A Two-Sided Discrete Concave Market

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Dedicated to my loving parents

and

caring husband.

Abstract

The marriage model and the assignment model are considered standard models in the theory of two-sided matching markets problems. These two matching problems are important because most of the daily life dealings are bilateral. Existence of a stable matching is the central solution concept of these models. Participants of these models are partitioned into two disjoint sets. Assignment model is one, in which indivisible goods are exchanged for money and each participant buys or sells exactly one item. Marriage model is related to the college admissions and the labour market for medical interns. Each participant of these markets has a preference list that depicts the order of his (her) choice. In the marriage model, exchange of money is not involved. That is why, participants of marriage model are called rigid. In the assignment game, exchange of money and negotiations are permitted. That is why, the participants of assignment model are called flexible.

We propose a two-sided matching market model with discrete concave utility functions and weighted incomes and payments. Our model includes the marriage model, the assignment model and several well-known hybrid models as special cases. We propose an algorithm to find a pairwise strictly stable outcome in our model.

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Introduction

Two-sided matching problems have a wide range of applications in practical life. We observe most of its uses in economics and business. This is due to the reason that we are faced with bilateral dealings in most of the situations. National Resident Matching Program (NRMP) of United States is a prominent application of two-sided matching models.

The concept of two-sided matching markets, was first introduced by Gale and Shapley in their pioneering work [12]. In two-sided matching markets, the set of participants is partitioned into two disjoint sets. We aim to match the participants of one set to that of the other set. A matching is a set consisting of pairs of participants from opposite sets and each participant appears at most once. A matching is said to be stable if each participant is acceptable to his (her) partner and there are no two participants who prefer each other to their partners.

The marriage model due to Gale and Shapley [12] and the assignment game by Shapley and Shubik [18] are considerd as standard models in the literature. In these two models, each participant is matched with at most one participant of the opposite side. Each participant has a strictly ordered preference list mentioning his (her) order of choice for each participant of the opposite side. Marriage model does not involve money transfer while the assignment game is based on monetary transfer. This is the main difference between the two models. This is why, the participants of marriage model are called rigid and that of the assignment game referred to as flexible.

Many extensions and developments to these two models [12, 18] have been added to the literature. The most prominent are the hybrid models by Eriksson and Karlander [5] and Sotomayor [19]. These models are the generalization of the above two models. Existence of stable outcome and the core is discussed in [5, 19]. Fujishige and Tamura [9] described generalization of the hybrid models by Eriksson and Karlander [5] and Sotomayor [19]. Fujishige and Tamura [10] used $M^\natural$-concave functions as value functions.

In first chapter of this thesis, we discuss some basic concepts about graphs, discrete convex analysis, linear programming and two-sided matching problems. Few results about M/ M$^\natural$-concave functions and the linear programming problems are mentioned in it. The second chapter constitutes the review of the two-sided matching problems: the marriage model due to Gale and Shapley [12] and the assignment game due to Shapley and Shubik [18]. The comparison of the two models is also discussed in this chapter. In chapter three, we extend the model of Fujishige and Tamura [10] by generalizing payoff functions. We introduce weighted income and weighted payments. We propose an algorithm to find a pairwise strictly stable outcome. We also discuss correctness and termination of the algorithm.

Contents

Chapter 1

Preliminaries

This chapter contains some introductory knowledge of graph theory, discrete convex analysis, linear programming and two-sided matching.

1.1 Graph Theory

Numerous real-world phenomenas are describable through plots composed of a set of points and lines that join pairs of these points (not necessarily all of them). For example, electrical network problems, utilities problem, travelling salesman problem, etc. Mathematical generalization of the description of such problems leads to the concept of a graph.

A *graph* is represented by $G = (V_G, E_G)$, composed of a set of points V_G and a set of lines (curves) E_G. The points in V_G are called *vertices* of the graph G and lines (curves) in E_G called the *edges* of the graph G. Each edge e *joins* an unordered pair of vertices u, v of G and is denoted by $e = \{u, v\}$. Vertices u and v are called *end vertices* of e if e joins u and v. If e joins a vertex u to itself then it is called a *loop*. If two edges are associated with same pair of vertices, they are referred to as *parallel edges*. A graph containing no loops or parallel edges is called a *simple graph*. Total number of vertices in a graph G is called the *order* of the graph G.

If a vertex u is one of the ends of an edge e then u and e are said to be *incident* on each other. Vertices u and v are called *adjacent* if they are ends of a same edge. Number of edges incident on a vertex u (where each loop is counted twice) is called *degree* of the vertex u and is denoted by $d(u)$.

The following theorems are evident from the definition of degree:

Theorem 1.1.1 (Bondy and Murty [2]). *In a graph $G = (V_G, E_G)$, the following holds:*

$$\sum_{v \in V_G} d(v) = 2m$$

where m is the number of edges in G.

Theorem 1.1.2 (Bondy and Murty [2]). *Vertices of odd degree in a graph are always even in number.*

If all vertices of a graph possess same degree, then it is called a *regular graph.* If $d(v) = k$ for each $v \in V_G$ then the graph is called a k-*regular* graph.

Let G and H be two graphs such that $V_H \subseteq V_G$ and $E_H \subseteq E_G$, then H is called a *subgraph* of the graph G. Every graph is considered its own subgraph.

A simple graph G is said to be a *complete graph* if it contains an edge between any pair of its distinct vertices. It is written by K_n where n is the order of the graph G. A graph G is *bipartite* if its vertex set V_G can be partitioned into two subsets X and Y such that each edge of G has one end in X and the other in Y. The pair (X, Y) is called *bipartition* of G. It is clear from the definition that a bipartite graph cannot contain a loop.

Theorem 1.1.3 (Bondy and Murty [2]). *A graph is bipartite if and only if no odd cycle is contained in it.*

A simple bipartite graph with bipartition (X, Y) such that every vertex in X is adjacent to every vertex in Y is called a *complete bipartite graph*, denoted by K_{n_1,n_2}, where $n_1 = |X|$ and $n_2 = |Y|$.

1.1.1 Walk, Trail, Path and Cycle

A finite sequence $v_0, e_1, v_1, e_2, v_2, \ldots, e_k, v_k$ alternating in vertices and edges of a graph G such that $e_i = \{v_{i-1}, v_i\}$ for each $1 \leq i \leq k$ is called a *walk* in a graph G. The starting and ending vertices of a walk are called *origin* and *terminus* of walk, respectively, and number of edges in the sequence is called *length* of walk. A walk is obviously a subgraph of the graph G that contains it.

If all edges of a walk in a graph are distinct then it is said to be a *trail*. If all vertices of a trail are distinct then it is called a *path* in the graph. A walk is usually denoted by W and a path by P. *Length* of a path is the total number of its edges. A path has no repeated vertex or edge and if P is considered as a graph itself then each of its vertex has degree atleast one and at the most two. The vertices of a path P other than the origin and terminus are called *internal vertices* of the path. If there exists a path between the two vertices u and v of a graph, then they are said to be *reachable* from each other in the graph.

If origin and terminus of a walk coincide then it is called a *closed* walk in graph. A closed trail with just its end points repeated is called a *cycle* in a graph G. A loop is the simplest example of a cycle in a graph. It is obvious that when a cycle is considered as a graph itself, then every vertex of cycle has degree exactly two. A graph G containing no cycle is called *acyclic*.

1.1.2 Connected and Disconnected Graphs

A graph G is *connected* if it contains a path between any pair of its vertices. If there exist vertices in a graph G that are not reachable from each other by a path then G is said to be *disconnected*. Walk, trail, path and cycle are connected graphs. Some important properties and results are given below:

Theorem 1.1.4 (Bondy and Murty [2]). *A graph G is disconnected if and only if there exists a partition (X, Y) of the vertex set V_G such that there exists no edge in E_G whose one end lies in X and the other in Y.*

Theorem 1.1.5 (Bondy and Murty [2]). *If a graph contains exactly two vertices u, v of odd degree, it must contain a path joining u and v (the graph may be connected or disconnected).*

1.1.3 Directed Graphs (Digraphs)

If the edges of a graph are given some direction then these are called *directed edges or arcs.* A graph with directed edges is called a *directed graph.* Most of the physical situations require directed graphs to be dealt with. Some basic definitions and results of directed graphs are discussed in this section.

A graph $G = (V_G, A_G)$ with set of vertices V_G and arc set A_G is called a *directed graph (digraph).* Each arc a joins an ordered pair of vertices u and v of G and is denoted by $a = (u, v)$. An arc $a = (u, v)$ is represented by a line (curve) with an arrow sign on it pointing from u to v and is said to be *leaving* vertex u and *entering* the vertex v.

The number of arcs leaving a vertex u is called *out-degree* of u and is denoted by $d^+(u)$. The number of arcs entering a vertex u is its *in-degree* and is denoted by $d^-(u)$. It is obvious that in a directed graph G, the sum of all in-degrees is equal to the sum of all out-degrees, that is,

$$\sum_{v \in V_G} d^+(v) = \sum_{v \in V_G} d^-(v).$$

A sequence $v_0, a_1, v_1, a_2, v_2, \ldots, a_k, v_k$ alternating in vertices and arcs in a directed graph G such that $a_i = (v_{i-1}, v_i)$ for each $1 \leq i \leq k$ is called a *directed walk* in G. A *directed walk* from u to v is an alternating sequence of vertices and arcs beginning from u and ending at v. v is said to be *reachable* from u if there is a directed walk from u to v. If both the vertices u and v are reachable from each other then they are called *mutually reachable.* Vertices and arcs may repeat in a walk. A *directed trail* is defined as a directed walk with distinct arcs. A walk with distinct arcs and vertices is a *directed path.* A closed directed trail with distinct vertices (except origin and terminus) is called a *directed cycle.*

A digraph G is said to be *weakly connected* if its underlying graph (a simple graph obtained by replacing arcs by the non-directed edges) is connected. A digraph G is said to be *strongly connected* if every two vertices in G are mutually reachable from each other. A digraph is acyclic if it contains no directed cycle.

1.1.4 Distance and Weight

Length of a shortest path between any two vertices u and v of a graph G is called the *distance* between u and v. It is denoted by $d(u, v)$. If there does not exist a path between vertices u and v then the distance is defined by $d(u, v) = +\infty$. Distance of a vertex u from itself is zero, that is, $d(u, u) = 0$ for each $u \in V_G$.

For digraphs, the length of shortest directed path from the vertex u to the vertex v is called the *directed distance* $d(u, v)$ from u to v.

Let G be a graph with a given function $w : E_G \to \mathbf{R}$ which maps an edge to a real number. Such a function w is called *weight function*. For each $e \in E_G$, the number $w(e)$ is called *weight* of the edge e. A graph with a weight function is called a *weighted graph*. The weight of a path P in G is defined by $w(P) = \sum_{e \in P} w(e)$. The shortest path from u to v in a weighted graph G is a path P with minimum $w(P)$.

1.2 Discrete Convex Analysis

For real valued functions defined on integer lattice, Murota [15] developed a theory called the *discrete convex analysis*. The theory analogues the ordinary convex analysis, duality and optimization theory. Submodular functions and exchange axioms form the technical basis for its development. The field of optimization is crucially influenced by the convexity concepts for sets and functions. Convex functions possess many properties. First of all, we formally define a convex set and a concave function.

A set $T \subseteq \mathbf{R}^n$ is said to be a *convex set* if a line segment joining any two points of T, lies in T. That means, if y and z are any two points of T then $\varepsilon y + (1 - \varepsilon)z$ is contained in T for all $0 \leq \varepsilon \leq 1$.

A function $f : T \to \mathbf{R}$ is called a *concave function* if it satisfies the inequality given below:

$$\varepsilon f(y) + (1 - \varepsilon)f(z) \leq f(\varepsilon y + (1 - \varepsilon)z) \qquad (\forall\, \varepsilon \in [0, 1]).$$

Remark 1.2.1. A real valued function $h : T \to \mathbf{R}$ is defined to be a *convex function* if $-h$ is concave.

A concave function f is also defined from $\mathbf{R}^n \to \mathbf{R} \cup \{-\infty\}$ with its *effective domain* S defined by:

$$S = \{z \in \mathbf{R}^n \mid f(z) > -\infty\}.$$

It is usually denoted by dom$f$.

For the sake of discreteness, we will consider concave functions defined on integer lattice. Let V be any finite set, say, of the form $V = \{v_1, v_2, \ldots, v_n\}$. Then $\mathbf{Z}^V$ is defined to be a set of orderd tuples of the form $z = (z(u) \in \mathbf{Z} \mid u \in V)$. A discrete concave function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{-\infty\}$ is then defined by

$$\varepsilon f(y) + (1-\varepsilon)f(z) \leq f(\varepsilon y + (1-\varepsilon)z) \qquad (\forall\ \varepsilon \in [0,1]).$$

Let U be a subset of V, then the *characteristic vector* χ_U of set U is defined by

$$\chi_U(v) = \begin{cases} 1 & \text{if } v \in U \\ 0 & \text{if } v \in V \setminus U. \end{cases}$$

If U consists of single element u then its characteristic vector is simply denoted by χ_u and is given by

$$\chi_u(v) = \begin{cases} 1 & \text{if } v = u \\ 0 & \text{if } v \neq u. \end{cases}$$

For convenience, we define χ_0 to be the zero vector on $\mathbf{Z}^V$. For a vector $z = (z(u) \mid u \in V) \in \mathbf{Z}^V$, the *negative* and the *positive support* of z are denoted by $\text{supp}^-(z)$ and $\text{supp}^+(z)$, respectively, and are defined by:

$$\text{supp}^-(z) = \{v \in V \mid z(v) < 0\} \quad \textit{and} \quad \text{supp}^+(z) = \{v \in V \mid z(v) > 0\}.$$

1.2.1 M-Concavity

This section contains some basic material on M-concave functions and M-convex sets.

A set $B \subseteq \mathbf{Z}^V$ is said to be *M-convex set* if it satisfies the exchange axiom (B-EXC[**Z**]) given as:

(B-EXC[**Z**]) For each y and z in B and u in $\text{supp}^+(y-z)$, there exists v in $\text{supp}^-(y-z)$ such that $y - \chi_u + \chi_v$ and $z + \chi_u - \chi_v$ belong to B.

An equivalent of the above axiom is given in the following proposition.

Proposition 1.2.1 (Murota [15]). *The following exchange axiom is equivalent to* (B-EXC[$\mathbf{Z}$]) *for a set* $B \subseteq \mathbf{Z}^V$*:*

(B-EXC$_+$[$\mathbf{Z}$]) *For each* y *and* z *in* B *and* u *in* $\text{supp}^+(y - z)$*, there exists* v *in* $\text{supp}^-(y - z)$ *such that* $z + \chi_u - \chi_v$ *belongs to* B*.*

For any $y \in \mathbf{Z}^V$ and $U \subseteq V$, we denote $y(U)$ by

$$y(U) = \sum_{u \in U} y(u).$$

The following proposition gives some more insight of an M-convex set.

Proposition 1.2.2 (Murota [15]). *For any* y *and* z *in an M-convex set* B*, we have* $y(V) = z(V)$*.*

A function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{-\infty\}$ with nonempty effective domain is said to be an *M-concave function* if the following axiom holds for it:

($-$M-EXC[$\mathbf{Z}$]) For each y and z in $\text{dom} f$ and u in $\text{supp}^+(y - z)$, there exists v in $\text{supp}^-(y - z)$ for which

$$f(y) + f(z) \leq f(y - \chi_u + \chi_v) + f(z + \chi_u - \chi_v).$$

1.2.2 M$^\natural$-Concavity

This section is devoted to the basics of M$^\natural$-concave functions. These functions have several properties due to which they are widely used. Following proposition about the effective domain of an M-concave function is a prompt outcome of ($-$M-EXC[$\mathbf{Z}$]).

Proposition 1.2.3 (Murota [15]). *The effective domain* $\text{dom} f$ *of an M-concave function* $f : \mathbf{Z}^V \to \mathbf{R} \cup \{-\infty\}$ *is an M-convex set. Hence, it lies on a hyperplane* $\{y \in \mathbf{R}^V \mid y(V) = p\}$ *for some* $p \in \mathbf{Z}$*.*

Since the efective domain of an M-concave function lies on a hyperplane (by the above proposition). So, instead of f in $|V|$ variables, we may consider its projection in $|V|-1$ variables taken along any chosen coordinate axis. We may define a function f' in $|V|-1$ variables by

$$f'(z') = f(z(u), z') \quad \text{with } z(u) = p - z'(V \setminus \{u\}) \quad (\text{where } z \in \mathbf{Z}^V \text{ and } z' \in \mathbf{Z}^{V\setminus\{u\}}),$$

where u is the chosen axis. Clearly, the effective domain domf' of the function f' is the projection of the effective domain dom$f$ of the function f along the axis u. Such a projection f' of an M-concave function f is called an M$^\natural$-concave function. Now we define M$^\natural$-concave functions formally.

