Approximate Closed-form Solution of an Ordinary Differential Equation

by

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Dedicated to

My beloved Mother and Siblings

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All praise to Almighty Allah, Lord of the Universe.

In the name of Allah, the sustainer of the worlds who blessed me to accomplish this work successfully and my humblest gratitude to the greatest symbol of guidance for humanity, the Holy Prophet **MUHAMMAD$^{\textbf{(SAW)}}$** (Peace be upon Him). In the first place, I feel honored to record my greatest appreciation towards my supervisor Prof. Azad Akhter Siddiqui, for his benevolent guidance, invaluable suggestions and critical appreciation which enabled me to successfully complete my work. Working under his guidance is a great source of inspiration for me. My work was further enhanced by working with Co-Supervisor Prof. Faiz Ahmed and his suggestions were extremely helpful throughout this thesis. I am also thankful to my GEC members Dr. Asif Farooq and Dr. Mujeeb Ur Rehman for their valuable advices.
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Abstract

The purpose of this thesis is obtaining approximate closed-form solution of the non linear ordinary differential equation. Finding the exact solution is very hard, when dealing with non-linear ordinary differential equations. Therefore, in such conditions, one looks for the approximate solutions. In this thesis, by using similarity transformation firstly the reduction of partial differential equation is carried out into an ordinary differential equation. Then the reduced ordinary differential equation is solved numerically and an approximate closed form solution using spectral collocation method is achieved. Both numerical and approximate solutions are compared by presenting plots. Residuals are used for analyzing the accuracy of the solution. The residual analysis describes that the obtained approximate solution is a good approximation of the exact solution.

Contents

Chapter 1

Introduction

Mathematical models describe important features and behavior of physical systems. Basically, a model is created for obtaining new knowledge and information about the world and system as well. Thus using reasonable approximations and by applying relevant physical principles a physical system can be modeled mathematically and as a result the equations obtained are often differential equations (DEs). Differential equations describe the relation between variables and their derivatives and play a very significant role in diverse fields of science, mathematics and many more. DEs are further categorized in two more types, partial differential equations (PDEs) and ordinary differential equations (ODEs) [1].
An ODE is defined as an equation which contains the derivatives of one or more dependent variables with respect to one or more independent variables [1–6] and PDE contains partial derivatives of one or more dependent variables of two or more independent variables. [1, 3]. DEs are further categorized depending on

the types of conditions; Initial-value problems (IVPs) and other is boundary value problems (BVPs). Equations in which conditions are specified at only one point are called as IVPs while the differential equations in which conditions are specified at more than one points are known as BVPs.

DEs are furthermore classified into linear DEs and the non-linear DEs [1]. If the power of dependent variable and all of its derivatives is one and their coefficients do not depend on dependent variable and its derivatives, then such equation are linear differential equation, otherwise the equation is defined as non linear differential equation. There are different techniques and methods available for finding the solution of linear DE, however, it is very difficult and also very limited theory is available for finding the analytic solutions of non-linear DEs. A technique which is used for the two linear and non-linear DEs is "symmetry method" [2]. In symmetry method, by using (if available) symmetry transformations of a partial differential equation, one can find the exact solution by first reducing the number of independent variables to obtain an ODE and then the order of the obtained ODE is reduced [2–5, 7].

Thesis plan is as follows. In next coming section, some basics terms are defined for better understanding. In Chapter 2, spectral method, for the solution of DE is discussed in detail. In Chapter 3, by using similarity transformation, reduction of a PDE to a non-linear ODE is carried out and then spectral collocation method is used for obtaining an approximate solution of the non-linear ODE. Residuals are used for analyzing the accuracy of the solution. The residual analysis shows that the obtained approximate closed form solution is a good choice.

1.1 Basic Definitions

Orthogonality

let h and i are two real-valued functions with respect to the weight function $w(x)>0$ defined on the interval (a, b) are called orthogonal if there inner-product is zero [8], that is

$$< h, i >= \int_a^b h(x)i(x)w(x)dx = 0, \qquad \text{whenever} \qquad h \neq i. \tag{1.1}$$

For $w(x)=1$, h and i are said to be simply orthogonal [8].

A polynomial $f_p(x)$ of degree p is given as

$$f_p(x) = a_0 + a_1x + a_2x^2 + ... + a_px^p, \qquad a_p \neq 0. \tag{1.2}$$

A sequence of polynomials $[f_p]_{p=0}^{\infty}$ is known as orthogonal set of polynomials, if the polynomials are pairwise orthogonal [8]. Two polynomials f_p and f_q are two polynomials they are orthogonal if

$< f_p, f_q >= 0, \qquad p \neq q,$

and if

$< f_p, f_p >= 1, \qquad p = 0, 1, ...$

then it is said to be an orthonormal set.

