega^1 + \cos t d\omega^1 - \cos t dt \wedge \omega^2 - \sin t d\omega^2, \\ &= -\sin t dt \wedge \omega^1 + \cos t d\omega^1 - \cos t dt \wedge \omega^2 - \sin t d\omega^2, \\ &= -\sin t dt \wedge (\cos t\theta^1 + \sin t\theta^2) - \cos t dt \wedge (-\sin t\theta^1 + \cos \theta^2), \\ &= -\sin t \cos t dt \wedge \theta^1 - \sin^2 t dt \wedge \theta^2) + \sin t \cos t dt \wedge \theta^1 - \cos^2 t dt \wedge \theta^2), \\ &= -(\sin^2 t + \cos^2 t) dt \wedge \theta^2 = -dt \wedge \theta^2, \end{aligned}$$

or

$$d\theta^1 = -\alpha \wedge \theta^2. \tag{2.54}$$

Here α is the Maurer-Cartan form defined by $\alpha = dt$. Similarly

$$\theta^2 = (\sin t)\omega^1 + (\cos t)\omega^2,$$

we have

$$d\theta^2 = \alpha \wedge \theta^1. \tag{2.55}$$

Thus the structure equations are

$$d\theta^1 = \alpha \wedge \theta^2, \quad d\theta^2 = \alpha \wedge \theta^1. \tag{2.56}$$

Since there is no torsion coefficient to normalize the structure group in this case consists of just a single parameter 't', which is not fully normalized. Next we have to figure out whether the symmetry group of the equivalence problem is infinite dimensional which indicate that the system (2.56) is in involution or finite dimensional. If the system (2.56) is not in involution, we must prolong. This will be decided by the Cartan test. Before we apply the Cartan Test we have to find the degree of indeterminacy and Cartan reduced characters.

For degree of indeterminacy replacing Maurer-Cartan form $\alpha = \pi$ by $z_1\theta^1 + z_2\theta^2$, in (2.56) and equating coefficient of $\theta^j \wedge \theta^k$ to zero. We get $z_1 = z_2 = 0$. Thus $r^{(1)} = 0$. Next we find Cartan characters for a given vector $\mathbf{v} = (v^1, v^2) \in \mathbb{R}^2$, we define $L[\mathbf{v}]$ to be 2×1 matrix

$$L[\mathbf{v}] \;=\; \begin{pmatrix} -v^2 \\ v^1 \end{pmatrix}. \tag{2.57}$$

The row of L corresponds to the two structure equations and the columns corresponds to the Maurer-Cartan forms π. The first reduce character $s_1^{'}$ is the maximum rank of $\mathbf{v} \in \mathbb{R}^2$, clearly $s_1^{'} = 1$, the second character is

$$s_1^{'} + s_2^{'} \;= \max \text{rank} \begin{pmatrix} -v^2 \\ v^1 \\ -\hat{v}^2 \\ \hat{v}^1 \end{pmatrix} = 1, \tag{2.58}$$

implies that $s_2^{'} = 0$. Applying the Cartan test

$$r^{(1)} \leq s_1^{'} + 2s_2^{'} \Rightarrow 0 < 1(\text{satisfied}).$$

Thus we must prolong by adjoining one-form α to our original lifted coframe. Thus the prolonged structure equations have the form

$$d\theta^1 = -\alpha \wedge \theta^2, \quad d\theta^2 = \alpha \wedge \theta^1, \quad d\alpha = 0. \tag{2.59}$$

In more abstract form

$$d\theta^1 = \theta^2 \wedge \theta^3, \quad d\theta^2 = -\theta^1 \wedge \theta^3, \quad d\theta^3 = 0, \tag{2.60}$$

with $T^1_{23} = 1, \; T^2_{13} = -1$. Thus the group associated with the equivalence problem has dimension three, and the associated group algebra can be found below, since we know that

$$[\frac{\partial}{\partial\theta^j}, \frac{\partial}{\partial\theta^k}] = -\sum_{i=1}^{3} T^i_{jk} \frac{\partial}{\partial\theta^i}, \qquad i,j,k = 1,2,3, \; j < k. \tag{2.61}$$

Therefore

$$[\frac{\partial}{\partial\theta^1}, \frac{\partial}{\partial\theta^2}] = -T^1_{12}\frac{\partial}{\partial\theta^1} - T^2_{12}\frac{\partial}{\partial\theta^2} - T^3_{12}\frac{\partial}{\partial\theta^3}, \tag{2.62}$$

$$= 0. \tag{2.63}$$

This implies

$$[V_1, V_2] = 0.$$

Similarly

$$[V_1, V_3] = V_2, \quad [V_2, V_3] = -V_1.$$

Therefore the symmetry group of the equivalence problem has dimension three, and the associated group algebra in this case is

$$[V_1, V_2] = 0, \quad [V_1, V_3] = V_2, \quad [V_2, V_3] = -V_1. \tag{2.64}$$

Chapter 3

Equivalence of scalar Lagrangians

Consider a first order variational problem [20, 24]

$$\mathfrak{L}[u] = \int L(x, u, p)dx, \quad x, u \in \mathbb{R}, \quad p = \frac{du}{dx}, \tag{3.1}$$

where the Lagrangian $L(x, u, p)$ is analytic on a domain $M \subset \mathbb{R}^3$. Two Lagrangians L and $\bar{L}$ are equivalent (under a fiber-preserving transformation) if and only if there exists a change of variables $\bar{x} = \Phi(x), \quad \bar{u} = \psi(x, u)$ mapping one to the other. We now want to obtain necessary and sufficient conditions for the equivalence of two scalar Lagrangians using Cartan approach. In order to do that we first transform our problem to the language of coframes on a cotangent space.

3.1 Coframe

The preliminary step in the Cartan algorithm is that we shift our equivalence problem on manifolds and associate differential one-forms to them. Introducing differential 1-forms (coframe) corresponding to the two Lagrangians on M and $\bar{M}$,

$$\begin{aligned} \omega_1 &= du - pdx, \quad \omega_2 = L(x, u, p)dx, \quad \omega_3 = dp, \\ \bar{\omega}_1 &= d\bar{u} - \bar{p}d\bar{x}, \quad \bar{\omega}_2 = \bar{L}(\bar{x}, \bar{u}, \bar{p})d\bar{x}, \quad \bar{\omega}_3 = d\bar{p}, \end{aligned} \tag{3.2}$$

which form the basis for the cotangent spaces T^*M and $T^*\bar{M}$ respectively.

3.2 Equivalence conditions

Recalling the equivalence of coframes we see that the two Lagrangians L and $\bar{L}$ are equivalent iff there exist a diffeomorphism $\Phi : M \to \bar{M}$ such that the pull back Φ^* transforms the coframes as (under fiber-preserving transformation)

$$\Phi^*(\bar{\omega}^1) = a_1\omega^1, \quad \Phi^*(\bar{\omega}^2) = \omega^2, \quad \Phi^*(\bar{\omega}^3) = a_3\omega^1 + a_4\omega^2 + a_5\omega^3, \tag{3.3}$$

where a_1, a_3, a_4, a_5 are functions of x, u, p. Introducing the lifted coframe

$$\theta^1 = a_1\omega^1,$$
$$\theta^2 = \omega^2,$$
$$\theta^3 = a_3\omega^1 + a_4\omega^2 + a_5\omega^3, \tag{3.4}$$

with the structure group,

$$g = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & 1 & 0 \\ a_3 & a_4 & a_5 \end{pmatrix}. \tag{3.5}$$

Therefore the basic equivalence conditions for the Lagrangian requires a structure group of dimension four, where $g \in GL(3, \mathbb{R})$. Calculating the Maurer-Cartan forms using the formula $\Pi = dg \cdot g^{-1}$, we get

$$\Pi = \begin{pmatrix} \alpha^1 = \frac{da_1}{a_1} & 0 & 0 \\ 0 & 0 & 0 \\ \alpha^2 = \frac{da_3}{a_1} - \frac{da_5 a_3}{a_1 a_5} & \alpha^3 = da_4 - \frac{da_5 a_4}{a_5} & \alpha^4 = \frac{da_5}{a_5} \end{pmatrix}, \tag{3.6}$$

where $\alpha^1, \alpha^2, \alpha^3, \alpha^4$, are the Maurer-Cartan forms which are dual to the Lie algebra of the structure group g. Now since

$$\omega^1 = du - pdx, \tag{3.7}$$

calculating exterior derivatives of the base coframe elements

$$d\omega^1 = -dp \wedge dx,$$
$$= -\omega^3 \wedge \frac{1}{L}\omega^2,$$

implies that

$$d\omega^1 = -\frac{1}{L}(\omega^3 \wedge \omega^2). \tag{3.8}$$

Similarly

$$d\omega^2 = \frac{L_u}{L}(\omega^1 \wedge \omega^2) - \frac{L_p}{L}(\omega^2 \wedge \omega^3),$$
$$d\omega^3 = 0. \tag{3.9}$$

Now since

$$\omega^1 = \frac{\theta^1}{a_1}, \quad \omega^2 = \theta^2, \quad \omega^3 = -\frac{a_3}{a_1 a_5}\theta^1 - \frac{a_4}{a_5}\theta^2 + \frac{\theta^3}{a_5}, \tag{3.10}$$

writing wedge product of base coframe elements $\omega' s$ interms of lifted coframe elements $\theta' s$,

$$\omega^1 \wedge \omega^2 = \frac{1}{a_1}(\theta^1 \wedge \theta^2),$$
$$\omega^2 \wedge \omega^3 = \frac{a_3}{a_1 a_5}\theta^1 \wedge \theta^2 + \frac{1}{a_5}\theta^2 \wedge \theta^3. \tag{3.11}$$

Thus

$$d\omega^1 = \frac{1}{L}(\frac{a_3}{a_1 a_5}\theta^1 \wedge \theta^2 + \frac{1}{a_5}\theta^2 \wedge \theta^3), \tag{3.12}$$

and

$$d\omega^2 = (\frac{L_u}{a_1 L} - \frac{a_3 L_p}{a_1 a_5 L})\theta^1 \wedge \theta^2 - \frac{L_p}{a_5 L}(\theta^2 \wedge \theta^3), \tag{3.13}$$
$$d\omega^3 = 0. \tag{3.14}$$

3.3 Structure equations

To find the structure equations, we compute the exterior derivative of the lifted coframe,

$$\begin{aligned} \theta^1 &= a_1 \omega^1, \\ d\theta^1 &= da_1 \wedge \omega^1 + a_1 d\omega^1, \\ &= da_1 \wedge \frac{\theta^1}{a_1} + \frac{a_1}{L}(\frac{a_3}{a_1 a_5}\theta^1 \wedge \theta^2 + \frac{1}{a_5}\theta^2 \wedge \theta^3), \end{aligned} \tag{3.15}$$

therefore,

$$d\theta^1 = \frac{da_1}{a_1} \wedge \theta^1 + \frac{a_3}{a_5 L}\theta^1 \wedge \theta^2 + \frac{a_1}{a_5 L}\theta^2 \wedge \theta^3, \tag{3.16}$$

Similarly

$$\begin{aligned} d\theta^2 &= (\frac{L_u}{a_1 L} - \frac{a_3 L_p}{a_1 a_5 L})\theta^1 \wedge \theta^2 + (-\frac{L_p}{a_5 L})\theta^2 \wedge \theta^3, \\ d\theta^3 &= (\frac{da_3}{a_1} - \frac{a_3}{a_1 a_5}da_5) \wedge \theta^1 + (da_4 - \frac{a_4}{a_5}da_5) \wedge \theta^2 + \frac{da_5}{a_5} \wedge \theta^3 \\ &\quad + (\frac{a_3^2}{a_1 a_5 L} + \frac{a_4 L_u}{a_1 L} - \frac{a_3 a_4 L_p}{a_1 a_5 L})\theta^1 \wedge \theta^2 + (\frac{a_3}{a_5 L} - \frac{a_4 L_p}{a_5 L})\theta^2 \wedge \theta^3. \end{aligned} \tag{3.17}$$

3.4 Absorption, normalization, invariant reduction

We are now in a position to implement the Cartan algorithm by applying successive loops as discussed in section (2.6), until we obtain an invariant coframe. Since the structure equations are

$$\begin{aligned} d\theta^1 &= \alpha^1 \wedge \theta^1 + T^1_{12}\theta^1 \wedge \theta^2 + T^1_{23}\theta^2 \wedge \theta^3, \\ d\theta^2 &= T^2_{12}\theta^1 \wedge \theta^2 + T^2_{23}\theta^2 \wedge \theta^3, \\ d\theta^3 &= \alpha^3 \wedge \theta^1 + \alpha^4 \wedge \theta^2 + \alpha^5 \wedge \theta^3 + T^3_{12}\theta^1 \wedge \theta^2 + T^3_{23}\theta^2 \wedge \theta^3, \end{aligned} \tag{3.18}$$

therefore the first loop follows.

Loop 1: From the structure equations it is clear that we can determine easily the inessential torsion coefficients, which can be absorbed by using the following freedom:

$\mathbf{a_1}$) **Absorption** :

$$\alpha^1 = \pi^1 + T^1_{12}\theta^2,$$
$$\alpha^3 = \pi^3 + T^1_{12}\theta^3,$$
$$\alpha^4 = \pi^4 + T^3_{23}\theta^3. \tag{3.19}$$

Thus the structure equations reduced to

$$d\theta^1 = \pi^1 \wedge \theta^1 + T^1_{23}\theta^2 \wedge \theta^3, \tag{3.20}$$
$$d\theta^2 = T^2_{12}\theta^1 \wedge \theta^2 + T^2_{23}\theta^2 \wedge \theta^3, \tag{3.21}$$
$$d\theta^3 = \pi^3 \wedge \theta^1 + \pi^4 \wedge \theta^2 + \pi^5 \wedge \theta^3, \tag{3.22}$$

with

$$T^1_{23} = \frac{a_1}{a_5 L}, \quad T^2_{12} = \frac{a_5 L_u - a_3 L_p}{a_1 a_5 L}, \quad T^2_{23} = -\frac{L_p}{a_1 a_5 L}. \tag{3.23}$$

As we have absorbed all the inessential torsion coefficients therefore we apply normalization.