If $0 \notin V$, then put $V' = \{0\} \cup V$. Let $f' : \mathbf{Z}^V \to \mathbf{R} \cup \{-\infty\}$ and $f : \mathbf{Z}^{V'} \to \mathbf{R} \cup \{-\infty\}$. If f defined by

$$f(z_0, z) = \begin{cases} f'(z) & \text{if } z_0 = -z(V) \\ -\infty & \text{otherwise} \end{cases} \qquad (z_0 \in \mathbf{Z}, z \in \mathbf{Z}^V).$$

is M-concave, then f' is called *M$^\natural$-concave function.*

The *exchange property* for an M$^\natural$-concave function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{-\infty\}$ is given as:

($-$M$^\natural$-EXC[$\mathbf{Z}$]) For each y and z in dom$f$ and u in supp$^+(y-z)$, there exists v in supp$^-(y-z) \cup \{0\}$ such that

$$\begin{aligned} f(y) + f(z) \ \leq \ & \min[f(y-\chi_u) + f(z+\chi_u), \min_{v \in \text{supp}^-(y-z)} \{f(y- \\ & \chi_u + \chi_v) + f(z + \chi_u - \chi_v)\}]. \end{aligned}$$

M-concave functions and the M$^\natural$-concave functions are conceptually equivalent to each other. But the class of M$^\natural$-concave functions is larger than the class of M-concave functions. This is evident from the following theorem:

Theorem 1.2.4 (Murota [15]). *An M-concave function is M$^\natural$-concave. But an M$^\natural$-concave function is M-concave if and only if the effective domain of M$^\natural$-concave function is contained in some hyperplane $\{y \in \mathbf{R}^V \mid y(V) = p\}$ for some $p \in \mathbf{Z}$.*

A set $B \subseteq \mathbf{Z}^V$ is called M$^\natural$-convex if it satisfies the following axiom:

(B$^\natural$-EXC[$\mathbf{Z}$]) For each y and z in B and u in supp$^+(y-z)$, a vector v exists in supp$^-(y-z) \cup \{0\}$ for which $y - \chi_u + \chi_v$ and $z + \chi_u - \chi_v$ are in B.

Following theorem describes some basic properties of M and M$^\natural$-concave functions:

Theorem 1.2.5 (Murota [15])**.** *M and M$^\natural$-concave functions have the following properties:*

- *For an M-concave function f and a point $y \in \mathrm{dom} f$ we have*

$$f(y) \geq f(z) \quad (\forall\, z \in \mathbf{Z}^V) \iff f(y) \geq f(y - \chi_u + \chi_v) \quad (\forall\, u, v \in V),$$

- *For an M$^\natural$-concave function f and a point $y \in \mathrm{dom} f$*

$$f(y) \geq f(z) \quad (\forall\, z \in \mathbf{Z}^V) \iff \begin{cases} f(y) \geq f(y - \chi_u + \chi_v) & (\forall\, u, v \in V), \\ f(y) \geq f(y \pm \chi_v) & (\forall\, v \in V). \end{cases}$$

For a function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{-\infty\}$ and $X \subseteq \mathbf{Z}^V$, the set of maximizers of f on X is defined by

$$\arg\max\{f(y) \mid y \in X\} = \{x \in X \mid \forall y \in X :\ f(x) \geq f(y)\}.$$

For convenience, we denote

$$AM(f, z) = \arg\max\{f(y) \mid y \leq z\}.$$

That is, $AM(f, z)$ is the set of maximizers of f on the set $\{y \in \mathbf{Z}^V \mid y \leq z\}$.

In the rest of the section, we give few known results. The next theorem characterizes the set of maximizers of an M$^\natural$-concave function.

Theorem 1.2.6 (Murota and Shioura [16])**.** *Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{-\infty\}$ be an M$^\natural$-concave function and $y \in \mathrm{dom} f$. Then $y \in \arg\max(f)$ if and only if $f(y) \geq f(y + \chi_u - \chi_v)$ for each $u, v \in \{0\} \cup V$.*

The following lemma describes an important property of M$^\natural$-convex sets.

Lemma 1.2.7 (Fujishige [8])**.** *Let B be an M$^\natural$-convex set and $y \in B$. For any distinct elements $u_1, v_1, u_2, v_2, \ldots, u_r, v_r \in \{0\} \cup V$, if $y + \chi_{v_i} - \chi_{u_i} \in B$ for all $i = 1, \ldots, r$ and $y + \chi_{v_j} - \chi_{u_i} \notin B$ for all i, j with $i < j$, then $z = y + \sum_{i=1}^{r} (\chi_{v_i} - \chi_{u_i}) \in B$.*

Next two lemmas give important properties of M$^\natural$-concave functions.

Lemma 1.2.8 (Fujishige and Tamura [9]). *Let* $f : \mathbf{Z}^V \to \mathbf{R} \cup \{-\infty\}$ *be an* $M^\natural$-*concave function and* $u \in V$. *Let* $z_1, z_2 \in (\mathbf{Z} \cup \{+\infty\})^V$ *be vectors such that* $z_1 = z_2 + \chi_u, AM(f, z_1) \neq \emptyset$ *and* $AM(f, z_2) \neq \emptyset$. *Then, the following two statements hold:*

(a) *For each* $y \in AM(f, z_1)$, *there exists* $v \in \{0\} \cup V$ *such that*

$$y + \chi_v - \chi_u \in AM(f, z_2).$$

(b) *For each* $y \in AM(f, z_2)$, *there exists* $v \in \{0\} \cup V$ *such that*

$$y - \chi_v + \chi_u \in AM(f, z_1).$$

Lemma 1.2.9 (Fujishige and Tamura [9]). *Let* $f : \mathbf{Z}^V \to \mathbf{R} \cup \{-\infty\}$ *be an* $M^\natural$-*concave function and* $z_2 \in (\mathbf{Z} \cup \{+\infty\})^V$. *Suppose that* $AM(f, z_2) \neq \emptyset$. *For any* $y \in AM(f, z_2)$ *and* $z_1 \in (\mathbf{Z} \cup \{+\infty\})^V$ *such that (i)* $z_1 \geq z_2$ *and (ii)* $y(u) = z_2(u) \Longrightarrow z_1(u) = z_2(u)$, *we have* $y \in AM(f, z_1)$.

1.3 Linear Programming (LP)

This section is devoted to some basic study of linear programming. Some basic definitions and results are stated here. Objective of a *linear programming problem* is to optimize a linear function that fulfils some conditions, linear in nature. These problems are very common and easy to be dealt with because computation is simpler for linear functions as compared to the nonlinear ones. We come accross many such problems in daily life. Budgeting problem, diet problem, transportation problem, warehousing problem and manufacturing problem are few of its applications. The problem called linear in the sense that each unknown appearing in the whole program has degree one. The function required to be optimized is called an *objective function* and the conditions or restrictions that the unknowns are required to satisfy are called *constraints*. These constraints may be linear equalities or inequalities.

A linear program is written in the following form:

$$\begin{aligned} \text{Minimize} \quad & \sum_{j=1}^{n} b_j y_j \\ \text{subject to} \quad & \sum_{j=1}^{n} a_{ij} y_j \;\geq\; c_i, \quad i = 1, 2, \ldots, m \\ & y_j \;\geq\; 0, \quad j = 1, 2, \ldots, n. \end{aligned}$$

Here $\sum_{j=1}^{n} b_j y_j$ is the *objective function*, a_{ij} are *coefficients* , $y_1, y_2, \ldots, y_n$ are *unknown varibles* to be found out and $\sum_{j=1}^{n} a_{ij} y_j \geq c_i$ is the i-th *constraint.* Set of values of y_j $(j = 1, 2, \ldots, n)$ that satisfies all the constraints is called a *feasible solution.* Region composed of the set of all feasible solutions is called *feasible region or space.* A feasible solution that minimizes the objective function is called an *optimal solution.* Corner points of the feasible region are called *extreme points.* If a finite and unique optimal solution exists for the problem, then it occurs at an extreme point and such extreme point is called an *optimal point.* From this we have the following theorem:

Theorem 1.3.1 (Bazaraa and Jarvis [1])**.** *If an optimal solution exists for a linear program, then an optimal extreme point also exists.*

In matrix form, a linear program is expressed by:

$$\begin{aligned} \text{minimize} \quad & \mathbf{by}, \\ \text{subject to} \quad & \mathbf{Ay} \geq \mathbf{c}, \\ & \text{and } \mathbf{y} \geq \mathbf{0}, \end{aligned}$$

where $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ is a row vector, $\mathbf{y}^t = (y_1, y_2, \ldots, y_n)$ is a vector of unknown variables, $\mathbf{A} = [a_{ij}]$ is an $m \times n$ matrix of coefficients and $\mathbf{c}^t = (c_1, c_2, \ldots, c_m)$.

Following theorems will describe that how can we guess for an optimal solution, when a solution exists and whether it is unique or the problem has infinitely many solutions.

Theorem 1.3.2 (Bazaraa and Jarvis [1])**.** *If the objective function has an optimal solution at two adjacent extreme points of the feasible region, then the problem has infinitely many solutions because objective function will have optimal value at all the points on the line segment joining these extreme points.*

Following theorem also gives an insight into solution of a linear program.

Theorem 1.3.3 (Bazaraa and Jarvis [1]). *For the given linear programming problem, no solution for the problem exists if the feasible region is empty.*

1.3.1 Dual Problems

With every linear problem, there is a related dual problem. Of these two, one is of minimization and the other of maximization. If optimal values of the objective functions of both of the problems are finite, then they are equal. In matrix form, a dual linear program is written as:

Primal Problem		*Dual Problem*	
minimize	$\mathbf{by}$	maximize	$\mathbf{wc}$
subject to	$\mathbf{Ay} \geq \mathbf{c}$,	subject to	$\mathbf{wA} \leq \mathbf{b}$,
	$\mathbf{y} \geq \mathbf{0}$,		$\mathbf{w} \geq \mathbf{0}$,

where $\mathbf{wc}$ is the objective function for the dual, $\mathbf{w} = (w_1, w_2, \ldots, w_m)$ is a row vector, $\mathbf{c}^t = (c_1, c_2, \ldots, c_m)$, $\mathbf{A}$ is the coefficient matrix and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ is a row vector.

The following lemma relates a linear program to its dual.

Lemma 1.3.4 (Luenberger [14]). *If* $\mathbf{y}$ *is a feasible solution of the primal problem and* $\mathbf{w}$ *that of the dual problem, then*

$$\mathbf{by} \geq \mathbf{wc}.$$

This lemma depicts that a feasible solution for one problem introduces a bound on the value of the other problem. The value of the objective function of the primal is larger than that of the dual. The primal problem is sought to obtain a minimum value of the objective function and the dual to obtain a maximum value of the objective function, each seeks to reach the other. Following is an important corollary from this.

Corollary 1.3.5 (Luenberger [14]). *Let* $\mathbf{y}$ *and* $\mathbf{w}$ *be the feasible solutions for the primal and the dual problems, respectively. Also,* $\mathbf{by} = \mathbf{wc}$. *Then* $\mathbf{y}$ *and* $\mathbf{w}$ *are the optimal solutions for the primal and the dual, respectively.*

The converse of the statement in above corollary is also true which is stated in the following theorem.

Theorem 1.3.6 (Theorem on Dual LP, [14]**).** *If any of the problem possesses a finite optimal solution, so the other has, and the values of objective functions associated with both of the problems are same. If the value of the associated objective function of any of the problems is unbounded, then the other problem has no feasible solution.*

1.4 Stable Matching

In this section, we will discuss the concept of two-sided matching market. Labour and auction markets are the examples of two-sided markets. A matching is established between agents of two finite and disjoint sets. Agents of these sets may be firms and workers, men and women or sellers and buyers.

To explain a stable matching, we consider the *marriage model*. Components of the conventional model are two disjoint and finite sets of individuals M and W. The set M denotes the set of men and W denotes the set of women. Both of men and women have preferences over each other. Basically, the preferences of an individual represent the choice that he or she would make when faced with different alternatives. These preferences may be such that a man (or a woman) may prefer to remain single instead of marrying a woman (or a man) whom he (or she) cannot bear at any cost. An individual m is said to *prefer* an individual w to the individual w' if m chooses w when faced with a choice between w and w'. An individual m is said to be *indifferent* between the two individuals w and w' if both of them are equally acceptable for him. If an individual m prefers w to w' or likes them equally then it is said that m *likes* w *at least as well as* w'.

For brief description, we represent the preferences of each individual in the form of an ordered list. Let us consider that $M = \{m_1, m_2, \ldots, m_p\}$ and $W = \{w_1, w_2, \ldots, w_q\}$. The preference list of a man m is then denoted by Pref(m) and is defined on the set $W \cup \{m\}$ and the preference list of a woman w is denoted by

Pref(w) and is defined on the set $M \cup \{w\}$. The preference list of m has the form

$$\text{Pref}(m) = w_1, w_3, w_p, m, \ldots, w_2.$$

This expresses that he first chooses to marry w_1, his second choice for marriage is w_3, third is w_p and fourth is to remain single. This means that he would like to marry w_1, w_3 or w_p if possible, otherwise, he would like to remain single instead of marrying any other woman. So, for our convenience, we will express the above preference list of m in the form

$$\text{Pref}(m) = w_1, w_3, w_q.$$

A man m may be indifferent between any two women. In this case his prefrence list will be of the form

$$\text{Pref}(m) = w_q, [w_4, w_2], w_8$$

indicating that m prefers w_q to w_4 but is indifferent between w_4 and w_2 and if he is unable to marry any of w_q, w_4, w_2 or w_8, he would like to be single instead of marrying any other woman.

We denote by μ_{mw} the rank (position) of woman w in the preference list of man m. So that, the rank of w_q in ms preference list is $\mu_{mw_q} = 1$ and that of w_4 and w_2 is two. Similarly, the rank of m in Pref(w) is denoted by μ_{wm}. So, we say that a man m *prefers* a woman w to a woman w' if $\mu_{mw'} > \mu_{mw}$ and is *indifferent* between them if $\mu_{mw'} = \mu_{mw}$. A man m *likes* w *at least as well as* w' if $\mu_{mw'} \geq \mu_{mw}$. Similarly, a woman w likes a man m more than the man m' if $\mu_{wm'} > \mu_{wm}$.

A woman w is *acceptable* to a man m if $\mu_{mm} > \mu_{mw}$, that is, if he ranks w higher than himself in Pref(m). Similarly, a man m is acceptable to a woman w if $\mu_{ww} > \mu_{wm}$. An individual is said to have *strict preferences* if he or she is not indifferent between any two individuals acceptable for him or her. A matching is a set of pairs of individuals from opposite sets such that each individual appears exactly once. Now we formally define a matching.

A *matching* $X : M \cup W \to M \cup W$ is a bijective mapping such that if for $m \in M$, $X(m) \neq m$ then $X(m) \in W$ and if for $w \in W$, $X(w) \neq w$ then $X(w) \in M$. We refer $X(m)$ to be the *mate* of m under the matching X. It can easily be noticed that this matching is of order two, that is, $X^2(m) = m$. For if m and w are matched to each other under the matching X then $X(m) = w$ which implies $X^2(m) = X(w) = m$. A matching X is said to be *unstable* if there is a pair of a man m and a woman w such that they are not matched to each other but $\mu_{wm} < \mu_{wX(w)}$ and $\mu_{mw} < \mu_{mX(m)}$. That is, a matching will be unstable if there is a man woman pair who prefer each other to their actual mates under the matching and the matching is said to be *blocked* by the pair (m, w). In such a case, (m, w) is called the *blocking pair* for X. If $X(m) = w$ then m and w are said to be *matched* under the matching X. If $X(m) = m$ then m is said to be *unmatched* or *self matched* in X. We consider the following example of matching.

Example 1.4.1. Consider four men m_1, m_2, m_3 and m_4 and the four women w_1, w_2, w_3 and w_4 with their preference lists given below:

$$\begin{aligned}
\text{Pref}(m_1) &= w_2, w_1, \\
\text{Pref}(m_2) &= w_1, w_3, w_2, \\
\text{Pref}(m_3) &= w_2, w_4, \\
\text{Pref}(m_4) &= w_4, w_1, w_3, w_2, \\
\text{Pref}(w_1) &= m_3, m_4, \\
\text{Pref}(w_2) &= m_1, m_3, m_2, \\
\text{Pref}(w_3) &= m_4, m_1, m_3, \\
\text{Pref}(w_4) &= m_1.
\end{aligned}$$

We consider two matchings X_1 and X_2 given by

$$X_1 = \begin{pmatrix} m_1 & m_2 & m_3 & m_4 \\ w_2 & w_1 & w_4 & w_3 \end{pmatrix} \quad \textit{and} \quad X_2 = \begin{pmatrix} m_1 & m_2 & m_3 & m_4 \\ w_3 & w_2 & w_4 & m_4 \end{pmatrix}.$$

Between these two mathings, X_1 is *stable* while X_2 is *unstble*. X_2 is blocked by the pairs (w_3, m_4) and (m_1, w_2). Because m_4 doesnot like to be single and w_3 prefers m_4 to m_1, i.e., $\mu_{w_3 m_1} > \mu_{w_3 m_4}$. Also, $\mu_{m_1 w_3} > \mu_{m_1 w_2}$ and $\mu_{w_2 m_2} > \mu_{w_2 m_1}$.

