Effect of Rotation on Speed of Surface Waves in Anisotropic Materials

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Dedicated

to

Most beloved Ammi and Abbu

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Abstract

This dissertation is the review of study of the effect of rotation on the speeds of surface waves propagating through anisotropic solids. Surface wave solutions are obtained by solving the governing equations for both cases (with and without rotation). These solutions satisfy the boundary conditions. The frequency equations are obtained for Rayleigh and Love waves propagating through hexagonal and orthotropic materials. Speed of waves in various materials is calculated using frequency equations under rotational and non-rotational effects. Various graphs are plotted for illustration purposes and analysis of results. The graphs are plotted between non-dimensional speed and non dimensional rotation for different choices of elastic constants. The results obtained are valid for wave of shorter wavelength. It is found that the speed of Rayleigh waves is affected considerably by rotation. The speed of Love wave however remains un-affected by rotation.

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Chapter 1

Introduction

Among the endless possible motions of matter, wave motion plays a particularly vital role. Waves or oscillations can be defined as the processes distinguished by a certain degree of repetition.

Waves appear widely in nature and every day life. Most information about our surroundings comes to us in the forms of waves. The sound that comes to our ears, light to our eyes, and electromagnetic signals to our radios and television sets are all through wave motion. Ultrasonic waves are used to detect cracks in machinery. They also have wide applications in medical field for the diagnostics of many fatal diseases like cancer and tumors etc. Another application of wave phenomenon is SONAR (sound navigation and ranging), a technique in which sound waves are used to navigate, detect and communicate under the surface of water. Geologists study the geological structure of earth by sending in waves. They are also used to detect oil and gas deposits under earth surface. Properties of materials are determined through the behavior of waves transmitted from them.

Explanation of many natural phenomena requires an understanding of waves. For instance, although skyscrapers and bridges appear to be rigid, they actually oscillate, a fact that architects and engineers who design and build them must take into account. Earthquakes are detected and investigated by studying properties of waves they generate. To understand how radio and television operate one must understand the origin and nature of electromagnetic waves and how they propagate through space. Waves or the oscillatory processes are at the very foundations of various branches of engineering. For example, radio engineering owes its existence to phenomenon of waves.

The history of study of wave phenomenon goes back hundreds of years. The earliest studies were naturally more observational than quantitative and frequently with musical tones or water waves, two of the most common associations with wave motion.

Applications of elastic waves in various fields such as geophysics was a stimulus for scientists and mathematicians to study the waves. The propagation of surface waves in isotropic elastic half space was first studied by Rayleigh [1] . He considered plane waves propagating through isotropic material. He made an assumption that the amplitude of these waves decays exponentially as they move away from the free surface. These waves were found to be non-dispersive and were later called Rayleigh waves after his name.

Love [2] proposed the idea of another type of surface waves, now called Love waves after his name. He found that these waves require an isotropic elastic half space covered with the layer of an isotropic elastic material for their propagation. These waves were dispersive in nature unlike Rayliegh waves, and were called Love waves after

Bouden and Datta [3] dedicated their study to the propagation of Rayleigh and Love waves in cladded anisotropic medium. It was reported that the relative material properties affects the behavior of surface waves.

In recent years, surface wave propagation through anisotropic elastic solids has been a subject of great interest for researchers because of its extensive applications in various branches of science and technology. Pham and Ogden [4] investigated the formulae for the speed of Rayleigh waves propagating through orthotropic compressible elastic solids. They explained the formulae explicitly by using the theory of cubic equations. Wave speed was expressed as continuous function of three dimensionless material parameters.

Abd-Allah [5] studied the propagation of Rayleigh waves in elastic half-space of orthotropic material. He studied the effect of initial stress and gravitational field on the propagation of the Rayleigh waves through orthotropic solid and frequency equation was obtained in this case.

Abd-Allah et al. [6] discussed propagation of Rayleigh waves in generalized magneto-thermo-elastic orthotropic material under initial stress and gravity field. The authors obtained the frequency equations which determine velocity of the waves. The solutions of generalized equations were obtained for thermo-elastic coupling by Helmoltz's theorem.

Baljit et al. [7] worked on the dispersion relation for Rayleigh waves in rotating orthotropic micropolar elastic solid half-space. A frequency equation was obtained with the help of which approximated speed of Rayleigh waves under rotational effects through orthotropic micropolar elastic solid was defined.

Recently, Abo-Dahab and Biswas [8] studied the effect of rotation on Rayleigh waves in magneto-thermoelastic transversely isotropic with thermal relaxation times. It was concluded that Rayleigh wave speed varies inversely with rotation and directly with magnetic field.

The focus of this thesis is to review the work of [15] and [16]. The effect of rotation on Rayleigh waves through anisotropic materials, particularly hexagonal and orthotropic elastic solids is explored. Various steps involved for the calculation of the wave speed under rotational effects have been detailed including graphical analysis. The summary of each chapter is given below:

Chapter 2 gives the readers a detailed description and understanding of the key concepts of theory of the elasticity. Definitions of strain, stress, Hooks law and the effects of symmetries on material properties of hexagonal and orthotropic solids are revised briefly. Wave motion, types of waves with their sub types are defined. Finally, wave equation is also derived in this chapter.

Chapter 3 is a review work of the effect of rotation on Love waves. In the first section, dispersion relation for Love waves traveling through hexagonal material is derived. It also includes graphs showing the phase velocity for lowest mode of Love waves through hexagonal materials. In the second section, dispersion relation for Love waves traveling through orthotropic elastic materials is derived along with the graphs. Finally the effect of rotation on Love waves is investigated. It is found that the rotations of orthotropic and hexagonal materials do not effect the speed of Love waves in these materials.

Chapter 4 includes the study of effect of rotation on Rayleigh surface waves. In the first section the effect of rotation on the speed of the waves traveling through hexagonal elastic solids is studied. Secular equation is established in case of the rotating hexagonal material. The wave speed for some solids is calculated numerically. Second section of the chapter contains the study on Rayleigh wave propagation through rotating orthotropic elastic solids. The secular equation thus obtained is almost similar to the one for hexagonal materials except for few material constants. Various numerical values of the elastic stiffness parameters are used to demonstrate the effects of rotation on the speed of Rayleigh waves. The plots of dimensionless speed verses rotation are finally discussed. Significant impact of rotation on speed of Rayleigh waves on contrary to Love waves is noticed.

In chapter 5 we conclude the results presented in the thesis.

Chapter 2

Fundamentals of the Theory of Elasticity

In this chapter we shall study the fundamentals of the theory of elasticity, basic concepts of stress, strain and Hook's law. Wave motion, types of waves and equation of propagation of waves are also briefly discussed.

2.1 Theory of elasticity

Matter is composed of molecules, which in turn consist of atomic and subatomic particles. These particles are bonded by forces of attraction. Significant distances exist between these bonded atoms, called inter-atomic or intermolecular spaces. Thus matter is not continuous, rather it consists of discrete particles. However, in everyday life we come across many aspects of materials that cannot be described and predicted accurately without neglecting the molecular structure of a material. In continuum mechanics, we regard matter as indefinitely divisible. We assume that it is distributed continuously so that we can think of density, volume etc as a continuous function of position. This approximation is called *continuum approximation*. For the motion of waves of small wavelength, this may not be a fine approximation but for long wavelength this approximation gives very good results [9].

2.2 Tensor analysis

Tensors are mathematical objects that can be used to describe physical properties of materials. Mathematically, a linear transformation $T : V \longrightarrow V$ is called tensor of order 2, where V is a vector space.
By definition the *n-rank tensor* components $T'_{ijk...}$ transform as follows;

$$T'_{...ijk...} = ...q_i^l q_j^m q_k^n ... T_{...lmn}, \ \ where\,\{i,j,k,..\} = \{1,2,3,..n\}\,. \qquad (2.2.1)$$

where q_ν^μ denotes the $n \times n$ transformation matrix from unprimed to primed coordinates. Generally the tensor of zero order is regarded as a scalar, and that of order one is a vector. Formally, tensors are represented by matrix notation. We will use tensors to represent the material properties such as stress, strain, elasticity etc.

2.3 Stress, strain and their relationship

2.3.1 Strain

Strain is the measure of deformation produced in a body as a result of applied forces. The deformation may be in length, shape or volume of a body. If **u(x,t)** is the displacement of particles, its gradient is deformation. The *strain* tensor **S** under the assumption of linear approximation can be defined as follows [10]

$$S_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}). \qquad (2.3.1)$$

It is evident that $S_{ij} = S_{ji}$. i.e. **S** is a symmetric tensor of order 2. In three dimensional space the strain tensor has 9 components out of which 6 are linearly independent.

2.3.2 Stress

When deformation occurs in a material body the arrangement of molecules of the body changes. This disturbs the original state of equilibrium. Forces therefore arise which tend to bring the body back to its equilibrium state. Stress is the physical quantity which describes these internal forces that are exerted by neighboring particles of continuous material on each other to regain equilibrium. Consider a surface Δs_k which has a normal in the x_k direction. Suppose the force due to stress be Δ **F**

$$T_{ik} = \lim_{\Delta s_k \to 0} \frac{\Delta F_i}{\Delta s_k}. \tag{2.3.2}$$

In Figure 2.1, τ_{ik} denote the components of stress tensor. The first index i specifies the direction in which the stress component acts and the second index k specifies the orientation of the surface upon which it is acting. Therefore the i^{th} component of force acting on a surface whose outward normal points in k^{th} direction is τ_{ik}.

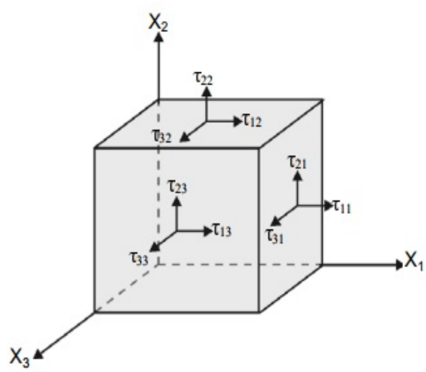

Figure 2.1: Components of Stress Tensor in 3-D

Like strain, the stress tensor is also symmetric. In 3-dimensional space it has 9 components out of which 6 are linearly independent.

2.3.3 Stress, strain relationship – Hook's Law

The relationship between the deformation of an elastic body and the forces responsible for it was stated by Robert Hooke in 1778. He stated that "*the extension is directly proportional to the force*". This law was generalized later by Cauchy as "*within the elastic limits each stress is a linear function of strain components*".

Cauchy called it generalized Hooks law [9]. Mathematically it can be written as

$$T_{ij} = C_{ijkl} S_{kl}, \tag{2.3.3}$$

where $\{i, j, k, l\} = \{1, 2, 3\}$.

It is clear that every component of stress is a linear combination of components of strain. In the above equation, C_{ijkl} is the fourth-rank elasticity tensor or the compliance tensor in the linear theory of elasticity. Its coefficients are called elastic parameters or elastic constants. In 3 dimensional space it has $3^4 = 81$ components. But these components reduce in number due to symmetries as explained in the section below.

2.4 Symmetries of components of the elasticity tensor

As mentioned earlier, T_{ij} is symmetric tensor i.e., $T_{ij} = T_{ji}$, so due to this symmetry the elasticity tensor is symmetric in first two indices. i.e.

$$C_{ijkl} = C_{jikl}. \tag{2.4.1}$$

This symmetry reduces the number of components of C_{ijkl} from 81 to 56. Also S_{kl} is symmetric i.e $S_{kl} = S_{lk}$. Because of this symmetry C_{ijkl} is symmetric in last two indices as well i.e.

$$C_{ijkl} = C_{ijlk}. \tag{2.4.2}$$

Due to the above symmetry the components of C_{ijkl} are reduced to 36. Further reduction of components from 36 to 21 can be done by using the relation for potential energy due to strain. i.e. the strain-energy function, which is defined as:

$$\epsilon = \frac{1}{2} C_{ijkl} S_{ij} S_{kl}. \tag{2.4.3}$$

It is evident from the above equation that C_{ijkl} is also symmetric pair wise i.e., $(ij) \leftrightarrow (kl)$. It implies

$$C_{ijkl} = C_{klij}. \tag{2.4.4}$$

For convenience, we use the famous two index Voigt's notation to express C_{ijkl} into 6×6 matrix. Thus each pair of indices corresponds to a single index. In this way,

$$\begin{aligned} (11) &\rightarrow (1) \\ (22) &\rightarrow (2) \\ (33) &\rightarrow (3) \\ (23) &\rightarrow (4) \\ (13) &\rightarrow (5) \\ (12) &\rightarrow (6) \end{aligned} \tag{2.4.5}$$

Now C_{ijkl} can be represented as 6×6 symmetric matrix, $C_{\alpha\beta}$ and is given below,

$$C_{\alpha\beta} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ * & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ * & * & C_{33} & C_{34} & C_{35} & C_{36} \\ * & * & * & C_{44} & C_{45} & C_{46} \\ * & * & * & * & C_{55} & C_{56} \\ * & * & * & * & * & C_{66} \end{bmatrix}. \tag{2.4.6}$$

In this matrix stars denote linearly dependent components of $C_{\alpha\beta}$.

2.5 Crystal structures and their symmetry

A crystalline solid or crystal is a substance the constituent particles of which possess a regular orderly arrangement. Structurally, in the ideal case, a crystal is bounded by plane surfaces, or faces. The morphological study of crystals of different symmetries showed that they could be classified into seven crystal systems based on the presence of certain crystallographic axes [12]. The focus of this study is on *hexagonal* and *orthorhombic* crystals .