$\mathbf{b_1}$) **Normalization**: The remaining torsion coefficients can not be absorbed and are known as essential torsion coefficients. Using Cartan's method we can normalize all such coefficients by equating these to appropriate constant values keeping in view that the inverse structure group exists. Therefore

$$T^1_{12} = 1, \quad T^2_{12} = 0, \quad T^2_{23} = -1, \tag{3.24}$$

thus equation (3.23) implies that

$$a_1 = L_p, \quad a_3 = \frac{L_u}{L}, \quad a_5 = \frac{L_p}{L}. \tag{3.25}$$

This complete one loop in the Cartan algorithm, thus g becomes

$$g = \begin{pmatrix} L_p & 0 & 0 \\ 0 & 1 & 0 \\ \frac{L_u}{L} & a_4 & \frac{L_p}{L} \end{pmatrix}. \tag{3.26}$$

Now we obtain the structure equations corresponding to the new structure group (3.26). Since

$$\begin{aligned} da_1 &= d(L_p) \\ &= L_{px}dx + L_{pu}du + L_{pp}dp, \end{aligned} \tag{3.27}$$

where again by replacing each 1-form (dx, du, dp) defined in terms of the new coframe elements (3.2) we obtain

$$\frac{da_1}{a_1} = \frac{1}{L_p}[L_{px}(\frac{\omega^2}{L}) + L_{pu}(\omega^2 + p(\frac{\omega^2}{L})) + L_{pp}\omega^3]. \tag{3.28}$$

We also have from (3.4)

$$\omega^1 = \frac{\theta^1}{L_p}, \omega^2 = \theta^2,$$
$$\omega^3 = -\frac{L_u}{L_p^2}\theta^1 - \frac{a_4 L}{L_p}\theta^2 + \frac{L}{L_p}\theta^3. \tag{3.29}$$

Thus we obtain from (3.28)

$$\frac{da_1}{a_1} = (\frac{L_{pu}}{L_p^2} - \frac{L_{pp}L_u}{L_p^3})\theta^1 + (\frac{L_{px}}{LL_p} + \frac{pL_{pu}}{LL_p} - \frac{a_4 LL_{pp}}{L_p^2})\theta^2 + \frac{LL_{pp}}{L_p^2}\theta^3, \tag{3.30}$$

$$\pi^1 \wedge \theta^1 = (\frac{L_{px}}{LL_p} + \frac{pL_{pu}}{LL_p} - \frac{a_4 LL_{pp}}{L_p^2})\theta^2 \wedge \theta^1 + \frac{LL_{pp}}{L_p^2}\theta^3 \wedge \theta^1. \tag{3.31}$$

thus

$$d\theta^1 = (\frac{a_4 LL_{pp}}{L_p^2} - \frac{L_{px}}{LL_p} - \frac{pL_{pu}}{LL_p})\theta^1 \wedge \theta^2 - \frac{LL_{pp}}{L_p^2}\theta^1 \wedge \theta^3 + \frac{\frac{L_u}{L}}{\frac{L_p}{L}.L}\theta^1 \wedge \theta^2 + \frac{L_p}{\frac{L_p}{L}.L}\theta^2 \wedge \theta^3$$

Therefore the structure equations become,

$$d\theta^1 = (\frac{L_p(L_u - L_{xp} - pL_{up}) + a_4 L^2 L_{pp}}{LL_p^2})\theta^1 \wedge \theta^2 - \frac{LL_{pp}}{L_p^2}\theta^1 \wedge \theta^3 + \theta^2 \wedge \theta^3.$$
$$d\theta^2 = -\theta^2 \wedge \theta^3,$$
$$d\theta^3 = \pi^4 \wedge \theta^2 + T^3_{12}\theta^1 \wedge \theta^2 + T^3_{23}\theta^2 \wedge \theta^3. \tag{3.32}$$

We now arrive at the second loop.

Loop 2: The structure equations are

$$d\theta^1 = T^1_{12}\theta^1 \wedge \theta^2 + T^1_{13}\theta^1 \wedge \theta^3 + \theta^2 \wedge \theta^3, \tag{3.33}$$
$$d\theta^2 = -\theta^2 \wedge \theta^3, \tag{3.34}$$
$$d\theta^3 = \pi^4 \wedge \theta^2 + T^3_{12}\theta^1 \wedge \theta^2 + T^3_{23}\theta^2 \wedge \theta^3. \tag{3.35}$$

There is no inessential torsion coefficient to absorb, thus we go on to normalization.

$\mathbf{b_2}$) Normalization: The only essential torsion coefficient to normalize is T^1_{12}, thus setting $T^1_{12} = 0$, implies that

$$\frac{L_p(L_u - L_{xp} - pL_{up}) + a_4 L^2 L_{pp}}{LL_p^2} = 0, \tag{3.36}$$

which gives

$$L_p(L_u - L_{xp} - pL_{up}) + a_4 L^2 L_{pp} = 0, \tag{3.37}$$

introducing a new variable

$$Q = \frac{\tilde{E}(L)}{L_{pp}}, \tag{3.38}$$

where

$$\tilde{E}(L) = L_u - L_{xp} - pL_{up},$$

we obtain the explicit form of a_4 from equation (3.37)

$$a_4 = -\frac{L_p Q}{L^2}. \tag{3.39}$$

3.5 Invariant coframe

We have normalized all the group parameters. Inserting their values, we get an invariant coframe for the standard, fiber-preserving Lagrangian equivalence problem,

$$\begin{aligned}
\theta^1 &= L_p(du - pdx), && (3.40)\\
\theta^2 &= Ldx, && (3.41)\\
\theta^3 &= \frac{L_u}{L}(du - pdx) + \frac{L_p}{L}(dp - Qdx),\\
&= d(\log L) - \hat{D}_x(\log L)dx, && (3.42)
\end{aligned}$$

where

$$\hat{D}_x := \frac{\partial}{\partial x} + p\frac{\partial}{\partial u} + Q(x,u,p)\frac{\partial}{\partial p}. \tag{3.43}$$

The structure equations for the invariant coframe are then found to be

$$\begin{aligned}
d\theta^1 &= -I_1\theta^1 \wedge \theta^3 + \theta^2 \wedge \theta^3,\\
d\theta^2 &= -\theta^2 \wedge \theta^3,\\
d\theta^3 &= I_2\theta^1 \wedge \theta^2 + I_3\theta^2 \wedge \theta^3,
\end{aligned} \tag{3.44}$$

where

$$\begin{aligned}
I_1 &= \frac{LL_{pp}}{L_p^2}, && (3.45)\\
I_2 &= \frac{1}{L^2L_p^2}(L_{ux}L_p - pL_{uu} - L_u^2 - L_uL_x + L_xL_{xp} + pL_xL_{ux} + L_u), && (3.46)\\
I_3 &= \frac{L_p}{L^2L_{pp}}(L_u - L_{xp} - pL_{up}) - \frac{L_{pp}}{L_p}. && (3.47)
\end{aligned}$$

Equation (3.44) provides us the structure equations on an invariant coframe where the invariants are determined explicitly with in terms of the Lagrangian and its derivatives. We can obtain some more information from these invariants by writing these in terms of the coframe derivatives. In order to do that we proceed as follows. First we have

$$\begin{aligned}
\theta^1 &= -pL_pdx + L_pdu + 0dp,\\
\theta^2 &= Ldx + 0du + 0dp,\\
\theta^3 &= (-p\frac{L_u}{L} - \frac{L_p}{L}Q)dx + \frac{L_u}{L}du + \frac{L_p}{L}dp.
\end{aligned} \tag{3.48}$$

The coframe elements in terms of coordinate coframe is given by,

$$\theta^i = \sum_j a^i_j dx^j, \tag{3.49}$$

or

$$\theta^1 = a^1_1 dx + a^1_2 du + a^1_3 dp, \tag{3.50}$$
$$\theta^2 = a^2_1 dx + a^2_2 du + a^2_3 dp, \tag{3.51}$$
$$\theta^3 = a^3_1 dx + a^3_2 du + a^3_3 dp, \tag{3.52}$$

with the coefficients matrix

$$A = \begin{pmatrix} -pL_p & L_p & 0 \\ L & 0 & 0 \\ -\frac{pL_u}{L} - \frac{QL_p}{L} & \frac{L_u}{L} & \frac{L_p}{L} \end{pmatrix}, \tag{3.53}$$

with

$$A^{-1} = B = b^i_j(x) \begin{pmatrix} 0 & \frac{1}{L} & 0 \\ \frac{1}{L_p} & \frac{p}{L} & 0 \\ -\frac{L_u}{L_p^2} & \frac{QL_p + pL_u - pL_u}{LL_p} & \frac{L}{L_p} \end{pmatrix}. \tag{3.54}$$

We can write the coframe derivatives in terms of the basic coframe elements by simply applying the chain rule.

$$\frac{\partial F}{\partial \theta^j} = \sum_i b^i_j \frac{\partial}{\partial x^i}, \tag{3.55}$$

$$\begin{aligned} \frac{\partial F}{\partial \theta^1} &= b^1_1 \frac{\partial F}{\partial x^1} + b^2_1 \frac{\partial F}{\partial x^2} + b^3_1 \frac{\partial F}{\partial x^3}, \\ &= b^1_1 \frac{\partial F}{\partial x} + b^2_1 \frac{\partial F}{\partial u} + b^3_1 \frac{\partial F}{\partial p}, \\ &= \frac{L_p F_u - L_u F_p}{L_p^2}. \end{aligned} \tag{3.56}$$

Similarly

$$\frac{\partial F}{\partial \theta^2} = \frac{1}{L} \tilde{D}_x F, \quad \frac{\partial F}{\partial \theta^3} = \frac{L}{L_p} \frac{\partial F}{\partial p}, \tag{3.57}$$

with

$$\tilde{D}_x = \frac{\partial}{\partial x} + p \frac{\partial}{\partial u} + Q \frac{\partial}{\partial p}. \tag{3.58}$$

The fundamental invariants can then be written in simple explicit form,

$$I_1 = \frac{LL_{pp}}{L_p^2} = \frac{1}{L_p} \frac{\partial L_p}{\partial \theta^3}, \tag{3.59}$$

$$I_2 = -\frac{1}{L} \frac{\partial^2 L}{\partial \theta^1 \partial \theta^2}, \tag{3.60}$$

$$I_3 = -\frac{1}{L} \frac{\partial^2 L}{\partial \theta^3 \partial \theta^2}, \tag{3.61}$$

$$\tag{3.62}$$

These formulas can be derived directly from the structure equation (3.44) and the basic definition for the coframe derivative. Applying (3.56), (3.57) when $F = L$, we first note that

$$\frac{\partial L}{\partial \theta^1} = 0, \quad \frac{\partial L}{\partial \theta^3} = L. \tag{3.63}$$

Therefore

$$\begin{aligned} dL &= \frac{\partial L}{\partial \theta^1}\theta^1 + \frac{\partial L}{\partial \theta^2}\theta^2 + \frac{\partial L}{\partial \theta^3}\theta^3, \\ dL &= (\frac{\partial L}{\partial \theta^2})\theta^2 + L\theta^3. \end{aligned} \tag{3.64}$$

Furthermore, using structure equation (3.44),

$$\begin{aligned} 0 = d^2L &= d(\frac{\partial L}{\partial \theta^2}) \wedge \theta^2 + dL \wedge \theta^3 + (\frac{\partial L}{\partial \theta^2})d\theta^2 + Ld\theta^3, \\ 0 &= (\frac{\partial^2 L}{\partial \theta^1 \partial \theta^2} + LI_2)\theta^1 \wedge \theta^2 + (-\frac{\partial^2 L}{\partial \theta^3 \partial \theta^2} + LI_3)\theta^2 \wedge \theta^3. \end{aligned} \tag{3.65}$$

Therefore

$$\frac{\partial^2 L}{\partial \theta^1 \partial \theta^2} + LI_2 = 0, \tag{3.66}$$

and

$$-\frac{\partial^2 L}{\partial \theta^3 \partial \theta^2} + LI_3 = 0. \tag{3.67}$$

Thus

$$I_2 = -\frac{1}{L}\frac{\partial^2 L}{\partial \theta^1 \partial \theta^2}, \tag{3.68}$$

$$I_3 = \frac{1}{L}\frac{\partial^2 L}{\partial \theta^3 \partial \theta^2}. \tag{3.69}$$

Now if these invariants are constants then the symmetry group has dimension three, and the group algebra can be found by

$$[\frac{\partial}{\partial \theta^j}, \frac{\partial}{\partial \theta^k}] = -\sum_{i=1}^{3} T^i_{jk}\frac{\partial}{\partial \theta^i}, \tag{3.70}$$

$i, j, k = 1,2,3,\ j < k$.

$$[\frac{\partial}{\partial \theta^1}, \frac{\partial}{\partial \theta^2}] = -T^1_{12}\frac{\partial}{\partial \theta^1} - T^2_{12}\frac{\partial}{\partial \theta^2} - T^3_{12}\frac{\partial}{\partial \theta^3}, \tag{3.71}$$

thus

$$\begin{aligned} [V_1, V_2] &= -I_2V_3, \\ [V_1, V_2] &= -c_2V_3, \end{aligned} \tag{3.72}$$

where $c_2 = I_2$. Similarly

$$[V_1, V_3] = -c_1V_1, \tag{3.73}$$

where $c_1 = I_1$, and

$$[V_2, V_3] = V_1 - V_2. \tag{3.74}$$

Thus we have the following group algebra

$$[V_1, V_2] = -c_2V_3, \quad [V_1, V_3] = -c_1V_1, \quad [V_2, V_3] = V_1 - V_2. \tag{3.75}$$

Chapter 4

Equivalence of scalar second order ODEs

In this chapter we present the equivalence problem for a second order linear ODE [18] with constant coefficients under fiber-preserving transformation using Cartan method of equivalence. We will also find the symmetry group and the group algebra associated to our equivalence problem. A linear second order ODE with constant coefficients is given by

$$u'' = au' + bu + c = Q(x, u, u'). \tag{4.1}$$

As before, we translate our problem in to geometrical terms.