Gram Schmidt Process:

Gram-Schmidt process with respect to an inner product is used to construct for each linearly independent functions a corresponding sequence of orthogonal functions. [1,9]. Let we have a set of linearly independent functions $[r_i]_{i=1}^{n}$ then it generates an set of orthogonal functions $[s_i]_{i=1}^{n}$ in the following manner.

$$s_1 = r_1,$$

$$s_2 = r_2 - \frac{< s_1, r_2 >}{< s_1, s_1 >} s_1,$$

$$s_3 = r_3 - \frac{< s_1, r_3 >}{< s_1, s_1 >} s_1 - \frac{< s_2, r_3 >}{< s_2, s_2 >} s_2,$$

:

:

$$s_n = r_n - \frac{< s_1, r_n >}{< s_1, s_1 >} s_1 - \frac{< s_2, r_n >}{< s_2, s_2 >} s_2, - - - - - - - - -, \frac{< s_{n-1}, r_n >}{< s_{n-1}, s_{n-1} >} s_{n-1}.$$

Example:

For generating a set of orthogonal polynomials, we take a linearly independent sequence $(1, x, x^2, ...)$ where $(a, b) = (0, 1)$ with $w(x) = 1$.

let us take

$$s_1 = r_1 = 1, \tag{1.3}$$

then we have

$$s_2 = r_2 - \frac{< s_1, r_2 >}{< s_1, s_1 >} s_1, \qquad = x - \frac{< 1, x >}{< 1, 1 >} 1, \tag{1.4}$$

since

$$< 1, x >= \int_0^1 x dx = 1/2. \qquad and \qquad < 1, 1 >= \int_0^1 dx = 1. \tag{1.5}$$

Using Eq. (1.5) in Eq. (1.4), we get

$$s_2 = x - 1/2. \tag{1.6}$$

Further

$$s_3 = r_3 - \frac{< s_1, r_3 >}{< s_1, s_1 >} s_1 - \frac{< s_2, r_3 >}{< s_2, s_2 >} s_2,$$

$$= x^2 - \frac{< 1, x^2 >}{< 1, 1 >} 1 - \frac{< x - 1/2, x^2 >}{< x - 1/2, x - 1/2 >} (x - 1/2). \tag{1.7}$$

$$< 1, x^2 >= \int_0^1 x^2 dx = 1/3. \tag{1.8}$$

$$< x - 1/2, x^2 >= \int_0^1 x^2(x - 1/2)dx = 1/12. \tag{1.9}$$

$$< x - 1/2, x - 1/2 >= \int_0^1 (x - 1/2)^2 dx = 1/12. \tag{1.10}$$

In Eq. (1.7) by substituting all Eq. (1.8) and Eq. (1.9) and Eq. (1.10), we get

$$s_3 = x^2 - x + 1/6. \tag{1.11}$$

Here we get the polynomials

$$s_1 = 1, \qquad s_2 = x - 1/2, \qquad and \qquad s_3 = x^2 - x + 1/6. \tag{1.12}$$

Next, we show that pairs of polynomials in Eq. (1.2) are orthogonal:

$$< s_1, s_2 >=< 1, x - 1/2 >= \int_0^1 (x - 1/2)dx = 0.$$

$$< s_1, s_3 >=< 1, x^2 - x + 1/6 >= \int_0^1 (x^2 - x + 1/6)dx = 0.$$

$$< s_2, s_3 >=< x - 1/2, x^2 - x + 1/6 >= \int_0^1 (x - 1/2)(x^2 - x + 1/6)dx = 0.$$

Zeros of Orthogonal Polynomials

Theorem: A sequence of orthogonal polynomials $[f_r(x)]_{r=0}^{\infty}$ has r distinct zeros in the interval (p, q) w.r.t the $w(x) > 0$.

Proof: Let f_0 is non-zero constant polynomial. For $r > 0$, we have

$$< f_0, f_r >= \int_p^q f_r(x) f_0 w(x) dx, \tag{1.13}$$

$$0 = f_0 \int_p^q f_r(x) w(x) dx,$$

$$\int_p^q f_r(x) w(x) dx = 0. \tag{1.14}$$

For the interval $[p, q]$ with $w(x) > 0$, $f_r(x)$ change its sign atleast one point in the given interval. Let us assume that $f_r(x)$ changes sign at the points $x_1, x_2, ..., x_k$, thus for $k \leq r$, $f_r(x)$ can have at most r distinct zeros. Let us suppose that $k < r$, then the polynomial $g_k(x)$ also changes sign at each points $x_1, x_2, ...x_k$. [6], where

$$g_k(x) = (x - x_1)(x - x_2).........(x - x_k), \tag{1.15}$$

Now

$$< f_r(x), g_k(x) >= \int_p^q f_r(x) g_k(x) w(x) dx = 0, \qquad k < r. \tag{1.16}$$

Also $g_k(x)$ changes sign at x_1 and $f_r(x)$ also changes sign at x_1 but product of $f_r(x)g_k(x)$ does not changes sign in the interval (p, q), similarly for $x_2, ..., x_k$, and

$$\int_p^q f_r(x)g_k(x)w(x)dx \neq 0, \tag{1.17}$$

which is contradiction and therefore we concluded that $k = r$. Thus in the interval (p, q), $f_r(x)$ has r distinct real zeros. [6, 9].

