The Study of Lattice Structure of Bipartite Stable Matchings

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Dedication

This thesis is dedicated to my respectable Supervisor, Teachers, Parents and Siblings for their endless devotion, support and encouragement.

Acknowledgement

All praises are for Almighty Allah, the most gracious and the most merciful, Who created this entire universe. I am highly grateful to Almighty Allah for showering His countless blessings upon me and giving me the ability and strength to complete this thesis successfully and blessing me more than I deserve. I am greatly indebted to my supervisor Dr. Quanita Kiran, for continuous support during my thesis. Without her guidance and expertise this work would not have been possible. My understanding and appreciation of the subject are entirely due to her efforts and kind attitude. I would like to express my deep gratitude to Dr. Yasir Ali for his help, guidance and contribution towards completion of my work. I would also like to thank Head of Department, GEC members and entire faculty of Mathematics Department who have assisted me during my time here.

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Bakhtawar Shaukat

Abstract

Theory of fixed points is of significant importance because it provides us the analogue results to the existence of solution as a fixed point of certain set function. The aim of this study is to investigate the lattice structure of stable matchings by using fixed point approach. Set of stable matchings behave as a fixed point under an appropriate function. The research incorporates the review of various literature mainly include the study of lattice structure of one to one and many to one bipartite stable matchings. Earlier, the lattice structures are studied with fixed preference lists with no price externality(price negotiation). In this present thesis a set of pairwise stable outcomes is obtained in two sided hybrid matching market with price externality. In this market the valuation of agents depends upon money and externality arises when they negotiate for the price. The most important feature is to devise an algorithm that characterize the stable matchings as fixed points of an increasing function T. Also the termination and correctness of this fixed point algorithm is proved. Furthermore, the lattice structure of the set of stable outcomes is studied as an application of Tarski's fixed point theorem.

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Chapter 1

Introduction

In a broad range of mathematical problems of existence of a solution is equivalently convertible to the problem of existence of a fixed point. The existence of a fixed point is therefore of prime importance in different areas of mathematics and other sciences. The results of fixed points provide conditions under which a problem may have solutions. There is an interconnection between fixed points methods and theory of stable matching. Matching is considered as one of the significant functions of Economic markets. Who gets which jobs, who marries whom, which school places, these assist in shaping careers and lives. Stable matching behave as a fixed point of certain set function. Establishing the lattice structure of set of stable matching, is an important result in the matching literature.

This present work is basically a study of an application of fixed point theory. In the literature, various work is available on the lattice structure of stable matchings by using Tarski's fixed point theorem. But their approach does not involve price negotiation. The current research aims at investigating the interlinks between fixed point theorem and theory of stable matchings. Also the study of lattice structure of stable matchings that involves price externality by using Tarski's fixed point theorem which no one did before. This research is conducted by the review of various literature.

In this thesis, the novel hybrid model of Ali and Farooq [2] and Echenique and Oviedo [6] is designed for finding a stable matchings. The obtained stable matchings behave as a fixed points of a certain set function. The lattice structure of these stable matchings is also studied. Moreover, a detailed review of the Adachi [1] and Echenique and Oviedo [6] on the lattice structure of stable matchings is also given.

Chapter 2 covers few basic concepts related to the field of study. It includes the study related to concepts of fixed point theory, bipartite matching theory and lattice theory. It also summarizes the main findings of certain studies conducted in this field of research. Chapter 3 is devoted to the detail study of one to one stable matchings of Gale and Shapley's marriage problem in an alternative way. The stable matching's lattice structure is proved as a direct implication of Tarski's fixed point theorem. Chapter 4 is a detailed review of many to one stable matchings. The devised T-algorithm characterizes the stable matchings. Stable matchings behave as a fixed point under the set function T. The stable matching's lattice structure is also given at the end. Chapter 5 includes the formation of novel hybrid model. An algorithm is devised that characterizes the set of stable matchings. It also comprises the lattice structure of stable matchings. Chapter 6 incorporate the conclusion.

Chapter 2

Preliminaries and history

In this chapter, some prerequisite ideas and concepts are discussed that reader should familiar with. It mainly includes study related to fixed point theory, bipartite matching theory and lattice theory.

Chapter 2 also provides some background of the current study. Moreover, Tarski's fixed point theorem is stated here, which will be use for the formation of lattice structure of stable matchings.

2.1 Functions and fixed points

Fixed point theory is considered as an interdisciplinary subject which can be applied in different disciplines of mathematics and mathematical sciences like game theory, optimization theory, mathematical economics, variational inequalities and approximation theory. Fixed point theory deals itself with a very simple and essential mathematical setting. A point is said to be a fixed point when it remains invariant, irrespective of the type of transformation it undergoes. Under appropriate conditions, the fixed point theorem states the existence of fixed points.

Definition 2.1.1. Consider the two sets S and B. R^* is a binary relation from S to B is defined as subset of $S \times B$. For an order pair (i, j) in $S \times B$, i is related to j by R^*, represent as:

$$iR^*j \Leftrightarrow (i, j) \in R^*.$$

And if i is not related to j written as:

$$i \not R^* j \Leftrightarrow (i, j) \notin R^*.$$

Relation consists of all those ordered pairs whose elements are related by given condition.

Definition 2.1.2. A function $T : S \longrightarrow B$ is defined as a relation R^* from set S to B that satisfies the following properties:

(i) every element of S is the first element of an ordered pair of T,

(ii) no two distinct ordered pairs in T have the same first element.

If T is a function from S to B, we write it as:

$$j = T(i) \Leftrightarrow (i, j) \in T.$$

Functions are basically used to describe the change in one variable as a result of change in other variable. They can be percieved as a rule which operates on input and produces an output.

Example 2.1.1. Consider $S = \{1, 2\}$ and $B = \{1, 2, 3\}$, and define a binary relation R^* from S to B as provided below:

$$(i, j) \in S \times B \Leftrightarrow i - j \text{ is even.}$$

The Cartesian product of S and B consists of ordered pairs:

$$S \times B = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3)\}.$$

and the ordered pairs which are in R^* stated as:

$$R^* = \{(1, 1), (1, 3), (2, 2)\}.$$

This given relation R^* is not a function since it is not satisfying the property (ii) of a function.

Example 2.1.2. Let $S = \{1, 2, 3\}$ and $B = \{1, 3, 5\}$, define a relation R^* from S to B as follows:

$$\forall (i, j) \in S \times B, \quad (i, j) \in R^* \Leftrightarrow i < j.$$

Here, a relation $R^* = \{(1, 3), (1, 5), (2, 3), (2, 5), (3, 5)\}$ from S (Domain) to B (Co-domain) in the form of Fig. 2.1.

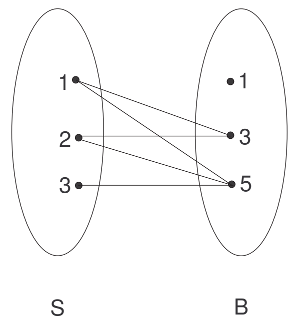

Figure 2.1: Relation

This given relation R^* is not a function since it is not satisfying the property (ii) of a function.

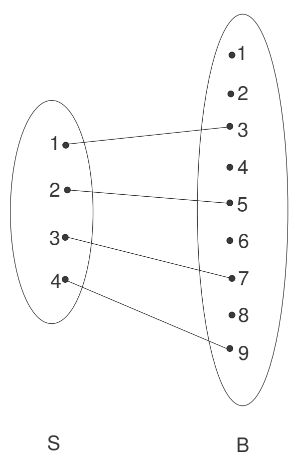

Figure 2.2: Function

Example 2.1.3. Consider $S = \{1, 2, 3, 4\}$ and $B = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Define a relation R^* as:

$$\forall\ (i,j) \in S \times B, \qquad (i,j) \in R^* \Leftrightarrow j = 2i + 1.$$

Here, the relation $R^* = \{(1,3),(2,5),(3,7),(4,9)\}$ is a function because it satisfies both properties (i) and (ii) of function. Graphical illustration is given in Fig. 2.2.

Example 2.1.4. Consider $S = \{2,4,6\}$ and $B = \{1,3,5\}$. Define a relation R^* as:

$$\forall (i,j) \in S \times B, \quad (i,j) \in R^* \Leftrightarrow j = i + 1.$$

Here the relation $R^* = \{(2,3),(4,5)\}$ is a not a function as it does not satisfy property (i) of function. Graphically present it as Fig 2.3.

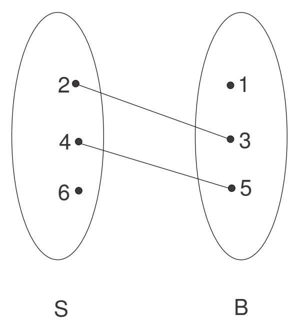

Figure 2.3: Not a function

Definition 2.1.3. The floor function $f(q)$ is defined for real numbers as the largest integer less than or equals to q. The notation $\lfloor q \rfloor$ is used for $f(q)$.

Example 2.1.5. $\lfloor 2.5 \rfloor = 2, \quad \lfloor -2.8 \rfloor = -3.$

Definition 2.1.4. The ceiling function $c(q)$ is defined for real numbers as the smallest integer greater than or equals to q. The notation $\lceil q \rceil$ is used for $c(q)$.

Example 2.1.6. $\lceil 1.7 \rceil = 2, \quad \lceil -2.7 \rceil = -2.$

Definition 2.1.5. (Epp [17]) A relation R^* on a set S is said to be a partial order if R^* is:

1. $i\, R^*\, i$ $\qquad$: $\forall\, i \in S$ (reflexive).

2. $i_1 \, R^* \, i_2 \wedge i_2 \, R^* \, i_1 \Rightarrow i_1 = i_2 \quad : \forall i_1, i_2 \in S$ (anti-symmetric).

3. $i_1 \, R^* \, i_2 \wedge i_2 \, R^* \, i_3 \Rightarrow i_1 \, R^* \, i_3 \quad : \forall \, i_1, i_2, i_3 \in S$ (transitive).

A set together with the partial order R^* is called a poset or a partially ordered set. Represent this poset by (S, R^*).

Example 2.1.7. Let $S = \{1, 2, 3, 4, 12\}$ be the set. Consider the relation R^* of divisibility on S as

$$i_1 \, R^* \, i_2 \Leftrightarrow i_1 | i_2 \quad \forall \, i_1, \, i_2 \in S. \tag{2.1}$$

R^* is a partial order on S. Fig. 2.4 is the Hasse Diagram(two-dimensional presentation of a directed acyclic graph all of whose edges are drawn without arrowheads but which are supposed to be directed upwards) of this defined relation R^*.

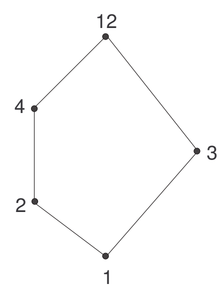

Figure 2.4: Partial order

Example 2.1.8. If we define a relation R^* on $S = \{1, 2, 3, 4, 12\}$ such that

$$i_1 \, R^* \, i_2 \Leftrightarrow i_1 < i_2 \quad \forall \, i_1, \, i_2 \in S. \tag{2.2}$$

Then this relation $<$ is not a partial order. Since $<$ is not reflexive.

Definition 2.1.6. Let (S, R^*) be the poset. A function

$$T : S \longrightarrow S$$

is said to be an increasing if for all $i_1, i_2 \in S$, we have

$$i_1 R^* i_2 \Rightarrow T(i_1) R^* T(i_2).$$

Example 2.1.9. Let $S = \{-1, 0, 1, 5, 8\}$ be the poset under the less than equals to relation as

$$i_1 \, R^* \, i_2 \Leftrightarrow i_1 \leq i_2 \quad \forall \, i_1, \, i_2 \in S. \tag{2.3}$$

. Consider a function

$$T : S \rightarrow S$$

such that

$$T(i) = i.$$

T is an increasing function on S. Graphically it can be viewed as Fig. 2.5:

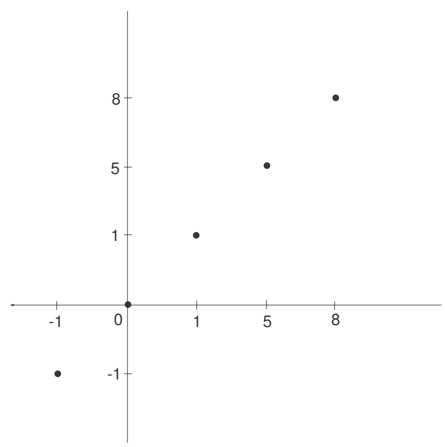

Figure 2.5: Increasing function under less than equals to relation.

Definition 2.1.7. If S is a set and a mapping

$$T : S \longrightarrow S$$

is a function then $i \in S$ is called fixed point of T if

$$T(i) = i.$$

Example 2.1.10. Let $S = \{-1, 0, 1, 5, 8\}$. Consider the function $T : S \rightarrow S$ defined as $T(i) = i$. Then this function has all elements of S as a fixed points. Graphically, presented in Fig. 2.6:

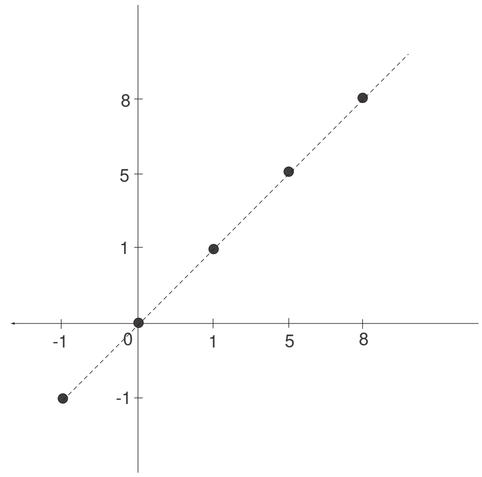

Figure 2.6: Fixed points of S.

Probably, Feder [10] and Subramanian [22] were the first one's who indicated a relationship among fixed points and stable matchings. Next section involves the relationship between fixed points and stable matchings.

2.2 Bipartite matchings

Since many years matching theory has been widely studied by economists, game theorist and mathematicians due to its extensive applications in many related fields. Decision-making in today's world requires coordination of a group of agents, which may comprise cyber, physical or human elements. These agents naturally engage in building an opinion matching on certain resources of interest that may include attitudes, prices or predictions about macroeconomic variables. In social networks, interacting agents can influence each other and gradually form an opinion matching. A number of physical models have been developed to explore human opinion propagation. Hence the matching theory is the study of resource allocation among the sets of agents with respect to the preferences of these agents, so that the allocation has important implications for their well-being.

The present section gives an introduction to the primary fundamentals of matching theory. Basically, in this thesis, our main results comprise lattice structure of one to one bipartite stable matchings. But the literature review also include lattice structure of many to one stable matchings.

Definition 2.2.1. (Echenique and Oviedo [6]). Let S and B be any two non-empty finite sets. A pair of functions $v = (v_S, v_B)$ is known as *pre-matching* if the functions $v_S : S \to S \cup B$ and $v_B : B \to S \cup B$ are such that:

1. for all $i \in S$, $v_i \in B \cup \{i\}$ where $v_S(i) := v_i$.
2. for all $j \in B$, $v_j \in S \cup \{j\}$ where $v_B(j) := v_j$.

Example 2.2.1. If $S = \{i_1, i_2, i_3\}$ and $B = \{j_1, j_2, j_3\}$ be the two disjoint and finite sets. Here, v be an arbitrary pre-matching shown in Fig. 2.7 such that $v_S = \{(i_1, j_1), (i_2, j_3), (i_3, j_2)\}$ and $v_B = \{(j_1, i_3), (j_2, i_1), (j_3, i_3)\}$.

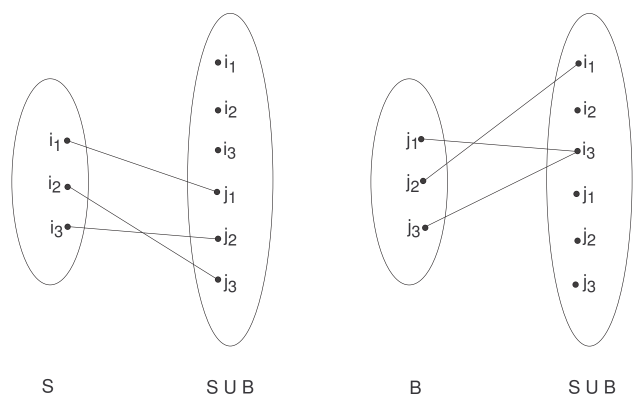

Figure 2.7: Pre-matching

Definition 2.2.2. A matching $\mu : S \cup B \longrightarrow S \cup B$ is one to one correspondence of order two (An order two matching is the one for which $\mu^2(y) = y$) such that if $\mu(i) \neq i$ then $\mu(i) \in B$ and if $\mu(j) \neq j$ then $\mu(j) \in S$.

Denote $\mu(i) = \mu_i$ and $\mu(j) = \mu_j$. Matchings and pre-matchings v have a close connection with each other. One can always define a pre-matching v from matching μ. But pre-matching v may not be a matching μ. Relation between them is present next.

Consider S and B be the two finite sets.

- μ defines v if for a given μ, a function $v = (v_S, v_B)$ define by

$$v_i := \mu_i \quad \text{and} \quad v_j := \mu_j \quad \forall i \in S \,\text{and}\, j \in B.$$

- v induces μ if for a given v, a function μ define by

$$\mu_i := v_i \quad \text{and} \quad \mu_j := v_j.$$

  is a matching.

- μ and v are equivalent if matching μ defines pre-matching v and v induces μ.

Definition 2.2.3. (Echenique and Oviedo [6]). A pre-matching $v = (v_S, v_B)$ is said to be a one to one *matching* μ if and only if v is one to one and self invertible ($v_j = i$ if and only if $j = v_i$).

Example 2.2.2. Let $S = \{i_1, i_2, i_3\}$ and $B = \{j_1, j_2, j_3\}$ be the two disjoint and finite sets. $\hat{\mu}_S = \{(i_1, j_1), (i_2, j_3), (i_3, j_2)\}$ and $\hat{\mu}_B = \{(j_1, i_1), (j_2, i_3), (j_3, i_2)\}$. be a matching defined in Fig. 2.8.

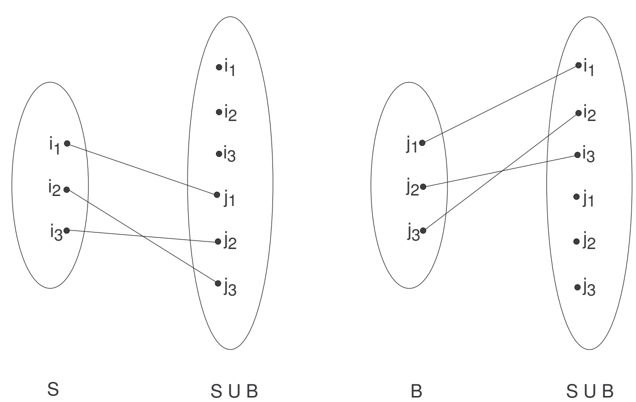

Figure 2.8: Matching

An alternative way of presenting matching $\hat{\mu} = \{(i_1, j_1), (i_2, j_3), (i_3, j_2)\}$ is given in Fig. 2.9.