To derive symmetries of crystal systems we first need to understand rotation axis. A rotation axis is an axis such that, if the cell is rotated about it through some angle, the cell remains invariant. This axis is called n-fold if the angle of rotation is $2\pi/n$.

2.5.1 Orthorhombic system

In orthorhombic system there are three mutually perpendicular 2-fold rotation axis, i.e. $n = 2$. Let a, b, and c are the lengths of basis vectors along x_1, x_2 and x_3 directions respectively. Also let α be the angle between (b, c), β be angle between (a, c) and γ be angle between (a, b) as shown in Figure 2.2.

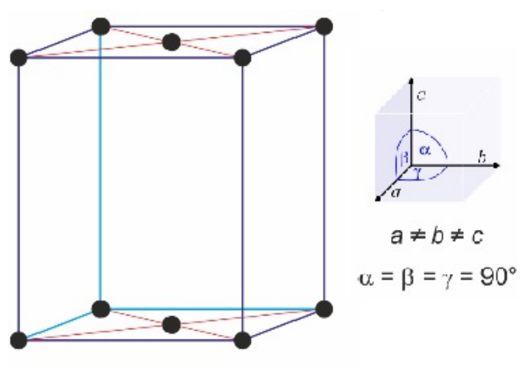

Figure 2.2: Orthorhomic structure.

For orthorhombic system,

$$a \neq b \neq c, \text{and } \alpha = \beta = \gamma = 90^o. \tag{2.5.1}$$

. General rotation matrix for rotation about x_3 for $3D$ is

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.5.2}$$

Assume that the axis of symmetry of the orthorhombic material is along x_3 axis and $\theta = \pi$, then the above matrix becomes

$$Q = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.5.3}$$

The transformation of C_{ijkl} after rotation along X_3-axis is as follows,

$$C'_{ijkl} = Q_{pi}Q_{qj}Q_{rk}Q_{sl}C_{pqrs}, \tag{2.5.4}$$

where, C'_{ijkl} denotes the stiffness constants after rotation, and C_{pqrs} denotes those before rotation. Using Eq. (2.5.3) in Eq. (2.5.4) we get

$$C'_{1111} = (-1)^4 C_{1111}, \tag{2.5.5}$$

or using Voigt's Notation,

$$C'_{11} = C_{11}. \tag{2.5.6}$$

Similarly, we get twelve non zero components of $C_{\alpha\beta}$. Nine of them are linearly independent and three are linearly dependent. The matrix representing elastic moduli for orthorhombic material is

$$C_{\alpha\beta} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix}. \tag{2.5.7}$$

2.5.2 Hexagonal system

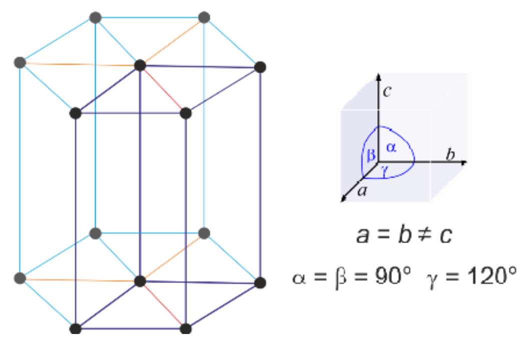

Figure 2.3: Hexagonal structure

The principal axis of hexagonal crystal has order six. The crystal, therefore, behaves as a combination of A_2 and A_3-axes. Let a, b, and c are the lengths of basis vectors along x_1, x_2 and x_3 directions respectively. Also let α be the angle between (b, c), β be angle between (a, c) and γ be angle between (a, b) as shown in the Figure 2.3.

The rotation is assumed to be along x_3-axis. In Eq. (2.5.2) we assume that $\theta = \frac{2\pi}{n} \neq \pi$ The invariant relation (2.5.4) is now more complicated to analyze since it involves many components. In order to deal with this we use the rotation matrix presented by Eq. (2.5.2) in Eq. (2.5.4). Therefore, $C_{\alpha\beta}$ finally becomes

$$C_{\alpha\beta} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \text{ with } C_{66} = \frac{C_{11} - C_{12}}{2}. \qquad (2.5.8)$$

The above matrix shows that there are only five independent components of elasticity tensor $C_{\alpha\beta}$ for hexagonal material.

A crystal which is independent of rotation about n-fold axis is called a *transversely isotropic crystal*. It can be proved easily that the hexagonal crystals are transversely isotropic. We will examine here the behavior of linearly independent moduli C_{11}, C_{12}, C_{13}, C_{33} and C_{44} under arbitrary rotation about x_3-axis. Using Eq. (2.5.2),

$$
\begin{aligned}
&C'_{1111} = Q_{p1}Q_{q1}Q_{r1}Q_{s1}C_{pqrs} \quad \text{where}, \{i,j,k,l\} = \{1,1,1,1\},\\
&\Rightarrow C'_{11} = Q_{11}^4 C_{11} + 2Q_{11}^2 Q_{21}^2 C_{12} + 4Q_{11}^2 Q_{21}^2 C_{66} + Q_{21}^4 C_{22},\\
&= (\cos^2\theta + \sin^2\theta)^2 C_{11},\\
&= C_{11}.
\end{aligned} \tag{2.5.9}
$$

$$
\begin{aligned}
&C'_{3333} = Q_{p3}Q_{q3}Q_{r3}Q_{s3}C_{pqrs} \text{ where}, \{i,j,k,l\} = \{3,3,3,3\},\\
&\Rightarrow C'_{33} = Q_{33}^4 C_{3333},\\
&= (1)^4 C_{33},\\
&= C_{33}.
\end{aligned} \tag{2.5.10}
$$

$$
\begin{aligned}
&C'_{2323} = Q_{p2}Q_{q3}Q_{r2}Q_{s3}C_{pqrs} \quad \text{where}, \{i,j,k,l\} = \{2,3,2,3\},\\
&\Rightarrow C'_{44} = (Q_{12}^2 Q_{33}^2 + Q_{22}^2 Q_{33}^2) C_{44},\\
&= (\sin^2\theta + \cos^2\theta) C_{44},\\
&= C_{44}.
\end{aligned} \tag{2.5.11}
$$

$$
\begin{aligned}
&C'_{1122} = Q_{p1}Q_{q1}Q_{r2}Q_{s2}C_{pqrs} \quad \text{where}, \{i,j,k,l\} = \{1,1,2,2\},\\
&\Rightarrow C'_{12} = 2\cos^2\theta\sin^2\theta C_{11}(\cos^4\theta + \sin^4\theta) C_{12}\\
&-2\cos^2\theta\sin^2\theta(C_{11} - C_{12}),\\
&= (\cos^2\theta + \sin^2\theta)^2 C_{12},\\
&= C_{12}.
\end{aligned} \tag{2.5.12}
$$

$$
\begin{aligned}
&C'_{1133} = Q_{p1}Q_{q1}Q_{r3}Q_{s3}C_{pqrs} \quad \text{where}, \{i,j,k,l\} = \{1,1,3,3\},\\
&\Rightarrow C'_{13} = (Q_{11}^2 Q_{33}^2 + Q_{21}^2 Q_{33}^2) C_{13},\\
&= (\cos^2\theta + \sin^2\theta) C_{13},\\
&= C_{13}.
\end{aligned} \tag{2.5.13}
$$

Similarly it can be shown for all other components of C_{ijkl}.

2.6 Wave motion

Wave is a disturbance or oscillation that travels through space and matter accompanied by transfer of energy. Wave motion transfers energy from one point to another, often with no permanent displacement of particles of the medium i.e. with little or no associated mass transport. The propagation of waves through different media and their effects like tidal waves, remote sensing and earthquakes are always a subject of interest. Minerals, inside the earth can be detected by sending in the elastic waves and by studying how they are reflected. Another important application of wave motion is *seismology* i.e the study of earthquakes.

2.6.1 Types of waves

Waves have been classified into two main categories,

- Mechanical waves.
- Electromagnetic waves.

Mechanical waves require medium for their propagation, whereas the electromagnetic waves do not require medium to travel. The mechanical waves can be further classified as body waves and surface waves.

Body waves

These are high frequency waves that can propagate into the interior of the material. They can be classified as,

- Longitudinal waves.
- Transverse waves.

Longitudinal waves

In these waves every particle of the medium, vibrates about its equilibrium position along the direction of propagation of waves. Longitudinal waves produce compressions and rarefaction in the material through which they are traveling, but they are not responsible for any rotation in the medium. These are also called compressional, or ir-rotational waves. In seismology they are known as P-waves or the primary waves, being the first waves appearing on the seismographs [14].

Transverse waves

In these waves the particles of medium vibrate perpendicular to the direction of propagation of waves. The material undergoes shearing and rotation when the transverse waves pass through it. These are also called shear or rotational waves. However, in seismology they are known as S-waves or the secondary waves [14].

Surface waves

These waves travel along the surface of material. The amplitude of these waves decreases exponentially as they move away from the surface.There are two types of surface elastic waves:

- Rayleigh waves.
- Love waves.

Rayleigh waves

Rayleigh waves travel near the surface of homogeneous half space. The particles of the medium vibrate in anti-clockwise elliptical paths as shown in the Figure 2.4. This fact was discovered by Lord Rayleigh [1] in 1885.

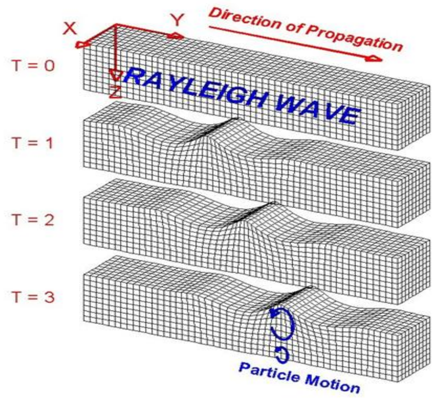

Figure 2.4: Rayleigh wave motion at different times.

Love waves

Love waves named after A. E. Love [2] are guided surface waves. These waves are transverse in nature and the particles vibrate parallel to the surface as shown in Figure 2.5. Unlike Rayleigh waves, Love waves do not travel in homogeneous half-space, rather they require a half-space covered with homogeneous layer [14].

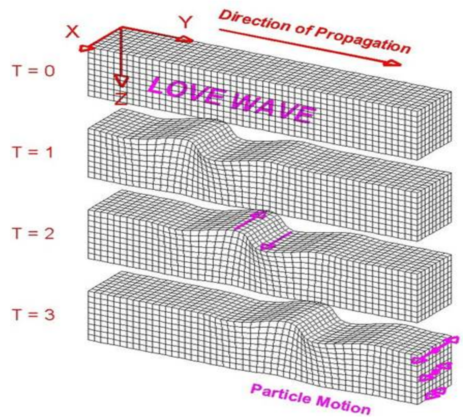

Figure 2.5: Love wave motion at different times.

2.6.2 Wave equation

The forces acting on a body can be categorized as body or surface forces. Body forces are intrinsic and act on the entire volume of the body, for example gravity, whereas the surface or contact forces act only on the surface of body. Examples are tension and pressure force. An arbitrary point in continuum experiences a force due to stress, which is

$$F_i = \frac{\partial T_{ij}}{\partial x_j} + \rho f_i, \ \ \{i, j\} = \{1, 2, 3\}. \tag{2.6.1}$$

Here $\frac{\partial T_{ij}}{\partial x_j}$ are pressure forces and ρf_i are body forces. Here, the usual summation convention is assumed on repeated indices. We have assumed a continuous solid (continuum approach) and when a disturbance is produced, it propagates through the solid so it is locally in motion [13]. If we consider an arbitrary point of the solid with coordinates x_k, having displacement u_i, where $u_i = u_i(x_k, t)$ and $\{i, k\} = \{1, 2, 3\}$. According to Newton's second law of motion

$$F_i = \rho \frac{\partial^2 u_i}{\partial t^2}. \tag{2.6.2}$$

Comparing Eqs. (2.6.1) and (2.6.2), we get

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial T_{ij}}{\partial x_j} + \rho f_i. \tag{2.6.3}$$

Ignoring the body forces in Eq. (2.6.3), we have

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial T_{ij}}{\partial x_j}. \tag{2.6.4}$$

The stress tensor T_{ij} is defined in Eq. (2.3.3) and by substituting the value of strain tensor from Eq. (2.3.1) into Eq. (2.3.3), we have

$$T_{ij} = C_{ijkl} \, \frac{1}{2}(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}). \tag{2.6.5}$$

We know that the strain tensor is symmetric, therefore

$$T_{ij} = C_{ijkl} \frac{\partial u_k}{\partial x_l}. \tag{2.6.6}$$

Substituting Eq. (2.6.6) in Eq. (2.6.4), we get

$$\rho \frac{\partial^2 u_i}{\partial t^2} = C_{ijkl} \frac{\partial^2 u_k}{\partial x_j \partial x_l}, \ \ \{i, j, k\} = \{1, 2, 3\}. \tag{2.6.7}$$

The set of these three second order partial differential equations represents the wave motion.

Chapter 3

Effect of Rotation on Love Waves

In this chapter the effect of rotation on speed of the Love waves is discussed. The dispersion relations for love waves traveling through hexagonal and orthotropic media first without rotation and than under rotational effects are presented.