Chapter 2

Spectral Method for Solving Differential Equation

Spectral methods were first introduced by Lanczos (1938) and later in 1970's Orszag introduced spectral methods for solving PDE's in fluid mechanics (Trefthen 1996) [10]. Orthogonal basis functions are the most crucial component of spectral methods and it is this very feature that makes them more useful than the finite-element method and the finite-difference method . The foundation of these methods are basis functions which are used locally over small sub domains. The spectral methods take superiority because they use global basis functions instead of local functions [11, 12].

Spectral methods more often exhibit the exponential convergence which yields the ability of achieving high accuracy or precision with a small number of data points (spectral accuracy). If the data defining the equation is smooth and on a single domain, then spectral method is considering the best tool for solving DEs to high accuracy [13]. On the other hand it is very difficult for spectral methods to work with irregular domain as it losses its accuracy and efficiency.

Spectral methods are divided into two sub categories.

1. Non-Interpolating,

2. Collocation.

Tau method and Galerkin methods fall into non-interpolating methods which belong to the weighted residual methods. There are no specific collocation points in these methods. Coefficients can be obtained by using the inner product of basis functions and unknown functions. In spectral collocation method, a set of grid points known as collocation points or nodes is choosen. For minimizing the error, residual must be reduced. Collocation method is a simple choice which gives accurate results and is recommended when high accuracy is needed.

Basically spectral methods approximates the function (solutions of DEs, PDEs, etc.) by using truncated series of orthogonal basis functions. A function $f(x)$ will be approximate by the finite sum where the coefficients a_n are unknown. [11]

$$f(x) = \sum_{k=0}^{N} a_k P_k(x) \tag{2.1}$$

Spectral methods are global approximations as the values of the expansion coefficients a_k influence the function and its derivative for all x.

Suppose we have a two point non-linear boundary value problem defined on infinite dimensional space of functions.

$$\begin{cases} L[y(x)] = g, & x \in (a,b), \\ y(a) = y(b) = 0, \end{cases} \tag{2.2}$$

where L is non-linear differential operator of any specified order.

In spectral collocation method, the above given equation is satisfied by using the nodes x_k of certain grid where $k = 1, 2, ..., N-1$, $x_0 = a$, $x_N = b$ and the two boundary conditions are enforced explicitly, in other words

$$\begin{cases} L[y^N(x_n)] - g(x_n) = 0, & x_n \in (a,b), \\ y^N(a) = y^N(b) = 0. \end{cases} \tag{2.3}$$

Now by using nodes and the two boundary conditions we obtain set of equations and then this algebraic system of equations is solved for finding the unknown coefficients a_n. Spectral methods mainly concern about the selection of basis functions. These functions should have following properties,

1. Easy computing,

2. Fast convergence,

3. Completeness, meaning that by taking the truncation N to arbitrarily high accuracy. For non-periodic domains or problems, Chebyshev polynomials (CP)

and Legendre polynomials are used as suitable orthogonal polynomials for approximation. Hermite polynomials are used for approximation on real line and in a semi-infinite domain, Laguerre polynomials are best suited.

Different approaches of spectral methods are used for the solution of DEs over semi-infinite domain [13]. One direct method is using spectral method along with orthogonal system like Hermite spectral method and Laguerre spectral method. Second technique is domain truncation, in which semi-infinite domain $[0, L]$ is replace by choosing L sufficiently large. One approach for solving problems in semi infinite domain is rational approximation which is obtained when algebraic mapping is applied to the Chebyshev polynomials [14].

We have used "Chebyshev polynomials" and "rational Chebyshev functions" (RCF) in solving our problem through Spectral method, so Chebyshev polynomials and rational Chebyshev functions are discussed in the next given section.

2.1 Chebyshev Polynomials:

The polynomials $T_r(x)$, $r \in N$ are given as

$$T_r(x) = \cos[r \cos^{-1}(x)], \qquad x \in [-1, 1], \tag{2.4}$$

where $T_r(x)$ denotes the Chebyshev polynomials of first kind (CPF) [9, 15, 16].

let

$$x = \cos(\theta), \qquad . \tag{2.5}$$

$$\theta = \cos^{-1}(x), \qquad \theta \in [0, \pi]. \tag{2.6}$$

Using Eq. (2.6) and Eq. (2.5) in Eq. (2.4) we transform an algebraic polynomial into trignometric polynomial as

$$T_r(x) = \cos[r\theta], \qquad \text{where} \qquad \theta \in [0, \pi]. \tag{2.7}$$

let we have trignometric identity

$$T_{r+1}(x) = \cos[(r + 1)\theta] = \cos[r\theta] \cos[\theta] - \sin[r\theta] \sin[\theta]. \tag{2.8}$$

$$T_{r-1}(x) = \cos[(r - 1)\theta] = \cos[r\theta] \cos[\theta] + \sin[r\theta] \sin[\theta]. \tag{2.9}$$