Consider a pre-matching v given in Fig. 2.7, it is not a matching because a pre-matching v such that $v_{i_1} = j_1$, $v_{i_2} = j_3$, $v_{i_3} = j_2$, $v_{j_1} = i_3$, $v_{j_2} = i_1$, $v_{j_3} = i_3$ does not

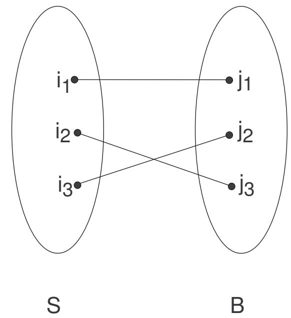

Figure 2.9: Alternative presentation of matching

induce a matching as $v_{i_1} = j_1$ but $v_{j_1} = i_3 \neq i_1$. While, a matching given in Fig. 2.8 defines a pre-matching such that $v_{i_1} = j_1$, $v_{i_2} = j_3$, $v_{i_3} = j_2$, $v_{j_1} = i_1$, $v_{j_2} = i_3$, $v_{j_3} = i_2$.

Definition 2.2.4. (Echenique and Oviedo [7]). A strict preference relation P on the set $S \cup B$ is a complete, anti-symmetric, and transitive binary relation on $S \cup B$. We denote by R the weak preference relation associated to P; so xRy if and only if $x = y$ or xPy.

Definition 2.2.5. *A matching μ is said to be stable if it is individually rational and there is no blocking pair in it.*

Individually rational means an individual (agent) is preferring its partner at least as much as to remain isolated (single). Whereas, no blocking pair means there is no such pair which strictly prefer each other over their current matched partners under μ.

Example 2.2.3. Let $S = \{i_1, i_2, i_3\}$ and $B = \{j_1, j_2, j_3\}$ be the two disjoint and finite sets of Men and women respectively, having the preferences given in the following Tables:

$P(i_1)$	j_2	j_1	j_3	i_1
$P(i_2)$	j_1	j_3	j_2	i_2
$P(i_3)$	j_1	j_2	j_3	i_3

Table 2.1: Set S preference list

$P(j_1)$	i_1	i_3	i_2	j_1
$P(j_2)$	i_3	i_1	i_2	j_2
$P(j_3)$	i_1	i_3	i_2	j_3

Table 2.2: Set B preference list

Matching $\mu = \{(i_1, j_2), (i_2, j_3), (i_3, j_1)\}$ given in Fig. 2.10 is a stable matching.

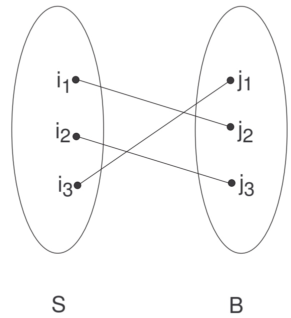

Figure 2.10: Stable matching

While, matching μ given in Fig. 2.11 is not a stable matching. Here, the pairs

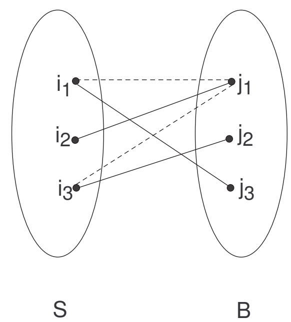

Figure 2.11: Not a stable matching

(i_1, j_1) and (i_3, j_1) blocks μ.

In the present work we deal with bipartite model and our work based upon the application on Tarski's fixed point theorem. The study of bipartite models originated with the work of David Gale and Lloyd Shapley. In [14] they solved the stable marriage problem for two equal and finite sets of men and women who have strict preferences over each other. Set of matchings among different men and women is a marriage scheme in this model. Such scheme is said to be stable if there does not exist a pair which is either preferring each other over their current partners or unmatched. A marriage scheme is unstable if it is not stable. Gale and Shapley [14] not only proposed the existence of a stable marriage scheme for any preference ranking of n men and n women, but they also devised a finite procedure, known as deferred acceptance algorithm(DAA)

to obtain that scheme. Shapley and Shubik [20] gave another standard model named as one-to-one buyer seller model, characterized as an assignment game that clarifies the role of money. They showed the formation of non-empty complete lattice structure of stable matchings under this assignment game. In literature several studies conducted on the adjuncts and variations of Gale and Shapley's marriage model and Shapley's and Shubik's assignment game.

Indivisible goods have been extensively studied with respect to the mathematical economics markets. A Gross Substitute (GS) condition was developed by Kelso and Crawford [16] through a two-sided matching model with money. With the help of this GS condition they also showed the existence of the stable matching. Above mentioned two-sided matching model is the combination of marriage model by Gale and Shapley [14] and assignment game by Shapley and Shubik [20]. Recently, Baroon et. al. [5] studied peer effects and stability in matching markets.

Ali and Farooq [2] has observed the presence of pairwise one to one stable matching in two sided market of sellers and buyers with externalities. In their model each seller possesses at most one indivisible good and each buyer possesses a finite amount of money which is an integer variable. Their four step algorithm assigns sellers the most optimal partners and buyers has to accept their best from those sellers who proposed them. The main feature of their algorithm is that the agents are flexible i.e; if the buyers have more than one choice and they rejected some sellers then the rejected sellers negotiate (modify) price to attract their favorite buyers. The process continues until no rejections left from the buyer's side.

2.3 Lattices

Lattice theory provides an elementary account of a noteworthy branch of contemporary mathematics concerning lattice theory. A lattice named as an abstract structure studied in the mathematical sub disciplines of abstract algebra and order theory.

Definition 2.3.1. Let (S, R^*) is a poset, the elements i_1 and i_2 of S are comparable if

$$i_1 \, R^* \, i_2 \quad \text{or} \quad i_2 \, R^* \, i_1.$$

Definition 2.3.2. Let (S, R^*) is a poset, the elements i_1 and i_2 of S are incomparable if

$$i_1 \not R^* \; i_2 \quad \text{and} \quad i_2 \not R^* \; i_1.$$

In poset, every pair of element need not to be comparable.

Definition 2.3.3. Let S be a poset, if every pair of elements in S is comparable, then S is called a linearly ordered set (or totally ordered set) and the partial order is said to be a linear order. In this case, S forms a chain.

Example 2.3.1. Let $S = \{1, 2, 3, 4, 12\}$ be the set. Consider the relation R^* of $\leq$ on S. Every two pairs are comparable under relation R^*. Therefore, it forms a linearly ordered set. Graphical presentation can be viewed in Fig. 2.12.

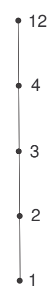

Figure 2.12: Totally ordered set.

Example 2.3.2. Consider the relation R^* of divisibility on S given in Example 2.1.7, then the elements $2 \not R^* \; 3$ and $3 \not R^* \; 2$. Similarly, $3 \not R^* \; 4$ and $4 \not R^* \; 3$. Therefore, the pairs $(2, 3)$ and $(3, 4)$ are incomparable. S is not linearly ordered set.

Definition 2.3.4. Let (S, R^*) is a partial ordered set and X be the subset of S. If there exists an element $\hat{i} \in S$ such that:

U.B

$$x \, R^* \, \hat{i} \; : \; \forall \, x \in X.$$

L.U.B

$$(x \, R^* \, y \; : \; \forall \, x \in X) \; : \; \forall \, y \in S \Rightarrow \hat{i} \, R^* \, y.$$

then $\hat{i}$ is called the least upper bound of X. It is represented by $\cup X$.

The first condition of Definition 2.3.4 states that $\cup X$ is an upper bound $(U.B)$ and the second states that it is least$(L.U.B)$.

Definition 2.3.5. Let (S, R^*) is a partial ordered set and X be the subset of S. If there exists an element $i \in S$ such that:

L.B

$$i\, R^*\, x \,:\, \forall\, x \in X.$$

G.L.B

$$(y\, R^*\, x \,:\, \forall\, x \in X) \,:\, \forall\, y \in S \Rightarrow y\, R^*\, i.$$

then i is called the greatest lower bound of X. It is represented by $\cap X$.

The first condition of Definition 2.3.5 states that $\cap X$ is an lower bound $(L.B)$ and the second states that it is greatest$(G.L.B)$.

Definition 2.3.6. Lattice is a partially ordered set (L, R^*) in which every subset $\{x, y\}$ consisting of two elements has a least upper bound (L.U.B) and a greatest lower bound(G.L.B). Represent $L.U.B(\{x, y\})$ by $x \vee y$ and say it the join of x and y. Similarly, represent $G.L.B(\{x, y\})$ by $x \wedge y$ and say it the meet of x and y.

Definition 2.3.7. L is said to be a complete lattice if every subset of lattice L has a least upper bound and greatest lower bound.

Example 2.3.3. Consider the set $S = \{2, 4, 8, 16\}$. Define a relation R^* such that

$$i_1\, R^*\, i_2 \Leftrightarrow i_1 | i_2 \quad \forall\, i_1,\, i_2 \in S.$$

Hasse diagram of the partial order on S is shown in the Fig. 2.13. For every subset X of S there exists a greatest lower bound and least upper bound in S. So S forms a complete lattice.

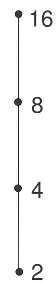

Figure 2.13: L.U.B and G.L.B of S.

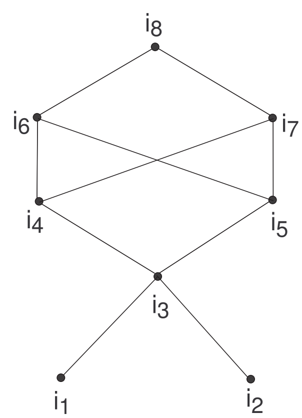

Figure 2.14: Not a lattice

Example 2.3.4. Consider a poset $S = \{i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8\}$, whose Hasse diagram is shown in Fig 2.14.

Assume two subsets of S, say $X_1 = \{i_1, i_2\}$ and $X_2 = \{i_3, i_4, i_5\}$. L.U.B of X_1 is i_3 and there is no G.L.B of it. And there is no *L.U.B* of X_2 while *G.L.B* is i_3. Therefore, S do not form a lattice.

Example 2.3.5. Let $S_1 = \{2, 4, 8, 12\}$ and $S_2 = \{2, 3, 6, 12\}$. Define a relation R^* of divisibility on S_1 and S_2. Hasse diagram is shown in Fig. 2.15.

In (a) subset $\{12, 8\}$ have no L.U.B while every subset of S_1 has a G.L.B under R^*. On the other hand, in (b) subset $\{2, 3\}$ have no G.L.B while every subset of S_2 has a L.U.B under R^*. Both S_1 and S_2 are not the lattices.

Example 2.3.6. Consider D_{20} be the set of all positive divisors of 20. Then D_{20} is a

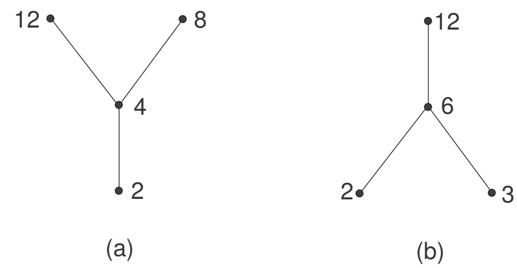

Figure 2.15: L.U.B and G.L.B of S_1 and S_2.

lattice under the divisibility relation. The Hasse diagram of D_{20} is shown in Fig. 2.16. Since, every pair in D_{20} under divisibility relation consists of least upper bound and greatest lower bound in D_{20}. Thus it forms a lattice. It is also a complete lattice since it is finite (every finite lattice is a complete lattice).

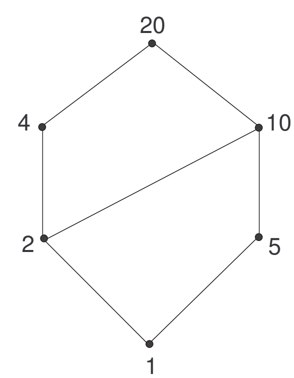

Figure 2.16: Lattice structure

Example 2.3.7. Given $P\{i_1, i_2, i_3\}$ under inclusion map $\subseteq$ forms a complete lattice. See the lattice structure in Fig. 2.17.

Following theorem is due to Tarski [23].

Theorem 2.3.1. *(Tarski [23]). Let*

1. *(A, R^*) be a complete lattice,*

2. *$f : A \to A$ be an increasing function.*

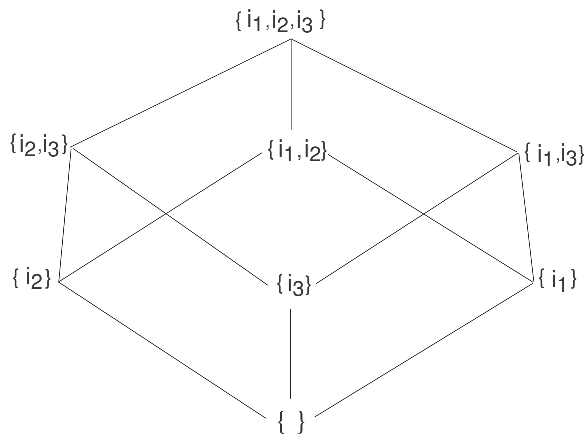

Figure 2.17: Complete lattice

3. P be the set consists of all fixed points of f.

Then, P is non-empty set and the system (P, R^*) is a complete lattice; in particular we have

$$\cup P = \cup E_x[xR^* f(x)] \in P.$$

$$\cap P = \cap E_x[f(x)R^* x] \in P.$$

Theorem 2.3.1 is an elementary lattice-theoretical fixed point theorem that holds in arbitrary complete lattices. It has a wide range of applications and extensions in the theories of simply ordered sets, topology, real functions, general set theory, as well as in Boolean algebras and matching theory.

Adachi [1] studied the stable matchings in Gale and Shapley [14] behave as fixed points of certain increasing function. By using Theorem 2.3.1 they showed the lattice structure of stable matchings. We will study their work in detail in subsequent Chapter 3. For more results on lattice structure one can also see Blair [4] and Alkan [3]. Fleiner [11, 12] studied the lattice structure of generalized stable matchings. Farooq, Flenier and Tamura [9] studied many to many matching model with contracts. They extended model of Hatfield and Milgrom [15]. The key to their results is Theorem 2.3.1. For more recent work one can see also Li [18] and Fleiner [13]. Recently Pycia and Yemnez [19] obtained matching with externalities. Their work also observed stable matchings as fixed points but their technique is not based upon Theoren 2.3.1 and hence no lattice structure of the matching is discussed. Very recently Uetake and Watanabe [24] devised an algorithm for two-sided matching model with externalities but again their work is not based on fixed point approach. Echenique and Oviedo in [6, 7] observed stable many to one and many to many matchings as fixed points of a certain function. Their characterization presents an algorithm for finding stable assignments and the lattice structure of stable matchings. Their algorithm is named as T-algorithm which is a procedure of iterating T, starting at some pre-matching v. And it stops when Tv is a matching. They also proved the obtained matching is indeed a fixed point of T. The detailed discussion on their work is present in Chapter 4. They further discussed the lattice structure of the set of fixed points of T. But the stable matchings attained in their model only deals with the fixed preference profile with no externalities.

In this work we apply Theorem 2.3.1 to study the lattice structure of two sided matching market with externalities i.e; the agents are flexible and can negotiate on price. We also present an algorithm that obtain the stable matching as a fixed point of an increasing function.

Chapter 3

Lattice structure of one to one stable matchings

This chapter is devoted to the detailed review of Adachi [1] that deals with the study of stable matchings as a fixed points of defined increasing function in the Gale-Shapley's marriage problem. The lattice property and existence of stable matchings are proved as a direct application of Theorem 2.3.1.

In Section 3.1, the brief introduction to the marriage model of Gale and Shapley and the key statement which guides to their formulation is presented. When agents have strict preferences, the alternative way of formulation that characterizes the stable matching's set and the formation of lattice structure is presented in Section 3.2.

3.1 Gale-Shapley marriage problem

Consider the two disjoint and finite sets of men and women represented as $S = \{i_1, i_2, \ldots, i_n\}$ and $B = \{j_1, j_2, \ldots j_k\}$ respectively, in the marriage market. Each agent on one side of the market has a preferences over the agent on the opposite side of the market. An agent may prefer to remain isolated/single than to get married. Therefore, man i's preference ordering $>_i$ is denoted by an ordered list over the set $W \cup \{i\}$ and similarly it is defined for the women. Assume that preferences are rational. Denote $j >_i \hat{j}$ to mean i prefers j to $\hat{j}$, and $j \geq_i \hat{j}$ to mean i likes j at least as much as $\hat{j}$. Define indifferent between two agents as $j =_i \hat{j}$, and write $j = \hat{j}$ means j is the same person as $\hat{j}$. Women j is acceptable to man i if $j \geq_i i$. An agent is said to have a strict preferences if it is not indifferent among two acceptable choices. Throughout this chapter, assume that preferences of agents are strict and matching μ is one to one.

Definition 3.1.1. A matching μ is a stable matching if the two conditions given below holds:

(IR) $\mu_i \geq_i i, \forall\, i \in S$ and $\mu_j \geq_j j, \forall\, j \in B$; i.e μ is individually rational matching.

(S) there does not exists a (i, j) such that $j >_i \mu_i$ and $i >_j \mu_j$; i.e there is no blocking pair in μ.

Lemma 3.1.1. *Suppose that the strict preference assumption holds. Let matching μ satisfies individually rational condition (**IR**). Then the matching μ satisfies stability condition (**S**) if and only if the condition given below get fulfilled:*

($\hat{S}$) *there does not exists a (i, j) such that $\{j >_i \mu_i$ and $i \geq_j \mu_j\}$ or $\{j \geq_i \mu_i$ and $i >_j \mu_j\}$.*

Example 3.1.2. Let $S = \{i_1, i_2, i_3\}$ and $B = \{j_1, j_2, j_3\}$ having the preferences:

$P(i_1)$	j_2	j_1^*	j_3	i_1
$P(i_2)$	j_1	j_3^*	j_2	i_2
$P(i_3)$	j_1	j_2^*	j_3	i_3

Table 3.1: Men preference list

$P(j_1)$	i_1^*	i_3	i_2	j_1
$P(j_2)$	i_3^*	i_1	i_2	j_2
$P(j_3)$	i_1	i_3	i_2^*	j_3

Table 3.2: Women preference list

Each agent has strict preferences over the agent of opposite side and preferring each other to remain single. For instance, man i_1 prefers j_2 the most, then j_1, and so on. Matching $\hat{\mu}$ is the women optimal stable matching among the two optimal matchings and it is denoted by *. The significant observation that guides to the formulation is that the partner of agent i_1 under $\hat{\mu}$, $\hat{\mu}_{i_1} = j_1$, is the greatest element (w.r.t i_1's preferences) between those potential partners who prefer i_1 at least as much as their partners under the $\hat{\mu}$, $\{j \in B : i_1 \geq_j \hat{\mu}_j\} \cup \{i_1\}$. For every agent this property holds. This observation indicates that when the preferences are strict, the set of stable matchings may be

represented by the solution to a set of individual maximization problems.
According to the Gale and Shapley [14] if the preferences are strict, there exists men and women optimal stable matching for every marriage problem. Further, set of stable matchings forms a complete lattice. Adachi [1] addresses the marriage problem in an alternative way through their formulation.

3.2 Formulation

This section involves an alternative formulation to address the marriage problem.

Let Υ_S and Υ_B represent the set of all such functions v_S and v_B respectively and

$$\Upsilon := \Upsilon_S \times \Upsilon_B = (\times_{i\in S}(B \cup \{i\})) \times (\times_{j\in B}(S \cup \{j\})). \tag{3.1}$$

represents the set of all pre-matchings v. This expresses Υ_S as the set of vectors in $\times_{i\in S}(B \cup \{i\})$ and of Υ_B as that in $\times_{j\in B}(S \cup \{j\})$. Adachi finds pre-matching as more suitable for their formulation than matching and have close connection among them. . This guides to the following observation, which will be helpful later.