4.1 Coframe

In this section we explain how to construct a runway for the Cartan algorithm, the first step is to shift the equivalence problem over a manifold ($M = J(x, u, p)$) and associate differential 1-forms (coframe) with it. Thus the associated 1-forms are

$$\omega^1 = du - u'dx, \quad \omega^2 = du - Q(x, u, u')dx, \quad \omega^3 = dx. \tag{4.2}$$

It is easy to see that the above 1-forms constitute the basis for cotangent space.

4.2 Equivalence conditions

Since our equivalence problem is shifted to a manifold and we have associated a coframe with it on M, now the coframe on $\bar{M}$ will come from the equivalence condition. Since under fiber-preserving transformations

$$\begin{aligned} d\overline{u} - \overline{p}d\overline{x} &= \lambda(du - pdx), \\ d\overline{p} - \overline{q}d\overline{x} &= \mu(du - pdx) + \nu(dp - qdx), \\ d\overline{x} &= \alpha dx. \end{aligned} \tag{4.3}$$

Therefore

$$\begin{pmatrix} \bar{\omega}^1 \\ \bar{\omega}^2 \\ \bar{\omega}^3 \end{pmatrix} = \begin{pmatrix} a_1 & 0 & 0 \\ a_2 & a_3 & 0 \\ 0 & 0 & a_4 \end{pmatrix} \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{pmatrix}, \tag{4.4}$$

(omitting the pull-back) so that the lifted coframe is

$$
\begin{aligned}
\theta^1 &= a_1\omega^1, \\
\theta^2 &= a_2\omega^1 + a_3\omega^2, \\
\theta^3 &= a_4\omega^3,
\end{aligned} \tag{4.5}
$$

with the structure group

$$
g = \left\{ \begin{pmatrix} a_1 & 0 & 0 \\ a_2 & a_3 & 0 \\ 0 & 0 & a_4 \end{pmatrix}, \text{ such that } \quad a_1 a_3 a_4 \neq 0 \right\}. \tag{4.6}
$$

Thus the structure group has dimension four, where $g \in GL(3, \mathbb{R})$.

$$
\begin{aligned}
\theta &= g \cdot \omega \\
&= \begin{pmatrix} a_1 & 0 & 0 \\ a_2 & a_3 & 0 \\ 0 & 0 & a_4 \end{pmatrix} \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{pmatrix}.
\end{aligned}
$$

4.3 Structure equations

To find the structure equations we first calculate the Maurer-Cartan forms using the formula $\Pi = dg \cdot g^{-1}$, since

$$
g = \begin{pmatrix} a_1 & 0 & 0 \\ a_2 & a_3 & 0 \\ 0 & 0 & a_4 \end{pmatrix}, \tag{4.7}
$$

and

$$
g^{-1} = \begin{pmatrix} \frac{1}{a_1} & 0 & 0 \\ -\frac{a_2}{a_1 a_3} & \frac{1}{a_3} & 0 \\ 0 & 0 & \frac{1}{a_4} \end{pmatrix}, \tag{4.8}
$$

also

$$
dg = \begin{pmatrix} da_1 & 0 & 0 \\ da_2 & da_3 & 0 \\ 0 & 0 & da_4 \end{pmatrix}. \tag{4.9}
$$

Therefore

$$
\Pi = \begin{pmatrix} \alpha^1 = \frac{da_1}{a_1} & 0 & 0 \\ \alpha^2 = \frac{da_2}{a_1} - \frac{a_2}{a_1 a_3} da_3 & \alpha^3 = \frac{da_3}{a_3} & 0 \\ 0 & 0 & \alpha^4 = \frac{da_4}{a_4} \end{pmatrix}. \tag{4.10}
$$

Now since

$$
\omega^1 = du - p dx, \quad \omega^2 = dp - q dx, \quad \omega^3 = dx. \tag{4.11}
$$

Calculating exterior derivative of $\omega' s$

$$\begin{aligned} d\omega^1 &= d(du) - dp \wedge dx, \\ d\omega^1 &= -\omega^2 \wedge \omega^3. \end{aligned} \tag{4.12}$$

Similarly

$$\begin{aligned} d\omega^2 &= -a(\omega^2 \wedge \omega^3) - b(\omega^1 \wedge \omega^3), \\ d\omega^3 &= 0. \end{aligned} \tag{4.13}$$

Writing $d\omega' s$ in term of the wedge product of $\theta' s$, since

$$\begin{aligned} \omega^1 &= \frac{\theta^1}{a_1}, \\ \omega^2 &= -\frac{a_2}{a_1 a_3}\theta^1 + \frac{\theta^2}{a_3}, \\ \omega^3 &= \frac{\theta^3}{a_4}, \end{aligned} \tag{4.14}$$

therefore

$$\begin{aligned} \omega^1 \wedge \omega^3 &= \frac{1}{a_1 a_4}\theta^1 \wedge \theta^3, \\ \omega^2 \wedge \omega^3 &= -\frac{a_2}{a_1 a_3 a_4}\theta^1 \wedge \theta^3 + \frac{1}{a_3 a_4}\theta^2 \wedge \theta^3. \end{aligned} \tag{4.15}$$

Thus we have

$$\begin{aligned} d\omega^1 &= -\omega^2 \wedge \omega^3, \\ &= -(-\frac{a_2}{a_1 a_3 a_4}\theta^1 \wedge \theta^3 + \frac{1}{a_3 a_4}\theta^2 \wedge \theta^3), \end{aligned} \tag{4.16}$$

or

$$d\omega^1 = \frac{a_2}{a_1 a_3 a_4}\theta^1 \wedge \theta^3 - \frac{1}{a_3 a_4}\theta^2 \wedge \theta^3. \tag{4.17}$$

Similarly

$$\begin{aligned} d\omega^2 &= [\frac{a.a_2}{a_1 a_3 a_4} - \frac{b}{a_1 a_4}]\theta^1 \wedge \theta^3 - \frac{a}{a_3 a_4}\theta^2 \wedge \theta^3, \\ d\omega^3 &= 0. \end{aligned} \tag{4.18}$$

Now since

$$\begin{aligned} \theta^1 &= a_1 \omega^1, \\ d\theta^1 &= da_1 \wedge \omega^1 + a_1 d\omega^1, \\ &= da_1 \wedge (\frac{\theta^1}{a_1}) + a_1(\frac{a_2}{a_1 a_3 a_4}\theta^1 \wedge \theta^3 - \frac{1}{a_3 a_4}\theta^2 \wedge \theta^3), \end{aligned} \tag{4.19}$$

or

$$d\theta^1 = \frac{da_1}{a_1} \wedge \theta^1 + \frac{a_2}{a_3 a_4}\theta^1 \wedge \theta^3 - \frac{a_1}{a_3 a_4}\theta^2 \wedge \theta^3. \tag{4.20}$$

Similarly

$$d\theta^2 = [\frac{da_2}{a_1} - \frac{a_2}{a_1 a_3} da_3] \wedge \theta^1 + \frac{da_3}{a_3}\theta^2 + [\frac{a_2^2}{a_1 a_3 a_4} + \frac{a.a_2}{a_1 a_4} - \frac{ba_3}{a_1 a_4}]\theta^1 \wedge \theta^3 + [-\frac{a_2}{a_3 a_4} - \frac{a}{a_4}]\theta^2 \wedge \theta^3,$$
$$d\theta^3 = \frac{da_4}{a_4} \wedge \theta^3. \tag{4.21}$$

4.4 Absorption, normalization, invariant reduction

Now we are ready to go for the important steps in the Cartan algorithm, the goal is to absorb as many torsion coefficients as we can and normalize the remaining torsion coefficients.

Loop 1: The structure equations are

$$\begin{aligned} d\theta^1 &= \alpha^1 \wedge \theta^1 + T^1_{13}\theta^1 \wedge \theta^3 + T^1_{23}\theta^2 \wedge \theta^3, \\ d\theta^2 &= \alpha^2 \wedge \theta^1 + \alpha^3 \wedge \theta^2 + T^2_{13}\theta^1 \wedge \theta^3 + T^2_{23}\theta^2 \wedge \theta^3, \\ d\theta^3 &= \alpha^4 \wedge \theta^3. \end{aligned} \tag{4.22}$$

Subsequently we absorb all the inessential torsion coefficients.

$\mathbf{a_1}$) Absorption : From the above equation, the inessional torsion coefficients can clearly be seen, they can easily be absorbed by using following freedom

$$\begin{aligned} \alpha^1 &= \pi^1 + T^1_{13}\theta^3, \\ \alpha^2 &= \pi^2 + T^2_{13}\theta^3, \\ \alpha^3 &= \pi^3 + T^2_{23}\theta^3. \end{aligned} \tag{4.23}$$

The structure equations are reduced to,

$$\begin{aligned} d\theta^1 &= \pi^1 \wedge \theta^1 + T^1_{23}\theta^2 \wedge \theta^3, \\ d\theta^2 &= \pi^2 \wedge \theta^1 + \pi^3 \wedge \theta^2, \\ d\theta^3 &= \pi^4 \wedge \theta^3, \end{aligned} \tag{4.24}$$

with

$$T^1_{23} = -\frac{a_1}{a_3 a_4}.$$

Since there are no more torsion coefficients to absorb therefore we apply normalization.

$\mathbf{b_1}$) Normalization: The remaing essentional torsion coefficients are then normalized by assigning to the an appropriate value. Thus normalizing $T^1_{23} = -1$, we determine

$$a_3 = \frac{a_1}{a_4}. \tag{4.25}$$

We now arrive at the second loop.

Loop 2

$$
\begin{aligned}
d\theta^1 &= \pi^1 \wedge \theta^1 - \theta^2 \wedge \theta^3, \\
d\theta^2 &= \pi^2 \wedge \theta^1 + (\pi^1 - \pi^4) \wedge \theta^2, \\
d\theta^3 &= \pi^4 \wedge \theta^3.
\end{aligned} \tag{4.26}
$$

In above system we have no essentional torsion to normalize, and thus the structure group is not fully reduced. Now we must figure out whether the symmetry group of the equivalence problem is infinite dimensional, which indicates that the system (4.26) is being in involution, or finite dimensional so that the system (4.26) is not in involution, If not we must prolong. To check the above cases we must apply the Cartan test.

4.5 Cartan test

First we will find the degree of indeterminacy, by replacing each Maurer's Cartan for π^i by a linear combination $z_1^i\theta^1 + z_2^i\theta^2 + z_3^i\theta^3$ of the lifted coframe element (4.26), equating the resulting coefficients of the basis of two form $\theta^j \wedge \theta^k$ to zero. Since

$$
d\theta^1 = \pi^1 \wedge \theta^1 - \theta^2 \wedge \theta^3, \tag{4.27}
$$

therefore

$$
\begin{aligned}
d\theta^1 &= (z_1^1\theta^1 + z_2^1\theta^2 + z_3^1\theta^3) \wedge \theta^1 - \theta^2 \wedge \theta^3, \\
&= z_2^1\theta^2 \wedge \theta^1 + z_3^1\theta^3 \wedge \theta^1 - \theta^2 \wedge \theta^3,
\end{aligned} \tag{4.28}
$$

which implies that

$$
z_2^1 = z_3^1 = 0. \tag{4.29}
$$

Similarly, from the other two equations we get

$$
\begin{aligned}
&z_3^2 = 0, \quad z_3^4 = 0, \quad z_1^1 = z_1^4 + z_2^2, \\
&z_1^4 = z_2^4 = 0.
\end{aligned} \tag{4.30}
$$

Thus we have

$$
z_1^1 = z_2^2, \quad z_1^2, \quad z_2^1 = z_3^1 = z_3^2 = z_1^4 = z_2^4 = z_3^4 = 0. \tag{4.31}
$$

The two parameters z_1^1, z_1^2, can be chosen arbitrarily, they are the free variables for the absorption equation. Thus

$$
r^{(1)} = 2. \tag{4.32}
$$

To continue to apply the Cartan test for involution, we next introduce Cartan character associated with the structure equation. For a given vector $\mathbf{v} = (v^1, v^2, v^3) \in \mathbb{R}^3$ define $L[\mathbf{v}]$ to be 3×3, matrix

$$
L[\mathbf{v}] = \begin{pmatrix} v^1 & 0 & 0 \\ v^2 & v^1 & -v^2 \\ 0 & 0 & v^3 \end{pmatrix}. \tag{4.33}
$$

The row of L corresponds to the three structure equation and columns corresponds to the Maurer's Cartan form π^1, π^2, π^4. The first reduce character $s_1^{'}$ is the maximal rank of $L[\mathbf{v}]$ for all possible vector $\mathbf{v} \in \mathbb{R}^3$. Clearly $s_1^{'} = 3$, the second Cartan character is defined as

$$s_1^{'} + s_2^{'} = \max \text{rank} \begin{pmatrix} v^1 & 0 & 0 \\ v^2 & v^1 & -v^2 \\ 0 & 0 & v^3 \\ \hat{v}^{'} & 0 & 0 \\ \hat{v}^2 & \hat{v}^{'} & -\hat{v}^2 \\ 0 & 0 & \hat{v}^3 \end{pmatrix} = 3. \tag{4.34}$$

Thus

$$\begin{aligned} s_1^{'} + s_2^{'} &= 3, \\ s_2^{'} &= 3 - 3 = 0, \\ s_2^{'} &= 0, \quad s_3^{'} = 0. \end{aligned} \tag{4.35}$$

Now applying the Cartan test to check whether the coframe is involutive or not

$$r^{(1)} \leq s_1^{'} + 2s_2^{'} + 3s_3^{'}, \tag{4.36}$$

$$2 < 3, \quad \text{(satisfied)}. \tag{4.37}$$

Thus the system (4.26), is not in involution, we must prolong.