Adding both Eq. (2.8) and Eq. (2.9) we get

$$T_{r+1}(x) + T_{r-1}(x) = 2 \cos[r\theta] \cos[\theta], \tag{2.10}$$

$$T_{r+1}(x) + T_{r-1}(x) = \cos[r\cos^{-1}(x)]x, \tag{2.11}$$

or

$$T_{r+1}(x) = \cos[r\cos^{-1}(x)]x - T_{r-1}(x), \tag{2.12}$$

or

$$T_{r+1}(x) = 2xT_r(x) - T_{r-1}(x). \qquad r > 0. \tag{2.13}$$

Thus Eq. (2.13) is the three term reccurence relation for CPF [14]. By def (2.1),

$T_0(x)$ =1,

then using the recurrence relation Eq. (2.13) we have

$T_1(x) = x$,

$T_2(x)$=$2x^2 - 1$,

$T_3(x) = 4x^3 - 3x$,

$T_4(x) = 8x^4 - 8x^2 + 1$.

Chebyshev polynomials are orthogonal with respect to the $w(x)$=$\frac{1}{\sqrt{1-x^2}}$.

$$\text{Let} \qquad x = \cos(\theta), \tag{2.14}$$

$$dx = -\sin(\theta)d\theta, \qquad dx = -\sqrt{1-x^2}d\theta. \tag{2.15}$$

Now

$$< T_g, T_h >= \int_{-1}^{1} T_g(x)T_h(x)\frac{1}{\sqrt{1-x^2}}dx, \tag{2.16}$$

when we use Eq. (2.14) and Eq. (2.15) in Eq. (2.16) it becomes

$$< T_g, T_h >= \int_0^\pi \cos(g\theta)\cos(h\theta)d\theta. \tag{2.17}$$

With the use of trigonometric identity,

$$\cos(g\theta)\cos(h\theta) = \frac{1}{2}[\cos(g+h)\theta + \cos(g-h)\theta)], \tag{2.18}$$

we have

$$< T_g, T_h >= \frac{1}{2}\int_0^\pi [\cos(g+h)\theta + \cos(g-h)\theta)]d\theta. \tag{2.19}$$

If $h = g$**.**

$$< T_g, T_g >= \frac{1}{2}\int_0^\pi cos(2g\theta)d\theta + \frac{1}{2}\int_0^\pi cos(0)\theta d\theta,$$
$$= \frac{1}{2}\frac{sin2g}{2g}|_0^\pi + \frac{1}{2}\int_0^\pi d\theta,$$
$$= \frac{1}{4g} \times [sin\pi - sin0] + \frac{1}{2}\pi,$$
$$< T_g, T_g >= \frac{\pi}{2}.$$

If $h \neq g$**.**

$$< T_g, T_h >= \frac{1}{2}\int_0^\pi cos(g+h)\theta d\theta + \frac{1}{2}\int_0^\pi cos(g-h)\theta d\theta,$$

$$= \frac{1}{2}\frac{sin(g+h)\theta}{(g+h)}|_0^\pi + \frac{1}{2}\frac{sin(g-h)\theta}{(g-h)}|_0^\pi,$$

$$< T_g, T_h >= 0.$$

if $h = g = 0$**.**

$$= \frac{1}{2}\int_0^\pi cos(0)\theta d\theta + \frac{1}{2}\int_0^\pi cos(0)\theta d\theta,$$

$$< T_0, T_0 >= \pi.$$

In summary, we have

$$(T_g, T_h) = \begin{cases} 0, & g \neq h, \\ \frac{\pi}{2}, & g = h \neq 0, \\ \pi, & g = h = 0. \end{cases} \tag{2.20}$$

2.1.1 Zeros of Chebyshev Polynomials:

In the interval [-1,1] there are exactly r distinct zeros x_p, which are given by

$$x_p = \cos\left[\frac{(2p+1)\pi}{2r}\right], \qquad p = 0, 1, 2, ..., r-1. \tag{2.21}$$

And there are $r+1$ extremas in the interval [-1,1],

$$x_k = \cos\left(\frac{k\pi}{r}\right). \tag{2.22}$$

2.2 Rational Chebyshev Functions:

By introducing an algebraic mapping to the CPF [14],

$$y = L\left(\frac{x+1}{1-x}\right), \qquad \longleftrightarrow \qquad x = \left(\frac{y-L}{y+L}\right). \tag{2.23}$$

Rational Chebyshev functions are defined as

$$R_r(x) = T_r\left[\frac{x-L}{x+L}\right], \tag{2.24}$$

where $T_r(x)$ is "Chebyshev polynomials" of first kind [14].