Remark 3.2.1. *A pre-matching v induces a matching μ if and only if v is such that*

$$v_i = j \quad \textit{iff} \quad i = v_j.$$

Therefore, if v induces a matching and $v_i = j$ (or equivalently, $v_j = i$), then $v_B \circ v_S(i) = i$ and $v_S \circ v_B(j) = j$.

Proposition 3.2.2. *Suppose that the assumption of strict preferences hold. Then*

(i) *If a matching μ is stable, then the pre-matching v defined by μ solves (3.2) and (3.3).*

$$v_i = \underset{>_i}{max}\{j \in B : i \geq_j v_j\} \cup \{i\} \quad \forall i \in S. \tag{3.2}$$

$$v_j = \underset{>_j}{max}\{i \in S : j \geq_i v_i\} \cup \{j\} \quad \forall j \in B. \tag{3.3}$$

(ii) *If a pre-matching v solves (3.2) and (3.3), then v induces a matching μ, which is stable.*

Proof. **(i)**. Let μ be a stable matching and suppose the pre-matching v defined by μ. Then by the definition of stability and Lemma 3.1.1, v satisfies

$$v_i \geq_i i \quad \text{and} \quad v_j \geq_j j \quad \forall\, i \in S, \forall\, j \in B. \tag{3.4}$$

and

$$\text{does not exists } (i,j) \text{ such that } \{j >_i v_i \,\text{and}\, i \geq_j v_j\} \text{ or } \{i >_j v_j \,\text{and}\, j \geq_i v_i\}. \tag{3.5}$$

When the preferences are strict, it is instant that these two above conditions are equivalent to v being a solution to the set of equations (3.2) and (3.3).

In (3.2) maximization is taken with respect to each man i's preference ordering $>_i$ over the set $B \cup \{i\}$ under the constraint $i \geq_j v_j$. Since we have assumed the preferences of agents are strict and they are of finite numbers, the RHS of (3.2) and (3.3) is well defined and singleton for each i and each j. This proves **(i)**.
The next Lemma will be use in order to prove the part **(ii)** of proposition.

Lemma 3.2.3. *Suppose that the presumption of strict preferences hold. If a pre-matching $v = (v_S, v_B)$ solves (3.2) and (3.3), then the conditions given below are equivalent:*

(1) $j \;\; \geq_i v_i$ *and* $i \;\; \geq_j v_j$,

(2) $j = v_i$ *and* $i = v_j$,

(3) $j = v_i$,

(4) $i = v_j$.

Proof. Suppose that (1) holds. Assume $j >_i v_i$ or $i >_j v_j$. Since $v = (v_S, v_B)$ is the solution of individual maximization problems. Either case contradicts the given assumption that v solves the (3.2) and (3.3). Therefore, it must be $j =_i v_i$ and $i =_j v_j$. But, with the strict preference assumption this indicate $j = v_i$ and $i = v_j$. So, (1) $\Rightarrow$ (2).
Suppose that (2) holds. Since, $j = v_i$ and $i = v_j$. This indicate $(2 \Rightarrow 1)$, $(2 \Rightarrow 3)$, $(2 \Rightarrow 4)$ are immediate.

Suppose that (3) holds. Assume $i >_j v_j$. But with this assumption it contradicts (3.3). Also assume that $i <_j v_j$. This also contradicts the (3.2). Therefore, it must be $i =_j v_j$, which indicate $i = v_j$ under the strict preference assumption. This implies (3) $\Rightarrow$ (4) $\Rightarrow$ (2). □

(ii). Suppose that the assumption of strict preferences hold. Let a pre-matching v solves (3.2) and (3.3). By the part (3) and (4) of Lemma 3.2.3 and Remark 3.2.1, pre-matching v induces a matching μ. Also, v satisfies the condition given in (3.4) and (3.5). This means that μ satisfies the conditions **(IR)** and $(\hat{S})$, which implies the conditions **(IR)** and **(S)**. □

For a lattice structure, the set of solutions to (3.2) and (3.3) is non-empty and is a complete lattice. In order to show this, firstly define a partial ordering on Υ_S, Υ_B and Υ.

Definition 3.2.1. Suppose $v \equiv (v_S, v_B) \in \Upsilon$. Define

1. A partial ordering $\geq_S$ on Υ_S by $v_S \geq_S \hat{v}_S$ if and only if $v_i \geq_i \hat{v}_i \; \forall \; i \in S$.
2. A partial ordering $\geq_B$ on Υ_B by $v_B \geq_B \hat{v}_B$ if and only if $v_j \geq_j \hat{v}_j \; \forall \; j \in B$.
3. A partial ordering $\geqq_S$ on Υ by $v_S \geq_S \hat{v}_S$ if and only if $v_S \geq_S \hat{v}_S$ and $\hat{v}_B \geq_B v_B$.

Consider a function $T \equiv (T_1, T_2)$, where $T_1 : \Upsilon \longrightarrow \Upsilon_S$ and $T_2 : \Upsilon \longrightarrow \Upsilon_B$ define as

$$T_1(v) = \underset{>_i}{max}\{j \in B : i \geq_j v_j\} \cup \{i\} \quad \forall i \in S. \tag{3.6}$$

$$T_2(v) = \underset{>_j}{max}\{i \in S : j \geq_i v_i\} \cup \{j\} \qquad \forall j \in B. \tag{3.7}$$

Proposition 3.2.4. *The set* $\hat{\Upsilon}$ *of solutions to (3.2) and (3.3) is non-empty and* $(\hat{\Upsilon}, \geqq_S)$ *forms a complete lattice.*

Proof. To show that set of fixed points of function T has this above property and this proposition will be prove by the application of Theorem 2.3.1. Firstly, to show $(\Upsilon, \geqq_S)$ is a complete lattice. Since, $\geqq_S$ be a partial ordering on Υ. For any two pre-matchings, either one is better than other(comparable) or if un-comparable then there exists a pre-matching (by consensus property i.e pre-matching form by giving the best partners among both of them) to which they are comparable. Thus for any two elements there is supremum and infimum. Hence, set of pre-matchings forms a lattice. Since, numbers of agents are finite. So, $(\Upsilon, \geqq_S)$ is a complete lattice. Now to show $T : \Upsilon \longrightarrow \Upsilon$ is an increasing function w.r.t $\geqq_S$. Consider any two pre-matchings $v = (v_S, v_B)$ and $\hat{v} = (\hat{v}_S, \hat{v}_B)$ such that $\hat{v} \geqq_S v$ $(i.e \quad \hat{v}_S \geq_S v_S \quad and \quad \hat{v}_B \leq_B v_B)$. Then

$$\begin{aligned} T_1(\hat{v}_i) &=_i \underset{>_i}{max}\{j \in B : i \geq_j \hat{v}_j\} \cup \{i\} \\ &\geq_i \underset{>_i}{max}\{j \in B : i \geq_j v_j\} \cup \{i\} \\ &=_i T_1(v_i). \end{aligned}$$

Since, $\{j \in B : i \geq_j \hat{v}_j\} \supseteq \{j \in B : i \geq_j v_j\}$. The above inequality $\geq_i$ follows from this fact. Hence, $T_1(\hat{v}_i) \geq_i T_1 v_i$. Similarly,

$$\begin{aligned} T_2(\hat{v}_j) &=_j \underset{>_j}{max}\{i \in S : j \geq_i \hat{v}_i\} \cup \{j\} \\ &\leq_j \underset{>_j}{max}\{i \in S : j \geq_i v_i\} \cup \{j\} \\ &=_j T_2(v_j). \end{aligned}$$

Since, $\{i \in S : j \geq_i \hat{v}_i\} \subseteq \{i \in S : j \geq_i v_i\}$. The above inequality $\leq_j$ follows from this fact. Hence, $T_2(\hat{v}_j) \leq_j T_2(v_j)$. Hence, $T\hat{v} \geqq_S Tv$. Now the proof follows from the application of Theorem 2.3.1. □

With the strict preference assumption, Proposition 3.2.2 and 3.2.4 indicate that we can identify stable matchings with the set $\hat{\Upsilon}$ of the solutions to (3.2) and (3.3). In this case, we call $v \in \hat{\Upsilon}$ itself a stable matching, and $\hat{\Upsilon}$ the set of stable matchings. Let represent the greatest and smallest elements with respect to $\geqq_S$ in $\hat{\Upsilon}$ by $\bar{v} \equiv (\bar{v}_S, \bar{v}_B)$ and $\underline{v} \equiv (\underline{v}_S, \underline{v}_B)$ respectively. A man optimal matching is $\bar{v}$ and every man likes it whereas every women dislike it. A women optimal matching is $\underline{v}$ and every women likes it whereas every man dislike it. The S- and B- equilibria, can be obtain by iterative procedure: to find $\bar{v}$, set $\bar{v}^0 \equiv (\bar{v}^0_S, \bar{v}^0_B)$ such a way $\bar{v}^0_S(i) := max_{>_i} j$ such that $j \in B \cup \{i\}$ for all i and $\underline{v}^0_B(j) := j$ for all j. Define a sequence $\bar{v} \equiv (\bar{v}^l_S, \bar{v}^l_B)$ by

$T\bar{v}^{l-1} := \bar{v}^l$ for $l \geq 1$. Since the sets S and B are finite, after a finite l the sequence $\bar{v}^l$ converges to $\bar{v} := \lim \bar{v}^l$. The limit $\overline{v}$ is the S-optimal stable matching. Similarly, to find B-optimal equilibrium, define a sequence $\underline{v}^l \equiv (\underline{v}^l_S, \underline{v}^l_B)$ by $T\underline{v}^{l-1} := \underline{v}^l$ starting with $\underline{v}^0_S(i) := i$ and $\bar{v}^0_B(j) := max_{>_j} i$ such that $i \in S \cup \{i\}$. Then $\underline{v} := \lim \underline{v}^l$ is the B-optimal stable matching.

Chapter 4

Lattice structure of many to one core matchings

This chapter is about the detailed review of Echenique and Oviedo [6]. It incorporates the study of core many to one matchings. An allocation of group of workers to each firm is said to be a many to one matching. For any given matching, if the workers are unhappy with the existing employers and the firms that are unhappy with their current group of workers may re-contract in some mutually favorable way, thus destroying the proposed matching. The formalization of such kind of matching that is robust to the re-contracting is said to be a core matching. In order to attain a core matchings non-empty, the structure impose on the firm preferences is substitutable. Thus when the preferences are substitutable, a matching is in the core iff it is stable. This chapter characterizes the core as a set of fixed points of a function T. By proving T a monotone increasing function, the lattice structure is obtained by using Theorem 2.3.1.

In Section 4.1 model and definitions are presented. Section 4.2 introduce the fixed point approach to the core. T-algorithm is given in Section 4.3. Section 4.4 includes the lattice structure.

4.1 Model

Consider the two finite and disjoint sets of firms B and workers S. Each worker $i \in S$ has a strict, transitive and complete preference relation $P(i)$ over $B \cup \emptyset$ and similarly each firm $j \in B$ has a strict, transitive and complete preference relation $P(j)$ over the set of all subsets of S. Preference profile are $(n+m)$-tuples of preference relations; we symbolize them by $P^* = (P(i_1), \ldots, P(i_n); P(j_1), \ldots, P(j_m))$. For a given $P(i)$, firm preferred by i to the empty set are called acceptable. In this case worker i may prefer to remain unemployed than working with an un-acceptable firm. Similarly, for a given $P(j)$, set of workers that j prefers to the empty set are called acceptable. In this case firm j may prefer not hiring any worker than hiring an un-acceptable set of workers. Thus only the acceptable partners matter, and preference relation is the list of acceptable partners. For example,

$$P(i_k) = j_1, j_3.$$

depicts that $j_1 P(i_k) j_3 P(i_k) \emptyset$.

$$P(j_l) = \{i_1, i_3\}, \{i_2\}, \{i_1\}, \{i_3\}.$$

depicts that $\{i_1, i_3\} P(j_l) \{i_2\} P(j_l) \{i_1\} P(j_l) \{i_3\}$.

The weak preference orders associated with P is denoted by R. Therefore, $j_k R(i) j_l$ if $j_k = j_l$ or $j_k P(i) j_l$. Similarly, define for $R(j)$. The study involves the matching of wokers to the firms and firms with the group of workers.

Definition 4.1.1. Consider the preference profile. For a set $W \subseteq S$, let $C(W, P(j))$ represent firm B's most preferred subset of W according to its preference ordering $P(j)$. Say $C(W, P(j))$ the choice set of W according to $P(j)$. That is $A = C(W, P(j))$ if and only if $A \subseteq W$ and $AP(j)D$ for all $D \subseteq W$ with $A \neq D$.

Definition 4.1.2. A *matching* μ is a mapping from set $S \cup B$ into the set of all subsets of set $S \cup B$ such that for all $i \in S$ and $j \in B$

(i) if $\mu_i \neq i$ then $\mid \mu_i \mid = 1$ and $\mu_i \in B$.

(ii) $\mu_j \in 2^S$.

(iii) $\mu_i = j$ if and only if $i \in \mu(j)$.

In words, workers can work at most one firm. Whereas, firm can hire more than one worker. $P(S)$ there is denoting the power set of S. μ is a matching if it is self invertible. Denote set of all matching by M.

Example 4.1.1. Consider two finite sets $S = \{i_1, i_2, i_3, i_4\}$ and $B = \{j_1, j_2\}$ where S represent the set of workers and B represent the set of firms. The preference profile P^* is given by Table 4.1 and 4.2.

$P(i_1)$	j_1	j_2
$P(i_2)$	j_2	j_1
$P(i_3)$	j_1	j_2
$P(i_4)$	j_1	j_2

Table 4.1: Worker's S preference list

$P(j_1)$	$\{i_1, i_2\}$	$\{i_3, i_4\}$	$\{i_1, i_3\}$	$\{i_2, i_4\}$	$\{i_1\}$	$\{i_2\}$	$\{i_3\}$	$\{i_4\}$
$P(j_2)$	$\{i_1, i_2\}$	$\{i_1, i_3\}$	$\{i_2, i_4\}$	$\{i_3, i_4\}$	$\{i_1\}$	$\{i_2\}$	$\{i_3\}$	$\{i_4\}$

Table 4.2: Firm's B preference list

Consider the matching $\mu = \{(\{i_1, i_3\}, j_1), (i_4, j_2)\}$ shown in the Fig. 4.1.

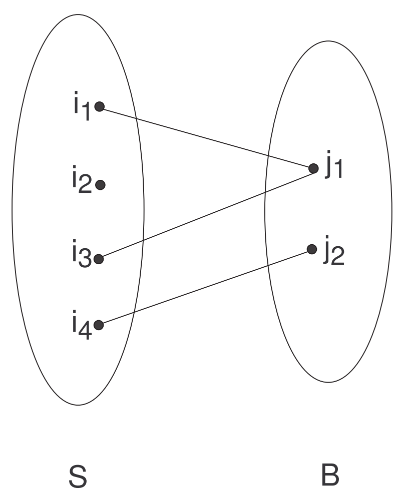

Figure 4.1: Set-wise matching

For a given preference profile P^* and a matching μ say,

(IRC) if

$$\mu_i R(i) \emptyset \qquad \forall i \in S,$$

and

$$\mu_j = C(\mu_j, P(j)) \quad \forall j \in B.$$

(BC) A worker firm pair (i, j) blocks μ if $i \notin \mu_j$

$$jP(i)\mu_i \quad \text{and} \quad i \in C(\mu_j \cup \{i\}, P(j)).$$

Matching μ is individually rational (IRC) if no agent can unilaterally improve over its assignment by μ, workers by choosing to remain un-employed and firms by firing some of its workers. Whereas, matching μ contains blocking pair (BC) if i and j are not in matching μ, worker i prefers firm j over the current match under μ and firm j wants to hire i possibly after firing some of its current workers under μ.

Definition 4.1.3. A matching μ is (pair-wise) stable if it is individually rational (IRC) and there is no worker firm pair that blocks μ (BC).

For a given preference profile P^* and a matching μ say,

(*BC*)** A pair $(F, j) \in 2^S \times B$ with $\emptyset \neq F \subseteq S$ *blocks** μ if

$$jP(i)\mu_i \qquad \forall i \in F,$$

and there is $G \subseteq \mu_j$ such that

$$[F \cup G]P(j)\mu_j.$$

In words, (F, j) *blocks** μ if all workers in F prefer j over the current partners under matching μ. And if firm j is willing to hire the workers in F, possibly after firing some of its current partners under the matching μ.

Definition 4.1.4. A matching is (set-wise) *stable** if it is individually rational (IRC) and there is no pair that *blocks** μ (BC^*).

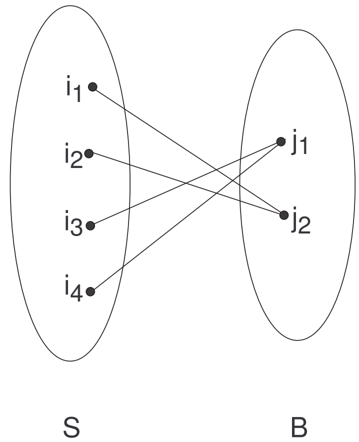

Figure 4.2: Set-wise stable matching

Example 4.1.2. Consider the preference profile given in Example 4.1.1, the matching $\mu = \{(\{i_1, i_2\}, j_2), (\{i_3, i_4\}, j_1)\}$ be the set-wise stable shown in Fig. 4.2.

While, if we consider matching μ given in Fig. 4.3 within the same preference profile given in Example 4.1.1, then it is not a set-wise stable matching. Dotted line here is the description of blocking pair in μ.

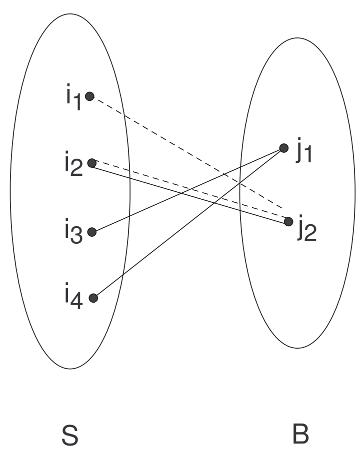

Figure 4.3: Not set-wise stable matching

Say the set of stable matchings by M_s, and set of *stable** matchings by M_{s^*}. M_{s^*} equals the core is proved in the subsequent results.

Lemma 4.1.3. *Let M_s be the pairwise stable matchings and M_{s^*} be the set-wise stable matchings. Then $M_{s^*} \subseteq M_s$ holds.*

Proof. Assume that $\mu \in M_{s^*}$, and contrary suppose that $\mu \notin M_s$. As μ is individually rational (IRC), and $\mu \notin M_s$, then there exists a pair $(i, j) \in S \times B$ such that

$$jP(i)\mu_i. \tag{4.1}$$

and

$$i \in C(\mu_j \cup \{i\}, P(j)). \tag{4.2}$$

(4.2) implies that

$$C(\mu_j \cup \{i\}, P(j))P(j)\mu_j. \tag{4.3}$$

Let consider $F = \{i\}$, and $G = C(\mu_j \cup \{i\}, P(j)) \cap \mu_j$. To show that (F, j) *blocks** μ. As $F = \{i\}$, (4.1) provides us the individually rational condition(IRC). Also

$$\begin{aligned} G &= C(\mu_j \cup \{i\}, P(j)) \cap \mu_j \\ &= C(\mu_j \cup \{i\}, P(j)) \setminus \{i\} \\ &= C(\mu_j \cup \{i\}, P(j)) \setminus F, \end{aligned}$$

Therefore, (4.3) implies that

$$[F \cup G] = C(\mu_j \cup \{i\}, P(j))P(j)\mu_j.$$

This shows that (F, j) is a blocking pair for μ. Thus $\mu \notin M_{s^*}$. This proves the result. □

Definition 4.1.5. Suppose P be a preference profile. The **core** is the set of matchings μ in which there is no $\hat{S} \subseteq S$, $\hat{B} \subseteq B$ with $\hat{S} \cup \hat{B} \neq \emptyset$ and $\hat{\mu} \in M$ such that for all $i \in \hat{S}$ and for all $j \in \hat{B}$

(i) $\hat{\mu}_i \subseteq \hat{B}$, and $\hat{\mu}_j \subseteq \hat{S}$,

(ii) $\hat{\mu}_i R(i) \mu_i$,

(iii) $\hat{\mu}_j R(j) \mu_j$,

(iv) and $\hat{\mu}_r P(r) \mu_r$ for at least one $r \in \hat{S} \cup \hat{B}$.