4.6 Prolongation

Since the associated symmetry group to our equivalence problem is not infinite dimensional, and the structure group is not fully reduced, which happen because the dimension of the associated symmetry group of the equivalence problem is greater than the dimension of the underlying manifold upon which the equivalence problem was defined. Therefore we must prolong the dimension of our manifold by adding additional 1-forms. Now since

$$L[\mathbf{v}] = A^i_{jk} v^j = \begin{pmatrix} z^1_1 & 0 & 0 \\ z^2_1 & z^2_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} r & 0 & 0 \\ s & r & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{4.38}$$

with $z^1_1 = z^2_2 = r$ and $z^2_1 = s$. Here

$$\begin{aligned} \pi^1 &= \varpi^1 + r\theta^1, \\ \pi^2 &= \varpi^2 + s\theta^1 + r\theta^2, \\ \pi^3 &= \varpi^4, \end{aligned} \tag{4.39}$$

are the modified Maurer-Cartan forms, and the additional 1-forms

$$\begin{aligned} \varpi^1 &= \alpha^1 - T^1_{13}\theta^3, \\ \varpi^1 &= \alpha^1 - \frac{a_2}{a_1}\theta^3, \\ \varpi^2 &= \alpha^2 + \left(\frac{ba_1^2 - a.a_1a_2a_4 - a_2^2a_4^2}{a_1^2a_4^2}\right)\theta^3, \\ \varpi^4 &= \alpha^4 - \left(\frac{a.a_1 + 2a_2a_4}{a_1a_4}\right)\theta^3. \end{aligned} \tag{4.40}$$

Now

$$\begin{aligned}
\varpi^1 &= \alpha^1 - \frac{a_2}{a_1}\theta^3, \\
d\varpi^1 &= d(\alpha^1) - d(\frac{a_2}{a_1}) \wedge \theta^3 - \frac{a_2}{a_1} d\theta^3, \\
d\varpi^1 &= -(\frac{da_2}{a_1} - \frac{a_2}{a_1^2} da_1) \wedge \theta^3 - \frac{a_2}{a_1}(\pi^4 \wedge \theta^3), \\
&= \frac{a_2}{a_1}(\frac{da_1}{a_1}) \wedge \theta^3 - \frac{da_2}{a_1} \wedge \theta^3 - \frac{a_2}{a_1}(\pi^4 \wedge \theta^3), \\
&= \frac{a_2}{a_1}\alpha^1 \wedge \theta^3 - \{\alpha^2 + \frac{a_2}{a_1}(\alpha^1 - \alpha^4)\} \wedge \theta^3 - \frac{a_2}{a_1}(\pi^4 \wedge \theta^3), \\
&= \frac{a_2}{a_1}\alpha^1 \wedge \theta^3 - \alpha^2 \wedge \theta^3 + \frac{a_2}{a_1}\alpha^1 \wedge \theta^3 - \frac{a_2}{a_1}\alpha^4 \wedge \theta^3 - \frac{a_2}{a_1}(\pi^4 \wedge \theta^3), \\
&= \varpi^2 \wedge \theta^3 + \frac{a_2}{a_1}\varpi^4 \wedge \theta^3 - \frac{a_2}{a_1}(\pi^4 \wedge \theta^3), \\
&= -(\pi^2 - s\theta^1 - r\theta^2) \wedge \theta^3 + \frac{a_2}{a_1}(\pi^4 \wedge \theta^3) - \frac{a_2}{a_1}(\pi^4 \wedge \theta^3),
\end{aligned}$$

or

$$d\varpi^1 = -\pi^2 \wedge \theta^3 + s\theta^1 \wedge \theta^3 + r\theta^2 \wedge \theta^3. \tag{4.41}$$

Similarly

$$\begin{aligned}
d\varpi^2 &= -s\theta^1 \wedge \pi^4 - r\theta^2 \wedge \pi^4 + \pi^2 \wedge \pi^4, \\
d\varpi^4 &= -2\pi^2 \wedge \theta^3 + 2s\theta^1 \wedge \theta^3 + 2r\theta^2 \wedge \theta^3.
\end{aligned} \tag{4.42}$$

Now

$$\begin{aligned}
\pi^1 &= \varpi^1 + r\theta^1, \\
d\pi^1 &= d\varpi^1 + dr \wedge \theta^1 + rd\theta^1, \\
&= (-\pi^2 \wedge \theta^3 + s\theta^1 \wedge \theta^3 + r\theta^2 \wedge \theta^3) + dr \wedge \theta^1 + r(\pi^1 \wedge \theta^1 - \theta^2 \wedge \theta^3), \\
&= dr \wedge \theta^1 + s\theta^1 \wedge \theta^3 + r\pi^1 \wedge \theta^1 - \pi^2 \wedge \theta^3,
\end{aligned}$$

which implies that

$$\begin{aligned}
d\pi^1 &= dr \wedge \theta^1 + s\theta^1 \wedge \theta^3 + r\pi^1 \wedge \theta^1 - \pi^2 \wedge \theta^3, \\
d\pi^1 &= \tilde{\rho} \wedge \theta^1 + s\theta^1 \wedge \theta^3 + r\pi^1 \wedge \theta^1 - \pi^2 \wedge \theta^3.
\end{aligned} \tag{4.43}$$

$\mathbf{a_2}$) **Absorption** : The inessional torsion coefficients can clearly be absorbed, they can be absorbed by using following freedom

$$\tilde{\rho} = \rho + s\theta^3 - r\pi^1, \tag{4.44}$$

and thus the structure equation $d\pi^1$ become

$$d\pi^1 = \rho \wedge \theta^1 - \pi^2 \wedge \theta^3. \tag{4.45}$$

Similarly

$$\begin{aligned}
d\pi^2 &= \delta \wedge \theta^1 + \rho \wedge \theta^2 - 2s\theta^2 \wedge \theta^3 + \pi^2 \wedge \pi^4, \\
d\pi^4 &= 2s\theta^1 \wedge \theta^3 + 2r\theta^2 \wedge \theta^3 - 2\pi^2 \wedge \theta^3.
\end{aligned} \tag{4.46}$$

Thus the structure equations are

$$\begin{aligned}
d\theta^1 &= \pi^1 \wedge \theta^1 - \theta^2 \wedge \theta^3,\\
d\theta^2 &= \pi^2 \wedge \theta^1 + (\pi^1 - \pi^4) \wedge \theta^2,\\
d\theta^3 &= \pi^4 \wedge \theta^3,\\
d\pi^1 &= \rho \wedge \theta^1 - \pi^2 \wedge \theta^3,\\
d\pi^2 &= \delta \wedge \theta^1 + \rho \wedge \theta^2 - A\theta^2 \wedge \theta^3 + \pi^2 \wedge \pi^4,\\
d\pi^4 &= A\theta^1 \wedge \theta^3 + B\theta^2 \wedge \theta^3 - 2\pi^2 \wedge \theta^3,
\end{aligned} \tag{4.47}$$

where

$$A = 2s, \quad B = 2r. \tag{4.48}$$

$\mathbf{b_2}$) **Normalization**: The only essentional torsion coefficients are A and B, which can be normalized by assigning to them some appropriate value, thus normalizing $A = B = 1$ implies that $s = \frac{1}{2}$ and $r = \frac{1}{2}$. Therefore we immediately reduce the prolonged equivalence problem to an equivalence problem for the coframe.

$$\begin{aligned}
d\theta^1 &= \pi^1 \wedge \theta^1 - \theta^2 \wedge \theta^3,\\
d\theta^2 &= \pi^2 \wedge \theta^1 + (\pi^1 - \pi^4) \wedge \theta^2,\\
d\theta^3 &= \pi^4 \wedge \theta^3,\\
d\pi^1 &= -\pi^2 \wedge \theta^3,\\
d\pi^2 &= -\theta^2 \wedge \theta^3 + \pi^2 \wedge \pi^4,\\
d\pi^4 &= \theta^1 \wedge \theta^3 + \theta^2 \wedge \theta^3 - 2\pi^2 \wedge \theta^3.
\end{aligned} \tag{4.49}$$

The zeroth order classifying manifold is a single point defined by $T^i_{jk} = -C^i_{jk}$. All derived invariants are automatically zero and hence every classifying manifold $\mathrm{C}^{(s)}$ is a single point. Therefore $\rho(s) = 0,\ s \geq 0$, therefore the coframe has rank zero. Now

$$\begin{aligned}
d\theta^1 &= -\theta^1 \wedge \theta^4 - \theta^2 \wedge \theta^3,\\
d\theta^2 &= -\theta^1 \wedge \theta^5 - \theta^2 \wedge \theta^4 + \theta^2 \wedge \theta^6,\\
d\theta^3 &= -\theta^3 \wedge \theta^6,\\
d\theta^4 &= \theta^3 \wedge \theta^5,\\
d\theta^5 &= -\theta^2 \wedge \theta^3 - \theta^5 \wedge \theta^6,\\
d\theta^6 &= \theta^1 \wedge \theta^3 + \theta^2 \wedge \theta^3 + 2\theta^3 \wedge \theta^5,
\end{aligned} \tag{4.50}$$

with the torsion coefficients

$$\begin{aligned}
&T^1_{14} = -1, \quad T^1_{23} = -1, \quad T^2_{15} = -1, \quad T^2_{24} = -1, \quad T^2_{26} = 1, \quad T^3_{36} = -1,\\
&T^4_{35} = 1, \quad T^5_{23} = -1, \quad T^5_{56} = 1, \quad T^6_{13} = 1, \quad T^6_{23} = 1, \quad T^6_{35} = 2.
\end{aligned} \tag{4.51}$$

Now since

$$[\frac{\partial}{\partial\theta^j}, \frac{\partial}{\partial\theta^k}] = -\sum_{i=1}^{6} T^i_{jk}\frac{\partial}{\partial\theta^i}, \tag{4.52}$$

where $i, j, k = 1...6; j < k$

$$[\frac{\partial}{\partial\theta^1}, \frac{\partial}{\partial\theta^2}] = -T^1_{12}\frac{\partial}{\partial\theta^1} - T^2_{12}\frac{\partial}{\partial\theta^2} - T^3_{12}\frac{\partial}{\partial\theta^3} - T^4_{12}\frac{\partial}{\partial\theta^4} - T^5_{12}\frac{\partial}{\partial\theta^5} - T^6_{12}\frac{\partial}{\partial\theta^6}, \tag{4.53}$$

thus

$$[V_1, V_2] = 0. \tag{4.54}$$

Similarly all others are zero, and the only non-zero commutators are,

$$[V_1, V_5] = [V_2, V_4] = -[V_2, V_6] = V_2, \quad [V_2, V_3] = V_1 + V_5 - V_6, \quad [V_3, V_5] = -V_4 - 2V_6,$$

$$[V_1, V_3] = -V_6, \quad [V_1, V_4] = V_1, \quad [V_3, V_6] = V_3, \quad [V_5, V_6] = -V_5.$$

Chapter 5

Equivalence of the system of second order linear ODEs

In the previous chapter we verified that for a scalar linear ODE with constant coefficients the algebra of fiber-preserving transformations is six dimensional. It is well known that there is more than one class of system of linear ODEs [30]. In this chapter our aim is to investigate the equivalence of systems of second order linear ODEs. The aim is to generalize the equivalence problem for a scalar linear ODE to a system of linear ODEs. For that purpose we consider a system of un-coupled linear ODEs with constant coefficients and prove that the algebra of fiber-preserving transformations is ten dimensional. In the last section we investigate a particular class of system of coupled ODEs with constant coefficients and establish that the dimension of algebra is six dimensional in this case. Therefore we have reconfirmed that there is more than one class of system of linear ODEs using Cartan approach.

5.1 Equivalence of uncoupled system of second order linear ODEs

Consider a system of second order linear ODE with constant coefficients a, b, c and d

$$\begin{aligned} y'' &= ay' + by, \\ z'' &= cz' + dz. \end{aligned} \tag{5.1}$$

Thus the Cartan algorithm can be applied by introducing the corresponding coframe on the cotangent space.

5.1.1 Coframe

In this section we explain how to construct a coframe for the Cartan algorithm to which the equivalence approach developed in the second chapter can be applied. The preliminary step is to shift the equivalence problem over a manifold ($M = J^2(x, y, z, y' = p_1, z' = q_1)$) and associate differential 1-forms (coframe) with it. Thus the associated 1-forms are

$$\begin{aligned} &\omega^1 = dy - y'dx, \quad \omega^2 = dy' - (ay' + by)dx, \\ &\omega^1 = dz - z'dx, \quad \omega^4 = dz' - (cz' + dz)dx, \\ &\omega^5 = dx. \end{aligned} \tag{5.2}$$

It is easy to see that the above 1-forms constitute the basis for cotangent space.

