Numerical Inversion of The Laplace Transform

Application to Fractional Differential Equations

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Contents

Abstract

Fractional Calculus is one of the branches of mathematical analysis which involves study of differential operator by taking the real number powers. Generalizing the differential equations in terms of fractional calculus, fractional differential equation evolves. Integral transforms are very well known tool for solving the differential equations, so as for fractional differential equations.

In this dissertation brief history, definitions and applications of fractional calculus is discussed. It also covers the Laplace integral definition and existence of integral and the fractional Laplace formulas. Numerical inversion of the Laplace transform using series solution include problems in which the solutions with integer orders and fractional orders are compared. Double Laplace transform is also discussed, with its basic properties and applications to fractional partial differential equations. Also, non-linearity of fractional differential equations is discussed and a humble effort is made to describe a method for finding the numerical solution of non-linear fractional ordinary and partial differential equations. The results were found using computer algebra system.

Chapter 1

Introduction

The typical derivative and integrals, to say the least about them, are a staple for technology professional and a fundamental mean of understanding while dealing with natural and non-natural systems. Fractional calculus is a subdivision of mathematics which grows away from typical definitions of these integrals and derivatives. It is the same outgrowth of definitions of these operators as of exponents with integer orders. Physical consideration of exponents, according to primary school teacher, the repeated multiplication of a numerical value by itself, exponent provides a short notation of it. To grasp this idea, no difficulty can be faced as it is very easy and straight forward. This clear physical definition turns into a huge confusion when exponents are of non-integer value. To explain the lines, one can easily verify $y^5 = y.y.y.y.y$, but what if one has to evaluate $y^{4.5}$, or moreover the transcendental exponent y^{π}, one can not conceive the idea how to multiply the same quantity 4.5 or π times by itself and yet a definite value of these expressions exist for any value of y, which can be verified by infinite series expansion or being more practical, with the help of a calculator.

In the similar approach we consider the derivatives and integrals as they are indeed the concept of more intricacy by nature but it is an easy job to represent their physical meaning. Once attaining the perfection over it, the idea of completing numerous of these operations, i.e, integration and differentiation follows naturally. Having a satisfaction of few restrictions like continuity of function, finding n derivatives are as methodical as multiplication. But being more curious about it, one may definitely ask about the derivative of non-integer order. Physical meaning of this become convoluted at first sight, but fractional calculus flows somewhat naturally from our typical definitions. It will become apparent that differentiation of fractional order is a matter of fact in many modern problems.

1.1 History and Mathematical Background

The birthday of fractional calculus cited by most authors is $30th$ September 1695. As in a letter from L'Hopital to Leibniz asking about his notation of nth derivative of a function in his publication,

$$\frac{D^n}{Dx^n}\left(f(x)\right).$$

L'Hopital inquired about the order of derivative if it would be a fraction, i.e, $n = \frac{1}{2}$, Leibniz answer to this was "An apparent paradox, from which one day useful consequences will be drawn...". This declaration gave birth to fractional calculus [1].

Following this first inquisition from L'Hopital, the study of fractional calculus was reserved primarily to best minds in mathematics. Mentioning a few famous names in the field of mathematics Fourier, Laplace and Euler who worked on fractional calculus and its mathematical consequences. Many mathematicians presented their own definitions using their personal line of attack that fit to the concept of fractional derivative or integral. Reimann-Liouville and Grunwald-Letnikov proposed their own definitions and these are famous in fractional calculus. Although the shear number of authentic definitions is as numerous as the mathematicians worked in this field but these all are some kind of variations of definitions by Reimann and Grunwald.

Most of mathematical study applied to fractional calculus was earlier to the turn of $20th$ century. However, what went before in last 100 years, the influence of fractional calculus in engineering and scientific applications have been found. To congregate the requirements of physical reality some changes in some cases are necessary to be made by mathematics.

Caputo transformed the definition of Reimann-Liouville fractional derivative for a purpose to use integer order initial conditions for solving fractional order differential equations [2]. In 1996, Kolwankar reformed again the Reimann-Liouville definition to differentiate no-where differentiable fractal functions. In his publication he discussed the fractional differentiability of "Weiestrass function" which has the property that it is continuous everywhere but differentiable nowhere.

$$W_\alpha(t) = \sum_{k=0}^{\infty} \alpha^{(s-2)k} \sin \alpha^k t, \qquad \alpha > 1.$$

Leibniz response, after the study of 300 years proved atleast half true. Study of the $20th$ century yet brought fractional calculus far from impossible by the help of physical manifestations and numerous applications. The physical meaning is arguably impossible to grasp because the definitions are no more rigorous then integer order counterparts.

To clear the understanding in the definitions of fractional calculus, first we discuss some necessary yet simple mathematical definitions that will go through to the sight in the study of these concepts.

1.1.1 Gamma Function

The bond of gamma function and fractional calculus is ionic by definition. Generalization of factorial is a very simple interpretation of gamma function for all real numbers. Gamma function can be defined by,

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt, \qquad \forall x \in \mathbb{R}^+.$$

To express the beauty of this function one cannot only rely on its properties. First,

$$\Gamma(x+1) = x\Gamma(x), \qquad x \in R$$

and

$$\Gamma(x) = (x-1)!, \qquad x \in N.$$

Observation of the above two equations, gamma function is unique in the value of any quantity and these can be shown by simple integration by parts. These relations for integer values of x are simple factorials. While the gamma function is undefined for negative integers but yet also defined for non-integers values. The figure demonstrates the behavior of gamma function at and around zero [1].

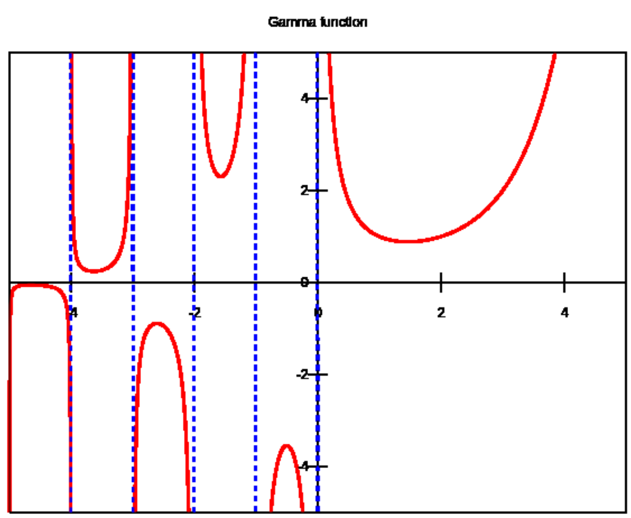

Figure 1.1: Plot of Gamma Function

1.1.2 Mittage-Leffler Function

Mittage-Leffler function finds extensive use on the planet of fractional calculus and owns alot of importance. Just as the exponential function which appears in the solution of integer order differential equations, however, Mittage-Leffler function plays parallel role in the solution of fractional differential equations [1].

The definition of Mittage-Leffler function is,

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \qquad \alpha > 0.$$

If $\alpha = 1$ the Mittage-Leffler function is simple exponential function. Using two arguments, α and β, Mittage-Leffler function has the form,

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \qquad \alpha, \beta > 0.$$

This is more general form but not always a necessity for fractional differential equations. Since Mittage-Lefller function is an entire function, the series of this function is absolutely convergent on the disk $|z| < R$, whenever z is on the complex plane.

1.2 Few Definitions of Fractional Integral and Derivative

Following the descending order first comes the definition of L. Euler (1730),

$$\frac{d^n}{dt^n} t^m = m(m-1)\dots(m-n+1)t^{m-n}, \qquad m > n$$

by using the following property of Gamma function,

$$\Gamma(m+1) = m(m-1)\dots(m-n+1)\Gamma(m-n+1),$$

to obtain

$$\frac{d^n}{dt^n} t^m = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} t^{m-n}.$$

Then coming down to J.B. Fourier $(1820-1822)$, by means of integral representation,

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(z)dz \int_{-\infty}^{\infty} \cos(\omega t - \omega z)d\omega,$$

he wrote,

$$\frac{d^n}{dt^n} f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(z)dz \int_{-\infty}^{\infty} \cos(\omega t - \omega z + \frac{n\pi}{2})d\omega.$$

N.H. Abel $(1823-1826)$ considered,

$$\phi(t) = \int_0^t \frac{p'(\mu)}{(t-\mu)^{\alpha}} d\mu,$$

for arbitrary α and derivative of p, he proposed,

$$p(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d^{-\alpha}}{dt^{-\alpha}} \phi(t).$$

J. Lioville $(1832-1855)$, in his first generalization, he used the exponential function and defined,

$$\frac{d^{\mu}}{dt^{\mu}} f(t) = \sum_{i=0}^{\infty} c_i a_i^{\mu} e^{a_i t},$$

and after that he defined fractional integral and derivative,

$$\int^{\nu} \psi(t) dt^{\nu} = \frac{1}{\Gamma(\nu)} \int_{-\infty}^{t} (t-y)^{\nu-1} \psi(y) dy,$$

$$\frac{d^{\nu}}{dt^{\nu}} f(t) = \frac{(-1)^{\nu}}{h^{\nu}} (f(t) \frac{\nu}{1} f(t+h) + \frac{\nu(\nu-1)}{1.2} f(t+2h) - \ldots).$$

G.F.B Riemann $(1847-1876)$ defined,

$$D^{-\mu} f(t) = \frac{1}{\Gamma(\mu)} \int_{a}^{t} (t-y)^{\mu-1} f(y) dy + \phi(y).$$

Next, N.Ya.Sonin (1869), A.V. Letnikov (1872), H. Laurent (1884), N. Nekrasove (1888) and K. Nishimoto (1987) , they all worked on Cauchy integral and proposed the following definition after few substitutions,

$$D^{\mu} f(z) = \frac{\Gamma(\mu+1)}{2\pi i} \int_{a}^{y^{+}} \frac{f(x)}{(x-z)^{\mu+1}}.$$

Reimann and Liouville defined the fractional derivative which became the basis of further research, they joined the two definitions and came up with their own,

$$D_t^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} (\frac{d}{dt})^m \int_0^t \frac{f(y)}{(t-y)^{\alpha-n+1}} dy.$$

Another joined definition was given by Grunwald and Letnikove,

$${}_{\beta}D_t^{\alpha} f(t) = \lim_{h \to 0} h^{-\alpha} \sum_{i=0}^{[\frac{t-c}{h}]} (-1)^i \binom{\alpha}{i} f(t-ih).$$

K.S. Miller and B. Ross (1993) used the differential operators of the form,

$$D^{\alpha_i} f(t) = D^{\alpha_1} D^{\alpha_2} \ldots D^{\alpha_n} f(t), \qquad i = 1, 2, 3, \ldots,$$

where D^{α_i} are Reimann-liouville or Caputo definitions of differential operators [3].

1.3 Fractional Integral and Derivative: Caputo's Approach

To describe the Caputo definition of fractional differential and integral, first one has to know about Reimann-Liouville definition which are refined by Caputo to come up with his own.

If α is positive real, then

$$J_a^{\alpha}[f(x)] = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt,$$

for $a \leq x \leq b$ is Reimann-Liouville fractional integral and considering α to be real positive and n is ceiling of α, then

$$D_x^{\alpha} f(x) = D^m J_a^{n-\alpha} f(x),$$

is the Reimann-Liouville fractional differential operator of α order.

Later the definitions faced a few disadvantages when discussion reached the real world phenomenon with fractional differential equations. Caputo's fractional derivative is defined as,

$${}_*D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^n(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau.$$

It is the case when $n-1 < \alpha \leq n$ and if α meets n in value then the fractional derivative is just the ordinary derivative for integer value [5].

1.3.1 Elementary Properties for Fractional Derivative

The idea of derivative is only associated to integer in general. For a given function $f(t)$ one can differentiate it one, two and three times and so on. Considering a few elementary properties of ordinary derivative with integer order also we would like fractional derivative to share them all [6].

Firstly, the linearity,

$${}_*D_t^n[cf(t)] = c{}_*D_t^n[f(t)],$$

where c is a constant. The second one is fulfilling the requirements of distributive law,

$${}_*D_t^n[f(t) \pm g(t)] = {}_*D_t^n[f(t)] \pm {}_*D_t^n[g(t)].$$

The last property is, the operator follows Leibniz rule for the product of functions,

$${}_*D_t^n[f(t)g(t)] = \sum_{k=0}^{n} \binom{n}{k} {}_*D_t^{n-k}[f(t)]{}_*D_t^k[g(t)] = \sum_{k=0}^{n} \binom{n}{k} {}_*D_t^{n-k}[g(t)]{}_*D_t^k[f(t)],$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, is binomial coefficient. Now in fractional calculus, fortunately the Leibniz rule have a replacement of factorial with gamma function.

$${}_*D_t^{\alpha}[f(t)g(t)] = \sum_{k=0}^{\infty} \binom{\alpha}{k} {}_*D_t^{\alpha-k}[f(t)]{}_*D_t^k[g(t)] = \sum_{k=0}^{\infty} \binom{\alpha}{k} {}_*D_t^{\alpha-k}[g(t)]{}_*D_t^k[f(t)],$$

where

$$\binom{\alpha}{k} = \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha+1-k)},$$

where α is real positive. Then,

$$\begin{aligned} {}_*D_t^{\alpha}[f(t)g(t)] &= \sum_{k=0}^{\infty}\binom{\alpha}{k}{}_*D_t^{\alpha-k}[t^0]{}_*D_t^{k}[f(t)\pm g(t)], \\ &= \sum_{k=0}^{\infty}\binom{\alpha}{k}{}_*D_t^{\alpha-k}[t^0]{}_*D_t^{k}[g(t)] \pm \sum_{k=0}^{\infty}\binom{\alpha}{k}{}_*D_t^{\alpha-k}[t^0]{}_*D_t^{k}[g(t)], \\ &= {}_*D_t^{\alpha}[f(t)] \pm {}_*D_t^{\alpha}[g(t)], \end{aligned}$$

and

$${}_*D_t^{\alpha}[f(t)C] = \sum_{k=0}^{\infty}\binom{\alpha}{k}{}_*D_t^{\alpha-k}[f(t)]{}_*D_t^{k}[C] = {}_*D_t^{\alpha}[f(t)]C.$$

Similarly,

$${}_*D_t^{\alpha}[f(at)] = a^{\alpha}[f(x)]; \quad x = at,$$

and for the monomial function $f(t) = t^{-m}$, where m is positive definite integer.

$${}_*D_t^{\alpha}[f(t)] = (-1)^{\alpha}\frac{\Gamma(m+\alpha)}{\Gamma(m)}t^{-(m+\alpha)}.$$

The above definition convert the real function to complex function and vice versa for the reason that there is complex factor $(-1)^{\alpha} = e^{i\alpha\pi}$.