$L > 0$ is a constant map parameter. Varying L may alter the width of the function without changing their shape.

2.2.1 Reccurence Relation:

Reccurence formula for rational chebyshev function is

$$R_{r+1}(x) = 2\left(\frac{x-L}{x+L}\right)R_r(x) - R_{r-1}(x). \tag{2.25}$$

Here by taking $L = 1$, first few terms are given as [14].

$R_0(x) = 1,$

$R_1(x) = \dfrac{x-1}{x+1},$

$R_2(x) = \dfrac{x^2-6x+1}{(x+1)^2},$

$$R_3(x) = \frac{x^3 - 15x^2 + 5x - 1}{(x+1)^3}.$$

The RCF are orthogonal with respect to $w(x) = \dfrac{1}{\sqrt{x}(x+1)}$ and their inner product is given as

$$< R_p, R_q >= \int_0^\infty R_p(x) R_q(x) \frac{1}{\sqrt{x}(x+1)} dx = \begin{cases} 0, & p \neq q, \\ \frac{\pi}{2}, & p = q \neq 0, \\ \pi, & p = q = 0. \end{cases} \tag{2.26}$$

Chapter 3

Approximate solution of Ordinary Differential Equation.

Mathematically designed models of many physical processes in different subject area such as wave mechanics, fluid mechanics, diffusion and many more is governed by non-linear PDEs and generally finding the analytic solution of those PDEs is very hard. Therefore the approach of reducing PDEs to ordinary differential equation is very important which helps us in different physical process. Thus a powerful technique "Lie-symmetry method" is used for analyzing such non-linear PDEs and reducing them to the ODEs. By using symmetry method one can reduce PDE to ODE but in most of the cases the exact solution of those reduced ODEs is not available. Thus in such type of cases, the reduced ordinary differential equation can be solved numerically. In the next given section, an approximation to the numerical solution of reduced ordinary differential equation is obtained with the help of Spectral method.

1st step: Reduction of PDE into ODE(s):

By using similarity transformation, if exists, a non-linear PDE can be transformed into an ODE.

2nd step: Numerical solution of the reduced ODE:

When a non-linear PDE is transformed into an ODE solution of that equation is required.

In the case of a linear ODE, it is not hard to find its solutions but that is generally very difficult to solve the non-linear ordinary differential equation for exact solution therefore one looks for the numerical solution.

3rd step: Approximating the numerical solution:

One can also try to find a closed form solution. Idea for finding closed form solution is to guess a function in order to approximate the numerical solution of reduced ODE and that guess can be improved untill the desired level of accuracy is reached.

In the next section, by using above methadology an example of diffusion equation is solved in order to find the approximate closed form solution.

3.1 Example

A one dimensional, non-linear PDE that describes the phenomena of gas flow through a semi infinite porous medium at uniform pressure when $t = 0$ has been considered [3], which is given as

$$\frac{\partial}{\partial z}\left(P\frac{\partial P}{\partial z}\right) = B\frac{\partial P}{\partial t}, \tag{3.1}$$

where B is constant. The B/C are

$$\begin{cases} P(z,0) = P_0, & 0 < z < \infty, \\ \\ P(0,t) = P_1(< P_0), & 0 \leqslant t < \infty, \\ \\ P(\infty,t) = P_0. & 0 < t < \infty. \end{cases} \tag{3.2}$$

using $P_0 = 1$,

$$\begin{cases} P(z,0) = 1, & 0 < z < \infty, \\ \\ P(0,t) = P_1(<1), & 0 \leqslant t < \infty, \\ \\ P(\infty,t) = 1. & 0 < t < \infty. \end{cases} \tag{3.3}$$

3.1.1 Reduction of PDE to ODE

Taking the following transformations derived in [2]

$$x = \frac{z}{\sqrt{t}}\sqrt{\frac{B}{4}}, \tag{3.4}$$

and

$$y = \alpha^{-1}(1 - P^2), \tag{3.5}$$

where

$$\alpha = 1 - {P_1}^2.$$

From Eq. (3.5), we have

$$P = \sqrt{1 - \alpha y}. \tag{3.6}$$

The partial derivative of the Eq. (3.4) w.r.t. "z" and also with "t", gives us

$$\frac{\partial x}{\partial z} = \frac{1}{\sqrt{t}}\sqrt{\frac{B}{4}}, \tag{3.7}$$

and

$$\frac{\partial x}{\partial t} = \frac{-z}{2t^{3/2}}\sqrt{\frac{B}{4}}. \tag{3.8}$$

The derivative of Eq. (3.6) w.r.t. "t" gives

$$\frac{\partial P}{\partial t} = \frac{\alpha z\sqrt{B}}{8t^{3/2}\sqrt{1 - \alpha y}}\frac{dy}{dx}, \tag{3.9}$$

and by taking the derivative of Eq. (3.6) w.r.t. "z" gives

$$\frac{\partial P}{\partial z} = \frac{-\alpha\sqrt{B}}{4\sqrt{t}\sqrt{1-\alpha y}}\frac{dy}{dx}. \tag{3.10}$$