3.1 Dispersion relation for Love waves traveling through hexagonal materials

Consider the half space $x_2 \geq 0$ of hexagonal elastic material with the mass density ρ, and the stiffness constant C_{44}. x_2 is considered positive in the downward direction throughout the thesis. We consider the Love waves traveling along x_3-axis, perpendicular to $(x_1 x_2)$ plane as shown in the Figure 3.1. The non zero component of displacement is $u_3(x_1, x_2, t)$. Thus

$$\mathbf{u} = u_3(x_1, x_2, t), \;\; u_1 = u_2 = 0. \tag{3.1.1}$$

For $i = 3$, Eq. (2.6.7) can be written as

$$\rho \frac{\partial^2 u_3}{\partial t^2} = C_{3jkl} \frac{\partial^2 u_k}{\partial x_j \partial x_l}. \tag{3.1.2}$$

After expanding the above expression and substituting the components of C_{ijkl} for hexagonal materials from Eq. (2.5.8) we get

$$\rho \frac{\partial^2 u_3}{\partial t^2} = C_{44} \left(\frac{\partial^2 u_3}{\partial x_1^2} + \frac{\partial^2 u_3}{\partial x_2^2} \right), \tag{3.1.3}$$

or

$$\frac{\rho}{C_{44}} \left(\frac{\partial^2 u_3}{\partial t^2} \right) = \frac{\partial^2 u_3}{\partial x_1^2} + \frac{\partial^2 u_3}{x_2^2}. \tag{3.1.4}$$

Since the amplitude of Love waves decreases exponentially when they move away from the surface, so we assume solution of the form

$$u_3 = Ae^{-bx_2} \exp\left[ik(x_1 - ct)\right], \tag{3.1.5}$$

where b must have positive real part. A is the amplitude of Love wave and c is the phase velocity. Differentiating above equation with respect to x_1, x_2 and t twice we get

$$\frac{\partial^2 u_3}{\partial x_1^2} = -k^2 Ae^{-bx_2} \exp\left[ik(x_1 - ct)\right], \tag{3.1.6}$$

$$\frac{\partial^2 u_3}{\partial x_2^2} = b^2 Ae^{-bx_2} \exp\left[ik(x_1 - ct)\right], \tag{3.1.7}$$

$$\frac{\partial^2 u_3}{\partial t^2} = -k^2 c^2 Ae^{-bx_2} \exp\left[ik(x_1 - ct)\right]. \tag{3.1.8}$$

Substitute Eqs. (3.1.6), (3.1.7) and (3.1.8) in Eq. (3.1.4) we obtain

$$-k^2 + b^2 = \frac{\rho}{C_{44}}(-k^2 c^2), \tag{3.1.9}$$

or

$$b = k\sqrt{1 - \frac{\rho c^2}{C_{44}}}. \tag{3.1.10}$$

Boundary condition for free surface implies that the stress component

$$T_{32} = C_{44} \frac{\partial u_3}{\partial x_2}, \tag{3.1.11}$$

vanishes at $x_2 = 0$ i. e. ,

$$T_{32} = C_{44} \frac{\partial u_3}{\partial x_2} = 0, \text{ at } x_2 = 0. \tag{3.1.12}$$

Equation (3.1.11) is obtained by substituting Eq. (2.3.3) in Eq. (2.6.6). Eq. (3.1.12) implies

$$-C_{44}bAe^{-bx_2}\exp[ik(x_1-ct)]=0. \tag{3.1.13}$$

Equation (3.1.13) can be satisfied only if either $b=0$ or $A=0$. Both these cases do not represent Love wave. This implies that Love wave propagation is not possible along free boundary. It was proposed by Love [2] that for Love wave propagation the half space should be covered by a layer of hexagonal elastic material having thickness H. Let the material constant in the layer be $C_{44}^{(l)}$ and mass density of material of layer be $\rho^{(l)}$ as shown in the Figure 3.1,

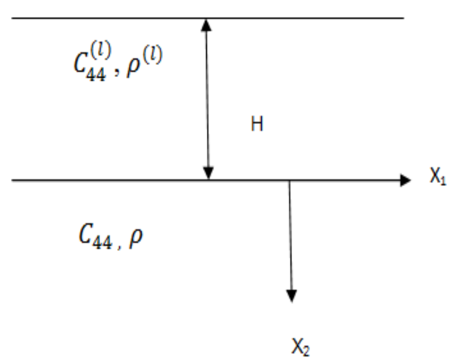

Figure 3.1: Geometery of the problem: Hexagonal elastic half space covered with a layer of another hexagonal material.

For layer, Eq. (3.1.4) becomes

$$\frac{\rho^{(l)}}{C_{44}^{(l)}}\left(\frac{\partial^2 u_3^{(l)}}{\partial t^2}\right)=\frac{\partial^2 u_3^{(l)}}{\partial x_1^2}+\frac{\partial^2 u_3^{(l)}}{x_2^2}, \tag{3.1.14}$$

where $u_3^{(l)}$ is the component of displacement in the layer. We assume the solution of the form

$$u_3^{(l)}=f(x_2)\exp\left[i(kx_1-\omega t)\right],\ \text{where}\ \omega=kc. \tag{3.1.15}$$

Differentiating Eq. (3.1.15) with respect to x_1, x_2 and t twice we get

$$\frac{\partial^2 u_3^{(l)}}{\partial x_1^2} = -k^2 f(x_2) \exp\left[i(kx_1 - \omega t)\right], \tag{3.1.16}$$

$$\frac{\partial^2 u_3^{(l)}}{\partial x_2^2} = f^{''}(x_2) \exp\left[i(kx_1 - \omega t)\right], \tag{3.1.17}$$

$$\frac{\partial^2 u_3^{(l)}}{\partial t^2} = -\omega^2 f(x_2) \exp\left[ik(x_1 - ct)\right]. \tag{3.1.18}$$

Substitution of Eqs. (3.1.16), (3.1.17) and (3.1.18) in Eq. (3.1.14) yields

$$-k^2 f(x_2) + f^{''}(x_2) = -\frac{\rho^{(l)}}{C_{44}^{(l)}}\omega^2 f(x_2), \tag{3.1.19}$$

or

$$f^{''}(x_2) + f(x_2)\left[\frac{\rho^{(l)}\omega^2}{C_{44}^{(l)}} - k^2\right] = 0. \tag{3.1.20}$$

Let

$$q^{(l)^2} = \frac{\rho^{(l)}\omega^2}{C_{44}^{(l)}} - k^2. \tag{3.1.21}$$

Equation (3.1.20) becomes

$$f^{''}(x_2) + q^{(l)^2} f(x_2) = 0. \tag{3.1.22}$$

The general solution of the above second order linear ordinary differential equation is

$$f(x_2) = k_1 \cos(q^{(l)} x_2) + k_2 \sin(q^{(l)} x_2). \tag{3.1.23}$$

Equation (3.1.15) becomes

$$u_3^{(l)} = \left[k_1 \cos(q^{(l)} x_2) + k_2 \sin(q^{(l)} x_2)\right] \exp(i(kx_1 - \omega t)). \tag{3.1.24}$$

The condition of continuity is that the stress in layer must be equal to stress in half space, at $x_2 = 0$

$$T_{23}^{(l)} = T_{23}, \text{ at } x_2 = 0. \tag{3.1.25}$$

Using Eq. (3.1.11) in Eq. (3.1.25), we get

$$C_{44}bA + C_{44}^{(l)}k_2q^{(l)} = 0. \tag{3.1.26}$$

The continuity of displacement at $x_2 = 0$ implies

$$u_3^{(l)} = u_3 \text{ at } x_2 = 0, \tag{3.1.27}$$

which gives

$$A - k_1 = 0. \tag{3.1.28}$$

The condition of vanishing shear stress at the free surface ($x_2 = -H$) implies

$$T_{23}^{(l)} = 0, \text{ at } x_2 = -H, \tag{3.1.29}$$

which gives

$$k_1 \sin(q^{(l)}H) + k_2 \cos(q^{(l)}H). \tag{3.1.30}$$

The matrix form of Eqs. (3.1.26), (3.1.28) and (3.1.30) is

$$\begin{bmatrix} bC_{44} & 0 & q^{(l)}C_{44}^{(l)} \\ 1 & -1 & 0 \\ 0 & \sin(q^{(l)}H) & \cos(q^{(l)}H) \end{bmatrix} \begin{bmatrix} A \\ k_1 \\ k_2 \end{bmatrix} = 0. \tag{3.1.31}$$

For non trivial solution the determinant of Eq. (3.1.31) should vanish.

$$\begin{vmatrix} bC_{44} & 0 & q^{(l)}C_{44}^{(l)} \\ 1 & -1 & 0 \\ 0 & \sin(q^{(l)}H) & \cos(q^{(l)}H) \end{vmatrix} = 0. \tag{3.1.32}$$

This implies

$$C_{44}^{(l)} \sin(q^{(l)}H) - bC_{44} \cos(q^{(l)}H) = 0. \tag{3.1.33}$$

Further simplification gives

$$\frac{\sin(q^{(l)H})}{\cos(q^{(l)}H)} = \frac{bC_{44}}{q^{(l)}C_{44}^{(l)}}, \tag{3.1.34}$$

or

$$\tan(q^{(l)}H) = \frac{bC_{44}}{q^{(l)}C_{44}^{(l)}}. \tag{3.1.35}$$

Substituting values of $q^{(l)}$ from Eq. (3.1.21) and b from Eq. (3.1.10) in Eq. (3.1.35) we get

$$\tan\left[\sqrt{\frac{\rho^{(l)}\omega^2}{C_{44}^{(l)}} - k^2}H\right] = \frac{kC_{44}\sqrt{1 - \frac{\rho c^2}{C_{44}}}}{C_{44}^{(l)}\sqrt{\frac{\rho^{(l)}c^2}{C_{44}^{(l)}} - 1}}. \tag{3.1.36}$$

For hexagonal material, the transverse wave velocity is [13]

$$c_T^2 = \frac{C_{44}}{\rho}, \quad c_T^{(l)^2} = \frac{C_{44}^{(l)}}{\rho^{(l)}}, \tag{3.1.37}$$

where, c_T is the transverse wave speed in a hexagonal elastic half space and $c_T^{(l)}$ is the transverse wave speed in layer. Equation (3.1.36) after substituting Eq. (3.1.37) becomes

$$\tan\left[kH\left(\sqrt{\frac{c^2}{c_T^{(l)^2}} - 1}\right)\right] - \frac{C_{44}\sqrt{1 - \frac{c^2}{c_T^2}}}{C_{44}^{(l)}\sqrt{\frac{c^2}{c_T^{(l)^2}} - 1}} = 0. \tag{3.1.38}$$

The above equation shows that Love waves are dispersive in nature, i.e. their speed depends upon wave number. The left hand side of Eq. (3.1.38) is positive for $c = c_T$ and is negative for $c = c_T^{(l)}$. Therefore, real roots are possible in the interval $c_T^{(l)} < c \leq c_T$. No real root will exist for $c_T < c_T^{(l)}$. If kH is considered as independent variable then for $kH = 0$, we have from Eq. (3.1.38)

$$\frac{C_{44}\sqrt{1 - \frac{c^2}{c_T^2}}}{C_{44}^{(l)}\sqrt{\frac{c^2}{c_T^{(l)^2}} - 1}} = 0,$$

which implies $c = c_T$. Let us introduce the following notation

$$x = \frac{2kH}{\pi} \Rightarrow \frac{x\pi}{2} = kH, \ \ y = \frac{c}{c_T}, \tag{3.1.39}$$

and

$$\frac{c}{c_T^{(l)}} = \frac{c}{c_T}\frac{c_T}{c_T^{(l)}} \Rightarrow \frac{c}{c_T^{(l)}} = y.\frac{c_T}{c_T^{(l)}}, \tag{3.1.40}$$

Using Eqs. (3.1.39) and (3.1.40) in Eq. (3.1.38), we have

$$\tan\left[\frac{x\pi}{2}\left(\sqrt{\frac{yc_T^2}{c_T^{(l)^2}}-1}\right)\right] - \frac{C_{44}\sqrt{1-y^2}}{C_{44}^{(l)}\sqrt{y^2\frac{c_T^2}{c_T^{(l)^2}}-1}} = 0. \tag{3.1.41}$$

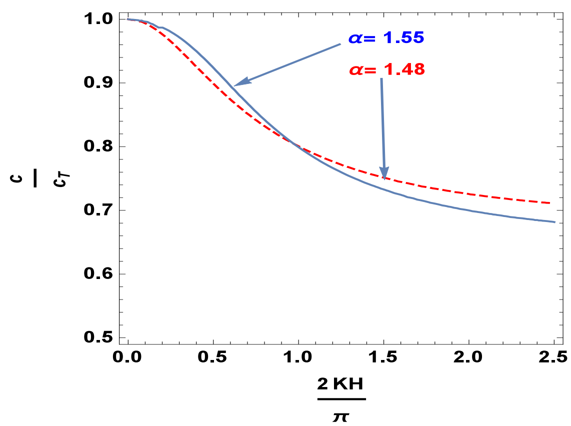

Figure 3.2: Phase velocity for lowest mode of Love waves through hexagonal materials. Here, $\alpha = \frac{c_T}{c_T^{(l)}}$

It can be observed from above graph that phase velocity decreases with the increase in kH values.