1.4 Physical and Geometrical Interpretation of Fractional Differentiation and Integration

An obvious lack of physical and geometrical interpretations of fractional integration and differentiation which is realized in many international conferences in USA and Tokyo. It is difficult to say something acceptable about geometrical interpretation if one can not see the picture some where. R. Rutman and F. Ben Adda suggested different approaches to fractional integration and differentiation but still they are not analogous with simple interpretations of their integer order which complements.

Igor Podlubny [7] considered the axes τ, g and f. He plotted the function $g_t(\tau)$ in a plane (τ, g) for $0 \le \tau \le t$. He built a fence of varying height along the obtained curve, so that the edge of fence is a three dimensional line $(\tau, g_t(\tau), f(\tau))$. His geometrical interpretation of the fractional integration is "**Shadows on the Wall**". His physical interpretation of fractional derivative (Reimann Liouville and Caputo) is "**Shadows of the Past**". In regard to this he described the two types of time, the

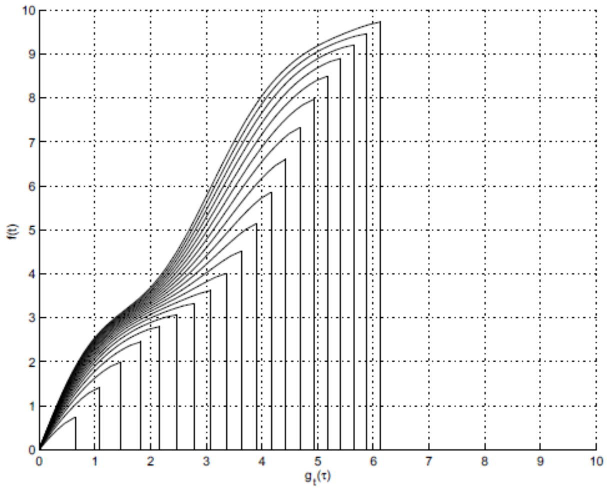

Figure 1.2: Snapshots of the changing "shadow" of changing "fence" for ${}_0I_t^\alpha f(t)$, $\alpha = 0.75$, $f(t) = t + 0.5\sin(t)$, $0 \leq t \leq 10$, with the time interval $\Delta t = 0.5$ between the snapshops.

cosmic time and the individual time. Cosmic time which is also known as time since the big bang, is the time coordinate frequently used in the Big Bang models of physical cosmology. Individual time is the time experienced by the object itself. If we follow the change of the "shadow" on the wall (g, f), which is changing simultaneously with the "fence" (see Fig. 1.2), then we have a dynamical geometric interpretation of the fractional integral as a function of the variable t.

1.5 Fractional Differential Equations

As, an equation involving the derivatives of a function is called a differential equation. In the same manner, an equation involving a fractional derivative is called a fractional differential equations [8]. For instance,

$$[D^{n\alpha} + b_{n-1}D^{(n-1)\alpha} + \ldots + b_0 D^0]f(t) = 0, \qquad t \geq 0,$$

is fractional differential equation with constant coefficients where α is real positive.

An example of fractional integral equation is Abel's integral equation,

$$\int_0^t (t-\tau)^{\alpha-1} f(t)dt = g(t), \qquad 0 < \alpha < 1,$$

where $g(t)$ is the given function. This equation can be expressed in fractional integral as,

$$\Gamma(\alpha) J_t^\alpha f(t) = g(t).$$

Fractional diffusion equation is an example of fractional partial differential equation,

$$_{*}D_t^{\alpha}u(x,t) = k\frac{\partial^2 u(x,t)}{\partial x^2}, \qquad x \in \mathbb{R} \;\; and \;\; t > 0,$$

with initial and boundary conditions,

$$u(x,t) \to 0 \qquad \text{as} \quad |x| \to \infty,$$

$$[u_t(x,t)]_{t=0} = f(x), \qquad \text{for} \;\; x \in \mathbb{R}$$

and k is the constant.

1.6 Applications of Fractional Calculus

Fractional calculus was first applied by Abel for the purpose of solving integral equations which occur in formulation of *tautochronous problem*, which deals with determination of shapes of frictionless plane curve through the origin in a vertical plane alongside a particle with some mass can drop in a time that does not depend on starting position.

In the last decades of $19th$ century , Heaviside was honored with success in development of his operational calculus, in 1892, he brought fractional derivatives in study of electric transmission lines. Heaviside used a letter p for differential operator and came up with the solution of diffusion equation. Fractional calculus plays a part in description of the viscous interactions between fluids and solid structures, reflections and transmission scattering operators which are derived from the slab of human cancellus bone in elastic frame using Bolt's theory. R.L Margin modeled the cardiac tissue electrode interface, using fractional calculus.

In engineering, fractional calculus is applied to sound wave propagation in Rigid Porous material, literal and longitudinal control of autonomous vehicle, in theory of visco elasticity, fractional differentiation for edge detection, wave propagation in viscoelastic horns using the fractional calculus rheology model. Last but not the least fractional calculus has a great influence in fluid mechanics as well.

These are few glimpse of where the fractional calculus is applied, yet its application ground is very vast [3].

Above are few mathematical models in which fractional calculus plays a part. To grasp the idea, we explain diffusion equation with integer and fractional order. Considering the general representation,

$$\frac{\partial^{\gamma}}{\partial t^{\gamma}}y(x,t) = D\frac{\partial^2}{\partial x^2}y(x,t),$$

where D is constant. If $\gamma < 1$ then it is sub diffusion, $\gamma = 1$ it represents normal diffusion and $\gamma > 1$ is super diffusion.

Choice of γ is the bridging regime between diffusion equation ($\gamma = 1$) and wave equation($\gamma = 2$). Extending the space derivative to fractional derivatives,

$$\frac{\partial}{\partial t} y(x,t) = D \frac{\partial^\alpha}{\partial x^\alpha} y(x,t),$$

is the same bridging regime between half wave equation($\alpha \to 1$) and diffusion equation($\alpha \to 2$) [4].

1.7 Laplace Transform; Definition and Existence

The Laplace transform is a famous integral transform used in mathematics with a lot of applications in physics as well as in engineering. The Laplace transform is originated from the Fourier integral. The Laplace transform has a vital use in solution of differential and integral equations as Fourier transform does. In an analysis, the Laplace transform is also interpreted as a transformation from the time-domain, that is inputs and outputs are time functions, to the frequency-domain where the same inputs and outputs are functions of complex angular frequency. Setting a simple mathematical description of an input or output of a system, the Laplace transform gives substitute another mathematical description that frequently simplifies the process of examine the behavior of the system.

Definition: Considering the Fourier integral,

$$g(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikt} dk \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x) dx \right],$$

replacing $f(t)$ by $H(t)e^{-at}f(t)$ for $t > 0$, where $H(t)$ is Heaviside function,

$$H(t)e^{-at}f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dk \int_{-\infty}^{\infty} e^{-t(a+ik)} f(t) dt,$$

or

$$H(t)f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{(a+ik)x} dk \int_{-\infty}^{\infty} e^{-t(a+i\ k)} f(t) dt.$$

Substituting $s = a + ik$ which implies $ds = idk$, we have for $t > 0$ and c to a,

$$H(t)f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{xs} ds \int_{0}^{\infty} e^{-st} f(t) dt. \tag{1.7.1}$$

Thus, we have the definition of the Laplace transform for all values of $t > 0$, where $\bar{f}(s)$ and $\mathcal{L}\{f(t)\}$ denotes the Laplace transform of the function $f(t)$,

$$\bar{f}(s) = \mathcal{L}\{f(t)\} = \int_{0}^{\infty} e^{-st} f(t) dt,$$

where s is real positive or complex number with positive real parts so that the integral is convergent. For convergence of integral function, it must be of exponential order.
Also from eq(1.7.1) we have,

$$f(t) = \pounds^{-1}\{\bar{f}(s)\} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{xs}\bar{f}(s)ds,$$

for all x. This complex integral defines the inverse Laplace transform denoted by $\pounds^{-1}\{\bar{f}(s)\} = f(t)$.
Sufficient Condition for Existence: Let $g(t)$ be piecewise continuous over the interval $0 \leq t < \infty$ and of exponential order for $t \geq T$, then $\pounds\{g(t)\}$ exists for $s > c$.
Proof:

$$\begin{aligned}
\pounds\{g(t)\} &= \int_0^\infty e^{-st}g(t)dt \\
&= \int_0^\tau e^{-st}g(t)dt + \int_\tau^\infty e^{-st}g(t)dt \\
&= I_0 + I_1.
\end{aligned}$$

The integral I_0 exists because it can be written as a sum of integrals over intervals for which $e^{-st}g(t)$ is continuous.

$$\begin{aligned}
|I_1| &\leq \int_\tau^\infty |e^{-st}g(t)|dt \leq M\int_\tau^\infty e^{-st}e^{ct}dt \\
&= M\int_\tau^\infty e^{-(s-c)t}dt = -M\frac{e^{-(s-c)t}}{s-c}|_\tau^\infty \\
&= M\frac{e^{-(s-c)\tau}}{s-c}, \qquad \text{for} \quad s > c.
\end{aligned}$$

Since $\int_\tau^\infty Me^{-(s-c)t}dt$ converges, then $\int_\tau^\infty |e^{-st}g(t)|$ converges by comparison test for improper integrals, which in return implies that I_1 exist for $s > c$. The existence of I_0 and I_1 implies the existence of Laplace integral.
Initial and Final Value Theorem: If

$$\pounds\{g(t)\} = \int_0^\infty e^{-st}g(t)dt,$$

is the Laplace transform of $g(t)$ then initial value theorem says,

$$\lim_{t\to 0} g(t) = \lim_{s\to\infty} sG(s).$$

Final value theorem states that,

$$\lim_{t\to\infty} g(t) = \lim_{s\to 0} sG(s),$$

where $G(s)$ is Laplace transform of $g(t)$.

1.8 Laplace Transform of Fractional Derivatives and Integrals

Considering the Reimann-Liouville fractional derivative definition,

$$D_t^{\alpha}[f(t)] = \frac{1}{\Gamma(\alpha)} \frac{d^m}{dt^m} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau,$$

is clearly a convolution. Applying Laplace transform,

$$\begin{aligned} \pounds\{D_t^{\alpha} f(t)\} &= \pounds\{f(t) * g(t)\} \\ &= \pounds\{f(t)\}\pounds\{g(t)\} \\ &= s^{-\alpha}\bar{f}(s), \end{aligned} \tag{1.8.1}$$

where $g(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ and $\bar{g}(s) = s^{-\alpha}$. The result is also true for $\alpha = 0$,

$$\pounds\left\{\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right\}_{\alpha=0} = [s^{-\alpha}]_{\alpha=0} = 1,$$

from the same result ,

$$\pounds\{D_t^{\alpha} t^{\beta}\} = \frac{\Gamma(\beta+1)}{s^{\beta-\alpha+1}}.$$

Equivalently,

$$D_t^{\alpha} t^{\beta} = \pounds^{-1}\left\{\frac{\Gamma(\beta+1)}{s^{\beta-\alpha+1}}\right\} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}.$$

A very important property of Laplace transform which is commonly used in this dissertation is [8],

$$\pounds\{_*D_t^{\alpha} f(t)\} = s^{\alpha}\bar{f}(s) - \sum_{k=0}^{m-1} s^{\alpha-k-1} f^k(0),$$

keeping in mind the convergence of the Laplace integral is conditional i.e function should be of exponential order as $t \to \infty$.

In next chapters we will numerically find the inverse Laplace transform of fractional differential equations and also try to find solution of non-linear fractional ordinary and partial differential equations by the help of the Laplace transform.

Chapter 2

Taylor Series Method for Inversion of the Laplace transform

The inverse Laplace transform is proved to be the most tough step in applying the Laplace transform technique. There are an inadequate number of the Laplace transforms and inverse Laplace transforms available in literature to provide inversion automatically. The lack of inverse transforms cut down the effectiveness of the Laplace transform method. However, numerical algorithms subsist for providing inverse Laplace transforms numerically and they are strong tools that can be used widen to the applicability of the Laplace transform method.

Method of finding the inverse Laplace transform using polynomial series is discussed in this chapter. It is obvious that any polynomial series basis vector can be converted into Taylor polynomial by the use of suitable substitutions.

The particular series which is discussed for functions of a certain family, is a somewhat generalized form of the Taylor expansion formula. It is important to have considerable sets of conditions on $\bar{y}(s)$ under which this series expansion is valid, for here too it frequently happens that a formal application of this series fails to give the inverse of Laplace transformation.

2.1 Method of Series Expansion

Generally, it is observed,

$$\pounds\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}},$$

inverse of which is,

$$\pounds^{-1}\left\{\frac{\Gamma(n+1)}{s^{n+1}}\right\} = t^n,$$

i.e, we approximate functions as linear combination of predefined expansion functions. Further is illustrated with following examples.