Denote the core by K.

Theorem 4.1.1. *Let M_{s^*} be the set-wise stable matchings and K is the core. Then $M_{s^*} = K$ holds.*

Proof. To prove this, it will show that $M_{s^*} \subseteq K$ and $K \subseteq M_{s^*}$. Consider, first $M_{s^*} \subseteq K$. Suppose $\mu \in M_{s^*}$, and let contrary assume that $\mu \notin K$. Then by definition, let $\hat{S} \subseteq S$, $\hat{B} \subseteq B$ with $\hat{S} \cup \hat{B} \neq \emptyset$, and suppose $\hat{\mu} \in M$ such that, for all $i \in \hat{S}$, and for all $j \in \hat{B}$

$$\hat{\mu}_i \subseteq \hat{B}, \quad \text{and} \quad \hat{\mu}_j \subseteq \hat{S}, \tag{4.4}$$

$$\hat{\mu}_i R(i) \mu_i, \tag{4.5}$$

$$\hat{\mu}_j R(j) \mu_j, \tag{4.6}$$

and

$$\hat{\mu}_r R(r) \mu_r \quad \text{for atleast one} \quad r \in \hat{S} \cup \hat{B}. \tag{4.7}$$

The next result will be helpful in proving the result.

Lemma 4.1.4. *There exists $j \in \hat{B}$, such that $\hat{\mu}_j P(j) \mu_j$, iff there is $i \in \hat{S}$ such that $\hat{\mu}_i P(i) \mu_i$.*

Proof. Suppose $\hat{\mu}_j P(j) \mu_j$. Since μ is individually rational, also $\hat{\mu}_j \nsubseteq \mu_j$, therefore let $\hat{i} \in \hat{\mu}_j \setminus \mu_j$. By (4.4), we have $\hat{i} \in \hat{\mu}_j \subseteq \hat{S}$; then $\hat{i} \notin \mu_j$ and (4.5) implies that

$$\hat{\mu}_{\hat{i}} P(\hat{i}) \mu_{\hat{i}}.$$

Now assume that there exists $i \in \hat{S}$ such that $\hat{\mu}_i P(i) \mu_i$. Let $\hat{j} = \hat{\mu}_i$. Then $\hat{j} \neq \mu_i$, so $i \notin \mu_{\hat{j}}$. Thus $\hat{\mu}_{\hat{j}} \neq \mu_{\hat{j}}$, and the (4.6) implies that $\hat{\mu}_{\hat{j}} P(\hat{j}) \mu_{\hat{j}}$. □

By the Lemma 4.1.4, one can assume that there exists $j \in \hat{B}$ such that $\hat{\mu}_j \neq \mu_j$. Let $F = \hat{\mu}_j \setminus \mu_j$ and $G = \hat{\mu}_j \cap \mu_j$ then

$$G \cup F = \hat{\mu}_j P(j) \mu_j. \tag{4.8}$$

Now, $F \subseteq \hat{\mu}_j \subseteq \hat{S}$, $F \cap \mu_j = \emptyset$. (4.5) implies that, for all $i \in S$,

$$j = \hat{\mu}_i R(i) \mu_i, \tag{4.9}$$

(4.8) and (4.9) contradicts $\mu \in M_{s^*}$. □

Now to prove $K \subseteq M_{s^*}$. Let $\mu \in K$, and contrary suppose that $\mu \notin M_{s^*}$. First to show μ is individually rational (IRC). Suppose that there exists $j \in B$ such that

$$\mu_j \neq C(\mu_j, P(j)).$$

Let $\hat{\mu}_j = C(\mu_j, P(j))$. $\hat{B} = \{j\} \subseteq B$, $\hat{S} = \hat{\mu}_j \subseteq S$. Then one can easily see that $\mu \notin K$, as $\hat{\mu}_i R(i) \mu_i$, for all $i \in \hat{S}$, while $\hat{\mu}_j R(j) \mu_j$. Thus

$$\mu_j = C(\mu_j, P(j)). \tag{4.10}$$

for all $j \in B$.

Now assume that there is $i \in S$ such that

$$\emptyset P(i) \mu_i \tag{4.11}$$

Let $\hat{\mu}_i = \emptyset$, $\hat{S} = \{i\}$, $\hat{B} = \emptyset$. As earlier, $\mu \notin K$ because $\hat{\mu}_i R(i) \mu_i$. Thus

$$\mu_i R(i) \emptyset. \tag{4.12}$$

for all $i \in S$. (4.10) and (4.12) shows that μ is individually rational.

Suppose there exists $(F, j) \in 2^S \times B$ with $F \neq \emptyset$, such that for all $i \in F$

$$j P(i) \mu_i.$$

and there exists $G \subseteq \mu_j$ such that

$$[G \cup F] P(j) \mu_j.$$

Denoting $\hat{\mu}_j = G \cup F$, $\hat{B} = \{j\}$ and $\hat{S} = F$, it implies that $\mu \notin K$. This is a contradiction to given statement.

4.2 Core as a set of fixed points

Now to construct a function T on the superset (Υ) of matchings M in such a way the set of fixed points of T is the core.

4.2.1 T and the core

Consider the pre-matching $v \in \Upsilon$, and denote

$$U(j, v) = \{i \in S : jR(i)\, v_i\}, \tag{4.13}$$

and

$$V(i, v) = \{j \in B : i \in C(v_j \cup \{i\}), P(j)\} \cup \{\emptyset\}. \tag{4.14}$$

The set $U(i, v)$ consists of the workers i that are willing to give up its partner v_i in exchange for the firm j. The set $V(j, v)$ consists of the firms j, that are willing to hire worker i, possibly after firing some of its workers it was assigned under v.

Now define a function $T : \Upsilon \longrightarrow \Upsilon$ such that for $v \in \Upsilon$

$$Tv_r = \begin{cases} C(U(r, v), P(r)) & \text{for} \quad r \in B, \\ \underset{P(r)}{max}\{V(r, v)\} & \text{for} \quad r \in S. \end{cases} \tag{4.15}$$

The function T has simple interpretation: Tv_i means the firm preferred by worker i among the firms that are willing to make partnership with i. Whereas Tv_j means the firm $j's$ optimal team of workers, among those workers who wants to work for j.

Denote ξ the set of fixed points of T, therefore, $\xi = \{v \in \Upsilon : v = Tv\}$.

Theorem 4.2.1. *Let ξ be the set of all fixed points under the T and M_{s^*} be the collection of all stable* matchings. Then $\xi = M_{s^*}$ holds.*

Proof. Let consider the pre-matching v such that $v \in \xi$ and first to show that for $v \in \xi \subseteq M_{s^*}$. Then it will prove that $M_{s^*} \subseteq \xi$. This will completes the proof.

Consider firstly $v = (v_S, v_B) \in \xi$. First to show v is a matching, it is divided into two parts.

1. Let $i \in v_j$, we will prove that $j = v_i$. Since, $v \in \xi$

$$i \in v_j = Tv_j = C(U(j,v), P(j)).$$

This implies $i \in U(j,v)$.

By the definition of (4.13)

$$jR(i)v_i. \tag{4.16}$$

Now, $v_j \cup \{i\} = v_j$ and $v \in \xi$, imply that

$$v_j = Tv_j = C(U(j,v), P(j)). \tag{4.17}$$

Therefore,

$$\begin{aligned} C(v_j, P(j)) &\overset{(i)}{=} C(C(U(j,v), P(j)), P(j)) \\ &\overset{(ii)}{=} C(U(j,v), P(j)) \\ &\overset{(iii)}{=} v_j. \end{aligned}$$

Equality (i) and (iii) follows from the (4.17). Equality (ii) is the property of choice sets: $C(C(W, P(j)), P(j)) = C(W, P(j))$. Therefore, we have that

$$v_j = C(v_j, P(j)) \tag{4.18}$$

Now for $i \in v_j$ implies that $C(v_j, P(j)) = C(v_j \cup \{i\}, P(j))$. So the (4.18) implies that $j \in V(i,v)$. But

$$v_i = Tv_i = \underset{P(i)}{max}\{V(i,v)\}.$$

So

$$v_i R(i) j. \tag{4.19}$$

(4.16) and (4.19) and by the anti-symmetry property of preference relations imply that, $j = v_i$.

2. Let $j = v_i$, we will prove that $i \in v_j$. Since $j = v_i$ implies that

$$i \in U(j,v). \tag{4.20}$$

Secondly, because of $v \in \xi$, we have

$$j = v_i = Tv_i = \underset{P(i)}{max}\{V(i,v)\}.$$

Here, we get

$$v_i R(i)\emptyset. \tag{4.21}$$

Now, $j \in V(i,v)$, then by definition of $V(i,v)$, we have

$$i \in C(v_j \cup \{i\}, P(j))R(j)v_j, \tag{4.22}$$

Since, $v \in \xi$

$$v_j = Tv_j = C(U(j,v), P(j)). \tag{4.23}$$

By the definition of choice set

$$v_j \subseteq U(j,v). \tag{4.24}$$

and

$$v_j R(j) U(j,v).$$

By (4.20) and (4.24) give

$$U(j,v) \supseteq C(v_j \cup \{i\}, P(j)).$$

By the definition of choice set implies that

$$v_j R(j) C(v_j \cup \{i\}, P(j)). \tag{4.25}$$

(4.22) and (4.25) and by the anti-symmetry property of preference relations imply that, $i \in v_j$.

By the parts 1 and 2, $i \in v_j$ iff $j = v_i$. Therefore, v is a matching. (4.18) and (4.21) imply that v is individually rational(IRC).

By the above results we have seen that v is an individuall rational matching. To show v is *stable**, assume that $j \in B$, $F \subseteq S$ such that $F \neq \emptyset$. We suppose that for all $i \in F$,

$$jP(i)v_i. \tag{4.26}$$

According to the definition of (4.13), we have

$$F \subseteq U(j, v). \tag{4.27}$$

Let $G \subseteq v_j$. Since v is a matching, we have for all $i \in G$, $j = v_i$. Therefore, the definition of (4.13) implies that

$$G \subseteq U(j, v). \tag{4.28}$$

Also, $v \in \xi$, so $v_j = Tv_j = C(U(j, v), P(j))$; (4.27) and (4.28) imply then

$$v_j R(j) C(G \cup F, P(j)) R(j) G \cup F. \tag{4.29}$$

(4.26) and (4.29) depicts that there is no (F, j) that *blocks** v. Thus $v \in \xi \subseteq M_{s^*}$.

Now, to prove $M_{s^*} \subseteq \xi$. For this, suppose $v \in M_{s^*}$ and contrary assume that $v \neq Tv$. Let suppose there exist $j \in B$ such that

$$v_j \neq (Tv_j) = C(U(j, v), P(j)) = H \subseteq U(j, v).$$

Let $G = H \cap v_j$, and $F = H \setminus v_j$. Since, v is an individually rational matching so we have $v_j \subseteq U(j, v)$ and $F \neq \emptyset$. Now,

$$G \cup F = HP(j)v_j. \tag{4.30}$$

Also,

$$jP(i)v_i. \tag{4.31}$$

for all $i \in G$, as $G \subseteq H \subseteq U(j, v)$. (4.30) and (4.31) depicts that (G, j) *blocks** v, which is a contradiction to the given condition $v \in M_{s^*}$. Therefore, for all $j \in B$,

$$v_j = Tv_{j.} \tag{4.32}$$

Let there exists $i \in S$ such that

$$v_i \neq Tv_i = \max_{P(i)}\{V(i, v)\} = \hat{j} \in V(i, v).$$

so by the definition of (4.14)

$$i \in C(v_{\hat{j}} \cup \{i\}, P(\hat{j})).$$

Since v is a matching therefore we have that $i \notin v_{\hat{j}}$ and $v_i \in V(i, v)$. This implies that

$$\hat{j}P(i)v_i. \tag{4.33}$$

and

$$i \in C(v_{\hat{j}} \cup \{i\}, P(\hat{j})) = H.$$

Let $G = H \cap v_{\hat{j}} = H \setminus \{i\}$, and $F = \{i\}$. Then $(F, \hat{j})$ *blocks** v because of

$$H = [G \cup F]P(\hat{j})v_{\hat{j}},$$

and the (4.33). Therefore, for all $i \in S$,

$$v_i = Tv_i. \tag{4.34}$$

(4.32) and (4.34) depicts that $v = Tv$. Hence, $v \in \xi$. □

By Theorem 4.1.1 and Theorem 4.2.1, we obtain the next result.

Corollary 4.2.1. *Let ξ be the collection of fixed points of T and K is the core. Then $\xi = K$ holds.*

4.3 T-algorithm

This algorithm is very simple as it starts from some pre-matching $v \in \Upsilon$ and iterate Tv until two the iterations are identical. It get stops when two iterations are identical.

It will be prove that when algorithm stops, it must be a core matching and the matching μ it attains will be in the core. Therefore, T-algorithm strolls around the pre-matching until it attains a matching. The algorithm must cycle, when the core is empty, and the cycle will only include the pre-matchings that are not the matching.

4.3.1 Algorithm

(1) Set $v^0 = v$. Set $v^1 = Tv^0$ and put $\kappa = 1$

(2) While $v^\kappa \neq v^{\kappa-1}$, do:

(i) Set $\kappa = \kappa + 1$,

(ii) Set $v^{\kappa} = Tv^{\kappa-1}$,

(3) Set $\mu = v^{\kappa}$. Stop.

Proposition 4.3.1. *If the T-algorithm stops at $\mu \in \Upsilon$, then μ is a stable matching and is in core. If v^{κ} is a core matching, for some iteration κ of T-algorithm, then algorithm stops at $\mu = v^{\kappa}$.*

Proof. If the T-algorithm stops at $\mu = Tv$ implies the two iterations are identical i.e. $\mu = v^{\kappa} = v^{k-1}$ *i.e*

$$\mu = v^{\kappa} = Tv^{\kappa-1} = T\mu.$$

So, $\mu \in \xi$. Then by Corollary 4.2.1, $\mu \in \xi = K$.

On the other hand, if v^{κ} is a core matching, then v^{κ} is a fixed point of T by the Corollary 4.2.1. Then

$$v^{\kappa} = Tv^{\kappa-1} = \mu.$$

implies that the T-algorithm stops at $\mu = v^{\kappa}$. □

Proposition 4.3.2. *For some iteration κ in the T-algorithm, if v^{κ} is matching, then the algorithm stops at $\mu = v^{\kappa}$, and thus v^{κ} is a core matching.*

Proof. The proof of Proposition is presented as two Steps:

Step 1 Firstly it will show that for a pre-matching $v \in \Upsilon$, if $v^1 = Tv$ is a matching, then v^1 is individually rational.

For all $j \in B$,

$$v_j^1 = Tv_j = C(U(j,v), P(j)).$$

So,

$$\begin{aligned} C(v_j^1, P(j)) &\overset{(i)}{=} C(C(U(j,v), P(j)), P(j)) \\ &\overset{(ii)}{=} C(U(j,v), P(j)) \\ &\overset{(iii)}{=} v_j^1. \end{aligned}$$

Equality (i) and (iii) follows from $v^1 = Tv$. Equality (ii) is the property of choice sets. Therefore, for all $j \in B$,

$$v_j^1 = C(v_j^1, P(j)). \tag{4.35}$$

For all $i \in S$,

$$v_i^1 = Tv_i = \max_{P(i)}\{V(i,v)\} \cup \{\emptyset\}.$$

So,

$$v_i^1 R(i) \emptyset. \tag{4.36}$$

(4.35) and (4.36) shows that v^1 is an individually rational matching.

Step 2 For $v \in \Upsilon$, if $v^1 = Tv$ is a matching, then Tv is a core matching. Thus the T-algorithm must stop at v^κ if $v^\kappa = Tv^{\kappa-1}$ is a matching, and thus v^κ is a core matching.

We shall prove this by showing if $v \in \Upsilon$ is a pre-matching, and $v^1 = Tv$ is a matching then $v^1 \in M_{s^*}$, then the result follows from Proposition 4.3.1. Since $M_{s^*} = \xi = K$.

By Step 1, v^1 is individually rational, contrary suppose that $v^1 \neq M_{s^*}$. Therefore, there exist a pair (F, j) that blocks v^1. Hence, for all $i \in F$

$$jP(i)v_i^1. \tag{4.37}$$

and there exist $G \subseteq v_j^1$ such that

$$[G \cup F]P(j)v_j^1. \tag{4.38}$$

But v^1 is a matching, and $v^1 = Tv$, therefore we have for all $i \in S$, $i \in v^1(v_i^1)$. But

$$v^1(v_i^1) = Tv(v_i^1) = C(U(v_i^1, v), P(v_i^1)).$$

So that, $i \in U(v_i^1, v)$, *i.e*

$$v_i^1 R(i) v_i. \tag{4.39}$$

(4.37) and (4.39) depict that, for all $i \in F$, $jR(i)v_i$, *i.e*

$$F \subseteq U(j, v). \tag{4.40}$$

Now, $G \subseteq v_j^1$ and $v_j^1 = Tv_j = C(U(j,v), P(j))$, so

$$G \subseteq U(j, v). \tag{4.41}$$

(4.38) ,(4.40) and (4.41) contradicts $v_j^1 = Tv_j = C(U(j,v), P(j))$. Thus $v^1 \in M_{s^*}$.

By Step 1 and Step 2 result of the proof follows. □

Example 4.3.3. Consider two finite sets $S = \{i_1, i_2, i_3, i_4\}$ and $B = \{j_1, j_2\}$ where S represent the set of workers and B represent the set of firms. The preference profile P^* is given by Table 4.1 and 4.2 in Example 4.1.1.

T-algorithm starting at

		j_1	j_2	i_1	i_2	i_3	i_4
v^0	$=$	$\{i_1, i_2\}$	$\{i_1, i_2\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

does:

		j_1	j_2	i_1	i_2	i_3	i_4
$U(j,v^0)/V(i,v^0)$	$=$	$\{i_1, i_2, i_3, i_4\}$	$\{i_1, i_2, i_3, i_4\}$	j_1, j_2	j_1, j_2	$\emptyset$	$\emptyset$
v^1	$=$	$\{i_1, i_2\}$	$\{i_1, i_2\}$	j_1	j_2	$\emptyset$	$\emptyset$
$U(j,v^1)/V(i,v^1)$	$=$	$\{i_1, i_3, i_4\}$	$\{i_2, i_3, i_4\}$	j_1, j_2	j_1, j_2	$\emptyset$	$\emptyset$
v^2	$=$	$\{i_3, i_4\}$	$\{i_2, i_4\}$	j_1	j_2	$\emptyset$	$\emptyset$
$U(j,v^2)/V(i,v^2)$	$=$	$\{i_1, i_3, i_4\}$	$\{i_2, i_3, i_4\}$	j_2	j_2	j_1	j_1, j_2
v^3	$=$	$\{i_3, i_4\}$	$\{i_2, i_4\}$	j_2	j_2	j_1	j_1
$U(j,v^3)/V(i,v^3)$	$=$	$\{i_1, i_3, i_4\}$	$\{i_1, i_2\}$	j_2	j_2	j_1	j_1, j_2
v^4	$=$	$\{i_3, i_4\}$	$\{i_1, i_2\}$	j_2	j_2	j_1	j_1

Now, v^4 is a matching, therefore by Proposition 4.3.2, v^4 is a core matching.