Such matchings described among men and women, resemble the situation of marriages in real life. So, it is called a *marriage*. It is also denoted by X. Formally, a one to one matching X with each man and woman appearing at the most once is called a *marriage* or a *set of marriages*.

Chapter 2

Two-Sided Matching Problems

In this chapter, we present two standard models of two-sided matching. Two-sided matching models are important as they give an insight into general economic models. Also, most of the real life dealings are bilateral in nature. We will first discuss the marriage problem by Gale and Shapley [12] and then the assignment game by Shapley and Shubik [18].

2.1 Marriage Problem

In 1962, Gale and Shapley [12] introduced the concept of two-sided matching markets. Since then many extensions of their model have been added to the literature. The main problem in their work [12] is the stable marriage problem. Further, they extended their model to the college admissions problem. First of all, we discuss the college admissions problem.

2.1.1 College Admissions

We consider a situation where, there are n students are to be admitted to m colleges. Each student has preferences over the colleges and each college has preferences over the students. We consider a situation where n' ($n' \leq n$) students apply to a college for admission but the college has a capacity for just q (where $q \leq n'$) students to be enrolled. The college admissions commission will decide that which q students out of n' are the best to be enrolled on the basis of some specific merit criteria of the college. But it cannot be assumed that all of these q students will accept the offer of the college, as they would not have just applied to one college. So, if the college offers admissions to just q students, it may not be able to fulfill its required quota of admissions as some of the q students may refuse its offer and join some other college to which they prefer more and has offered them admission. So, to fulfill the admissions quota, the college must offer admissions to the q best students and should put the names of those students in the waiting list whom the college can consider for enrollment. But the introduction of such waiting lists may also be problematic for colleges as well as for the students. For example, if a student's name is on waiting list of first college but at the same time he is offered admission by the second college which is ranked lower in his preference list than the first college. The student will be confronted with a state of confusion. Let he accepts the offer of the second college and gets enrolled but later on the first college, on not fulfilling its quota, offers him admission. Then the student if accepts the offer and quits the second college to join the first one, will be responsible for the displeasure of the second college. On the other hand, if he does not accept the admission offer of the second college and hopes to get offer from the first college. Then it can create problem for the student in case when the first college has acquired its quota and does not need to offer any admissions to the students on its waiting list.

These uncertainties and problems may be avoided by following some specific procedure for assignment of students to colleges which is acceptable for both, the colleges and the students at the same time.

2.1.2 Assignment Procedure

Assume that there are n students that are to be admitted to m colleges with quota of the i-th college being q_i. Each student is required to rank the colleges in his preference list omitting those that are unacceptable for him at any cost. The preferences must be strict. Similarly, each college also ranks those students in its preference list who have applied to it, omitting all those who are unacceptable for it even if its admissions quota remains unfulfilled. Taking this data into consideration, we wish to frame a unanimous and fair criterion of assigning students to colleges.

Apparently, this kind of assignment seems very simple but this may not be the case in actual. Complications may appear even after all this effort. For example, consider two colleges A and B and two students a and b, where A prefers a to b but a prefers B to A. In this case, any of the assignments will be unsatisfactory. But if we form the principle that: the colleges are there to serve the students, that is, students should get preference over colleges then a will be assigned to B but not to A. This principle too, may not help much.

Whatever the assignment may be, still this kind of situation must not occur that is defined in the following definition:

Definition 2.1.1. Let two students a and b are assigned to the colleges A and B, respectively. If b prefers A to B and A prefers b to a then such an assignment is called *unstable*.

This kind of assignment is unstable because A and b may develop a state of mutual consent between themselves such that A and b may act together to form a coalition and upset the assignment (b will join A on getting an offer for admission from it and will upset quota of B, by leaving it). Still, it benefits both of A and b. This depicts the sense in which the assignment is said to be unstable.

An assignment that is not unstable or that does not tend to get upset by any college and student pair (or pairs), is called a *stable assignment*.

We are always interested in finding an assignment, which is not unstable. But first we need to know that either this is possible to find a stable assignment in every case or not. If we assume that this may happen then what should be the criteria of finding that which one of these stable assignments is preferable. For this we turn to the following definition:

Definition 2.1.2. Consider the set of all stable assignments for a problem. Then the one among them under which each applicant is atleast as well off than under any other assignment, is called an *optimal* assignment.

The existence of stable assignment does not imply the existence of optimal assignment. It is obvious indeed that if an optimal assignment exists then it is unique.

For if there are two different optimal assignments, then it was possible that there exists at least one applicant who is in better position under one assignment than under the other. Depicting that one of the assignments is not optimal.

To find whether stable assignments exist, we consider a special case where the number of the students and the colleges is the same and each college having quota equal to unity. But this kind of assumption is quite unnatural, so we consider another situation into which this assumption fits well.

2.1.3 Marriage Model

Consider a certain community with men and women equal in number, say n. They are to be married. Each of them is required to rank the members of opposite sex in order of his or her preferences for marriage. We make assignments by taking these rankings into account. Any assignment of this form is a one to one matching of the given two sets and the assignment of this form is called a *marriage* as is described in Section 1.4. We denote a marriage by X. As defined in the previous section, a marriage X is said to be *unstable* if under it two men m and m' are matched with the women w and w', respectively, but m prefers w' to w and w' prefers m to m'. It can be noted easily that in this case, where the number of men and the women is same, total number of possible assignments is $n!$. But all of them may not be stable.

We wish to settle down the question that either a set of stable marriages exists for any form of the preferences. For this we first consider some examples.

Example 2.1.1. Consider three men m_1, m_2 and m_3 and three women w_1, w_2 and w_3 with their preference lists given below.

$$\text{Pref}(m_1) = w_2, w_1, w_3,$$

$$\text{Pref}(m_2) = w_1, w_3, w_2,$$

$$\text{Pref}(m_3) = w_2, w_3, w_1,$$

$$\text{Pref}(w_1) = m_1, m_3, m_2,$$

$$\text{Pref}(w_2) = m_2, m_1, m_3,$$

$$\text{Pref}(w_3) = m_1, m_2, m_3.$$

Here six sets of marriages are possible:

$$X_1 = \{(m_1, w_2), (m_2, w_3), (m_3, w_1)\},$$
$$X_2 = \{(m_1, w_1), (m_2, w_3), (m_3, w_2)\},$$
$$X_3 = \{(m_1, w_3), (m_2, w_2), (m_3, w_1)\},$$
$$X_4 = \{(m_1, w_1), (m_2, w_2), (m_3, w_3)\},$$
$$X_5 = \{(m_1, w_2), (m_2, w_1), (m_3, w_3)\},$$
$$X_6 = \{(m_1, w_3), (m_2, w_1), (m_3, w_2)\}.$$

Out of these six sets of marriages X_1, X_2, X_3 and X_5 are stable while X_4 is unstable due to w_2 and m_3 as they prefer each other to their partners and similarly, X_6 is unstable because of m_1 and w_2.

We consider another example.

Example 2.1.2. Consider four men, m_1, m_2, m_3 and m_4 and the four women w_1, w_2, w_3 and w_4 with their preference lists given below.

$$\text{Pref}(m_1) = w_2, w_1, w_4, w_3,$$
$$\text{Pref}(m_2) = w_3, w_4, w_2, w_1,$$
$$\text{Pref}(m_3) = w_1, w_3, w_2, w_4,$$
$$\text{Pref}(m_4) = w_3, w_2, w_4, w_1,$$
$$\text{Pref}(w_1) = m_2, m_1, m_3, m_4,$$
$$\text{Pref}(w_2) = m_2, m_4, m_1, m_3,$$
$$\text{Pref}(w_3) = m_1, m_3, m_2, m_4,$$
$$\text{Pref}(w_4) = m_1, m_2, m_4, m_3.$$

Obviously there are twenty four possible sets of marriages but out of all these the only stable set of marriages is:

$$X = \{(m_1, w_1), (m_2, w_4), (m_3, w_3), (m_4, w_2)\}.$$

Note that in this marriage, no one gets his or her first preference. However, the marriage is still stable.

The following is an example of matching from a set onto itself which is similar to the marriage problem.

Example 2.1.3. Consider a situation where $2n$ (we take even number $2n$ for our convenience) students are to be divided into pairs of roommates. Each one is asked to give his preference list in which he ranks the other $2n-1$ students to be his roommate. A set of couplings is said to be *unstable* if there are students who prefer each other to their roommmates under this coupling. There is a simple example that shows that stability may not exist in some situations. Consider four students s_1, s_2, s_3 and s_4 with their preference lists:

$$\text{Pref}(s_1) = s_3, s_2, s_4,$$
$$\text{Pref}(s_2) = s_1, s_3, s_4,$$
$$\text{Pref}(s_3) = s_2, s_1, s_4,$$
$$\text{Pref}(s_4) = s_3, s_1, s_2.$$

Since each of s_1, s_2 and s_3 ranks s_4 last and s_2 ranks s_1 first, s_1 ranks s_3 first and s_3 ranks s_2 first, so, no stable coupling exists in this case. For whoever is to share room with s_4, prefers someone else who also prefers him to his own roommate. This situation is independent of the preferences of s_4.

All of these examples predict that stability may be achieved but it may not be simple or clear at first sight. We have the following theorem on stability of marriages.

Theorem 2.1.1 (Gale and Shapley [12])**.** *For every marriage market with strict preferences, there exists a stable matching.*

To prove this theorem, Gale and Shapley [12] designed an algorithm. Their algorithm has two versions: When the men are proposers and when the women are proposers. In either case, the proposers get their best partners. We recall from Section 1.4 that μ_{mw} denotes the rank of woman w in preference list of the man m (position of w in Pref(m)). Similarly, μ_{wm} denotes the rank of man m in preference list of the woman w (position of m in Pref(w)). The woman w is said to prefer man m to the man m' if $\mu_{wm'} > \mu_{wm}$ and vice versa. The number of men and women is the same and is equal to n. The preferences of each of the individuals are required to be strict. The procedure to be adopted is as follows: Let $F(W)$ denote the set of all free women and $F(M)$ be that of all free men. X be the set of man-woman pairs. A man or woman will be free if it is not appeared in X.

2.1.4 Algorithm for a Stable Marriage

Step 0: Set $F(M) = M$, $F(W) = W$, $X = \emptyset$.

Step 1: If $F(M) = \emptyset$ then stop.

Step 2: Take $m \in F(M)$ and $w \in \text{Pref}(m)$ with $\mu_{mw} = 1$.

Step 3: **(i)** If $w \in F(W)$ then set

$$\begin{aligned} X &:= X \cup \{(m, w)\}, \\ F(M) &:= F(M) \setminus \{m\}, \\ F(W) &:= F(W) \setminus \{w\}. \end{aligned}$$

(ii) If $(m', w) \in X$ for some $m' \in M$ and $\mu_{wm} < \mu_{wm'}$ then set

$$\begin{aligned} X &:= X \setminus \{(m', w)\} \cup \{(m, w)\}, \\ F(M) &:= F(M) \setminus \{m\} \cup \{m'\}, \\ F(W) &:= F(W). \end{aligned}$$

(iii) If $(m', w) \in X$ for some $m' \in M$ and $\mu_{wm} > \mu_{wm'}$ then set

$$\begin{aligned} X &:= X, \\ F(M) &:= F(M), \\ F(W) &:= F(W). \end{aligned}$$

Step 4: Set $\text{Pref}(m) := \text{Pref}(m) \setminus \{w\}$. Go to Step 1.

The termination of the algorithm, the stability and optimality of the set of marriages can easily be verified in the following lines. No man is allowed to propose a woman more than once. So, the procedure will terminate as soon as all the women have been appeared in X. This marriage is called the man-optimal stable set of marriages. Following lemma is enough to prove the termination of the algorithm.

Lemma 2.1.2. *In each iteration of the algorithm, following hold:*

(i) $|F(M)|$ *either decreases or remains the same.*

(ii) $|X|$ *either increases or remains the same.*

(iii) $|\mathrm{Pref}(m)|$ *strictly decreases for m taken at Step 2.*

Since all the sets are finite so, procedure terminates after a finite number of steps.

Remark 2.1.1 (Gale and Shapley [12])**.** The complexity of the above described algorithm is of order $O(n^2)$.

In the above algorithm, if we reverse the roles of men and women (that is, women acting as proposers), we obtain an algorithm to find a woman-optimal stable set of marriages. The following theorem about the Gale and Shapley algorithm briefly states the situation when an optimal set of marriage is attainable.

Theorem 2.1.3 (Gale and Shapley [17])**.** *If each individual has strict preferences then a man-optimal and a woman-optimal stable set of marriages always exist. Also, the set of marriages obtained by the men as proposers is man-optimal and that obtained by women as proposers is woman-optimal stable set of marriages.*

This procedure may be applied in the case even when the number of men and the number of women is not the same. In such case the Step 1 will be modified as: if either $F(M) = \emptyset$ or $F(W) = \emptyset$ then stop. Consider that there are p men and q women. If $p < q$ then the procedure terminates when all the p men are in X. If $p > q$ then the procedure terminates when all the q women are in X. The above mentioned procedure may be opted to find a stable set of marriages in previously discussed examples. We discuss the following example to implement the above described algorithm.

Example 2.1.4. Consider four men, m_1, m_2, m_3 and m_4 and four women w_1, w_2, w_3

and w_4 with their preference lists given below:

$$\text{Pref}(m_1) = w_2, w_1, w_3, w_4,$$
$$\text{Pref}(m_2) = w_3, w_2, w_1, w_4,$$
$$\text{Pref}(m_3) = w_3, w_1, w_2, w_4,$$
$$\text{Pref}(m_4) = w_1, w_3, w_4, w_2,$$
$$\text{Pref}(w_1) = m_1, m_3, m_2, m_4,$$
$$\text{Pref}(w_2) = m_4, m_1, m_2, m_3,$$
$$\text{Pref}(w_3) = m_4, m_3, m_1, m_2,$$
$$\text{Pref}(w_4) = m_3, m_1, m_4, m_2.$$

Now we start the procedure. At Step 0 all the men and the women are free. In first iteration, at Step 2, we start from m_1. At Step 3, since $\mu_{m_1w_2} = 1$ and $w_2 \in F(W)$ so $X = \{(m_1, w_2)\}$, $F(M) = \{m_2, m_3, m_4\}$, $F(W) = \{w_1, w_3, w_4\}$ and at Step 4 $\text{Pref}(m_1) = w_1, w_3, w_4$.

In second iteration, at Step 2, we take $m_2 \in F(M)$. At Step 3, since $\mu_{m_2w_3} = 1$ and $w_3 \in F(W)$ so $X = \{(m_1, w_2), (m_2, w_3)\}$, $F(m) = \{m_3, m_4\}$, $F(W) = \{w_1, w_4\}$ and at Step 4 $\text{Pref}(m_2) = w_2, w_1, w_4$.

In third iteration, at Step 2, we take $m_3 \in F(M)$. At Step 3, since $\mu_{m_3w_3} = 1$, $(m_2, w_3) \in X$ and $\mu_{w_3m_2} > \mu_{w_3m_3}$ hence $X = \{(m_1, w_2), (m_3, w_3)\}$, $F(M) = \{m_2, m_4\}$, $F(W) = \{w_1, w_4\}$ and at Step 4 $\text{Pref}(m_3) = w_1, w_2, w_4$.

In fourth iteration, at Step 2, take $m_2 \in F(M)$. At Step 3, since $\mu_{m_2w_2} = 1$, $(m_1, w_2) \in X$ and $\mu_{w_2m_1} < \mu_{w_2m_2}$ hence $X = \{(m_1, w_2), (m_3, w_3)\}$, $F(M) = \{m_2, m_4\}$, $F(W) = \{w_1, w_4\}$ and at Step 4 $\text{Pref}(m_2) = w_1, w_4$.

In fifth iteration, at Step 2, take $m_2 \in F(M)$. At Step 3, since $\mu_{m_2w_1} = 1$ in $\text{Pref}(m_2)$ and $w_1 \in F(W)$ hence $X = \{(m_1, w_2), (m_2, w_1), (m_3, w_3)\}$, $F(M) = \{m_4\}$, $F(W) = \{w_4\}$ and at Step 4 $\text{Pref}(m_2) = w_4$.

In sixth iteration, at Step 2, take $m_4 \in F(M)$. At Step 3, since $\mu_{m_4w_1} = 1$ in $\text{Pref}(m_4)$, $(m_2, w_1) \in X$ and $\mu_{w_1m_2} < \mu_{w_1m_4}$ hence $X = \{(m_1, w_2), (m_2, w_1), (m_3, w_3)\}$, $F(M) = \{m_4\}$, $F(W) = \{w_4\}$ and at Step 4 $\text{Pref}(m_4) = w_3, w_4, w_2$.