Example 1.1.1: If,

$$\bar{f}(s) = \frac{e^{\frac{-a}{s}}}{s^{\frac{3}{2}}},$$

then it is known that

$$e^{-t} = \sum \frac{(-t)^n}{n!}.$$

So,

$$\bar{f}(s) = \frac{1}{s^{3/2}}\left(1 - \frac{a}{s} + \frac{a^2}{2!s^2} - \frac{a^3}{3!s^3} - \dots\right),$$

$$\bar{f}(s) = \frac{1}{s^{3/2}} - \frac{a}{s^{5/2}} + \frac{a^2}{2!s^{7/2}} - \frac{a^3}{3!s^{9/2}} - \dots.$$

We know that,

$$\pounds^{-1}\left\{\frac{1}{s^{3/2}}\right\} = \frac{2t^{1/2}}{\sqrt{\pi}},$$

$$\pounds^{-1}\left\{\frac{1}{s^{5/2}}\right\} = \frac{4}{3}\frac{t^{3/2}}{\sqrt{\pi}}.$$

Generalizing to nth term,

$$\pounds^{-1}\{s^{-(n+\frac{1}{2})}\} = \frac{2^n}{(2n-1)\cdots 3.1}\frac{t^{n-\frac{1}{2}}}{\sqrt{\pi}}, \qquad n = 1, 2, \cdots,$$

$$f(t) = \frac{2t^{1/2}}{\sqrt{\pi}} - \frac{4a}{3.1}\frac{t^{3/2}}{\sqrt{\pi}} + \frac{8a^2}{2!5.3.1}\frac{t^{5/2}}{\sqrt{\pi}} - \dots + \frac{(-1)^n 2^{n+1} a^n}{n!(2n+1)\cdots 3.1}\frac{t^{n+\frac{1}{2}}}{\sqrt{\pi}}, \qquad n = 0, 1, \cdots,$$

series is convergent for all t>0, where the exact solution of problem is $(a\pi)^{-(\frac{1}{2})}\sin 2\sqrt{at}$.

Example 1.1.2: If,

$$\bar{f}(s) = \frac{1}{1+\sqrt{s}},$$

we can write,

$$\bar{f}(s) = \frac{1}{\sqrt{s}(1+\frac{1}{\sqrt{s}})}.$$

Expanding,

$$\bar{f}(s) = \frac{1}{\sqrt{s}}\left(1 - \frac{1}{\sqrt{s}} + \frac{1}{\sqrt{s}^2} + \frac{1}{\sqrt{s}^3} - \dots\right),$$

$$\bar{f}(s) = \frac{1}{\sqrt{s}} - \frac{1}{\sqrt{s}^2} + \frac{1}{\sqrt{s}^3} + \frac{1}{\sqrt{s}^4} - \dots,$$

$$\bar{f}(s) = \left(\frac{1}{\sqrt{s}} + \frac{1}{\sqrt{s}^3} + \frac{1}{\sqrt{s}^5} \dots\right) - \left(\frac{1}{s} + \frac{1}{s^2} + \frac{1}{s^3} + \dots\right).$$

Taking term by term inverse of the series,

$$\pounds^{-1}\left\{\frac{1}{s} + \frac{1}{s^2} + \frac{1}{s^3} + \dots\right\} = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots,$$

which is the form of exponential series, and

$$\pounds^{-1}\left\{\frac{1}{\sqrt{s}} + \frac{1}{\sqrt{s}^3} + \frac{1}{\sqrt{s}^5} \dots\right\} = \frac{1}{\sqrt{t\pi}}\left(1 + 2t + \frac{2t^2}{3} + \dots + \frac{(4t)^n}{(2n)!} + \dots\right).$$

Above series clearly converges for all $t > 0$, but as $t \to 0$, the series goes infinite which restricts the range of t.

Before moving on to next section first understanding few concepts are requirement of topic.

Hilbert Matrix: Hilbert matrix was introduced by Hilbert in 1894, on the query in Approximation theory, "Assume that I = [a, b] is a real interval. Is it then possible to find a non-zero polynomial P with integral coefficients, such that the integral

$$\int_a^b P(x)^2 dx$$

is smaller than any given bound e > 0, taken arbitrarily small?" Hilbert derived the formula for finding the determinant of Hilbert matrices and investigated about their asymptotic. So, Hilbert matrices are square matrices with all entries being the unit fractions

$$H = \frac{1}{i+j-1}, \qquad i, j = 1, 2, 3, n.$$

These matrices can be regarded as derived from integral

$$H_{ij} = \int_0^1 x^{i+j} dx.$$

Hilbert matrices are symmetric and positive definite as well as totally positive i.e determinant of every sub matrix is positive [9].

2.2 Taylor Series Expansion

Chung and Sun [10] reported the technique of Taylor series expansion for inverting the Laplace transform. Method includes the inversion of Hilbert matrix. A function f(t), analytic in the neighborhood of $t_0 = 0$ might be expanded in power series (using Maclaurin formula)

$$f(t) = \sum_{n=0}^{\infty} a_n y_n(t),$$

where $y_n(t) = t^n$ and $a_n = \frac{1}{n!} f^n(0)$. Assuming f(t) to be analytic, restricting the series upto finite terms say m, we have,

$$f(t) \backsimeq \sum_{0}^{m-1} a_n y_n(t),$$

$$f(t) \backsimeq a^T y(t),$$

where a^T is a vector of the form,

$$a^T = [a_0, a_1, a_2, \ldots, a_{m-1}],$$

and

$$y^T(t) = [y_0(t), y_1(t), \ldots . y_{m-1}(t)],$$

$y(t)$ represents the basis function vector. For any z, $y(z)$ can be written as

$$y_0(z) = 1,$$

$$y_1(z) = z,$$

$$y_2(z) = z^2,$$

$$\vdots$$

$$y_i(z) = z^i.$$

Now integrating $y_i(z)y_j(z)$ over the interval $[0, 1]$

$$\int_0^1 y_i(z)y_j(z)dz = \frac{1}{i+j+1}. \tag{2.2.1}$$

For the purpose of finding inverse Laplace transform domain has to be expanded to $[0, \infty)$, for this purpose a substitution is introduced,

$$z = e^{-\frac{t}{c}}.$$

So that the basis function becomes $y_i(t) = e^{-it/c}$, inserting in (2.2.1),

$$\int_0^\infty y_i(t)y_j(t)e^{-\frac{t}{c}}dt = \frac{c}{i+j+1}.$$

For $c = 1$

$$\int_0^\infty y_i(t)y_j(t)e^{-\frac{t}{c}}dt = \frac{1}{i+j+1} \triangleq H_{ij}, \tag{2.2.2}$$

Finding the inverse Laplace transform, of any function $f(t)$ can be represented in terms of series expansion necessary to be finite. Performing the steps of this method, $f(t)$ is represented in Taylor polynomial,

$$f(t) \backsimeq \sum_{i=0}^{m-1} a_i y_i(t), \tag{2.2.3}$$

where, a_i are the coefficients, whose values are to be determined. Second step is to multiply (2.2.2) $y_j(t)e^{-t/c}$ and integrating over the interval $[0, \infty)$,

$$\int_0^\infty f(t)y_j(t)e^{-t/c}dt = \sum_{i=0}^{m-1} a_i \int_0^\infty y_i(t)y_j(t)e^{-t/c}dt, \tag{2.2.4}$$

where $j = 0, 1, 2, \ldots, m-1$ inserting (2.2.2), (2.2.4) transforms into,

$$\sum_{i=0}^{m-1} H_{ij}a_i = \int_0^\infty f(t)e^{-t/c}y_j(t)dt. \tag{2.2.5}$$

The Laplace transform is defined as,

$$\int_0^\infty f(t)e^{-st}dt = \pounds\{f(t)\} = \bar{f}(s). \tag{2.2.6}$$

Now for different variation of j,

$$\int_0^\infty f(t)e^{-t/c}y_0(t)dt = \bar{f}(1/c),$$

$$\int_0^\infty f(t)e^{-t/c}y_1(t)dt = \bar{f}(2/c),$$

$$\int_0^\infty f(t)e^{-t/c}y_2(t)dt = \bar{f}(3/c),$$

$$\vdots$$

$$\int_0^\infty f(t)e^{-t/c}y_j(t)dt = \bar{f}\left((j+1)/c\right). \tag{2.2.7}$$

So, (2.2.5) and (2.2.7) yields a consistent system of m linear equations which helps to determine the values of coefficients a_i. Functional estimation for inverse Laplace can be obtained by re-inserting the coefficients in (2.2.3).

Example 2.2.1: If $f(t) = t^{-1}(1 - \cos t)$ and

$$\bar{f}(s) = \frac{1}{2}\log(1 + \frac{1}{s^2}),$$

then by Taylor series method

$$\sum_{i=0}^{m-1} H_{ij}a_i = \bar{f}\left(\frac{j+1}{c}\right)$$

considering $c = 1$ and $m = 5$

$$a_0 + \frac{1}{2}a_1 + \frac{1}{3}a_2 + \frac{1}{4}a_3 + \frac{1}{5}a_4 = \frac{1}{2}\log(2),$$

$$\frac{1}{2}a_0 + \frac{1}{3}a_1 + \frac{1}{4}a_2 + \frac{1}{5}a_3 + \frac{1}{6}a_4 = \frac{1}{2}\log\left(\frac{3}{2}\right),$$
$$\frac{1}{3}a_0 + \frac{1}{4}a_1 + \frac{1}{5}a_2 + \frac{1}{6}a_3 + \frac{1}{7}a_4 = \frac{1}{2}\log\left(\frac{4}{3}\right),$$
$$\frac{1}{4}a_0 + \frac{1}{5}a_1 + \frac{1}{6}a_2 + \frac{1}{7}a_3 + \frac{1}{8}a_4 = \frac{1}{2}\log\left(\frac{5}{4}\right),$$
$$\frac{1}{5}a_0 + \frac{1}{6}a_1 + \frac{1}{7}a_2 + \frac{1}{8}a_3 + \frac{1}{9}a_4 = \frac{1}{2}\log(3).$$

Solving the above system of linear equations, we get

$a_0 = 0.4244$, $a_1 = 3.6200$, $a_2 = -15.8660$, $a_3 = 20.9806$, $a_4 = -9.2215$.

So the inverse Laplace,

$$f(t) \simeq 0.4244 + 3.6200e^{-t} - 15.8660e^{-2t} + 20.9806e^{-3t} - 9.2215e^{-4t}.$$

Varying the values of m and comparing for estimation of error in Table(2.1).

		$m = 5$		$m = 10$		$m = 50$	
t	*exact*	Taylor	error	Taylor	error	Taylor	error
0.1	0.0500	0.0714	0.0214	0.0545	0.0045	0.0483	0.0017
0.5	0.2448	0.2167	0.0281	0.2503	0.0055	0.2467	0.0019
0.9	0.4204	0.4316	0.0112	0.4186	0.0018	0.4235	0.0031
1.3	0.5635	0.6063	0.0428	0.5677	0.0042	0.5618	0.0017
1.7	0.6640	0.6738	0.0098	0.6527	0.0113	0.6665	0.0025
2.1	0.7166	0.6662	0.0504	0.7222	0.0056	0.7102	0.0064
2.5	0.7205	0.6258	0.0947	0.7391	0.0019	0.7250	0.0045
10	0.1839	0.4245	0.2406	0.0478	0.1361	-0.0081	0.192
50	0.1284	0.4244	0.296	0.0467	0.0817	-0.0094	0.1375

Table 2.1: Comparison of numerical solution with exact solution and estimation of absolute error

Example 2.2.2: If,

$$\bar{f}(s) = \left(\frac{1}{\sqrt{s}}\right)e^{-\sqrt{5s}},$$

which has approximated solution for $m = 3$ and $c = 1$.

$$f(t) \simeq 0.5385 - 1.7025e^{-t} + 1.1905e^{-2t},$$

whereas exact solution is,

$$f(t) = \frac{1}{\sqrt{t\pi}} \sin(\frac{5}{2t}).$$

See Table(2.2),

t	*exact*	$m=5$		$m=10$		$m=50$	
		Taylor	error	Taylor	error	Taylor	error
0.1	-0.2361	0.0958	1.1941	-0.1479	0.0882	-0.1871	0.049
0.5	-0.7651	-0.2329	0.5322	-0.4773	0.2878	-0.9958	0.2307
0.9	0.2116	-0.0007	0.2123	0.2058	0.0058	0.1074	0.1042
1.3	0.4644	0.3801	0.0843	0.4735	0.0091	0.3593	0.1051
1.7	0.4305	0.5457	0.1152	0.3824	0.0481	0.5202	0.0897
2.1	0.3615	0.5191	0.1576	0.4027	0.0412	0.4253	0.0638
2.5	0.3003	0.4020	0.1017	0.3408	0.0405	0.1090	0.1913
10	0.0441	-0.1860	0.2301	0.2243	0.1802	-0.4794	0.5235
50	0.0040	-0.1865	0.1905	0.2248	0.2208	-0.4834	0.04874

Table 2.2: Comparison of numerical solution with exact solution and estimation of absolute error

2.3 Application to Fractional Differential Equations

Solving an ordinary fractional differential equation by the help of Laplace transform may not yield the exact solution due to the difficulty in finding the inverse of Laplace transform. In the section fractional differential equation is converted into algebraic equation using Laplace transform and for the purpose to find numerical solution, Taylor series method is applied to the following examples.

Example 2.3.1: Consider the inhomogeneous fractional differential equation

$$_{*}D_t^{\alpha} f(t) = te^{\frac{t}{2}},$$

where $1 < \alpha \leq 2$ and

$$f(0) = f^{'}(0) = 0.$$

Applying the Laplace transform we get,

$$\pounds\{_{*}D_t^{\alpha} f(t)\} = \pounds\{te^{\frac{t}{2}}\},$$

$$s^{\alpha}\bar{f}(s) - s^{\alpha-1}f(0) - s^{\alpha-2}f'(0) = \frac{1}{(s-\frac{1}{2})^2},$$
$$s^{\alpha}\bar{f}(s) = \frac{1}{(s-\frac{1}{2})^2},$$
$$\bar{f}(s) = \frac{1}{(s^{\alpha})(s-\frac{1}{2})^2}.$$

For different values of "α",comparison of results are shown in Fig.(2.1) represents the behavior of solutions of fractional α to its integer value solution.