3.2 Dispersion relation for Love waves traveling through orthotropic materials

The dispersion relation for Love waves traveling through the half space of orthotropic elastic material will be discussed in this section. It is assumed that the Love waves are traveling along x_3-axis, perpendicular (x_1x_2) plane with the component of displacement u_3 as in Eq. (3.1.1). For orthotropic materials, Eq. (3.1.2) after substituting the values of elastic parameters from Eq. (2.5.7) becomes

$$\rho \frac{\partial^2 u_3}{\partial t^2} = C_{55} \frac{\partial^2 u_3}{\partial x_1^2} + C_{44} \frac{\partial^2 u_3}{x_2^2}. \tag{3.2.1}$$

Assuming the solution of the form given in Eq. (3.1.5) and substituting it in Eq. (3.2.1) we get

$$b = k\sqrt{\frac{C_{55} - \rho c^2}{C_{44}}}. \tag{3.2.2}$$

The condition for free boundary comes out to be the same as that for hexagonal material, as expressed in Eq. (3.1.12). For the existence of Love wave there should be a layer of orthotropic material over the half space having thickness H. Let the material constants in the layer be $C_{44}^{(l)}$, $C_{55}^{(l)}$ and $\rho^{(l)}$ as shown in the Figure 3.3

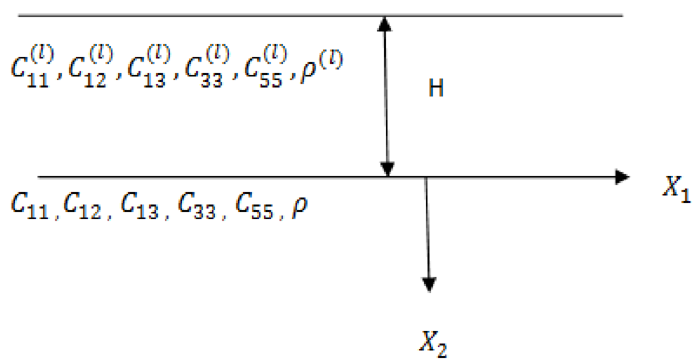

Figure 3.3: Geometry of the problem: Orthotropic elastic half space covered with a layer of another orthotropic material.

For orthotropic layer Eq. (3.2.1) becomes

$$\rho \frac{\partial^2 u_3^{(l)}}{\partial t^2} = C_{55}^{(l)} \frac{\partial^2 u_3^{(l)}}{\partial x_1^2} + C_{44}^{(l)} \frac{\partial^2 u_3^{(l)}}{x_2^2}. \tag{3.2.3}$$

We assume the solution of the form as expressed in Eq. (3.1.15), which on substitution in Eq. (3.2.3) yields

$$-k^2 C_{55}^{(l)} f(x_2) + C_{44}^{(l)} f''(x_2) = -\rho^{(l)} \omega^2 f(x_2), \tag{3.2.4}$$

or

$$f''(x_2) + f(x_2)\left[\frac{-k^2 C_{55}^{(l)} + \omega^2 \rho^{(l)}}{C_{44}^{(l)}}\right] = 0. \tag{3.2.5}$$

We get the general solution as expressed in Eq. (3.1.23), which gives similar expression for u_3 as mentioned in Eq. (3.1.24). But here the value of $q^{(l)}$ is slightly different because of different parameters i.e.

$$q^{(l)} = k\sqrt{\frac{\rho^{(l)} c^2 - C_{55}^{(l)}}{C_{44}^{(l)}}}. \tag{3.2.6}$$

By imposing the boundary conditions expressed in Eq. (3.1.25), (3.1.27) and (3.1.29)we get the equation similar to Eq. (3.1.31) whose determinant should vanish for non trivial solution. We finally get the following expression

$$\tan\left[kH\left(\sqrt{\frac{\rho^{(l)} c^2 - C_{55}^{(l)}}{C_{44}^{(l)}}}\right)\right] - \sqrt{\frac{C_{44}}{C_{44}^{(l)}}\left(\frac{C_{55} - \rho c^2}{\rho^{(l)} c^2 - C_{55}^{(l)}}\right)} = 0. \tag{3.2.7}$$

For orthotropic materials, the transverse wave velocity is [13]

$$c_T^2 = \frac{C_{55}}{\rho}, \quad c_T^{(l)^2} = \frac{C_{55}^{(l)}}{\rho^{(l)}}, \tag{3.2.8}$$

where c_T is the transverse wave speed in orthotropic elastic half space and $c_T^{(l)}$ is the transverse wave speed in orthotropic layer. Equation (3.2.7) after substituting Eq. (3.2.8) becomes

$$\tan\left[kH\sqrt{\frac{\rho^{(l)} c^2}{C_{44}^{(l)}}\left(1 - \frac{c_T^{(l)^2}}{c^2}\right)}\right] - \sqrt{\frac{\rho C_{44}}{\rho^{(l)} C_{44}^{(l)}}\left(\frac{\frac{c_T^2}{c^2} - 1}{1 - \frac{c_T^{(l)^2}}{c^2}}\right)} = 0 \tag{3.2.9}$$

Equation (3.2.9) is again a dispersive relation for phase velocity of Love waves in which the real roots are possible only for the interval $c_T^{(l)} < c \leq c_T$. For simplification consider

$$\alpha = \frac{\rho c_T^2}{C_{44}},\ \ \alpha^{(l)} = \frac{\rho^{(l)} c_T^{(l)^2}}{C_{44}^{(l)}},\ \ \beta = \frac{C_{44}}{C_{44}^{(l)}},\ \ kH = \frac{x\pi}{2},\ y = \frac{c}{c_T},\ \ \eta = \frac{c_T}{c_T^{(l)}}. \quad (3.2.10)$$

Substituting all the above values in Eq. (3.2.9), we get

$$\tan\left[\frac{x\pi}{2}\sqrt{\alpha^{(l)}\left[(\eta y)^2 - 1\right]}\right] - \beta\sqrt{\frac{\alpha}{\alpha^{(l)}}\left(\frac{1-y^2}{(\eta y)^2 - 1}\right)} = 0 \quad (3.2.11)$$

Where, $\eta = \frac{c_T}{c_T^2}$.

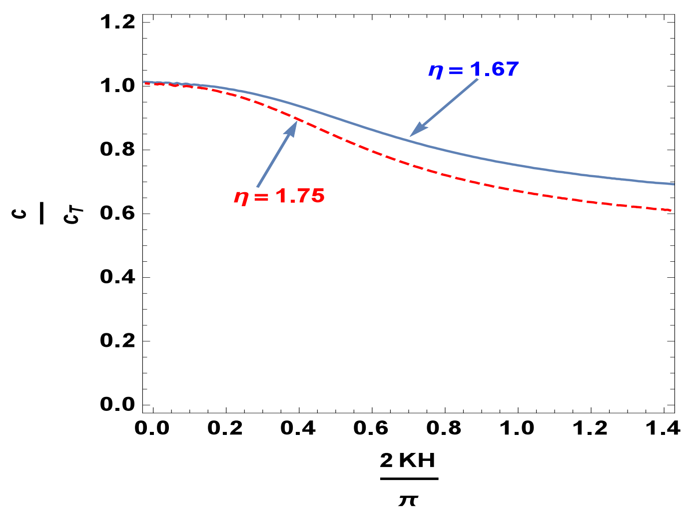

Figure 3.4: Phase velocity for lowest mode of Love waves through orthorhombic materials.

The graph in the Figure 3.4 is displaying the lowest modes of Love waves through two orthorhombic materials having different stiffness constants and mass densities. It displays that the phase velocity shows the similar behavior with change in kH values as in hexagonal materials given in Figure (3.2) i.e. the phase velocity decreases with the increase in kH values.

It can be concluded from the graphs shown in the Figures 3.2 and 3.4 that the phase velocity of Love waves decreases with the increase in dimensionless wave number kH irrespective of the type of crystal they are passing through.

3.3 Effect of rotation on Love waves traveling through hexagonal and orthotropic materials

In this section the Love wave propagation through a rotating hexagonal material will be discussed. The half space $x_3 \leq 0$ of hexagonal elastic solid is considered. If the material is rotated along x_3-axis with constant angular velocity $\mathbf{\Omega}$, the time rate of change of displacement vector $\mathbf{u}$ is $(\dot{\mathbf{u}} + \mathbf{\Omega} \times \mathbf{u})$. This expression can be rephrased as $(\ddot{u}_i + \epsilon_{ijk}\Omega_j u_k)$. Where ϵ_{ijk} is Levi-Civita tensor defined as

$$\epsilon_{ijk} = \begin{cases} 1, & \text{for even permutations of ijk,} \\ -1, & \text{for odd permutations of ijk,} \\ 0, & \text{otherwise.} \end{cases}$$

Similarly the second derivative of the displacement vector $\mathbf{u}$ with respect to time becomes $(\ddot{u}_i + \Omega_j u_j \Omega_i - \Omega^2 u_i + 2\epsilon_{ijk}\Omega_j \dot{u}_i)$. The wave equation given in Eq. (2.6.7) in a rotating medium can be written as

$$T_{ij,j} = \rho(\ddot{u}_i + \Omega_j u_j \Omega_i - \Omega^2 u_i + 2\epsilon_{ijk}\Omega_j \dot{u}_k), \tag{3.3.1}$$

where $\Omega = \Omega(0, 0, 1)$.

For $i = 3$, Eq. (3.3.1) becomes

$$T_{3j,j} = \rho(\ddot{u}_3 + \Omega_j u_j \Omega_3 - \Omega^2 u_3 + 2\epsilon_{3jk}\Omega_j \dot{u}_k) \tag{3.3.2}$$

$$T_{31,1} + T_{32,2} = \rho\left[\ddot{u}_3 + \Omega^2 u_3 - \Omega^2 u_3\right], \tag{3.3.3}$$

or

$$T_{31,1} + T_{32,2} = \rho\ddot{u}_3 \tag{3.3.4}$$

The rotation terms disappear in Eq. (3.3.4). This shows that rotation does not affect the speed of Love waves traveling through hexagonal medium if the medium is rotated in the direction of propagation of the waves.

If the axis of rotation is perpendicular to the direction of propagation of the waves, i.e. the rotation is along x_1 axis then the angular velocity vector will be $\mathbf{\Omega} = \Omega(1, 0, 0)$. For $i = 1$ and rotation along x_1 axis Eq. (3.3.1) becomes

$$T_{11,1} + T_{12,2} + T_{13,3} = 2\rho\dot{u}_3 \tag{3.3.5}$$

Left hand side of Eq. (3.3.5) is zero for hexagonal material, which implies

$$2\rho\dot{u}_3 = 0 \tag{3.3.6}$$

For $i = 2$ in Eq. (3.3.1) we obtain the similar expression as in Eq. (3.3.6). For $i = 3$ and rotation along x_1 axis Eq. (3.3.1) becomes

$$T_{31,1} + T_{32,2} = \rho\left[\ddot{u}_3 - \Omega^2 u_3\right] \tag{3.3.7}$$

For a hexagonal material

$$T_{31,1} + T_{32,2} = C_{44}\left(\frac{\partial^2 u_3}{\partial x_1^2} + \frac{\partial^2 u_3}{\partial x_2^2}\right) \tag{3.3.8}$$

Comparing Eqs. (3.3.7) and (3.3.8) we get

$$C_{44}\left(\frac{\partial^2 u_3}{\partial x_1^2} + \frac{\partial^2 u_3}{\partial x_2^2}\right) = \rho\left[\ddot{u}_3 - \Omega^2 u_3\right] \tag{3.3.9}$$

But Eq. (3.3.6) implies $\ddot{u}_3 = 0$, so Eq. (3.3.9) becomes

$$C_{44}\left(\frac{\partial^2 u_3}{\partial x_1^2} + \frac{\partial^2 u_3}{\partial x_2^2}\right) = -\rho\Omega^2 u_3 \tag{3.3.10}$$

Equation (3.3.10) shows that there is no term having time derivative of u_3 which implies that there is no phase velocity and hence Love wave propagation is not possible.

Similarly, if we consider Love wave propagation through rotating orthotropic material we will obtain the same expressions as given in Eqs. (3.3.4) and (3.3.10). From this it can be concluded that the speed of Love waves is not affected if they travel through rotating half space. And this fact remains unchanged, irrespective of the medium through which the waves propagate.

Chapter 4

Effect of Rotation on Rayleigh Waves

In this chapter the effect of rotation on the speed of Rayleigh waves is studied. In the first section the effect of rotation on the speed of the waves traveling through hexagonal material is studied. Second section is dedicated to the study of rotational effects on the speed of Rayleigh waves traveling through orthotropic materials.

4.1 Effect of rotation on the speed of Rayleigh waves traveling through hexagonal elastic solids

Consider the infinite free surface of transversely isotropic elastic solid as shown in Figure 4.1. The rectangular coordinate system is chosen in such a way that x_3-axis is perpendicular to the surface and the material is placed in the plane $x_3 \leq 0$. Rayleigh waves are considered to be traveling in x_1-direction in x_1x_3-plane with the components of displacement $(u_1, 0, u_3)$. The displacement vector is,

$$u_i = u_i(x_1, x_3, t) \text{ where}, i = \{1, 3\} \text{ and } u_2 = 0. \tag{4.1.1}$$

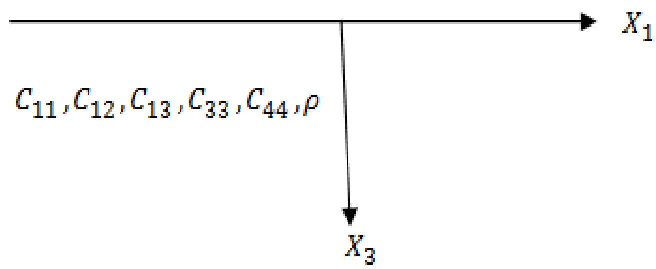

Figure 4.1: Hexagonal elastic half space.

