Some Formulas for the Rayleigh Wave Speed

By

Attiya Imtiaz

Supervised by

Dr. Moniba Shams

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Examination Committee Members

1. Name: Prof. Faiz Ahmad Signature: Faiz Ahmad

2. Name: Dr. Tooba Feroze Signature:

3. Name: Signature:

External Examiner: Dr. Adnan Jahangir Signature:

Supervisor's Name: Dr. Moniba Shams Signature:

Head of Department

Date: 01-03-18

COUNTERSINGED

Date: 01/03/18

Dean/Principal

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Signature: ______________________

Name of Supervisor Dr. Moniba Shams

Date: 01-03-2018

Signature (HoD): ______________________

Date: 01-03-18

Signature (Dean/Principal): ______________________

Date: 01/03/18

Dedicated

to

My Beloved Father and Mother

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Attiya Imtiaz

Contents

Abstract

In this thesis, a review is presented on the formula for Rayleigh wave speed for compressible isotropic linear elastic solids by Vinh and Ogden [6]. The thesis is primarily based on the discussion carried out in the above cited paper. Vinh and Ogden [6] utilized the theory of cubic equations and Cardan's formulas to present an explicit formula for the speed of Rayleigh waves. A comparison of this formula is also presented with the one given by Malischewsky [5] for the same material. The two formulas are equivalent, however, the one presented by Vinh and Ogden [6] is better explained and elaborated. The two formulas are shown to be equivalent by graphing the results using MATHEMATICA.

Chapter 1

Introduction

Waves are disturbance in a medium which carry information from one place to the other. Being carriers of information, waves find primary importance in our daily life. Sound, heat and light are all types of waves and no one can deny how significant these waves are for all of us. All the wireless communication uses electromagnetic waves for transmitting and receiving information. Electromagnetic waves are not only fastest but also reliable source of carrying information. Heat comes from the sun to earth in the form of electromagnetic radiations, thus enabling life to sustain on earth. Radars and ultrasonics use sound waves for communication. Even we can listen to a person when the sound waves from his/her mouth reaches our ears. Waves are indeed very important for nature to sustain its periodicity.

For compressible isotropic elastic solids, Rayleigh waves were first discovered by Lord Rayleigh [1]. These waves have been widely studied and find their applications in a vast range of fields such as seismology, material science, nondestructive testing, geophysics and so on [2]. Rayleigh waves are very important to be studied from seismological point of view because the most devastation in earthquakes is carried out due to Rayleigh waves [3]. The propagation of Rayleigh waves is specified by secular equation in an elastic half space. The roots of this secular equation gives the Rayleigh wave speed. Being of vital importance, Rayleigh waves have motivated many researchers to find the formulae for Rayleigh wave speed.

Rahman and Barber [4] made use of the theory of cubic equation to propose a solution for the secular equation of Rayleigh waves for compressible isotropic elastic solid. However, the expressions proposed by them were limited to specific ranges of poisson ratio.

Nkemzi [3] formulated an alternative solution for the Rayleigh wave equation which satisfied for whole range of material parameter in linearly elastic isotropic solid. Later, Malischewsky [5] pointed out that the formula presented by Nkemzi [3] was incorrect. He also pointed out that it was unnecessary to impose such complexities to find the formula for Rayleigh wave speed. Malischewsky [5] used the Cardan's formula along with the use of trigonometric formulas to provide a correct formula for Rayleigh wave speed which satisfied for all ranges of material parameter, and it also contained a signum function. Malischewsky believed that for compressible isotropic medium, his formula is the simplest expression available for the wave speed of Rayleigh waves. However, Malichewsky did not show in detail, the procedure through which Cardan's formula is used alongside trigonometric formulas and with the help of MATHEMATICA to give the solution of the Rayleigh wave equation.

Vinh and Ogden [6] provided a detailed explanation of Malischewsky's formula. They also made use of the Cardan's formula to give an alternative expression for the Rayleigh wave speed in compressible isotropic elastic solid without the inclusion of signum function. Furthermore, Vinh and Ogden showed that their formula is equivalent to Malischewsky's formula for the given range of material parameter.

Basing on the theory of cubic equations, Ogden and Vinh [7] also provided a formula for incompressible orthotropic material. For compressible orthotropic elastic materials, Vinh and Ogden [8] and Vinh and Ogden [9] provided a solution for Rayleigh wave speed also based on cubic equations theory.

The formula for Rayleigh wave speed in an isotropic half-space as well as in an incompressible orthotropic half-space was provided by Rahman and Michelitsch [10].

Nkemzi [11] gave a simple formula for Rayleigh wave speed in an isotropic linearly elastic solid. However, this formula contained irrational denominator.

Liu and Fan [12] derived a new formula for Rayleigh wave speed in isotropic elastic half-space, using a form of Cardano's formula.

Recently, Sudheer et al. [13] used numerical and approximate techniques to give a formula for wave speed of Rayleigh waves in isotropic and anisotropic media.

In this thesis, Rayleigh wave speed formulas given by Malischewsky [5], Vinh and Ogden [6] are reviewed and discussed.

Chapter 2 is composed of some basic concepts, such as complex numbers, formula for finding their roots, their principal values. The concepts of stress in an isotropic material are briefly given here. Secular equation for Rayleigh waves in compressible isotropic elastic solids to determine the Rayleigh wave speed is also reviewed. Furthermore, Cardan's fomulas to find the roots of cubic equation are derived in detail.

Chapter 3 is devoted to discuss the formula obtained by Malischewsky [5] for the speed of Rayleigh waves in compressible isotropic elastic solids. It is based on the theory of cubic equations and Cardan's formula. Formula for complex roots of Rayleigh wave equation given by Malischewsky [5] is also reviewed in this chapter.

In chapter 4, Rayleigh wave speed formula given by Vinh and Ogden [6] using the theory of cubic equations and Cardan's formula is discussed in detail. An equivalent form of secular equation for Rayleigh waves and its conversion into cubic equation along with the uniqueness of its solution related to Rayleigh wave speed is given here. The solutions of this cubic equation in different cases of discriminant are studied in detail. This study is used to analyze the formula obtained by Malischewsky [5], and its connection with the formula given by Vinh and Ogden [6] is explained.

Chapter 5 consists of conclusions after reviewing papers of Malischewsky [5] and Vinh and Ogden [6].

Chapter 2

Preliminaries

In this chapter, some basic concepts required in the subsequent chapters are given.

2.1 Complex numbers

In this section, complex numbers, their roots and principal values are explained as mentioned by Zill and Shanahan [14].

2.1.1 Notation and modulus of complex number

If a number is written in the form $w = p + iq$ then w is called complex number where p and q are real numbers and i is an imaginary number with value $\sqrt{-1}$. Modulus $|w|$, of this complex number is defined as

$$|w| = r = \sqrt{p^2 + q^2}, \tag{2.1.1}$$

which is a positive real number.

2.1.2 Polar form of a complex number

The complex number $w = p + iq$ in its polar form is given by

$$w = r(\cos\theta + i\sin\theta), \tag{2.1.2}$$

where the value of r is given by Eq. (2.1.1), and θ is the **argument** of w or **phase angle** of w. The values of p and q in polar form are given by

$$p = r\cos\theta, \qquad q = r\sin\theta. \tag{2.1.3}$$

The phase angle or argument of complex number w is represented by $\theta = \arg(w) = \tan^{-1}(q/p)$. Since 2π is the period of $\cos\theta$ and $\sin\theta$, therefore, the phase angle or argument of complex number w is not unique. If argument of w is denoted by θ_1, then $\theta_1 \pm 2\pi h$ where $h = \pm 1, \pm 2, \pm 3, ...,$ must be the arguments of w. The argument θ_1 of w is called **principal argument** or **principal value** of $\arg(w)$, denoted as $\text{Arg}(w)$, is distinctive and given by $-\pi < \text{Arg}(w) \leq \pi$.

2.1.3 Roots of complex numbers

The formula for finding the nth roots of complex number $w = p + iq$ is given by

$$w_h^{1/n} = \sqrt[n]{r}[\cos(\frac{\theta + 2h\pi}{n}) + i\sin(\frac{\theta + 2h\pi}{n})], \tag{2.1.4}$$

or

$$w_h^{1/n} = |w|^{1/n} e^{i(\frac{\theta + 2h\pi}{n})}, \tag{2.1.5}$$

where $h = 0, 1, 2, ..., n - 1$ and n is the number of roots.

Example 1: Find the three cube roots of $w = i$.

Solution: Since $r = 1$, $\theta = \text{Arg}(i) = \pi/2$, hence from Eq. (2.1.4), we have

$$w_h^* = w_h^{1/3} = \sqrt[3]{1}[\cos(\frac{\pi/2 + 2h\pi}{3}) + \sin(\frac{\pi/2 + 2h\pi}{3})], \tag{2.1.6}$$

where $h = 0, 1, 2$. Therefore the three cube roots of i are

$$w_0^* = \frac{\sqrt{3}}{2} + \frac{1}{2}i, \quad w_1^* = -\frac{\sqrt{3}}{2} + \frac{1}{2}i, \quad w_2^* = -i. \tag{2.1.7}$$

2.1.4 Principal nth root

The distinctive root of a complex number w (acquired by using Arg(w) with $h = 0$) is called principal nth root. In Example 1, the principal cube root of ι is $w_0^* = \frac{\sqrt{3}}{2} + \frac{1}{2}\iota$, where $\text{Arg}(\iota) = \pi/2$ and $h = 0$. The use of Arg(w) with $h = 0$ gives us the assurance that if the complex number w is a positive real number r, then the principal nth root is $\sqrt[n]{r}$, i.e. if $w = 2 + 0\iota$ then $|w| = 2$ and $\text{Arg}(w) = 0$, hence by using Eq. (2.1.5), the three cube roots of w are given by $w_0^* = \sqrt[3]{2}$, $w_1^* = \sqrt[3]{2}e^{\iota(\frac{2\pi}{3})}$ and $w_2^* = \sqrt[3]{2}e^{\iota(\frac{4\pi}{3})}$, where $w_0^* = \sqrt[3]{2}$ is principal cube root of complex number $w = 2 + 0\iota$.

2.2 Unique root of a function in an interval

In this section, the unique root of a function in an interval is explained as mentioned by Thomas and Finney [15]. To find the unique root in a given interval, first derivative test for increasing and decreasing of a function helps in this regard. For this purpose, suppose $f(\psi)$ be a function, then if f is continuous on the interval $[p_1, q_1]$, where p_1 and q_1 are real numbers, and f' exists in the interval (p_1, q_1), and $f(p_1)f(q_1) < 0$, and either $f' > 0$ or $f' < 0$ on the interval (p_1, q_1), then the root of the equation $f(\psi) = 0$, is unique in the interval (p_1, q_1). Since $f(p_1)f(q_1) < 0$, so there exists at least one real root in the interval (p_1, q_1), and function is either increasing or decreasing in the interval (p_1, q_1), which ensures that there is no turning point in the interval (p_1, q_1), so root will be unique in this interval.

2.3 Stress, Strain and their relationship

Stress and strain have important role in the theory of elasticity. In this section, stress, strain and their relation is explained according to the study provided by Royer and Dieulesaint [16].

2.3.1 Strain:

If a force is applied to a material, it deforms in shape, size or volume. The amount of this deformation is called strain and this quantity is free of dimensions. Assuming $\mathbf{d}(\mathbf{z}, t)$ be the displacement vector, then strain tensor of rank two S_{ij} in linear approximation is defined by

$$S_{ij} = \frac{1}{2}(\frac{\partial d_i}{\partial z_j} + \frac{\partial d_j}{\partial z_i}), \quad i, j = 1, 2, 3. \tag{2.3.1}$$

From Eq. (2.3.1), it is obvious that strain tensor S_{ij} of rank two is symmetric, i.e. $S_{ij} = S_{ji}$.

2.3.2 Stress:

Stress is a force acting per unit area of the body. If an orthonormal frame of reference is considered, then stress tensor is defined by

$$T_{ik} = lim_{\Delta s_k \to 0}(\frac{\Delta F_i}{\Delta s_k}), \tag{2.3.2}$$

where ΔF_i is the $i-th$ component of force $\mathbf{\Delta F}$ applied on the surface element Δs_k, where Δs_k is orthogonal to $k-$axis, through the medium in the positive direction. It is observed that stress tensor of rank two is symmetric under equilibrium conditions, i.e. $T_{ik} = T_{ki}$.

2.3.3 Stress and strain relation:

If a medium regains its original form after the removal of external forces, then it is elastic. The elasticity of this medium is because of internal stress. If stress is applied then it causes strain. It is observed by experiments that, most of the materials show elastic behavior for small deformations. The relation of stress and strain in linear theory of elasticity, can be explained by considering stress as a function of strain and by expanding Taylor series about origin as follows

$$T_{ij}(S_{kl}) = T_{ij}(0) + (\frac{\partial T_{ij}}{\partial S_{kl}})_{S_{kl}=0} S_{kl} + ..., \tag{2.3.3}$$

where in Eq. (2.3.3), $T_{ij}(0) = 0$, because if there is no strain then no stress and vice versa. Since strain is sufficiently small in linear theory of elasticity, so ignoring second and higher order terms, then Eq. (2.3.3) results in

$$T_{ij}(S_{kl}) = c_{ijkl} S_{kl}, \tag{2.3.4}$$

where $c_{ijkl} = (\frac{\partial T_{ij}}{\partial S_{kl}})_{S_{kl}=0}$, are the components of elastic stiffness tensor of rank four.