5.1.2 Equivalence conditions

Since under fiber-preserving transformation

$$\begin{aligned} d\overline{u} - \overline{p}d\overline{x} &= \lambda(du - pdx), \\ d\overline{p} - \overline{q}d\overline{x} &= \mu(du - pdx) + \nu(dp - qdx), \\ d\overline{x} &= \alpha dx. \end{aligned} \tag{5.3}$$

Applying above transformation we get

$$\overline{\omega}^1 = a_1\omega^1, \quad \overline{\omega}^2 = a_2\omega^1 + a_3\omega^2, \quad \overline{\omega}^3 = a_4\omega^3,$$

$$\overline{\omega}^4 = a_5\omega^3 + a_6\omega^4, \quad \overline{\omega}^5 = a_7\omega^5, \tag{5.4}$$

$$\begin{pmatrix} \overline{\omega}^1 \\ \overline{\omega}^2 \\ \overline{\omega}^3 \\ \overline{\omega}^4 \\ \overline{\omega}^5 \end{pmatrix} = \begin{pmatrix} a_1 & 0 & 0 & 0 & 0 \\ a_2 & a_3 & 0 & 0 & 0 \\ 0 & 0 & a_4 & 0 & 0 \\ 0 & 0 & a_5 & a_6 & 0 \\ 0 & 0 & 0 & 0 & a_7 \end{pmatrix} \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \\ \omega^5 \end{pmatrix}, \tag{5.5}$$

with the structure group

$$g = G_F = \begin{pmatrix} a_1 & 0 & 0 & 0 & 0 \\ a_2 & a_3 & 0 & 0 & 0 \\ 0 & 0 & a_4 & 0 & 0 \\ 0 & 0 & a_5 & a_6 & 0 \\ 0 & 0 & 0 & 0 & a_7 \end{pmatrix}; \quad a_1a_3a_4a_6a_7 \neq 0, \tag{5.6}$$

thus structure group has dimension seven, where $g \in GL(5, \mathbb{R})$. Introducing the lifted coframe

$$\theta = g\omega = \begin{pmatrix} a_1 & 0 & 0 & 0 & 0 \\ a_2 & a_3 & 0 & 0 & 0 \\ 0 & 0 & a_4 & 0 & 0 \\ 0 & 0 & a_5 & a_6 & 0 \\ 0 & 0 & 0 & 0 & a_7 \end{pmatrix} \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \\ \omega^5 \end{pmatrix}. \tag{5.7}$$

5.1.3 Structure equations

For the structure equations let us compute $d\theta$

$$\begin{aligned} d\theta &= dg \wedge \omega + gd\omega, \\ &= dgg^{-1} \wedge g\omega + gd\omega, \\ &= \Pi \wedge \theta + T_{ij}\theta^i \wedge \theta^j. \end{aligned} \tag{5.8}$$

Where Π are the Maurer's Cartan forms given by

$$\Pi = dg \cdot g^{-1} = \begin{pmatrix} \frac{da_1}{a_1} & 0 & 0 & 0 & 0 \\ \frac{da_2}{a_1} - \frac{a_2}{a_1a_3}da_3 & \frac{da_3}{a_3} & 0 & 0 & 0 \\ 0 & 0 & \frac{da_4}{a_4} & 0 & 0 \\ 0 & 0 & \frac{da_5}{a_4} - \frac{a_5}{a_4a_6}da_6 & \frac{da_6}{a_6} & 0 \\ 0 & 0 & 0 & 0 & \frac{da_7}{a_7} \end{pmatrix}. \tag{5.9}$$

Now since

$$\begin{aligned}\omega^1 &= dy - p_1 dx,\\ d\omega^1 &= d(dy) - dp_1\wedge dx - 0,\\ d\omega^1 &= -dp_1\wedge dx = -\omega^2\wedge\omega^5,\\ d\omega^1 &= -\omega^2\wedge\omega^5.\end{aligned} \tag{5.10}$$

Similarly

$$\begin{aligned}d\omega^2 &= -a(\omega^2\wedge\omega^5) - b(\omega^1\wedge\omega^5),\\ d\omega^3 &= -\omega^4\wedge\omega^5,\\ d\omega^4 &= -c(\omega^4\wedge\omega^5) - d(\omega^3\wedge\omega^5),\\ d\omega^5 &= 0.\end{aligned} \tag{5.11}$$

Writing wedge product of $\omega's$ in terms of wedge product of $\theta's$

$$\omega^2\wedge\omega^5 = (\frac{1}{a_3}\theta^2 - \frac{a_2}{a_1a_3}\theta^1)\wedge(\frac{\theta^5}{a_7}),$$

or

$$\omega^2\wedge\omega^5 = \frac{1}{a_3a_7}\theta^2\wedge\theta^5 - \frac{a_2}{a_1a_3a_7}\theta^1\wedge\theta^5. \tag{5.12}$$

Similarly

$$\begin{aligned}\omega^1\wedge\omega^5 &= \frac{1}{a_1a_7}\theta^1\wedge\theta^5,\\ \omega^3\wedge\omega^5 &= \frac{1}{a_4a_7}\theta^3\wedge\theta^5,\\ \omega^4\wedge\omega^5 &= \frac{1}{a_6a_7}\theta^4\wedge\theta^5 - \frac{a_5}{a_4a_6a_7}\theta^3\wedge\theta^5.\end{aligned} \tag{5.13}$$

In order to get the structure equations we calculate the differential of the lifted coframe, since

$$\begin{aligned}\theta^1 &= a_1\omega^1,\\ d\theta^1 &= d(a_1\omega^1) = da_1\wedge\omega^1 + a_1 d\omega^1,\\ &= da_1\wedge(\frac{\theta^1}{a^1}) + a_1(-\omega^2\wedge\omega^5),\\ &= \frac{da_1}{a_1}\wedge\theta^1 + a_1(\frac{1}{a_3a_7}\theta^2\wedge\theta^5 - \frac{a_2}{a_1a_3a_7}\theta^1\wedge\theta^5),\\ d\theta^1 &= \frac{da_1}{a_1}\wedge\theta^1 - \frac{a_1}{a_3a_7}\theta^2\wedge\theta^5 + \frac{a_2}{a_3a_7}\theta^1\wedge\theta^5.\end{aligned} \tag{5.14}$$

Similarly

$$\begin{aligned}d\theta^2 &= (\frac{da_2}{a_1} - \frac{a_2}{a_1a_3}da_3)\wedge\theta^1 + \frac{da_3}{a_3}\wedge\theta^2 + (\frac{a.a_2}{a_1a_7} + \frac{a_2^2}{a_1a_3a_7} - \frac{b.a_3}{a_1a_7})\theta^1\wedge\theta^5 - (\frac{a_2}{a_3a_7} + \frac{a}{a_7})\theta^2\wedge\theta^5,\\ d\theta^3 &= \frac{da_4}{a_4}\wedge\theta^3 - \frac{a_4}{a_6a_7}\theta^4\wedge\theta^5 + \frac{a_5}{a_6a_7}\theta^3\wedge\theta^5,\\ d\theta^4 &= (\frac{da_5}{a_4} - \frac{a_5}{a_4a_6}da_6)\wedge\theta^3 + \frac{da_6}{a_6}\wedge\theta^4 + (\frac{c.a_5}{a_4a_7} + \frac{a_5^2}{a_4a_6a_7} - \frac{d.a_6}{a_4a_7})\theta^3\wedge\theta^5 - (\frac{a_5}{a_6a_7} + \frac{c}{a_7})\theta^4\wedge\theta^5,\\ d\theta^5 &= \frac{da_7}{a_7}\wedge\theta^5.\end{aligned} \tag{5.15}$$

5.1.4 Absorption, normalization, invariant reduction

Now we are ready to go for the first loop of the Cartan algorithm, the goal is to absorb as many torsion coefficients as we can, and normalize the remaining torsion coefficients called the essential torsion.

Loop 1: The structure equations are

$$\begin{aligned}
d\theta^1 &= \pi^1 \wedge \theta^1 + T^1_{25}\theta^2 \wedge \theta^5 + T^1_{15}\theta^1 \wedge \theta^5, \\
d\theta^2 &= \pi^2 \wedge \theta^1 + \pi^3 \wedge \theta^2 + T^2_{15}\theta^1 \wedge \theta^5 + T^2_{25}\theta^2 \wedge \theta^5, \\
d\theta^3 &= \pi^4 \wedge \theta^3 + T^3_{35}\theta^3 \wedge \theta^5 + T^3_{45}\theta^4 \wedge \theta^5, \\
d\theta^4 &= \pi^5 \wedge \theta^3 + \pi^6 \wedge \theta^4 + T^4_{35}\theta^3 \wedge \theta^5 + T^4_{45}\theta^4 \wedge \theta^5, \\
d\theta^5 &= \pi^7 \wedge \theta^5.
\end{aligned} \tag{5.16}$$

We now absorb as many torsion coefficients as possible in the following step.

$\mathbf{a_1}$) **Absorption** : From the structure equations we can easily determine the inessential torsion coefficients, they can be absorbed by using the freedom $\pi^i \to \pi^i + \lambda^i_j \theta^j$. Many torsion coefficients can be absorbed obtaining

$$\begin{aligned}
d\theta^1 &= \pi^1 \wedge \theta^1 + T^1_{25}\theta^2 \wedge \theta^5, \\
d\theta^2 &= \pi^2 \wedge \theta^1 + \pi^3 \wedge \theta^2, \\
d\theta^3 &= \pi^4 \wedge \theta^3 + T^3_{45}\theta^4 \wedge \theta^5, \\
d\theta^4 &= \pi^5 \wedge \theta^3 + \pi^6 \wedge \theta^4, \\
d\theta^5 &= \pi^7 \wedge \theta^5,
\end{aligned} \tag{5.17}$$

with

$$T^1_{25} = -\frac{a_1}{a_3 a_7}, \quad T^3_{45} = -\frac{a_4}{a_6 a_7}. \tag{5.18}$$

$\mathbf{b_1}$) **Normalization**: The essential torsion coefficients that we are left with, are then normalized by assigning to them an appropriate value, thus normalizing them to,

$$\begin{gathered}
T^1_{25} = -1 = T^3_{45}, \\
\frac{a_1}{a_3 a_7} = 1, \frac{a_4}{a_6 a_7} = 1.
\end{gathered}$$

$$a_3 = \frac{a_1}{a_7}, a_6 = \frac{a_4}{a_7}. \tag{5.19}$$

The structure group g becomes

$$g = \begin{pmatrix} a_1 & 0 & 0 & 0 & 0 \\ a_2 & \frac{a_1}{a_7} & 0 & 0 & 0 \\ 0 & 0 & a_4 & 0 & 0 \\ 0 & 0 & a_5 & \frac{a_4}{a_7} & 0 \\ 0 & 0 & 0 & 0 & a_7 \end{pmatrix}, \tag{5.20}$$

and the Maurer-Cartan forms become

$$\Pi = \begin{pmatrix} \pi^1 = \frac{da_1}{a_1} & 0 & 0 & 0 & 0 \\ \pi^2 = (\frac{da_2}{a_1} - \frac{d(\frac{a_1}{a_7}).a_2a_7}{a_1^2}) & \pi^3 = \frac{d(\frac{a_1}{a_7}).a_7}{a_1} & 0 & 0 & \\ 0 & 0 & \pi^4 = \frac{da_4}{a_4} & 0 & 0 \\ 0 & 0 & \pi^5 = (\frac{da_5}{a_4} - \frac{d(\frac{a_4}{a_7}).a_5a_7}{a_4^2}) & \pi^6 = \frac{d(\frac{a_4}{a_7}).a_7}{a_4} & 0 \\ 0 & 0 & 0 & 0 & \pi^7 = \frac{da_7}{a_7} \end{pmatrix}. \tag{5.21}$$

We now arrive at the second loop.

Loop 2: Applying the same procedure we arrive at the following structure equations

$$\begin{aligned} d\theta^1 &= \pi^1 \wedge \theta^1 - \theta^2 \wedge \theta^5, \\ d\theta^2 &= \pi^2 \wedge \theta^1 + (\pi^1 - \pi^7) \wedge \theta^2, \\ d\theta^3 &= \pi^4 \wedge \theta^3 - \theta^4 \wedge \theta^5, \\ d\theta^4 &= \pi^5 \wedge \theta^3 + (\pi^4 - \pi^7) \wedge \theta^4, \\ d\theta^5 &= \pi^7 \wedge \theta^5. \end{aligned} \tag{5.22}$$

In the above system we have no torsion to absorb/normalize. Thus we apply the Cartan test which will indicate whether the symmetry group associated to our equivalence problem is infinite dimensional or finite. If it is finite we prolong the dimension of underlying manifold.

5.1.5 Cartan test

Now we must figure out whether the symmetry group of the equivalence problem is infinite dimensional which is indicated by the system (5.22) being in involution, or finite dimensional, so that the system (5.22) is not in involution, we must prolong. Now to discuss above cases and proceed the problem we apply Cartan's test. For Cartan test first we will find the degree of indeterminacy by replacing each Maurer's Cartan form π^i by linear combination $z^i_1\theta^1 + z^i_2\theta^3 + z^i_3\theta^3 + z^i_4\theta^4 + z^i_5\theta^5$ of the lifted coframe elements (5.22). Equating the resulting coefficients of the basis of two forms $\theta^j \wedge \theta^k$ to zero.

$$\begin{aligned} d\theta^1 &= \pi^1 \wedge \theta^1 - \theta^2 \wedge \theta^5, \\ &= (z^1_1\theta^1 + z^1_2\theta^3 + z^1_3\theta^3 + z^1_4\theta^4 + z^1_5\theta^5) \wedge \theta^1 - \theta^2 \wedge \theta^5, \\ &= z^1_2\theta^3 \wedge \theta^1 + z^1_3\theta^3 \wedge \theta^1 + z^1_4\theta^4 \wedge \theta^1 + z^1_5\theta^5 \wedge \theta^1, \end{aligned} \tag{5.23}$$

which implies that

$$z^1_2 = z^1_3 = z^1_4 = z^1_5 = 0. \tag{5.24}$$

Similarly from

$$d\theta^2 = \pi^2 \wedge \theta^1 + (\pi^1 - \pi^7) \wedge \theta^2,$$

we get

$$z_1^1 = z_2^2, \quad z_3^2 = z_4^2 = z_5^2 = z_3^7 = z_4^7 = z_5^7 = 0,$$

and from

$$d\theta^3 = \pi^4 \wedge \theta^3 - \theta^4 \wedge \theta^5, \tag{5.25}$$

we get

$$z_1^4 = z_2^4 = z_4^4 = z_5^4 = 0. \tag{5.26}$$

Similarly from

$$d\theta^4 = \pi^5 \wedge \theta^3 + (\pi^4 - \pi^7) \wedge \theta^4, \tag{5.27}$$

we get

$$z_1^5 = z_2^5 = z_5^5 = z_1^7 = z_2^7 = 0, \ z_3^4 = z_4^5, \tag{5.28}$$

and from

$$d\theta^5 = \pi^7 \wedge \theta^5, \tag{5.29}$$

we get

$$z_1^7 = z_2^7 = z_3^7 = z_4^7 = 0. \tag{5.30}$$

Thus

$$\begin{aligned} z_1^1 = z_2^2, \quad z_3^4 = z_4^5, \quad z_2^1 = z_3^1 = z_4^1 = z_5^1 = z_3^2 = z_4^2 = z_5^2 = z_1^4 = z_2^4 \\ = z_4^4 = z_5^4 = z_1^5 = z_2^5 = z_5^5 = z_1^7 = z_2^7 = z_3^7 = z_4^7 = z_5^7 = 0. \end{aligned} \tag{5.31}$$

