THE CLASSIFICATION OF A CLASS OF PLANE SYMMETRIC STATIC SPACETIMES ACCORDING TO THEIR NOETHER SYMMTERIES

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Dedicated to

my loving parents

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Abstract

Scientists use symmetries to solve the real-world problems and to attain new understandings of the world around us. They provide a shortcut based on geometry for getting at some of Nature's innermost secrets. In this dissertation, the symmetry methods have been used to classify a class of plane symmetric static spacetimes according to their Noether symmetries and metrics. The method adopted here provides all those plane symmetric static spacetimes which we obtain during the classification of these spacetimes according to their isometries. The Lie algebra of infinitesimal generators corresponding to each metric has also been discussed here in terms of their structure constants.

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Chapter 1

Introduction

The dissertation is devoted to "The Classification of a Class of Plane Symmetric Static Spacetimes according to the Noether Symmetries". As the title indicates, this dissertation concerns notion from different mathematical disciplines. Our main purpose is to find the solutions of the Einstein field equations (EFEs).

$$R_{ab} - \frac{1}{2} g_{ab} R + g_{ab} \Lambda = \kappa T_{ab}, \ (a, b.... = 0, 1, 2, 3) \tag{1.0.1}$$

where R_{ab}, R, and Λ are respectively the Ricci tensor, the Ricci scalar, the cosmological constant and $\kappa = \frac{8\pi G}{c^4}$ is the coupling constant. The cosmological constant is now often considered as a part of the stress-energy tensor. The solutions of the EFEs depend on the stress-energy tensor, which in turn depends on the dynamics of matter and energy. So if one is looking for the exact solutions, the evolution of the metric and stress-energy tensor must be solved together. Many techniques have been developed to find the solutions to the EFEs. In this respect, by solving the Noether symmetry governing equations, we are able to find some exact solutions to the EFEs.

1.1 Background

Differential equations (DEs) have been the center of attention ever since their development. They gained the interest of many researchers, because of their vast range of applications in most scientific and engineering problems. From a physical point of view a DE represents the dynamics of some phenomenon. Many real-world models are formulated by using DEs. Therefore, finding methods to solve DEs is an important problem that has been studied by various people. Methodologies have been developed for finding the solutions to DEs, but still little is known about them. Various methods have been used to find the exact and numerical solutions, but there are some kind of DEs which cannot be solved by using these methods. These equations include non-linear DEs whose solutions are difficult to be found due to their behavior under different initial or boundary conditions. We see that some methods of finding the solutions of DEs, work well under some condition while others do not. So there is a need of finding such a method that provides solutions for all kinds of DEs under any condition.

The concept of symmetry plays a key role in gaining an understanding of the physical laws governing the behavior of matter because the laws of Physics are the same at any time and any location. It is the key to solving DEs. A symmetry is a transformation applied to a structure such that the transformation preserves the properties of the structure.

In the nineteenth century, the Norwegian mathematician Sophous Lie, discovered symmetries hidden in the solutions to DEs. He constructed exact solutions with the symmetry methods and linearized certain DEs. In all the cases in which one can find an exact solution, the underlying property is a symmetry of the equation. For finding the solutions of EFEs, symmetry methods not only provide sufficient tools but also gives a way of classifying the solutions as well.

The role of symmetry in fundamental physics is very essential. Until the twentieth century, principles of symmetry played little part in theoretical physics, the Greeks and others were fascinated by the symmetries of objects and believed that these would be mirrored in the structures of nature. This enchantment lead Kepler to impose his opinion of symmetry on the planetary motion.

Over forty years or more, the concept of symmetries was not fully appreciated. In the beginning of the twentieth century, Einstein regarded the symmetry principle as the primary feature of nature. Ten years later, with the advancement in the concept of General Relativity, this point of view became very popular. With the development of Quantum Mechanics in the 1920s, symmetry principles came to play even more vital role. In the latter half of the twentieth century, symmetry has been the most dominant concept in the exploration and formulation of the fundamental laws of physics. Today it serves as a guiding principle in the search for further unification and progress.

The symmetry groups that arise most often in the applications to geometry and DEs are Lie groups of transformations acting on a finite-dimensional manifold. Lie considered the group of geometric transformations: translations; rotations; and the group of scaling symmetries; among others that were defined on the independent and dependent variables, mapping one solution to the other. Consequently, the problem was to find such symmetries and to use them to transform our problem into a simple form.

If one is able to find such group of symmetries, then immediately it is possible to get new solutions from the known solutions. Symmetry groups also provide an opportunity to characterize solutions of the DEs in terms of the group elements that transform it from one form to another. A different approach is to classify those differential equations that admit some given symmetry; the problem can be studied using infinitesimal methods and using the theory of differential invariants.