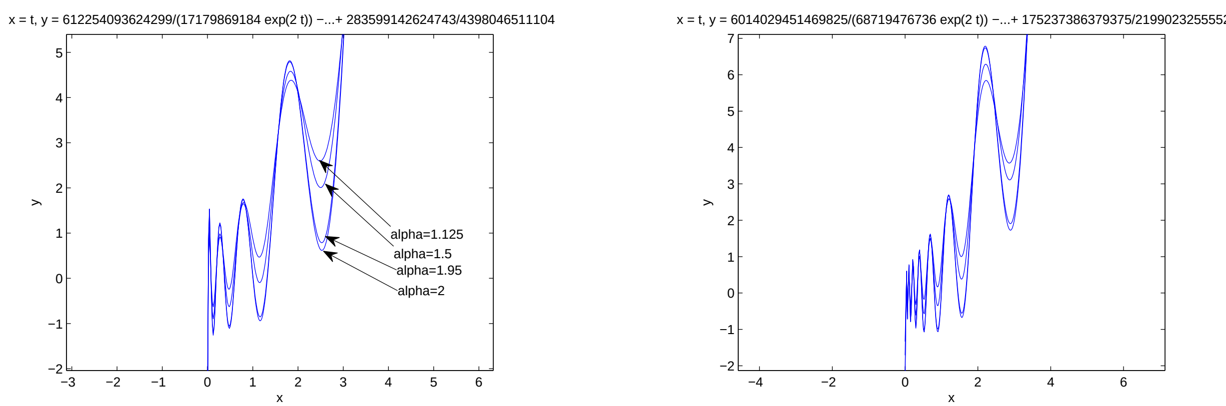

Figure 2.1: Behaviour of numerical solution for different fractional values of "α" at $m = 10$ and $m = 50$

Example 2.3.2: Considering the fractional differential equation

$$Df(t) - {}_*D_t^{\alpha}f(t) + f(t) = e^t,$$

where $0 < \alpha < 1$ and

$$f(0) = 1.$$

Solution: Applying Laplace transform, we have

$$\pounds\{Df(t)\} - \pounds\{{}_*D_t^{\alpha}f(t)\} + \pounds\{f(t)\} = \pounds\{e^t\},$$
$$(s\bar{f}(s) - f(0)) - (s^{\alpha}\bar{f}(s) - s^{\alpha-1}f(0)) + \bar{f}(s) = \frac{1}{s-a},$$
$$(s - s^{\alpha} + 1)\bar{f}(s) = \frac{1}{s-a} + \frac{1}{s} - \frac{s^{\alpha-1}}{s},$$

$$\bar{f}(s) = \frac{1}{(s-a)(s-s^{\alpha}+1)} + \frac{1}{s(s-s^{\alpha}+1)} - 2\frac{s^{\alpha-1}}{s(s-s^{\alpha}+1)},$$

for $\alpha = \frac{1}{2}$ the exact solution is obtained in terms of Mittage Lefller function and insertion of values of these functions we get the solution,

$$f(t) = \frac{1}{16}[4(3t^2+12t+4)e^t + (15+48t+16t^2-3t^3)erfc(\sqrt{t}) + 2\sqrt{\frac{t}{\pi}}(3t^{\frac{3}{2}}+8t+17)].$$

yet the method is not at all suitable for this example.

2.4 Conclusion

In the letter pointing out that the execution of Taylor series method involves the inversion of an Hilbert matrix. This results in numerical instability and makes the algorithm unreliable, example(2.3.2) clearly support this argument. Alternative approaches which give a least-square fit for the solution are also identified. The exponential functions make a good combination with this method as the results are at a level of accuracy.

Chapter 3

Method of Weeks for Inversion of Laplace Transform

Numerical inversion of Laplace transform by the help of Laguerre polynomials can be found in text books (Lancoz, 1956, $Ch.4$; Luke, 1969, $Ch.16$; Krylov *et al.*, 1969, $Ch.5$). Beatman followed the idea back to 1835 introduced to the method adopted by Romanovsky (1924). Initially Ward(1954) concluded that if the Laplace integral equation,

$$\bar{y}(s) = \int_0^\infty e^{-st} y(t)\, dt,$$

contain six to twelve terms of the form $(s-c)^{-1}$,the error might be in a typical case of order 0.05 if ten terms of the series were used. Papoulis(1956) used eleven terms of the series,

$$y(t) = e^{-at} t^\alpha \sum_{k=0}^{\infty} \alpha_k L_k(bt), \tag{3.0.1}$$

with $a = 0$. Khamis(1960) calculated three values of beta probability function using the first nine terms of

$$y(t) = e^{-at} t^\alpha \sum_{k=0}^{\infty} \alpha_k L_k(bt),$$

but in integrated form, with $b = a$ and α by equating mean and variance of distribution to the mean and variance of leading term (3.0.1).

A similar method was used by Tiku(1965) for the approximation of non-central χ^2 distribution. The term of his tenth zero converged to zero up to accuracy of four decimal places between three and sixteen terms, depending upon the size of non-centrally parameter λ and accurate result up to four decimal for $y(t)$.

The Laguerre method of numerical inversion of the Laplace transform is an old set method based on Tricomi-Wildder theorem which approves the possibility of expansion of needed function as weighted sum of Laguerre functions. A lot of variance to this method can be found in literature. For example, Fourier series method for calculation of Laguerre weight functions, systematic methods of scaling etc.

Week's method is one of the most well known and comparatively more effective then series method. It yields an explicit expression dependant on time domain as expansion of Laguerre polynomials. Method was introduced by Picone [11] and was rigourously used by Ticomi [12]. Since then many results of theoretical and practical nature have followed. Firstly a few results are given to establish the series expansion of Weeks method [13,14]. Finally, application to fractional differential equations highlight the utility of Weeks method.

Theorem 3.1: If $\bar{y}(s)$ can be expanded in absolutely convergent series of the form,

$$\bar{y}(s) = \sum_{k=0}^{\infty} \frac{\alpha_k}{s^{\mu_k}}, \qquad |s| > R, \tag{3.0.2}$$

where μ_k form an increasing sequence of numbers $0 < \mu_0 < \mu_1 < \ldots, \to \infty$ then,

$$f(t) = \sum_{k=0}^{\infty} \frac{\alpha_k t^{\mu_k - 1}}{\Gamma(\mu_k)}, \tag{3.0.3}$$

the series is convergent for all t, real and complex.

Theorem 3.2: If $\bar{y}(s)$ can be expanded in the neighborhood of $s = \beta_k$, in the complex s-plane, in the form of an absolutely convergent power series with arbitrary exponents,

$$\bar{y}(s) = \sum_{k=0}^{\infty} b_k (s - \beta_k)^{\mu_k}, \qquad -k < \mu_0 < \mu_1 < \ldots, \to \infty, \tag{3.0.4}$$

then there exist a contribution to the asymptotic expansion of $y(t)$ as $t \to \infty$ of the form,

$$e^{\beta_k t} \sum_{k=0}^{\infty} \frac{\beta_k t^{\mu_k - 1}}{\Gamma(-\mu_k)}, \tag{3.0.5}$$

where $\frac{1}{\Gamma(-\mu_k)} = 0$ if μ_k is a positive integer or zero.

Watson's Lemma: Suppose that the function $f(t)$ satisfies,

$$f(t) \sim \sum_{k=0}^{\infty} \beta_k t^{\beta_k}, \qquad as \quad t \to 0+ \tag{3.0.6}$$

where $-1 < Re(\beta_0) < Re(\beta_1) < \ldots$ and $\lim(Re\beta_k) = \infty$ whenever $k \to \infty$, then for any $\delta > 0$,

$$\bar{y}(s) \sim \sum_{k=0}^{\infty} \frac{\beta_k \Gamma(\beta_k + 1)}{s^{\beta_k + 1}} \qquad as \qquad s \to \infty, |arg(s)| < \frac{\pi}{2} - \delta. \tag{3.0.7}$$

Parseval's Theorem: A generalization of Parseval's theorem for the Laplace transform is,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}|y(a+it)|^2\,dt = \int_0^{\infty} e^{-2a\omega}|f(\omega)|^2\,d\omega. \tag{3.0.8}$$

Although the relation needs some arguments for its equivalence.

3.1 Weeks Method

The method combine the idea of Papoulis and Lancoz [13, 14]. Purpose of this method is to find the series expansion of $y(t)$ in terms of orthonormal Leguerre function,

$$\phi_k(t) = e^{-\frac{t}{2}}L_k(t), \qquad k = 0, 1, 2, \ldots, \tag{3.1.1}$$

where $L_k(t)$ represent the kth Leguerre polynomial. Orthogonality of a function can be confirmed if a function satisfies the following condition,

$$\int_0^{\infty}\phi_k(t)\phi_l(t)\,dt = 0, \tag{3.1.2}$$

whenever $k \neq l$.

If the same function posses unit norm along with fulfilling the orthogonality condition is known to be orthonormal function. Particularly $\phi(0) = 1$.
For any function $y(t)$ satisfying,

$$|y(t)| = O(e^{\gamma t}),$$

along with $Theorem$ 3.1 and $Theorem$ 3.2 also $Watson's\ Lemma$, own an approximation by the function $y_N(t)$ of the form,

$$y_N(t) = e^{at}\sum_{k=0}^{\infty}\alpha_k\phi_k(\frac{t}{T}), \tag{3.1.3}$$

for $T > 0$, being the scale factor and

$$\alpha_k = \frac{1}{T}\int_0^{\infty}e^{-at}y(t)\phi_k(\frac{t}{T})\,dt, \tag{3.1.4}$$

the approximation of $y(t)$ ensures the convergence of integral i.e for any $\epsilon > 0$, there exist an N_0 such that,

$$\int_0^{\infty}e^{-2at}|y(t) - y_N(t)|^2\,dt < \epsilon, \quad \text{whenever} \quad N > N_0. \tag{3.1.5}$$

The Laplace transform of $y_N(t)$ is,

$$\begin{aligned} y_N(t) &= \int_0^\infty e^{-st} y_N(t)\,dt \\ &= \int_0^\infty e^{-(s-a)t} \sum_0^N \alpha_k \phi_k(\frac{t}{T})\,dt \\ &= \sum_0^N \alpha_k \int_0^\infty e^{-(s-a)t} e^{-\frac{t}{2T}} L_k(\frac{t}{T})\,dt \\ &= \sum_0^N \alpha_k \frac{(s-a-\frac{1}{2T})^k}{(s-a+\frac{1}{2T})^{k+1}}. \end{aligned} \tag{3.1.6}$$

As a consequence of Parseval's theorem,

$$\int_0^\infty e^{-2at} |y(t) - y_N(t)|^2\,dt = \frac{1}{2\pi} \int_{-\infty}^\infty |\psi(a+i\omega) - \psi_N(a+i\omega)|^2\,d\omega, \tag{3.1.7}$$

where the convergence of $|\bar{y}_N(a+i\omega)|$ is obvious with increasing N to $y(a+i\omega)$. Since both $y(t)$ and $y_N(t)$ are of exponential order. If $a > \gamma$ then for any $\epsilon > 0$, there exist N_0 such that,

$$\frac{1}{2\pi} \int_{-\infty}^\infty |\bar{\psi}(a+i\omega) - \bar{\psi}_N(a+i\omega)|^2\,d\omega < \varepsilon, \tag{3.1.8}$$

whenever $N > N_0$.

Weeks expanded the Laplace transform trigonometrically. A substitution of (3.1.6) in (3.1.8) for $s = a + i\omega$ yields,

$$\frac{1}{2\pi} \int_{-\infty}^\infty \left| \psi(a+i\omega) - \sum_{k=0}^N \alpha_k \frac{(i\omega - \frac{1}{2T})^k}{(i\omega + \frac{1}{2T})^{k+1}} \right|^2 d\omega < \epsilon, \tag{3.1.9}$$

whenever $N > N_0$.

Suppose $\omega = \frac{1}{2T} \cot \frac{1}{2}\theta$ and the identities,

$$\frac{i \cos \frac{1}{2}\theta - \sin \frac{1}{2}\theta}{i \cos \frac{1}{2}\theta + \sin \frac{1}{2}\theta} = e^{i\theta},$$

$$\left| i \cos \frac{1}{2}\theta + \sin \frac{1}{2}\theta \right|^2 = 1,$$

we have,

$$\frac{T}{2\pi} \int_{-\pi}^{\pi} \left\{ \left| (\frac{1}{2T} + \frac{i}{2T} \cot \frac{1}{2}\theta) \bar{\psi}(a + i \cot \frac{1}{2}\theta) - \sum_{k=0}^N \alpha_k e^{ik\theta} \right|^2 \right\} d\theta < \epsilon, \tag{3.1.10}$$

whenever $N > N_0$, the inequality implies,

$$(\frac{1}{2T} + \frac{i}{2T} \cot \frac{1}{2}\theta) \bar{\psi}(a + \frac{1}{2T} \cot \frac{1}{2}\theta) \approx \sum_{k=0}^N \alpha_k e^{ik\theta}, \tag{3.1.11}$$

the right hand side of above equation converges in the mean to left hand side as N increases. Defining the two separate functions of $\psi(a+i\omega)$ as $\psi_1(a+i\omega)$ and $\psi_2(a+i\omega)$ such that,

$$\psi(a+i\omega) = \psi_1(a+i\psi) + i\psi_2(a+i\psi), \tag{3.1.12}$$

then the real part of (3.1.11)is restricted to expansion and is reduced to,

$$G(\theta) \approx \sum_{k=0}^{N} \alpha_k \cos k\theta, \tag{3.1.13}$$

and

$$G(\theta) = \frac{1}{2T}(\psi_1(a,\omega) + \psi_2(a,\omega)\cot\frac{1}{2}\theta).$$

Trigonometric interpolation assist in approximation of the coefficients α_k,

$$\alpha_0 = \frac{1}{N+1}\sum_{i=0}^{N} G(\theta_i), \tag{3.1.14}$$

$$\alpha_k = \frac{2}{N+1}\sum_{i=0}^{N} G(\theta_i)\cos k\theta_i, \qquad k > 0 \tag{3.1.15}$$

where,

$$\theta_i = (\frac{2i+1}{N+1})\frac{\pi}{2}. \tag{3.1.16}$$

The interpolation formulas from (3.1.13) to (3.1.16) become the usual Tchebychef interpolation as one introduces the substitution of $z = \cos k\theta$, then $k\theta = T_k(z)$ is the Tchebychef polynomial of degree k.Proper and credible choice of the scale factor T and the constant "a" will make the convergence process faster. Although the convergence in the series (3.0.5) was established for any positive T and a. Considering the scalar factor T, we know that the function $\phi_k(t)$ is the product of Leguerre polynomial and *death function*(i.e the decreasing exponential function). The Leguerre polynomial has n real and positive zeros, however the largest zero t_k of $L_k(t)$ satisfies the inequality,

$$t_k \leq 4k-6, \qquad k \geq 4.$$

Thus the function $\phi_k(t)$ do not drop down to zero and never exceed $4k-6$ i.e oscillation is within the interval $0 < t < 4k-6$, function decreases monotonically and approaches to the limiting value of zero for $t > 4k-6$. Hence the expectation will be an oscillatory function representable in linear combination of ϕ's over the interval $(0, t_{max})$.

$$\frac{t_{max}}{T} < 4N-6,$$

Weeks suggested, the choice of parameter T is obtained by taking,

$$T = \frac{t_{max}}{N},$$

which ensures the good approximation of $y(t)$ in $(0, t_{max})$ depending upon the proper choice of "a" and taking N to be sufficiently large.