The four parameters $z_1^2, z_2^2, z_3^5, z_4^5$ can be chosen arbitrarily. These are the free variables for the absorption equations. Thus

$$r^{(1)} = 4. \tag{5.32}$$

To continue to describe Cartan's test for involutivity, we next introduce Cartan characters associated with the structure equations. For a given vector $\mathbf{v} = (v^1, v^2, v^3, v^4, v^5) \in \mathbb{R}^5$ define $L[\mathbf{v}]$ to be the 5×5 matrix

$$L[\mathbf{v}] = \begin{pmatrix} v^1 & 0 & 0 & 0 & 0 \\ v^2 & v^1 & 0 & 0 & -v^2 \\ 0 & 0 & v^3 & 0 & 0 \\ 0 & 0 & v^4 & v^3 & -v^4 \\ 0 & 0 & 0 & 0 & v^5 \end{pmatrix}. \tag{5.33}$$

The rows of L corresponds to the five structure equation and columns corresponds to the Maurer-Cartan forms $\pi^1, \pi^2, \pi^4, \pi^5, \pi^7$. The first reduced character is just the maximal rank of

$L[\mathbf{v}]$ for all possible vectors $v \in \mathbb{R}^5$. Clearly $s_1^{'} = 5$ and the second Cartan reduced character is defined by

$$s_1^{'} + s_2^{'} = \max \text{rank} \begin{pmatrix} v^1 & 0 & 0 & 0 & 0 \\ v^2 & v^1 & 0 & 0 & -v^2 \\ 0 & 0 & v^3 & 0 & 0 \\ 0 & 0 & v^4 & v^3 & -v^4 \\ 0 & 0 & 0 & 0 & v^5 \\ \hat{v}^1 & 0 & 0 & 0 & 0 \\ \hat{v}^2 & \hat{v}^1 & 0 & 0 & -\hat{v}^2 \\ 0 & 0 & \hat{v}^3 & 0 & 0 \\ 0 & 0 & \hat{v}^4 & \hat{v}^3 & -\hat{v}^4 \\ 0 & 0 & 0 & 0 & \hat{v}^5 \end{pmatrix}. \tag{5.34}$$

The maximum rank of this matrix is 5. Thus

$$s_2^{'} = 5 - s_1^{'} = 5 - 5 = 0 \tag{5.35}$$

which implies that

$$s_2^{'} = 0. \tag{5.36}$$

Similarly

$$s_3^{'} = 0 = s_4^{'} = s_5^{'}. \tag{5.37}$$

Now we apply Cartan test to check whether the coframe is involutive or not

$$\begin{aligned} r^{(1)} &\leq s_1^{'} + 2s_2^{'} + 3s_3^{'} + 4s_4^{'} + 5s_5^{'}, \\ 4 &< 5. \end{aligned} \tag{5.38}$$

Thus the system is not in involution, we must prolong the dimension of underlying manifold .

5.1.6 Prolongation

As we have seen that the associated symmetry group to our equivalence problem is not infinite dimensional and the structure group is not fully reduced. It happened because the dimension of the associated symmetry group was greater than the dimension of the underlying manifold upon which the equivalence problem was constructed. Therefore we must prolong the dimension of our manifold by adjoining additional 1-forms. Now since

$$L[\mathbf{v}] = A^i_{jk} v^j = \begin{pmatrix} z_1^1 & 0 & 0 & 0 & 0 \\ z_1^2 & z_2^2 & 0 & 0 & 0 \\ 0 & 0 & z_3^4 & 0 & 0 \\ 0 & 0 & z_3^5 & z_4^5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{5.39}$$

with

$$z_1^1 = z_2^2 = r, z_3^4 = z_4^5 = u, z_1^2 = s, z_3^5 = t. \tag{5.40}$$

Thus

$$L[\mathbf{v}] = \begin{pmatrix} r & 0 & 0 & 0 & 0 \\ s & r & 0 & 0 & 0 \\ 0 & 0 & u & 0 & 0 \\ 0 & 0 & t & u & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.41}$$

Here

$$\begin{aligned} &\pi^1 = \varpi^1 + r\theta^1, \quad \pi^2 = \varpi^2 + s\theta^1 + r\theta^2, \quad \pi^4 = \varpi^4 + u\theta^3, \\ &\pi^5 = \varpi^5 + t\theta^3 + u\theta^4, \quad \pi^7 = \varpi^7, \end{aligned} \tag{5.42}$$

are the Modified Maurer-Cartan forms and where

$$\begin{aligned} \varpi^1 &= \alpha^1 - \frac{a_2}{a_1}\theta^5, \\ \varpi^2 &= \alpha^2 + (\frac{b}{a_7^2} - \frac{aa_2}{a_1a_7} - \frac{a_2^2}{a_1^2})\theta^5, \\ \varpi^4 &= \alpha^4 - \frac{a_5}{a_4}\theta^5, \\ \varpi^5 &= \alpha^5 + (\frac{d}{a_7^2} - \frac{ca_5}{a_4a_7} - \frac{a_5^2}{a_4^2})\theta^5, \\ \varpi^7 &= \alpha^4 - [\frac{2a_2a_7 + a.a_1}{a_1a_7}]\theta^5, \end{aligned} \tag{5.43}$$

are the additional 1-forms. Now calculating the exterior derivative of $\varpi' s$, since

$$\varpi^1 = \alpha^1 - \frac{a_2}{a_1}\theta^5, \tag{5.44}$$

thus

$$\begin{aligned} d\varpi^1 &= d(\alpha^1) - d(\frac{a_2}{a_1}) \wedge \theta^5 - \frac{a_2}{a_1}d\theta^5 - \{\frac{da_2}{a_1} - \frac{a_2}{a_1^2}da_1\} \wedge \theta^5 - \frac{a_2}{a_1}(\pi^7 \wedge \theta^5), \\ &= (\frac{a_2}{a_1}\frac{da_1}{a_1}) \wedge \theta^5 - \frac{da_2}{a_1} \wedge \theta^5 - \frac{a_2}{a_1}(\pi^7 \wedge \theta^5), \\ &= \frac{a_2}{a_1}\alpha^1 \wedge \theta^5 - \{\alpha^2 + \frac{a_2}{a_1}(\alpha^1 - \alpha^7)\} \wedge \theta^5 - \frac{a_2}{a_1}(\pi^7 \wedge \theta^5), \\ &= \frac{a_2}{a_1}\alpha^1 \wedge \theta^5 - \alpha^2 \wedge \theta^5 - \frac{a_2}{a_1}\alpha^1 \wedge \theta^5 + \frac{a_2}{a_1}\alpha^7 \wedge \theta^5 - \frac{a_2}{a_1}(\pi^7 \wedge \theta^5), \\ &= -\varpi^2 \wedge \theta^5 = -(\pi^2 - s\theta^1 - r\theta^2) \wedge \theta^5, \end{aligned} \tag{5.45}$$

or

$$d\varpi^1 = -\pi^2 \wedge \theta^5 + s\theta^1 \wedge \theta^5 + r\theta^2 \wedge \theta^5. \tag{5.46}$$

Similarly we obtain

$$\begin{aligned} d\varpi^2 &= \pi^2 \wedge \theta^7 - s\theta^1 \wedge \pi^7 - r\theta^2 \wedge \pi^7, \\ d\varpi^4 &= t\theta^3 \wedge \theta^5 - \pi^5 \wedge \theta^5, \\ d\varpi^5 &= \pi^5 \wedge \pi^7 + t\pi^7 \wedge \theta^3 + C\pi^5 \wedge \theta^5 - Ct\theta^3 \wedge \theta^5 - F\pi^5 \wedge \theta^5 + Ft\theta^3 \wedge \theta^5, \\ d\varpi^7 &= -2\alpha^2 \wedge \theta^5. \end{aligned} \tag{5.47}$$

Now to find the structure equations we take exterior derivative of $\pi's$, now since

$$\pi^1 = \varpi^1 + r\theta^1, \tag{5.48}$$

therefore

$$\begin{aligned} d\pi^1 &= d\varpi^1 + dr \wedge \theta^1 + rd\theta^1, \\ &= -\pi^2\theta^5 + s\theta^1 \wedge \theta^5 + r\theta^2 \wedge \theta^5 + dr \wedge \theta^1 + r(\pi^1 \wedge \theta^1 - \theta^2 \wedge \theta^5), \\ &= dr \wedge \theta^1 + s\theta^1 \wedge \theta^5 + r\pi^1 \wedge \theta^1 - \pi^2 \wedge \theta^5. \end{aligned} \tag{5.49}$$

We now absorb all the inessential torsion coefficients in the above structure equation.

$\mathbf{a_2}$) **Absorption** : The inessential torsion coefficients can be seen, they can easily be absorbed by using following freedom of $\tilde{\rho}$

$$\tilde{\rho} = \rho + s\theta^5 - r\pi^1,$$

implies that

$$d\pi^1 = \rho \wedge \theta^1 - \pi^2 \wedge \theta^5. \tag{5.50}$$

Similarly

$$\begin{aligned} d\pi^2 &= \delta \wedge \theta^1 + \rho \wedge \theta^2 - 2s\theta^2 \wedge \theta^5 + \pi^2 \wedge \pi^7, \\ d\pi^4 &= \sigma \wedge \theta^3 - u\theta^4 \wedge \theta^5 - \pi^5 \wedge \theta^5, \\ d\pi^5 &= \Omega \wedge \theta^3 + \sigma \wedge \theta^4 + \pi^5 \wedge \pi^7 + C\pi^5 \wedge \theta^5 - F\pi^5 \wedge \theta^5 - 2t\theta^4 \wedge \theta^5 - u\pi^7 \wedge \theta^4, \\ d\pi^7 &= 2s\theta^1 \wedge \theta^5 + 2r\theta^2 \wedge \theta^5 - 2\pi^2 \wedge \theta^5, \end{aligned} \tag{5.51}$$

Thus the structure equations (after absorption) becomes

$$\begin{aligned} d\pi^1 &= \rho \wedge \theta^1 - \pi^2 \wedge \theta^5, \\ d\pi^2 &= \delta \wedge \theta^1 + \rho \wedge \theta^2 - 2s\theta^2 \wedge \theta^5 + \pi^2 \wedge \pi^7, \\ d\pi^4 &= \sigma \wedge \theta^3 - u\theta^4 \wedge \theta^5 - \pi^5 \wedge \theta^5, \\ d\pi^5 &= \Omega \wedge \theta^3 + \sigma \wedge \theta^4 + \pi^5 \wedge \pi^7 + C\pi^5 \wedge \theta^5 - F\pi^5 \wedge \theta^5 - 2t\theta^4 \wedge \theta^5 - u\pi^7 \wedge \theta^4, \\ d\pi^7 &= 2s\theta^1 \wedge \theta^5 + 2r\theta^2 \wedge \theta^5 - 2\pi^2 \wedge \theta^5, \end{aligned} \tag{5.52}$$

or

$$\begin{aligned} d\pi^1 &= \rho \wedge \theta^1 - \pi^2 \wedge \theta^5, \\ d\pi^2 &= \delta \wedge \theta^1 + \rho \wedge \theta^2 - A\theta^2 \wedge \theta^5 + \pi^2 \wedge \pi^7, \\ d\pi^4 &= \sigma \wedge \theta^3 - K\theta^4 \wedge \theta^5 - \pi^5 \wedge \theta^5, \\ d\pi^5 &= \Omega \wedge \theta^3 + \sigma \wedge \theta^4 + \pi^5 \wedge \pi^7 + C\pi^5 \wedge \theta^5 - F\pi^5 \wedge \theta^5 - L\theta^4 \wedge \theta^5 - K\pi^7 \wedge \theta^4, \\ d\pi^7 &= A\theta^1 \wedge \theta^5 + B\theta^2 \wedge \theta^5 - 2\pi^2 \wedge \theta^5, \end{aligned} \tag{5.53}$$

where $A = 2s, \quad B = 2r, \quad K = u, \quad L = 2t, \quad C = \frac{2a_5a_7 + a.a_1}{a_1a_7}, \quad F = \frac{2a_5a_7 + c.a_4}{a_4a_7}$.

Subsequently the essential torsion coefficients can be normalized in the next step.