In 1918, Emmy Noether proved two statements that merged symmetry groups with equations arising from variational systems [1]. In her first statement, it is stated that for every symmetry of the Lagrangian, there is a corresponding conservation law. The symmetries of Lagrangians usually include the underlying symmetries of space and time and these lead to standard conservation laws as follows:

- symmetry under translation in space implies conservation of momentum;
- symmetry under rotation in space implies conservation of angular momentum;
- symmetry under translation in time implies conservation of energy.

The second statement relates the symmetries of action with a DE. It states that if a Langragian admits symmetries depending on parameters, its variational derivatives obey certain relations, called the Noether identity. With these studies by Noether, symmetry methods became a strong tool to solve many practical problems arising in shock waves, scattering theory, fluid mechanics, elasticity and relativity.

In the early years of research in general relativity, only a small number of exact solutions of EFEs were discussed. These mostly arose from highly idealized physical problems. As examples, one may cite the well-known spherically symmetric solutions of Schwarzschild, Reissner and Nordstrom, Tolman and Friedmann and the axially symmetric and static solution of Kerr, etc.

Moreover, most of the classification techniques now in use, were unknown to the researchers [2]. The first to become popular was the groups of motion (isometries and homotheties). The next, which was in part motivated by the study of gravitational radiations, was the algebraic classification of the Weyl tensor into Petrov types [3] and the understanding of the properties of the algebraically special metrics. The other schemes for the classification of the exact solutions are the algebraic classification of the Ricci tensor (Plebanski or Segre types) [2] and the physical characterization of the energy-momentum tensor, the existence and structure of preferred vector fields. Up till now a lot of work has been done in this field. We will not go in details, rather we will give a brief introduction to some of the work cited.

In 1987, Bokhari and Qadir [4] considered Killing vectors (KVs) of spherically symmetric static spacetimes. They concluded that spherically symmetric static spacetimes have either ten KVs (corresponding to de Sitter, Minkowski, and anti de Sitter metrics), or seven KVs (corresponding to Einstein and anti-Einstein metrics), or six KVs (incorporating the Bertotti-Robinson and two other metrics), or four KVs (the minimal isometry). In 1988, Qadir and Ziad [5] reviewed the work done by Bokhari and Qadir [4]. They studied the symmetry group $SO(1,2)\otimes SO(3)$. One of the metric of those spacetimes was a generalization of a metric given by Petrov [3]. In 1997, Ahmed and Ziad [6] discover the homotheties of spherically symmetric spacetimes admitting maximal isometry groups larger than $SO(3)$ along with their metrics, without imposing any restriction on the energy-stress tensor. They analyzed that there are either 11 or 7 or 5 homotheties. In 2001, Feroze, Qadir, and Ziad [7] gave a complete classification of the plane symmetric spacetimes according to their isometries. They obtained metrics by solving Killing equations. By considering isometries, the metrics obtained, admit the group of motions G_r (where $r = 3, 4, 5, 6, 7$, and 10).

1.2 Objective of the Dissertation

In this dissertation, we employed symmetry methods to classify the plane symmetric static spacetimes. The main tool to be used in classifying the spacetimes is the Noether symmetry. To start the classification, we consider the metric given as

$$ds^2 = e^{2\nu(x)}dt^2 - dx^2 - dy^2 - dz^2.$$

The simplification of the Noether symmetry governing equations for the above metric yields a system of partial differential equations (PDEs). We used suitable integration and differentiation techniques to solve the equations of the system. In this dissertation, a complete solution of the system of PDEs is given, providing all the possible metrics along with the Noether symmetry generators. We have also calculated the Lie algebra of the Noether symmetry generators corresponding to each metric.

1.3 Scheme of Work

The work has been divided into four chapters.

The Chapter 1 contains the review of the relevant background presented in this dissertation. This chapter lays out the style in which the subsequent chapters are being arranged in the dissertation and also includes a brief overview of the contents of each chapter.

In Chapter 2, basic concepts and definitions are presented along with examples that are necessary to provide the background for the later work. Afterwards, the work of Feroze [8] has been reviewed. There the Killing equations were used to obtain a complete classification of the spacetimes according to their isometries and metrics. Our focus is only on the static case of the problem.

In Chapter 3, our main emphasis is on the study of the classification of plane symmetric static spacetimes. The Noether symmetry condition is used to obtain the system of the PDEs that are solved analytically by applying methods known to us. In the end, we get the Noether symmetry generators corresponding to each metric. For compactness, we have calculated the Lie algebra for each existing case.

In addition, the last Chapter of this dissertation is dedicated to the conclusion of our work.

Chapter 2

Preliminaries

This chapter is devoted to some elementary concepts and results used throughout this dissertation. We also provide examples to make the definitions easily understandable. Necessary notations and terminologies used in the sequel are also introduced. We begin our exposition with a brief review of the basic concepts related to differential geometry. Afterwards, definitions relating to the symmetry methods have been introduced. In the end, the classification of the plane symmetric spacetimes according to their isometries is given.