We obtain the following set of equations for transversely isotropic solid by substituting Eq. (2.3.1) in Eq. (2.3.3)

$$T_{11} = C_{11}u_{1,1} + C_{33}u_{3,3}, \tag{4.1.2}$$

$$T_{33} = C_{13}u_{1,1} + C_{33}u_{3,3}, \tag{4.1.3}$$

$$T_{13} = C_{44}(u_{1,3} + u_{3,1}). \tag{4.1.4}$$

where the stiffness constants $C_{11}, C_{33}, C_{13}, C_{44}$ satisfy the following inequalities [4],

$$C_{ii} > 0, i = 1, 3, 4, \quad C_{11}C_{33} - C_{13}^2 > 0. \tag{4.1.5}$$

Equation (4.1.5) represents the conditions for positive definiteness of material energy which means that the strain energy is equal to or greater than zero for all states of strain and is zero only if all the components of strain are zero. If the material is rotated along x_3-axis with constant angular velocity $\mathbf{\Omega} = (0, 0, \Omega)$, then for $i = \{1, 3\}$, Eq. (3.3.1) can be written as

$$T_{11,1} + T_{13,3} = \rho(\ddot{u}_1 - \Omega^2 u_1), \tag{4.1.6}$$

$$T_{31,1} + T_{33,3} \quad = \rho\ddot{u}_3. \tag{4.1.7}$$

Substituting Eqs .(4.1.2)-(4.1.4) in Eqs. (4.1.6) and (4.1.7) we get

$$\begin{aligned} C_{11}u_{1,11} + C_{13}u_{3,31} + C_{44}(u_{1,33} + u_{3,13}) &= \rho(\ddot{u}_1 - \Omega^2 u_1), \\ C_{13}u_{1,13} + C_{33}u_{3,33} + C_{44}(u_{1,31} + u_{3,11}) &= \rho\ddot{u}_3. \end{aligned} \tag{4.1.8}$$

The boundary conditions for free surface are

$$T_{3i} = 0;\ i = 1, 3 \ \text{ on the plane } x_3 = 0. \tag{4.1.9}$$

By definition of the surface wave, wave displacement and the components of stress decay when it moves away from the surface. This implies

$$u_i \to 0,\ T_{ij} \to 0\ (i, j = 1, 3) \text{ as } x_3 \to -\infty. \tag{4.1.10}$$

For the waves propagating in x_1-direction, by following Pham and Ogden [4], we assume solution of the form

$$u_j = \Psi(kx_3)\exp[ik(x_1 - ct)];\ j = 1, 3. \tag{4.1.11}$$

Where, c is the speed of Rayleigh wave and k is the wave number, $\Psi, j = 1, 3$ are the functions to be determined. For $j = 1$,

$$u_1 = \Psi_1(kx_3)exp[ik(x_1 - ct)], \tag{4.1.12}$$

and for $j = 3$

$$u_3 = \Psi_3(kx_3)exp[ik(x_1 - ct)]. \tag{4.1.13}$$

Differentiating Eq. (4.1.12) w.r.t x_1 and x_3 gives

$$u_{1,11} = -k^2\Psi_1(kx_3)exp[ik(x_1 - ct)], \tag{4.1.14}$$

$$u_{1,13} = ik^2\Psi_1'(kx_3)exp[ik(x_1 - ct)], \tag{4.1.15}$$

$$u_{1,33} = k^2\Psi_1''(kx_3)exp[ik(x_1 - ct)]. \tag{4.1.16}$$

Differentiating Eq. (4.1.13) w.r.t x_1 and x_3 gives,

$$u_{3,11} = -k^2\Psi_3(kx_3)exp[ik(x_1 - ct)], \tag{4.1.17}$$

$$u_{3,13} = ik^2\Psi_3'(kx_3)exp[ik(x_1 - ct)], \tag{4.1.18}$$

$$u_{3,33} = k^2\Psi_3''(kx_3)exp[ik(x_1 - ct)]. \tag{4.1.19}$$

Substituting Eqs. (4.1.12)-(4.1.19) in Eq. (4.1.8) and (4.1.8) gives

$$C_{44}k^2\Psi_1^{''} + ik^2\Psi_3^{'}(C_{13}+C_{44}) + [k^2(\rho c^2 - C_{11}) + p\Omega^2]\Psi_1 = 0, \qquad (4.1.20)$$

$$C_{33}\Psi_3^{''} + i(C_{13}+C_{44})\Psi_1^{'} + \Psi_3(\rho c^2 - C_{44}) = 0. \qquad (4.1.21)$$

By considering Eqs. (4.1.2), (4.1.3) ,(4.1.4) and Eq. (4.1.11), the boundary conditions at $x_3 = 0$ given in Eq. (4.1.9) give

$$\Psi_1^{'} + i\Psi_3 = 0, \qquad (4.1.22)$$

and

$$C_{33}\Psi_3^{'} + iC_{13}\Psi_1 = 0, \qquad (4.1.23)$$

also Eqs. (4.1.10) and (4.1.11) give

$$\Psi_j^{'} \to 0 \text{ as } x_3 \to -\infty. \qquad (4.1.24)$$

The Laplace transform of Eqs. (4.1.20) and (4.1.21) using Eq. (4.1.22) and (4.1.23) yields

$$\begin{aligned}
&[k^2(C_{44}s^2 + \rho c^2 - C_{11}) + \rho\Omega^2]\bar{\Psi}_1(s) + ik^2(C_{44}+C_{13})s\bar{\Psi}_3(s) \qquad (4.1.25)\\
&= C_{44}k^2[s\Psi_1(0) + \Psi_1^{'}(0)] + ik^2(C_{44}+C_{13})\Psi_3(0),\\
&i(C_{13}+C_{44})s\bar{\Psi}_1(s) + (C_{33}s^2 - C_{44} + \rho c^2)\bar{\Psi}_3(s) \qquad (4.1.26)\\
&= i(C_{44}+C_{13})\Psi_1(0) + C_{33}[\Psi_3(0)s + \Psi_3^{'}(0)].
\end{aligned}$$

Solving Eqs. (4.1.25) and (4.1.26) we get

$$\bar{\Psi}_1(s) = \frac{\begin{vmatrix} C_{44}k^2 s\Psi_1 + \Psi_1^{'} + ik^2(C_{13}+C_{44})\Psi_3 & ik^2(C_{13}+C_{44})s \\ i(C_{13}+C_{44})\Psi_1 + C_{33}\left(s\Psi_3 + \Psi_3^{'}\right) & (C_{33}s^2 - C_{44} + \rho c^2) \end{vmatrix}}{Q}, \qquad (4.1.27)$$

where

$$\begin{aligned}
Q &= k^2C_{33}C_{44}s^4 + [k^2\{(C_{44}+C_{13})^2 + C_{33}(\rho c^2 - C_{11}) \qquad (4.1.28)\\
&+C_{44}(\rho c^2 - C_{44})\}\rho\Omega^2C_{33}]s^2 + (\rho c^2 - C_{44})\{k^2(\rho c^2 - C_{11}) + \rho\Omega^2\}.
\end{aligned}$$

For a non trivial solution the determinant in numerator of Eq. (4.1.27) must be zero. Since $\bar{\Psi}_1(s) \neq 0$, it implies that the expression in denominator should be zero, which is 4^{th} order equation in 's'. It implies

$$k^2 C_{33} C_{44} s^4 + [k^2 \left\{(C_{13} + C_{44})^2 + C_{33}(\rho c^2 - C_{44})\right\} + C_{33}\rho\Omega^2] s^2 + (\rho c^2 - C_{44}) \left\{k^2(\rho c^2 - C_{11}) + \rho\Omega^2\right\} = 0. \tag{4.1.29}$$

Let the roots of Eq (4.1.29), which is quadratic in s^2 , be s_1^2 and s_2^2. Since s is the Laplace variable so the roots s_1^2 and s_2^2 must have positive real parts. Equation. (4.1.27) after factorization of Eq. (4.1.29) becomes

$$\bar{\Psi}_1(s) = \frac{A_1}{s - s_1} + \frac{A_2}{s - s_2} + \frac{A_3}{s + s_1} + \frac{A_4}{s + s_2}. \tag{4.1.30}$$

In the above equation A_1, A_2, A_3 and A_4 are constants to be determined. The inverse Laplace transformation of Eq. (4.1.30) using (4.1.24) yields

$$\Psi_1(y) = A_1 \exp(s_1 y) + A_2 \exp(s_2 y). \tag{4.1.31}$$

By Eqs. (4.1.25), (4.1.26) and (4.1.27), it is clear that $\bar{\Psi}_1(s)$ and $\bar{\Psi}_3(s)$ are linearly dependent, so they can be written in linear combination of each other.

$$\Psi_3(y) = \alpha_1 A_1 \exp(s_1 y) + \alpha_2 A_2 \exp(s_2 y). \tag{4.1.32}$$

Since Ψ_1 and Ψ_3 are linearly dependent, Eq. (4.1.27) implies,

$$\alpha_j = i \frac{[C_{44} k^2 s_j^2 + k^2(\rho c^2 - C_{11}) + \rho\Omega^2]}{k^2(C_{44} + C_{13}) s_j}\ , \ \ j = 1, 3. \tag{4.1.33}$$

Let s_1^2 and s_2^2 be the roots of Eq. (4.1.29), the sum of roots is

$$s_1^2 + s_2^2 = -\frac{k^2[(C_{44}+C_{13})^2 + C_{33}(\rho c^2 - C_{11}) + C_{44}(\rho c^2 - C_{44})] + C_{33}\rho\Omega^2}{k^2 C_{33} C_{44}}. \tag{4.1.34}$$

and the product of roots of Eq. (4.1.29) is

$$s_1^2 s_2^2 = \frac{(\rho c^2 - C_{44})[k^2(\rho c^2 - C_{11}) + \rho\Omega^2]}{k^2 C_{33} C_{44}}. \tag{4.1.35}$$

Substituting values of Ψ_1 and Ψ_3 from Eqs. (4.1.31) and (4.1.32) into Eqs. (4.1.22) and (4.1.23)

$$\begin{aligned} &iC_{13}[A_1 \exp(s_1 y) + A_2 \exp(s_2 y)] + \\ &C_{33}[\alpha_1 A_1 s_1 \exp(s_1 y) + \alpha_2 A_2 s_2 \exp(s_2 y)] = 0, \end{aligned} \tag{4.1.36}$$

which gives, at $y = 0$

$$[iC_{13} + C_{33}\alpha_1 s_1]A_1 + [iC_{13} + C_{33}\alpha_2 s_2]A_2 = 0. \tag{4.1.37}$$

Equation. (4.1.23) becomes,

$$A_1(s_1 + i\alpha_1) + A_2(s_2 + i\alpha_2) = 0. \tag{4.1.38}$$

Equations (4.1.37) and (4.1.38) represent homogeneous system, in which A_1 and A_2 are to be determined. The matrix of coefficients must be singular, i.e. the determinant must vanish for non trivial solution.

$$\begin{vmatrix} iC_{13} + C_{33}\alpha_1 s_1 & iC_{13} + C_{33}\alpha_2 s_2 \\ s_1 + i\alpha_1 & s_2 + i\alpha_2 \end{vmatrix} = 0. \tag{4.1.39}$$

It implies,

$$(iC_{13} + C_{33}\alpha_1 s_1)(s_2 + i\alpha_2) - (iC_{13} + C_{33}\alpha_2 s_2)(s_1 + i\alpha_1) = 0. \tag{4.1.40}$$

Using Eqs. (4.1.33), (4.1.34) and (4.1.35) in Eq. (4.1.40), after simplification we get

$$\begin{aligned} &(\rho c^2 - C_{44})[k^2 C_{13}^2 + C_{33}\{k^2(\rho c^2 - C_{11}) + \rho\Omega^2\}] - \\ &k\rho c^2 \sqrt{C_{33}C_{44}} \sqrt{\{k^2(\rho c^2 - C_{11}) + \rho\Omega^2\}(\rho c^2 - C_{44})} = 0. \end{aligned} \tag{4.1.41}$$

Simplifying it further

$$\begin{aligned} &\sqrt{\frac{C_{33}}{C_{44}}(\frac{\rho c^2}{C_{11}} - \frac{C_{44}}{C_{11}})}[\frac{k^2 C_{13}^2}{C_{11}C_{33}} + \frac{k^2 \rho c^2}{C_{11}} - k^2 + \frac{\rho\Omega^2}{C_{11}}] \\ &-k\rho c^2 \sqrt{k^2(\frac{\rho c^2}{C_{11}} - 1) + \frac{\rho\Omega^2}{C_{11}}} = 0, \end{aligned} \tag{4.1.42}$$

or

$$\sqrt{\frac{C_{33}}{C_{44}}\frac{(\frac{\rho c^2}{C_{11}}-\frac{C_{44}}{C_{11}})}{(\frac{\rho c^2}{C_{11}}-1+\frac{\rho\Omega^2}{k^2C_{11}})}}[\frac{C_{13}^2}{C_{11}C_{33}}+\frac{\rho c^2}{C_{11}}-1+\frac{\rho\Omega^2}{C_{11}k^2}]-\frac{\rho c^2}{C_{11}}=0. \quad (4.1.43)$$

If we let

$$u=\frac{\rho c^2}{C_{11}},\ a=\frac{C_{44}}{C_{33}},\ b=\frac{C_{44}}{C_{11}},\ p=\frac{C_{13}^2}{C_{11}C_{33}},\text{and } r=\frac{\rho\Omega^2}{k^2C_{11}}. \quad (4.1.44)$$

Equation (4.1.43) becomes

$$\sqrt{\frac{1}{a}\frac{(u-b)}{(u+r-1)}}\,[p+u+r-1]-u=0, \quad (4.1.45)$$

or

$$u=\sqrt{\frac{1}{a}\frac{(u-b)}{(u+r-1)}}\,[p+u+r-1]\,. \quad (4.1.46)$$

Squaring above Eq. (4.1.46) yields

$$u^2=\frac{1}{a}\frac{(u-b)}{(u+r-1)}(p+u+r-1)^2, \quad (4.1.47)$$

or

$$(u-b)(p+u+r-1)^2=au^2(u+r-1). \quad (4.1.48)$$

After simplification we get cubic equation in u as follows

$$\begin{aligned}&(1-a)u^3+\{2p-b+(2-a)(r-1)\}\,u^2 \quad (4.1.49)\\&+(p+r-1)(p+r-1-2b)u-b(p+r-1)^2=0.\end{aligned}$$

One can solve this equation by using MATLAB or MATHEMATICA. Three values are obtained after solving it by MATHEMATICA, out of which one is real and two are complex conjugates. But of course we are interested in real roots of non dimensional wave speed u.