In seventh iteration, at Step 2, take $m_4 \in F(M)$. At Step 3, since $\mu_{m_4w_3} = 1$ in $\text{Pref}(m_4)$, $(m_3, w_3) \in X$ and $\mu_{w_3m_4} < \mu_{w_3m_3}$ hence $X = \{(m_1, w_2), (m_2, w_1), (m_4, w_3)\}$,

$F(M) = \{m_3\}$, $F(W) = \{w_4\}$ and at Step 4 Pref$(m_4) = w_4, w_2$.

In eighth iteration, at Step 2, take $m_3 \in F(M)$. At Step 3, since $\mu_{m_3w_1} = 1$ in Pref(m_3), $(m_2, w_1) \in X$ and $\mu_{w_1m_3} < \mu_{w_1m_2}$ hence $X = \{(m_1, w_2), (m_3, w_1), (m_4, w_3)\}$, $F(M) = \{m_2\}$, $F(W) = \{w_4\}$ and at Step 4 Pref$(m_3) = w_2, w_4$.

In ninth iteration, at Step 2, take $m_2 \in F(M)$. At Step 3, since $\mu_{m_2w_4} = 1$ in Pref(m_2) and $w_4 \in F(W)$ hence $X = \{(m_1, w_2), (m_2, w_4), (m_3, w_1), (m_4, w_3)\}$, $F(W) = \emptyset$, $F(M) = \emptyset$ also at Step 4 Pref$(m_2) = \emptyset$.

Since $F(M) = \emptyset$ hence the procedure is stopped and the man-optimal stable marriage is

$$X = \{(m_1, w_2), (m_2, w_4), (m_3, w_1), (m_4, w_3)\}.$$

By reversing the roles of men and women, we apply the same procedure to the problem. Then the woman-optimal stable marriage is

$$X = \{(m_1, w_2), (m_2, w_4), (m_3, w_1), (m_4, w_3)\}.$$

In the above example, man-optimal solution is just the same as that of the woman-optimal solution. But, this may not happen in each case. The solutions are the same only when a unique stable set of marriages exists.

2.1.5 Application to the College Admissions Problem

The algorithm described above can be easily applied to find a stable assignment of students to the colleges. Let the number of the students is n. Each student applies to the college he prefers the most. Each college places top q (q is the quota of the college) students among the students applied to it, on its waiting list and rejects all of the others. If the number of students who have applied to the college is less than q, it place all of them on waiting list. The rejected students apply to the college of their second choice. Each college places top q students from among on its waiting list or have now applied to it, and rejects all of the others. The process terminates when each student is either on the waiting list of some college or has been rejected by all the colleges in his preference list. Each college now allows admission to all the students on its waiting list. This assignment is surely stable and can be verified easily. The matching such obtained is not only stable but also optimal. This process yields student-optimal assignment. On reversing the role of students and the colleges in the above process, that is, colleges offerring admissions to the students. We will get a college-optimal assignment. We have the following lemma regarding a stable college admissions problem:

Lemma 2.1.4 (Gale and Shapley [17])**.** *A matching of the college admissions problem is stable if and only if the corresponding matching of the related marriage problem is stable.*

2.2 Assignment Game

In this section, we give a review of the assignment game proposed by Shapley and Shubik [18]. The Shapley-Shubik model [18] is related to one of the classes of two-sided game theoretic models. It deals with the problem in which an assignment of buyers and sellers is to be established and the indivisibility of goods for sale is its distinctive feature. The goods are of the form like houses, vehicles, automobiles and etc. We are mainly concerned and have devoted ourselves to the core of the game.

Definition 2.2.1. A *core* of a game is the assignment of buyers to sellers that is an optimal assignment.
It means that, it cannot be improved by any other assignment.

A stable matching is said to be *weak pareto optimal* if there exists any matching (stable or not) that is preferred by all the participants of a game. Shapley and Shubik [18] imposed several restrictions on the model, which are as follows:

(i) identification of utility with money,

(ii) permitting third-party payments,

(iii) indivisibility of goods of trade,

(iv) inflexibility of supply and demand functions.

2.2.1 Game Description

Consider a situation where, there are m homeowners (sellers) and n purchasers (buyers). The value of the i-th home for its seller is c_i and that for the j-th buyer is h_{ij}. There exists a favourable price for both of the seller and the buyer only if $h_{ij} > c_i$. But it cannot be guaranteed that this situation occurs in all the cases.

A move in the game is the case, when a house is transferred from a seller to a buyer and the money from the buyer to the seller of the home. Here we are only concerned with the prices and their suitability to the buyers and the sellers. Let i-th house is sold for p_i, to the j-th buyer. Let no arbitrator is involved (that is, there are no third party payments). Then the profit gained by the seller i is

$$p_i - c_i$$

and the gain of the buyer j is

$$h_{ij} - p_i.$$

Now we can define the *characteristic function* or the *value function* for a game. To find a stable assignment in the game, we wish to maximize the value of characteristic function.

2.2.2 The Characteristic Function

Any subset C of the set $S \cup B$ (where S is the set of sellers and B is the set of buyers) is called an *alliance* or *coalition*. The value or worth of any alliance of a game is called the *characteristic function* of the game. We denote an alliance by C and its worth by $\nu(C)$. If there is no member or just a single member in C then obviously $\nu(C) = 0$, as no individual can have a profitable deal with himself. Also, it is obvious that an alliance is possible only if it is formed between the partners who are not from the same set. Now we can define $\nu(\{i, j\})$ for $i, j \in C$ as

$$\nu(\{i,j\}) = \begin{cases} \max\{0, h_{ij} - c_i\} & \text{if } i \in S \text{ and } j \in B \\ 0 & \text{if } \{i,j\} \subseteq S \text{ or } \{i,j\} \subseteq B \end{cases} \qquad (\forall\ i, j \in C).$$

We denote this value by a_{ij} (that is, $\nu(i, j) = a_{ij}$). It can be noted that this number is independent of the sale price p_i. For larger mixed alliance, that is, for $|C| = k > 2$

where $k = \min\{|C \cap S|, |C \cap B|\}$ and the players of the game be $i_1, i_2, \ldots, i_k \in C \cap S$ and $j_1, j_2, \ldots, j_k \in C \cap B$. The value of the alliance is obtained by maximizing the total gain of all possible $2k$ arrangements. Symbolically, we write

$$\nu(C) = \max[a_{i_1j_1} + a_{i_2j_2} + \ldots + a_{i_kj_k}].$$

This gives an optimal assignment for the game. We consider the following examples for an optimal assignment:

Example 2.2.1. Consider the following matrix (a_{ij}) (where $i \in C \cap S$ and $j \in C \cap B$) for an assignment game:

0	3	0
3	0	3
0	3	0

Here value of the characteristic function is $\nu(C) = 6$ and there are four optimal assignments shown by the circled entries in the following tables:

⓪	3	0
3	0	③
0	③	0

0	③	0
③	0	3
0	3	⓪

0	③	0
3	0	③
⓪	3	0

0	3	⓪
③	0	3
0	③	0

In the following example, there is only one optimal assignment.

Example 2.2.2. Consider the following matrix (a_{ij}) (where $i \in C \cap S$ and $j \in C \cap B$)

4	7	1
6	8	5
1	2	1

The only optimal assignment is shown in the following table with circled entries.

4	⑦	1
⑥	8	5
1	2	①

The value of the characteristic function $\nu(C) = 14$.

We can observe that the characteristic function $\nu(C)$ is symmetric when described in terms of a_{ij} in the assignment game, though the model itself is not symmetric. So we form an alternative model for our game that is symmetric from the beginning.

Consider an economic environment with two types of players or agents. For example, men and women or manufacturers and cutomers. They are restricted to have unshared (one good is sold to a single person as goods are indivisible) and two-sided (mixed) dealings with each other. After some negotiations, pairs are formed. Any kind of trade (of goods and services but not the money) is made between the partners. Here we are not concerned about the type of the goods or the services. With each pair $\{i, j\}$, there is the number a_{ij} (that is, $\nu(\{i, j\})$) that describes the worth of the alliance of i and j, if it is formed). If we treat this system as our game then its characteristic function has all those properties that are satisfied by the original game model.

Any gain is divided into the two partners in the alliance. A deal will be finalized when each partner will have the best possible terms for him that could not be attained in any other alliance. The contracts would be settled when the prices are in harmony for both of the partners of a coalition. It means that the prices not only satisfy both of the partners but their sum should be equal to the worth of the partnership. However, we wish to settle down the question that either such prices (harmonious) exist or not. In a mathematical form, we may express this question as: do the numbers $q_i \geq 0$ and $r_j \geq 0$ exist such that

$$\begin{cases} q_i + r_j = a_{ij} & \text{if } i \text{ and } j \text{ are partners,} \\ q_i + r_j \geq a_{ij} & \text{if } i \text{ and } j \text{ are not partners.} \end{cases}$$

2.2.3 Linear Programming Form of the Model

Shapley and Shubik [18] used the linear programming to solve the problem, as it is usefull and is simpler.

Consider the alliance of all players of the game. Then we wish to determine the worth of $S \cup B$. We introduce $m+n$ constraints on mn real variables y_{ij} with $i \in S$ and $j \in B$. Then the linear program is:

$$\begin{array}{ll} \text{maximize} & \sum_{i\in S}\sum_{j\in B} a_{ij}y_{ij} \\ \text{subject to} & \sum_{i\in S} y_{ij} \le 1 \qquad \forall\, j \in B, \\ & \sum_{j\in B} y_{ij} \le 1 \qquad \forall\, i \in S, \\ & x_{ij} = 0 \text{ } or \text{ } 1 \quad \forall\, i \in S,\, j \in B. \end{array}$$

The variables y_{ij} represent the probability of forming a partnership of i and j in the case of the partnership game (since the sum of probabilities of all the possible outcomes is always one. That is why, the sum of variables y_{ij} is less than or equal to 1). Maximum value of $\sum_{i\in S}\sum_{j\in B} a_{ij}y_{ij}$ is obtained when all the variables y_{ij} are either 0 or 1 (reader is reffered to Dantzig [4] for a detailed study). So that the continuous linear programming problem coincides with the discrete assignment game and we get that

$$\max \sum_{i\in S}\sum_{j\in B} a_{ij}y_{ij} = \nu(S \cup B).$$

We know that with every linear programming problem, there is associated a dual linear programming problem. Also, both of the objective functions converge to the same value. Here in our case, the dual problem is as follows:

$$\begin{array}{ll} \text{minimize} & \sum_{i\in S} u_i + \sum_{j\in B} v_j \\ \text{subject to} & u_i + v_j \ge a_{ij} \quad (\forall\, i \in S,\, j \in B). \end{array}$$

From the Fundamental Duality Theorem 1.3.6, we have

$$\min(\sum_{i\in S} u_i + \sum_{j\in B} v_j) = \max \sum_{i\in S}\sum_{j\in B} a_{ij}y_{ij}.$$

Let the vector $(u, v) = (u_1, \ldots, u_m, v_1, \ldots, v_n)$ be the solution of the dual problem. Then we have

$$\sum_{i \in S} u_i + \sum_{j \in B} v_j = \min(\sum_{i \in S} u_i + \sum_{j \in B} v_j) = \max \sum_{i \in S} \sum_{j \in B} a_{ij} y_{ij} = \nu(S \cup B). \quad (2.2.1)$$

This means that the vector (u, v) is a nonnegative, pareto optimal payoff vector for the game. Also, the constraints for the dual problem show that for each pair $\{i, j\}$ with $i \in S$, $j \in B$, we have

$$u_i + v_j \geq a_{ij} = \nu(i, j).$$

So that for an alliance C we get that

$$\sum_{i \in C \cap S} u_i + \sum_{j \in C \cap B} v_j \geq \nu(C). \quad (2.2.2)$$

The feasibility of (u, v) is evident from (2.2.1). The constraint $u_i + v_j \geq \alpha_{ij}$ ($\forall\, i \in S$, $j \in B$) assures the stability of the outcome. The optimality of the solution for any alliance is ensured by (2.2.2). So, the feasibility and the optimality of solution demonstrates that, it is the *core* of the game.

On the other hand, any vector in the core satisfies (2.2.1) and (2.2.2) and thus satisfy all the conditions of a solution to the dual linear programming problem.

Now, we can summarize the above discussion in the form of the following theorem:

Theorem 2.2.1 (Shapley and Shubik [18]). *The core of the assignment game and the solutions to the dual linear programming problem associated with the game, coincide with each other.*

2.2.4 Mathematical Description

Now we describe the model mathematically. Let $E = S \times B$ and for each $(i, j) \in E$ β_{ij} denotes the value of j-th buyer for the i-th seller (that is, profit of the seller i) and β_{ji} the value of i-th seller for the j-th buyer (that is, profit of the buyer j). Then $\beta_{ij} + \beta_{ji}$ (here $\beta_{ij} + \beta_{ji}$ is the same as a_{ij}) is called the total value of the pair (i, j) if they form a coalition (or alliance). An assignment (or matching) A is called pairwise stable if there exists $(q, r) \in \mathbf{R}^S \times \mathbf{R}^B$ such that

A1: $\sum_{i \in S} q_i + \sum_{j \in B} r_j = \sum_{(i,j) \in A} (\beta_{ij} + \beta_{ji})$.

A2: $q \geq 0$ and $r \geq 0$.

A3: $q_i + r_j \geq \beta_{ij} + \beta_{ji}$ for all $(i, j) \in A$.

It can be easily inferred from **A1** $-$ **A3** that $q_i + r_j = \beta_{ij} + \beta_{ji}$ for all $(i, j) \in A$ and $q_i = 0$ when i is unmatched and $r_j = 0$ when j is unmatched. Here q_i is the payoff of the seller i and r_j is the payoff of the buyer j.

2.3 Comparison of the Two Models

In this section, we will try to figure out the similarities or dissmilarities between the two models: the marriage model [12] and the assignment game [18].

In both of the games, the matching we wish to obtain is one to one and the players are partitioned into two disjoint sets. A partnership is possible only if each pair in the matching constitutes one player from each set. Both allow the players to be self matched (in marriage model a player can be self matched only in the case when the number of men and women is not the same).

The basic difference between the marriage model [12] and the assignment game [18] lies in the fact that the latter involves the exchange of money while the former does not allow any monetary exchange. The players of the marriage model are rigid as no negotiations are allowed in the game while the players of the assignment game are referred to as the flexible. The equality $q_i + r_j = \beta_{ij} + \beta_{ji}$ reflects that the players can negotiate and distribute the amount $\beta_{ij} + \beta_{ji}$ between each other without disturbing the total value of their partnership.

Chapter 3

A Two-Sided Discrete Concave Market

In this chapter, we extend the model of Fujishige and Tamura [10] by generalizing payoff functions. We introduce weighted income and weighted payments. We propose an algorithm to find a pairwise strictly stable outcome. We also discuss correctness and termination of the algorithm. At first, we give a brief description of our model.

3.1 Our Model

We extend the model of Fujishige and Tamura [10] by introducing general payoff functions. We give here a brief description of the model.

Consider two disjoint and finite sets P and Q of agents. The agents in P are called workers and the agents in Q are called firms. The set of all ordered pairs (i, j) of agents $i \in P$ and $j \in Q$ is denoted by $E = P \times Q$. Each worker $i \in P$ can supply multi units of labor time and each firm $j \in Q$ can employ workers with multi units of labor time and pay a salary to worker i if j hires i. For each $i \in P$ and $j \in Q$, we define $E_{(i)} = \{i\} \times Q$ and $E_{(j)} = P \times \{j\}$. The number of units of labor time for a firm j that hires a worker i is denoted by $x(i, j)$. The labor allocation is represented by a vector $x = (x(i, j) \in \mathbf{Z} \mid (i, j) \in E)$. We assume that each pair

$(i, j) \in E$ may have a lower and an upper bound on a salary per unit of labor time. The lower and upper bounds of salaries per unit of labor time are denoted by two vectors $\underline{\pi} \in (\mathbf{R} \cup \{-\infty\})^E$ and $\overline{\pi} \in (\mathbf{R} \cup \{+\infty\})^E$ where $\underline{\pi} \leq \overline{\pi}$. For a vector $y \in \mathbf{R}^E$ and $k \in P \cup Q$, the restriction of y on $E_{(k)}$ is denoted by $y_{(k)}$. For each $k \in P \cup Q$, the value function f_k is defined as $f_k : \mathbf{Z}^{E_{(k)}} \to \mathbf{R} \cup \{-\infty\}$. Moreover, we assume that for each $k \in P \cup Q$, the value function f_k satisfies the following:
(A) The effective domain $\mathrm{dom} f_k$ is bounded and hereditary, and contains $\mathbf{0} \in \mathbf{Z}^{E_{(k)}}$ as a lower bound.