Weeks also proposed a suitable value of "a". In principle, "a" could be any non-negative number sufficiently large to guarantee the convergence of Laplace integrals, the smallest number a_0 is *abassica* of the Laplace integral for which,

$$\int_0^\infty e^{-ct}|y(t)|\,dt \int_0^\infty e^{-ct}|y(t)|^2\,dt,$$

converge.
Marking a position of a line through right most singularity for $s = a_0$, then

$$a = (a_0 + \frac{1}{t_{max}})u(a_0 + \frac{1}{t_{max}}), \tag{3.1.17}$$

where $u(x)$ is the unit step function.
Examining $\bar{\psi}(s)$ to find position of right most singularity in the order to obtain the *abscissa* of convergence a_0.

Computational process to find the inverse Laplace $\bar{\psi}(s)$ depends upon largest value to which one desire to evaluate function $y(t)$. Suppose t_{max} stands for this value , step two is locating the right most singularity of $\bar{\psi}(s)$. Further, supposing s_0 to be the singularity(right most) then $Re(s_0) = a_0$ is *abscissa* of convergence of the Laplace integral will give us the value of constant "a". Choosing N i.e the number of terms for the expansion of series of $y(t)$. The choice of t_{max} and N will help to compute T.

Calculation of coefficients α_k, involves the evaluation of function $\psi_1(a, \omega)$ and $\psi_2(a, \omega)$ defined in (3.1.12). $G(\theta)$ should be evaluated only for $N + 1$ values of θ. Thus,

$$G(\theta_i) = \frac{1}{2T}\psi_1(a + \iota\omega_i), \quad for \quad i = 0, 1, 2, \ldots, \tag{3.1.18}$$

where,

$$\omega_i = \frac{1}{2T}\cot\frac{1}{2}\theta_i, \qquad i = 0, 1, 2, \ldots, N. \tag{3.1.19}$$

θ_i given by eq.(3.1.16), ω_j can be recursively calculated by,

$$\omega_i = \frac{\gamma\omega_i - 1 - \delta}{\omega_i - 1 + \gamma},$$

where $\gamma = \frac{1}{2T}\cot\theta_0$ and $\delta = \frac{1}{4T^2}$ starting value can be defined as,

$$\omega_0 = \frac{1}{2T}\cot\frac{1}{2}\theta_0,$$

$$\omega_0 = \frac{1}{2T}\left(\frac{1+\cos\theta_0}{\sin\theta_0}\right),$$

and

$$\cos\theta_i = a\cos\theta_{i-1} - b\sin\theta_{i-1},$$

$$\sin\theta_i = b\cos\theta_{i-1} + a\sin\theta_{i-1},$$

where,

$$a = \cos 2\theta_0 = 2\cos^2\theta - 1,$$

$$\theta = \sin 2\theta_0 = 2\sin\theta_0\cos\theta_0.$$

Furthermore, we can calculate $\cos n\theta_i$ for $n = 2$, using identity,

$$\cos n\theta_i = 2\cos\theta_i, \cos(n-1)\theta_i - \cos(n-2)\theta_i.$$

Thus collecting all the trigonometric functions required to calculate α_n according to (3.1.15), we have

$$\phi_0(t) = e^{\frac{-t}{2}},$$

$$\phi_1(t) = (1-t)\phi_0(t),$$

$$k\phi_k(t) = (2k-1-t)\phi_{k-1}(t) - (k-1)\phi_{k-2}(t), \quad k > 1.$$

3.2 Examples

Example 3.2.1: Let the function whose inverse Laplace transform has to be evaluated is given as,

$$\bar{f}(s) = \frac{2}{s(s^2+4)}$$

which is the Laplace transform of,

$$f(t) = \sin^2 t.$$

Table(3.1) shows the exact values along with the numerical value to calculate the accuracy of Weeks method.

Example 3.2.2: If,

$$f(t) = \csc t \sin 5t,$$

when applied Laplace transform to above function it results in a function of the form,

$$\bar{f}(s) = \frac{1}{s} + \frac{2s}{s^2+4} + \frac{2s}{s^2+16},$$

finding the numerical solution by Weeks method and results are analyzed in Table(3.2),

t	*exact*	$m = 10$		$m = 50$		$m = 100$	
		numeric	error	numeric	error	numeric	error
0.1	0.0117	0.0103	0.0014	0.0100	0.0017	0.0483	0.0017
2	0.8268	0.8273	0.0005	0.8268	0.0000	0.2467	0.0000
3	0.0199	0.0290	0.0091	0.0199	0.0000	0.4235	0.0000
4	0.5728	0.4470	0.1258	0.5728	0.0000	0.5618	0.0000
5	0.9195	1.3751	0.4556	0.9195	0.0000	0.6665	0.0000
6	0.0781	-0.0059	0.084	0.0781	0.0000	0.7102	0.0000
7	0.4316	-2.5670	2.9986	0.4316	0.0000	0.7250	0.0000
8	0.9788	1.4940	0.5152	0.9788	0.0000	-0.0081	0.0000
9	0.1698	17.3656	17.1958	0.1698	0.0000	-0.0094	0.0000

Table 3.1: Comparison of numerical solution with exact solution and estimation of absolute error

3.3 Application to Fractional Differential Equations

Difference Fractional Differential Equation: Finding the numerical solution of,

$$ {}_{*}D_t^{\alpha} f(t) - f(t-1) = 1, $$

where $0 < \alpha \leq 1$ and $f(0) = 0$.
Application of Laplace transform to above fractional differential equation yields,

$$ \bar{f}(s) = \frac{1}{s(s^{\alpha} - e^{-s})}. $$

Fig.(3.1) indicates that the solution of fractional order differential equation follows the same behavior as the solution of ordinary differential equation for integer order α and gives the numerical values.
Transfer and Impulse Response Function: Considering,

$$ f^{\alpha}(t) + 2f^{\alpha-1}(t) + 5f(t) = g(t), $$

function along with conditions $1 < \alpha \leq 2$ and

$$ f(0) = 0, \qquad f'(0) = -1. $$

Applying the fractional Laplace transform formulas, we get

$$ \bar{f}(s) = \frac{1}{s^{\alpha} + s^{\alpha+1} + 5} - \frac{s^{\alpha-1}}{2s^{\alpha} + s^{\alpha-1} + 5}. $$

t	$exact$	$m = 10$		$m = 50$		$m = 100$	
		numeric	error	numeric	error	numeric	error
0.1	4.8023	5.0558	0.2536	4.8028	0.0006	4.8023	0.0000
2	-0.5983	1.1842	1.7825	-0.5996	0.0013	-0.5983	0.0000
3	4.6080	-0.0095	4.6175	4.6011	0.0069	4.6081	0.0000
4	-1.2063	4.5893	5.7956	-1.1627	0.0436	-1.2061	0.0002
5	0.1380	4.6798	4.5418	0.3660	0.2280	0.1385	0.0005
6	3.5361	-20.7244	24.2604	3.0481	0.4880	3.5375	0.0014
7	-0.6517	-28.6729	28.0211	-1.0442	0.3925	-0.6482	0.0035
8	0.7531	93.1598	92.4067	4.5504	3.7973	0.7529	0.0003
9	2.0647	332.2628	0330.1981	-9.3677	11.4324	2.0320	0.0327

Table 3.2: Comparison of numerical solution with exact solution and estimation of absolute error

Considering different fractional values of α and observing the behavior of solution in Fig.(3.2).
Example 3.3.3: Considering a fractional differential equation with a restriction $1 < \alpha \leq 2$,

$$ {}_*D_t^{\alpha} f(t) + {}_*D_t^{\alpha-1} f(t) + f(t) = \cos t - \sin t, $$

where,

$$ f(0) = f'(0) = 1. $$

Application of fractional Laplace formulas and implementation of initial conditions result in,

$$ (s^{\alpha} + s^{\alpha-1} + 1)\bar{f}(s) = \frac{s-1}{s^2+1} + s^{\alpha-1} + 2s^{\alpha-2}, $$

$$ \bar{f}(s) = \frac{s-1}{(s^2+1)(s^{\alpha} + s^{\alpha-1} + 1)} + \frac{s^{\alpha-1}}{s^{\alpha} + s^{\alpha-1} + 1} + \frac{2s^{\alpha-2}}{s^{\alpha} + s^{\alpha-1} + 1}, $$

the exact solution for $\alpha = 2$ is $f(t) = \cos t + \sin t$.
Fig.(3.3) represents the behavior of fractional differential equation.
Example 3.3.4: Considering two dimensional heat equation,

$$ \frac{\partial^{\alpha}}{\partial t^{\alpha}} f(x,t) = \frac{\partial^2}{\partial x^2} f(x,t), \qquad 0 < \alpha \leq 1, \tag{3.3.1} $$

along with boundary conditions,

$$ f(x,0) = g(x), $$

$$f(0,t) = f(t), \qquad |f(x,t)| < \infty,$$

where $g(x) = 1$,$f(t) = 0$ and $c = 1$.
Applying the Laplace transform we get the ordinary differential equation of the form,

$$(D^2 - s^\alpha)\bar{f}(x,s) = -1.$$

Using method of undetermined coefficient and applying the boundary conditions we get,

$$\bar{f}(x,s) = \frac{1}{s}(1 - e^{xs^{-\frac{\alpha}{2}}}).$$

Fig.(3.4) shows the fixed value of x and varying t. Weeks method MATLAB algorithm is used for plotting the graph of numerical solution and to check the behavior of solution by comparing it with classical case i.e $\alpha = 1$.

Example 3.3.5: Considering the partial differential equation,

$$\frac{\partial^{1+\alpha}}{\partial x \partial t^\alpha} f(x,t) = \sin x \sin t,$$

where $0 < \alpha \leq 1$ initial conditions,

$$f(x,0) = 1 + \cos x, \qquad f_t(0,t) = 2\sin t.$$

Assuming that $f_x(x,t)$ and $f_t(x,t)$ are differentiable in the domain of defining function $f(x,t)$, i.e

$$\frac{\partial^2}{\partial x \partial t} f(x,t) = \frac{\partial^2}{\partial t \partial x} f(x,t).$$

Now substituting $\frac{\partial^\alpha}{\partial t^\alpha} = F$,

$$\frac{\partial}{\partial x}\frac{\partial^\alpha}{\partial t^\alpha} f(x,t) = \sin x \sin t,$$

$$\frac{\partial}{\partial x} F(x,t) = \sin x \sin t,$$

taking the Laplace transform w.r.t x and putting the initial conditions,

$$F(s,t) = -\frac{-2\sin t}{s} + \sin t(\frac{1}{s} - \frac{s}{1+s^2}),$$

and the inverse Laplace yields,

$$F(x,t) = -2\sin t + \sin t(1 - \cos x).$$

Now,

$$\frac{\partial^\alpha}{\partial t^\alpha} f(x,t) = -2\sin t + \sin t(1 - \cos x).$$

Application of Laplace transform to above equation and of initial condition yields,

$$f(x,s) = (1+\cos x)(\frac{1}{s} - \frac{1}{s^{\alpha}(1+s^2)}).$$

Whereas the solution for classical case i.e $\alpha = 1$ is,

$$f(x,t) = -\cos t(1+\cos x).$$

Fig.(3.5) represents the behavior of fractional numerical solution in comparison with integer order derivative.

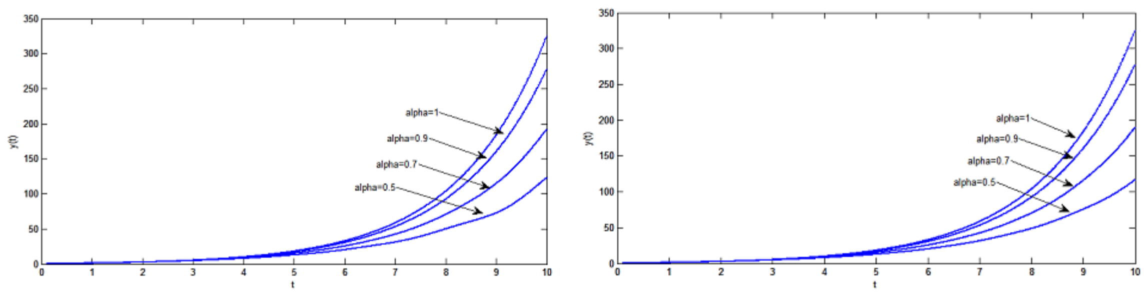

Figure 3.1: Behaviour of numerical solution for different fractional values of "α" at $m = 50$ and $m = 100$

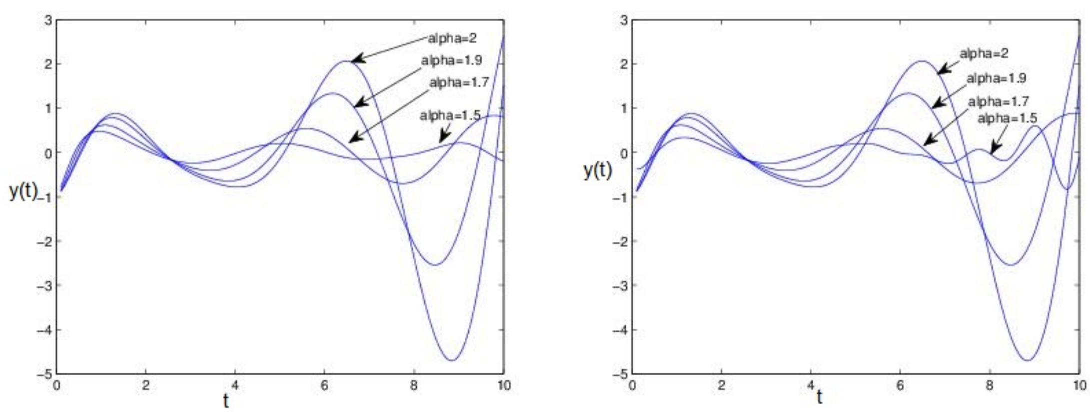

Figure 3.2: Behaviour of numerical solution for different fractional values of "α" at $m = 50$ and $m = 100$

3.4 Conclusion

Weeks method in comparison to Taylor series method is more efficient and fast convergent yet accuracy level is not very high. Drawback of Weeks method, up to a specified extent it repeats the values rather refining it on increment of iterations. The parameter "b" in

$$f(t) = \sum_{k=0}^{\infty} \alpha_k L_k(bt),$$

is an element to accelerate the convergence of series. The efficiency factor of this method is highly dependant upon convergence of this series. So, change of parameters can also help to find numerical inversion of the Laplace transform for few problems.