$\mathbf{b_2}$) **Normalization**: The remaining torsion coefficients are to be normalized by assigning them some appropriate value, thus normalizing them to $A = B = K = L = 1$ implies that $s = \frac{1}{2}, \quad r = \frac{1}{2}, \quad u = 1, \quad t = \frac{1}{2}$ and $ds = dr = du = dt = 0$ so that $\sigma = \rho = \delta = \Omega = 0$. Therefore we immediately reduce the prolonged equivalence problem to an equivalence problem for the coframe. Thus the final structure equations are

$$\begin{aligned}
d\theta^1 &= \pi^1 \wedge \theta^1 - \theta^2 \wedge \theta^5,\\
d\theta^2 &= \pi^1 \wedge \theta^2 - \pi^7 \wedge \theta^2,\\
d\theta^3 &= \pi^4 \wedge \theta^3 - \theta^4 \wedge \theta^5,\\
d\theta^4 &= \pi^4 \wedge \theta^4 - \pi^7 \wedge \theta^4,\\
d\theta^5 &= \pi^7 \wedge \theta^5,\\
d\pi^1 &= -\pi^2 \wedge \theta^5,\\
d\pi^2 &= -\theta^2 \wedge \theta^5 + \pi^2 \wedge \pi^7,\\
d\pi^4 &= -\theta^4 \wedge \theta^5 - \pi^5 \wedge \theta^5,\\
d\pi^5 &= \pi^5 \wedge \pi^7 - \theta^4 \wedge \theta^5 - \pi^7 \wedge \theta^4,\\
d\pi^7 &= \theta^1 \wedge \theta^5 + \theta^2 \wedge \theta^5 - 2\pi^2 \wedge \theta^5.
\end{aligned} \tag{5.54}$$

The zeroth order classifying manifold is a single point defined by $T^i_{jk} = -C^i_{jk}$. All derived invariants are automatically zero and hence every classifying manifold $\mathrm{C}^{(s)}$ is a single point. Therefore $\rho(s) = 0, \quad s \geq 0$, therefore the coframe has rank zero. In more abstract notation

$$\begin{aligned}
d\theta^1 &= -\theta^1 \wedge \theta^6 - \theta^2 \wedge \theta^5,\\
d\theta^2 &= -\theta^2 \wedge \theta^6 + \theta^2 \wedge \theta^{10},\\
d\theta^3 &= -\theta^3 \wedge \theta^8 - \theta^4 \wedge \theta^5,\\
d\theta^4 &= -\theta^4 \wedge \theta^8 + \theta^4 \wedge \theta^{10},\\
d\theta^5 &= -\theta^5 \wedge \theta^{10},\\
d\theta^6 &= \theta^5 \wedge \theta^7,\\
d\theta^7 &= -\theta^2 \wedge \theta^5 + \theta^7 \wedge \theta^{10},\\
d\theta^8 &= -\theta^4 \wedge \theta^5 + \theta^5 \wedge \theta^9,\\
d\theta^9 &= -\theta^4 \wedge \theta^5 + \theta^4 \wedge \theta^{10} + \theta^9 \wedge \theta^{10},\\
d\theta^{10} &= \theta^1 \wedge \theta^5 - \theta^2 \wedge \theta^5 + 2\theta^5 \wedge \theta^7,
\end{aligned} \tag{5.55}$$

where $(\pi^1 = \theta^6, \quad \pi^2 = \theta^7, \quad \pi^4 = \theta^8, \quad \pi^5 = \theta^9, \quad \pi^7 = \theta^{10})$

$$\begin{aligned}
T^1_{16} &= -1, \quad T^1_{25} = -1, \quad T^2_{26} = -1, \quad T^2_{210} = 1, \quad T^3_{38} = -1, \quad T^3_{45} = -1,\\
T^4_{48} &= -1, \quad T^4_{410} = 1, \quad T^5_{510} = -1, \quad T^6_{57} = 1, \quad T^7_{25} = -1, \quad T^7_{710} = 1\\
T^8_{45} &= -1, \quad T^8_{59} = 1, \quad T^9_{45} = -1, \quad T^9_{410} = 1, \quad T^9_{410} = 1, \quad T^9_{910} = 1,\\
T^{10}_{15} &= 1, \quad T^{10}_{25} = -1, \quad T^{10}_{57} = -2,
\end{aligned} \tag{5.56}$$

Now since

$$[\frac{\partial}{\partial\theta^j}, \frac{\partial}{\partial\theta^k}] = -\sum_{i=1}^{10} T^i_{jk} \frac{\partial}{\partial\theta^i}, \tag{5.57}$$

for $i = 1, j = 2$, we get

$$[\frac{\partial}{\partial\theta^1}, \frac{\partial}{\partial\theta^2}] = -T^1_{12}\frac{\partial}{\partial\theta^1} - T^2_{12}\frac{\partial}{\partial\theta^2} - T^3_{12}\frac{\partial}{\partial\theta^3} - T^4_{12}\frac{\partial}{\partial\theta^4} - T^5_{12}\frac{\partial}{\partial\theta^5} - T^6_{12}\frac{\partial}{\partial\theta^6}, -T^7_{12}\frac{\partial}{\partial\theta^7}, -T^8_{12}\frac{\partial}{\partial\theta^8},$$
$$-T^9_{12}\frac{\partial}{\partial\theta^9}, -T^{10}_{12}\frac{\partial}{\partial\theta^{10}}, \tag{5.58}$$

which implies that

$$[V_1, V_2] = 0. \tag{5.59}$$

Similarly all other commutators relations are zero, and the only non-zero commutators relations are,

$$\begin{aligned}
&[V_1, V_5] = -V_{10}, \quad [V_1, V_6] = V_1, \quad [V_2, V_5] = V_1 + V_7 + V_{10}, \quad [V_2, V_6] = V_2, \\
&[V_2, V_{10}] = -V_2, \quad [V_3, V_8] = -V_3, \quad [V_4, V_5] = V_3 + V_8 + V_9, \quad [V_4, V_8] = V_4, \\
&[V_4, V_{10}] = -V_4 - V_9, \quad [V_5, V_7] = -V_6 - 2V_{10}, \quad [V_5, V_9] = -V_8, \quad [V_5, V_{10}] = V_5, \\
&[V_7, V_{10}] = -V_7, \quad [V_9, V_{10}] = -V_9.
\end{aligned} \tag{5.60}$$

5.2 Equivalence of coupled systems of second order linear ODEs

Consider a coupled system of second order linear ODE,

$$\begin{aligned}
y'' &= az' + bz, \\
z'' &= cy' + dy,
\end{aligned} \tag{5.61}$$

where a, b, c and d are arbitrary constants. The above system is a special case of a coupled linear system. Both equations in the above system contains velocity and displacement terms.

5.2.1 Coframe

In this section we explain how to construct a coframe for the Cartan algorithm to which the equivalence approach developed in the second chapter can be applied. The preliminary step is to shift the equivalence problem over a manifold ($M = J^2(x, y, z, p_1, q_1)$) and associate differential 1-forms (coframe) with it. Thus the associated 1-forms are

$$\begin{aligned}
&\omega^1 = dy - p_1 dx, \quad \omega^2 = dp_1 - q_1 dx, \\
&\omega^3 = dz - p_2 dx, \quad \omega^4 = dp_2 - q_2 dx, \\
&\omega^5 = dx.
\end{aligned} \tag{5.62}$$

such that their wedge products do not vanish.

5.2.2 Equivalence conditions

Under a (subcase) of fiber-preserving transformation in which

$$(x, y, z) = (\bar{x} \to (\Phi(x), \bar{y} = \Psi(x, z), \bar{z} = \eta(x, y)). \tag{5.63}$$

Thus the coframe on $\bar{M}$, using above transformation

$$\begin{aligned} &\overline{\omega}^1 = a_1\omega^3, \quad \overline{\omega}^2 = a_2\omega^3 + a_3\omega^4, \quad \overline{\omega}^3 = a_4\omega^1, \\ &\overline{\omega}^4 = a_5\omega^1 + a_6\omega^2, \quad \overline{\omega}^5 = a_7\omega^5, \end{aligned} \tag{5.64}$$

or

$$\begin{pmatrix} \overline{\omega}^1 \\ \overline{\omega}^2 \\ \overline{\omega}^3 \\ \overline{\omega}^4 \\ \overline{\omega}^5 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a_1 & 0 & 0 \\ 0 & 0 & a_2 & a_3 & 0 \\ a_4 & 0 & 0 & 0 & 0 \\ a_5 & a_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_7 \end{pmatrix} \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \\ \omega^5 \end{pmatrix}, \tag{5.65}$$

with the structure group

$$g = G_F = \begin{pmatrix} 0 & 0 & a_1 & 0 & 0 \\ 0 & 0 & a_2 & a_3 & 0 \\ a_4 & 0 & 0 & 0 & 0 \\ a_5 & a_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_7 \end{pmatrix}; \quad a_1a_3a_4a_6a_7 \neq 0, \tag{5.66}$$

the structure group in this case has dimension seven, where $g \in GL(5, \mathbb{R})$. The lifted coframe are given by

$$\theta = g \cdot \omega = \begin{pmatrix} 0 & 0 & a_1 & 0 & 0 \\ 0 & 0 & a_2 & a_3 & 0 \\ a_4 & 0 & 0 & 0 & 0 \\ a_5 & a_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_7 \end{pmatrix} \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \\ \omega^5 \end{pmatrix}. \tag{5.67}$$

5.2.3 Structure equations

To find the structure equation we take exterior derivative of θ, therefore

$$\begin{aligned} d\theta &= dg \wedge \omega + gd\omega = dgg^{-1} \wedge g\omega + gd\omega, \\ &= \Pi \wedge \theta + T^i_{jk}\theta^j \wedge \theta^k. \end{aligned} \tag{5.68}$$

Where Π are the Maurer's Cartan forms, and is given by

$$\Pi = dg \cdot g^{-1} = \begin{pmatrix} \frac{da_1}{a_1} & 0 & 0 & 0 & 0 \\ \frac{da_2}{a_1} - \frac{a_2}{a_1a_3}da_3 & \frac{da_3}{a_3} & 0 & 0 & 0 \\ 0 & 0 & \frac{da_4}{a_4} & 0 & 0 \\ 0 & 0 & \frac{da_5}{a_4} - \frac{a_5}{a_4a_6}da_6 & \frac{da_6}{a_6} & 0 \\ 0 & 0 & 0 & 0 & \frac{da_7}{a_7} \end{pmatrix}. \tag{5.69}$$

Now let us calculate $d\omega$'s

$$\begin{aligned}
\omega^1 &= dy_1 - p_1 dx, \\
d\omega^1 &= d(dy_1) - dp_1 \wedge dx - 0, \\
d\omega^1 &= -dp_1 \wedge dx = -\omega^2 \wedge \omega^5, \\
d\omega^1 &= -\omega^2 \wedge \omega^5. \qquad (5.70)
\end{aligned}$$

Since

$$\omega^2 \wedge \omega^5 = -\frac{a_5}{a_4 a_6}\theta^3 \wedge \theta^5 + \frac{1}{a_6 a_7}\theta^4 \wedge \theta^5. \qquad (5.71)$$

Therefore

$$d\omega^1 = \frac{a_5}{a_4 a_6}\theta^3 \wedge \theta^5 - \frac{1}{a_6 a_7}\theta^4 \wedge \theta^5. \qquad (5.72)$$

Similarly

$$\begin{aligned}
d\omega^2 &= [\frac{a.a_2}{a_1 a_3 a_7} - \frac{b}{a_1 a_7}]\theta^1 \wedge \theta^5 - \frac{a}{a_3 a_7}\theta^2 \wedge \theta^5, \\
d\omega^3 &= \frac{a_2}{a_1 a_3 a_7}\theta^1 \wedge \theta^5 - \frac{1}{a_3 a_7}\theta^2 \wedge \theta^5, \\
d\omega^4 &= [\frac{c.a_5}{a_4 a_6} - \frac{d}{a_4 a_7}]\theta^3 \wedge \theta^5 - \frac{c}{a_6 a_7}\theta^4 \wedge \theta^5, \\
d\omega^5 &= 0. \qquad (5.73)
\end{aligned}$$

Now since

$$\theta^1 = a_1 \omega^3.$$

Calculating the exterior derivative of θ^1, we get

$$\begin{aligned}
d\theta^1 &= d(a_1\omega^3) = da_1 \wedge \omega^3 + a_1 d\omega^3, \\
&= \frac{da_1}{a_1} \wedge \theta^1 + a_1(\frac{a_2}{a_1 a_3 a_7}\theta^1 \wedge \theta^5 - \frac{1}{a_3 a_7}\theta^2 \wedge \theta^5), \\
d\theta^1 &= \frac{da_1}{a_1} \wedge \theta^1 + \frac{a_2}{a_3 a_7}\theta^1 \wedge \theta^5 - \frac{a_1}{a_3 a_7}\theta^2 \wedge \theta^5. \qquad (5.74)
\end{aligned}$$

Similarly

$$\begin{aligned}
d\theta^2 &= (\frac{da_2}{a_1} - \frac{a_2}{a_1 a_3}da_3) \wedge \theta^1 + \frac{da_3}{a_3} \wedge \theta^2 + \frac{a_2^2}{a_1 a_3 a_7}\theta^1 \wedge \theta^5 \\
&\quad - \frac{a_2}{a_3 a_7}\theta^2 \wedge \theta^5 - (\frac{c.a_3 a_5}{a_4 a_6} - \frac{a_3.d}{a_4 a_7})\theta^3 \wedge \theta^5 - \frac{c.a_3}{a_6 a_7})\theta^4 \wedge \theta^5, \\
d\theta^3 &= \frac{da_4}{a_4} \wedge \theta^3 + \frac{a_5}{a_6}\theta^3 \wedge \theta^5 - \frac{a_4}{a_6 a_7}\theta^4 \wedge \theta^5, \\
d\theta^4 &= (\frac{da_5}{a_4} - \frac{a_5}{a_4 a_6}da_6) \wedge \theta^3 + \frac{da_6}{a_6} \wedge \theta^4 + (\frac{a.a_2 a_6}{a_1 a_3 a_7} - \frac{b.a_6}{a_1 a_7})\theta^1 \wedge \theta^5 \\
&\quad - \frac{a.a_6}{a_3 a_7}\theta^2 \wedge \theta^5 + \frac{a_5^2}{a_4 a_6}\theta^3 \wedge \theta^5 - \frac{a_5}{a_6 a_7}\theta^4 \wedge \theta^5, \\
d\theta^5 &= \frac{da_7}{a_7} \wedge \theta^5. \qquad (5.75)
\end{aligned}$$

5.2.4 Absorption, normalization, invariant reduction

Now we are ready to go for the first loop of the Cartan algorithm, the goal is to absorb as many torsion coefficients as we can, and normalize the remaining essential torsion.