Next example is an illustration of an empty core in the T-algorithm.

Example 4.3.4. Consider two finite sets $S = \{i_1, i_2, i_3\}$ and $B = \{j_1, j_2, j_3\}$ where S represent the set of workers and B represent the set of firms. The preference profile P^* is given by Table 4.3 and 4.4.

$P(i_1)$	j_1	j_3	j_2
$P(i_2)$	j_2	j_1	j_3
$P(i_3)$	j_3	j_2	j_1

Table 4.3: Worker's preference list

$P(j_1)$	$\{i_1, i_2\}$	$\{i_3\}$
$P(j_2)$	$\{i_2, i_3\}$	$\{i_1\}$
$P(j_3)$	$\{i_1, i_3\}$	$\{i_2\}$

Table 4.4: Firm's preference list

T-algorithm starts at

$$\begin{array}{ccccccccc} & & j_1 & j_2 & j_3 & i_1 & i_2 & i_3 \\ v^0 & = & \emptyset & \emptyset & \emptyset & j_1 & j_2 & j_3 \end{array}$$

The T-algorithm does

$$\begin{array}{rcrrrccc} & & j_1 & j_2 & j_3 & i_1 & i_2 & i_3 \\ U(j,v^0)/V(i,v^0) & = & \{i_1\} & \{i_2\} & \{i_3\} & j_2 & j_3 & j_1 \\ v^1 & = & \emptyset & \emptyset & \emptyset & j_2 & j_3 & j_1 \\ U(j,v^1)/V(i,v^1) & = & \{i_1,i_2,i_3\} & \{i_1,i_2,i_3\} & \{i_1,i_2,i_3\} & j_2 & j_3 & j_1 \\ v^2 & = & \{i_1,i_2\} & \{i_2,i_3\} & \{i_1,i_3\} & j_2 & j_3 & j_1 \\ U(j,v^2)/V(i,v^2) & = & \{i_1,i_2,i_3\} & \{i_1,i_2,i_3\} & \{i_1,i_2,i_3\} & j_1,j_3 & j_1,j_2 & j_2,j_3 \\ v^3 & = & \{i_1,i_2\} & \{i_2,i_3\} & \{i_1,i_3\} & j_1 & j_2 & j_3 \\ U(j,v^3)/V(i,v^3) & = & \{i_1\} & \{i_2\} & \{i_3\} & j_1,j_3 & j_1,j_2 & j_2,j_3 \\ v^4 & = & \emptyset & \emptyset & \emptyset & j_1 & j_2 & j_3 \end{array}$$

Note here $v^\kappa, \kappa = 0, \cdots, 4$ in Υ are not matchings. Algorithm cycles because $v^0 = v^4$. In this case core is empty.

4.4 The lattice structure of the core

Now a partial order on pre-matchings Υ will be introduce in such a way that if the preferences of the agents are substitutable, T is a monotone increasing function. Then the Theorem 2.3.1 implies a lattice structure on ξ, and thus on K.

Definition 4.4.1. Define the partial orders on Υ_B,Υ_S and Υ given below:

1. $P(B)$ on Υ_B by $v_B P(B)\hat{v}_B$ if and only if $\hat{v}_B \neq v_B$ and for all $j \in B$

$$v_j = C(v_j \cup \hat{v}_j, P(j))$$

2. $P(S)$ on Υ_S by $v_S P(S)\hat{v}_S$ if and only if $\hat{v}_S \neq v_S$ and

$$v_i R(i)\hat{v}_i \qquad \forall i \in S$$

3. The weak partial order associated to $P(B)$ and $P(S)$ are denoted $R(B)$ and $R(S)$, defined as: $v_B R(B)\hat{v}_B$ if $v_B = \hat{v}_B$ or $v_B P(B)\hat{v}_B$ and $v_S R(S)\hat{v}_S$ if $v_S = \hat{v}_S$ or $v_S P(S)\hat{v}_S$

4. $P(B)$ on Υ by $vP(B)\hat{v}$ iff $v_B P(B)\hat{v}_B$ and $\hat{v}_S R(S) v_S$.

5. $P(S)$ on Υ by $vP(S)\hat{v}$ iff $v_S P(S)\hat{v}_S$ and $\hat{v}_B R(B) v_B$.

Definition 4.4.2. A firm $j's$ preference ordering $P(j)$ satifies substitutability condition if for any $W \subseteq S$ containing workers i and $\hat{i}$ ($i \neq \hat{i}$), if $i \in C(W, P(j))$ then $i \in (C(W \setminus \{\hat{i}\}, P(j))$. A preference profile P^* is substitutable if, for each firm j, the preference ordering satisfies subtitutability.

Theorem 4.4.1. *Consider the preference profile P be the substitutable. Then $(K_i(P), P(B))$ and $(K_i(P), P(S))$ forms the non empty complete lattices, and*

(i) $sup_{P(B)} K_i(P) = inf_{P(S)} K_i(P)$.

(ii) $inf_{P(B)} K_i(P) = sup_{P(S)} K_i(P)$.

According to Theorem 4.4.1 there are two core matchings, μ_S and μ_B in such a way that: For workers $\mu_S = sup_{P(S)}K_i(P)$is better, and $\mu_B = inf_{P(S)}K_i(P)$ is worse than other core matchings. For firms, $\mu_B = sup_{P(B)}K_i(P)$ is better and $\mu_S = inf_{P(B)}K_i(P)$ is worse, than any other core matchings.

Let $\hat{\Upsilon} = \{v \in \Upsilon : v(s)R(s)\emptyset, \forall s \in B \cup S\}$. This means $\forall v \in \Upsilon$, $Tv \in \hat{\Upsilon}$. Consider P be the substitutable.

Proof. Theorem 4.4.1 is presented by four lemmas.

Lemma 4.4.3. $(\hat{\Upsilon}, P(B))$ *is a complete lattice.*

Proof. For each i, let $X_i = \{j \in B : jR(i)\emptyset\}$; $(X_i, R(i))$ is a totally ordered finite set, so is a complete lattice.

Similarly, for each j, let $X_j = \{G \subseteq S : GR(j)\emptyset\}$; and suppose $\geq_B$ be the partial order on X_j defined by $G \geq F$ if and only if $G = F$ or $G = C(G \cup F, P(j))$. By Blair [4], $(X_j, \geq_B)$ is a complete lattice.
Now, $\hat{\Upsilon} = (\times_{i \in S} X_i) \times (\times_{j \in B} X_j)$, and $P(B)$ is the product order of the partial orders introduced above. Hence $(\hat{\Upsilon}, P(B))$ is a complete lattice. □

Lemma 4.4.4. *Let μ and $\hat{\mu}$ be pre-matchings. If $\hat{\mu}R(B)\mu$ then, $\forall i \in S$ and $j \in B$,*

$$U(j, \mu) \subseteq U(j, \hat{\mu}),$$

$$V(i, \hat{\mu}) \subseteq V(i, \mu).$$

Proof. Let $i \in U(j, \mu)$. We have that $jR(i)\mu(i)$, but the definition of $\mu R(S)\hat{\mu}$ implies $\mu(i)R(i)\hat{\mu}(i)$. So, by using transitive property we have $jR(i)\hat{\mu}(i)$. This implies $i \in U(j, \hat{\mu})$. This proves $U(j, \mu) \subseteq U(j, \hat{\mu})$.
Now to show $V(i, \hat{\mu}) \subseteq V(i, \mu)$. Firstly, if $V(i, \hat{\mu}) = \emptyset$, there there is nothing to prove, because $\emptyset = V(i, \hat{\mu}) \subseteq V(i, \mu)$. Consider if $V(i, \hat{\mu}) \neq \emptyset$, and suppose $j \in V(i, \hat{\mu})$. Then by definition we have that

$$i \in C(\hat{\mu}_j \cup \{i\}, P(j)). \tag{4.42}$$

but the definition of $\hat{\mu}R(B)\mu$ implies that for all $j \in B$, either $\hat{\mu}_j = \mu_j$ therefore we have from (4.42) we have $i \in C(\mu_j \cup \{i\}, P(j))$, or

$$\hat{\mu}_j = C(\hat{\mu}_j \cup \mu_j, P(j)).$$

Then by the (4.42) implies that

$$\begin{aligned} i &\in C(\hat{\mu}_j \cup \{i\}, P(j)) \\ &= C(C(\hat{\mu}_j \cup \mu_j, P(j)) \cup \{i\}, P(j)) \\ &\overset{(1)}{=} C(\hat{\mu}_j \cup \mu_j \cup \{i\}, P(j)). \end{aligned}$$

From Blair [4], the equality (1) holds according to the choice set property. He showed, if preference profile P is substitutable, then $C(G \cup F, P(j)) = C(C(G, P(j)) \cup F, P(j))$ for all G and F. Now by substitutability of P implies that

$$i \in C(\mu_j \cup \{i\}, P(j)).$$

This implies $j \in V(i, \mu)$. Thus proves $V(i, \hat{\mu}) \subseteq V(i, \mu)$. This completes the proof. □

Lemma 4.4.5. *ξ is non-empty and a complete lattice.*

Proof. Firstly, we will show that $T\mid_{\hat{\Upsilon}}$ is a monotone increasing. Let consider the two pre-matchings. $\mu = (\mu_B, \mu_S)$ and $\hat{\mu} = (\hat{\mu_B}, \hat{\mu}_S)$. We have to show, if $\hat{\mu}R(B)\mu$, then $T\hat{\mu}R(B)T\mu$. Consider $\hat{\mu}R(B)\mu$, let $j \in B$ and $i \in S$. Since by Lemma 4.4.4 says $U(j, \mu) \subseteq U(j, \hat{\mu})$. Firstly we will show that

$$C(U(j, \hat{\mu}), P(j)) = C([C(U(j, \hat{\mu}), P(j)) \cup C(U(j, \mu), P(j))], P(j)). \tag{4.43}$$

In order to show this, suppose $Y \subseteq C(U(j, \hat{\mu}), P(j)) \cup C(U(j, \mu), P(j))$. Then $Y \subseteq U(j, \mu) \cup U(j, \hat{\mu}) = U(j, \hat{\mu})$, therefore $C(U(j, \hat{\mu}), P(j))R(j)Y$. But $C(U(j, \hat{\mu}, P(j)) \subseteq C(U(j, \hat{\mu}), P(j)) \cup C(U(j, \mu), P(j))$, so we get Eq.(4.43).

Now, $T\hat{\mu}_j = C(U(j, \hat{\mu}), P(j))$ and $T\mu_j = C(U(j, \mu), P(j))$, therefore (4.43) implies that

$$T\hat{\mu}_j = C([T\hat{\mu}_j \cup T\mu_j], P(j)). \tag{4.44}$$

Now to show that $T\mu_i R(i)(T\hat{\mu}_i)$. Since, Lemma 4.4.4 says that $V(i,\hat{\mu}) \subseteq V(i,\mu)$.

$$\begin{array}{rcl} T\mu_i & = & \underset{P(j)}{max}\{V(i,\mu)\} \quad for \quad i \in S \\ & R(i) & \underset{P(j)}{max}\{V(i,\hat{\mu})\} \\ & = & T\hat{\mu}_i. \end{array}$$

Hence

$$T\mu_i R(i) T\hat{\mu}_i. \tag{4.45}$$

i.e from (4.44) and (4.45) imply that $T\mu R(B) T\hat{\mu}$.

Finally, $T(\Upsilon) \subseteq \hat{\Upsilon}$ so $\xi \subseteq \hat{\Upsilon}$, and ξ equals the set of fixed points of $T\mid_{\hat{\Upsilon}}$. $T(\Upsilon) \subseteq \hat{\Upsilon}$ also implies that the restricted map $T\mid_{\hat{\Upsilon}}$ has a range in $\hat{\Upsilon}$.
Now, $(\hat{\Upsilon}, P(B))$ be a complete lattice by Lemma 4.4.3. Define T on $\hat{\Upsilon}$, such that T be an increasing function on $\hat{\Upsilon}$ to $\hat{\Upsilon}$ as shown above. ξ be the set of all the fixed points of T. Then ξ be not empty and the system $(\xi, P(B))$ is a complete lattice by the Tarski's fixed point theorem. Hence, proved □

Lemma 4.4.6. *$(K_i(P), P(B))$ and $(K_i(P), P(S))$ forms the non-empty complete lattices, and*

(i) $sup_{P(B)} K_i(P) = inf_{P(S)} K_i(P)$.

(ii) $inf_{P(B)} K_i(P) = sup_{P(S)} K_i(P)$.

Proof. By Corollary 4.2.1 and Lemma 4.4.5 we get $(K_i(P), P(B))$. The orders $P(S)$ and $P(B)$ are the order-duals, hence $(K_i(P), P(S))$ is a non empty complete lattice and (i) and (ii) are immediate in the Lemma. □

□

Chapter 5

The study of lattice structure of bipartite stable matchings with flexible agents

In the currrent chapter we follow the idea presented by Ali and Farooq [2] and Echenique and Oviedo [6] to devise an algorithm that characterizes the set of stable matchings for hybrid model. At the end we study the lattice structure as a prompt application of Theorem 2.3.1. We also give an example to support our results. This chapter based work is published [21].

This chapter is structured as follows. We divide our main results into following sections: In Section 5.1, we demonstrate the model, some basic notations and definitions. Subsection 5.1.1 involves the mathematical presentation of modified T-algorithm and its termination. The working of algorithm with the help of example is also given at the end of this section. Moreover, it includes a precise description of Ali and Farooq [2] work through an example along with the motivation and comparison of it with our modified T-algorithm. Subsection 5.1.2 consists of lattice structure of the set of stable one to one matchings obtained through the hybrid algorithm. Throughout this chapter we consider two finite and disjoint sets of same cardinality n, the set of sellers S and the set of buyers B.

5.1 Seller-Buyer hybrid model

In this model, we consider two sets of agents. Namely, sellers (set S) that possess some commodity to sell and buyers (set B) that possess finite amount of money. Each agent can trade with at most one agent of opposite side. Also assume that the money is bounded and has a discrete/integer values. We present our model in mathematical terms as follows. For each $(i,j) \in E$, we define set of all possible seller-buyer pairs by $E = S \times B$. We define the increasing valuations ω_{ij} and ω_{ji} by:

$$\omega_{ij}(y) = \lambda_{ij} y + \delta_{ij} \;; \qquad \omega_{ji}(-y) = -\lambda_{ji} y + \delta_{ji}. \tag{5.1}$$

where λ_{ij} , $\lambda_{ji} \in \mathbb{R}^+$, $\delta_{ij}, \delta_{ji} \in \mathbb{R}$ and $y \in \mathbb{Z}$ for each $(i,j) \in E$. Here, $\omega_{ji}(-y_{ij})$ represent the utility to buyer j if he/she trades with the seller i and pays a price/money y_{ij}. It means that i is always considered as a payee. The negative sign in $\omega_{ji}(-y_{ij})$ means j is always considered as a payer. Now, price vector p is defined by[1] Ali and Farooq [2] as:

$$p_{ij} = \begin{cases} \overline{\pi}_{ij} & if\, \omega_{ji}(-\overline{\pi}_{ij}) \geq 0, \\ max\left\{\underline{\pi}_{ij}, \left\lfloor \frac{\delta_{ji}}{\lambda_{ji}} \right\rfloor\right\} & \text{otherwise.} \end{cases} \tag{5.2}$$

.

Furthermore, for each $(i,j) \in E$. The lower and upper bounds of price are given by the two vectors $\underline{\pi}, \overline{\pi} \in \mathbb{Z}^E$ and $\underline{\pi}_{ij} \leq \overline{\pi}_{ij}$ for each $(i,j) \in E$ [2]. A vector $p = (p_{ij} \in \mathbb{Z} \mid (i,j) \in E)$ is called feasible money/price vector if $\underline{\pi}_{ij} \leq p_{ij} \leq \overline{\pi}_{ij}$ for each $(i,j) \in E$. In the proceeding chapter, the term c_{ij} denotes any feasible price.

Definition 5.1.1. An agent is said to have a strict preference relation P if it is not indifferent among two alternatives. We say $jP(i)\hat{j}$ at some y_{ij} and $y_{i\hat{j}}$ means i is strictly preferring j at price y_{ij} to $\hat{j}$ at price $y_{i\hat{j}}$ if $\omega_{ij}(y_{ij}) > \omega_{i\hat{j}}(y_{i\hat{j}})$. We denote it by $\{jP(i)\hat{j}\}_{y_{ij}}^{y_{i\hat{j}}}$. Similarly, we define for buyers $iP(j)\hat{i}$ at some y_{ij} and $y_{\hat{i}j}$ means j is strictly preferring i at price y_{ij} to $\hat{i}$ at price $y_{\hat{i}j}$ if $\omega_{ij}(-y_{ij}) > \omega_{\hat{i}j}(-y_{\hat{i}j})$ and denote it by $\{iP(j)\hat{i}\}_{y_{ij}}^{y_{\hat{i}j}}$.

[1] $\lfloor y \rfloor = sup\{n \in \mathbb{Z} : y \geq n\}$

[2] $\mathbb{Z}^E$ stands for integer lattice whose points are indexed by E

Definition 5.1.2. An agent is said to have a weak preference relation R, if it is either indifferent among two alternatives or strictly preferring its partner. In this case, we say $jR(i)\hat{j}$ at some y_{ij} and $y_{i\hat{j}}$ means i is weakly preferring j at price y_{ij} to $\hat{j}$ at price $y_{i\hat{j}}$ if $\omega_{ij}(y_{ij}) \geq \omega_{i\hat{j}}(y_{i\hat{j}})$ and denote it by $\{jR(i)\hat{j}\}_{y_{ij}}^{y_{i\hat{j}}}$. Similarly we can define it for buyers.

Note that R is a total order on $S \cup B$. Also from Definition 5.1.2 we have:

$\{\mu_i R(i) j\}_{p_{i\mu_i}}^{c_{ij}}$ for price $c_{ij} \leq p_{i\mu_i}$ means i is preferring its partner under μ over j if $\omega_{ij}(c_{ij}) \leq \omega_{i\mu_i}(p_{i\mu_i})$.

Similarly, $\{\mu_j R(j) i\}_{p_{\mu_j j}}^{c_{ij}}$ for price $-p_{\mu_j j} > -c_{ij}$ means j is preferring its partner under μ over i if $\omega_{ji}(-c_{ij}) < \omega_{\mu_j j}(-p_{\mu_j j})$.