For isotropic material (the material whose physical properties are same in every direction), the elasticity tensor c_{ijkl} is given by [17]

$$c_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{2.3.5}$$

where $\delta_{ij}, \delta_{kl}, \delta_{ik}, \delta_{jl}, \delta_{il}$ and δ_{jk} are rank 2 symmetric isotropic tensors (the components of which remain unchanged with the change of basis). Here λ, μ are Lamé parameters. So for isotropic material, the stress-strain relation defined in Eq. (2.3.4) can be expressed as

$$T_{ij} = \lambda \delta_{ij} S_{kk} + 2\mu S_{ij}, \tag{2.3.6}$$

where $S_{kk} = S_{11} + S_{22} + S_{33}$ and S_{ij} is defined by Eq. (2.3.1). Equation (2.3.6) represents stress in terms of strain. Similarly one can express strain in terms of stress.

2.4 Waves and their types

A wave is a disturbance that travels through space and matter, accompanied by transfer of energy. If this disturbance is in elastic medium then these waves are called elastic waves. Elastic waves that propagate in solids can be of many types. Some of them are discussed here.

2.4.1 Longitudinal waves

The waves in which particle displacement is parallel to the direction of motion of the waves are called longitudinal waves.

2.4.2 Transverse waves

The waves in which particle displacement is perpendicular to the direction of motion of the waves are called transverse waves.

2.4.3 Surface waves

These waves travel through the surface of the solid material. These waves are called surface waves because the displacement decay exponentially as distance from the surface increases. Rayleigh waves are type of surface waves, and a brief discussion about these waves in linear elastic isotropic solid is given in the following section.

2.4.4 Secular equation for Rayleigh waves in compressible isotropic elastic solids

In this section, secular equation of Rayleigh waves is derived according to study provided in [17]. Rayleigh waves are considered to be traveling beside the surface of an elastic half space in z_1 direction, while the propagation of these waves is occurring only in $z_1 z_2-$plane, and z_2 is positive in the downward direction as shown in Fig 2.1.

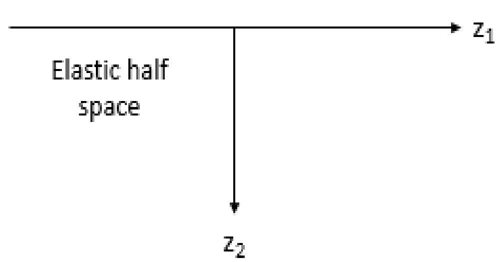

Figure 2.1: Isotropic elastic half space

Displacement equation for waves in an isotropic material is

$$\mu\nabla^2\mathbf{d} + (\lambda+\mu)\nabla(\nabla.\mathbf{d}) = \rho\frac{\partial^2\mathbf{d}}{\partial t^2}, \tag{2.4.1}$$

where $\mathbf{d}$ is displacement vector and ρ denotes mass density of the material and $\nabla^2 = (\frac{\partial^2}{\partial z_1^2} + \frac{\partial^2}{\partial z_2^2})$ is the Laplace operator. Since Rayleigh wave is a surface wave so components of displacement vector $\mathbf{d}$ in z_1z_2-plane are

$$d_1(z_1, z_2, t) = Ae^{-akz_2}\exp[ik(z_1 - ct)], \tag{2.4.2a}$$

$$d_2(z_1, z_2, t) = Be^{-akz_2}\exp[ik(z_1 - ct)], \tag{2.4.2b}$$

$$d_3(z_1, z_2, t) \equiv 0, \tag{2.4.2c}$$

where A, B are wave amplitudes, a has positive real part, c represents Rayleigh wave speed and k is the wave number (the distance between two peaks of a wave correspond to wave length and reciprocal of this wave length is called wave number). The components d_1 and d_2 satisfy Eq. (2.4.1) as

$$\mu(\frac{\partial^2}{\partial z_1^2} + \frac{\partial^2}{\partial z_2^2})d_1 + (\lambda+\mu)\frac{\partial}{\partial z_1}(\frac{\partial d_1}{\partial z_1} + \frac{\partial d_2}{\partial z_2}) = \rho\frac{\partial^2 d_1}{\partial t^2}, \tag{2.4.3a}$$

$$\mu(\frac{\partial^2}{\partial z_1^2} + \frac{\partial^2}{\partial z_2^2})d_2 + (\lambda+\mu)\frac{\partial}{\partial z_2}(\frac{\partial d_1}{\partial z_1} + \frac{\partial d_2}{\partial z_2}) = \rho\frac{\partial^2 d_2}{\partial t^2}. \tag{2.4.3b}$$

After substituting the values of components d_1 and d_2 from Eqs. (2.4.2a) and (2.4.2b) in Eqs. (2.4.3a) and (2.4.3b), we have

$$(c_1^2a^2 - c_2^2 + c^2)A - (c_2^2 - c_1^2)iaB = 0, \tag{2.4.4a}$$

$$(c_2^2a^2 - c_1^2 + c^2)B - (c_2^2 - c_1^2)iaA = 0, \tag{2.4.4b}$$

where

$$c_1^2 = \frac{\mu}{\rho}, \qquad c_2^2 = \frac{\lambda+2\mu}{\rho}, \tag{2.4.5}$$

where c_1 and c_2 represents transverse wave speed and longitudinal wave speed in isotropic material, respectively. Equations (2.4.4a) and (2.4.4b) represents a system of homogeneous equations in A and B. For nontrivial solution of this system

$$\begin{vmatrix} (c_1^2a^2 - c_2^2 + c^2) & -(c_2^2 - c_1^2)ia \\ -(c_2^2 - c_1^2)ia & (c_2^2a^2 - c_1^2 + c^2) \end{vmatrix} = 0. \tag{2.4.6}$$

After simplification, Eq. (2.4.6) yields

$$c_2^2c_1^2a^4 - [c_1^2(c_2^2 - c^2) + c_2^2(c_1^2 - c^2)]a^2 + [(c_2^2 - c^2)(c_1^2 - c^2)] = 0, \tag{2.4.7}$$

which may be written as

$$(a^2 - (1 - \frac{c^2}{c_1^2}))(a^2 - (1 - \frac{c^2}{c_2^2})) = 0. \tag{2.4.8}$$

The roots of Eq. (2.4.8) are given by

$$a_1 = \sqrt{1 - \frac{c^2}{c_2^2}}, \qquad a_2 = \sqrt{1 - \frac{c^2}{c_1^2}}. \tag{2.4.9}$$

Since from Eq. (2.4.5), $c_1 < c_2$, it is observed that if $c < c_1 < c_2$ and positive value of square roots are used, then a_1 and a_2 are positive and real. Using Eq. (2.4.9) and Eqs. (2.4.2a) and (2.4.2b), we have

$$d_1(z_1, z_2, t) = [A_1e^{-a_1kz_2} + A_2e^{-a_2kz_2}]\exp[ik(z_1 - ct)], \tag{2.4.10a}$$

$$d_2(z_1, z_2, t) = [B_1e^{-a_1kz_2} + B_2e^{-a_2kz_2}]\exp[ik(z_1 - ct)]. \tag{2.4.10b}$$

Since there are two expressions with four unknowns, we reduce Eqs. (2.4.10a) and (2.4.10b) with two unknowns. For this purpose, replace A by A_2, B by B_2 and a by a_2 in Eq. (2.4.4a), and by using Eq. $(2.4.9)_2$, we get

$$B_2 = i\frac{A_2}{a_2}. \tag{2.4.11}$$

Similarly, by replacing A by A_1, B by B_1 and a by a_1 in Eq. (2.4.4b), and by using

Eq. $(2.4.9)_1$, we have

$$B_1 = ia_1A_1. \tag{2.4.12}$$

At free surface $z_2 = 0$, the stress components $T_{21} = 0$, $T_{22} = 0$, $T_{23} \equiv 0$, for all z_1, $t > 0$. Since $T_{21} = 0$, by Eq. (2.3.6), we have

$$T_{21} = \mu(\frac{\partial d_2}{\partial z_1} + \frac{\partial d_1}{\partial z_2}) = 0. \tag{2.4.13}$$

By using Eqs. (2.4.9)– (2.4.12) and simplifying, Eq. (2.4.13) yields

$$2\sqrt{1-\frac{c^2}{c_1^2}}\sqrt{1-\frac{c^2}{c_2^2}}A_1 + (2-\frac{c^2}{c_1^2})A_2 = 0. \tag{2.4.14}$$

Since $T_{22} = 0$, by using Eq. (2.3.6), we have

$$T_{22} = \lambda(\frac{\partial d_1}{\partial z_1} + \frac{\partial d_2}{\partial z_2}) + 2\mu\frac{\partial d_2}{\partial z_2} = 0. \tag{2.4.15}$$

Again by using Eqs. (2.4.9) – (2.4.12) and simplifying, Eq. (2.4.15) becomes

$$(2-\frac{c^2}{c_1^2})A_1 + 2A_2 = 0. \tag{2.4.16}$$

Equations (2.4.14) and (2.4.16) shows system of homogeneous equations in A_1, A_2. For nontrivial solution

$$\begin{vmatrix} 2\sqrt{1-\frac{c^2}{c_1^2}}\sqrt{1-\frac{c^2}{c_2^2}} & (2-\frac{c^2}{c_1^2}) \\ (2-\frac{c^2}{c_1^2}) & 2 \end{vmatrix} = 0. \tag{2.4.17}$$

Simplification yields

$$(2-\frac{c^2}{c_1^2})^2 = 4\sqrt{1-\frac{c^2}{c_2^2}}\sqrt{1-\frac{c^2}{c_1^2}}. \tag{2.4.18}$$

Since Eq. (2.4.18) is independent of wave number, therefore Rayleigh waves are dispersion-less in an isotropic elastic half-space and hence, Eq. (2.4.18) is called **secular** equation for Rayleigh wave in compressible isotropic elastic solids.

To find the interval in which Rayleigh wave is lying, one can take square on both sides of Eq. (2.4.18)

$$(2-\frac{c^2}{c_1^2})^4 = 16(1-\frac{c^2}{c_2^2})(1-\frac{c^2}{c_1^2}). \tag{2.4.19}$$

By simplification, Eq. (2.4.19) becomes

$$\frac{c^8}{c_1^8} - 8\frac{c^6}{c_1^6} + 24\frac{c^4}{c_1^4} - 16\frac{c^2}{c_1^2} = 16\frac{c^4}{c_1^2 c_2^2} - 16\frac{c^2}{c_2^2}. \tag{2.4.20}$$

Multiplying Eq. (2.4.20) by a factor $\frac{c_1^2}{c^2}$, it becomes

$$f(c) = (\frac{c}{c_1})^6 - 8(\frac{c}{c_1})^4 + (24 - 16\frac{c_1^2}{c_2^2})\frac{c^2}{c_1^2} - 16(1-\frac{c_1^2}{c_2^2}) = 0, \tag{2.4.21}$$

where

$$f(c) = (\frac{c}{c_1})^6 - 8(\frac{c}{c_1})^4 + (24 - 16\frac{c_1^2}{c_2^2})\frac{c^2}{c_1^2} - 16(1-\frac{c_1^2}{c_2^2}). \tag{2.4.22}$$

Since from Eq. (2.4.9), $c < c_1$. On substituting $c = 0$ and $c = c_1$ in Eq. (2.4.22), we have

$$\begin{aligned} f(0) &= -16(1-\frac{c_1^2}{c_2^2}) < 0, \qquad (\because c_1 < c_2) \\ f(c_1) &= 1 > 0. \end{aligned} \tag{2.4.23}$$

Thus there is at least one real root of Eq. (2.4.18) in the interval $0 < c < c_1$. To see, if there is a unique real root in the interval $0 < c < c_1$, a discussion is presented later in this section.

Rayleigh wave speed is the root of Eq. (2.4.18). To convert Eq. (2.4.18) into dimensionless form is an appropriate way to find the root of Rayleigh wave equation [3]. For this purpose, introducing a dimensionless variable

$$y = \frac{c^2}{c_1^2}. \tag{2.4.24}$$

So Eq. (2.4.18) becomes

$$(2-y)^2 = 4\sqrt{1-\Gamma y}\sqrt{1-y}, \tag{2.4.25}$$

where

$$0 < y < 1, \quad 0 < \Gamma = \frac{c_1^2}{c_2^2} = \frac{\mu}{\lambda + 2\mu} < \frac{3}{4}, \tag{2.4.26}$$

where Γ is dimensionless parameter. Since $0 < c < c_1$, dividing the interval by a factor c_1 results in $0 < \frac{c}{c_1} < 1$, and hence Eq. (2.4.24) implies, $0 < y < 1$.