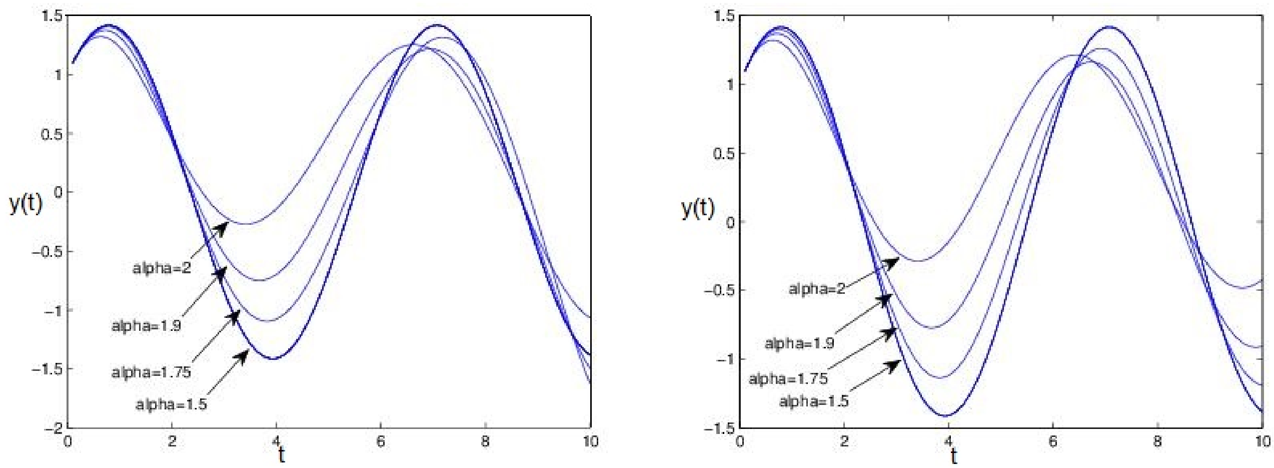

Figure 3.3: Behaviour of numerical solution for different fractional values of "α" at $m = 10$ and $m = 50$

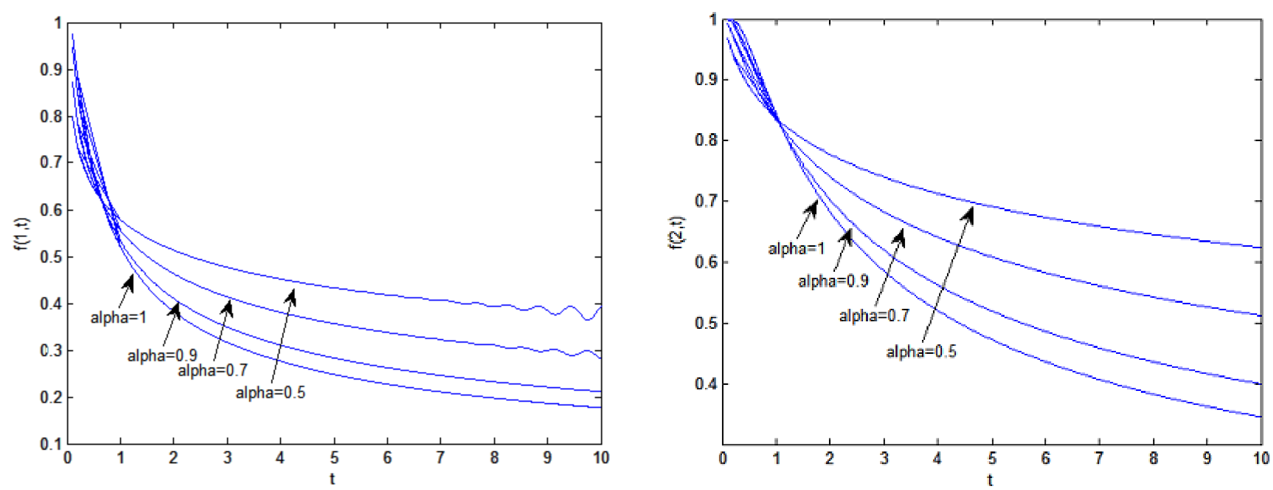

Figure 3.4: Behavior of numerical solution for different fractional values of "α" for fixed $x = 1$ and $x = 2$

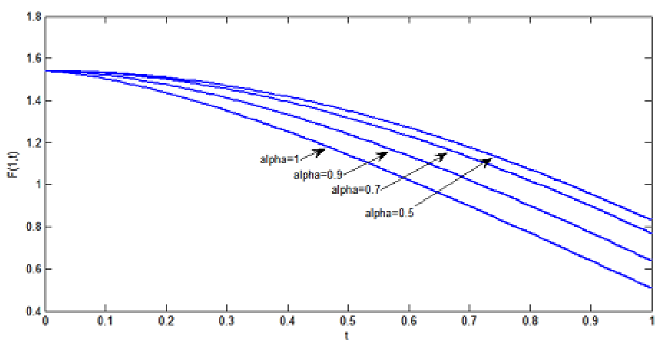

Figure 3.5: Behavior of numerical solution for different fractional values of "α" for fixed $x = 1$.

Chapter 4

Double Laplace Transform

Describing the importance of ordinary differential equation one can not deny the importance of partial differential equations. These are frequently used in mathematical physics, branches of physics, applied mathematics and engineering as well. These equations are of two types, one is homogeneous equation with constant coefficients which posses many method for its solution like separation of variable and method of characteristics etc. The other type is non homogeneous equation with constant coefficient and the non-linear partial differential equations are also a part of this field to deal with. Double Laplace transform is very useful method to solve partial differential equations and also has a very effective role in simplification of calculation in many branches of engineering and mathematics.

The theory of double laplace transform originated from a paper by Humbert [15]. Applications are also described in books by Volker and Doetsch [16], Sneddon [17] and also by Ditkin and Prudnikov [18,19]. The technique is yet not very liked in recent years because of the difficulty in inversion of integral. This approach has its worth because it is very straight forward method with a flexibility of using a numerical method for inversion of this transform [20].

Understanding the importance of partial differential equation in mathematics and physics , many mathematicians worked on methods to find the solution of these equations. Eltayeb and Kilicman [21, 22] worked on a relationship between double Samudu transform and double Laplace transform. In a line of this, Singh and Mandia [23] developed the relation between double Mellin transform and double Laplace transform. Recently, Eltayeb and Kilicman [24] implemented the technique of double Laplace transform for the solution of general linear telegraph and partial integro differential equations.

In the chapter we will go through the properties of double Laplace transform and then will apply this technique to fractional partial differential equations.

4.1 Double Laplace Transform; Definition and Existence

Let $f(y,t)$ be a function of two variables y and t, where $y,t > 0$. The double Laplace transform of $f(y,t)$ is defined as,

$$\pounds_t \pounds_y \{f(y,t)\} = \int_0^\infty e^{-s_1 t} dt \int_0^\infty e^{-s_0 y} f(y,t) dy, \tag{4.1.1}$$

whenever the integral(improper) converges [25].

The existence or convergence of double Laplace integral is strongly bonded to the function to be of exponential order. Hence the existence criteria for the double Laplace transform is same for single Laplace transform as well.

4.2 Basic Properties of Double Laplace Transform

Some properties of double Laplace transform are described as follows, [25]

Linearity:

If $p(y,t)$ and $q(y,t)$ be the functions of y and t such that,

$$\pounds_t \pounds_y \{p(y,t)\} = \bar{p}(s_0, s_1),$$

and

$$\pounds_t \pounds_y \{q(y,t)\} = \bar{q}(s_0, s_1),$$

then

$$\pounds_t \pounds_y \{ap(y,t) + bq(y,t)\} = a\pounds_t \pounds_y \{p(y,t)\} + b\pounds_t \pounds_y \{q(y,t)\}.$$

Scale Property:

If,

$$\pounds_t \pounds_y \{p(y,t)\} = \bar{p}(s_0, s_1),$$

then

$$\pounds_t \pounds_y \{p(\alpha y, \beta t)\} = \frac{1}{\alpha\beta} \bar{p}(\frac{s_0}{\alpha}, \frac{s_1}{\beta}),$$

where α and β are constants.

Proof: From the definition (4.1.1) we have,

$$\pounds_t \pounds_y \{p(\alpha y, \beta t)\} = \int_0^\infty e^{-s_1 t} \int_0^\infty e^{-s_0 y} p(\alpha y, \beta t) dx dt, \tag{4.2.1}$$

assuming $\alpha y = \mu$ and $\beta t = \nu$ in (4.2.1), we get

$$\begin{aligned}
\pounds_t\pounds_y\{p(\alpha y, \beta t)\} &= \int_0^\infty e^{-s_1(\frac{\nu}{\beta})}\int_0^\infty e^{-s_0(\frac{\mu}{\alpha})}p(\mu,\nu)\frac{\mu}{\alpha}\frac{\nu}{\beta} \\
&= \frac{1}{\alpha\beta}\int_0^\infty e^{-s_1(\frac{\nu}{\beta})}\int_0^\infty e^{-s_0(\frac{\mu}{\alpha})}p(\mu,\nu)d\mu d\nu \\
&= \frac{1}{\alpha\beta}\bar{p}(\frac{s_0}{\alpha}, \frac{s_1}{\beta}).
\end{aligned}$$

First Shifting Property:

If,

$$\pounds_t\pounds_y\{p(y,t)\} = \bar{p}(s_0, s_1),$$

then ,

$$\pounds_t\pounds_y\{e^{\alpha y+\beta t}p(y,t)\} = \bar{p}(s_0-\alpha, s_1-\beta),$$

where α and β are constants.

Proof: From the definition (4.1.1) we have

$$\begin{aligned}
\pounds_t\pounds_y\{e^{\alpha y+\beta t}p(y,t)\} &= \int_0^\infty e^{-s_1 t}\int_0^\infty e^{-s_0 y}e^{\alpha y+\beta t}p(y,t)dxdt \\
&= \int_0^\infty e^{-(s_1-\beta)t}\int_0^\infty e^{-(s_0-\alpha)y}p(\alpha y, \beta t)dxdt \\
&= \bar{p}(s_0-a, s_1-b).
\end{aligned}$$

Double Laplace of Partial Derivatives: If,

$$\pounds_t\pounds_y\{p(y,t)\} = \bar{p}(s_0, s_1),$$

then,

$$\pounds_t\pounds_y\{p_{yt}(y,t)\} = s_0 s_1\bar{p}(s_0,s_1) - s0\bar{p}(s_0,0) - s_1\bar{p}(0,s_2) + p(0,0).$$

Proof: From definition (4.1.1),

$$\begin{aligned}
\pounds_t\pounds_y\{e^{\alpha y+\beta t}p_{yt}(y,t)\} &= \int_0^\infty e^{-s_1 t}\int_0^\infty e^{-s_0 y}p_{yt}(y,t)dydt \\
&= \int_0^\infty e^{-s_0 y}[\int_0^\infty e^{-s_1 t}p_{y,t}(y,t)dt]dy \\
&= \int_0^\infty e^{-s_0 y}[e^{-s_1 t}p_y(y,t)|_0^\infty - \int_0^\infty -s_1 e^{-s_1 t}p_y(y,t)dt]dy \\
&= \int_0^\infty e^{-s_0 y}[-p_y(y,0) + s_1\int_0^\infty e^{-s_1 t}p_y(y,t)dt]dy \\
&= -\int_0^\infty e^{-s_0 y}p_y(y,0)dy + s_1\int_0^\infty e^{-s_0 y}\int_0^\infty e^{-s_1 t}p_y(y,t)dtdy \\
&= -[e^{-s_1 t}p_y(y,0)|_0^\infty - \int_0^\infty e^{-s_0 y}(-s_0)p(y,0)dt] + s_1\int_0^\infty e^{-s_1 t}\int_0^\infty e^{-s_0 y}p_y(y,t)dydt
\end{aligned}$$

$$\begin{aligned}
\pounds_t\pounds_y\{e^{\alpha y+\beta t}p_{yt}(y,t)\} &= [p(0,0)+s_1\int_0^\infty e^{-s_0y}p(y,0)dx]+s_1\int_0^\infty e^{-s_1t}[e^{-s_0y}p(y,t)|_0^\infty \\
&\quad -\int_0^\infty e^{-s_1t}(-s_0)p(y,t)dy]dt \\
&= p(0,0)-s_0\bar{p}(s_0,0)+s_1\int_0^\infty e^{-s_1t}[-p(0,t)+s_0\int_0^\infty e^{-s_0y}f(y,t)dy]dt \\
&= p(0,0)-s_0\bar{p}(s_0,0)-s_1\int_0^\infty s^{-s_1t}p(0,t)dt+s_0s_1\int_0^\infty e^{-s_1t}\int_0^\infty e^{-s_0y}f(y,t)dydt \\
&= s_0s_1\bar{p}(s_0,s_1)-s_1\bar{p}(s_1,0)-s_1\bar{p}(0,s_1)+f(0,0).
\end{aligned}$$