Definition 5.1.3. Given a seller's strict preference relation $P(i)$, the buyers j preferred by i to remain isolated (single) at some amount y_{ij} are called *acceptable* if $\omega_{ij}(y_{ij}) \geq 0$ for $j \in B$. This signifies that i is willing to trade with j at price y_{ij}. We denote it as $\{jP(i)\emptyset\}_{y_{ij}}$. Similarly, given a preference relation of buyer $P(j)$, the seller i preferred by j to remain isolated (single) at some price y_{ij} are called *acceptable* if $\omega_{ji}(-y_{ij}) \geq 0$ for $i \in S$. This signifies that the seller i is acceptable to buyer j at price y_{ij}. We denote it as $\{iP(j)\emptyset\}_{y_{ij}}$.

In the Definition 5.1.3, if the strict preference relation P is replaced by weak preference relation R, then it is assume that an agent is either indifferent (i.e; an agent *may* prefer to remain isolated/single/unmatched) or strictly preferring its partner (i.e; an agent always prefer to be matched).

Definition 5.1.4. A 2-tuple (μ, p) is said to be pairwise stable outcome if the two conditions mentioned below are hold:

IR (μ, p) is individually rational if: for all $(i, j) \in E$

$$\{\mu_i R(i)\emptyset\}_{p_{i\mu_i}} \quad \text{and} \quad \{\mu_j R(j)\emptyset\}_{p_{\mu_j j}}.$$

N.B.P Any $(i, j) \in E$ for $c_{ij} \in [\underline{\pi}_{ij}, \overline{\pi}_{ij}]$ does not block (μ, p) if:

$$\{\mu_i R(i) j\}_{p_{i\mu_i}}^{c_{ij}} \quad \text{or} \quad \{\mu_j R(j) i\}_{p_{\mu_j j}}^{c_{ij}}.$$

Condition (**IR**) says that the matching (μ, p) at a price vector p is individually rational. Condition (**N.B.P**) means (μ, p) is not blocked by any seller-buyer pair. A matching μ is called pairwise stable if (μ, p) is pairwise stable.

5.1.1 Mathematical presentation of modified T-algorithm

With the help of various mathematical tools and a constructive evidence, we show that the model described in Section 5.1 always depicts an existence of pairwise stable outcome. Initially, we define $p \in \mathbb{Z}$ by (5.2). At the start of algorithm we will exclude all those pairs that are not acceptable to each other and define a function T on the set of pre-matchings Υ that assigns each seller a better partner. By setting any random pre-matching $v = v^0$ we are looking for a matching by iterative applications of T on v^0. During this process there might be some mutually acceptable seller-buyer pairs in which seller is not matched with the buyer. For such pairs we will modify price vector. This price externality results in new preference profile in which a matched agent may change his partner if some better potential partner appears in the preference profile. i.e; an agent on the accepting side might want to go back to a proposer that is already rejected, or an agent on the proposing side might want to withdraw a partner that already made. Throughout the procedure, we will exclude two types of unmatched pairs, if they exist. Firstly, those seller-buyer pairs in which the buyer is not acceptable to the seller and secondly, those seller-buyer pairs in which price vector becomes less than its lower bound. It is beneficial to take note that the price vector is non-increasing and the size of the set of acceptable seller-buyer pairs is non-decreasing at each step of the algorithm. As long as, the price vector is bounded and discrete and the number of agents is finite, the algorithm will terminate after a finite number of iterations and a stable outcome is achieved as a fixed point of a function T.

We now state our algorithm in mathematical terms. We first define two subsets L_0 and E_0 of E by that will be helpful to find a matching satisfying **N.B.P** as follows:

$$L_0 = \{(i, j) \in E : \omega_{ji}(-p_{ij}) < 0\} . \tag{5.3}$$

and E_0 as:

$$E_0 = \{(i,j) \in E : \omega_{ij}(p_{ij}) < 0\}\,. \tag{5.4}$$

Here, L_0 consists of set of those seller-buyer pairs in which the seller is not acceptable to the buyer. Whereas, E_0 consists of the set of those seller-buyer pairs in which buyer is not acceptable to the seller. Note that, when certain agent becomes unacceptable then that agent may prefer to remain un-matched (isolated) rather than to match with an un-acceptable partner. It reveals that only acceptable partners matter, so we shall write preference relation briefly as lists of acceptable partners. Both of these two sets enables us to define the set of mutually acceptable seller-buyer pairs as follows:

$$\tilde{E} = E \setminus \{L_0 \cup E_0\}. \tag{5.5}$$

Let $v \in \Upsilon$ be a pre-matching and price vector p defined by (5.2). We define:

$$U(i,v) = \left\{ j \in B : \{iR\,(j)\,v_j\}_{p_{ij}}^{p_{v_j j}} \right\}. \tag{5.6}$$

and

$$V(j,v) = \left\{ i \in S :\ \{jR\,(i)\,v_i\}_{p_{ij}}^{p_{i v_i}} \right\}. \tag{5.7}$$

The set $U(i,v)$ consists of those buyers j that prefers i at price p_{ij} atleast as much as they prefer their partners under pre-matching v. Similarly we can define the set $V(j,v)$. Now define a function $T : \Upsilon \longrightarrow \Upsilon$ such that for $v \in \Upsilon$

$$Tv = \begin{cases} Tv_i = \underset{R(i)}{max}\,\{U(i,v)\} & \text{for} \quad i \in S, \\ Tv_j = \underset{R(j)}{max}\,\{V(j,v) : Tv_i = j\} & \text{for} \quad j \in B. \end{cases} \tag{5.8}$$

The function T has simple interpretation: Tv_i means the buyer preferred by i among the buyers that are willing to make partnership with i. Whereas Tv_j means the seller preferred by j among those sellers who prefer j as their partner under T. A matching $\mu \subseteq \tilde{E}$ is then consists of matched members of Tv. So we define $(\mathbf{q},\mathbf{r}) \in \mathbb{R}^S \times \mathbb{R}^B$

$$q_i = \begin{cases} \omega_{iTv_i}(p_{iTv_i}) & \text{if} \quad (i,Tv_i) \in \mu, \\ 0 & \text{otherwise.} \end{cases} \qquad (i \in S) \tag{5.9}$$

and

$$r_j = \begin{cases} \omega_{jTv_j}(-p_{Tv_j j}) & \text{if } (Tv_j, j) \in \mu, \\ 0 & \text{otherwise.} \end{cases} \qquad (j \in B) \tag{5.10}$$

Now the 2-tuple (μ, p) obviously satisfies **I.R** but still **N.B.P** may not hold. To ensure **N.B.P**, we will modify price vector. Particularly, we will modify the price vector for pairs in Y defined below:

$$Y = \{(i, Tv_i) \in (Tv \setminus \mu) : \{Tv_i R(i) \emptyset\}_{p_{iTv_i}}\}. \tag{5.11}$$

Note that $Y \subseteq \tilde{E}$ that consists of all those seller-buyer pairs in which buyer is most preferred by the seller but the seller is unmatched in μ. During price modification we will make sure that ***IR*** and the feasibility of p are preserved. To modify p, we find an integer η_{iTv_i} for each $(i, Tv_i) \in Y$, by [3]

$$\eta_{iTv_i} = max\left\{1, \left\lceil \frac{r_{Tv_i} - \omega_{Tv_i i}(-p_{iTv_i})}{\lambda_{Tv_i i}} \right\rceil \right\}. \tag{5.12}$$

A subset L of Y is defined by

$$L = \{(i, Tv_i) \in Y : p_{iTv_i} - \eta_{iTv_i} < \underline{\pi}_{iTv_i}\}. \tag{5.13}$$

Now we modify the price vector p that is denoted by $\tilde{p}$. For each $(i, j) \in E$ modified price vector $\tilde{p}$ is defined by

$$\tilde{p}_{ij} := \begin{cases} max\left\{\underline{\pi}_{iTv_i}, p_{iTv_i} - \eta_{iTv_i}\right\} & \text{if} \quad (i, Tv_i) \in Y, \\ p_{ij} & \text{otherwise.} \end{cases} \tag{5.14}$$

A subset $\tilde{E}_0$ of Y is defined by:

$$\tilde{E}_0 := \{(i, Tv_i) \in Y : \omega_{iTv_i}(\tilde{p}_{iTv_i}) < 0\}. \tag{5.15}$$

We now finally present our algorithm.

Modified T-Algorithm:

Step-I Initially define p, L_0, E_0, $\tilde{E}$, any random pre-matching v, $U(i, v) \neq \emptyset$, $V(j, v)$ and Tv by (5.2)-(5.8), respectively. Set $v^0 = v$. Set $v^1 = Tv^0$ and $k = 1$. Find a matching μ within Tv consists of matched members of Tv. Define $\mathbf{r}$ and Y by (5.10) and (5.11), respectively.

[3] $\lceil y \rceil = inf\{n \in \mathbb{Z} : y \leq n\}$

Step-II If $\mu = Tv$ implies $Y = \emptyset$ then stop($i.e \quad T\mu = \mu$).

Step-III For each pair $(i, Tv_i) \in Y$, calculate η_{iTv_i} by (5.12) and find $\tilde{p}$ by (5.14). Define L and $\tilde{E}_0$ by (5.13) and (5.15) respectively. Update E_0 by $E_0 := E_0 \cup \tilde{E}_0$ and L_0 by $L_0 := L_0 \cup L$.

Step-IV Put $p := \tilde{p}$ and modify $\tilde{E}$ by (5.5). Set $k = k + 1$. Set $v^k = Tv^{k-1}$ for updated $\tilde{E}$ and p. Find a matching μ in Tv which consists of matched members of Tv. Define $\mathbf{r}$ and Y by (5.10) and (5.11) respectively. Go to **Step II.**

Note that we can always obtain a seller optimal stable matching by taking $v \in \Upsilon$ with

$$v_i = \underset{R(i)}{max}\, B \quad \text{and} \quad v_j = \{\emptyset\}.$$

For buyer optimal stable matching we have the following two cases:

Case 1 A buyer optimal stable matching can be found by taking $v \in \Upsilon$ with

$$v_i = \{\emptyset\} \quad \text{and} \quad v_j = \underset{R(j)}{max}\, S.$$

provided that all buyers have different optimal preference in their preference list, (see Example 5.1.1). Hence the set of stable matchings M_s contains at least two matchings and we can define a lattice structure in this case.

Case 2 Unlike in Case 1, if some buyers have equally likely optimal preferences, then we cannot obtain a buyer optimal matching. Hence, in this case we claim that we always get a unique stable matching.

Let $\xi = \{v \in \Upsilon : v = Tv\}$ be the set of all fixed points of function T and M_s be the collection of pairwise stable matchings. We show that $\xi \neq \emptyset$ and the above algorithm terminates.

Lemma 5.1.5. *Let v be a pre-matching such that $v \in \xi$, then v is an individually rational matching.*

Proof: As we are considering pairs from $\tilde{E}$ thus it is trivial to show v is individually rational. Let v = (v_S ,v_B) $\in \xi$. Now we show that v is a matching i.e

$$v_i = j \Leftrightarrow i = v_j.$$

First suppose that $j = v_i$ then (5.7) implies that

$$i \in \max_{R(j)} \{V(j, v) : Tv_i = j\}, \tag{5.16}$$

we may write

$$i = Tv_j. \tag{5.17}$$

and

since $v \in \xi$ implies that $Tv_j = v_j$ thus (5.17) *gives $i = Tv_j = v_j$. Next suppose that $i = v_j$ then we shall prove that $j = v_i$. First we note that $i = v_j$, (5.6) implies that*

$$j \in U(i, v).$$

Secondly, due to $v \in \xi$ note that

$$i = v_j = Tv_j = \max_{R(j)} \{V(j, v) : Tv_i = j\} \quad for \quad j \in B. \tag{5.18}$$

Since by defined condition partner of j is such kind of element whose image under T is j i.e $Tv_i = j$ this proves the result.

Theorem 5.1.1. *Let ξ be the set of all fixed points of T and M_s be the collection of all stable matchings. Then $\xi = M_s$.*

Proof: To do this, we will first show $\xi \subseteq M_s$ and $M_s \subseteq \xi$. Firstly, we will show $v \in \xi \subseteq M_s$. By Lemma 5.1.5 we know that v is individually rational matching. We have to show that $v \in M_s$ means no blocking pair exists in v. On contrary assume that there exist $(i, j) \in E$, and $c_{ij} \in [\underline{\pi}_{ij}, \overline{\pi}_{ij}]$ such that

$$\{iR(j)Tv_j\}_{c_{ij}}^{p_{Tv_jj}} \quad for \quad c \leq p_{iTv_i}. \tag{5.19}$$

Then by definition of $U(i, v)$ by (5.6)

$$j \in U(i, v). \tag{5.20}$$

Since, $v \in \xi$ so,

$$v_i = Tv_i = \max_{R(i)} \{U(i,v)\} \quad \text{for} \quad i \in S. \tag{5.21}$$

By (5.21) $v_i \neq j$, because with $c \leq p_{iTv_i}$

$$\omega_{ij}(c) \leq \omega_{iTv_i}(p_{iTv_i}).$$

Hence we write

$$\{Tv_i R(i) j\}^{c_{ij}}_{p_{iTv_i}}. \tag{5.22}$$

(5.19) and (5.22) show that there is no such pair (i,j) that blocks v. Thus

$$v \in M_s \quad \text{for} \quad c \leq p_{iTv_i}.$$

Similarly, on contrary suppose

$$\{jR(i)Tv_i\}^{p_{iTv_i}}_{c_{ij}} \quad \text{for} \quad c > p_{Tv_j j}. \tag{5.23}$$

By definition of $V(j,v)$ from (5.7) implies

$$i \in V(j,v).$$

Since $v \in \xi$, so

$$v_j = Tv_j = \max_{R(j)} \{V(j,v) : Tv_i = j\} \quad \text{for} \quad j \in B.$$

This means j is preferring its partner over i that is

$$\omega_{ji}(-c) \leq \omega_{jTv_j}(-p_{Tv_j j}) \quad \text{for } c > p_{iTv_i}.$$

Therefore, we obtain

$$\{Tv_j R(j) i\}^{c_{ij}}_{p_{Tv_j j}}. \tag{5.24}$$

(5.23) and (5.24) shows that there is no pair (i,j) that block v.
Thus,

$$v \in M_s \quad \text{for} \quad c > p_{Tv_j j}.$$

Hence, $v \in M_s$ *for* $c_{ij} \in [\underline{\pi}_{ij}, \overline{\pi}_{ij}]$.

Let $v \in M_s$ *and suppose that* $v \neq Tv$. *First we suppose that there exist* $i \in S$ *such that*

$$v_i \neq Tv_i = \max_{R(i)} \{U(i,v)\} = \hat{j} \in U(i,v). \tag{5.25}$$

From (5.25)

$$\{\hat{j}R(i)v_i\}^{c_{iv_i}}_{p_{i\hat{j}}}. \tag{5.26}$$

By the definition of $U(i,v)$ *from (5.6)*

$$\{iR(\hat{j})v_{\hat{j}}\}^{c_{v_{\hat{j}}\hat{j}}}_{p_{i\hat{j}}}. \tag{5.27}$$

(5.26) and (5.27) imply that $(i, \hat{j})$ *blocks* v *which contradicts that* $v \in M_s$. *Therefore, for all* $i \in S$

$$v_i = Tv_i. \tag{5.28}$$

Similarly, suppose that there exists j *such that for* $j \in B$

$$v_j \neq Tv_j = \max_{R(j)} \{V(j,v) : Tv_i = j\} = \hat{i} \in V(j,v). \tag{5.29}$$

So, by definition of $V(j,v)$ *(5.7)*

$$\{jR(\hat{i})v_{\hat{i}}\}^{c_{\hat{i}v_{\hat{i}}}}_{p_{\hat{i}j}}. \tag{5.30}$$

By (5.29) indicates that

$$\{\hat{i}R(j)v_j\}^{c_{v_j j}}_{p_{\hat{i}j}}. \tag{5.31}$$

Then $(\hat{i}, j)$ *blocks* v *because of (5.30) and (5.31) which contradicts that* $v \in M_s$. *Hence, for all* $j \in B$

$$v_j = Tv_j. \tag{5.32}$$

(5.28) and (5.32) imply that

$$v = Tv.$$

This proves $v \in \xi$.

Next, we state two Lemma's which will be helpful in the termination of algorithm.

Lemma 5.1.6 (Ali and Farooq [2]). *In each iteration of the algorithm, $\tilde{E}$ remains the same or reduces. In particular, if $L \neq \emptyset$ or $\tilde{E}_0 \neq \emptyset$ at Step III then $\tilde{E}$ reduces at Step IV.*

Lemma 5.1.7 (Ali and Farooq [2]). *In each iteration of the algorithm at Step IV, if $\omega_{ji}(-c_{ij}) > \omega_{jTv_j}(-p_{Tv_j j})$ for some $(i,j) \in E$ then c_{ij} is the maximum integer in $[\underline{\pi}_{ij}, \overline{\pi}_{ij}]$ for which the inequality holds.*

Theorem 5.1.2. *For some iteration k in the modified T-algorithm, if v^k is matching, then the algorithm stops at $\mu = v^k$, and thus v^k is a stable matching.*

Proof: We will present the proof of this Theorem 5.1.2 as two Steps:

Step-1 *First we will show that for a pre-matching $v^k \in \Upsilon$, if $\mu = v^k = Tv^{k-1}$ is a matching, then v^k is individually rational. In order to show this, for $v^{k-1} \in \Upsilon$ we have $Tv^{k-1} = v^k = \mu$, where μ is a matching. Suppose that algorithm terminates at Step I i.e $Tv^0 = v^1$. This implies $Y = \emptyset$ and (v^1, p) be the 2-tuple obtained at termination where we find a matching v^1.*

Initially we define $\tilde{E}$ by (5.5). Therefore

$$\omega_{ij}(p_{ij}) \geq 0 \quad \textit{and} \quad \omega_{ji}(-p_{ij}) \geq 0 \quad ((i,j) \in \tilde{E}). \tag{5.33}$$

at Step-I of algorithm. For all $i \in S$

$$v_i^1 = Tv_i^0 = \underset{R(i)}{max}\,\{U(i, v^0)\},$$

Since $U(i, v^0) \neq \emptyset$, so

$$\{v_i^1 R(i) \emptyset\}_{p_{iv_i^1}}. \tag{5.34}$$

For all $j \in B$

$$v_j^1 = Tv_j^0 = \underset{R(j)}{max}\left\{V(j, v^0) : Tv_i^0 = j\right\}.$$

This implies

$$\{v_j^1 R(j) \emptyset\}_{p_{v_j^1 j}}. \tag{5.35}$$

As $v^1 \subseteq \tilde{E}$. The (5.34) and (5.35) imply that v^1 is individually rational. Since in each iteration we modify $\tilde{E}$ by (5.5) at Step IV of algorithm. So, (5.33) holds in each iteration at Step IV by Lemma 5.1.6. Thus if $\mu = v^k$ is a matching then $v^k \subseteq \tilde{E}$ and **IR** *hold's for all pairs at the termination of algorithm.*

Step-2 *For $v^{k-1} \in \Upsilon$, if $v^k = Tv^{k-1}$ is a matching, then the modified T-algorithm must stop at v^k, and v^k is a stable matching.*

We shall prove that, if $v^{k-1} \in \Upsilon$ is a pre-matching and $Tv^{k-1} = v^k$ is a matching, then $v^k \in M_s$. By ***Step-1***, *v^k is individually rational. Assume that $v^k \notin M_s$. So there must exist (i, j) at $c_{ij} \in [\underline{\pi}_{ij}, \overline{\pi}_{ij}]$ that blocks v^k. We will prove this by considering the two cases:*

Case 1: *Assume that $\{iR(j)Tv_j\}_{c_{ij}}^{p_{Tv_j j}}$ for $c_{ij} \leq p_{iTv_i}$. We will show $\{Tv_i R(i) j\}_{p_{iv_i}}^{c_{ij}}$. Contrary suppose there exists (i, j) that blocks v^k i.e for $c_{ij} \leq p_{iTv_i}$*

$$\{iR(j)v_j^k\}_{c_{ij}}^{p_{v_j^k j}}. \tag{5.36}$$

and

$$\{jR(i)v_i^k\}_{c_{ij}}^{p_{iv_i^k}}. \tag{5.37}$$

But v^k is a matching and $v^k = Tv^{k-1}$ by the given condition, so we have that for $j = v^k(v_j^k) \in B$. But

$$v^k(v_j^k) = Tv^{k-1}(v_j^k) = \max_{R(v_j^k)} \left\{U(v_j^k, v^{k-1})\right\}$$

for $v_j^k \in S$. So that $v^k(v_j^k) = j \in U(v_j^k, v^{k-1})$. This implies

$$\{v_j^k R(j) v_j^{k-1}\}_{p_{v_j^k j}}^{c_{v_j^{k-1} j}} \tag{5.38}$$

By transitive property, (5.36) and (5.38) imply that,

$$\{iR(j)v_j^{k-1}\}_{c_{ij}}^{c_{v_j^{k-1} j}}.$$

By definition of $U(i, v)$ from (5.6)

$$j \in U(i, v_j^{k-1}). \tag{5.39}$$

and by given condition

$$v_i^k = Tv_i^{k-1} = \max_{R(i)} \left\{ U(i, v_j^{k-1}) \right\}. \tag{5.40}$$

(5.37) and (5.39) contradicts (5.40). Thus $v^k \in M_s$. *Since we are giving the most optimal partners to the seller* i *among the potential partner in each iteration of algorithm, this implies that the following equation*

$$\{Tv_i^{k-1} R(i) j\}_{p_{iTv_i^{k-1}}}^{c_{ij}}.$$

holds at each Step.