Poisson's ratio: The ratio of transverse contraction strain to longitudinal extension strain in the direction of applied force is called poisson ratio. It is given by $\nu = -S_{trans}/S_{longitudinal}$, where strain in longitudinal direction is taken positive while the strain in transverse direction is taken negative. Most of the materials have positive poisson ratio i.e. they contract in transverse direction if stretched in longitudinal direction, e.g. rubber has poisson ratio $\nu = \frac{1}{2}$, while some materials show negative poisson ratio because when they stretched in longitudinal direction, they expand in transverse direction like re-entrant honey comb which has poisson ratio $\nu \approx -1$ [18]. For stable, isotropic, linear elastic solid, the range of poissons's ratio is $-1 < \nu < 1/2$ [19]. For this range of poisson's ratio, it can be shown that it is sufficient to consider the parameter Γ for the range $0 < \Gamma < 3/4$. To show this considering the relation of ν with Lamé parameters, we have

$$\nu = \frac{\lambda}{2(\lambda + \mu)} = \frac{\lambda + 2\mu - 2\mu}{2(\lambda + 2\mu - \mu)}. \tag{2.4.27}$$

Multiplying and dividing Eq. (2.4.27) by $\frac{1}{\rho}$ and using Eq. (2.4.5), we have

$$\nu = \frac{(\lambda + 2\mu)/\rho - 2\mu/\rho}{2((\lambda + 2\mu)/\rho - \mu/\rho)} = \frac{c_2^2 - 2c_1^2}{2(c_2^2 - c_1^2)}. \tag{2.4.28}$$

Using Eq. $(2.4.26)_2$, Eq. (2.4.28) becomes

$$\nu = \frac{1 - 2\Gamma}{2(1 - \Gamma)}. \tag{2.4.29}$$

Rearranging Eq. (2.4.29) to get the relation of Γ in terms of ν, we have

$$\Gamma = \frac{1 - 2\nu}{2 - 2\nu}. \tag{2.4.30}$$

By substituting $\nu = -1$ and $\nu = 1/2$ in Eq. (2.4.30) results in $\Gamma = 3/4$ and $\Gamma = 0$ respectively. Thus for $-1 < \nu < 1/2$, the range of parameter Γ is $0 < \Gamma < 3/4$.

2.5 Cardan's formulas for finding the roots of cubic equation

In 1545, Gerolamo Cardano published the solution to cubic equations [20]. Cardano, however, was not the pioneer in this field. Niccolo Tartaglia had also provided a hint for the solution of cubic equations. It is thought that Tartaglia himself had got the idea for the solution from Scipione del Ferro, who did not publish his solution but disclosed it to his students.

Cardan's formulas are of great importance in order to find the roots of cubic equation. To derive these formulas, consider the cubic equation in general form as

$$g(z) = z^3 + m_2 z^2 + m_1 z + m_0 = 0. \tag{2.5.1}$$

To simplify Eq. (2.5.1), presenting a new unknown χ, hence, resulting in the absence of the square of unknown in the equation, where the unknown χ is defined by the relation

$$z = \chi + h_1, \tag{2.5.2}$$

where h_1 is chosen randomly. By applying Taylor's series on the right side of Eq. (2.5.2), we have

$$g(\chi + h_1) = g(h_1) + g'(h_1)\chi + \frac{g''(h_1)}{2}\chi^2 + \frac{g'''(h_1)}{6}\chi^3, \tag{2.5.3}$$

where

$$\begin{aligned} &g(h_1) = h_1^3 + m_2 h_1^2 + m_1 h_1 + m_0, \qquad g'(h_1) = 3h_1^2 + 2m_2 h_1 + m_1, \\ &\frac{1}{2}g''(h_1) = 3h_1 + m_2, \qquad \frac{1}{6}g'''(h_1) = 1. \end{aligned} \tag{2.5.4}$$

By setting $3h_1 + m_2 = 0$, the resulting equation in χ will be free of χ^2 term, and hence

$$h_1 = -\frac{m_2}{3}. \tag{2.5.5}$$

After substituting the value of h_1 in Eq. (2.5.4), we get

$$g(-\frac{m_2}{3}) = \frac{2m_2^3}{27} - \frac{m_1 m_2}{3} + m_0, \qquad g'(-\frac{m_2}{3}) = -\frac{m_2^2}{3} + m_1. \tag{2.5.6}$$

Using Eq. (2.5.5) in Eq. (2.5.2), we have

$$z = \chi - \frac{m_2}{3}. \tag{2.5.7}$$

So by using Eq. (2.5.7), Eq. (2.5.1) is converted into

$$g(\chi) = \chi^3 + r_1 \chi + r_0 = 0, \tag{2.5.8}$$

where

$$r_1 = -\frac{m_2^2}{3} + m_1, \quad r_0 = \frac{2m_2^3}{27} - \frac{m_1 m_2}{3} + m_0. \tag{2.5.9}$$

In order to solve Eq. (2.5.8), the unknowns α and γ are introduced through

$$\chi = \alpha + \gamma \ . \tag{2.5.10}$$

Substitution of Eq. (2.5.10) in Eq. (2.5.8) results in

$$\alpha^3 + \gamma^3 + (3\alpha\gamma + r_1)(\alpha + \gamma) + r_0 = 0. \tag{2.5.11}$$

Defining another relation between α and γ to solve Eq. (2.5.11), for this purpose taking

$$3\alpha\gamma + r_1 = 0, \tag{2.5.12}$$

$$\text{or} \quad \alpha\gamma = -\frac{r_1}{3}. \tag{2.5.13}$$

Using Eq. (2.5.13) in Eq. (2.5.11), we have

$$\alpha^3 + \gamma^3 + r_0 = 0. \tag{2.5.14}$$

By substituting the value of γ from Eq. (2.5.13), Eq. (2.5.14) will be converted into quadratic equation in α^3

$$\alpha^6 + r_0\alpha^3 - \frac{r_1^3}{27} = 0. \tag{2.5.15}$$

Roots of Eq. (2.5.15) are given by quadratic formula as

$$\alpha^3 = -\frac{r_0}{2} \pm \sqrt{\frac{r_0^2}{4} + \frac{r_1^3}{27}}. \tag{2.5.16}$$

By using Eq. (2.5.13), with out loss of generality we have

$$\alpha^3 = -\frac{r_0}{2} + \sqrt{\frac{r_0^2}{4} + \frac{r_1^3}{27}}, \quad \gamma^3 = -\frac{r_0}{2} - \sqrt{\frac{r_0^2}{4} + \frac{r_1^3}{27}}. \tag{2.5.17}$$

Let

$$M = \sqrt[3]{-\frac{r_0}{2} + \sqrt{\frac{r_0^2}{4} + \frac{r_1^3}{27}}}, \quad N = \sqrt[3]{-\frac{r_0}{2} - \sqrt{\frac{r_0^2}{4} + \frac{r_1^3}{27}}}. \tag{2.5.18}$$

So α and γ have three possible values. The values of α are calculated by using Eq. $(2.5.17)_1$ and Eq. $(2.5.18)_1$ as

$$\begin{aligned}
&\alpha^3 - [(-\frac{r_0}{2} + \sqrt{\frac{r_0^2}{4} + \frac{r_1^3}{27}})^{\frac{1}{3}}]^3 = 0,\\
\implies &\alpha^3 - M^3 = 0,\\
\implies &(\alpha - M)(\alpha^2 + \alpha M + M^2) = 0.
\end{aligned} \tag{2.5.19}$$

Hence from Eq. (2.5.19), the three possible values of α are given by

$$\alpha_1 = M, \quad \alpha_2 = \omega M, \quad \alpha_3 = \omega^2 M, \tag{2.5.20}$$

where

$$\omega = e^{i\frac{2\pi}{3}} = \frac{-1 + i\sqrt{3}}{2}, \quad \omega^2 = e^{-i\frac{2\pi}{3}} = \frac{-1 - i\sqrt{3}}{2}. \tag{2.5.21}$$

Similarly by adopting the same procedure, the three possible values of γ by using Eqs. $(2.5.17)_2$ and $(2.5.18)_2$ are given by

$$\gamma_1 = N, \quad \gamma_2 = \omega N, \quad \gamma_3 = \omega^2 N. \tag{2.5.22}$$

Because of the restriction, given by Eq. (2.5.13), there are only three possible combinations of α and γ from Eqs. (2.5.20) and (2.5.22), that are satisfying the relation (2.5.10), which is shown as follows

$$\begin{aligned}\chi_1 = \alpha_1\gamma_1 = MN &= \sqrt[3]{-\frac{r_0}{2} + \sqrt{\frac{r_0^2}{4} + \frac{r_1^3}{27}}} \times \sqrt[3]{-\frac{r_0}{2} - \sqrt{\frac{r_0^2}{4} + \frac{r_1^3}{27}}}, \\ &= \sqrt[3]{-\frac{r_1}{27}} = -\frac{r_1}{3}.\end{aligned} \tag{2.5.23}$$

$$\begin{aligned}\chi_2 = \alpha_2\gamma_3 = \omega M \times \omega^2 N &= \omega\sqrt[3]{-\frac{r_0}{2} + \sqrt{\frac{r_0^2}{4} + \frac{r_1^3}{27}}} \times \omega^2\sqrt[3]{-\frac{r_0}{2} - \sqrt{\frac{r_0^2}{4} + \frac{r_1^3}{27}}}, \\ &= [(-\frac{1}{2})^2 - (i\frac{\sqrt{3}}{2})^2]\sqrt[3]{-\frac{r_1}{27}} = -\frac{r_1}{3}.\end{aligned} \tag{2.5.24}$$

Similarly

$$\begin{aligned}\chi_3 = \alpha_3\gamma_2 = \omega^2 M \times \omega N &= \omega^2\sqrt[3]{-\frac{r_0}{2} + \sqrt{\frac{r_0^2}{4} + \frac{r_1^3}{27}}} \times \omega\sqrt[3]{-\frac{r_0}{2} - \sqrt{\frac{r_0^2}{4} + \frac{r_1^3}{27}}}, \\ &= -\frac{r_1}{3}.\end{aligned} \tag{2.5.25}$$

Hence, these three possible combinations are given by

$$\chi_1 = M + N, \tag{2.5.26}$$

$$\chi_2 = \omega M + \omega^2 N, \tag{2.5.27}$$

$$\chi_3 = \omega^2 M + \omega N, \tag{2.5.28}$$

where χ_1, χ_2 and χ_3 are the roots of Eq. (2.5.8), and the values of M and N are given by Eq. (2.5.18). The formulas given by Eqs. (2.5.26) – (2.5.28), are called Cardan's formulas. The roots of Eq. (2.5.1) are thus found through Eq. (2.5.2) and the values of χ_1, χ_2, and χ_3.

Chapter 3

Malischewsky's Formula for the Rayleigh Wave Speed

In this chapter, a review of the paper [5] is presented. In the first section, Malischewsky's formula for the real root of secular equation of Rayleigh waves is explained. In the next section, Malischewsky's formula for the complex roots of secular equation of Rayleigh waves is discussed.

3.1 Malischewsky's formula for real root of Rayleigh wave equation for compressible isotropic elastic solids

Malischewsky [5] used renowned form of Rayleigh wave equation for compressible isotropic elastic solids [1] to find Rayleigh wave speed, which is obtained by taking square of Eq. (2.4.25) and some reshuffling gives

$$f(y) = y^3 + b_2 y^2 + b_1 y + b_0 = 0, \tag{3.1.1}$$

where

$$b_2 = -8, \quad b_1 = 8(3-2\Gamma), \quad b_0 = -16(1-\Gamma), \quad 0 < y = \frac{c^2}{c_1^2} < 1, \quad 0 < \Gamma = \frac{c_1^2}{c_2^2} < 3/4\,, \tag{3.1.2}$$

where Γ is defined by Eq. (2.4.30). Malischewsky derived the formula for Rayleigh wave speed by using Cardan's formulas and trigonometric formulas along with MATHEMATICA, but it was not explained in his paper [5] as to how he did it. Malischewsky's formula for real root of Eq. (3.1.1) to determine Rayleigh wave speed c is

$$y = \frac{2}{3}\left[4 - \sqrt[3]{s_3(\Gamma)} + \mathrm{sgn}[s_4(\Gamma)]\sqrt[3]{\mathrm{sgn}[s_4(\Gamma)]s_2(\Gamma)}\right], \tag{3.1.3}$$

where the functions $s_i(\Gamma)$,$i = 1, 2, 3, 4$ are given as

$$\begin{aligned} &s_1(\Gamma) = 3\sqrt{3}\sqrt{11 - 62\Gamma + 107\Gamma^2 - 64\Gamma^3}, \qquad s_2(\Gamma) = 45\Gamma - 17 + s_1(\Gamma), \\ &s_3(\Gamma) = 17 - 45\Gamma + s_1(\Gamma), \qquad s_4(\Gamma) = -\Gamma + \frac{1}{6}. \end{aligned} \tag{3.1.4}$$

Malischewsky [5] used signum function (where signum function is defined as $\mathrm{sgn}(\psi) = -1$ if $\psi < 0$, $\mathrm{sgn}(\psi) = 0$ if $\psi = 0$, and $\mathrm{sgn}(\psi) = 1$ if $\psi > 0$) in his formula (3.1.3) to avoid zero-crossing of function $s_2(\Gamma)$, which is shown by following graphs

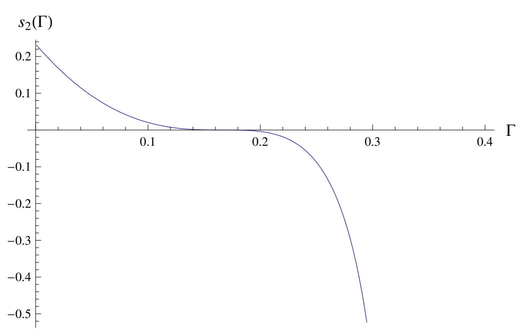

Figure 3.1: Graph of $s_2(\Gamma)$

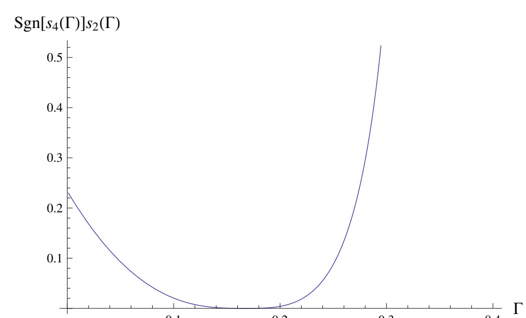

Figure 3.2: Graph of $sgn[s_4(\Gamma)]s_2(\Gamma)$

3.1.1 Discussion about principal values of cube roots in Malischewsky's formula

Malischewsky [5] chose that one value of cube roots in Eq. (3.1.3) which is located in first and fourth quadrant, depicts that Malischewsky took the principal values of complex cube roots in (3.1.3). To show it, we take different ranges of the parameter $\Gamma \in (0, 3/4 = 0.75)$.