When Eq. (3.6), Eq. (3.9) and Eq. (3.10) are substituted in Eq. (3.1), we get

$$y'' + \frac{2x}{\sqrt{1-\alpha y}}y' = 0, \tag{3.11}$$

where derivatives are with respect to x. The boundary conditions are now transformed into

$$\begin{cases} y = 1, \quad \text{at} \qquad x = 0, \\ y = 0. \quad \text{when} \qquad x \to \infty. \end{cases} \tag{3.12}$$

3.1.2 Numerical solution of the non-linear ODE

By using similarity transformation, PDE given by Eq. (3.1) is transformed into a non-linear ODE but finding its exact solution is very hard. Therefore **NDSolve** in Mathematica is used for solving the non-linear ODE numerically given by the Eq. (3.11). Graph of numerical solution for $\alpha = 0.22471722182$ is presented in Figure 3.1.

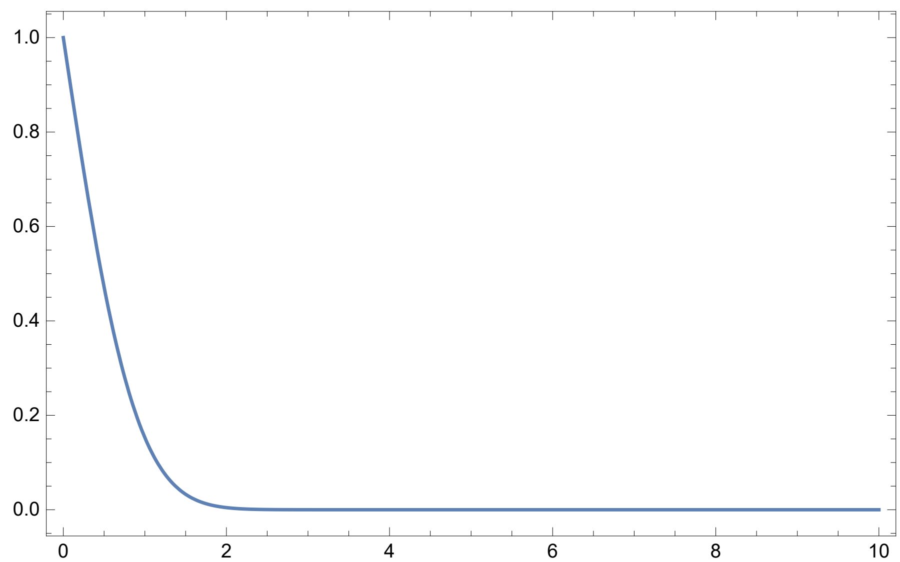

Figure 3.1: "Numerical solution of the reduced ordinary differential equation".

3.1.3 Approximation of the Reduced ODE using Spectral Method:

We used a function $f(x)$ for approximating the solution of the non-linear ODE which can be seen in Figure (3.1), given as:

$$f(x) = \sum_{r=0}^{N} m_r R_r(x), \tag{3.13}$$

here $R_r(x)$ are rational Chebyshev functions. Expanding the above function $f(x)$ for $N = 11$, Eq. (3.13) is written as

$$f(x) = y_{11}(x) = \sum_{r=0}^{11} m_r R_r(x_r). \tag{3.14}$$

$$f(x) = m_0 R_0(x) + m_1 R_1(x) + m_2 R_2(x) + ... + m_{11} R_{11}(x). \tag{3.15}$$

Now by using the values of RCF in Eq. (3.15), where the first few terms are given in from section 2.2. We get the following equation

$$
\begin{aligned}
f(x) = {} & m_0 + m_1(\frac{x-1}{1+x}) + m_2(\frac{x^2-6x+1}{(1+x)^2}) + m_3(\frac{x^3-15x^2+15x-1}{(1+x)^3}) + m_4(\frac{x^4-28x^3+70x^2-28x+1}{(1+x)^4}) + \\
& m_5(\frac{x^5-45x^4+210x^3-210x^2+45x-1}{(1+x)^5}) + m_6(\frac{x^6-66x^5+495x^4-924x^3+495x^2-66x+1}{(1+x)^6}) + \\
& m_7(\frac{x^7-91x^6+1001x^5-3003x^4+3003x^3-1001x^2+91x-1}{(1+x)^7}) + \\
& m_8(\frac{x^8-120x^7+1820x^6-8008x^5+12870x^4-8008x^3+1820x^2-120x+1}{(1+x)^8}) + \\
& m_9(\frac{x^9-153x^8+3060x^7-18564x^6+43758x^5-43758x^4+18564x^3-3060x^2+153x-1}{(1+x)^9}) + \\
& m_{10}(\frac{x^{10}-190x^9+4845x^8-38760x^7+125970x^6-184756x^5+125970x^4-38760x^3+4845x^2-190x+1}{(1+x)^{10}}) + \\
& m_{11}(\frac{x^{11}-231x^{10}+7315x^9-74613x^8+319770x^7-646646x^6+646646x^5-319770x^4+74613x^3-7315x^2}{(1+x)^{11}} \\
& \frac{231x-1}{(1+x)^{11}})
\end{aligned}
\tag{3.16}
$$