Here heredity means that $[\mathbf{0}, y]_{\mathbf{Z}} \subseteq \mathrm{dom} f$ for each $y \in \mathrm{dom} f$.[1] A vector $x \in \mathbf{Z}^E$ is called *feasible allocation* if $x_{(k)} \in \mathrm{dom} f_k$ for each $k \in P \cup Q$. A vector $s \in \mathbf{R}^E$ is called a *feasible salary vector* if $\underline{\pi}(i, j) \leq s(i, j) \leq \overline{\pi}(i, j)$ for each $(i, j) \in E$. For a feasible allocation $x \in \mathbf{Z}^E$ and a feasible salary vector $s \in \mathbf{R}^E$, the pair (x, s) is called an *outcome*.

Let $\alpha = (\alpha(i, j) \in \mathbf{R}_+ \mid (i, j) \in E)$ be a given positive vector. For each $i \in P$, the sum $\sum_{j \in Q} s(i, j)\alpha(i, j)x(i, j)$ is called *weighted income* of i from the firms that hire i. Similarly, for each $j \in Q$, the sum $\sum_{i \in P} s(i, j)\alpha(i, j)\ x(i, j)$ is called *weighted payment* of the firm j to the workers that firm j hires. We define payoff functions more general than the payoff functions defined by Fujishige and Tamura [10]. The payoff of a worker i on an outcome (x, s) is defined by $f_i[+s_{(i)}, \alpha_{(i)}](x_{(i)}) = f_i(x_{(i)}) + \sum_{j \in Q} s(i, j)\alpha(i, j)x(i, j)$. This means that the payoff of a worker i on an outcome (x, s) is the sum of value of i on x and the weighted income from the firms that hire i. Similarly, the payoff of a firm j on an outcome (x, s) is defined by $f_j[-s_{(j)}, \alpha_{(j)}](x_{(j)}) = f_j(x_{(j)}) - \sum_{i \in P} s(i, j)\alpha(i, j)x(i, j)$. This means that the payoff of a firm j on an outcome (x, s) is the difference of the value of firm j on x and the weighted payments to the workers that firm j hires. If α is all-one vector then our model coincides with the model of Fujishige and Tamura [10].

An outcome (x, s) is said to satisfy the *incentive constraints* if no agent can

[1] For $x, y \in \mathrm{dom} f$, we define $[x, y]_{\mathbf{Z}} = \{z \in \mathbf{Z}^E \mid x(e) \leq z(e) \leq y(e)\ \forall e \in E\}$.

unilaterally decrease the units of labour time x at the salary s. That is, if it satisfies

$$f_i[+s_{(i)}, \alpha_{(i)}](x_{(i)}) = \max\{f_i[+s_{(i)}, \alpha_{(i)}](y) \mid y \leq x_{(i)}\} \quad (\forall\ i \in P), \tag{3.1.1}$$

$$f_j[-s_{(j)}, \alpha_{(j)}](x_{(j)}) = \max\{f_j[-s_{(j)}, \alpha_{(j)}](y) \mid y \leq x_{(j)}\} \quad (\forall\ j \in Q). \tag{3.1.2}$$

An outcome (x, s) is said to be *pairwise unstable* if it does not satisfy the incentive constraints (3.1.1) and (3.1.2) or there exist $i \in P$, $j \in Q$, $\gamma \in [\underline{\pi}(i,j), \overline{\pi}(i,j)]$, $y' \in \mathbf{Z}^{E(i)}$ and $y'' \in \mathbf{Z}^{E(j)}$ such that

$$\begin{aligned} &f_i[+s_{(i)}, \alpha_{(i)}](x_{(i)}) < f_i[+s^j_{(i)}[\gamma], \alpha_{(i)}](y'),\\ &y'(i,j') \leq x(i,j') \qquad (\forall\ j' \in Q \setminus \{j\}), \end{aligned} \tag{3.1.3}$$

$$\begin{aligned} &f_j[-s_{(j)}, \alpha_{(j)}](x_{(j)}) < f_j[-s^i_{(j)}[\gamma], \alpha_{(j)}](y''),\\ &y''(i',j) \leq x(i',j) \qquad (\forall\ i' \in P \setminus \{i\}), \end{aligned} \tag{3.1.4}$$

$$y'(i,j) = y''(i,j). \tag{3.1.5}$$

An outcome (x, s) is said to be *pairwise stable* if it does not satisfy (3.1.3)−(3.1.5) and is said to be *pairwise quasi-unstable* if it does not satisfy the incentive constraints (3.1.1) and (3.1.2) or there exist $i \in P$, $j \in Q$, $\gamma \in [\underline{\pi}(i,j), \overline{\pi}(i,j)]$, $y' \in \mathbf{Z}^{E(i)}$ and $y'' \in \mathbf{Z}^{E(j)}$ that satisfy (3.1.3) and (3.1.4) ((3.1.5) is not required to be satisfied).

Now we give the concept of pairwise strictly stable outcome. For any $s \in \mathbf{R}^E, \gamma \in \mathbf{R}, i \in P$ and $j \in Q$, the vector $s^j_{(i)}[\gamma]$ is defined as a vector obtained from $s_{(i)}$ by replacing (i,j)-component by γ. Similarly, we define $s^i_{(j)}[\gamma]$.

An outcome (x, s) is *pairwise strictly stable* if and only if for each $i \in P$, $j \in Q$ and $\gamma \in \mathbf{R}$ with $\underline{\pi}(i,j) \leq \gamma \leq \overline{\pi}(i,j)$, the following hold:

(PSS1) $f_i[+s_{(i)}, \alpha_{(i)}](x_{(i)}) = \max\{f_i[+s_{(i)}, \alpha_{(i)}](y) \mid y \leq x_{(i)}\}$,

(PSS2) $f_j[-s_{(j)}, \alpha_{(j)}](x_{(j)}) = \max\{f_j[-s_{(j)}, \alpha_{(j)}](y) \mid y \leq x_{(j)}\}$.

(PSS3) Either

(i): $f_i[+s_{(i)}, \alpha_{(i)}](x_{(i)}) \geq$
$\max\{f_i[+s^j_{(i)}[\gamma], \alpha_{(i)}](y) \mid y(i,j') \leq x(i,j'), \forall j' \neq j\}$

or

(ii): $f_j[-s_{(j)}, \alpha_{(j)}](x_{(j)}) \geq$
$\max\{f_j[-s^i_{(j)}[\gamma], \alpha_{(j)}](y) \mid y(i',j) \leq x(i',j), \forall i' \neq i\}$.

Remark 3.1.1. It can be easily noticed that an outcome (x, s) is pairwise strictly stable if it is not pairwise quasi-unstable. As, a pairwise unstable outcome is also pairwise quasi-unstable. Hence a pairwise strictly stable outcome is also pairwise stable.

The following lemma is the modification of Lemma 3.1 [10]. It states the conditions for which a pairwise stable outcome becomes pairwise strictly stable.

Lemma 3.1.1. *If $f_k (k \in P \cup Q)$ are M$^\natural$-concave functions satisfying (A) and if one of the following conditions:*

(i) $\underline{\pi} = \overline{\pi}$,

(ii) $\mathrm{dom} f_k \subseteq \{0, 1\}^{E(k)}$ *for all* $k \in P \cup Q$,

(iii) *there exists a vector* $u \in \mathbf{Z}^E$ *such that for each* $k \in P \cup Q$, *we have* $\mathrm{dom} f_k = \{y \in \mathbf{Z}^{E(k)} \mid \mathbf{0} \leq y \leq u_{(k)}\}$ *and* f_k *is linear over* $\mathrm{dom} f_k$

holds, then any pairwise stable outcome is pairwise strictly stable.

Proof. Contrarily, suppose that (x, s) is not pairwise strictly stable outcome. Then by the Remark 3.1.1, it is pairwise quasi-unstable. We show that it is also pairwise unstable. Since the outcome is pairwise stable. So, it satisfies incentive constraints (3.1.1) and (3.1.2). By our supposition, since the outcome is pairwise quasi-unstable. So, there exist $i \in P$, $j \in Q$, $\gamma \in [\underline{\pi}(i, j), \overline{\pi}(i, j)]$, $y' \in \mathbf{Z}^{E(i)}$, and $y'' \in \mathbf{Z}^{E(j)}$ satisfying (3.1.3) and (3.1.4).

(i): In this case $s(i, j) = \gamma$ (because $s(i, j), \gamma \in [\underline{\pi}, \overline{\pi}]$). Since (x, s) satisfies incentive constraints, so we must have $y'(i, j) > x(i, j)$ and $y''(i, j) > x(i, j)$. Assume $y'(i, j), y''(i, j)$ as small as possible and satisfying (3.1.3) and (3.1.4). By ($-$M$^\natural$-EXC[$\mathbf{Z}$]) for $e = (i, j) \in \mathrm{supp}^+(y' - x_{(i)})$, there exists $e' \in \mathrm{supp}^-(y' - x_{(i)}) \cup \{0\}$ such that

$$\begin{aligned} f_i[+s^j_{(i)}[\gamma], \alpha_{(i)}](y') &+ f_i[+s_{(i)}, \alpha_{(i)}](x_{(i)}) \\ &= f_i[+s^j_{(i)}[\gamma], \alpha_{(i)}](y') + f_i[+s^j_{(i)}[\gamma], \alpha_{(i)}](x_{(i)}) \\ &\leq f_i[+s^j_{(i)}[\gamma], \alpha_{(i)}](y' - \chi_e + \chi_{e'}) + f_i[+s^j_{(i)}[\gamma], \alpha_{(i)}](x_{(i)} + \chi_e - \chi_{e'}). \end{aligned}$$

By choice of y', we get $f_i[+s^j_{(i)}[\gamma], \alpha_{(i)}](y') > f_i[+s^j_{(i)}[\gamma], \alpha_{(i)}](y' - \chi_e + \chi_{e'})$. This gives

$$f_i[+s_{(i)}, \alpha_{(i)}](x_{(i)}) < f_i[+s^j_{(i)}[\gamma], \alpha_{(i)}](x_{(i)} + \chi_e - \chi_{e'}).$$

This implies that $y'(i,j) = x(i,j) + 1$ (as $x_{(i)} + \chi_e - \chi_{e'}$ can be chosen as y'). By the same procedure, we can show that $y''(i,j) = x(i,j)+1$. This implies that (x, s) also satisfies (3.1.5). This proves that (x, s) is a pairwise unstable outcome, which is a contradiction. Hence (x, s) is pairwise strictly stable outcome.

(ii): The case for $s(i,j) = \gamma$ can be treated like the Case (i). The case for $s(i,j) > \gamma$ can be dealt by rearranging roles of firms and workers in the following arguments. So, we only deal with the case where $s(i,j) < \gamma$. The case for $x(i,j) = 0$ can be proceeded like Case (i). So, we suppose that $x(i,j) > 0$. We get

$$\begin{array}{c} f_i[+s_{(i)}, \alpha_{(i)}](x_{(i)}) < f_i[+s^j_{(i)}[\gamma], \alpha_{(i)}](x_{(i)}), \\ f_j[-s_{(j)}, \alpha_{(j)}](y) \geq f_j[-s^i_{(j)}[\gamma], \alpha_{(j)}](y) \end{array} \qquad (\forall\ y \leq x_{(j)}).$$

Hence $y''(i,j)$ must be greater than $x(i,j)$. So that $y''(i,j) \geq 2$, which is not possible as $\mathrm{dom} f_k \subseteq \{0,1\}^{E_{(k)}}$. So, we deal with Case (iii).

(iii): Replace y' by $x_{(i)} + (y''(i,j) - x(i,j))\chi_{(i,j)} \in \mathrm{dom} f_i$. Then

$$f_i[+s^j_{(i)}[\gamma], \alpha_{(i)}](y') > f_i[+s_{(i)}, \alpha_{(i)}](y').$$

As $x(i,j) > 0$, f_i is linear over $\mathrm{dom} f_i$ and (x, s) satisfies incentive constraints. So, we have

$$f_i[+s_{(i)}, \alpha_{(i)}](y') \geq f_i[+s_{(i)}, \alpha_{(i)}](x_{(i)}).$$

Hence (3.1.3) and (3.1.5) are satisfied by the new y'. This is again a contradiction. So, we conclude that if any of the three cases hold, any pairwise stable outcome is pairwise strictly stable under given conditions. □

Generally, the condition of hereditary on the domain of a value function is useful. However, when the value functions are M$^\natural$-concave, we may relax the condition of hereditary because of the following lemma.

Lemma 3.1.2. *Let $f : \mathbf{Z}^E \to \mathbf{R} \cup \{-\infty\}$ be an M$^\natural$-concave function. Then $[x, y]_{\mathbf{Z}} \subseteq \mathrm{dom} f$ for each $x, y \in \mathrm{dom} f$ with $x \leq y$.*

Proof. Let $\tilde{y} \in [x, y]_{\mathbf{Z}}$ with $\tilde{y} \neq y$. We show that $\tilde{y} \in \mathrm{dom} f$. Then note that $\tilde{y} = y - \sum_{e \in \mathrm{supp}^+(y-\tilde{y})} (y(e) - \tilde{y}(e))\chi_e$. Let $e_1 \in \mathrm{supp}^+(y - \tilde{y})$. Then $e_1 \in \mathrm{supp}^+(y - x)$ and since f is M$^\natural$-concave, we have

$$f(x) + f(y) \leq f(x + \chi_{e_1}) + f(y - \chi_{e_1}).$$

This shows that $y' = y - \chi_{e_1} \in \mathrm{dom} f$. If $\mathrm{supp}^+(y' - \tilde{y}) = \emptyset$ then $y' = \tilde{y}$ and the assertion is true. Otherwise, take $e_2 \in \mathrm{supp}^+(y' - \tilde{y}) (\subseteq \mathrm{supp}^+(y - \tilde{y}))$. Then $e_2 \in \mathrm{supp}^+(y' - x)$ and since f is M$^\natural$-concave, we have

$$f(x) + f(y') \leq f(x + \chi_{e_2}) + f(y' - \chi_{e_2}).$$

This shows that $y'' = y' - \chi_{e_2} \in \mathrm{dom} f$. That is, $y - \chi_{e_1} - \chi_{e_2} \in \mathrm{dom} f$ for $e_1, e_2 \in \mathrm{supp}^+(y - \tilde{y})$. By continuing this process, we get $\tilde{y} = y - \sum_{e \in \mathrm{supp}^+(y-\tilde{y})} (y(e) - \tilde{y}(e))\chi_e \in \mathrm{dom} f$. □

Corollary 3.1.3. *Let $f : \mathbf{Z}^E \to \mathbf{R} \cup \{-\infty\}$ be an M$^\natural$-concave function. Then* $\mathrm{dom} f$ *is hereditary.*

Thus, when the value functions are assumed to be M$^\natural$-concave, we can re-write the condition (A) by dropping the condition of hereditary in the following way: For each $k \in P \cup Q$, the value function f_k satisfies the following:
(A$'$) The effective domain $\mathrm{dom} f_k$ of f_k is bounded and contains $\mathbf{0} \in \mathbf{Z}^{E_{(k)}}$ as a lower bound.

In the following theorem, we see that if value functions are M$^\natural$-concave satisfying (A$'$) then there exists a pairwise strictly stable outcome.

Theorem 3.1.4 (Fujishige and Tamura [10])**.** *For M$^\natural$-concave functions $f_k (k \in P \cup Q)$ satisfying* (A$'$) *and for vectors $\underline{\pi} \in (\mathbf{R} \cup \{-\infty\})^E$ and $\overline{\pi} \in (\mathbf{R} \cup \{+\infty\})^E$ with $\underline{\pi} \leq \overline{\pi}$, there exists a pairwise strictly stable outcome (x, s).*

The next theorem characterizes the pairwise strictly stable outcome. This theorem is the modification of the Theorem 3.3 [10].