For $0 < \Gamma \leq 0.32$, $s_3(\Gamma)$ and $\mathrm{sgn}[s_4(\Gamma)]s_2(\Gamma)$ are positive and real numbers, therefore, according to section 2.1.4, the principal cube root of $s_3(\Gamma)$ is $\sqrt[3]{s_3(\Gamma)}$ and the principal cube root of $\mathrm{sgn}[s_4(\Gamma)]s_2(\Gamma)$ is $\sqrt[3]{\mathrm{sgn}[s_4(\Gamma)]s_2(\Gamma)}$.

For $0.32 < \Gamma \leq 0.37$, the value of the expression $(11 - 62\Gamma + 107\Gamma^2 - 64\Gamma^3)$ is negative and hence by Eq. $(3.1.4)_1$, $s_1(\Gamma) = i3\sqrt{3}\sqrt{-(11 - 62\Gamma + 107\Gamma^2 - 64\Gamma^3)}$. Also for this range, the value of the expression $(17 - 45\Gamma)$ is positive while the value of the expression $(45\Gamma - 17)$ is negative and the value of signum function $\mathrm{sgn}[s_4(\Gamma)]$ is equal to -1. Therefore, the complex cube roots in Eq.(3.1.3) for this range of Γ are represented as, say X and Y,

$$
\begin{aligned}
X &= \sqrt[3]{s_3(\Gamma)} = \sqrt[3]{(17 - 45\Gamma) + i3\sqrt{3}\sqrt{-(11 - 62\Gamma + 107\Gamma^2 - 64\Gamma^3)}},\\
Y &= \sqrt[3]{\mathrm{sgn}[s_4(\Gamma)]s_2(\Gamma)} = \sqrt[3]{-(45\Gamma - 17) - i3\sqrt{3}\sqrt{-(11 - 62\Gamma + 107\Gamma^2 - 64\Gamma^3)}},
\end{aligned}
\tag{3.1.5}
$$

respectively. The radicand $(s_3(\Gamma))$ of the cube root in Eq. $(3.1.5)_1$ lies in the first quadrant with the phase angle contained in $(0, \pi/2)$. If θ is the phase angle of X, then by using Eq. (2.1.5), the three values of X are given by

$$
X_1 = |s_3(\Gamma)|^{\frac{1}{3}} e^{i\theta}, \quad X_2 = |s_3(\Gamma)|^{\frac{1}{3}} e^{i(\theta + 2\pi/3)}, \quad X_3 = |s_3(\Gamma)|^{\frac{1}{3}} e^{i(\theta + 4\pi/3)}, \quad \theta \in (0, \pi/6).
\tag{3.1.6}
$$

According to section 2.1.4, X_1 is the principal value in Eq. (3.1.6) and is lying in the first quadrant. Since the radicand of cube root in Eq. $(3.1.5)_2$ is the complex conjugate of first one, so by using Eq. (2.1.5), the three possible values of Y are given by

$$
\begin{aligned}
&Y_1 = |\text{sgn}[s_4(\Gamma)]s_2(\Gamma)|^{\frac{1}{3}}e^{-i\theta}, \quad Y_2 = |sign[s_4(\Gamma)]s_2(\Gamma)|^{\frac{1}{3}}e^{i(-\theta+2\pi/3)},\\
&Y_3 = |\text{sgn}[s_4(\Gamma)]s_2(\Gamma)|^{\frac{1}{3}}e^{i(-\theta+4\pi/3)}, \qquad \theta \in (0, \pi/6).
\end{aligned} \tag{3.1.7}
$$

Through section 2.1.4, Y_1 is the principal value in Eq. (3.1.7) and is lying in the fourth quadrant.

For $0.37 < \Gamma < 0.75$, the value of the expression $(11 - 62\Gamma + 107\Gamma^2 - 64\Gamma^3)$ is negative and hence by Eq. $(3.1.4)_1$, $s_1(\Gamma) = i3\sqrt{3}\sqrt{-(11 - 62\Gamma + 107\Gamma^2 - 64\Gamma^3)}$. For this range, the value of the expression $(17 - 45\Gamma)$ is negative while the value of the expression $(45\Gamma - 17)$ is positive and the value of signum function is equal to -1. Therefore, the representation of complex cube roots in Eq. (3.1.3) for this case will be same as given in Eq. (3.1.5). In this case the radicand of the cube root in Eq. $(3.1.5)_1$ lies in second quadrant with the phase angle contained in $(\pi/2, \pi)$. The three possible values of complex cube root X in this case will be given by Eq. (3.1.6) where $\theta \in (\pi/6, \pi/3)$ and hence, the principal value of X in this case will be X_1 and lies in the first quadrant. Since the radicand of cube root in Eq. $(3.1.5)_2$ is complex conjugate of first one, so the three possible values of complex cube root Y are given by Eq. (3.1.7) where $\theta \in (\pi/6, \pi/3)$ and hence, the principal value of Y in this case will be Y_1 and lies in the fourth quadrant. Hence it is proved that Malischewsky [5] took the principal values of complex cube roots in his formula for Rayleigh wave speed.

It is observed that for $\Gamma = 1/6$, Eq. $(3.1.4)_4$ implies $\text{sgn}[s_4(\Gamma)] = 0$, and hence from Eq. (3.1.3)

$$
y(1/6) = \frac{2}{3}[4 - \sqrt[3]{19}], \tag{3.1.8}
$$

which depicts that the third term on the right hand side of Eq. (3.1.3) disappears for $\Gamma = 1/6$. Also, for cubic function in Eq. (3.1.1), the point of inflection $y = 8/3$ is called horizontal point of inflection for $\Gamma = 1/6$, i.e. $f'(8/3) = 0$ and $f''(8/3) = 0$.

3.2 Malischewsky's formula for complex roots of Rayleigh wave equation for compressible isotropic elastic solids

Malischewsky [5] derived the formula for complex roots of cubic equation (3.1.1), that are usually taken as irrelevant. The formula is given by

$$y_c = \frac{1}{3}[8 + (1 \pm i\sqrt{3})\sqrt[3]{s_3(\Gamma)} + (-1 \pm i\sqrt{3})\text{sgn}[s_4(\Gamma)]\sqrt[3]{\text{sgn}[s_4(\Gamma)]s_2(\Gamma)}]. \quad (3.2.1)$$

For $\Gamma = 1/6$, Eq. $(3.1.4)_4$ implies $sign[s_4(\Gamma)] = 0$, consequently, the third term of Eq. (3.2.1) disappears

$$y_c(1/6) = \frac{1}{3}[8 + (1 \pm i\sqrt{3})\sqrt[3]{19}]. \quad (3.2.2)$$

Malischewsky [5] considered his formula (3.1.3) as simplest one to find Rayleigh wave speed and it is for any range of Γ as well as for ν, and unlike to Rahman and Barber [4] formulae that are not applicable for whole range of ν. Opposing to Nkemzi [3], Malischewsky [5] considered his formula (3.1.3) as correct form of formula for Rayleigh wave speed.

Chapter 4

Vinh and Ogden's Formula for the Rayleigh Wave Speed

In this chapter a detailed review of the paper [6] is presented. In the first section, Rayleigh wave speed equation for compressible isotropic elastic solids is discussed. In the next section, formula for Rayleigh wave speed given by Vinh and Ogden [6] is explained and analyzed in detail. In the last section, connection of Vinh and Ogden's formula with Malischewsky's formula is shown.

4.1 Analysis of the Rayleigh wave equation for compressible isotropic elastic solids

Vinh and Ogden [6] used the form of secular equation of Rayleigh waves for compressible isotropic elastic solids which is given by Eq. (2.4.25), and found its equivalent form for the related real root of Rayleigh wave equation.

4.1.1 An equivalent form of the secular equation for Rayleigh wave

An equivalent form of Eq. (2.4.25) is derived in [6], given by Eq. (4.1.1)

$$(1-y)[4(1-\Gamma)-y]-y\sqrt{1-y}\sqrt{1-\Gamma y}=0, \tag{4.1.1}$$

where $0<y=\frac{c^2}{c_1^2}<1$ and $0<\Gamma=\frac{c_1^2}{c_2^2}<3/4$. To derive Eq. (4.1.1), rewriting Eq. (2.4.25) as

$$4\sqrt{1-\Gamma y}\sqrt{1-y}-4(1-y)-y^2=0. \tag{4.1.2}$$

Multiplying Eq. (4.1.2) by $\sqrt{1-\Gamma y}\sqrt{1-y}$, we get

$$(1-y)[4(1-\Gamma y)-4\sqrt{1-\Gamma y}\sqrt{1-y}]-y^2\sqrt{1-\Gamma y}\sqrt{1-y}=0. \tag{4.1.3}$$

Using Eq. (2.4.25) in Eq. (4.1.3), we have

$$\begin{aligned}
&(1-y)[4(1-\Gamma y)-(2-y)^2]-y^2\sqrt{1-\Gamma y}\sqrt{1-y}=0,\\
\implies\ &(1-y)[4-4+4y-4\Gamma y-y^2]-y^2\sqrt{1-\Gamma y}\sqrt{1-y}=0,\\
\implies\ &(1-y)[4y-4\Gamma y-y^2]-y^2\sqrt{1-\Gamma y}\sqrt{1-y}=0. \qquad (4.1.4)
\end{aligned}$$

Dividing Eq. (4.1.4) by a factor y results in Eq. (4.1.1).

Equation (4.1.1) can also be derived by the generalized equation

$$\begin{aligned}
&(\beta_{11}-\rho c^2)[(\sigma_1-\rho c^2)\sigma_2-(\sigma_2-\tau)^2]+(\sigma_2/\beta_{22})^{1/2}(\beta_{11}-\rho c^2)^{1/2}(\sigma_1-\rho c^2)^{1/2}\\
&\times[\beta_{22}(\beta_{11}-\rho c^2)-\beta_{12}^2]=0, \qquad (4.1.5)
\end{aligned}$$

where $\beta_{11},\beta_{22},\beta_{12},\sigma_1,\sigma_2$ are the components of fourth-order elastic modulus tensor in the non linear theory, ρ denotes the mass density of material, c denotes the Rayleigh wave speed, τ is the pre-stress. Equation (4.1.5) is known as secular equation for Rayleigh waves in pre-stressed compressible elastic materials. This equation was obtained by Dowaikh and Ogden [21] using the non linear theory of elasticity. Since for the derivation of Eq. (4.1.1), we are taking the case of no pre-stress, so by putting $\tau = 0$ in Eq. (4.1.5), we have

$$(\beta_{11} - \rho c^2)[(\sigma_1 - \rho c^2)\sigma_2 - (\sigma_2)^2] + (\sigma_2/\beta_{22})^{1/2}(\beta_{11} - \rho c^2)^{1/2}(\sigma_1 - \rho c^2)^{1/2} \times [\sigma_{22}(\beta_{11} - \rho c^2) - \sigma_{12}^2] = 0. \tag{4.1.6}$$

In the absence of the pre-stress, the material is assumed to be a linear isotropic material and the values of components of fourth-order elastic modulus tensor used in Eq. (4.1.5), are given by Eq. (4.8) in [21] as

$$\beta_{11} = \beta_{22} = \lambda + 2\mu, \quad \beta_{12} = \lambda, \quad \sigma_1 = \sigma_2 = \mu. \tag{4.1.7}$$

By substituting various elastic constants from Eq. (4.1.7) in Eq. (4.1.6) results in

$$((\lambda + 2\mu) - \rho c^2)[(\mu - \rho c^2)\mu - \mu^2] + (\mu/(\lambda + 2\mu))^{1/2}((\lambda + 2\mu) - \rho c^2)^{1/2}(\mu - \rho c^2)^{1/2} \times[(\lambda + 2\mu)((\lambda + 2\mu) - \rho c^2) - \lambda^2] = 0, \tag{4.1.8}$$

which can be written in the form of Γ as

$$((\lambda + 2\mu) - \rho c^2)[(\mu - \rho c^2)\mu - \mu^2] + (\Gamma)^{1/2}((\lambda + 2\mu) - \rho c^2)^{1/2}(\mu - \rho c^2)^{1/2} \times [(\lambda + 2\mu)((\lambda + 2\mu) - \rho c^2) - \lambda^2] = 0. \tag{4.1.9}$$

where Γ is given by Eq. (2.4.26). After simplifications, Eq. (4.1.9) yields

$$((\lambda + 2\mu) - \rho c^2)[(\mu - \rho c^2)\mu - \mu^2] + (\Gamma)^{1/2}((\lambda + 2\mu) - \rho c^2)^{1/2}(\mu - \rho c^2)^{1/2} \times [4\mu(\mu + \lambda) - \rho c^2(\lambda + 2\mu)] = 0. \tag{4.1.10}$$

Dividing Eq. (4.1.10) through by μ^2 and also by using Eqs. (2.4.24) and (2.4.26),we get

$$((\lambda+2\mu)-\rho c^2)[(1-y)-1]+(\Gamma)^{1/2}((\lambda+2\mu)-\rho c^2)^{1/2}(\mu-\rho c^2)^{1/2}[4\frac{(\mu+\lambda)}{\mu} - \frac{y}{\Gamma}] = 0. \tag{4.1.11}$$