With the use of linear transformation, the differential equation on any arbitrary domain can be converted to [-1,1]. Thus in order to find the zeros of rational Chebyshev functions, we have

$$
x_l = \left[\frac{(b+a)+(b-a)x_p}{2} \right]. \tag{3.17}
$$

Here x_l denote the lth zero called the collocation points and by using this Eq. (3.17) we can get collocation points of 11th rational chebyshev functions $R_{11}(x)$, which are given as

$x_0 = -0.991445, \quad x_1 = -0.92388, \quad x_2 = -0.793353,$

$x_3 = -0.608761, \quad x_4 = -0.382683, \quad x_5 = -0.130526,$

$x_6 = 0.130526, \quad x_7 = 0.382683, \quad x_8 = 0.608761,$

$x_9 = 0.793353, \quad x_{10} = 0.92388, \quad x_{11} = 0.991445.$

By using the boundary conditions from Eq. (3.12) in Eq. (3.16), we get two equations and ten more equations can be obtained by taking the residual zero at these collocations points $x_1, x_2, x_3, ..., x_{12}$. On further by solving this algebraic system of 11 equations simultaneously in mathematica, we acquire the values of the coefficients $m_0, m_1, m_2, ..., m_{11}$.

$m_0 = 0.364362, \quad m_1 = -0.539632, \quad m_2 = 0.164749,$

$m_3 = 0.0467228, \quad m_4 = -0.0379996, \quad m_5 = -0.012027,$

$m_6 = 0.0105087, \quad m_7 = 0.00582588, \quad m_8 = -0.0020889,$

$m_9 = -0.00213712, \quad m_{10} = 0.000469001, \quad m_{11} = 0.00124698.$

By substituting the values of these given coefficients in Eq. (3.16) and on further simplification, we get the approximate solution of the reduced ODE given as

$$f(x) = \frac{1 + 10.0018x + 37.1589x^2 + 163.064x^3 - 174.317x^4 + 892.673x^5 - 941.741x^6}{(1+x)^{11}}$$

$$\frac{+386.694x^7 - 77.1672x^8 + 7.23166x^9 - 0.371938x^{10} - 5.59448 * 10^{-17}x^{11}}{(1+x)^{11}}. \tag{3.18}$$

The graph of approximate solution is shown in Figure 3.2.

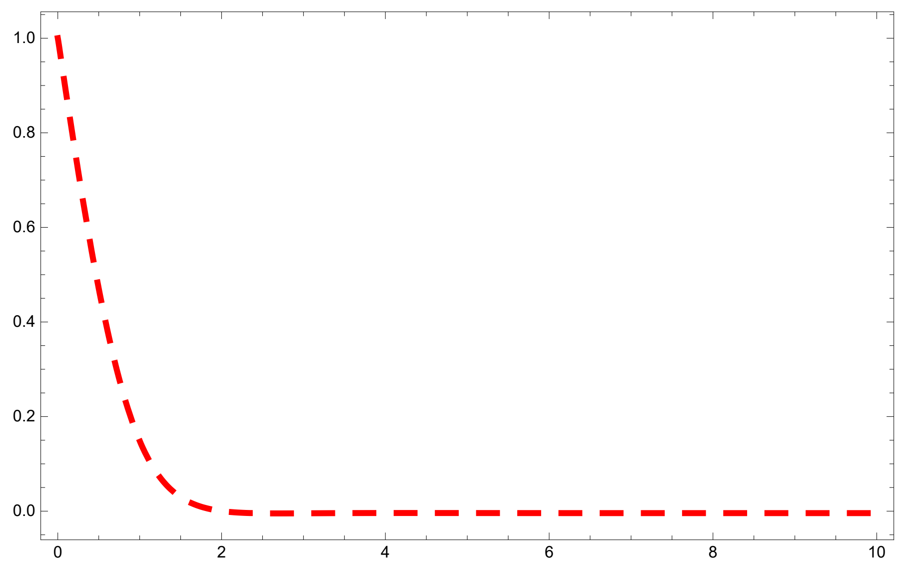

Figure 3.2: Graph of approximate closed form solution

Figure 3.3. gives the approximate and the numerical solutions together. Notice that the approximate closed form solution matches quite well with the numerical solution.