Theorem 3.1.5. *For each $k \in P \cup Q$, assume that f_k is an M$^\natural$-concave function satisfying* (A$'$)*. Let x be a feasible allocation. Then there exists a feasible salary* *vector s forming a pairwise strictly stable outcome (x, s) if and only if there exist $p \in \mathbf{R}^E$, $z_P = (z_{(i)} \mid i \in P) \in (\mathbf{Z} \cup \{+\infty\})^E$ and $z_Q = (z_{(j)} \mid j \in Q) \in (\mathbf{Z} \cup \{+\infty\})^E$ such that*

$$x_{(i)} \in AM(f_i[+p_{(i)}, \alpha_{(i)}], z_{(i)}) \quad (\forall i \in P), \tag{3.1.6}$$

$$x_{(j)} \in AM(f_j[-p_{(j)}, \alpha_{(j)}], z_{(j)}) \quad (\forall j \in Q), \tag{3.1.7}$$

$$\underline{\pi} \leq p \leq \overline{\pi}, \tag{3.1.8}$$

$$e \in E, z_P(e) < +\infty \Rightarrow p(e) = \underline{\pi}(e), z_Q(e) = +\infty, \tag{3.1.9}$$

$$e \in E, z_Q(e) < +\infty \Rightarrow p(e) = \overline{\pi}(e), z_P(e) = +\infty. \tag{3.1.10}$$

Moreover, for any x, p, z_P, and z_Q satisfying (3.1.6)–(3.1.10)*, (x, p) is a pairwise strictly stable outcome.*

Proof. The Only-If Part: Let (x, s) be a pairwise strictly stable outcome. Then for each pair $(i, j) \in E$ with $x(i, j) = 0$, we define $r(i, j)$ as the supremum of the set of γs satisfying (PSS3(i)) without the constraint $\gamma \leq \overline{\pi}(i, j)$. (we have $r(i, j) \neq +\infty$ if there exists $y \in \mathrm{dom} f_i$ such that $y(i, j) > 0$ and $y(i, j') \leq x(i, j')$ for all $j' \in Q \setminus \{j\}$.) If $r(i, j) = +\infty$, then redefine $r(i, j)$ as the infimum of the set of γs satisfying (PSS3(ii)) without the constraint $\underline{\pi}(i, j) \leq \gamma$. (We have $r(i, j) \neq -\infty$ if there exists $y \in \mathrm{dom} f_j$ such that $y(i, j) > 0$ and $y(i', j) \leq x(i', j)$ for all $i' \in P \setminus \{i\}$.) If $r(i, j) = -\infty$, then we redefine $r(i, j) = -\varepsilon$ for a sufficiently large positive number ε. For each $(i, j) \in E$, we redefine $p \in \mathbf{R}^E$, $z_P, z_Q \in (\mathbf{Z} \cup \{+\infty\})^E$ by

$$p(i, j) = \begin{cases} s(i, j) & \text{if } x(i, j) > 0, \\ r(i, j) & \text{else if } \underline{\pi}(i, j) \leq r(i, j) \leq \overline{\pi}(i, j), \\ \underline{\pi}(i, j) & \text{else if } r(i, j) < \underline{\pi}(i, j), \\ \overline{\pi}(i, j) & \text{else if } \overline{\pi}(i, j) < r(i, j). \end{cases} \tag{3.1.11}$$

$$z_P(i, j) = \begin{cases} x(i, j) & \text{if (PSS3(i)) does not hold for } \gamma = p(i, j), \\ +\infty & \text{otherwise.} \end{cases} \tag{3.1.12}$$

$$z_Q(i, j) = \begin{cases} x(i, j) & \text{if (PSS3(ii)) does not hold for } \gamma = p(i, j), \\ +\infty & \text{otherwise.} \end{cases} \tag{3.1.13}$$

Since s is feasible hence (3.1.8) is satisfied by p. It follows from pairwise strict stability of (x, s) that $z_P(i, j) = +\infty$ holds. We consider the case where $z_P(i, j) < +\infty$.

In this case, there exists $y' \in \mathbf{Z}^{E_{(j)}}$ such that $f_i[+p_{(i)}, \alpha_{(i)}](x_{(i)}) < f_i[+p_{(i)}, \alpha_{(i)}](y')$ and $y'(i, j') \le x(i, j')$ for all $j' \in Q \setminus \{j\}$. Here we note that $f_i[+p_{(i)}, \alpha_{(i)}](x_{(i)}) = f_i[+s_{(i)}, \alpha_{(i)}](x_{(i)})$ and $f_i[+p_{(i)}, \alpha_{(i)}](y') = f_i[+s^j_{(i)}[p(i,j)], \alpha_{(i)}](y')$. We will show (3.1.9). On contrary, suppose that $p(i,j) > \underline{\pi}(i,j)$. If $x(i,j) > 0$ then for a sufficiently small number $\delta > 0$, we get

$$f_i[+p_{(i)}, \alpha_{(i)}](x_{(i)}) < f_i[+p_{(i)} - \delta\chi_{(i,j)}, \alpha_{(i)}](y'),$$
$$f_j[-p_{(j)}, \alpha_{(j)}](x_{(j)}) < f_j[-p_{(j)} - \delta\chi_{(i,j)}, \alpha_{(j)}](x_{(j)}).$$

This implies that (PSS3) does not hold for $\gamma = p(i,j) - \delta \ge \underline{\pi}(i,j)$, which is a contradiction. So, we assume that $x(i,j) = 0$. Since $p(i,j) > \underline{\pi}(i,j)$, we get $p(i,j) \le r(i,j)$ by (3.1.11). Since (PSS3(i)) is nondecreasing in γ and $p(i,j) \le r(i,j)$. So, the definition of $r(i,j)$ guarantees that (PSS3(i)) holds for $\gamma = p(i,j)$. This contradicts $z_P(i,j) < +\infty$. So, (3.1.9) is satisfied.

Now we consider the case $z_Q(i,j) < +\infty$ and prove that (3.1.10) holds. On the contrary, suppose that $p(i,j) < \overline{\pi}(i,j)$. If $x(i,j) > 0$, we reach a contradiction in the same way as above. Hence, assume that $x(i,j) = 0$. By (3.1.11) and by the assumption, we have $p(i,j) \ge r(i,j)$. If $r(i,j)$ is defined by (PSS3(i)) then for a sufficiently small number $\delta > 0$, (PSS3) does not hold for $\gamma = p(i,j) + \delta \le \overline{\pi}(i,j)$. This is a contradiction.

In the other case (where $r(i,j)$ is either defined by (PSS3(ii)) or set to be sufficiently small $-\varepsilon$), (PSS3(ii)) holds for $\gamma = p(i,j)$ because $p(i,j) \ge r(i,j)$. This is a cotradiction to the assumption that $z_Q(i,j) < +\infty$. Hence, we get (3.1.10).

Now we have to prove (3.1.6). On the contrary, suppose that (3.1.6) does not hold. Then for some $i \in P$ there exists $y' \in AM(f_i[+p_{(i)}, \alpha_{(i)}], z_{(i)})$ with $f_i[+p_{(i)}, \alpha_{(i)}](x_{(i)}) < f_i[+p_{(i)}, \alpha_{(i)}](y')$. We choose $y' \in AM(f_i[+p_{(i)}, \alpha_{(i)}], z_{(i)})$ with $f_i[+p_{(i)}, \alpha_{(i)}](x_{(i)}) < f_i[+p_{(i)}, \alpha_{(i)}](y')$ that minimizes $\sum\{y'(e) - x_{(i)}(e) \mid e \in \mathrm{supp}^+(y' - x_{(i)})\}$. Since $f_i[+s_{(i)}, \alpha_{(i)}](y) = f_i[+p_{(i)}, \alpha_{(i)}](y)$ holds for all $y \in \mathbf{Z}^{E_{(i)}}$ with $\mathbf{0} \le y \le x_{(i)}$. So, (PSS1) implies the existence of $e \in E_{(i)}$ with $y'(e) > x_{(i)}(e)$. By ($-$M$^\natural$-EXC[$\mathbf{Z}$]), there exists $e' \in \mathrm{supp}^-(y' - x_{(i)}) \cup \{0\}$ such that

$$f_i[+p_{(i)}, \alpha_{(i)}](y') + f_i[+p_{(i)}, \alpha_{(i)}](x_{(i)}) \le f_i[+p_{(i)}, \alpha_{(i)}](y' - \chi_e + \chi_{e'})$$
$$+ f_i[+p_{(i)}, \alpha_{(i)}](x_{(i)} + \chi_e - \chi_{e'}).$$

By definition of y' , we have

$$f_i[+p_{(i)}, \alpha_{(i)}](y') > f_i[+p_{(i)}, \alpha_{(i)}](y' - \chi_e + \chi_{e'}).$$

Using the above two inequalities, we have $f_i[+p_{(i)}, \alpha_{(i)}](x_{(i)}) < f_i[+p_{(i)}, \alpha_{(i)}](x_{(i)} + \chi_e - \chi_{e'})$. This yields $z_P(e) = x_{(i)}(e)$ by (3.1.12), which contradicts that $y' \leq z_{(i)}$. Hence (3.1.6) holds. ((3.1.7) has a similar proof).

The If Part: Let $p \in \mathbf{R}^E$ and $z_P, z_Q \in (\mathbf{Z} \cup \{+\infty\})^E$ be vectors that satisfy (3.1.6) – (3.1.10). We take $s = p$ and prove that (x, s) is a pairwise strictly stable outcome. Since for all $k \in (P \cup Q)$ $x_{(k)} \leq z_{(k)}$, hence (PSS1) and (PSS2) are evident directly from (3.1.6) and (3.1.7). Contrarily, suppose that there exist $i \in P$, $j \in Q$, $\gamma \in [\underline{\pi}(i,j), \overline{\pi}(i,j)]$, $y' \in \mathbf{Z}^{E_{(i)}}$ and $y'' \in \mathbf{Z}^{E_{(j)}}$ such that (3.1.3) and (3.1.4) hold.

Since $y' \geq \mathbf{0}$. So, using (3.1.6), condition (3.1.3) implies that either (Case i) $y'(i,j) > z_{(i)}(i,j)$ or (Case ii) $y'(i,j) \leq z_{(i)}(i,j)$ and $p(i,j) < \gamma$ holds. Similarly, (3.1.7) and (3.1.4) imply that either (Case iii) $y''(i,j) > z_{(j)}(i,j)$ or (Case iv) $y''(i,j) \leq z_{(j)}(i,j)$ and $\gamma < p(i,j)$ holds. But the (Case ii) and the (Case iv) are trivially inconsistent. Also, (3.1.9) or (3.1.10) assures that the (Case i) and the (Case iii) cannot hold simultaneously. Further, (Case i) and (3.1.9) together imply that $p(i,j) = \underline{\pi}(i,j)$ is inconsistent with the (Case iv). Similarly, due to (3.1.10), (Case ii) is inconsistent with the (Case iii). This means that (3.1.3) and (3.1.4) cannot hold at a time, which is a contradiction to our supposition. Hence proved that (x, s) is a pairwise strictly stable outcome. $\square$

For each $x \in \mathbf{Z}^E$, we define two aggregated functions as follows:

$$f_P(x) = \sum_{i \in P} f_i(x_{(i)}), \quad f_Q(x) = \sum_{j \in Q} f_j(x_{(j)}). \tag{3.1.14}$$

Then one can see that f_P is an M$^\natural$-concave function on $\mathbf{Z}^E$ if f_i is an M$^\natural$-concave function on $\mathbf{Z}^{E_{(i)}}$ for each $i \in P$. Similarly, f_Q is an M$^\natural$-concave function on $\mathbf{Z}^E$ if f_j is an M$^\natural$-concave function on $\mathbf{Z}^{E_{(j)}}$ for each $j \in Q$.

Lemma 3.1.6 (Fujishige and Tamura [10]). *For each $i \in P$, $x_{(i)} \in AM(f_i\ [+p_{(i)}, \alpha_{(i)}]$ $, z_{(i)})$ if and only if $x \in AM(f_P[+p, \alpha],\ z_P)$. Similarly, for each $j \in Q$, $x_{(j)} \in AM(f_j[-p_{(j)}, \alpha_{(j)}], z_{(j)})$ if and only if $x \in AM(f_Q[-p, \alpha], z_Q)$.*

For aggregated functions f_P and f_Q defined by (3.1.14), we now re-write (A′) in the following way:

(A″) The effective domains $\mathrm{dom} f_P$ and $\mathrm{dom} f_Q$ of f_P and f_Q, respectively, are bounded and contain $\mathbf{0} \in \mathbf{Z}^E$ as a lower bound.

Theorem 3.1.7. *Let $f_P, f_Q : \mathbf{Z}^E \to \mathbf{R} \cup \{-\infty\}$ be $M^\natural$-concave functions satisfying (A″) and, $\underline{\pi} \in (\mathbf{R} \cup \{-\infty\})^E$ and $\overline{\pi} \in (\mathbf{R} \cup \{+\infty\})^E$ with $\underline{\pi} \leq \overline{\pi}$. Then there exist $x \in \mathbf{Z}^E, p \in \mathbf{R}^E$ and $z_P, z_Q \in (\mathbf{Z} \cup \{+\infty\})^E$ such that*

$$x \in AM(f_P[+p, \alpha], z_P) \cap AM(f_Q[-p, \alpha], z_Q), \tag{3.1.15}$$

$$\underline{\pi} \leq p \leq \overline{\pi}, \tag{3.1.16}$$

$$e \in E, z_P(e) < +\infty \Rightarrow p(e) = \underline{\pi}(e), z_Q(e) = +\infty, \tag{3.1.17}$$

$$e \in E, z_Q(e) < +\infty \Rightarrow p(e) = \overline{\pi}(e), z_P(e) = +\infty. \tag{3.1.18}$$

By Lemma 3.1.6 and Theorem 3.1.5, we see that Theorem 3.1.4 is a direct consequence of Theorem 3.1.7. Thus, to find a pairwise strictly stable outcome in our model, it is enough to prove Theorem 3.1.7. To prove Theorem 3.1.7, we will give an algorithm in the next section.

In view of Theorem 3.1.7, we say that a pair (x, p) of $x \in \mathbf{Z}^E$ and $p \in \mathbf{R}^E$ is *pairwise strictly stable* if there exist vectors $z_P, z_Q \in (\mathbf{Z} \cup \{+\infty\})^E$ that satisfy (3.1.15)−(3.1.18).

3.2 An Algorithm

In this section, we prove Theorem 3.1.7 by proposing an algorithm. Our algorithm and the algorithm proposed by Fujishige and Tamura [10] have the same essence. However, the technique used by Fujishige and Tamura [10] to modify the salary vector does not work in our case due to generality of our model. We will employ linear programming to modify the salary vector in each iteration of the algorithm. Moreover, our presentation of the algorithm is much simpler than that of Fujishige and Tamura [10].

First, we give few lemmas that will play a crucial role in devising algorithm. At the end of this section, we present the algorithm and discuss its correctness and termination. We remark that Lemmas 3.2.1 and 3.2.2 are extracted from Section 5 of Fujishige and Tamura [10] with substantial modifications.

Lemma 3.2.1. *For* $M^\natural$*-concave functions* $f_P, f_Q : \mathbf{Z}^E \to \mathbf{R} \cup \{-\infty\}$ *satisfying* (A′′) *and for vectors* $\underline{\pi} \in (\mathbf{R} \cup \{-\infty\})^E$ *and* $\overline{\pi} \in (\mathbf{R} \cup \{+\infty\})^E$ *with* $\underline{\pi} \leq \overline{\pi}$*, there exist* $x_P, x_Q \in \mathbf{Z}^E$*,* $z_P, z_Q \in (\mathbf{Z} \cup \{+\infty\})^E$ *and* $p \in \mathbf{R}^E$ *such that*

$$x_Q \leq x_P, \tag{3.2.1}$$

$$x_P \in AM(f_P[+p, \alpha], z_P), \tag{3.2.2}$$

$$x_Q \in AM(f_Q[-p, \alpha], z_Q), \tag{3.2.3}$$

$$\underline{\pi} \leq p \leq \overline{\pi}, \tag{3.2.4}$$

$$e \in E, z_P(e) < +\infty \Rightarrow p(e) = \underline{\pi}(e), z_Q(e) = +\infty, \tag{3.2.5}$$

$$\begin{aligned} e \in E, z_Q(e) < +\infty \Rightarrow p(e) = \overline{\pi}(e), z_P(e) = +\infty \ \textit{and} \\ x_P(e) = x_Q(e) = z_Q(e). \end{aligned} \tag{3.2.6}$$

Proof. We will find vectors $x_P, x_Q \in \mathbf{Z}^E$, $z_P, z_Q \in (\mathbf{Z} \cup \{+\infty\})^E$ and $p \in \mathbf{R}^E$ that satisfy (3.2.1)−(3.2.6). For each $e \in E$, we define $p(e)$ and $z_P(e)$ by:

$$\begin{aligned} p(e) &= \begin{cases} \overline{\pi}(e) & \text{if } \overline{\pi}(e) < +\infty \\ b & \text{otherwise,} \end{cases} \\ z_P(e) &= +\infty, \end{aligned}$$

where b is a sufficiently large positive number. Then p satisfies (3.2.4). Let $x_P \in AM(f_P[+p, \alpha], z_P)$ and define $z_Q(e)$, for each $e \in E$, by:

$$z_Q(e) = \begin{cases} x_P(e) & \text{if } \overline{\pi}(e) < +\infty \\ +\infty & \text{otherwise.} \end{cases}$$

Let $x_Q \in AM(f_Q[-p, \alpha], z_Q)$. If there is $e \in E$ with $x_Q(e) < x_P(e)$ and $p(e) = \overline{\pi}(e)$, we set $z_Q(e) = +\infty$. Lemma 1.2.9 guarantees that $x_Q \in AM(f_Q[-p, \alpha], z_Q)$ for the updated z_Q. Then x_P, x_Q, z_P, z_Q and p satisfy (3.2.5) and (3.2.6). Moreover, since b is chosen sufficiently large positive number and by (A′′), $\mathrm{dom} f_Q$ is bounded below by $\mathbf{0} \in \mathbf{Z}^E$, we have $x_Q(e) = 0$ for each $e \in E$ with $\overline{\pi}(e) = +\infty$. Therefore $x_Q(e) \leq x_P(e)$ for each $e \in E$. Thus (3.2.1) is also true. □