Taking $((\lambda+2\mu)-\rho c^2)^{1/2}$ common, Eq. (4.1.11) becomes

$$((\lambda+2\mu)-\rho c^2)^{1/2}[-y]+(\Gamma)^{1/2}(\mu-\rho c^2)^{1/2}[4\frac{(\mu+\lambda)}{\mu}-\frac{y}{\Gamma}]=0. \tag{4.1.12}$$

Dividing Eq. (4.1.12) by $(\mu)^{1/2}$, and using Eqs. (2.4.24) and (2.4.26), Eq. (4.1.12) takes the form

$$(\frac{1}{\Gamma}-y)^{1/2}(-y)+(\Gamma)^{1/2}(1-y)^{1/2}[4\frac{(\mu+\lambda)}{\mu}-\frac{y}{\Gamma}]=0. \tag{4.1.13}$$

Multiplying through by $(\Gamma)^{1/2}$, we have from Eq. (4.1.13)

$$(1-\Gamma y)^{1/2}(-y)+(\Gamma)(1-y)^{1/2}[4\frac{(\mu+\lambda)}{\mu}-\frac{y}{\Gamma}]=0. \tag{4.1.14}$$

Multiplying Eq. (4.1.14) through with $(1-y)^{1/2}$, we get

$$\begin{aligned}
& -y(1-y)^{1/2}(1-\Gamma y)^{1/2}+(1-y)[4\Gamma\frac{(\lambda+2\mu-\mu)}{\mu}-y]=0,\\
\implies & -y(1-y)^{1/2}(1-\Gamma y)^{1/2}+(1-y)[4\Gamma(\frac{1}{\Gamma}-1)-y]=0,\\
\implies & -y(1-y)^{1/2}(1-\Gamma y)^{1/2}+(1-y)[4-4\Gamma-y]=0,\\
\implies & (1-y)[4(1-\Gamma)-y]-y(1-y)^{1/2}(1-\Gamma y)^{1/2}=0,
\end{aligned}$$

or

$$(1-y)[4(1-\Gamma)-y]-y\sqrt{1-y}\sqrt{1-\Gamma y}=0,$$

which is the same equation as Eq. (4.1.1), derived through Eq. (4.1.5) by taking $\tau=0$.

4.2 A cubic equation for Rayleigh wave speed

To convert Eq. (4.1.1) to a cubic equation, Vinh and Ogden [6] introduced the notation

$$\xi=\sqrt{\frac{1-\Gamma y}{1-y}}, \tag{4.2.1}$$

where y in terms of variable ξ is given by

$$y = \frac{1-\xi^2}{\Gamma - \xi^2}. \tag{4.2.2}$$

Equation (4.1.1) after division through with $(1-y)$ gives

$$[4(1-\Gamma) - y] - y\sqrt{\frac{1-\Gamma y}{1-y}} = 0. \tag{4.2.3}$$

Using Eqs. (4.2.1) and (4.2.2), Eq. (4.2.3) becomes

$$4 - 4\Gamma - \frac{1-\xi^2}{\Gamma-\xi^2} - (\frac{1-\xi^2}{\Gamma-\xi^2})\xi = 0, \tag{4.2.4}$$

or

$$4(\Gamma-\xi^2) - 4\Gamma(\Gamma-\xi^2) - (1-\xi^2) - (1-\xi^2)\xi = 0. \tag{4.2.5}$$

Equation (4.2.5) can now be rewritten as

$$g(\xi) = \xi^3 + b_2\xi^2 - \xi + b_0 = 0, \tag{4.2.6}$$

where

$$b_2 = 4\Gamma - 3, \quad b_0 = -(1-2\Gamma)^2, \text{ and } 1 < \xi < \infty. \tag{4.2.7}$$

From Eq. (4.1.1), $y \in (0,1)$, and hence from Eq. (4.2.1), $\xi \in (1,\infty)$. It is important to note that, the coefficients b_0 and b_2 are different in values from the coefficients defined in Eq. (3.1.2).

4.2.1 Uniqueness of solution of the cubic equation (4.2.6) in the interval $(1,\infty)$

Since from Eq. (4.2.6), $g(1) = -4(1-\Gamma)^2 < 0$, and $g(\xi) \longrightarrow \infty$ as $\xi \longrightarrow \infty$, there must be at least one real root of Eq. (4.2.6) that lies in the interval $(1,\infty)$, also from Eq. (4.2.6), $g(0) < 0$.

Through Descarte's rule of signs [22], maximum number of possible positive and negative real roots of Eq. (4.2.6) can be determined. Let m be the total number of sign changes that occur in Eq. (4.2.6). Since from Eq. (4.2.7), $b_2 < 0$ and $b_0 < 0$ for $\Gamma \in (0, 3/4)$ and hence from Eq. (4.2.6), $m = 1$. So the maximum number of possible positive real roots of Eq. (4.2.6) is one. To find negative real roots replacing ξ by $-\xi$ in Eq. (4.2.6), we have

$$g(-\xi) = -\xi^3 + b_2\xi^2 + \xi + b_0 = 0. \tag{4.2.8}$$

Hence from Eq. (4.2.8), $m = 2$. So the maximum number of possible negative real roots of Eq. (4.2.6) are two.

Through second derivative test and product of roots of $g'(\xi)$, one can find the uniqueness of the solution of Eq. (4.2.6) in the interval $(1, \infty)$. For this purpose taking the first derivative of $g(\xi)$ and putting it equal to zero, we have

$$g'(\xi) = 3\xi^2 + 2b_2\xi - 1 = 0. \tag{4.2.9}$$

The equation $g'(\xi) = 0$ has discriminant equal to $4(b_2^2+3) > 0$, which is representing that there are two distinct real roots of Eq. (4.2.9) and are given by the quadratic formula as

$$\xi = \frac{-(2b_2) \pm \sqrt{(2b_2)^2 - 4(3)(-1)}}{2(3)},$$

$$\xi_1 = \frac{-b_2 + \sqrt{b_2^2+3}}{3}, \qquad \xi_2 = \frac{-b_2 - \sqrt{b_2^2+3}}{3}. \tag{4.2.10}$$

By taking the second derivative of $g(\xi)$, we have

$$g''(\xi) = 6\xi + 2b_2. \tag{4.2.11}$$

By substituting ξ_1 in Eq. (4.2.11), we get

$$g''(\xi) = 6(\tfrac{-b_2+\sqrt{b_2^2+3}}{3}) + 2b_2,$$

$$g''(\xi) = \sqrt{b_2^2+3} > 0. \tag{4.2.12}$$

Therefore, by second derivative test $\xi_1 = \xi_{min}$, which is denoting the point of minimum value of the function $g(\xi)$. After substituting ξ_2 in Eq. (4.2.11), we have

$$\begin{aligned} g''(\xi) &= 6(\tfrac{-b_2-\sqrt{b_2^2+3}}{3}) + 2b_2, \\ g''(\xi) &= -\sqrt{b_2^2+3} < 0. \end{aligned} \tag{4.2.13}$$

Therefore, through second derivative test $\xi_2 = \xi_{max}$, which is denoting the point of maximum value of the function $g(\xi)$.

In order to analyze the nature of the roots of Eq. (4.2.6), one can take the product of two roots of Eq. (4.2.9) as

$$\xi_{min}\xi_{max} = (\frac{-b_2+\sqrt{b_2^2+3}}{3})(\frac{-b_2-\sqrt{b_2^2+3}}{3}) = \frac{-1}{3} < 0. \tag{4.2.14}$$

Equation (4.2.14) shows that at least one of the turning points (ξ_{min} and ξ_{max}) is negative. Since from Eq. (4.2.7), $b_2 < 0$ and hence from Eq. (4.2.10), $\xi_{min} > \xi_{max}$, therefore, $\xi_{max} < 0 < \xi_{min}$. Hence the real root of Eq. (4.2.6) is unique in the interval $(1, \infty)$.

It is observed that the unique real root that lies in interval $(1, \infty)$ is the largest one, therefore, the largest real root will relate to Rayleigh wave speed in the case of two or three distinct real roots of Eq. (4.2.6).

4.3 Rayleigh wave speed formula for compressible isotropic elastic solids by Vinh and Ogden

It is stated in the last section that the largest (real) root of cubic Eq. (4.2.6) will relate to Rayleigh wave speed. So for finding the largest real root of Eq. (4.2.6), it is convenient to simplify this equation by presenting a new unknown x, hence, resulting in the absence of square of unknown in the equation, where

$$x = \xi + \frac{1}{3}b_2. \tag{4.3.1}$$

Substitution of the value of ξ from Eq. (4.3.1) in the Eq. (4.2.6) results in

$$(x - \frac{1}{3}b_2)^3 + b_2(x - \frac{1}{3}b_2)^2 - (x - \frac{1}{3}b_2) + b_0 = 0, \tag{4.3.2}$$

which after simplification becomes

$$x^3 + x(-\frac{b_2^2}{3} - 1) + \frac{1}{27}(2b_2^3 + 9b_2 + 27b_0) = 0. \tag{4.3.3}$$

Equation (4.3.3) can written in the form

$$x^3 - 3[\frac{1}{9}(b_2^2 + 3)]x + \frac{1}{27}(2b_2^3 + 9b_2 + 27b_0) = 0. \tag{4.3.4}$$

Hence Eq. (4.2.6) is converted into depressed cubic equation as

$$x^3 - 3u^2x + v = 0, \tag{4.3.5}$$

where the coefficients u and v are defined by

$$u = \frac{1}{3}\sqrt{b_2^2 + 3}\ , \qquad v = \frac{1}{27}(2b_2^3 + 9b_2 + 27b_0). \tag{4.3.6}$$

The coefficient u can also be written as

$$u = \frac{1}{2}(\xi_{min} - \xi_{max}), \tag{4.3.7}$$

where ξ_{min} and ξ_{max} are given by Eq. (4.2.10). Equation (4.3.5) can also be written in the form

$$x^3 + 3Ux - 2V = 0, \tag{4.3.8}$$

where

$$U = -u^2, \qquad V = -\frac{1}{2}v\ . \tag{4.3.9}$$

In order to solve the Eq. (4.3.8), the unknowns η and ζ are introduced using the relation

$$x = \eta + \zeta. \tag{4.3.10}$$

So Eq. (4.3.8) becomes

$$(\eta + \zeta)^3 + 3U(\eta + \zeta) - 2V = 0. \tag{4.3.11}$$

After simplification, we get

$$\eta^3 + \zeta^3 + 3(\eta\zeta + U)(\eta + \zeta) - 2V = 0. \tag{4.3.12}$$

Defining another relation between η and ζ to solve the Eq. (4.3.12), for this purpose taking

$$3(\eta\zeta + U) = 0. \tag{4.3.13}$$

$$\text{Since} \quad 3 \neq 0, \quad \text{so} \quad \eta\zeta + U = 0, \tag{4.3.14}$$

$$\eta\zeta = -U. \tag{4.3.15}$$

Equation (4.3.12) becomes

$$\eta^3 + \zeta^3 - 2V = 0, \tag{4.3.16}$$

which after using Eq. (4.3.15) becomes

$$\eta^6 - 2V\eta^3 - U^3 = 0. \tag{4.3.17}$$

Equation (4.3.17) is quadratic equation in η^3 and its roots are given by quadratic formula as

$$\eta^3 = \frac{-(-2V) \pm \sqrt{(-2V)^2 - 4(1)(-U^3)}}{2(1)},$$

$$\implies \eta^3 = V \pm \sqrt{V^2 + U^3}. \tag{4.3.18}$$

If $\eta^3 = V + \sqrt{V^2 + U^3}$, then from Eq. $(4.3.15)_2$, we have

$$\begin{aligned}\zeta^3 &= -\frac{U^3}{V + \sqrt{V^2 + U^3}}, \\ &= -\frac{U^3}{V + \sqrt{V^2 + U^3}} \times \frac{V - \sqrt{V^2 + U^3}}{V - \sqrt{V^2 + U^3}}, \\ &= -\frac{U^3(V - \sqrt{V^2 + U^3})}{V^2 - (V^2 + U^3)},\end{aligned}$$

$$\zeta^3 = V - \sqrt{V^2 + U^3}. \tag{4.3.19}$$

If $\eta^3 = V - \sqrt{V^2 + U^3}$, then by following the same steps we get $\zeta^3 = V + \sqrt{V^2 + U^3}$. Without loss of generality we assume

$$\eta^3 = V + \sqrt{V^2 + U^3}, \qquad \zeta^3 = V - \sqrt{V^2 + U^3}. \tag{4.3.20}$$

Let

$$P = \sqrt[3]{V + \sqrt{V^2 + U^3}}, \qquad Q = \sqrt[3]{V - \sqrt{V^2 + U^3}}. \tag{4.3.21}$$

Then η and ζ each has three possible values. The values of η from Eq. $(4.3.20)_1$ are calculated by

$$\begin{aligned}&\eta^3 - (V + \sqrt{V^2 + U^3}) = 0, \\ \Longrightarrow\ &\eta^3 - [(V + \sqrt{V^2 + U^3})^{\frac{1}{3}}]^3 = 0, \\ \Longrightarrow\ &\eta^3 - (P)^3 = 0, \\ \Longrightarrow\ &(\eta - P)(\eta^2 + \eta P + P^2) = 0, \\ \text{where}\ \ &(\eta - P) = 0 \implies \eta = P, \\ \text{and}\ \ &(\eta^2 + \eta P + P^2) = 0 \implies \eta = \frac{-P \pm \sqrt{P^2 - 4P^2}}{2} \\ &\qquad\qquad\qquad\qquad\qquad\ \ = P(\frac{-1 \pm i\sqrt{3}}{2}).\end{aligned}$$

So the three possible values of η are given by

$$\eta_1 = P, \quad \eta_2 = (\frac{-1}{2} + i\frac{\sqrt{3}}{2})P, \quad \eta_3 = (\frac{-1}{2} - i\frac{\sqrt{3}}{2})P. \tag{4.3.22}$$

By adopting the same procedure, the three possible values of ζ from Eq. $(4.3.20)_2$ are given by

$$\zeta_1 = Q, \quad \zeta_2 = (\frac{-1}{2} + i\frac{\sqrt{3}}{2})Q, \quad \zeta_3 = (\frac{-1}{2} - i\frac{\sqrt{3}}{2})Q \; . \tag{4.3.23}$$

The restriction $\eta\zeta = -U$ allows only three possible combinations of η and ζ to satisfy the relation (4.3.10).