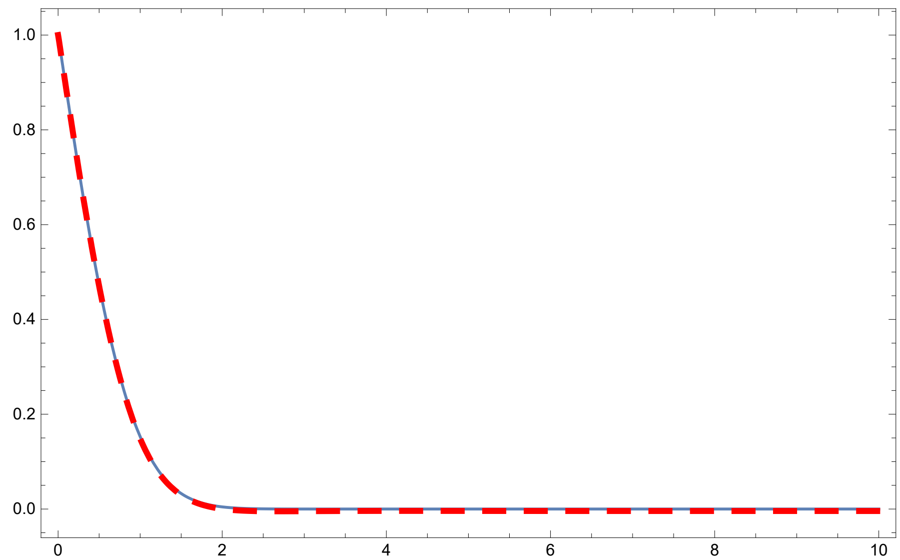

Figure 3.3: Graph of approximate and numerical solutions. Solid line gives numerical solution while red dashed line represents approximate soution.

3.1.4 Residual Analysis of ODE

The residual of the reduced non-linear ODE given by Eq. (3.11) can be obtained with respect to the approximate closed-form solution $f(x)$ Eq. (3.18), as

$$R = \frac{f^{''}(x) + 2xf^{'}(x)}{\sqrt{1 - (0.22471721)f(x).}} \tag{3.19}$$

Here first we have to find f', f''. Differentiation gives us,

$$f'(x) = \frac{-0.998159 - 25.7005x + 154.761x^2 - 2001.78x^3 + 5683.58x^4 - 11006.5x^5}{(1+x)^{12}}$$
$$\frac{+7415.57x^6 - 2164.11x^7 + 296.587x^8 - 18.1827x^9 + 0.371938x^{10}}{(1+x)^{12}}, \tag{3.20}$$

and

$$f''(x) = \frac{-13.7226 + 592.227x - 7552.94x^2 + 40750.3x^3 - 100501.0x^4 + 121539.0x^5 -}{(1+x)^{13}}$$
$$\frac{59642.2x^6 + 13193.3x^7 - 1349.99x^8 + 58.2675x^9 - 0.743877x^{10}}{(1+x)^{13}}. \tag{3.21}$$

When Eq. (3.18), with Eq. (3.20) along with Eq. (3.21) is used in Eq. (3.19), we get residual equation. The graph of residual is given in Figure 3.4.

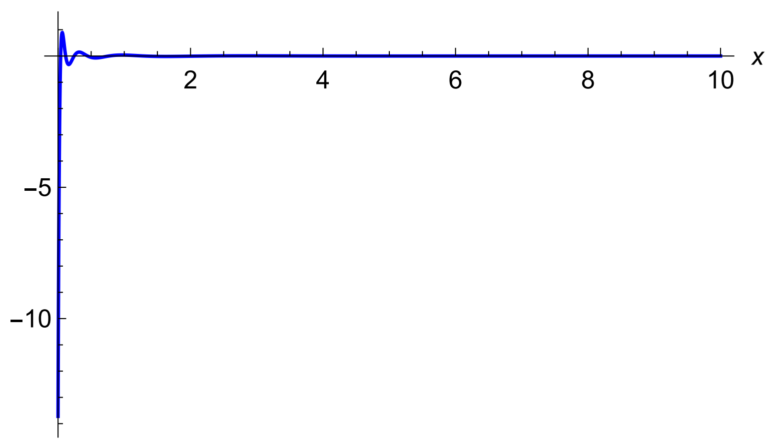

Figure 3.4: Graph of the residual.

In figure 3.4, our approximation does not give good result in the start as the curve show fluctuations but later on we can see that it become more stable and closely matches with the exact solution.

Chapter 4

Conclusion

The purpose of this thesis was the construction of an approximate closed form solution of a non-linear ODE on semi-infinite interval. In order to achieve this goal, spectral collocation method using rational Chebyshev function is employed. We also obtained the numerical solution of the non-linear ODE. Residual is used in order to analyze the accuracy of approximate closed-form solution, which shows that our approximation is a good choice of exact solution. The maximum absolute error observed for approximate closed form solution of non-linear ODE is very small and it is found that the given approximate closed-form solution closely matches with the numerical solution.

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