For a given feasible salary vector $s \in \mathbf{R}^E$ and for vectors $x, y \in \mathbf{Z}^E$, we denote

$$L_s = \{e \in E \mid s(e) = \underline{\pi}(e)\}, \tag{3.2.7}$$

$$T_{xy} = \mathrm{supp}^+(x - y). \tag{3.2.8}$$

Lemma 3.2.2. *Let $f_P, f_Q : \mathbf{Z}^E \to \mathbf{R} \cup \{-\infty\}$ be $M^\natural$-concave functions satisfying* (A$''$) *and $x_P, x_Q \in \mathbf{Z}^E, z_P, z_Q \in (\mathbf{Z} \cup \{+\infty\})^E$ and $p \in \mathbf{R}^E$ are the vectors that satisfy* (3.2.1)–(3.2.6)*. Assume that $L_p \cap T_{x_P x_Q} \neq \emptyset$ and $\tilde{e} \in L_p \cap T_{x_P x_Q}$. Then there exist vectors x'_P, x'_Q, z'_P, z'_Q and $p' = p$ that satisfy* (3.2.1)–(3.2.6) *and*[2]

$$z'_P \leq z_P, \quad z_Q \leq z'_Q, \tag{3.2.9}$$

$$\sum_{e \in E}(x'_P(e) - x'_Q(e)) \leq \sum_{e \in E}(x_P(e) - x_Q(e)), \tag{3.2.10}$$

$$z'_P(\tilde{e}) < z_P(\tilde{e}), \tag{3.2.11}$$

$$x_Q(\tilde{e}) = x'_Q(\tilde{e}) \leq x'_P(\tilde{e}) = x_P(\tilde{e}) - 1. \tag{3.2.12}$$

Proof. We will find vectors x'_P, x'_Q, z'_P and z'_Q by modification of x_P, x_Q, z_P and z_Q. From (3.2.6), we see that $z_Q(e) = +\infty$ for each $e \in L_p \cap T_{x_P x_Q}$. Therefore, we can define $z'_P(e)$ for each $e \in E$ by

$$z'_P(e) = \begin{cases} x_P(e) & \text{if } e \in L_p \cap T_{x_P x_Q} \\ z_P(e) & \text{otherwise.} \end{cases}$$

Then one can readily see that $x_P \in AM(f_P[+p, \alpha], z'_P)$. Let $z'_P := z'_P - \chi_{\tilde{e}}$. By Lemma 1.2.8 there exists $e' \in \{0\} \cup E \setminus \{L_p \cap T_{x_P x_Q}\}$ such that $x'_P := x_P + \chi_{e'} - \chi_{\tilde{e}} \in AM(f_P[+p, \alpha], z'_P)$. Obviously, $z'_P \leq z_P$ and $z'_P(\tilde{e}) < z_P(\tilde{e})$.

If $e' = 0$ or $z_Q(e') = +\infty$ then no modification is needed for e' and we take $x'_Q := x_Q$ and $z'_Q := z_Q$.

If $z_Q(e') < +\infty$ then $x'_P(e') = x_Q(e') + 1 = z_Q(e') + 1$. In this case, we set $z'_Q := z_Q + \chi_{e'}$. By Lemma 1.2.8, there exists $e'' \in E \cup \{0\}$ such that $x'_Q := x_Q - \chi_{e''} + \chi_{e'} \in AM(f_Q[-p, \alpha], z'_Q)$. If $e' = e''$ then $x'_Q(e') < x'_P(e') = z'_Q(e')$ and we set $z'_Q(e') = +\infty$. In this case, Lemma 1.2.9 implies $x'_Q \in AM(f_Q[-p, \alpha], z'_Q)$. If $e' \neq e''$ then $x'_P(e') = x'_Q(e') = z'_Q(e')$ and no further modification is needed for e'.

[2] This means that (3.2.1)–(3.2.6) remain true if x_P, x_Q, z_P, z_Q and p are replaced by x'_P, x'_Q, z'_P, z'_Q and p', respectively.

Now, if $z'_Q(e'') < +\infty$ then we put $z'_Q(e'') = +\infty$. Lemma 1.2.9 gives $x'_Q \in AM(f_Q[-p, \alpha], z'_Q)$. If $z'_Q(e'') = +\infty$ then no modification is needed for e''. Then one can see that (3.2.1)−(3.2.6) and (3.2.9)−(3.2.12) hold for the vectors x'_P, x'_Q, z'_P, z'_Q and p. □

We see that if $L_p \cap T_{x_P x_Q} \neq \emptyset$ in Lemma 3.2.2 then p remains unchanged. In the next lemma, we will see that if $L_p \cap T_{x_P x_Q} = \emptyset$ then we need a modification of the salary vector p.

Lemma 3.2.3. *Let $f_P, f_Q : \mathbf{Z}^E \to \mathbf{R} \cup \{-\infty\}$ be $M^\natural$-concave functions satisfying* (A″) *and $x_P, x_Q \in \mathbf{Z}^E$, $z_P, z_Q \in (\mathbf{Z} \cup \{+\infty\})^E$ and $p \in \mathbf{R}^E$ are the vectors that satisfy* (3.2.1)−(3.2.6)*. Assume that $x_P \neq x_Q$ and $L_p \cap T_{x_P x_Q} = \emptyset$. Then there exist vectors x'_P, x'_Q, z'_P, z'_Q and $p' \leq p$ that satisfy* (3.2.1)−(3.2.6)*,* (3.2.9) *and* (3.2.10)*. Moreover, at-least one of the following holds:*

$$L_p \subset L_{p'}, \tag{3.2.13}$$

$$\sum_{e \in E} x'_P(e) < \sum_{e \in E} x_P(e), \tag{3.2.14}$$

$$\sum_{e \in E} x_Q(e) < \sum_{e \in E} x'_Q(e), \tag{3.2.15}$$

$$z_Q \neq z'_Q, \tag{3.2.16}$$

$$L_{p'} \cap T_{x'_P x'_Q} \neq \emptyset. \tag{3.2.17}$$

Proof. Define a subset $U \subseteq E$ by

$$U = \{e \in E \mid z_Q(e) < +\infty\} \tag{3.2.18}$$

and[3] let

$$S = \{0\} \cup L_p \cup U. \tag{3.2.19}$$

We will find vectors x'_P, x'_Q, z'_P, z'_Q and p' by modification of x_P, x_Q, z_P, z_Q and p.

[3]Note that $L_p \subseteq U$ at this stage. However, it may not be true in the subsequent iterations of the algorithm.

For each $x \in \mathbf{Z}^E$, we define

$$\begin{aligned} F_P(x) &= \begin{cases} f_P(x) & \text{if } x \leq z_P \\ -\infty & \text{otherwise,} \end{cases} \\ F_Q(x) &= \begin{cases} f_Q(x) & \text{if } x \leq z_Q \\ -\infty & \text{otherwise.} \end{cases} \end{aligned} \quad (3.2.20)$$

Obviously, F_P and F_Q are M$^\natural$-concave, $x_P \in \arg\max F_P[+p, \alpha]$ and $x_Q \in \arg\max F_Q[-p, \alpha]$.

Let $\widehat{E} = \{0\} \cup E$. Construct a directed graph $G = (\widehat{E}, A)$, where A is the union of two sets A_P and A_Q defined by:

$$\begin{aligned} A_P &= \{(e, e') \in \widehat{E} \times \widehat{E} \mid e \neq e', x_P + \chi_e - \chi_{e'} \in \mathrm{dom} F_P\}, \\ A_Q &= \{(e, e') \in \widehat{E} \times \widehat{E} \mid e \neq e', x_Q - \chi_e + \chi_{e'} \in \mathrm{dom} F_Q\}. \end{aligned} \quad (3.2.21)$$

Define weight $\ell(e, e')$ of each arc $(e, e') \in A$ by:

$$\ell(e, e') = \begin{cases} F_P[+p, \alpha](x_P) - F_P[+p, \alpha](x_P + \chi_e - \chi_{e'}) & \text{if } (e, e') \in A_P \\ F_Q[-p, \alpha](x_Q) - F_Q[-p, \alpha](x_Q - \chi_e + \chi_{e'}) & \text{if } (e, e') \in A_Q. \end{cases} \quad (3.2.22)$$

By Theorem 1.2.6, $\ell(e, e') \geq 0$ for each $(e, e') \in A$. Also, $\mathbf{0} \in \mathrm{dom} F_P$ by (A$''$). Lemma 3.1.2 implies $[\mathbf{0}, x_P] \subseteq \mathrm{dom} F_P$ which further implies that $x_P - \chi_e \in \mathrm{dom} F_P$ for each $e \in T_{x_P x_Q}$. Hence, $(\mathbf{0}, e) \in A_P$ for each $e \in T_{x_P x_Q}$. This shows that there is a path from S to $T_{x_P x_Q}$ in G. Let $\mathbf{P}$ be the set of paths from S to $T_{x_P x_Q}$ in the graph G. Denote the terminal arc of each path $P \in \mathbf{P}$ by e_P. Define

$$\tilde{m} = \min\{\frac{\sum_{a \in P} \ell(a)}{\alpha(e_P)} \mid P \in \mathbf{P}\}. \quad (3.2.23)$$

Let $\tilde{P}$, denoted by $(e_0, a_1, e_1, \ldots, a_n, e_n)$, be a path from S to $T_{x_P x_Q}$ which satisfies (3.2.23). Furthermore, suppose that $\tilde{P}$ has the minimum number of arcs among those paths that satisfy (3.2.23). We define a linear program by:

$$\text{Maximize} \sum_{e \in E} d(e) \tag{3.2.24}$$

subject to

$$\ell(e,e') + \alpha(e)d(e) - \alpha(e')d(e') \geq 0 \quad \forall (e,e') \in A,$$

$$d(e) = 0 \quad \forall e \in S,$$

$$0 \leq \alpha(e)d(e) \leq \alpha(e_n)\tilde{m} \quad \forall e \in E \setminus S,$$

$$0 \leq d(e) \leq p(e) - \underline{\pi}(e) \quad \forall e \in E \setminus S.$$

Note that the feasible region is bounded and zero vector is a feasible solution of the linear program (3.2.24). Thus there is an optimal solution $\tilde{d} = (\tilde{d}(e) \mid e \in E)$, say, of the linear program (3.2.24). We shall write $\tilde{\ell}(a) = \ell(a) + \alpha(e)\tilde{d}(e) - \alpha(e')\tilde{d}(e')$ for each $a = (e,e') \in A$. Since $\tilde{\ell}(e,e') \geq 0$ for each $(e,e') \in A$, it holds that $\tilde{\ell}(e,e') \geq 0$ for each $e, e' \in \widehat{E}$. This together Theorem 1.2.6 gives

$$x_P \in \arg\max F_P[+(p - \tilde{d}), \alpha], \quad x_Q \in \arg\max F_Q[-(p - \tilde{d}), \alpha]. \tag{3.2.25}$$

Let $p' = p - \tilde{d}$. Then x_P, x_Q, z_P, z_Q and p' satisfy (3.2.1)–(3.2.6), (3.2.9) and (3.2.10). If $\tilde{d}(e) = p(e) - \underline{\pi}(e)$ for some $e \in E \setminus S$ then obviously $L_p \subset L_{p'}$, that is, (3.2.13) is true.

Next, we consider the case when $\tilde{d}(e) < p(e) - \underline{\pi}(e)$ for each $e \in E \setminus S$. We first prove the following claim:

Claim: If $\tilde{d}(e) < p(e) - \underline{\pi}(e)$ for each $e \in E \setminus S$ then $\tilde{d}(e_n) = \tilde{m}$.

[Proof of Claim:] Obviously, $\tilde{d}(e_n) \leq \tilde{m}$. On contrary, suppose that $\tilde{d}(e_n) < \tilde{m}$. Let

$$\tilde{A} = \{(e,e') \in A \mid \tilde{\ell}(e,e') = 0\}. \tag{3.2.26}$$

We define a subset $\widetilde{E} \subseteq E$ by:

$$\widetilde{E} = \{e \in E \mid \ e_n \text{ is reachable from } e \text{ by a path of arcs in } \tilde{A}\}. \tag{3.2.27}$$

Then there is no path from S to e_n of arcs in $\tilde{A}$. For otherwise, if P is a path from S to e_n of arcs in $\tilde{A}$ then $\tilde{\ell}(e,e') = 0$ for each $(e,e') \in P$. This gives $\sum_{a \in P} \ell(a)/\alpha(e_n) = \tilde{d}(e_n) < \tilde{m}$ which contradicts (3.2.23). Consequently, we have

$$S \cap \widetilde{E} = \emptyset. \tag{3.2.28}$$

Since $\tilde{\ell}(e,e') = 0$ and $\ell(e,e') \geq 0$ for each $(e,e') \in \tilde{A}$, it holds that $\alpha(e)\tilde{d}(e) \leq \alpha(e')\tilde{d}(e')$. By assumption, $\alpha(e_n)\tilde{d}(e_n) < \alpha(e_n)\tilde{m}$ which implies

$$\alpha(e)\tilde{d}(e) < \alpha(e_n)\tilde{m} \quad \forall e \in \widetilde{E}.$$

Observe that if there is an arc $(e,e') \in A$ with $e \notin \widetilde{E}$ and $e' \in \widetilde{E}$ then $\tilde{\ell}(e,e') > 0$. Thus one can choose a positive real number ε which satisfies the following:

$$\begin{aligned}
&\tilde{d}(e) + \frac{\varepsilon}{\alpha(e)} \leq p(e) - \underline{\pi}(e) \quad \forall e \in \widetilde{E},\\
&\alpha(e)\left(\tilde{d}(e) + \frac{\varepsilon}{\alpha(e)}\right) \leq \alpha(e_n)\tilde{m} \quad \forall e \in \widetilde{E},\\
&\ell(e,e') + \alpha(e)\tilde{d}(e) - \alpha(e')\left(\tilde{d}(e') + \frac{\varepsilon}{\alpha(e')}\right) \geq 0\\
&\forall (e,e') \in A \text{ with } e \notin \widetilde{E} \text{ and } e' \in \widetilde{E}.
\end{aligned} \tag{3.2.29}$$

We define

$$\hat{d} = \tilde{d} + \sum_{e \in \widetilde{E}} \frac{\varepsilon}{\alpha(e)} \chi_e. \tag{3.2.30}$$

The definition of $\hat{d}$ together with (3.2.28) and (3.2.29) imply that

$$\begin{aligned}
&\ell(e,e') + \alpha(e)\hat{d}(e) - \alpha(e')\hat{d}(e') \geq 0 \quad \forall (e,e') \in A,\\
&\hat{d}(e) = \tilde{d}(e) = 0 \quad \forall e \in S,\\
&\alpha(e)\hat{d}(e) \leq \alpha(e_n)\tilde{m} \quad \forall e \in E \setminus S,\\
&\hat{d}(e) \leq p(e) - \underline{\pi}(e) \quad \forall e \in E \setminus S.
\end{aligned} \tag{3.2.31}$$

This shows that $\hat{d}$ is a feasible solution of the linear program (3.2.24). However, $\hat{d} \geq \tilde{d}$ and $\hat{d}(e_n) > \tilde{d}(e_n)$ which contradicts to the fact that $\tilde{d}$ is an optimal solution of the linear program (3.2.24). Hence, $\tilde{d}(e_n) = \tilde{m}$. [end of proof of Claim]

By the constraints in the linear program (3.2.24) and the above claim, we have

$$\tilde{d}(e_0) = 0, \tag{3.2.32}$$

$$\tilde{m} = \frac{\sum_{a \in \tilde{P}} \ell(a)}{\alpha(e_n)}, \tag{3.2.33}$$

$$\alpha(e_i)\tilde{d}(e_i) \leq \tilde{m}\alpha(e_n) \qquad (\forall\ e_i \in \tilde{P}, i \in \{1,2,\ldots,n-1\}), \tag{3.2.34}$$

$$\tilde{d}(e_n) = \tilde{m}. \tag{3.2.35}$$

Since last arc in $\tilde{P}$ is $a_n = (e_{n-1}, e_n)$. Using (3.2.32) – (3.2.35) we have

$$\begin{aligned}
0 \le \tilde{\ell}(a_n) &= \ell(a_n) + \alpha(e_{n-1})\tilde{d}(e_{n-1}) - \alpha(e_n)\tilde{d}(e_n) \\
&= \ell(a_n) + \alpha(e_{n-1})\tilde{d}(e_{n-1}) - \sum_{a\in\tilde{P}} \ell(a) \\
&\le - \sum_{a\in\{\tilde{P}\setminus\{a_n,a_{n-1}\}\}} \ell(a) + \alpha(e_{n-2})\tilde{d}(e_{n-2}) \\
&\;\vdots \\
&\le - \sum_{a\in\{\tilde{P}\setminus\{a_n,a_{n-1},\ldots,a_1\}\}} \ell(a) + \alpha(e_0)\tilde{d}(e_0) - \ell(a_1) \\
&\le \alpha(e_0)\tilde{d}(e_0) = 0.
\end{aligned}$$