Case 2: *Assume that* $\{jR(i)Tv_i\}_{c_{ij}}^{p_{iTv_i}}$ *for* $c_{ij} > p_{Tv_j j}$. *We will show* $\{Tv_j R(j) i\}_{p_{v_j j}}^{c_{ij}}$. *If the algorithm terminates in the first iteration then (5.2) is the maximum price so it hold's trivially. We get stability by* ***Case 1***. *Otherwise we divide our argument in two parts:* $\omega_{ji}(-c_{ij}) \le \omega_{jTv_j}(-p_{Tv_j j})$ *or* $\omega_{ji}(-c_{ij}) > \omega_{jTv_j}(-p_{Tv_j j})$.
To show, for $\{jR(i)Tv_i\}_{c_{ij}}^{p_{iTv_i}}$ *for* $c_{ij} > p_{jTv_j}$, *we have* $\{Tv_j R(j) i\}_{p_{Tv_j j}}^{c_{ij}}$. *This means* $\omega_{ji}(-c_{ij}) \le \omega_{jTv_j}(-p_{Tv_j j})$. *For this we assume that there exist a* (i, j) *that blocks* v^k *i.e for* $c > p_{jTv_j}$

$$\{iR(j)v_j^k\}_{c_{ij}}^{p_{v_j^k j}}. \tag{5.41}$$

$$\{jR(i)v_i^k\}_{c_{ij}}^{p_i v_i^k}. \tag{5.42}$$

But v^k *is a matching,* $v^k = Tv^{k-1}$, *so we have that* $i = v^k(v_i^k) \in S$. *Hence* $v^k(v_i^k) = Tv^{k-1}(v_i^k)$

$$= \max_{R(v_i^k)} \left\{ V(v_i^k, v^{k-1}) : Tv^{k-1}(v^k(v_i^k)) = v_i^k \right\}.$$

for $v^k(v_i^k) \in S$.
So that $v^k(v_i^k) = i \in V(v_i^k, v^{k-1})$. *This implies by (5.7)*

$$\{v_i^k R(i) v_i^{k-1}\}_{p_i v_i^k}^{c_i v_i^{k-1}}. \tag{5.43}$$

By transitive property, (5.42) and (5.43) imply that,

$$\{jR(i)v_i^{k-1}\}_{c_{ij}}^{c_i v_i^{k-1}}.$$

From (5.7)

$$i \in V(j, v^{k-1}). \tag{5.44}$$

and by given condition, for $j \in B$, $v_j^k = Tv_j^{k-1}$

$$= \max_{R(j)} \left\{ V(j, v^{k-1}) : Tv^{k-1}(v_j^k) = j \right\}. \tag{5.45}$$

(5.41) and (5.44) contradicts (5.45). Thus $v^k \in M_s$. *And if* $\omega_{ji}(-c_{ij}) > \omega_{jTv_j}(-p_{Tv_j j})$ *then by Lemma 5.1.7* c_{ij} *is the maximum price. Thus we get stability by above **Case 1**.*

*By **Step-1** and **Step-2**, we deduce that if* v^k *is a matching for some iteration* k *in modified* T*-algorithm, then* v^k *is a stable matching. If* $v^k = \mu$ *is a stable matching, then* v^k *is a fixed point of modified* T *by the Theorem 5.1.1. Then*

$$v^k = v^{k+1} = Tv^k = \mu.$$

implies that the T*-algorithm stops at* $\mu = v^k$. *Thus this proves Theorem 5.1.2.*

Theorem 5.1.1 shows that the stable matching is a fixed point of T and Theorem 5.1.2 shows the termination of the algorithm.

Example 5.1.1. Consider two finite and disjoint sets $S = \{i_0, i_1, i_2, i_3\}$ and $B = \{j_0, j_1, j_2, j_3\}$ where S represent the set of sellers and B represent the set of buyers. The set of all possible seller-buyer pair is given by $E = S \times B$. Define the lower and upper bounds for all $(i, j) \in E$ as given below:

$$\begin{aligned} \underline{\pi}_{ij} &= -1 \qquad \forall (i,j) \in E, \\ \overline{\pi}_{ij_0} &= 3 = \overline{\pi}_{ij_1} \ \forall i \in S, \\ \overline{\pi}_{ij_2} &= 2 = \overline{\pi}_{ij_3} \ \ \forall i \in S. \end{aligned}$$

We assume valuations given by (5.1), where λ_{ij}, λ_{ji}, δ_{ij}, and δ_{ji} for each $(i, j) \in E$ are given in the Tables 5.1-5.4 respectively.

We begin from the Step I of the modified T-algorithm. We find price vector

λ_{ij}	j_0	j_1	j_2	j_3
i_0	2	2.5	1	1
i_1	2	2.5	1	1
i_2	2	2.5	1	1
i_3	2	2.5	1	1

Table 5.1: λ_{ij} for $(i,j) \in E$

λ_{ji}	i_0	i_1	i_2	i_3
j_0	2	2	2	2
j_1	2.5	2.5	2.5	2.5
j_2	1	1	1	1
j_3	1	1	1	1

Table 5.2: λ_{ji} for $(i,j) \in E$

δ_{ij}	j_0	j_1	j_2	j_3
i_0	2.5	-3	2	10.5
i_1	8.5	2.5	5.5	-3
i_2	4.5	1.5	6.5	5.5
i_3	2.5	10.5	8.5	3.5

Table 5.3: δ_{ij} for $(i,j) \in E$

δ_{ji}	i_0	i_1	i_2	i_3
j_0	-0.5	12.5	7.5	-6.5
j_1	10.5	12.5	20.5	0.5
j_2	11.5	10.5	3.5	0
j_3	10.5	11.5	12.5	13.5

Table 5.4: δ_{ji} for $(i,j) \in E$

$p = (p_{i_0j_0}, p_{i_0j_1}, p_{i_0j_2}, p_{i_0j_3}, p_{i_1j_0}, p_{i_1j_1}, \cdots, p_{i_3j_3})$ by (5.2). We have $\omega_{ji}(-\overline{\pi}_{ij}) \geq 0$ for $(i,j) \in E \setminus \{(i_1, j_3), (i_3, j_0)\}$ and $\omega_{ji}(-\overline{\pi}_{ij}) < 0$ for $(i,j) \in \{(i_1, j_3), (i_3, j_0)\}$. Therefore, we have $p = (-1, 3, 2, 2, 3, 3, 2, 2, 3, 3, 2, 2, -1, 0, 0, 2)$. Using the values given in Tables 5.1-5.4 and price vector p from (5.2), we get $\omega_{ij}(p_{ij})$ and $\omega_{ji}(-p_{ij})$ from (5.1) for each $(i,j) \in E$ as follows in Tables 5.5 and 5.6.

$\omega_{ij}(p_{ij})$	j_0	j_1	j_2	j_3
i_0	0.5	4.5	4	12.5
i_1	14.5	10	7.5	-1
i_2	10.5	9	8.5	7.5
i_3	0.5	10.5	8.5	5.5

Table 5.5: $\omega_{ij}(p_{ij})$ for $(i,j) \in E$

$\omega_{ji}(-p_{ij})$	i_0	i_1	i_2	i_3
j_0	1.5	6.5	1.5	-4.5
j_1	3	5	13	0.5
j_2	9.5	8.5	1.5	0
j_3	8.5	9.5	10.5	11.5

Table 5.6: $\omega_{ji}(-p_{ij})$ for $(i,j) \in E$

By (5.3) and (5.4), we have $L_0 = \{(i_3, j_0)\}$ and $E_0 = \{(i_1, j_3)\}$. By (5.5), we obtain the set of mutually acceptable seller-buyer pairs which is presented by:

$$\tilde{E} = E \setminus \{(i_1, j_3), (i_3, j_0)\}.$$

(Here we will consider the two different pre-matchings to get the two extreme(optimal) matchings of hybrid model). Let consider a pre-matching

$$\begin{array}{rcccccccccc} & & i_0 & i_1 & i_2 & i_3 & j_0 & j_1 & j_2 & j_3 \\ v^0 & = & j_3 & j_0 & j_0 & j_1 & \emptyset & \emptyset & \emptyset & \emptyset \\ U(i,v^0)|V(j,v^0) & = & j_0,j_1,j_2,j_3 & j_0,j_1,j_2 & j_0,j_1,j_2,j_3 & j_1,j_2,j_3 & i_1,i_2 & i_3 & \emptyset & i_0 \\ Tv^0 = v^1 & = & j_3 & j_0 & j_0 & j_1 & i_1 & i_3 & \emptyset & i_0 \end{array}$$

and define μ which consists of matched member of $Tv^0 = v^1$

$$\mu = \{(i_0, j_3), (i_1, j_0), (i_3, j_1)\}.$$

We find $\mathbf{r}$ by (5.10) and get:

$$r_{j_0} = 6.5, \quad r_{j_1} = 0.5, \quad r_{j_2} = 0, \quad r_{j_3} = 8.5.$$

By (5.11) we have $Y = \{(i_2, j_0)\}$. This finishes the Step I. After this we switch to the Step II since $Y \neq \emptyset$. Finding $\eta_{i_2 j_0}$ by (5.12) we get $\eta_{i_2 j_0} = 3$. Modifying price vector $\tilde{p}$ by (5.14) we get:

$$\tilde{p} = (-1, 3, 2, 2, 3, 3, 2, 2, 0, 3, 2, 2, -1, 0, 0, 2).$$

For the obtained modified price vector $\tilde{p}$, we have the values $\omega_{i_2 j_0}(\tilde{p}_{i_2 j_0}) = 4.5$, $\omega_{j_0 i_2}(-\tilde{p}_{i_2 j_0}) = 7.5$ and the remaining values given in the Table 5.5 and 5.6 remain unchanged. Both L and $\tilde{E}_0$ are empty for the obtained modified price vector. Therefore, E_0 and L_0 remain unchanged for the modified vector $\tilde{p}$.
At Step 4, we modify $\tilde{E}$ by (5.5). As both L and $\tilde{E}_0$ are empty, it implies that $\tilde{E}$ remains unchanged.

$$\begin{array}{rcccccccccc} & & i_0 & i_1 & i_2 & i_3 & j_0 & j_1 & j_2 & j_3 \\ v^1 & = & j_3 & j_0 & j_0 & j_1 & i_1 & i_3 & \emptyset & i_0 \\ U(i,v^1)|V(j,v^1) & = & j_1,j_2,j_3 & j_0,j_1,j_2 & j_0,j_1,j_2,j_3 & j_1,j_2,j_3 & i_1,i_2 & i_2,i_3 & i_2 & i_0,i_2 \\ Tv^1 = v^2 & = & j_3 & j_0 & j_1 & j_1 & i_1 & i_2 & \emptyset & i_0 \end{array}$$

$$\mu = \{(i_0, j_3), (i_1, j_0), (i_2, j_1)\}.$$

We find $\mathbf{r}$ by (5.10) and get:

$$r_{j_0} = 6.5, \quad r_{j_1} = 13, \quad r_{j_2} = 0, \quad r_{j_3} = 8.5.$$

By (5.11) we have $Y = \{(i_3, j_1)\}$. This finishes the Step I. Since $Y \neq \emptyset$ we move to Step III. Finding $\eta_{i_3 j_1}$ by (5.12) we get $\eta_{i_3 j_1} = 5$. Modifying price vector $\tilde{p}$ by (5.14) we get:

$$\tilde{p} = (-1, 3, 2, 2, 3, 3, 2, 2, 0, 3, 2, 2, -1, -1, 0, 2).$$

For the obtained modified price vector $\tilde{p}$, we have the values $\omega_{i_3 j_1}(\tilde{p}_{i_3 j_1}) = 8$, $\omega_{j_1 i_3}(-\tilde{p}_{i_3 j_1}) = 3$ and the remaining values given in the Table 5.5 and 5.6 remain unchanged. For the obtained modified price vector, by (5.13) we have $L = \{(i_3, j_1)\}$. Therefore, E_0 and L_0 get changed for $\tilde{p}$.
At Step IV, we modify $\tilde{E}$ by (5.5). Thus $\tilde{E} = E \setminus \{(i_1, j_3), (i_3, j_0), (i_3, j_1)\}$. For this updated $\tilde{E}$, we have now

$$\begin{array}{rcccccccccc} & & i_0 & i_1 & i_2 & i_3 & j_0 & j_1 & j_2 & j_3 \\ v^2 & = & j_3 & j_0 & j_1 & j_1 & i_1 & i_2 & \emptyset & i_0 \\ U(i, v^2)|V(j, v^2) & = & j_2, j_3 & j_0, j_2 & j_0, j_1, j_2, j_3 & j_2, j_3 & i_1 & i_2 & i_3 & i_0, i_3 \\ Tv^2 = v^3 & = & j_3 & j_0 & j_1 & j_2 & i_1 & i_2 & i_3 & i_0 \end{array}$$

and

$$\mu = \{(i_0, j_3), (i_1, j_0), (i_2, j_1), (i_3, j_2)\}.$$

Since $Y = \emptyset$. Algorithm at this point terminates and matching

$$\mu = \{(i_0, j_3), (i_1, j_0), (i_2, j_1), (i_3, j_2)\}.$$

is a pairwise seller optimal matching.

Let consider another pre-matching

$$\begin{array}{rcccccccccc} & & i_0 & i_1 & i_2 & i_3 & j_0 & j_1 & j_2 & j_3 \\ v^0 & = & \emptyset & \emptyset & \emptyset & \emptyset & i_1 & i_2 & i_0 & i_3 \\ U(i, v^0)|V(j, v^0) & = & j_2 & j_0 & j_1 & j_3 & i_0, i_1, i_2 & i_0, i_1, i_2, i_3 & i_0, i_1, i_2, i_3 & i_0, i_2, i_3 \\ Tv^0 = v^1 & = & j_2 & j_0 & j_1 & j_3 & i_1 & i_2 & i_0 & i_3 \end{array}$$

and

$$\mu = \{(i_0, j_2), (i_1, j_0), (i_2, j_1), (i_3, j_3)\}.$$

Since $Y = \emptyset$. Our algorithm terminates at this point and the matching μ is a pairwise buyer optimal matching. Thus we can say that M_s consists of at least two stable optimal matchings.

Next we will give an example to show the working of Ali and Farooq [2] algorithm for finding a stable matching.

Example 5.1.2. We consider two finite and disjoint sets $S = \{i_0, i_1, i_2, i_3\}$ and $B = \{j_0, j_1, j_2, j_3\}$ where S represent the set of sellers and B represent the set of buyers. The set of all possible seller-buyer pair is given by $E = S \times B$. Define the lower and upper bounds for all $(i, j) \in E$ as follows:

$$\begin{aligned} \underline{\pi}_{ij} &= -1 \qquad \forall\ (i,j) \in E, \\ \overline{\pi}_{ij_0} &= 3 = \overline{\pi}_{ij_1} \quad \forall i \in S, \\ \overline{\pi}_{ij_2} &= 2 = \overline{\pi}_{ij_3} \quad \forall i \in S. \end{aligned}$$

We assume valuations given by (5.1), where λ_{ij}, λ_{ji}, δ_{ij}, and δ_{ji} for each $(i, j) \in E$ are given as in the Tables 5.1 - 5.4 respectively.

We begin from the first step of the algorithm given by Ali and Farooq [2]. Here we set $r = 0$ and $\tilde{Q} = 0$. Whereas, define r as:

$$r_j = \begin{cases} \omega_{ji}(-p_{ij}) & if \quad (i,j) \in \ \mu, \text{ for some } i \in \ S, \\ 0 & \text{otherwise.} \end{cases} \qquad (j \in B) \tag{5.46}$$

and $\tilde{Q}$ as:

$$\tilde{Q} = \ \{j \in B : j \text{ is matched in } \mu\}. \tag{5.47}$$

We find price vector $p = (p_{i_0j_0}, p_{i_0j_1}, p_{i_0j_2}, p_{i_0j_3}, p_{i_1j_0}, p_{i_1j_1}, \cdots, p_{i_3j_3})$ by Eq.(5.2). We have $\omega_{ji}(-\overline{\pi}_{ij}) \geq 0$ for $(i,j) \in E \setminus \{(i_1, j_3), (i_3, j_0)\}$ and $\omega_{ji}(-\overline{\pi}_{ij}) < 0$ for $(i,j) \in \{(i_1, j_3), (i_3, j_0)\}$. Therefore, we have $p = (-1, 3, 2, 2, 3, 3, 2, 2, 3, 3, 2, 2, -1, 0, 0, 2)$. Using the values given in Tables 5.1 - 5.4 and price vector p from (5.2), we get $\omega_{ij}(p_{ij})$ and $\omega_{ji}(-p_{ij})$ by (5.1) as follows in Table 5.5 and 5.6.