Loop 1: Since the structure equations are

$$\begin{aligned} d\theta^1 &= \alpha^1 \wedge \theta^1 + T^1_{15}\theta^1 \wedge \theta^5 + T^1_{25}\theta^2 \wedge \theta^5, \\ d\theta^2 &= \alpha^2 \wedge \theta^1 + \alpha^3 \wedge \theta^2 + T^2_{15}\theta^1 \wedge \theta^5 + T^2_{25}\theta^2 \wedge \theta^5 + T^2_{35}\theta^3 \wedge \theta^5 + T^2_{45}\theta^4 \wedge \theta^5, \\ d\theta^3 &= \alpha^4 \wedge \theta^3 + T^3_{35}\theta^3 \wedge \theta^5 + T^3_{45}\theta^4 \wedge \theta^5, \\ d\theta^4 &= \alpha^5 \wedge \theta^3 + \alpha^6 \wedge \theta^4 + T^4_{15}\theta^1 \wedge \theta^5 + T^4_{25}\theta^2 \wedge \theta^5 + T^4_{35}\theta^3 \wedge \theta^5 + T^4_{45}\theta^4 \wedge \theta^5, \\ d\theta^5 &= \alpha^7 \wedge \theta^5. \end{aligned} \tag{5.76}$$

a₁) Absorption : We will absorb as many inessional torsion coefficients as we can, by using the following freedom

$$\begin{aligned} \alpha^1 &= \pi^1 + T^1_{15}\theta^5, \\ \alpha^2 &= \pi^2 + T^2_{15}\theta^5, \\ \alpha^3 &= \pi^3 + T^2_{25}\theta^5, \\ \alpha^4 &= \pi^4 + T^3_{35}\theta^5, \\ \alpha^5 &= \pi^5 + T^4_{35}\theta^5, \\ \alpha^6 &= \pi^6 + T^4_{45}\theta^5. \end{aligned} \tag{5.77}$$

These torsion coefficients can be absorbed and thus the structure equations reduced to

$$\begin{aligned} d\theta^1 &= \pi^1 \wedge \theta^1 + T^1_{25}\theta^2 \wedge \theta^5, \\ d\theta^2 &= \pi^2 \wedge \theta^1 + \pi^3 \wedge \theta^2 + T^2_{35}\theta^3 \wedge \theta^5 + +T^2_{45}\theta^4 \wedge \theta^5, \\ d\theta^3 &= \pi^4 \wedge \theta^3 + T^3_{45}\theta^4 \wedge \theta^5, \\ d\theta^4 &= \pi^5 \wedge \theta^3 + \pi^6 \wedge \theta^4 + T^4_{15}\theta^1 \wedge \theta^5 + T^4_{25}\theta^2 \wedge \theta^5, \\ d\theta^5 &= \pi^7 \wedge \theta^5, \end{aligned} \tag{5.78}$$

with

$$\begin{aligned} T^1_{25} = -\frac{a_1}{a_3 a_7}, \quad T^2_{35} = \frac{c.a_3 a_5}{a_4 a_6} - \frac{d.a_3}{a_4 a_7}, \quad T^2_{45} = -\frac{c.a_3}{a_6 a_7}, \quad T^3_{45} = -\frac{a_4}{a_6 a_7}, \\ T^4_{15} = \frac{a.a_2 a_6}{a_1 a_3 a_7} - \frac{b a_6}{a_1 a_7}, \quad T^4_{25} = -\frac{a.a_6}{a_3 a_7}. \end{aligned} \tag{5.79}$$

b₁) Normalization: The unabsorbed torsion coefficients called the essential torsion coefficients are get to normalize, thus assigning them to

$$\begin{aligned} T^1_{25} = -1 \Rightarrow \frac{a_1}{a_7} = a_3, T^2_{45} = -1 \Rightarrow \frac{c.a_1}{a_7^2} = a_6, \\ T^3_{45} = -1 \Rightarrow \frac{c.a_1}{a_7} = a_4, T^2_{35} = 0 \Rightarrow \frac{d.a_1}{a_7^3} = a_5, \\ T^4_{15} = 0 \Rightarrow \frac{b.a_1}{a a_7} = a_2, \quad T^4_{25} = -1 \Rightarrow a_7 = \pm\sqrt{ac}. \end{aligned}$$

Therefore

$$a_2 = \frac{b}{a\sqrt{ac}}(a_1), \quad a_3 = \frac{1}{\sqrt{ac}}(a_1), \quad a_4 = \frac{c}{a\sqrt{ac}}(a_1),$$
$$a_5 = \frac{d}{ac\sqrt{ac}}(a_1), \quad a_6 = \frac{1}{a}(a_1), \quad a_7 = \sqrt{ac}. \tag{5.80}$$

Thus g becomes

$$g = \begin{pmatrix} 0 & 0 & a_1 & 0 & 0 \\ 0 & 0 & \frac{ba_1}{a\sqrt{ac}} & \frac{a_1}{\sqrt{ac}} & 0 \\ \frac{c.a_1}{\sqrt{ac}} & 0 & 0 & 0 & 0 \\ \frac{d.a_1}{ac\sqrt{ac}} & \frac{a_1}{a} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{ac} \end{pmatrix}, \tag{5.81}$$

and the Maurer-Cartan forms become

$$\Pi = \begin{pmatrix} \pi^1 = \frac{da_1}{a_1} & 0 & 0 & 0 & 0 \\ 0 & \pi^1 & 0 & 0 & \\ 0 & 0 & \pi^1 & 0 & 0 \\ 0 & 0 & 0 & \pi^1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.82}$$

Loop 2: Now we are in a position to apply the second loop, therefore

$$\begin{aligned} d\theta^1 &= \pi^1 \wedge \theta^1 - \theta^2 \wedge \theta^5, \\ d\theta^2 &= \pi^1 \wedge \theta^2 - \theta^4 \wedge \theta^5, \\ d\theta^3 &= \pi^1 \wedge \theta^3 - \theta^4 \wedge \theta^5, \\ d\theta^4 &= \pi^1 \wedge \theta^4 - \theta^2 \wedge \theta^4, \\ d\theta^5 &= 0. \end{aligned} \tag{5.83}$$

Since there are no essential torsion coefficients to normalize and thus the structure group is left with a single parameter. Now we must figure out whether the symmetry group of the equivalence problem is infinite dimensional or finite dimensional, which will be decided by following Cartan test.

5.2.5 Cartan test

In order to apply the Cartan test we find the degree of indeterminacy by replacing each Maurer's Cartan form π^i by linear combination $z^i_1\theta^1 + z^i_2\theta^2 + z^i_3\theta^3 + z^i_4\theta^4 + z^i_5\theta^5$ of the lifted co-frame elements (5.83). Equating the resulting coefficients of the basis of two forms $\theta^j \wedge \theta^k$ to zero. Now since

$$\begin{aligned} d\theta^1 &= \pi^1 \wedge \theta^1 - \theta^2 \wedge \theta^5, \\ &= (z^1_1\theta^1 + z^1_2\theta^2 + z^1_3\theta^3 + z^1_4\theta^4 + z^1_5\theta^5) \wedge \theta^1 - \theta^2 \wedge \theta^5, \\ &= z^1_2\theta^3 \wedge \theta^1 + z^1_3\theta^3 \wedge \theta^1 + z^1_4\theta^4 \wedge \theta^1 + z^1_5\theta^5 \wedge \theta^1, \end{aligned} \tag{5.84}$$

which implies that

$$z_2^1 = z_3^1 = z_4^1 = z_5^1 = 0. \tag{5.85}$$

Similarly

$$z_1^1 = z_3^1 = z_4^1 = z_5^1 = 0. \tag{5.86}$$

Since there is no free parameter in the absorption equation, thus

$$r^{(1)} = 0. \tag{5.87}$$

To apply Cartan's test of involutivity, we next introduce the Cartan characters associated with the structure equations. For a given vector $v = (v^1, v^2, v^3, v^4, v^5) \in \mathbb{R}^5$ define $L[\mathbf{v}]$ to be 5×1 matrix

$$L[\mathbf{v}] = \begin{bmatrix} v^1 & v^2 & v^3 & v^4 & 0 \end{bmatrix}^T. \tag{5.88}$$

The row of L corresponds to the five structure equation and column corresponds to the Maurer's Cartan form π^1. The first reduced character is just the maximal rank of $L[\mathbf{v}]$ for all possible vectors $v \in \mathbb{R}^5$. Clearly $s_1^{'} = 1$, the 2nd Cartan reduced character is defined by

$$s_1^{'} + s_2^{'} = \text{max rank} \begin{bmatrix} v^1 & v^2 & v^3 & v^4 & 0 & \hat{v}^1 & \hat{v}^2 & \hat{v}^3 & \hat{v}^4 & 0 \end{bmatrix}^T, \tag{5.89}$$

thus

$$s_1^{'} + s_2^{'} = 1, \quad s_2^{'} = 0, \tag{5.90}$$

Similarly

$$s_3^{'} = s_4^{'} = s_5^{'} = 0. \tag{5.91}$$

Now we apply the Cartan test to check whether the coframe is involutive or not

$$\begin{aligned} r^{(1)} &\leq s_1^{'} + 2s_2^{'} + 3s_3^{'} + 4s_4^{'} + 5s_5^{'}, \\ 0 &< 1 + 0 + 0 + 0 + 0, \\ 0 &< 1. \end{aligned} \tag{5.92}$$

Thus the system is not in involution, we must prolong.

5.2.6 Prolongation

Now proceeding to the next step by adding additional one-forms, since

$$L[\mathbf{v}] = A^i_{jk} v^j = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T. \tag{5.93}$$

Now

$$\pi^1 = \varpi^1, \tag{5.94}$$

and

$$\varpi^1 = \alpha^1 - \frac{b}{a\sqrt{ac}}\theta^5,$$
$$d\varpi^1 = d\alpha^1 - d(\frac{b}{a\sqrt{ac}}) \wedge \theta^5 - \frac{b}{a\sqrt{ac}} d\theta^5,$$
$$d\varpi^1 = 0 + 0 - \frac{b}{a\sqrt{ac}}(0),$$
$$d\varpi^1 = 0. \tag{5.95}$$

Thus the final structure equations become

$$\begin{aligned} d\theta^1 &= \pi^1 \wedge \theta^1 - \theta^2 \wedge \theta^5, \\ d\theta^2 &= \pi^1 \wedge \theta^2 - \theta^4 \wedge \theta^5, \\ d\theta^3 &= \pi^1 \wedge \theta^3 - \theta^4 \wedge \theta^5, \\ d\theta^4 &= \pi^1 \wedge \theta^4 - \theta^2 \wedge \theta^5, \\ d\theta^5 &= 0, \\ d\pi^1 &= 0. \end{aligned} \tag{5.96}$$

Consider $\theta^6 = \pi^1$, therefore the invariant coframe is obtained and the structure equations become

$$\begin{aligned} d\theta^1 &= -\theta^1 \wedge \theta^6 - \theta^2 \wedge \theta^5, \\ d\theta^2 &= -\theta^2 \wedge \theta^6 - \theta^4 \wedge \theta^5, \\ d\theta^3 &= -\theta^3 \wedge \theta^6 - \theta^4 \wedge \theta^5, \\ d\theta^4 &= -\theta^4 \wedge \theta^6 - \theta^2 \wedge \theta^5, \\ d\theta^5 &= 0, \\ d\theta^6 &= 0, \end{aligned} \tag{5.97}$$

where

$$\begin{aligned} &T^1_{16} = -1, \quad T^1_{25} = -1, \quad T^2_{26} = -1, \quad T^2_{45} = -1, \\ &T^3_{36} = -1, \quad T^3_{45} = -1, \quad T^4_{46} = -1, \quad T^4_{25} = -1. \end{aligned} \tag{5.98}$$

Now since we know that

$$[\frac{\partial}{\partial\theta^j}, \frac{\partial}{\partial\theta^k}] = -\sum_{i=1}^{6} T^i_{jk} \frac{\partial}{\partial\theta^i}, \tag{5.99}$$

thus we have

$$[\frac{\partial}{\partial\theta^1}, \frac{\partial}{\partial\theta^2}] = -T^1_{12}\frac{\partial}{\partial\theta^1} - T^2_{12}\frac{\partial}{\partial\theta^2} - T^3_{12}\frac{\partial}{\partial\theta^3} - T^4_{12}\frac{\partial}{\partial\theta^4} - T^5_{12}\frac{\partial}{\partial\theta^5} - T^6_{12}\frac{\partial}{\partial\theta^6},$$

which implies that

$$[\frac{\partial}{\partial\theta^1}, \frac{\partial}{\partial\theta^2}] = 0.$$

Therefore

$$[V_1, V_2] = 0, \tag{5.100}$$

where all the other commutators are zero except

$$\begin{aligned} &[V_1, V_6] = V_1, \quad [V_2, V_5] = V_1 + V_4, \quad [V_2, V_6] = V_2, \\ &[V_3, V_6] = V_3, \quad [V_4, V_5] = V_2 + V_3, \quad [V_4, V_6] = V_4. \end{aligned} \tag{5.101}$$

Note that the symmetry algebra of fiber-preserving transformation is six dimensional for a coupled system of ODEs (5.61). However the algebra in the case of an un-coupled system of ODEs (5.1) was ten dimensional. Since both systems (5.1) and (5.61) are linear therefore Cartan approach successfully confirms that there is more than one class of system of linear ODEs.

Chapter 6

Conclusion

Cartan's method of equivalence is a powerful technique to unveil hidden invariant properties of two geometric objects. It involves algorithmic steps of absorption, normalization and prolongation. The main aim of the thesis was to study the equivalence of scalar Lagrangians, scalar linear ODEs and their systems. The equivalence problem for the two scalar Lagrangians was investigated and it was verified that the maximum dimensions of the symmetry group associated to the equivalence problem is three. Besides, it was shown that a scalar second order linear ODE with constant coefficients is equivalent to a free particle equation because the dimension of symmetry algebra of fibre-preserving transformations for the former is six, i.e., equivalent to the dimension of the symmetry algebra for free particle equation. These results were already known in the literature which we extended to systems of two linear ODEs. In particular it was shown that the equivalence of systems of two linear ODEs give us the identification of two different symmetry algebras of fibre-preserving transformations. It turns out that we obtain a six-dimensional algebra if the system is coupled and ten-dimensional algebra if the system is uncoupled. Interestingly the coupling between the two ODEs in the system involves both velocity as well as displacement terms. Therefore, we have reconfirmed the claim that there is more than one class of systems of linear ODEs using Cartan approach which was known from another approach. The Cartan approach can be extended to identify all classes of systems of linear ODEs under point transformations which is the main subject of study in a forthcoming paper [31].

Chapter 6

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