4.1.1 Rayleigh waves speed in some transversely isotropic materials for an angular frequency.

In Eq. (4.1.43), Ω is arbitrary, so its value can be chosen at random. For our convenience, we set

$$(\frac{\Omega}{k})^2 = \frac{C_{11}}{\rho}, \tag{4.1.50}$$

which on substituting in Eq. (4.1.44) gives

$$u = \frac{c^2\Omega^2}{k^2}. \tag{4.1.51}$$

Using Eq. (4.1.51) in Eq. (4.1.44), we get $r = 1$. Substituting $r = 1$ in Eq. (4.1.49)

$$(1-a)u^3 + (2p-b)u^2 + p(p-2b)u - bp^2 = 0. \tag{4.1.52}$$

Equation (4.1.52) can be solved for u for different materials. As an example we will solve it here for Cadmium, following [15]. Stiffness constants and mass density for Cadmium are as follows:

$$\begin{aligned} &\rho = 4824\, Kg\, m^{-3},\ C_{11} = 11.6 \times 10^{10}\ Nm^{-2},\ \ C_{13} = 4.1 \times 10^{10}\ Nm^{-2}. \\ &C_{33} = 5.09 \times 10^{10}\ Nm^{-2},\ C_{44} = 1.96 \times 10^{10}\ Nm^{-2}. \end{aligned} \tag{4.1.53}$$

Substituting all these values in Eq. (4.1.44) we get,

$$\begin{aligned} a &= \frac{C_{44}}{C_{33}} = 0.385069, \\ b &= \frac{C_{44}}{C_{11}} = 0.168966,\ p = \frac{C_{13}^2}{C_{11}C_{33}} = 0.284703. \end{aligned} \tag{4.1.54}$$

Using a, b, p and r in Eq. (4.1.52) we get

$$0.614931\, u^3 + 0.40044\, u^2 - 0.015154456\, u - 0.013695674 = 0. \tag{4.1.55}$$

Solving this equation by MATHEMATICA we obtain three values of u, two of which are negative, but we will choose only positive real value, because we are interested in real, positive roots of c. Eq. (4.1.55) implies,

$$u = 0.178648,\ \ c = 1780.46\,\text{m/s}. \tag{4.1.56}$$

The material properties and mass densities for various other hexagonal materials [13] used for the calculation of speed are given in Table 4.1.

Stiffness (10^{10}N/m^2)	**Be**	**Ceramic PZT**	**ZnO**	**Ti**
C_{11}	29.2	13.9	20.97	16.24
C_{12}	2.67	7.8	12.11	9.20
C_{13}	1.4	7.4	10.51	6.90
C_{33}	33.64	11.5	21.09	18.07
C_{44}	16.25	2.56	4.25	4.67
Mass density (ρ) Kg/m^3	1848	7500	5676	4506

Table 4.1: Elastic stiffness constants and mass densities of various materials.

The speed of Rayleigh waves in some other rotating hexagonal materials is given in Table 4.2.

Material	u	**Rayleigh wave speed(m/s)**
Beryllium(Be)	1.07281	13019.7
Ceramic PZT-4	0.189525	3775.63
Zinc Oxide(ZnO)	0.230412	2917.56
Cadmium Sulphide (CdS)	0.178648	1780.46
Titanium (Ti)	0.324916	3422.02

Table 4.2: Rayleigh wave speeds for rotating hexagonal materials.

We now consider the non-rotating case i.e let us set $\Omega = 0$. In Eq. (4.1.44), putting $\Omega = 0$ implies $r = 0$. Eq. (2.5.4) becomes,

$$(1-a)u^3 + \{2p - b - 2 + a\}u^2 + (p-1)(p-1-2b)u - b(p-1)^2 = 0. \tag{4.1.57}$$

Considering Eq. (4.1.44), we calculate the speed of Rayleigh wave through non-rotating solids. As an example we will consider Cadmium Sulphide again. Using Eq. (4.1.54) in Eq. (4.1.57) we get the following cubic equation

$$0.8408u^3 - 1.48208u^2 + 0.79355u - 0.093676 = 0, \tag{4.1.58}$$

which on solving for u by Mathematica gives three values for u, two of which are negative, but for real roots of c only positive real value of u will be considered. Equation (4.1.58) implies,

$$u = 0.163166, \; c = 1701.56. \tag{4.1.59}$$

This value of c is satisfying

$$\rho c^2 < \min\{C_{11}, C_{44}\}. \tag{4.1.60}$$

This condition was proposed by Pham and Ogden [4]. In the similar way the speed of Rayleigh waves in some other non-rotating hexagonal materials is calculated as shown in Table 4.3

Material	$\boldsymbol{u}$	**Rayleigh wave speed(m/s)**
Beryllium(Be)	0.414131	4726.996
Ceramic PZT-4	0.163705	3509.03
Zinc Oxide(ZnO)	0.185062	2614.79
Cadmium Sulphide (CdS)	0.163166	1701.56
Titanium (Ti)	0.25184	3012.73

Table 4.3: Rayleigh wave speed for non-rotating hexagonal materials.

4.1.2 Numerical results for Rayleigh wave speed through some hexagonal materials

Now we shall discuss the numerical results for the wave speed through orthotropic elastic materials. For this purpose, the expression represented by Eq. (4.1.44) will be written with simple notations. Let

$$u = \frac{\rho c^2}{C_{11}}, \quad r = \frac{\rho \Omega^2}{C_{11} k^2}. \tag{4.1.61}$$

Equation (4.1.43) will become

$$\sqrt{\left[\frac{C_{33}}{C_{44}}\left(\frac{u - \frac{C_{44}}{C_{11}}}{u - 1 + r}\right)\right]}\left[\frac{(C_{13})^2}{C_{11}C_{33}} + u - 1 + r\right] - u = 0. \tag{4.1.62}$$

Using the values of material constants and mass density for Beryllium from Table 4.1 we obtain the following graph as shown in Figure 4.2

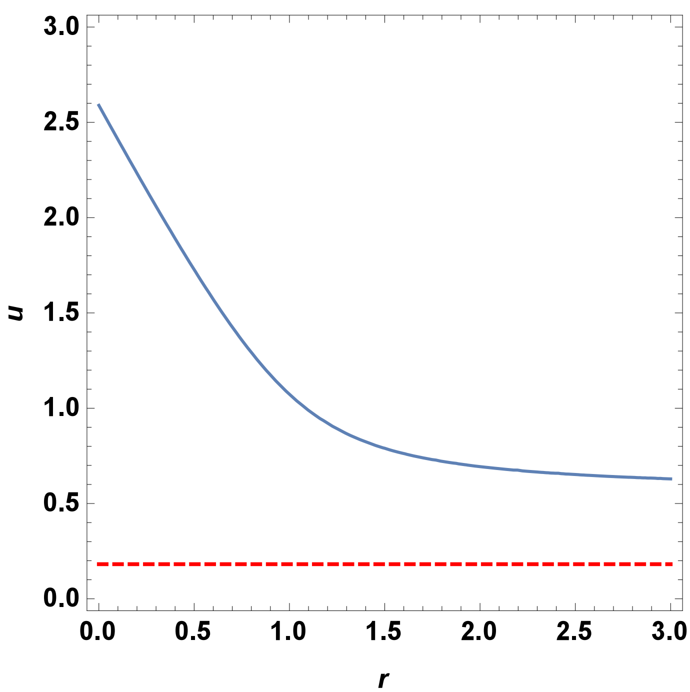

Figure 4.2: Variation in speed of Rayleigh wave with rotation in Beryllium. Continuous curve indicates the variation of dimensionless speed with rotation, where as the dashed curve shows the behavior of the speed without rotational effects in Beryllium.

The graph shown in Figure 4.2 represents the variation of the Rayleigh wave speed with rotation in Beryllium. It is evident that as rotation r increases, the dimensionless speed u decreases initially but after some time it turns out to be constant which illustrates that further increase in rotation does not effect the wave speed. The dotted curve represents the non dispersive case. The dashed line is plotted by substituting $\Omega = 0$ in Eq. (4.1.43). The wave speed is calculated at $\Omega = 0$ and plotted as constant horizontal line for the particular value of u. The plots of few more hexagonal elastic materials are shown in the following figures.

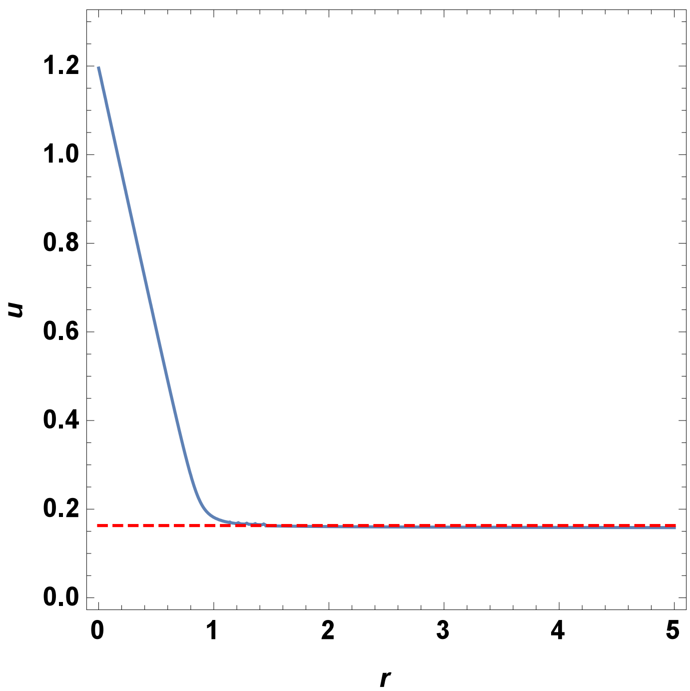

Figure 4.3: Variation in speed of Rayleigh wave with rotation in Cadmium Sulphide.

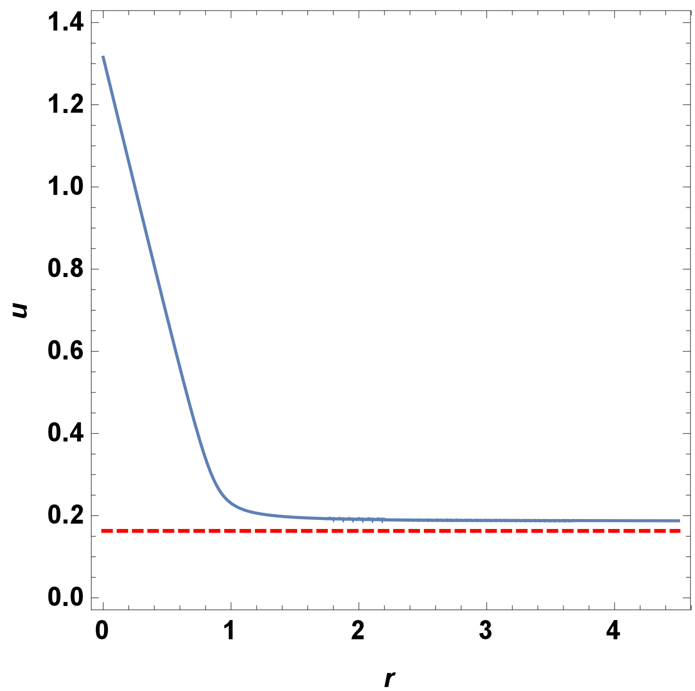

Figure 4.4: Variation in speed of Rayleigh wave with rotation in Ceramic-PZT.