In general,

$$\pounds_t\pounds_y\left\{\frac{\partial^n}{\partial x^n}f(y,t)\right\}=s_0^n\pounds_t\pounds_y\{f(y,t)\}-\sum_{k=0}^{n-1}s_1^{n-1-k}\pounds_t\left\{\frac{\partial^k}{\partial y^k}\right\}f(0,t),$$

$$\pounds_t\pounds_y\left\{\frac{\partial^n}{\partial t^n}f(y,t)\right\}=s_1^n\pounds_t\pounds_y\{f(y,t)\}-\sum_{i=0}^{n-1}s_1^{n-1-i}\pounds_x\left\{\frac{\partial^i}{\partial t^i}\right\}f(y,0),$$

$$\begin{aligned}
\pounds_t\pounds_y\left\{\frac{\partial^{n+m}}{\partial t^n\partial y^m}f(y,t)\right\} &= s_0^m s_1^n[\pounds_t\pounds_y\{f(y,t)\}-\sum_{i=0}^{n-1}s_0^{n-1-i}\pounds_x\left\{\frac{\partial^i}{\partial t^i}\right\}f(y,0)-\sum_{k=0}^{n-1}s_1^{n-1-k}\pounds_t\left\{\frac{\partial^k}{\partial y^k}\right\}f(0,t) \\
&\quad +\sum_{k=0}^{n-1}\sum_{i=0}^{m-1}s_0^{-k-1}s_1^{-i-1}\frac{\partial^{k+i}}{\partial y^k\partial t^i}f(0,0)].
\end{aligned}$$

4.3 Solution of Fractional Partial Differential Equations using Double Laplace Transform

Fractional Wave Equation: Considering the fractional wave equation,

$$_*D_t^\alpha f(y,t)-\frac{1}{\pi^2}\frac{\partial f(y,t)}{\partial x^2}=0, \tag{4.3.1}$$

with initial conditions,

$$f(y,0)=\sin\pi x, \qquad f_t(y,0)=0,$$

$$f(0,t)=0, \qquad f_y(0,t)=\pi E_\alpha(-t^\alpha),$$

where $0<\alpha\le 2$ and $c=-\frac{1}{\pi^2}$.

First applying the double Laplace transform over initial conditions for the ease in further solution.

$$\pounds_y\{f(y,0)\}=\bar{f}(s_0,0)=\frac{\pi}{s_0^2+\pi^2},$$

$$\pounds_y\{f_t(y,0)\}=\bar{f}_y(s_0,0)=0,$$

$$\pounds_t\{f(0,t)\} = \bar{f}(0,s_1) = 0,$$

$$\pounds_t\{f_y(0,t)\} = \bar{f}_t(0,s_1) = \frac{\pi s_2^{\alpha-1}}{1+s_1^{\alpha}}.$$

Applying double laplace to (4.3.1) ,

$$\pi^2[s_1^{\alpha}\bar{f}(s_0,s_1) - s_1^{\alpha-1}\bar{f}(s_0,0) - s_1^{\alpha-2}\bar{f}(s_0,s_1)\bar{f}_x(s_0,0)] - [s_0^2\bar{f}(s_0,s_1) - s_0\bar{f}(0,s_1) - \bar{f}_t(0,s_1)] = 0.$$

Putting the conditions and arranging the equation we get,

$$(\pi^2 s_1^{\alpha} - s_0^2)\bar{f}(s_0,s_1) = \frac{\pi^3 s_1^{\alpha-1}}{s_0^2+\pi^2} - \frac{\pi s_1^{\alpha-1}}{1+s_1^{\alpha}},$$

after simplification,

$$\bar{f}(s_0,s_1) = \frac{\pi s_1^{\alpha-1}}{(s_0^2+\pi^2)(1+s_1^{\alpha})}.$$

Applying the inverse double Laplace we get the solution of the form,

$$f(y,t) = \sin \pi y E_{\alpha}(-t^{\alpha}).$$

Fractional Heat Equation: Considering the fractional form of heat equation,

$$_*D_t^{\alpha} f(y,t) - \frac{\partial^2}{\partial y^2} f(y,t) = 0, \tag{4.3.2}$$

with initial conditions,

$$f(y,0) = 1,$$

$$f(0,t) = 1, \qquad f_y(0,t) = 0,$$

which transforms in the way,

$$\pounds_y\{f(y,0)\} = \bar{f}(s_0,0) = \frac{1}{s_0},$$

$$\pounds_t\{f(0,t)\} = \bar{f}(0,s_1) = \frac{1}{s_1},$$

$$\pounds_t\{f_y(0,t)\} = 0.$$

Applying Double Laplace to (4.3.2),

$$(s_1^{\alpha} - s_0^2)\bar{f}(s_0,s_1) = \frac{s_1^{\alpha-1}}{s_0} - \frac{s_0}{s_1}.$$

After simplifying,

$$\bar{f}(s_0,s_1) = \frac{1}{s_0 s_1},$$

inversion will result in,

$$f(y,t) = 1.$$

The examples readily depicts that the solution by this transformation is yet possess a high level of dependency on the conditions.

Fractional Kleinn-Gordon Equation: Let us consider

$$f_{tt} - c^2 \Delta f + m^2 f = 0, \tag{4.3.3}$$

is called Kleinn-Gordan equation which appears in quantum field theory where m denotes the mass and considering the dissipation or damping term in (4.3.3),

$$f_{tt} - c^2 \Delta f + a f_t + m^2 f = 0,$$

which in one dimension is called the *Telegrapher's equation* because it governs in electrical transmission in telegraph cable when possibility of current leakage to ground is expected.

Now considering fraction Klein-Gordon equation in one dimension with $c^2 = 1$ and $m^2 = 1$,

$$f_{tt} - {}_*D_t^\alpha f + f = 0, \tag{4.3.4}$$

with initial conditions,

$$f(y,0) = 0, \qquad f_t(y,0) = E_\alpha(y^\alpha),$$
$$f(0,t) = t, \qquad f_y(0,t) = 0,$$

which on application of the Laplace transform results in,

$$\pounds_y\{f(y,0)\} = \bar{f}(s_0,0) = 0,$$
$$\pounds_y\{f_t(y,0)\} = \bar{f}_t(s_0,0) = \frac{s_0^{\alpha-1}}{s_0^\alpha - 1},$$
$$\pounds_t\{f(0,t)\} = \bar{f}(0,s_1) = \frac{1}{s_1^2},$$
$$\pounds_t\{f_y(0,t)\} = \bar{f}_y(0,s_1) = 0.$$

Applying double Laplace on (4.3.4),

$$\pounds_t \pounds_y\{f_{tt} - {}_yD_*^\alpha f + f\} = 0,$$

and putting the conditions, we get

$$\begin{aligned} [(s_1^2 - s_0^\alpha + 1)]\bar{f}(s_0,s_1) &= \frac{s_0^{\alpha-1}}{s_0^\alpha - 1} - \frac{s_0^{\alpha-1}}{s_1^2}, \\ &= \frac{s_0^{\alpha-1}(s_1^2 - s_0^\alpha + 1)}{s_1^2(s_0^\alpha - 1)}, \end{aligned}$$

$$\bar{f}(s_0,s_1) = \frac{s_0^{\alpha-1}}{s_1^2(s_0^\alpha - 1)},$$

inverse double Laplace yields,

$$f(y,t) = tE_\alpha(y^\alpha).$$

4.4 Conclusion

Double Laplace transform possess the same properties of single Laplace integral and the convergence condition is same as well. Introducing this technique spares from an extra effort of applying single Laplace integral twice on two different variables, which can be sometimes a cause of confusion.

Chapter 5

Adomian Decomposition Method For Solution Of Non-linear Fractional Differential Equations

Discussion of non-linear problems of differential equations, Adomian decomposition method has been developed to solve with accuracy, ease and with efficiency, a large family of linear and non-linear ordinary, partial, deterministic or stochastic differential equations [26]. The method comes out very well to physical problems since linearization , perturbation or any other restriction is not a necessary requirement and also it does not involve any kind of assumptions which may modify the problem being solved, sometimes seriously.

Khuri [27] was the first who proposed Laplace decomposition method (LDM) for approximation of solutions for a family of non-linear ordinary differential equations. E. Yosufoglu [28] in the year of 2006 worked on this method to solve Duffing equations. Further exploitation to this method was done by S. Nasser [29] for the solution of Falkner-Skan equation. Laplace decomposition method was shown to be more compatible with the flexible nature of physical problems and was applied to a bigger family of functional equations [30, 31].

Non-linear phenomenon owns a crucial part in applied mathematics and physics. Unlike classical techniques, Laplace decomposition method solves the non-linear equations easily and elegantly without transforming the equation. Laplace decomposition method provides a proficient, explicit and numerical solution with high order of accuracy, numerical calculations and avoidance of physically unrealistic assumptions [32].

Various methods have been projected to deal with non linearities such as Laplace decomposition method and Adomian decomposition method [31]. Furthermore a combination of Laplace transform method with well known homotopy perturbation method [33] and iteration method [34], to produce highly effective techniques for solving non-linear problems. Modified Laplace decomposition method is also an easier to implement method in comparison to Adomian decomposition method.

In this chapter the method of Adomian decomposition [27] is illustrated and applied to fractional differential equations in order to find the absolute and relative error.

5.1 Adomian Decomposition Method

Considering a non-linear second order initial value problem,

$$y'' + a(t)y' + b(t)y = f(y), \tag{5.1.1}$$

$$y(0) = \alpha, \qquad y'(0) = \beta, \tag{5.1.2}$$

where $f(y)$ is non-linear operator and $a(t)$ and $b(t)$ are the functions from functions space. The method initiates with the first step of applying Laplace transform to the both sides of (5.1.1),

$$\pounds\{y''\} + \pounds\{a(t)y'\} + \pounds\{b(t)y\} = \pounds\{f(y)\}. \tag{5.1.3}$$

Applying the Laplace formulas and initial conditions,

$$s^2\pounds\{y\} - sy(0) - y'(0) + \pounds\{a(t)y'\} + \pounds\{b(t)y\} = \pounds\{f(y)\}, \tag{5.1.4}$$

$$s^2\pounds\{y\} = \beta + s\alpha - \pounds\{a(t)y'\} - \pounds\{b(t)y\} + \pounds\{f(y)\}, \tag{5.1.5}$$

or

$$\pounds\{y\} = \frac{\beta}{s^2} + \frac{\alpha}{s} - \frac{1}{s^2}\pounds\{a(t)y'\} - \frac{1}{s^2}\pounds\{b(t)y\} + \frac{1}{s^2}\pounds\{f(y)\}. \tag{5.1.6}$$

The Laplace transform decomposition method moves further to another step which is, symbolizes the solution as an infinite series of the form,

$$y = \sum_{k=0}^{\infty} y_k, \tag{5.1.7}$$

where the terms y_k are recursively computed.

The non-linear factor $f(y)$ goes through the process of decomposition as follows,

$$f(y) = \sum_{k=0}^{\infty} A_k, \tag{5.1.8}$$

where $A_k = A_k(y_0, y_1, \ldots, y_k)$ are called *Adomian* polynomials. The first few polynomials are

$$\begin{aligned}
A_0 &= f(y_0), \\
A_1 &= y_1 f'(y_0), \\
A_2 &= y_2 f'(y_0) + \frac{1}{2!} y_1^2 f''(y_0), \\
A_3 &= y_3 f'(y_0) + y_1 y_2 f''(y_0) + \frac{1}{3!} y_1^3 f'''(y_0).
\end{aligned} \tag{5.1.9}$$

Substituting (5.1.7) and (5.1.8) in (5.1.6) results in,

$$\pounds\left\{\sum_{k=0}^{\infty} y_k\right\} = \frac{\alpha}{s} + \frac{\beta}{s^2} - \frac{1}{s^2}\pounds\left\{a(t)\sum_{k=0}^{\infty} y_k'\right\} - \frac{1}{s^2}\pounds\left\{b(t)\sum_{k=0}^{\infty} y_k\right\} + \frac{1}{s^2}\pounds\left\{\sum_{k=0}^{\infty} A_k\right\}. \tag{5.1.10}$$

Using the linearity property of Laplace transform,

$$\sum_{k=0}^{\infty}\pounds\{y_k\} = \frac{\alpha}{s} + \frac{\beta}{s^2} - \frac{1}{s^2}\sum_{k=0}^{\infty}\pounds\{a(t)y_k'\} - \frac{1}{s^2}\sum_{k=0}^{\infty}\pounds\{b(t)y_k\} + \frac{1}{s^2}\sum_{k=0}^{\infty}\pounds\{A_k\}. \tag{5.1.11}$$

Matching both sides of (5.1.11) yields the following iterative algorithm,

$$\pounds\{y_0\} = \frac{\alpha}{s} + \frac{\beta}{s^2}, \tag{5.1.12}$$

$$\pounds\{y_1\} = -\frac{1}{s^2}\pounds\{a(t)y_0'\} - \frac{1}{s^2}\pounds\{b(t)y_0\} + \frac{1}{s^2}\pounds\{A_0\}, \tag{5.1.13}$$

$$\pounds\{y_2\} = -\frac{1}{s^2}\pounds\{a(t)y_1'\} - \frac{1}{s^2}\pounds\{b(t)y_1\} + \frac{1}{s^2}\pounds\{A_1\}. \tag{5.1.14}$$

In general,

$$\pounds\{y_{k+1}\} = -\frac{1}{s^2}\pounds\{a(t)y_k'\} - \frac{1}{s^2}\pounds\{b(t)y_k\} + \frac{1}{s^2}\pounds\{A_k\}. \tag{5.1.15}$$

Applying the inverse Laplace transform to (5.1.12) yields,

$$y_0 = \beta t + \alpha, \tag{5.1.16}$$

substituting the value of y_0 into (5.1.13) gives,

$$\pounds\{y_1\} = -\frac{1}{s^2}\pounds\{\beta a(t)\} - \frac{1}{s^2}\pounds\{b(t)(\alpha + \beta t)\} + \frac{1}{s^2}\pounds\{A_0\}. \tag{5.1.17}$$

Evaluating the Laplace transform of the quantities on the right-hand side of (5.1.17) then application of inverse Laplace transform, we get the value of y_1. The other terms can also be recursively obtained by using (5.1.15).