From above inequalities, it is easily seen that $\tilde{\ell}(a) = 0$ for each $a \in \tilde{P}$. This together with (3.2.25) gives

$$x_P + \chi_e - \chi_{e'} \in \arg\max F_P[+p', \alpha] \quad \forall (e, e') \in A_P \cap \tilde{P}, \tag{3.2.36}$$

$$x_Q - \chi_e + \chi_{e'} \in \arg\max F_Q[-p', \alpha] \quad \forall (e, e') \in A_Q \cap \tilde{P}. \tag{3.2.37}$$

Now, we prove that the arcs of $\tilde{P}$ alternate in arcs of A_P and A_Q. suppose contrarily that there are two consecutive arcs $(e, e'), (e', e'') \in \tilde{P}$ that belong to A_P. Then $x_P + \chi_{e'} - \chi_{e''}$ and $x_P + \chi_e - \chi_{e'}$ belong to domF_P (by (3.2.21)). Since $\tilde{P}$ has minimum number of arcs and, F_P and F_Q are M$^\natural$-concave. So, applying ($-$M$^\natural$-EXC[$\mathbf{Z}$]), we have

$$\begin{aligned}
&F_P[+p, \alpha](x_P + \chi_{e'} - \chi_{e''}) + F_P[+p, \alpha](x_P + \chi_e - \chi_{e'}) \\
&\qquad \le \max \left\{ \begin{array}{c} F_P[+p, \alpha](x_P + \chi_e - \chi_{e''}) + F_P[+p, \alpha](x_P) \\ F_P[+p, \alpha](x_P + \chi_e + \chi_{e'} - \chi_{e''}) + F_P[+p, \alpha](x_P - \chi_{e'}) \end{array} \right\}.
\end{aligned}$$

Again applying ($-$M$^\natural$-EXC[$\mathbf{Z}$]), we get

$$\begin{aligned}
&F_P[+p, \alpha](x_P + \chi_{e'} - \chi_{e''}) + F_P[+p, \alpha](x_P + \chi_e - \chi_{e'}) \\
&\qquad \le \max \left\{ \begin{array}{c} F_P[+p, \alpha](x_P + \chi_e - \chi_{e''}) + F_P[+p, \alpha](x_P) \\ F_P[+p, \alpha](x_P + \chi_e) + F_P[+p, \alpha](x_P - \chi_{e''}) \end{array} \right\}.
\end{aligned}$$

Applying ($-$M$^\natural$-EXC[$\mathbf{Z}$]) again on the last inequality, we get

$$\begin{aligned}
&F_P[+p, \alpha](x_P + \chi_{e'} - \chi_{e''}) + F_P[+p, \alpha](x_P + \chi_e - \chi_{e'}) \\
&\qquad \le F_P[+p, \alpha](x_P + \chi_e - \chi_{e''}) + F_P[+p, \alpha](x_P).
\end{aligned}$$

Using (3.2.22), the last inequality implies that

$$\ell(e, e') + \ell(e', e'') \geq \ell(e, e'').$$

This contradicts that $\tilde{P}$ has minimum number of arcs. Consequently, we have $\{e_1, e_1'\} \cap \{e_2, e_2'\} = \emptyset$ for each two distinct arcs $(e_1, e_1'), (e_2, e_2') \in A_P \cap \tilde{P}$. Similarly, $\{e_1, e_1'\} \cap \{e_2, e_2'\} = \emptyset$ for each two distinct arcs $(e_1, e_1'), (e_2, e_2') \in A_Q \cap \tilde{P}$. This together with Lemma 1.2.7 gives

$$x_P' := x_P + \sum_{(e,e') \in A_P \cap \tilde{P}} (\chi_e - \chi_{e'}) \in \arg\max F_P[+p', \alpha], \qquad (3.2.38)$$

$$x_Q' := x_Q - \sum_{(e,e') \in A_Q \cap \tilde{P}} (\chi_e - \chi_{e'}) \in \arg\max F_P[-p', \alpha]. \qquad (3.2.39)$$

With this modification, we see that

$$x_P' \in AM(f_P[+p', \alpha], z_P), \quad x_Q' \in AM(f_Q[-p', \alpha], z_Q).$$

Moreover, $x_Q' \leq x_P'$ and $\underline{\pi} \leq p' \leq \overline{\pi}$. Let $z_P' := z_P$ and $z_Q' := z_Q$. Then at this stage, (3.2.1)–(3.2.5), (3.2.9) and (3.2.10) are attained for x_P', x_Q', z_P', z_Q' and p'. $a_1 = (e_0, e_1)$ is the first arc of $\tilde{P}$. If $e_0 \notin U$ (i.e., $e_0 = 0$ or $e_0 \in L_{p'}$) then (3.2.6) is also true for x_P', x_Q', z_P', z_Q' and p'. If $e_0 = 0$ then either (3.2.14) or (3.2.15) is attained. If $e_0 \in L_{p'}$ then $x_Q'(e_0) < x_P'(e_0)$, that is, (3.2.17) is true.

Finally, we consider the case when $e_0 \in U$. By (3.2.38) and (3.2.39), we see that either $x_Q'(e_0)+1 = x_P'(e_0) = z_Q'(e_0)+1$ (i.e., $a_1 \in A_P$) or $x_Q'(e_0)+1 = x_P'(e_0) = z_Q'(e_0)$ (i.e., $a_1 \in A_Q$). We modify vectors x_P', x_Q' and z_Q' to attain (3.2.6) while preserving (3.2.1)–(3.2.5), (3.2.9) and (3.2.10).

If $x_Q'(e_0) + 1 = x_P'(e_0) = z_Q'(e_0)$, we remove e_0 from U by setting $z_Q'(e_0) := +\infty$. Lemma 1.2.9 guarantees that $x_Q' \in AM(f_Q[-p', \alpha], z_Q')$ for the modified z_Q'. Thus, (3.2.6) and (3.2.16) are obtained.

If $x_Q'(e_0) + 1 = x_P'(e_0) = z_Q'(e_0) + 1$ then we fix $z_Q'(e_0) := z_Q'(e_0) + 1$, i.e., $z_Q'(e_0)$ is increased by one. Then by Lemma 1.2.8(b), there exists $e' \in \widehat{E}$ such that $x_Q' - \chi_{e'} + \chi_{e_0} \in AM(f_Q[-p', \alpha], z_Q')$ for the modified z_Q'. We fix $x_Q' := x_Q' - \chi_{e'} + \chi_{e_0}$. If $e' \in U$, we remove it from U by fixing $z_Q'(e') := +\infty$. Lemma 1.2.9 implies that $x_Q' \in AM(f_Q[-p', \alpha], z_Q')$ for the modified z_Q'. If $e' \notin U$ then we keep $z_Q'(e')$ the same. Hence, (3.2.6) and (3.2.16) are obtained. □

An algorithm for finding a pairwise strictly stable outcome:

Input: Two disjoint and finite sets P and Q, the set of ordered pairs $E = P \times Q$, a vector $\alpha \in \mathbf{R}_{+}^{E}$, two vectors $\underline{\pi} \in (\mathbf{R} \cup \{-\infty\})^{E}$ and $\overline{\pi} \in (\mathbf{R} \cup \{+\infty\})^{E}$ with $\underline{\pi} \leq \overline{\pi}$, two aggregated M$^{\natural}$-concave functions defined by (3.1.14).

Output: Vectors $x_P, x_Q \in \mathbf{Z}^E$, $z_P, z_Q \in (\mathbf{Z} \cup \{+\infty\})^E$ and $p \in \mathbf{R}^E$ that satisfy (3.2.1)$-$(3.2.6) and $x_P = x_Q$.

Step 0: Find x_P, x_Q, z_P, z_Q and p that satisfy (3.2.1)$-$(3.2.6) by Lemma 3.2.1.

Step 1: If $x_P = x_Q$ then stop.

Step 2: Define L_p and $T_{x_P x_Q}$ by (3.2.7) and (3.2.8), respectively. If $L_p \cap T_{x_P x_Q} \neq \emptyset$ then go to Step 3; else go to Step 4.

Step 3: Find x'_P, x'_Q, z'_P, z'_Q and $p' = p$ that satisfy (3.2.1)$-$(3.2.6) and (3.2.9) $-$(3.2.12) by Lemma 3.2.2. Go to Step 5.

Step 4: Find x'_P, x'_Q, z'_P, z'_Q and $p' \leq p$ that satisfy (3.2.1)$-$(3.2.6), (3.2.9) and (3.2.10) by Lemma 3.2.3. Go to Step 5.

Step 5: Set $x_P := x'_P$, $x_Q := x'_Q$, $z_P := z'_P$, $z_Q := z'_Q$ and $p := p'$. Go to Step 1.

We now show correctness and termination of the Algorithm.

Lemma 3.2.4. *In each iteration of the Algorithm, the following statements hold:*

(i) *z_P decreases or remains the same. z_Q increases or remains the same.*

(ii) $\sum_{e \in E} (x_P(e) - x_Q(e))$ *decreases or remains the same.*

(iii) *L_p enlarges or remains the same.*

Proof. Note that the vectors x_P, x_Q, z_P, z_Q and p are modified at Step 5 of the Algorithm. When the Algorithm goes from Step 3 to Step 5 then (3.2.9) and (3.2.10) hold by Lemma 3.2.2, which implies that (i) and (ii) hold. If the Algorithm goes from Step 4 to Step 5 then (3.2.9) and (3.2.10) hold by Lemma 3.2.3, which implies that

(i) and (ii) hold. Also, if the Algorithm goes from Step 3 to Step 5 then p remains unchanged at Step 5 and consequently, L_p remains the same. If the Algorithm goes from Step 4 to Step 5 then (3.2.13) implies that L_p remains the same or enlarges at Step 5. This proves the lemma. □

Lemma 3.2.5. *In each iteration of the Algorithm at Step 5, at-least one of the following holds:*

(i) *Some component of z_P decreases.*

(ii) *Some component of z_Q increases.*

(iii) $\sum_{e\in E}(x_P(e) - x_Q(e))$ *decreases.*

(iv) L_p *enlarges.*

(v) $L_p \cap T_{x_P x_Q} \neq \emptyset$.

Proof. If the Algorithm goes from Step 3 to Step 5 then (3.2.11) holds by Lemma 3.2.2 so that,(i) holds. If the Algorithm goes from Step 4 to Step 5 then at-least one from (3.2.13)−(3.2.17) holds by Lemma 3.2.3, so that, (i)-(iii) and atleast one of (iv) and (v) hold. Therefore, the lemma is proved. □

It is noteworthy to mention that if $L_p \cap T_{x_P x_Q} \neq \emptyset$ in some iteration at Step 5 then in next iteration at Step 5, (i) of Lemma 3.2.5 is true.

If the Algorithm terminates at Step 1 then the vectors $x_P, x_Q \in \mathbf{Z}^E$, $z_P, z_Q \in (\mathbf{Z} \cup \{+\infty\})^E$ and $p \in \mathbf{R}^E$ satisfy (3.2.1)−(3.2.6) and $x_P = x_Q$. Hence, by letting $x = x_P = x_Q$, we see that (3.1.15)−(3.1.18) hold true.

Lemma 3.2.6. *If the Algorithm terminates then the output satisfies* (3.1.15)−(3.1.18) *with $x = x_P = x_Q$. Consequently, there exists a pairwise strictly stable outcome.*

Lemma 3.2.7. *The Algorithm terminates after a finite number of iterations.*

Proof. The termination depends on $\sum_{e\in E}(x_P(e) - x_Q(e)), z_P, z_Q$ and L_p. By (A″) and by definitions of x_P, x_Q, z_P, z_Q and L_p, we have the following:

- $\sum_{e\in E}(x_P(e) - x_Q(e))$ is bounded below by 0 and has an upper bound.

- $\mathbf{0} \leq x_P \leq z_P$, that is, z_P is bounded below by $\mathbf{0}$.
- If $z_Q(e) < +\infty$ for some $e \in E$ then $z_Q(e) = x_Q(e)$ by (3.2.6), that is, $z_Q(e)$ has an upper bound.
- $|L_p| \leq |E|$.

The termination follows from Lemmas 3.2.4 and 3.2.5. □

3.3 Open Problems

In this chapter, we extended the economic model of Fujishige and Tamura [10] by defining the payoff functions in more general way. We proposed an algorithm to find a pairwise strictly stable outcome in our model.

1. Suppose that $\alpha = (\alpha(i,j) \in \mathbf{R}_+ \mid (i,j) \in E)$ and $\beta = (\beta(i,j) \in \mathbf{R}_+ \mid (i,j) \in E)$ be two given positive vectors. Let us define payoff of a worker i on an outcome (x,s) by

$$f_i[+s_{(i)}, \alpha_i](x_{(i)}) = f_i(x_{(i)}) + \sum_{j \in Q} s(i,j)\alpha(i,j)x(i,j). \tag{3.3.1}$$

Similarly, define payoff of a firm j on an outcome (x,s) by

$$f_j[-s_{(j)}, \beta_j](x_{(j)}) = f_j(x_{(j)}) - \sum_{i \in P} s(i,j)\beta(i,j)x(i,j). \tag{3.3.2}$$

It is an open problem to find a pairwise strictly stable outcome in our model by defining the payoff of workers and firms by (3.3.1) and (3.3.2), respectively.

2. In our model, we assume that the feasible salary vector s is a real vector, that is, $s \in \mathbf{R}^E$. It is an open problem to find a pairwise strictly stable outcome in our model by assuming $s \in \mathbf{Z}^E$.

Bibliography

[1] M. S. Bazaraa and J. J. Jarvis, Linear Programming and Network Flows, John Wiley & Sons, New-York, (1990).

[2] J. A. Bondy and U. S. R. Murty, Graph Theory with Applications, Macmillan London, Canada, (1982).

[3] V. P. Crawford and E. M. Knoer, Job Matching with Heterogeneous Firms and Workers, Econometrica, **49** (1981), 437–450.

[4] G. B. Dantzig, Linear Programming and Extensions, Princeton University Press, Princeton, 1963.

[5] K. Eriksson and J. Karlander, Stable Matching in a Common Generalization of the Marriage and Assignment Models, Discrete Math. **217** (2000), 135 − 156.

[6] R. Farooq and A. Shioura, A Note on the Equivalence Between Substitutability and $M^\natural$-Convexity, Pacific J. Optim.,**1** (2005), 243 − 252.

[7] R. Farooq and A. Tamura, A New Characterization of $M^\natural$-Convex Set Functions by Substitutability, J. Oper. Res. Soc. Japan, **47** (2004), 18 − 24.

[8] S. Fujishige, Submodular Functions and Optimization, Annals of Discrete Mathematics, North-Holland, Amsterdam, **58** (2005).

[9] S. Fujishige and A. Tamura, A General Two-Sided Matching Market with Discrete Concave Utility Fucntions, Discrete Appl. Math., **154** (2006), 950–970.

[10] S. Fujishige and A. Tamura, A Two-Sided Discrete-Concave Market with Possibly Bounded Side Payments: An Approach by Discrete Convex Analysis, Math. Oper. Res., **32** (2007), 136–155.

[11] S. Fujishige and Z. Yang, A Note on Kelso and Crawford's Gross Substitutes Condition, Math. Oper. Res., **28** (2003), 463–469.

[12] D. Gale and L. S. Shapley, College Admissions and the Stability of Marriage, Amer. Math. Monthly, **69** (1962), 9 − 15.

[13] F. Gul and E. Stacchetti, Walrasian Equilibrium with Gross Substitutes, J. Econom. Theory, **87** (1999), 95–124.

[14] D. G. Luenberger, Linear and Nonlinear Programming, International Series in Operations Research and Management, 2nd Ed., 516 , (2003).

[15] K. Murota, Discrete Convex Analysis, Society for Industrial and Applied Mathematics, Philadelphia, (2003).

[16] K. Murota and A. Shioura, M-Convex Function on Generalized Polymatroid, Math. Oper. Res., **24** (1999), 95–105.

[17] A. E. Roth and M. Sotomayor, Two-Sided Matching, A Study in Game-Theoretic Modeling and Analysis, Cambridge University Press, (1990).

[18] L. S. Shapley and M. Shubik, The Assignment Game I: The Core, International Journal of Game Theory, **1** (1972), 111-130.

[19] M. Sotomayor, Existence of Stable Outcomes and the Lattice Property for a Unified Matching Market, Math. Social Sci., **39** (2000), 119–132.