Define L_0 and E_0 by (5.3) and (5.4) respectively. We get $L_0 = \{(i_3, j_0)\}$ and $E_0 = \{(i_1, j_3)\}$. By

$$\tilde{E} := E \setminus \{L_0 \cup E_0\}. \tag{5.48}$$

we obtain the set of mutually acceptable seller-buyer pairs which is given by:

$$\tilde{E} = E \setminus \{(i_1, j_3), (i_3, j_0)\}.$$

Define

$$\tilde{q}_i = \ max\{\omega_{ij}(p_{ij}) : \ (i,j) \in \ \tilde{E}\}. \tag{5.49}$$

By (5.49), we find

$$\tilde{q}_{i_0} = 12.5 \quad \tilde{q}_{i_1} = 14.5 \quad \tilde{q}_{i_2} = 10.5 \quad \tilde{q}_{i_3} = 10.5. \tag{5.50}$$

Now,

$$\tilde{E}_S = \{(i,j) \in \tilde{E} : \omega_{ij}(p_{ij}) = \tilde{q}_i\}. \tag{5.51}$$

we get $\tilde{E}_S = \{ \ (i_0, j_3), \ (i_1, j_0), \ (i_2, j_0), \ (i_3, j_1)\}$. Since $r = 0$, we have $\tilde{E}_S = \hat{E}_S$. Where

$$\hat{E}_S = \{(i,j) \in \tilde{E}_S : \omega_{ji}(-p_{ij}) \geq r_j\}. \tag{5.52}$$

and define μ be a matching in bipartite graph $(S, B, \hat{E}_S)$ which satisfies the two conditions. First one is μ matches all members of $\tilde{Q}$. Second one is it maximizes $\sum_{(i,j)\in\mu}(\omega_{ji}(-p_{ij}))$ among the matchings satisfying first condition. So, we obtain

$$\mu = \{ \ (i_0, j_3), \ (i_1, j_0), \ (i_3, j_1)\}.$$

We find $\mathbf{r}$ by (5.46) and get:

$$r_{j_0} = 6.5, \quad r_{j_1} = 0.5, \quad r_{j_2} = 0, \quad r_{j_3} = 8.5.$$

Updating $\tilde{Q}$ by (5.47), we obtain

$$\tilde{Q} = \{j_0, j_1, j_3\}. \tag{5.53}$$

Now,

$$Y = \{(i,j) \in \tilde{E}_S : i \text{ is unmatched in } \mu\}. \tag{5.54}$$

We get $Y = \{(i_2, j_0)\}$. This finishes the first step. After this we switch to the second step of their algorithm because $Y \neq \emptyset$. By

$$\eta_{ij} = max\left\{1, \left\lceil \frac{r_j - \omega_{ji}(-p_{ij})}{\lambda_{ji}} \right\rceil\right\}. \tag{5.55}$$

finding $\eta_{i_2 j_0}$, we get $\eta_{i_2 j_0} = 3$. Modifying price vector $\tilde{p}$ as:

$$\tilde{p}_{ij} := \begin{cases} max\{\ \underline{\pi}_{ij},\ p_{ij} - \eta_{ij}\} & \text{if} \quad (i,j) \in Y, \\ \quad p_{ij} & \text{otherwise.} \end{cases} \qquad (i,j) \in E \tag{5.56}$$

we get:

$$\tilde{p} = (-1, 3, 2, 2, 3, 3, 2, 2, 0, 3, 2, 2, -1, 0, 0, 2).$$

For the obtained modified price vector $\tilde{p}$, we have the values $\omega_{i_2 j_0}(\tilde{p}_{i_2 j_0}) = 4.5$, $\omega_{j_0 i_2}(-\tilde{p}_{i_2 j_0}) = 7.5$ and the remaining values given in the Table 5.5 and Table 5.6 remain unchanged. Both L and $\tilde{E}_0$ are empty for the obtained modified price vector. Therefore, E_0 and L_0 remain unchanged for the modified vector $\tilde{p}$.
At forth step of their algorithm, modify $\tilde{E}$ by (5.48). As both L and $\tilde{E}_0$ are empty, it implies that $\tilde{E}$ remains unchanged. By (5.49), we get

$$\tilde{q}_{i_0} = 12.5 \quad \tilde{q}_{i_1} = 14.5 \quad \tilde{q}_{i_2} = 9 \quad \tilde{q}_{i_3} = 10.5.$$

By updating and $\tilde{E}_S$ and $\hat{E}_S$ by (5.51) and (5.52), respectively, we obtain

$$\tilde{E}_S = \{\ (i_0, j_3),\ (i_1, j_0),\ (i_2, j_1),\ (i_3, j_1)\}.$$

and

$$\hat{E}_S = \{\ (i_0, j_3),\ (i_1, j_0),\ (i_2, j_1),\ (i_3, j_1)\}.$$

In bipartite graph $(S, B, \hat{E}_S)$, a matching

$$\mu = \{\ (i_0, j_3),\ (i_1, j_0),\ (i_2, j_1)\}.$$

We find $\mathbf{r}$ by (5.46) and get:

$$r_{j_0} = 6.5, \quad r_{j_1} = 13, \quad r_{j_2} = 0, \quad r_{j_3} = 8.5.$$

By (5.54) we have $Y = \{(i_3, j_1)\}$. This finishes the second step. Since $Y \neq \emptyset$ we move to the third step of their algorithm. Finding $\eta_{i_3j_1}$ by (5.55) we get $\eta_{i_3j_1} = 5$. Modifying price vector $\tilde{p}$ by (5.56) we get:

$$\tilde{p} = (-1, 3, 2, 2, 3, 3, 2, 2, 0, 3, 2, 2, -1, -1, 0, 2).$$

For the obtained modified price vector $\tilde{p}$, we have the values $\omega_{i_3j_1}(\tilde{p}_{i_3j_1}) = 8$, $\omega_{j_1i_3}(-\tilde{p}_{i_3j_1}) = 3$ and the remaining values given in the Table 5.5 and 5.6 remain unchanged. For the obtained modified price vector, by (5.56) we have $L = \{(i_3, j_1)\}$. Therefore, E_0 and L_0 get changed for $\tilde{p}$.
At forth step, we modify $\tilde{E}$ by (5.48). Thus $\tilde{E} = E \setminus \{(i_1, j_3), (i_3, j_0), (i_3, j_1)\}$. For this updated $\tilde{E}$, we have now

$$\tilde{q}_{i_0} = 12.5 \quad \tilde{q}_{i_1} = 14.5 \quad \tilde{q}_{i_2} = 9 \quad \tilde{q}_{i_3} = 8.5.$$

Again by updating $\tilde{E}_S$ and $\hat{E}_S$ by (5.51) and (5.52), respectively, we obtain

$$\tilde{E}_S = \{(i_0, j_3), (i_1, j_0), (i_2, j_1), (i_3, j_2)\}.$$

and

$$\hat{E}_S = \{(i_0, j_3), (i_1, j_0), (i_2, j_1), (i_3, j_2)\}.$$

In bipartite graph $(S, B, \hat{E}_S)$, a matching

$$\mu = \{(i_0, j_3), (i_1, j_0), (i_2, j_1), (i_3, j_2)\}.$$

Since $Y = \emptyset$. Our algorithm terminates at this point and matching

$$\mu = \{(i_0, j_3), (i_1, j_0), (i_2, j_1), (i_3, j_2)\}.$$

is a pairwise seller optimal matching.

A limitation of Ali and Farooq [2] model is that it suggests the seller optimal matching only.

Remark 5.1.3. *If we use the algorithm given by Ali and Farooq [2], we obtain only one optimal stable matching as working given in Example 5.1.2. While if we use modified T-algorithm using the same given conditions as in Example 5.1.2, then one can obtain more than one stable matching as working given in Example 5.1.1.*

The motivation of this work is to find more stable matchings within their model.

5.1.2 Lattice structure of stable one to one matching

If the preferences of the agents are flexible, by using Theorem 2.3.1, we obtain a lattice structure on ξ.

Definition 5.1.8. Define the following complete partial orders on Υ_S,Υ_B and Υ:

1. $R^{\otimes}(S)$ on Υ_S by $v_S R^{\otimes}(S)\hat{v}_S$ if and only if:
$$\{v_i R(i)\hat{v}_i\}_{p_{iv_i}}^{p_{i\hat{v}_i}} \qquad \forall i \in S.$$
2. $R^{\otimes}(B)$ on Υ_B by $v_B R^{\otimes}(B)\hat{v}_B$ if and only if:
$$\{v_j R(j)\hat{v}_j\}_{p_{v_j j}}^{p_{\hat{v}_j j}} \qquad \forall j \in B.$$
3. The strict partial order associated to $R^{\otimes}(S)$ and $R^{\otimes}(B)$ are represented by $P^{\otimes}(S)$ and $P^{\otimes}(B)$ can be defined similarly.
4. $\preceq_{\otimes}$ on Υ by $v \preceq_{\otimes} \hat{v}$ if and only if:
$$v_S R^{\otimes}(S)\hat{v}_S \quad \text{and} \quad \hat{v}_B R^{\otimes}(B)v_B.$$
5. The strict partial order associated to $\preceq_{\otimes}$ on Υ is represented by $\prec_{\otimes}$ and can be defined similarly.

Υ defined in (3.1) contains all pre-matchings either having acceptable or un-acceptable partners. Since, modified T-algorithm only involves acceptable partners so we define the following sub collection.

$$\hat{\Upsilon} = \{v \in \Upsilon : \{v_s R(s)\emptyset\}_{p_{sv_s}}, \forall s \in S \cup B\}. \tag{5.57}$$

$\hat{\Upsilon}$ consists of all such pre-matchings in which agents are weakly preferring their partners to remain isolated at a feasible price. This implies $\hat{\Upsilon} \subseteq \Upsilon$. Thus, $\forall v \in \Upsilon$, $Tv \in \hat{\Upsilon}$. It is worth noting that the function $T : \hat{\Upsilon} \to \hat{\Upsilon}$ is a self map.

Lemma 5.1.9. *Let $\hat{\Upsilon}$ be the collection of pre-matchings defined in (5.57). Then $(\hat{\Upsilon}, \preceq_{\otimes})$ forms a complete lattice.*

Proof: For each i, let $\hat{\Upsilon}_i = \{j \in B : \{jR(i)\emptyset\}_{p_{ij}}\}$. Then, $(\hat{\Upsilon}_i, R(i))$ is a totally ordered finite set and so is a complete lattice. Similarly $(\hat{\Upsilon}_j, R(j))$, for each j is a complete lattice where $\hat{\Upsilon}_j = \{i \in S : \{iR(j)\emptyset\}_{p_{ij}}\}$.

Note that $\hat{\Upsilon}$ defined in (5.57) can be viewed as $\hat{\Upsilon} = (\times_{i\in S}\hat{\Upsilon}_i) \times (\times_{j\in B}\hat{\Upsilon}_j)$. Hence $(\hat{\Upsilon}, \preceq_{\otimes})$ is also a totally ordered finite set and forms a complete lattice where $\preceq_{\otimes}$ is the complete partial order given in Definition 5.1.8.

Lemma 5.1.10. *Let μ and $\hat{\mu}$ be pre-matchings. If $\hat{\mu} \preceq_{\otimes} \mu$ then, $T\hat{\mu} \preceq_{\otimes} T\mu$ i.e; T defined in* (5.8) *is monotone increasing function on $\hat{\Upsilon}$.*

Proof: Let $j \in U(i,\mu)$. We have that $\{iR(j)\mu_j\}_{p_{ij}}^{c_{\mu_j j}}$, but the definition of $\hat{\mu} \preceq_{\otimes} \mu$ implies $\{\mu_j R(j)\hat{\mu}_j\}_{c_{\mu_j}j}^{c_{\hat{\mu}_j j}}$. So, by using transitive property we have $\{iR(j)\hat{\mu}_j\}_{p_{ij}}^{c_{\hat{\mu}_j j}}$. This implies $j \in U(i,\hat{\mu})$. This proves $U(i,\mu) \subseteq U(i,\hat{\mu})$. Now,

$$\begin{aligned} T\hat{\mu}_i &= \max_{R(i)}\{U(i,\hat{\mu})\} \quad \text{for} \quad i \in S \\ &R(i) \ \max_{R(i)}\{U(i,\mu)\} \\ &= T\mu_i. \end{aligned}$$

This implies $T\hat{\mu}R^{\otimes}(S)T\mu$.

Similarly let $i \in V(j,\hat{\mu})$. Then by definition we have that $\{jR(i)\hat{\mu}_i\}_{p_{ij}}^{c_{i\hat{\mu}_i}}$, but the definition of $\hat{\mu} \preceq_{\otimes} \mu$ implies $\{\hat{\mu}_i R(i)\mu_i\}_{c_{i\hat{\mu}_i}}^{c_{i\mu_i}}$. So, by using transitive property we have $\{jR(i)\mu_i\}_{p_{ij}}^{c_{i\mu_i}}$. This implies $i \in V(j,\mu)$. Thus proves $V(j,\hat{\mu}) \subseteq V(j,\mu)$. Now

$$\begin{aligned} T\mu_j &= \max_{R(j)}\{V(j,\mu) : Tv_i = j\} \quad \text{for} \quad j \in B \\ &R(j) \ \max_{R(j)}\{V(j,\hat{\mu}) : Tv_i = j\} \\ &= T\hat{\mu}_i. \end{aligned}$$

This implies $T\hat{\mu}R^{\otimes}(B)T\mu$. The above arguments show that $T\mu \preceq_{\otimes} T\hat{\mu}$. This completes the proof that T is an increasing function on Υ.

Since $T(\Upsilon) \subseteq \hat{\Upsilon}$, the restriction $T\mid_{\hat{\Upsilon}}$ is an increasing function on $\hat{\Upsilon}$. Also, the set of fixed points of T equals the set of fixed points of $T\mid_{\hat{\Upsilon}}$.

Theorem 5.1.3. *The set of fixed points ξ of T is non-empty and forms a complete lattice.*

Proof: Since $(\hat{\Upsilon}, \preceq_{\otimes})$ is a complete lattice by Lemma 5.1.9 and T is an increasing function from $\hat{\Upsilon}$ to $\hat{\Upsilon}$ from Lemma 5.1.10. If ξ is the set of all the fixed points of T then from Theorem 2.3.1 ξ is non-empty and the system $(\xi, \preceq_{\otimes})$ is a complete lattice.

Remark 5.1.4. *If we switch the role of sellers with buyers and define a price vector by:*

$$p_{ij} = \begin{cases} \underline{\pi}_{ij} & \text{if} \quad \omega_{ij}(\underline{\pi}_{ij}) \geq 0 \\ \min\left\{\overline{\pi}_{ij}, \left\lceil \frac{-\delta_{ij}}{\lambda_{ij}} \right\rceil\right\} & \text{otherwise.} \end{cases} \tag{5.58}$$

Then, by following the same strategy of the above hybrid model, we obtain another set of stable matchings. All the above results will hold in this case too.

Here we will demonstrate this above fact within an Example 5.1.1.

Example 5.1.5. We will solve Example 5.1.1 by taking price vector as given in (5.58). We begin from the Step I of the algorithm. We find price vector

$p = (p_{i_0j_0}, p_{i_0j_1}, p_{i_0j_2}, p_{i_0j_3}, p_{i_1j_0}, p_{i_1j_1}, \cdots, p_{i_3j_3})$ by Eq.(5.58). We have $\omega_{ji}(-\overline{\pi}_{ij}) \geq 0$ for $(i,j) \in E \setminus \{(i_1, j_3), (i_3, j_0)\}$ and $\omega_{ji}(-\overline{\pi}_{ij}) < 0$ for $(i,j) \in \{(i_1, j_3), (i_3, j_0)\}$. Therefore, we have $p = (-1, 2, -1, -1, -1, -1, -1, 2, -1, 0, -1, -1, -1, -1, -1, -1)$. Using the values given in Tables (5.1)-(5.4) and price vector p from (5.58), we get $\omega_{ij}(p_{ij})$ and $\omega_{ji}(-p_{ij})$ from (5.1) for each $(i,j) \in E$ as follows in Tables 5.7 and 5.8.

$\omega_{ij}(p_{ij})$	j_0	j_1	j_2	j_3
i_0	0.5	2	1	9.5
i_1	6.5	0	4.5	-1
i_2	2.5	1.5	5.5	4.5
i_3	0.5	8	7.5	2.5

Table 5.7: $\omega_{ij}(p_{ij})$ for $(i,j) \in E$

$\omega_{ji}(-p_{ij})$	i_0	i_1	i_2	i_3
j_0	1.5	14.5	9.5	-4.5
j_1	5.5	15	20.5	3
j_2	12.5	11.5	4.5	1
j_3	11.5	9.5	13.5	14.5

Table 5.8: $\omega_{ji}(-p_{ij})$ for $(i,j) \in E$

By (5.3) and (5.4), we have $L_0 = \{(i_3, j_0)\}$ and $E_0 = \{(i_1, j_3)\}$. By (5.5), we obtain the set of mutually acceptable seller-buyer pairs which is given by:

$$\tilde{E} = E \setminus \{(i_1, j_3), (i_3, j_0)\}.$$

(Here we will consider the two different pre-matchings to get the two extreme(optimal) matchings of hybrid model). Let consider a pre-matching

$$\begin{array}{rcccccccccc} & & i_0 & i_1 & i_2 & i_3 & j_0 & j_1 & j_2 & j_3 \\ v^0 & = & \emptyset & \emptyset & \emptyset & \emptyset & i_1 & i_2 & i_0 & i_3 \\ U(i,v^0)|V(j,v^0) & = & j_2 & j_0 & j_1 & j_3 & i_0,i_1,i_2 & i_0,i_1,i_2,i_3 & i_0,i_1,i_2,i_3 & i_0,i_2,i_3 \\ Tv^0 = v^1 & = & j_2 & j_0 & j_1 & j_3 & i_1 & i_2 & i_0 & i_3 \end{array}$$

$$\mu = \{(i_0,j_2),(i_1,j_0),(i_2,j_1),(i_3,j_3)\}.$$

Since $Y = \emptyset$. Algorithm at this point terminates and the matching μ is a pairwise buyer optimal matching.

Let consider another pre-matching

$$\begin{array}{rcccccccccc} & & i_0 & i_1 & i_2 & i_3 & j_0 & j_1 & j_2 & j_3 \\ v^0 & = & j_3 & j_0 & j_2 & j_1 & \emptyset & \emptyset & \emptyset & \emptyset \\ U(i,v^0)|V(j,v^0) & = & j_0,j_1,j_2,j_3 & j_0,j_1,j_2 & j_0,j_1,j_2,j_3 & j_1,j_2,j_3 & i_1 & i_3 & i_2 & i_0 \\ Tv^0 = v^1 & = & j_3 & j_0 & j_2 & j_1 & i_1 & i_3 & i_2 & i_0 \end{array}$$

and define μ which consists of matched member of $Tv^0 = v^1$

$$\mu = \{(i_0,j_3),(i_1,j_0),(i_2,j_2),(i_3,j_1)\}.$$

Since $Y = \emptyset$. Our algorithm terminates at this point and the matching μ is a pairwise seller optimal matching. Thus we can say that M_s for the price vector given by (5.58) consists of at least two stable optimal matchings.

Remark 5.1.6. *The lattice structure obtain from using the price vector (5.58) is incomparable as obtained with the price vector (5.2) because due to the different price vectors, valuations get change and results in a different optimal stable matchings.*

Chapter 6

Conclusion

This chapter finishes up the research by expressing and summarizing the inferences and findings. The knowledge assists the reader to understand the essence of the study and parting ways for future undertakings identified with this territory of research.

The main aim of this thesis is to study the lattice structure of stable matchings. We initiated by taking Theorem 2.3.1, and see its application in the formation of lattice structure by reviewing various papers, mainly reviewed papers are stated in Chapter 3 and 4. Then finally, we present a hybrid model to the theory of stable matchings. This model designed with the help of modified T-Algorithm and it differs from the two existing models presented in [2] and [6] in the following way:

1. The modified T-Algorithm not only gives the seller optimal stable matching as obtained by Ali and Farooq [2] but because of the fact that it starts with any random pre-matching, more than one stable matchings can be achieved including the buyer and the seller optimal matchings. Moreover, these stable matchings can be characterize as fixed points of T and hence forms a complete lattice.

2. The modified T-Algorithm is better from T-algorithm by Echenique and Oviedo [6] because this hybrid algorithm involves flexible agents and price externalities (modification) unlike T-algorithm that involves no price negotiation.

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