Consider the first combination as $\eta_1 + \zeta_1 = x_1$ and it is satisfying the condition $\eta_1\zeta_1 = -U$ which is shown as

$$\begin{aligned} \eta_1\zeta_1 &= \sqrt[3]{V + \sqrt{V^2 + U^3}} \times \sqrt[3]{V - \sqrt{V^2 + U^3}}, \\ &= \sqrt[3]{V^2 - (V^2 + U^3)}, \\ &= -U. \end{aligned} \tag{4.3.24}$$

The second combination of $\eta_2 + \zeta_3 = x_2$ is also satisfying the condition $\eta_2\zeta_3 = -U$ which is shown as

$$\begin{aligned} \eta_2\zeta_3 &= (\tfrac{-1}{2} + i\tfrac{\sqrt{3}}{2})P \times (\tfrac{-1}{2} - i\tfrac{\sqrt{3}}{2})Q, \\ &= (-\tfrac{1}{2})^2 - i^2(\tfrac{\sqrt{3}}{2})^2 PQ, \\ &= PQ = -U. \end{aligned} \tag{4.3.25}$$

Similarly, the third possible combination is given by $\eta_3 + \zeta_2 = x_3$. So the three possible roots of Eq. (4.3.5) and are given by

$$x_1 = P + Q, \quad x_2 = -\frac{1}{2}(P+Q) + i\frac{\sqrt{3}}{2}(P-Q), \quad x_3 = -\frac{1}{2}(P+Q) - i\frac{\sqrt{3}}{2}(P-Q), \tag{4.3.26}$$

where

$$P = \sqrt[3]{V + \sqrt{D}}, \quad Q = \sqrt[3]{V - \sqrt{D}}, \quad D = V^2 + U^3, \tag{4.3.27}$$

where D is the discriminant of Eq. (4.3.5), and V and U are given by Eq. (4.3.9).

4.3.1 Discussion about cases of discriminant of cubic equation (4.3.5)

In this section, the nature of the solutions (4.3.26) of Eq. (4.3.5) is determined according to the sign of its discriminant D. While discussing these cases, u and v are considered as real numbers in the solutions (4.3.26). Vinh and Ogden [6] showed that in case of ($D > 0$, $D = 0$, $D < 0$) the largest real root of Eq. (4.3.5) is represented by

$$x_0 = \sqrt[3]{V + \sqrt{D}} + \sqrt[3]{V - \sqrt{D}}, \tag{4.3.28}$$

where roots are generally assumed as complex roots taking their principal values.

Case 1: D > 0

When $D > 0$, there are one real and two complex conjugates roots of Eq. (4.3.5), given by Eq. (4.3.26). Denoting this real root by x_0, given by x_1 in Eq. $(4.3.26)_1$, where the cube roots are real. In the case of $D > 0$, $\sqrt{D}$ is taken real and positive. Using Eqs. (4.2.7), $(4.3.6)_2$ and $(4.3.9)_2$, we find that $V > 0$. Also, from Eqs. $(4.3.9)_1$ and $(4.3.27)_3$, we have $U < 0$, and hence $V > \sqrt{D}$. Therefore,

$$V + \sqrt{D} > 0, \qquad V - \sqrt{D} > 0. \tag{4.3.29}$$

Equation (4.3.29) shows that $V + \sqrt{D}$ and $V - \sqrt{D}$ are real and positive numbers, so according to the section 2.1.4, their principal cube roots are $\sqrt[3]{V + \sqrt{D}}$ and $\sqrt[3]{V - \sqrt{D}}$ respectively. Hence Eq. (4.3.28) is valid in this case.

Case 2: D = 0

When $D = 0$, then all the three roots of Eq. (4.3.5) are real and at least two of them are equal. As $V > 0$, and also from Eqs. $(4.3.6)_1$ and (4.3.7), $u > 0$, therefore, Eqs. (4.3.9) and $(4.3.27)_3$ implies $v = -2u^3$. Equation (4.3.5) then becomes

$$x^3 - 3u^2x - 2u^3 = 0. \tag{4.3.30}$$

As $x = 2u$ is satisfying the Eq. (4.3.30), therefore, $(x - 2u)$ is the first factor, while the other two factors are given by synthetic division

$$\begin{array}{c|cccc} 2u & 1 & 0 & -3u^2 & -2u^3 \\ & & 2u & 4u^2 & 2u^3 \\ \hline & 1 & 2u & u^2 & 0 \end{array}$$

Here, quotient $= x^2 + 2ux + u^2$ and remainder$= 0$. Hence, the other two factors are given by $(x+u)^2$. Therefore, the three real roots of Eq. (4.3.30) are

$$x_1 = 2u, \quad x_2 = -u, \quad x_3 = -u, \tag{4.3.31}$$

where x_2 and x_3 are equal. Since $u > 0$, therefore, $x_1 = 2u$ is the largest real root. Lets denote this largest real root by x_0. Equation $(4.3.9)_2$ implies $V = -\frac{1}{2}(-2u^3) = u^3$ and consequently Eq. (4.3.28) gives

$$x_0 = \sqrt[3]{u^3} + \sqrt[3]{u^3} = 2u. \tag{4.3.32}$$

Hence Eq. (4.3.28) is valid.

Case 3: $\mathbf{D < 0}$

When $D < 0$, then all the three roots of Eq. (4.3.5) are real and distinct. In the case of $D < 0$, we can take $\sqrt{D} = i\sqrt{-D}$ and hence Eq. (4.3.27) implies that

$$\begin{aligned} P &= \sqrt[3]{V + i\sqrt{-D}}, \\ Q &= \sqrt[3]{V - i\sqrt{-D}}, \end{aligned} \tag{4.3.33}$$

where P and Q have three possible values. To find these values, $|P|$ is calculated as

$$|V + i\sqrt{-D}| = \sqrt{-U^3} = u^3, \qquad (\because\ U = -u^2). \tag{4.3.34}$$

Hence

$$|P| = |V + i\sqrt{-D}|^{\frac{1}{3}} = (u^3)^{\frac{1}{3}} = u. \tag{4.3.35}$$

Similarly by following the same procedure the value of $|Q|$ is

$$|Q| = |V - i\sqrt{-D}|^{\frac{1}{3}} = (u^3)^{\frac{1}{3}} = u. \tag{4.3.36}$$

Since $V > 0$, thus the phase angle of $V + i\sqrt{-D}$ belongs to $(0, \pi/2)$. Assuming θ as the phase angle of complex cube root P which belongs to $(0, \pi/6)$. From Equation $(4.3.33)_2$, as Q is complex conjugate of P, so the phase angle of complex cube root Q is equal to $-\theta$. By using equation (2.1.5) and Eq. (4.3.35), the three possible values of complex cube root P are given by

$$P_1 = ue^{i\theta}, \quad P_2 = ue^{i(\theta+\frac{2\pi}{3})}, \quad P_3 = ue^{i(\theta+\frac{4\pi}{3})}, \quad \theta \in (0, \pi/6). \tag{4.3.37}$$

According to section 2.1.4, P_1 is the principal value of the complex cube root P. The three possible values of complex cube root Q by using Eq. (2.1.5) and Eq. (4.3.36) are given by

$$Q_1 = ue^{-i\theta}, \quad Q_2 = ue^{i(-\theta+\frac{2\pi}{3})}, \quad Q_3 = ue^{i(-\theta+\frac{4\pi}{3})}, \quad \theta \in (0, \pi/6). \tag{4.3.38}$$

Similarly through section 2.1.4, Q_1 is the principal value of the complex cube root Q.

From the possible values of complex cube roots P and Q, Vinh and Ogden [6] only take their principal values in this case. On substituting these principal values from Eq. $(4.3.37)_1$ and Eq. $(4.3.38)_1$ of complex cube roots P and Q, respectively, in Eq. (4.3.26), we have

$$x_1 = u(e^{i\theta} + e^{-i\theta}) = 2u\cos\theta, \tag{4.3.39}$$

$$x_2 = u(e^{i(\theta+\frac{2\pi}{3})} + e^{-i(\theta+\frac{2\pi}{3})}) = 2u\cos(\theta + \frac{2\pi}{3}), \tag{4.3.40}$$

$$x_3 = u(e^{i(\theta-\frac{2\pi}{3})} + e^{-i(\theta-\frac{2\pi}{3})}) = 2u\cos(\theta - \frac{2\pi}{3}) = 2u\cos(\theta + \frac{4\pi}{3}), \tag{4.3.41}$$

where Eqs. (4.3.39), (4.3.40) and (4.3.41) represents the three distinct real roots of Eq. (4.3.5). By plotting the graphs of these roots in MATHEMATICA it is observed that for $\theta \in (0, \pi/6)$, $x_1 > x_3 > x_2$, shows that $x_1 = x_0$ is the largest real root of Eq. (4.3.5) in this case. Hence Eq. (4.3.28) is valid.

The value of V can be expressed as a function of parameter Γ with the help of Eqs. (4.2.7), $(4.3.6)_2$ and $(4.3.9)_2$ as

$$\begin{aligned} V &= -\tfrac{1}{2}v = -\tfrac{1}{2}[\tfrac{1}{27}(2b_2^3 + 9b_2 + 27b_0)], \\ &= -\tfrac{1}{2}[\tfrac{1}{27}(2(4\Gamma - 3)^3 + 9(4\Gamma - 3) + 27(-(1 - 2\Gamma)^2))]. \end{aligned} \tag{4.3.42}$$

After simplification, we have

$$V = \frac{2}{27}(27 - 90\Gamma + 99\Gamma^2 - 32\Gamma^3) \ . \tag{4.3.43}$$

Similarly the value of D can also be expressed as a function of parameter Γ with the help of Eqs. (4.2.7), (4.3.6), (4.3.9) and $(4.3.27)_3$ as

$$\begin{aligned} D &= [-\tfrac{1}{2}[\tfrac{1}{27}(2b_2^3 + 9b_2 + 27b_0)]]^2 + [-(\tfrac{1}{9}(b_2^2 + 3))]^3, \\ &= [\tfrac{1}{4}(\tfrac{1}{729}(4b_2^6 + 36b_2^4 + 81b_2^2 + 108b_2^3b_0 + 486b_2b_0 + 729b_0^2))] \\ &\quad -[\tfrac{1}{729}(b_2^6 + 27 + 9b_2^4 + 27b_2^2)]. \end{aligned} \tag{4.3.44}$$

After simplification, Eq. (4.3.44) yields

$$D = \frac{4}{27}(-64\Gamma^5 + 235\Gamma^4 - 340\Gamma^3 + 242\Gamma^2 - 84\Gamma + 11). \tag{4.3.45}$$

By synthetic division, we have

$$D = \frac{4}{27}(1 - \Gamma)^2(11 - 62\Gamma + 107\Gamma^2 - 64\Gamma^3) \ . \tag{4.3.46}$$

The Rayleigh wave speed formula for compressible isotropic elastic solids by using

Eqs. (2.4.24), (4.1.1), (4.2.1), (4.3.1) and (4.3.28) is derived by the following steps

$$
\begin{aligned}
&4(1-\Gamma) - y - y\xi = 0,\\
&4(1-\Gamma) - \frac{\rho c^2}{\mu}(1+\xi) = 0,\\
&\frac{\rho c^2}{\mu} = 4(1-\Gamma)(1+\xi)^{-1},\\
&\quad\ = 4(1-\Gamma)(1+x-\frac{1}{3}b_2)^{-1},\\
&\quad\ = 4(1-\Gamma)(1+x-\frac{1}{3}(4\Gamma-3))^{-1},
\end{aligned}
$$

$$
\frac{\rho c^2}{\mu} = 4(1-\Gamma)\big(2-\frac{4}{3}\Gamma + \sqrt[3]{V+\sqrt{D}} + \sqrt[3]{V-\sqrt{D}}\big)^{-1}, \tag{4.3.47}
$$

where the roots in Eq. (4.3.47) are complex cube roots taking their principal values, and V and D are specified by Eq. (4.3.43) and Eq. (4.3.46), respectively.

4.4 Connection with Malischewsky's formula

In this section, Rayleigh wave speed formula given by Malischewsky [5] for real root of Rayleigh wave equation (3.1.1), represented by Eq. (3.1.3), is connected with Vinh and Ogden [6] formula. Before explaining the formula (3.1.3), its uniqueness in the interval $(0, 1)$ is determined.