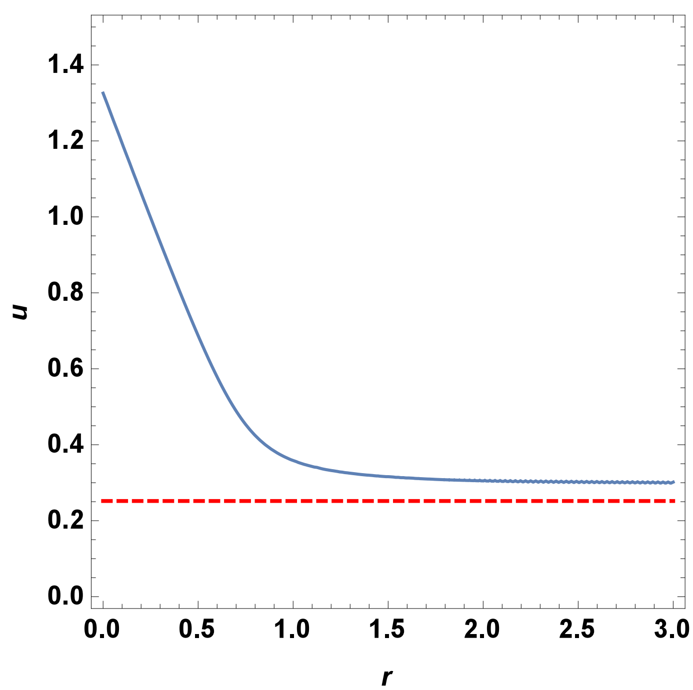

Figure 4.5: Variation in speed of Rayleigh wave with rotation in Titanium.

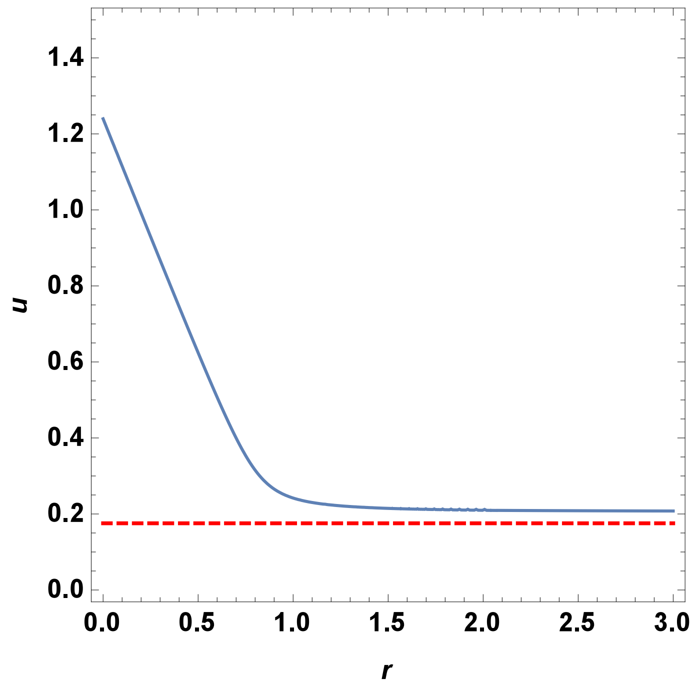

Figure 4.6: Variation in speed of Rayleigh wave with rotation in Zinc Oxide.

The graph shown in Figure 4.3 shows that the Rayleigh wave speed in rotating Cadmium Sulphide varies with rotation. It is evident that like Beryllium it decreases initially when the rotation r increases and become constant after some time. The dotted line lying exactly on the continuous line indicates that when there is no rotation the speed of the Rayleigh wave can be less than or equal to the speed with certain rotation. Figure 4.4, 4.5, and 4.6 shows the effect of rotation on Rayleigh wave speed through Ceramic-PZT, Titanium and Zinc Oxide respectively. They show nearly the same behavior as in Beryllium i.e. the speeds without rotation are strictly less than the ones under rotational effects.

4.2 Rayleigh waves propagation through rotating orthotropic medium

In this section the effect of rotation on the speed of Rayleigh waves traveling through orthotropic materials will be studied. Consider a semi infinite stress free elastic half space of orthotropic material as shown in the Figure 4.7

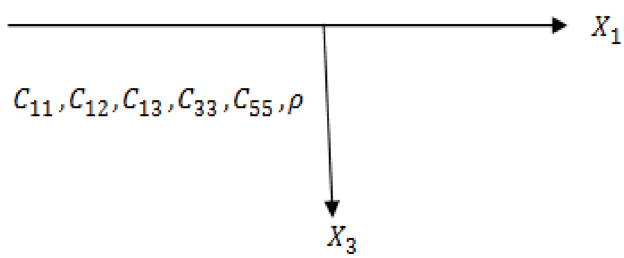

Figure 4.7: Orthotropic elastic half space.

It is evident from the above figure that the material is placed in the plane $X_3 \leq 0$. C_{11}, C_{12}, C_{13}, C_{33} and C_{55} are the material properties and ρ is the mass density of orthotropic material. The rectangular coordinate is chosen in such a way that X_3-axis is perpendicular to the surface. Rayleigh waves are again considered to be traveling along X_1 direction in X_1X_3-plane. The displacement vector is same as expressed in Eq. (4.1.1). By substituting Eq. (2.5.7) in generalized Hook's law represented by Eq. (2.3.3), the following set of equations is obtained

$$T_{11} = C_{11}\frac{\partial u_1}{\partial x_1} + C_{33}\frac{\partial u_3}{\partial x_3}, \quad (4.2.1)$$

$$T_{33} = C_{13}\frac{\partial u_1}{\partial x_1} + C_{33}\frac{\partial u_3}{\partial x_3}, \quad (4.2.2)$$

$$T_{13} = C_{55}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right), \quad (4.2.3)$$

where the elastic parameters $C_{11}, C_{13}, C_{33}, \text{and} C_{55}$ satisfy inequalities mentioned in Eq. (4.1.5). These equations represent the necessary and sufficient conditions for the positive definiteness of strain energy function of the material as mentioned earlier. If a homogenous orthotropic elastic body is considered to be rotating along x_3-axis with a constant angular velocity $\mathbf{\Omega} = \Omega(0,0,1)$. The equations of motion for infinitesimal deformation in the absence of body forces in rotating orthotropic medium are similar to those represented by Eq. (3.3.1). For $i = (1,3)$ Eq. (3.3.1) for orthotropic material becomes exactly similar to the expression given by Eq. (4.1.6). Substituting Eqs .(4.2.1), (4.2.2) and (4.2.3) in Eqs. (4.1.6) and (4.1.7) we get,

$$C_{11}\frac{\partial^2 u_1}{\partial x_1^2} + C_{13}\frac{\partial^2 u_3}{\partial x_3 \partial x_1} + C_{55}\left(\frac{\partial^2 u_1}{\partial x_3^2} + \frac{\partial^2 u_3}{\partial x_1 \partial x_3}\right) = \rho\left(\frac{\partial^2 u_1}{\partial t^2} - \Omega^2 u_1\right), \quad (4.2.4)$$

$$C_{13}\frac{\partial^2 u_3}{\partial x_1 \partial x_3} + C_{33}\frac{\partial^2 u_3}{\partial x_3^2} + C_{55}\left(\frac{\partial^2 u_1}{\partial x_3 \partial x_1} + \frac{\partial^2 u_3}{\partial x_1^2}\right) = \rho\frac{\partial^2 u_3}{\partial t^2}. \quad (4.2.5)$$

The boundary conditions for zero traction are represented by Eq. (4.1.9). Usual requirements that the displacement and the stress components of surface waves decay when they move away from the boundary implies Eq. (4.1.10). We consider the solution of the form

$$u_j = \xi_j(kx_3)\exp[ik(x_1 - ct)];\ j = 1,3, \quad (4.2.6)$$

where, c is the speed of Rayleigh wave and k is the wave number, $\xi_j, j = 1,3$ are the functions to be determined. Substituting Eq. (4.2.6) in Eqs. (4.2.4) and (4.2.5) we get

$$C_{44}k^2\xi_1^{''} + ik^2\xi_3^{'}(C_{13} + C_{55}) + [k^2(\rho c^2 - C_{11}) + \rho\Omega^2]\xi_1 = 0, \quad (4.2.7)$$

$$C_{33}\xi_3^{''} + i(C_{13} + C_{55})\xi_1^{'} + \xi_3(\rho c^2 - C_{55}) = 0. \quad (4.2.8)$$

By considering Eqs. (4.2.1), (4.2.2) ,(4.2.3) and Eq. (4.2.6), the boundary conditions given in Eq. (4.1.9) become

$$\xi_1^{'} + i\xi_3 = 0, \quad (4.2.9)$$

and

$$C_{33}\xi_3^{'} + iC_{13}\xi_1 = 0,\ \text{on the plane } x_3 = 0, \quad (4.2.10)$$

also Eqs. (4.1.10) and (4.1.11) gives

$$\xi_j, \xi_j^{'} \to 0 \text{ as } x_3 \to -\infty. \tag{4.2.11}$$

The Laplace transformation of Eqs. (4.2.7) and (4.2.8) using Eq. (4.2.9) and (4.2.10) yields

$$\begin{aligned}
&\left[k^2\left(C_{55}s^2+\rho c^2-C_{11}\right)+\rho\Omega^2\right]\bar{\xi_1}(s)+ik^2\left(C_{55}+C_{13}\right)s\bar{\xi_3}(s)= \\
&C_{55}k^2\left(s\xi_1(0)+\xi_1^{'}(0)\right)+ik^2\left(C_{55}+C_{13}\right)\xi_3(0), \qquad (4.2.12)\\
&i\left(C_{13}+C_{55}\right)s\bar{\xi_1}(s)+\left(C_{33}s^2-C_{55}+\rho c^2\right)\bar{\xi_3}(s)= \\
&i\left(C_{55}+C_{13}\right)\xi_1(0)+C_{33}\left(\xi_3(0)s+\xi_3^{'}(0)\right). \qquad (4.2.13)
\end{aligned}$$

Solving Eqs. (4.2.12) and (4.2.13) we get

$$\bar{\xi_1}(s)=\frac{\begin{vmatrix} C_{55}k^2s\xi_1+\xi_1^{'}+ik^2(C_{13}+C_{55})\xi_3 & ik^2(C_{13}+C_{55})s \\ i(C_{13}+C_{55})\xi_1+C_{33}\left(s\xi_3+\xi_3^{'}\right) & (C_{33}s^2-C_{55}+\rho c^2)\end{vmatrix}}{R}, \tag{4.2.14}$$

where

$$\begin{aligned}
R=\;&k^2C_{33}C_{55}s^4+[k^2\{(C_{55}+C_{13})^2+C_{33}(\rho c^2-C_{11}) \\
&+C_{55}(\rho c^2-C_{55})\}\rho\Omega^2C_{33}]s^2+(\rho c^2-C_{55})\{k^2(\rho c^2-C_{11})+\rho\Omega^2\}. \qquad (4.2.15)
\end{aligned}$$

For non trivial solution the determinant in numerator of Eq. (4.2.14) must be zero. Since $\bar{\xi_1}(s)\neq 0$, it implies that the expression in denominator should be zero, which is 4^{th} order equation in 's'. It implies

$$\begin{aligned}
&k^2C_{33}C_{55}s^4+[k^2\left\{(C_{13}+C_{55})^2+C_{33}(\rho c^2-C_{55})\right\}+ \\
&C_{33}\rho\Omega^2]s^2+(\rho c^2-C_{55})\left\{k^2(\rho c^2-C_{11})+\rho\Omega^2\right\}=0. \qquad (4.2.16)
\end{aligned}$$

Let the roots of Eq (4.2.16), which is quadratic in s^2 , be s_1^2 and s_2^2. Since s is the Laplace variable so the roots s_1^2 and s_2^2 must have positive real parts. Eq. (4.2.14) after factorization of Eq. (4.2.16) becomes

$$\bar{\xi_1}(s)=\frac{B_1}{s-s_1}+\frac{B_2}{s-s_2}+\frac{B_3}{s+s_1}+\frac{B_4}{s+s_2}. \tag{4.2.17}$$

In the above equation B_1, B_2, B_3 and B_4 are constants to be determined.The inverse Laplace transformation of Eq. (4.2.17) using (4.2.11) yields

$$\xi_1(y) = B_1 \exp(s_1 y) + B_2 \exp(s_2 y). \tag{4.2.18}$$

By Eqs. (4.2.12), (4.2.13) and (4.2.14), it is clear that $\bar{\xi_1}(s)$ and $\bar{\xi_3}(s)$ are linearly dependent, so they can be written in linear combination of each other.

$$\xi_3(y) = \beta_1 B_1 \exp(s_1 y) + \beta_2 B_2 \exp(s_2 y). \tag{4.2.19}$$

Since ξ_1 and ξ_3 are linearly dependent, Eq. (4.2.14) implies

$$\beta_j = i\frac{[C_{55}k^2 s_j^2 + k^2(\rho c^2 - C_{11}) + \rho\Omega^2]}{k^2(C_{55} + C_{13})s_j}\ ,\ \ j = 1, 3. \tag{4.2.20}$$

Let s_1^2 and s_2^2 be the roots of Eq. (4.2.16), the sum of roots is

$$s_1^2 + s_2^2 = -\frac{k^2[(C_{55}+C_{13})^2 + C_{33}(\rho c^2 - C_{11}) + C_{55}(\rho c^2 - C_{55})] + C_{33}\rho\Omega^2}{k^2 C_{33} C_{55}}, \tag{4.2.21}$$

and the product of roots of Eq. (4.2.16) is

$$s_1^2 s_2^2 = \frac{(\rho c^2 - C_{55})[k^2(\rho c^2 - C_{11}) + \rho\Omega^2]}{k^2 C_{33} C_{55}}. \tag{4.2.22}$$