5.2 Application to Fractional Differential Equations

Duffing Equation: George Duffing is the person who first introduced "Duffing Equation" , which is non-linear, second order differential equation which is commonly used in damped and driven oscillators. Duffing equation is of the form,

$$y^{''} + \delta y^{'} + ay + by^3 = \gamma \cos \omega_0 t$$

and it describes the motion of damped oscillator with more complicated potential then in simple harmonic motion.

We consider the fractional Duffing equation when $\gamma = 0$ which means there is no driving force in the system but damping coefficient(δ) is present.

$$_tD_*^{\alpha} y(t) + 0.2y^{'}(t) + 1.2y(t) + y^3(t) = 1, \tag{5.2.1}$$

where $0 < \alpha \leq 2$ and initial conditions are,

$$y(0) = 0.3, \qquad y\prime(0) = -2.8.$$

Applying Laplace Transform on (5.2.1),

$$[s^{\alpha}\pounds\{y(t)\} - s^{\alpha-1}y(0) - s^{\alpha-2}y^{'}(0)] = -0.2\pounds\{y^{'}(t)\} - 1.2\pounds\{y(t)\} - \pounds\{y^3(t)\} + \pounds\{1\}. \tag{5.2.2}$$

According to Adomian Decomposition method, suppose

$$y(t) = \sum_{i=0}^{\infty} y_i(t),$$

$$f(y) = y^3 = \sum_{i=0}^{\infty} A_i.$$

Now

$$\pounds\{y_i\} = \frac{0.3}{s} - \frac{2.8}{s^2} - 0.2\pounds\{y^{'}(t)\} - 1.2\pounds\{y(t)\} - \pounds\{y^3(t)\} + \pounds\{1\}, \tag{5.2.3}$$

then

$$\pounds\{y_0\} = \frac{0.3}{s} - \frac{2.8}{s^2},$$

$$y_0 = -2.8t + 0.3. \tag{5.2.4}$$

Using the mathematica code for generation of Adomian Polynomials, we generated upto 15 Polynomials,

$$A_0 = 0.027 + 0.756t + 7.056t^2 - 21.925t^3,$$

$$A_1 = (0.756 + 14.112t - 65.775t^2)y_1,$$

$$A_2 = \frac{1}{2}(14.112 - 131.55t)y_1^2 + (0.756 + 14.112t - 65.775t^2)y_2,$$

$$\vdots$$

$$\begin{aligned} A_{15} &= -131.55(y_1^{\frac{3}{2}} + y_1^2 y_2 + y_1^2 y_3 + y_1^2 y_4 + y_1^2 y_5 + 1/2 y_1^2 y_6 + y_1^2 y_7 + \\ &\frac{1}{2}y_1^2 y_8 + \frac{1}{2}y_1^2 y_9 + \frac{1}{2}y_1^2 y_{10} + \frac{1}{2}y_1^2 y_{11} + \frac{1}{2}y_1^2 y_{13}) + (14.112 - 131.55t)(y_1 y_2 + y_1 y_4 + y_1 y_6 \\ &+ y_1 y_8 + y_1 y_{10} + y_1 y_{12} + y_1 y_{14}) + (0.756 + 14.112t - 65.775t^2)y_{15}. \end{aligned}$$

Summing the assumed series of solution upto fifteen we get,

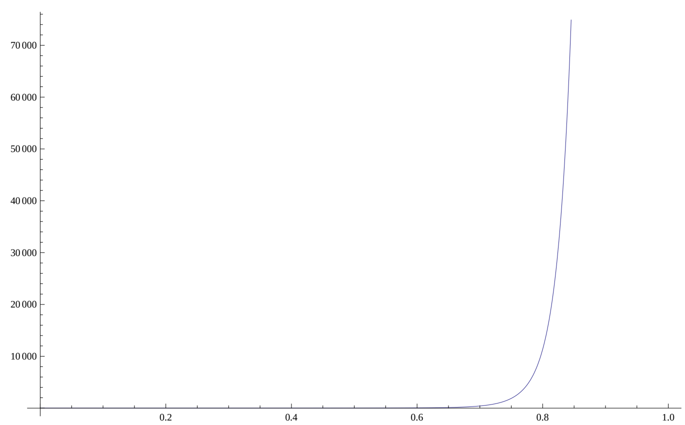

Figure 5.1: Numerical approximation of fractional Duffing equation with no driving force

The graph of the duffing equation with no driving force is maximum and high damping can be clearly seen in the solution which is some how the desired behavior but still the error rate is very high which cannot be neglected as well as can be reduced by refined series solution.

Example 5.2.2: Considering a non-linear partial differential equation with no fractional derivative,

$$\frac{\partial}{\partial t}y(x,t) = \frac{\partial^2}{\partial x^2}y(x,t) + y(x,t)\frac{\partial}{\partial x}y(x,t),$$

with initial condition,

$$y(x,0) = 1 + \frac{2}{x}.$$

Application of Adomian decomposition method upto only nine iterations, we get the Adomian polynomials of the form,

$$A_0 = f(y_0(x,t)),$$

$$A_1 = y_1(x,t)f^{(1,0)}(y_0(x,t)),$$

$$A_2 = y_2(x,t)f^{(1,0)}(y_0(x,t)) + \frac{1}{2}y_1^2(x,t)f^{(2,0)}(y_0(x,t)),$$

$$A_3 = y_3(x,t)f^{(1,0)}(y_0(x,t)) + y_1(x,t)y_2(x,t)f^{(2,0)}(y_0(x,t)) + \frac{1}{6}y_1^3(x,t)f^{(3,0)}(y_0(x,t)),$$

$$\begin{aligned} A_4 = {} & y_4(x,t)f^{(1,0)}(y_0(x,t)) + \tfrac{1}{2}y_2^2(x,t)f^{(2,0)}(y_0(x,t)) + y_1(x,t)y_3(x,t)f^{(2,0)}(y_0(x,t)) \\ & + \tfrac{1}{2}y_1^2(x,t)y_2(x,t)f^{(3,0)}(y_0(x,t)) + \tfrac{1}{24}y_1^4(x,t)f^{(4,0)}(y_0(x,t)), \end{aligned}$$

$$\begin{aligned} A_5 = {} & y_5 f^{(1,0)}(y_0(x,t)) + y_2(x,t)y_3(x,t)f^{(2,0)}(y_0(x,t)) + y_1(x,t)y_4(x,t)f^{(2,0)}(y_0(x,t)) \\ & + \tfrac{1}{2}y_1(x,t)y_2^2(x,t)f^{(3,0)}(y_0(x,t)) + \tfrac{1}{2}y_1^2(x,t)y_3(x,t)f^{(3,0)}(y_0(x,t)) \\ & + \tfrac{1}{6}y_1^3(x,t)y_2(x,t)f^{(4,0)}(y_0(x,t)) + \tfrac{1}{120}y_1^5(x,t)f^{(5,0)}(y_0(x,t)), \end{aligned}$$

$$\vdots$$

$$\begin{aligned} A_9 = {} & y_9(x,t)f^{(1,0)}(y_0(x,t)) + y_4(x,t)y_5(x,t)f^{(2,0)}(y_0(x,t)) + y_3(x,t)y_6(x,t)f^{(2,0)}(y_0(x,t)) \\ & + y_2(x,t)y_7(x,t)f^{(2,0)}(y_0(x,t)) + y_1(x,t)y_8(x,t)f^{(2,0)}(y_0(x,t)) + \tfrac{1}{6}y_3^3(x,t)f^{(3,0)}(y_0(x,t)) \\ & + y_2(x,t)y_3(x,t)y_4(x,t)f^{(3,0)}(y_0(x,t)) + \tfrac{1}{2}y_1(x,t)y_4^2(x,t)f^{(3,0)}(y_0(x,t)) \\ & + \tfrac{1}{2}y_2^2(x,t)y_5(x,t)f^{(3,0)}(y_0(x,t)) + y_1(x,t)y_3(x,t)y_5(x,t)f^{(3,0)}(y_0(x,t)) \\ & + y_1(x,t)y_2(x,t)y_6(x,t)f^{(3,0)}(y_0(x,t)) + \tfrac{1}{2}y_1^2(x,t)y_7(x,t)f^{(3,0)}(y_0(x,t)) \\ & + \tfrac{1}{6}y_2^3(x,t)y_3(x,t)f^{(4,0)}(y_0(x,t)) + \tfrac{1}{2}y_1(x,t)y_2(x,t)y_3^2(x,t)f^{(4,0)}(y_0(x,t)) \\ & + \tfrac{1}{2}y_1(x,t)y_2^2(x,t)y_4(x,t)f^{(4,0)}(y_0(x,t)) + \tfrac{1}{2}y_1^2(x,t)y_3(x,t)y_4(x,t)f^{(4,0)}(y_0(x,t)) \\ & + \tfrac{1}{24}y_1(x,t)y_2^4(x,t)f^{(5,0)}(y_0(x,t)) + \tfrac{1}{4}y_1^2(x,t)y_2^2(x,t)y_3(x,t)f^{(5,0)}(y_0(x,t)) \\ & + \tfrac{1}{12}y_1^3(x,t)y_3^2(x,t)f^{(5,0)}(y_0(x,t)) + \tfrac{1}{6}y_1^3(x,t)y_2(x,t)y_4(x,t)f^{(5,0)}(y_0(x,t)) \\ & + \tfrac{1}{24}y_1^4(x,t)y_5(x,t)f^{(5,0)}(y_0(x,t)) + \tfrac{1}{36}y_1^3(x,t)y_2^3(x,t)f^{(6,0)}(y_0(x,t)) \\ & + \tfrac{1}{24}y_1^4(x,t)y_2(x,t)y_3(x,t)f^{(6,0)}(y_0(x,t)) + \tfrac{1}{120}y_1^5(x,t)y_4(x,t)f^{(6,0)}(y_0(x,t)) \\ & + \tfrac{1}{240}y_1^5(x,t)y_2^2(x,t)f^{(7,0)}(y_0(x,t)) + \tfrac{1}{720}y_1^6(x,t)y_3(x,t)f^{(7,0)}(y_0(x,t)) \\ & + \tfrac{1}{5040}y_1^7(x,t)y_2(x,t)f^{(8,0)}(y_0(x,t)) + \tfrac{1}{362880}y_1^9(x,t)f^{(9,0)}(y_0(x,t)). \end{aligned}$$

The approximated series is of the form,

$$
\begin{aligned}
y(x,t) \simeq (\tfrac{1}{x^{19}})(2+x)(&2489344t^9 + 4978688t^9x + 366080t^8x^2 + 4100096t^9x^2 \\
&+ 640640t^8x^3 + 1793792t^9x^3 + 54912t^7x^4 + 448448t^8x^4 + 448448t^9x^4 \\
&+ 82368t^7x^5 + 160160t^8x^5 + 64064t^9x^5 + 8448t^6x^6 + 47520t^7x^6 \\
&+ 30800t^8x^6 + 4928t^9x^6 + 10560t^6x^7 + 13200t^7x^7 + 3080t^8x^7 \\
&+ 176t^9x^7 + 1344t^5x^8 + 4800t^6x^8 + 1800t^7x^8 + 140t^8x^8 \\
&+ 2t^9x^8 + 1344t^5x^9 + 960t^6x^9 + 108t^7x^9 + 2t^8x^9 \\
&+ 224t^4x^{10} + 448t^5x^{10} + 80t^6x^{10} + 2t^7x^{10} + 168t^4x^{11} \\
&+ 56t^5x^{11} + 2t^6x^{11} + 40t^3x^{12} + 36t^4x^{12} + 2t^5x^{12} \\
&+ 20t^3x^{13} + 2t^4x^{13} + 8t^2x^{14} + 2t^3x^{14} + 2t^2x^{15} \\
&+ 2tx^{16} + x^{18}).
\end{aligned}
$$

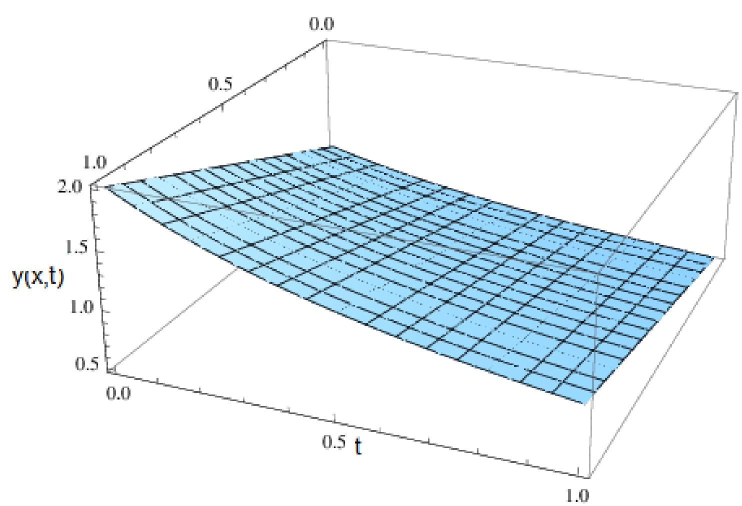

Figure 5.2: Numerical approximation of partial differential equation in example 5.2.2

5.3 Conclusion

The technique is very useful to solve non-linear differential equations, yet mathematica software is not very supportive for fetching the updated values, using them for calculation of new values and repeating the process again and again. That is why solutions are approximated for a small series.

Further Workable Content

I have worked on few techniques for finding the numerical inversion of the Laplace transform. Theses techniques are yet effective but there is always a chance of improvement.

Third chapter of this dissertations is about Weeks method, application of this method is found very useful but when we desired for applying this method to n-dimensional Laplace transform we realized that this technique lacks numerical coding for dimension more then one. In future, we will put our efforts on developing an algorithm for this in matlab software.

Adomian decomposition method is discussed for the solution of non-linear fractional differential equations and applied numerically using mathematica scheme. If the algorithm is applied using matlab the results would be more satisfying.

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