4.4.1 Uniqueness of solution of the cubic equation (3.1.1) in the interval $(0, 1)$

From Eq. (3.1.1), $f(0) < 0$ and $f(1) > 0$, therefore, there must be at least one real root of Eq. (3.1.1) that lies in the interval $(0, 1)$. Through Descartes' rule of signs, the maximum number of positive and negative real roots of cubic equation (3.1.1) can be determined. Let m be the total number of sign change that occur in Eq. (3.1.1). Since from Eq. (3.1.2), $b_2 < 0$, $b_0 < 0$ and $b_1 > 0$ for $\Gamma \in (0, 3/4)$. Therefore, from Eq. (3.1.1), $m = 3$. Hence, maximum number of possible positive real roots of Eq. (3.1.1) are three. Replacing y by $-y$ for finding the maximum number of negative real roots

$$f(-y) = -y^3 + b_2y^2 - b_1y + b_0 = 0. \tag{4.4.1}$$

Equation (4.4.1) shows that $m = 0$, i.e., there is no negative real root of Eq. (3.1.1) and hence, $f(y)$ lies on the positive side of y-axis. By using the second derivative test and the product of roots of $f'(y)$, one can find the uniqueness of solution of Eq. (3.1.1) in the interval $(0, 1)$. For this purpose, taking the first derivative of $f(y)$ of Eq. (3.1.1) and putting it equal to zero, we have

$$f'(y) = 3y^2 - 16y + 8(3 - 2\Gamma) = 0. \tag{4.4.2}$$

For $\Gamma \leq 1/6$, the discriminant of Eq. (4.4.2) is $-32+192\Gamma \leq 0$, and hence $f'(y) \geq 0$, which means that there is no turning point of the graph of cubic Eq. (3.1.1) and function is increasing, therefore, real root of Eq. (3.1.1) is unique in the interval $(0, 1)$ for $\Gamma \leq 1/6$. For $\Gamma > 1/6$, the discriminant of Eq. (4.4.2) is $-32 + 192\Gamma > 0$, then the two roots of Eq. (4.4.2) are real and distinct, therefore, there are two turning points of the graph of cubic Eq. (4.4.2) for $\Gamma > 1/6$. These two real roots of Eq. (4.4.2) are given by quadratic formula as

$$y_1 = \frac{8 + 2\sqrt{12\Gamma - 2}}{3}, \qquad y_2 = \frac{8 - 2\sqrt{12\Gamma - 2}}{3}. \tag{4.4.3}$$

By taking second derivative of $f(y)$, we have

$$f''(y) = 6y - 16. \tag{4.4.4}$$

Now by substituting y_1 in $f''(y)$, we get

$$f''(y_1) = 4\sqrt{12\Gamma - 2}. \tag{4.4.5}$$

For $\Gamma > 1/6$

$$f''(y_1) = 4\sqrt{12\Gamma - 2} > 0, \tag{4.4.6}$$

and hence $y_1 = y_{min}$ which is denoting the point of minimum value of the function $f(y)$. After substituting y_2 in $f''(y)$, we have

$$f''(y_2) = -4\sqrt{12\Gamma - 2}. \tag{4.4.7}$$

For $\Gamma > 1/6$

$$f''(y_2) = -4\sqrt{12\Gamma - 2} < 0, \tag{4.4.8}$$

and consequently $y_2 = y_{max}$ which is denoting the point of maximum value of the function $f(y)$. The product of two roots (turning points) of Eq. (4.4.2) by using Eq. (4.4.3) is given by

$$y_{min}y_{max} = \tfrac{8}{3}(3 - 2\Gamma). \tag{4.4.9}$$

Since for $\Gamma > 1/6$, $(3 - 2\Gamma) > 1$, we have

$$y_{min}y_{max} = \frac{8}{3}(3 - 2\Gamma) > \frac{8}{3}. \tag{4.4.10}$$

Since from Descarte's method, it is observed that function lies on the positive side of y-axis, therefore, both turning points (y_{min} and y_{max}) are positive. From Eq. (4.4.3), it is obvious that $y_{min} > y_{max}$ for $\Gamma \in (1/6, 3/4)$. There exists two cases

Case 1: $\mathbf{y_{max} < 1}$

From Eq. (4.4.10)

$$\begin{aligned} & y_{min} > \tfrac{8}{3}\tfrac{1}{y_{max}}, \\ \implies & y_{min} > \tfrac{8}{3} > 1. \end{aligned} \tag{4.4.11}$$

Hence

$$0 < y_{max} < 1 < y_{min} \tag{4.4.12}$$

Case 2: $\mathbf{y_{max} \geq 1}$

Since $y_{min} > y_{max}$, therefore

$$1 \leq y_{max} < y_{min} \tag{4.4.13}$$

From both these cases and by using the fact that $f(0) < 0$ and $f(1) > 0$, it is obvious that there is a unique solution of Eq. (3.1.1) in the interval $(0, 1)$. Since function is on the positive side of y-axis and other real roots are lying outside the interval $(0, 1)$, therefore, if there are two or three distinct real roots of Eq. (3.1.1), then the smallest real one will relate to Rayleigh wave speed.

To explain the Malischewsky's formula (3.1.3) on the basis of study provided in section (4.3), Vinh and Ogden [6] introduced the variable x

$$x = y + \frac{1}{3}b_2 \tag{4.4.14}$$

where b_2 is given by Eq. $(3.1.2)_1$. The expressions for x in Eqs. (4.3.1) and (4.4.14) are not same. By substituting the value of variable y from Eq. (4.4.14) in Eq. (3.1.1) and by using Eq. (3.1.2), we have

$$x^3 + (\frac{8}{3}(1 - 6\Gamma))x + \frac{1}{27}(16(17 - 45\Gamma)) = 0. \tag{4.4.15}$$

Equation (4.4.15) can also be written in the form

$$x^3 - 3(\frac{8}{9}(6\Gamma - 1))x + \frac{1}{27}(16(17 - 45\Gamma)) = 0. \tag{4.4.16}$$

so Eq. (3.1.1) is converted into

$$x^3 - 3u^2x + v = 0, \tag{4.4.17}$$

where

$$u^2 = \frac{8}{9}(6\Gamma - 1), \quad v = \frac{1}{27}(16(17 - 45\Gamma)). \tag{4.4.18}$$

It is important to note that the values of coefficients u and v in Eq. (4.3.6) are different from the values given by Eq. (4.4.18).

In order to find the roots of cubic Eq. (4.4.17), we are following the same procedure as given from Eq. (4.3.8) to Eq. (4.3.26). So the three possible roots of Eq. (4.4.17) are

$$x_1 = P + Q, \quad x_2 = -\frac{1}{2}(P+Q) + i\frac{\sqrt{3}}{2}(P-Q), \quad x_3 = -\frac{1}{2}(P+Q) - i\frac{\sqrt{3}}{2}(P-Q), \tag{4.4.19}$$

where

$$\begin{aligned} &P = \sqrt[3]{V+\sqrt{D}}, \qquad Q = \sqrt[3]{V-\sqrt{D}}, \\ &D = V^2 + U^3 = \frac{1}{27}(64(11 - 62\Gamma + 107\Gamma^2 - 64\Gamma^3)), \\ &V = -\frac{1}{2}v = \frac{1}{27}(8(45\Gamma - 17)), \qquad U = -u^2 = \frac{1}{9}(8(1-6\Gamma)), \end{aligned} \tag{4.4.20}$$

where D is the discriminant of Eq. (4.4.17). Note that the roots given in Eq. (4.3.26) are unlike the roots given in Eq. (4.4.19), also the values of the expressions P, Q, D, V, U, given in Eqs. (4.3.27) and (4.3.9) are not same as given by Eq. (4.4.20). The values of $\sqrt{D}$, V and U in terms of $s_i(\Gamma)$ functions defined in Eq. (3.1.4) are given by

$$\begin{aligned} &\sqrt{D} = \tfrac{8}{27}s_1(\Gamma), \qquad V_1 = \tfrac{1}{27}(8(s_2(\Gamma) - s_1(\Gamma)), \\ &V_2 = -\tfrac{8}{27}(s_3(\Gamma) - s_1(\Gamma), \qquad U = 16(s_4(\Gamma)). \end{aligned} \tag{4.4.21}$$

Equation $(4.4.20)_3$ implies that D is a function of Γ, where $D(\Gamma) = 0$ is written as

$$D(\Gamma) = \Gamma^3 - \frac{107}{64}\Gamma^2 + \frac{31}{32}\Gamma - \frac{11}{64} = 0, \qquad \Gamma \in (0, 3/4). \tag{4.4.22}$$

First derivative of $D(\Gamma)$ is given by

$$D'(\Gamma) = 3\Gamma^2 - \frac{107}{32}\Gamma + \frac{31}{32}. \tag{4.4.23}$$

Discriminant of Eq. (4.4.23) is $(\frac{-107}{32})^2 - 12(\frac{31}{32}) < 0$, and hence $D'(\Gamma) > 0$. From Eq. (4.4.22), $D(0) < 0$ and $D(3/4) > 0$, so according to the section 2.2, the real root of Eq. (4.4.22) is unique in the interval $(0, 3/4)$. To find this real root Eq. (4.4.22) can be converted into the simplified form by introducing the variable x^* [4]

$$x^* = \Gamma - \frac{107}{192}, \tag{4.4.24}$$

By substituting the value of Γ from Eq. (4.4.24) into the Eq. (4.4.22), we have

$$(x^* + \frac{107}{192})^3 - \frac{107}{64}(x^* + \frac{107}{192})^2 + \frac{31}{32}(x + \frac{107}{192}) - \frac{11}{64} = 0. \tag{4.4.25}$$

After simplification, Eq. (4.4.25) becomes equal to

$$x^{*3} + \frac{455}{12288}x^* + \frac{77293}{3538944} = 0. \tag{4.4.26}$$

Equation (4.4.26) can be written in the form

$$x^{*3} + u^* x + v^* = 0, \tag{4.4.27}$$

where

$$u^* = \frac{455}{12288}, \quad v^* = \frac{77293}{3538944}. \tag{4.4.28}$$

The discriminant of the Eq. (4.4.27) is given as

$$D^* = \frac{v^{*2}}{4} + \frac{u^{*3}}{27}. \tag{4.4.29}$$

Substitution of the values of u^* and v^* from Eq. (4.4.28) into the Eq. (4.4.29) results in $D^* > 0$, and hence, Eq. (4.4.27) has one real and two complex conjugate roots, consequently, Eq. (4.4.22) has one real root and two complex conjugate roots. As real root is of interest only, so by using Eqs. (2.5.18) and (2.5.26) of section 2.5, the real root of Eq. (4.4.27) is given by

$$x^*{}_1 = \sqrt[3]{-\frac{v^*}{2} + \sqrt{D}} + \sqrt[3]{-\frac{v^*}{2} - \sqrt{D}}. \tag{4.4.30}$$

From equation (4.4.28), (4.4.29) and (4.4.30), $x^*{}_1 = -0.2357955391$. By substituting the value of $x^*{}_1$ in Eq. (4.4.24), the corresponding real root of Eq. (4.4.22) is given by

$$\Gamma^* = \frac{107}{192} - 0.2357955391 = 0.3214984... \tag{4.4.31}$$

hence $\Gamma^* = 0.3214984$ is the real root of Eq. (4.4.22) at which $D(\Gamma) = 0$.

4.4.2 Discussion about different cases of parameter Γ to explain Malischewsky's formula

In this section, the smallest real root of Eq. (4.4.17) is denoted by $x_0{}^*$ by Vinh and Ogden [6], in which the roots are generally assumed as complex roots taking their principal values, and Malischewsky's formula is explained with respect to different cases of parameter Γ.

Case 1: $0 < \Gamma \leq 1/6$.

For this range of Γ, Eq. (4.4.20) implies that $U \geq 0$, $V < 0$ and $D > 0$. Since $D > 0$, therefore, Eq. (4.4.17) has one real and two complex conjugate roots and are given by Eq. (4.4.19). Let this real root is represented by $x_0{}^*$, given by x_1 from Eq. $(4.4.19)_1$, where roots are assumed to be real. In the case of $D > 0$, $\sqrt{D}$ is considered as real and positive. Since $V < 0$ and $U \geq 0$ which implies that

$$V + \sqrt{D} \geq 0, \qquad V - \sqrt{D} < 0, \qquad (\because\ V < \sqrt{D} \ \text{ when } U > 0). \tag{4.4.32}$$

Therefore $x_0{}^*$ can be written in the form

$$x_0{}^* = -\sqrt[3]{-V + \sqrt{D}} + \sqrt[3]{V + \sqrt{D}}. \tag{4.4.33}$$

As $-V + \sqrt{D} > 0$ and $V + \sqrt{D} \geq 0$, so according to section 2.1.4, $\sqrt[3]{-V + \sqrt{D}}$ and $\sqrt[3]{V + \sqrt{D}}$ are principal cube roots.

For $0 < \Gamma < 1/6$, Eq. $(3.1.4)_4$ implies $s_4(\Gamma) > 0$, consequently $sign[s_4(\Gamma)] = 1$, hence Eq. (3.1.3) results in

$$y = \frac{2}{3}[4 - \sqrt[3]{s_3(\Gamma)} + \sqrt[3]{s_2(\Gamma)}]. \tag{4.4.34}$$

From Eq. (4.4.21) and Eq. (4.4.33), we have

$$x_0{}^* = -\sqrt[3]{-(-\frac{8}{27}(s_3(\Gamma) - s_1(\Gamma))) + \frac{8}{27}s_1(\Gamma)} + \sqrt[3]{-\frac{8}{27}(s_3(\Gamma) - s_1(\Gamma)) + \frac{8}{27}s_1(\Gamma)}. \tag{4.4.35}$$

After simplification, we get

$$x_0{}^* = -\frac{2}{3}\sqrt[3]{s_3(\Gamma)} + \frac{2}{3}\sqrt[3]{s_2(\Gamma)}. \tag{4.4.36}$$

Substitution of Eq. (4.4.36) in Eq. (4.4.14) results in Eq. (4.4.34), and hence Eq. (3.1.3) is valid.