Substituting values of ξ_1 and ξ_3 from Eqs. (4.2.18) and (4.2.19) into Eqs. (4.2.9) and (4.2.10)

$$\begin{aligned} &iC_{13}[B_1 \exp(s_1 y) + B_2 \exp(s_2 y)] + \\ &C_{33}[\beta_1\ B_1 s_1 \exp(s_1 y) + \beta_2\ B_2\ s_2 \exp(s_2 y)] = 0, \end{aligned} \tag{4.2.23}$$

at $y = 0$,

$$[iC_{13} + C_{33}\beta_1 s_1]B_1 + [iC_{13} + C_{33}\beta_2 s_2]B_2 = 0. \tag{4.2.24}$$

Equation (4.2.10) becomes,

$$B_1(s_1 + i\beta_1) + B_2(s_2 + i\beta_2) = 0 \tag{4.2.25}$$

Equations (4.2.24) and (4.2.25) represent homogeneous system, in which B_1 and B_2 are to be determined. The matrix of coefficients must be singular, i.e. the determinant must vanish for non trivial solution.

$$\begin{vmatrix} iC_{13} + C_{33}\beta_1 s_1 & iC_{13} + C_{33}\beta_2 s_2 \\ s_1 + i\beta_1 & s_2 + i\beta_2 \end{vmatrix} = 0. \tag{4.2.26}$$

It implies,

$$(iC_{13} + C_{33}\beta_1 s_1)(s_2 + i\beta_2) - (iC_{13} + C_{33}\beta_2 s_2)(s_1 + i\beta_1) = 0. \tag{4.2.27}$$

Using Eqs. (4.2.20), (4.2.22) and (4.2.23) in Eq. (4.2.27), after simplification we get,

$$\begin{aligned}&(\rho c^2 - C_{55})[k^2 C_{13}^2 + C_{33}\{k^2(\rho c^2 - C_{11}) + \rho\Omega^2\}] - \\ &k\rho c^2 \sqrt{C_{33}C_{55}}\sqrt{\{k^2(\rho c^2 - C_{11}) + \rho\Omega^2\}(\rho c^2 - C_{55})} = 0.\end{aligned} \tag{4.2.28}$$

Simplifying it further,

$$\begin{aligned}&\sqrt{\frac{C_{33}}{C_{55}}(\frac{\rho c^2}{C_{11}} - \frac{C_{55}}{C_{11}})}[\frac{k^2 C_{13}^2}{C_{11}C_{33}} + \frac{k^2 \rho c^2}{C_{11}} - k^2 + \frac{\rho\Omega^2}{C_{11}}] \\ &-k\rho c^2 \sqrt{k^2(\frac{\rho c^2}{C_{11}} - 1) + \frac{\rho\Omega^2}{C_{11}}} = 0,\end{aligned} \tag{4.2.29}$$

or

$$\sqrt{\frac{C_{33}}{C_{55}}\frac{(\frac{\rho c^2}{C_{11}} - \frac{C_{55}}{C_{11}})}{(\frac{\rho c^2}{C_{11}} - 1 + \frac{\rho\Omega^2}{k^2 C_{11}})}}[\frac{C_{13}^2}{C_{11}C_{33}} + \frac{\rho c^2}{C_{11}} - 1 + \frac{\rho\Omega^2}{C_{11}k^2}] - \frac{\rho c^2}{C_{11}} = 0. \tag{4.2.30}$$

Equation (4.2.30) gives the speed of Rayleigh waves propagating through orthotropic materials.

4.2.1 Rayleigh waves speed in some orthotropic materials for an angular frequency

The angular velocity Ω can be chosen randomly but for convenience let

$$(\frac{\Omega}{k})^2 = \frac{C_{11}}{\rho}. \tag{4.2.31}$$

Substituting Eq. (4.2.31) in Eq. (4.2.30) we get

$$\sqrt{\frac{C_{33}}{C_{55}}\frac{(\frac{\rho c^2}{C_{11}}-\frac{C_{55}}{C_{11}})}{(\frac{\rho c^2}{C_{11}})}}[\frac{C_{13}^2}{C_{11}C_{33}}+\frac{\rho c^2}{C_{11}}]-\frac{\rho c^2}{C_{11}}=0. \tag{4.2.32}$$

If we let

$$u=\frac{\rho c^2}{C_{11}},\ \lambda=\frac{C_{55}}{C_{33}},\ \mu=\frac{C_{55}}{C_{11}},\ \ p=\frac{C_{13}^2}{C_{11}C_{33}},\text{and}\ \ r=\frac{\rho\Omega^2}{k^2C_{11}}. \tag{4.2.33}$$

Equation (4.2.32) becomes

$$\sqrt{\frac{1}{\lambda}\frac{(u-\mu)}{(u+r-1)}}\left[p+u+r-1\right]-u=0, \tag{4.2.34}$$

or

$$u=\sqrt{\frac{1}{\lambda}\frac{(u-\mu)}{(u+r-1)}}\left[p+u+r-1\right]. \tag{4.2.35}$$

Squaring above Eq. (4.2.35) yields

$$u^2=\frac{1}{\lambda}\frac{(u-\mu)}{(u+r-1)}(p+u+r-1)^2, \tag{4.2.36}$$

or

$$(u-\mu)(p+u+r-1)^2=\lambda u^2(u+r-1). \tag{4.2.37}$$

After simplification we get cubic equation in u as follows

$$\begin{aligned}&(1-\lambda)u^3+\{2p-\mu+(2-\lambda)(r-1)\}\,u^2\\&+(p+r-1)(p+r-1-2\mu)u-\mu(p+r-1)^2=0.\end{aligned} \tag{4.2.38}$$

It is evident from Eq. (4.2.38) that there will be three values for u but here only those positive real values are considered which satisfy the following expression [4]

$$\rho c^2\leq\min\left\{C_{11},C_{55}\right\}. \tag{4.2.39}$$

Other values of u will be treated as extraneous roots. We will now consider some orthotropic materials and calculate the wave speed in them by using Mathematica.

The values of elastic parameters of orthotropic materials along with their mass densities are given in Table 4.4

Stiffness (10^{10}**N/m^2**)	**Iodic acid** (HIO_3)	**Barium Sodium Nioabte** ($Ba_2NaNb_5O_{15}$)
C_{11}	3.01	23.9
C_{13}	1.11	5.0
C_{33}	4.29	13.5
C_{55}	2.06	6.6
Density (ρ) **Kg/m^3**	4640	5300

Table 4.4: Elastic stiffness constants and mass densities of some orthotropic materials.

The wave speeds for two rotating orthotropic solids are given in the Table 4.5.

Material	u	c (m/s)
Iodic acid(HIO_3)	1.1599	3966.36
Barium Sodium Niobate($Ba_2NaNb_5O_{15}$)	0.424613	4375.8

Table 4.5: Rayleigh wave speed for rotating orthotropic materials.

We now consider the non-rotating case i.e let us set $\Omega = 0$ in Eq. (4.2.30) The speed of Rayleigh waves through some materials without rotation effect is calculated by solving Eq. (4.2.37) using Mathematica software. The values of elastic parameters and densities are substituted from the Table 4.5. Three values for c will be obtained through each orthotropic material but only those will be considered which satisfy (4.2.39). Wave speeds for two non rotating orthotropic solids are given in Table 4.6.

Material	u	c m/s)
Iodic acid(HIO_3)	0.441176	1691.73
Barium Sodium Niobate($Ba_2NaNb_5O_{15}$)	0.233231	3243.05

Table 4.6: Rayleigh wave speed for non rotating orthotropic materials.

It is evident from the Table 4.6 that the rotation effect has enhanced the speed of Rayleigh wave propagating through orthotropic material.

4.2.2 Numerical results for Rayleigh wave speed through orthotropic materials

Now we shall discuss the numerical results for the wave speed through orthotropic elastic materials. Let

$$y = \frac{\rho c^2}{C_{11}}, \;\; x = \frac{\rho \Omega^2}{C_{11} k^2}. \tag{4.2.40}$$

The expression represented by Eq. (4.2.32) will become simpler.

$$\sqrt{\left[\frac{C_{33}}{C_{55}}\left(\frac{y - \frac{C_{55}}{C_{11}}}{y - 1 + x}\right)\right]}\left[\frac{(C_{13})^2}{C_{11}C_{33}} + y - 1 + x\right] - y = 0. \tag{4.2.41}$$

Using the values of material constants and mass density for Iodic acid from Table 4.4 we obtain the following graph as shown in Figure 4.8.

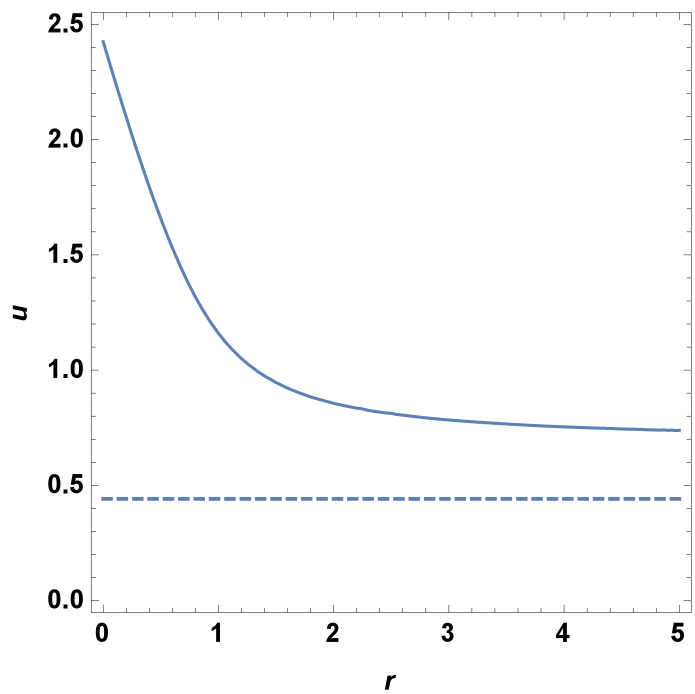

Figure 4.8: Variation in speed of Rayleigh wave with rotation in Iodic Acid.

It is evident from the graph shown in Figure 4.8 that as rotation r increases, the dimensionless speed decreases initially but after some time it turns out to be constant which illustrates that further increase in rotation does not effect the wave speed. The dotted curve represents the non dispersive case.

Similarly by using the values of material constants and mass density for the material Barium Sodium Niobate from Table 4.5 we obtain the following graph as shown in Figure 4.9.

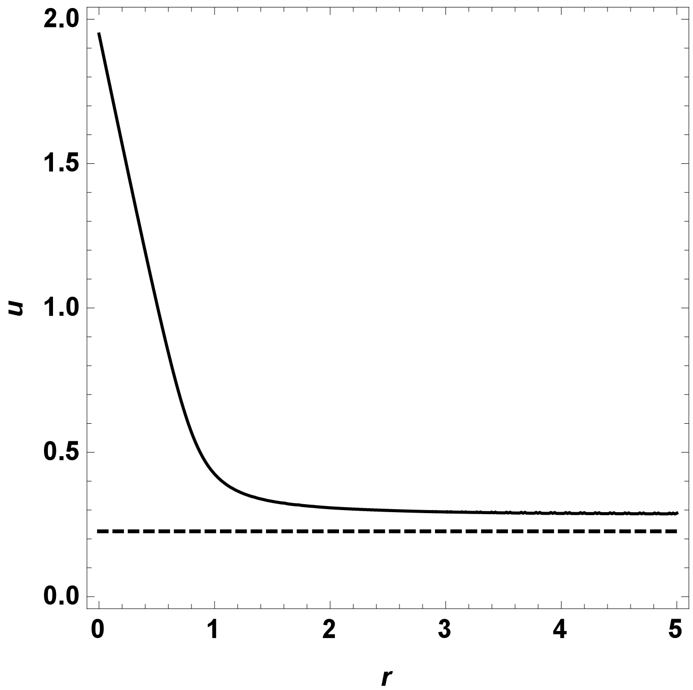

Figure 4.9: Variation in speed of Rayleigh wave with rotation in Barium Sodium Niobate.

It can be seen from the plot shown in Figure 4.8 and 4.9 that the speed of Rayleigh waves passing through these particular materials also decreases to some extent like hexagonal materials but as we go on increasing the angular velocity further it becomes constant. Here too the dotted curve represents zero rotation. The value of u is calculated for $\Omega = 0$ and plotted.

Moreover it is also clear from both the plots shown in Figures 4.8 and 4.9 that the speed of the waves is always greater if the rotation is added. The speed have less values when rotation is zero.

Chapter 5

Conclusions

In this dissertation the effect of rotation on the speeds of surface waves particularly Love and Rayleigh waves propagating through anisotropic media has been discussed. The brief summary of results and conclusion is as follows.

It is noted that the Love waves are dispersive in anisotropic elastic half-spaces, particularly those of hexagonal and orthotropic materials unlike Rayleigh waves. The propagation of Love waves requires a half-space covered with a layer. The Love wave speed is independent of rotation.

The effect of rotation on the speed of Rayleigh waves traveling through anisotropic solids is observed. Rayleigh waves show dispersive nature under the effect of rotation. Ω can be any arbitrary number but in this thesis a special case is considered i.e. $(\frac{\Omega}{k})^2 = \frac{C_{11}}{\rho}$. For this case it is observed that the rotation can reduce the wave speed to some extent. The wave speed however becomes independent of rotation for higher values of Ω.

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