For $\Gamma = 1/6$, Eq. $(3.1.4)_4$ implies $s_4(\Gamma) = 0$, hence $sign[s_4(\Gamma)] = 0$, therefore, Eq. (3.1.3) results in

$$y = \frac{2}{3}[4 - \sqrt[3]{s_3(\Gamma)}]. \tag{4.4.37}$$

since $\Gamma = 1/6$, therefore, Eq. $(3.1.4)_2$ implies $s_2(\Gamma) = 0$. On substituting $s_2(\Gamma) = 0$ in Eq. (4.4.36), we have

$$x_0{}^* = -\frac{2}{3}\sqrt[3]{s_3(\Gamma)}. \tag{4.4.38}$$

With the help of Eqs. (4.4.14), (4.4.37) and (4.4.38), we infer that Eq. (3.1.3) is once more valid.

Case 2: $1/6 < \Gamma < \Gamma^*$

Γ^* is given by Eq. (4.4.31). In this case $V < 0$, $U < 0$ and $D > 0$. Since $D > 0$, therefore as mentioned in Case 1, there is only one real root of Eq. (4.4.17), represented by $x_0{}^*$, given by Eq. $(4.4.19)_1$. Since $V < 0$ and $U < 0$ and $\sqrt{D}$ is considered as real and positive, therefore

$$V - \sqrt{D} < 0, \qquad V + \sqrt{D} < 0. \tag{4.4.39}$$

Hence $x_0{}^*$ can be written in the form

$$x_0{}^* = -\sqrt[3]{-V + \sqrt{D}} - \sqrt[3]{-(V + \sqrt{D})}, \tag{4.4.40}$$

where

$$-V+\sqrt{D}>0, \qquad -(V+\sqrt{D})>0. \tag{4.4.41}$$

With the help of section 2.1.4, $\sqrt[3]{-V+\sqrt{D}}$ and $\sqrt[3]{-(V+\sqrt{D})}$ are principal cube roots in Eq. (4.4.40).

In this case Eq. $(3.1.4)_4$ implies that $s_4(\Gamma)<0$, therefore, $sign[s_4(\Gamma)]=-1$, so from Eq. (3.1.3)

$$y=\frac{2}{3}[4-\sqrt[3]{s_3(\Gamma)}-\sqrt[3]{-s_2(\Gamma)}]. \tag{4.4.42}$$

From Eq. (4.4.21) and Eq. (4.4.40), we have

$$x_0{}^*=-\sqrt[3]{-(-\frac{8}{27}(s_3(\Gamma)-s_1(\Gamma)))+\frac{8}{27}s_1(\Gamma)}-\sqrt[3]{-(-\frac{8}{27}(s_3(\Gamma)-s_1(\Gamma))+\frac{8}{27}s_1(\Gamma))}. \tag{4.4.43}$$

By simplifying, Eq. (4.4.43) becomes

$$x_0{}^*=-\frac{2}{3}\sqrt[3]{s_3(\Gamma)}-\frac{2}{3}\sqrt[3]{-s_2(\Gamma)}. \tag{4.4.44}$$

Substitution of Eq. (4.4.44) in Eq. (4.4.14) leads to Eq. (4.4.42), hence Eq. (3.1.3) is valid.

Case 3: $\Gamma=\Gamma^*$

For $\Gamma=0.3214984$, we have $D=0$ and $V<0$. Since $D=0$, therefore all three roots of Eq. (4.4.17) are real and at least two of them are equal. As $D=0$, then Eq. $(4.4.20)_3$ implies $V^2+U^3=0$, and hence $v=2u^3$. Equation (4.4.17) then becomes

$$x^3-3u^2x+2u^3=0, \tag{4.4.45}$$

Since $x=-2u$ is satisfying the Eq. (4.4.45), therefore, $(x+2u)$ is the factor of Eq. (4.4.45). The other two factors are given by synthetic division

$$\begin{array}{r|rrrr} -2u & 1 & 0 & -3u^2 & 2u^3 \\ & & -2u & 4u^2 & -2u^3 \\ \hline & 1 & -2u & u^2 & 0 \end{array}$$

Here, quotient= $x^2 - 2ux + u^2$ and remainder= 0. Hence, the other two factors are as $(x-u)^2$. Therefore, the three real roots of Eq. (4.4.45) are given by

$$x_1 = -2u, \quad x_2 = u, \quad x_3 = u, \tag{4.4.46}$$

where x_2 and x_3 are equal. As $V < 0$ and $v = 2u^3$,hence from Eq. $(4.4.20)_4$, $u > 0$. Therefore, x_1 is the smallest real root in this case. Lets denote this smallest real root by ${x_0}^*$. Equation $(4.4.20)_4$ implies $V = -\frac{1}{2}(2u^3) = -u^3$. Since from Eq. $(3.1.4)_4$, $s_4(\Gamma) < 0$, therefore, $sign[s_4(\Gamma)] = -1$, and hence Eq. (3.1.3) results in

$$y = \frac{2}{3}[4 - \sqrt[3]{17 - 45(\Gamma) + s_1(\Gamma)} - \sqrt[3]{-s_2(\Gamma)}], \tag{4.4.47}$$

since $V = -u^3$, Eq. (4.4.21) implies that $s_1 = \frac{27}{8}\sqrt{D}$ and $s_2(\Gamma) = \frac{27}{8}(-u^3)$, so Eq. (4.4.47) takes the form

$$y = \frac{2}{3}[4 - \sqrt[3]{17 - 45(\Gamma) + \frac{27}{8}\sqrt{D}} - \sqrt[3]{-(\frac{27}{8}(-u^3))}], \tag{4.4.48}$$

since $D = 0$, so after simplification, we have

$$y = \frac{8}{3} - 2u\ . \tag{4.4.49}$$

Substitution of ${x_0}^* = -2u$ in Eq. (4.4.14) results in Eq. (4.4.49), therefore, Eq. (3.1.3) is valid.

Case 4: $\Gamma^* < \Gamma < 3/4$

The case $\Gamma > \Gamma^*$ implies that $D < 0$, therefore, we have three distinct real roots of Eq. (4.4.17). In the case of $D < 0$, $\sqrt{D} = i\sqrt{-D}$, hence Eq. (4.4.20) implies that P and Q are complex cube roots and each of them has three possible values. In this case, the phase angle of $V + i\sqrt{-D}$ lies in the interval $(0, \pi)$, therefore, the phase angle of complex cube root P confined in the interval $(0, \frac{\pi}{3})$. By following the same procedure as given from Eq. (4.3.33) to Eq. (4.3.41), we have

$$x_0{}^* = 2u\cos(\theta + \frac{2\pi}{3}), \qquad \theta \in (0, \pi/3). \tag{4.4.50}$$

Note that the expressions for smallest real root in both equations (4.3.40) and (4.4.50) are different in values with respect to u and phase angle θ. From Eq. $(3.1.4)_4$, $s_4(\Gamma) < 0$, consequently, $sign[s_4(\Gamma)] = -1$, and by using Eq. (3.1.4), Eq. (3.1.3) can be written in the form

$$y = \frac{2}{3}[4 - \sqrt[3]{s_3(\Gamma)} - \sqrt[3]{17 - 45(\Gamma) + s_1(\Gamma) - 2s_1(\Gamma)}]. \tag{4.4.51}$$

Equation (4.4.51) can also be written as

$$y = \frac{8}{3} - \sqrt[3]{-[-\frac{8}{27}(s_3(\Gamma) - s_1(\Gamma))] + \frac{8}{27}s_1(\Gamma)} - \sqrt[3]{-[-\frac{8}{27}(s_3(\Gamma) - s_1(\Gamma)) + \frac{8}{27}s_1(\Gamma)]}. \tag{4.4.52}$$

After using Eq. $(4.4.21)_{(1,2)}$, Eq. (4.4.52) takes the form

$$y = \frac{8}{3} - \sqrt[3]{-V + \sqrt{D}} - \sqrt[3]{-(V + \sqrt{D})}. \tag{4.4.53}$$

By using Eqs. (4.4.14), (4.4.50) and (4.4.53), we have

$$-\sqrt[3]{-V + \sqrt{D}} - \sqrt[3]{-(V + \sqrt{D})} = 2u\cos(\theta + \frac{2\pi}{3}), \qquad \theta \in (0, \pi/3). \tag{4.4.54}$$

For the validity of Eq. (3.1.3), we show that left side of Eq. (4.4.54) is equal to its right side. If 3θ is the phase angle of $V + \sqrt{D} = V + i\sqrt{-D}$, we have

$$Arg(V + \sqrt{D}) = 3\theta, \quad Arg(V - \sqrt{D}) = -3\theta, \quad \theta \in (0, \pi/3), \tag{4.4.55}$$

where $V - \sqrt{D} = V - i\sqrt{-D}$ is complex conjugate of $V + \sqrt{D}$. Hence

$$Arg[-(V + \sqrt{D})] = 3\theta - \pi, \quad Arg[-(V - \sqrt{D})] = -3\theta + \pi, \quad \theta \in (0, \pi/3). \tag{4.4.56}$$

Since from Eq. (4.3.35) and Eq. (4.3.36), $|V+\sqrt{D}|^{\frac{1}{3}}=u$ and also $|V-\sqrt{D}|^{\frac{1}{3}}=u$, therefore

$$|-(V+\sqrt{D})|^{\frac{1}{3}}=u, \quad |-(V-\sqrt{D})|^{\frac{1}{3}}=u. \tag{4.4.57}$$

By using Eqs. (2.1.5), (4.4.56) and (4.4.57), we have

$$\sqrt[3]{-(V+\sqrt{D})}=ue^{i(\theta-\pi/3)}, \quad \sqrt[3]{-(V-\sqrt{D})}=ue^{i(-\theta+\pi/3)}, \tag{4.4.58}$$

where roots are principal cube roots. After substitution of Eq. (4.4.58) in Eq. (4.4.54), we get

$$\begin{aligned} & -ue^{i(-\theta+\pi/3)}-ue^{i(\theta-\pi/3)}=2u\cos(\theta+\tfrac{2\pi}{3}), \\ \Longrightarrow\quad & -2u\cos(\pi+(\theta-\tfrac{\pi}{3}))=2u\cos(\theta+\tfrac{2\pi}{3}), \\ \Longrightarrow\quad & 2u\cos(\theta+\tfrac{2\pi}{3})=2u\cos(\theta+\tfrac{2\pi}{3}), \quad (\because\ \cos(\theta+\pi)=-\cos\theta). \end{aligned} \tag{4.4.59}$$

Hence Eq. (3.1.3) holds in this case.

It is noticed that, all above cases are expressed by a single formula y which is given by Eq. (3.1.3).

4.4.3 Equivalence through graphs

It is really difficult to prove the indistinguishability of both formulae by hand. Therefore, we plot the solutions of both as a function of parameter Γ, and the plots show the equivalence of both formulae as follows

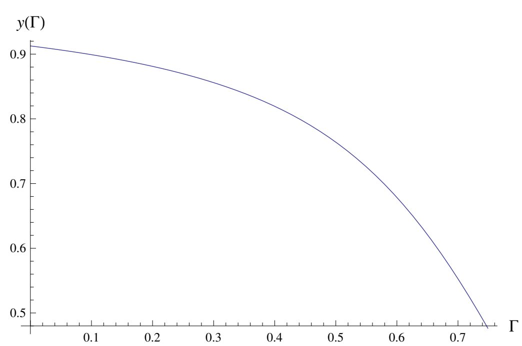

Figure 4.1: Graph of solution (3.1.3) for $\Gamma \in (0, 3/4)$

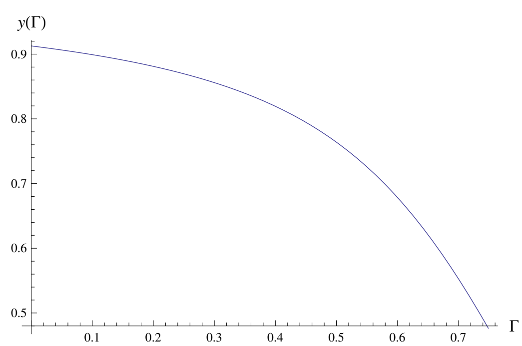

Figure 4.2: Graph of solution (4.3.47) for $\Gamma \in (0, 3/4)$

Chapter 5

Conclusions

In this thesis, two different formulas for Rayleigh wave speed in compressible isotropic linear elastic solids proposed by Malischewsky [5] and Vinh and Ogden [6], have been reviewed and commented.

It is concluded that Malischewsky [5] in his paper did not explain the use of MATHEMATICA using Cardan's formulas alongside trigonometric formulas. Whereas Vinh and Ogden [6] have comprehensively discussed the application of the theory of cubic equations and Cardan's formulas with trigonometric formulas for deriving the Rayleigh wave speed formula.

The function $\mathrm{sgn}(-\Gamma + 1/6)$ appears in the formula presented by Malischewsky [5] (see Eq. (3.1.3)) and holds significant role for conversion of real roots to complex ones with their principal values, while the formula presented by Vinh and Ogden [6] (see Eq. (4.3.47)) is explicit and does not need this signum function.

Both the formulas, given by Malischewsky [5] as well as Vinh and Ogden [6], are equivalent, however the latter formula is well-derived and better explained.

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