2.0.1 Isometry or Killing Vector

A *Killing vector field* [9] is a metric preserving map, i.e. it provides a direction along which the metric can be transported unchanged. These are the infinitesimal generators of the isometries. A Killing vector field $\mathbf{k}$ fulfills

$$\mathfrak{L}_{\mathbf{k}}\mathbf{g} = 0, \tag{2.0.1}$$

where $\mathfrak{L}$ is a Lie derivative. We can write eq.(2.0.1) as

$$\mathbf{k_{a;b}} + \mathbf{k_{b;a}} = 0 \tag{2.0.2}$$

Thus, from eq.(2.0.2) we obtain the Killing equations as

$$k^c g_{ab,c} + g_{ac} k^c_{,b} + g_{bc} k^c_{,a} = 0. \tag{2.0.3}$$

where g_{ab} is a metric and k^c is an arbitrary vector field.

2.1 Basic Concepts in Symmetry Methods

2.1.1 Point Transformation

An *nth*-order ODE is of the form [10]

$$F(x, y, y', y'', ..., y^{(n)}) = 0, \tag{2.1.1}$$

where x is the independent variable, y is the dependent variable, and $y', y'', ..., y^{(n)}$ are the derivatives of y with respect to x.

When dealing with DE of the form given by eq.(2.1.1), one tries to simplify the equation by appropriate change of variables, x and y,

$$\tilde{x} = \tilde{x}(x, y), \qquad \tilde{y} = \tilde{y}(x, y), \tag{2.1.2}$$

where $\tilde{x}$ and $\tilde{y}$ are continuous functions. This is called a *point transformation*. When we talk about the Lie point symmetry, we have to consider an invertible point transformation that depends on at least one arbitrary parameter ϵ,

$$\tilde{x} = \tilde{x}(x, y; \epsilon), \qquad \tilde{y} = \tilde{y}(x, y; \epsilon), \tag{2.1.3}$$

where $\tilde{x}$ and $\tilde{y}$ are infinitely differentiable with respect to x, y, and the parameter epsilon, ϵ.

Symmetry Transformation

A *symmetry transformation* [10] of an nth-order ODE of the form eq.(2.1.1) is an invertible transformation of the form eq.(2.1.3) that leaves the ordinary differential eq.(2.1.1) invariant in $\tilde{x}$, $\tilde{y}$, i.e.

$$F(\tilde{x}, \tilde{y}, \tilde{y'}, \tilde{y''}, ..., \tilde{y^{(n)}}) = 0, \tag{2.1.4}$$

whenever,

$$F(x, y, y', y'', ..., y^{(n)}) = 0.$$

2.1.2 Generators and their Prolongation

By considering infinitesimal transformations given by eq.(2.1.3), we take an arbitrary point (x, y) such that

$$\left.\begin{aligned} \tilde{x}(x, y; \epsilon) &= x + \epsilon\xi(x, y) + ... = x + \epsilon\mathbf{X}x + ..., \\ \tilde{y}(x, y; \epsilon) &= y + \epsilon\eta(x, y) + ... = y + \epsilon\mathbf{X}y + ..., \end{aligned}\right\} \tag{2.1.5}$$

where the functions ξ and η are defined by

$$\xi(x, y) = \left.\frac{\partial\tilde{x}}{\partial\epsilon}\right|_{\epsilon=0}, \qquad \eta(x, y) = \left.\frac{\partial\tilde{y}}{\partial\epsilon}\right|_{\epsilon=0}, \tag{2.1.6}$$

and the operator $\mathbf{X}$ is given by

$$\mathbf{X} = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}.$$

The operator $\mathbf{X}$ is called the *infinitesimal generator* [10] of the transformation.

If we want to apply a point transformation to a DE, we must know how to get the transformed derivatives of y. For this, define

$$\left.\begin{aligned} &\tilde{y}' = \tfrac{d\tilde{y}}{d\tilde{x}} = \tfrac{d\tilde{y}(x,y;\epsilon)}{d\tilde{x}(x,y;\epsilon)} = \tfrac{y'(\frac{\partial\tilde{y}}{\partial y}) + (\frac{\partial\tilde{y}}{\partial x})}{y'(\frac{\partial\tilde{x}}{\partial y}) + (\frac{\partial\tilde{x}}{\partial x})} = \tilde{y}'(x, y, y'; \epsilon) \\ &\tilde{y}'' = \tfrac{d\tilde{y}'}{d\tilde{x}} = \tilde{y}''(x, y, y', y''; \epsilon); \end{aligned}\right\} \tag{2.1.7}$$

i.e. the transformed derivatives are the derivatives of the transformed variables. We now need the extensions (or prolongations) of the infinitesimal generator $\mathbf{X}$. We write

$$\left.\begin{aligned}
&\tilde{x} = x + \epsilon\xi(x,y) + ... = x + \epsilon\mathbf{X}x + ...,\\
&\tilde{y} = y + \epsilon\eta(x,y) + ... = y + \epsilon\mathbf{X}y + ...,\\
&\tilde{y}' = y' + \epsilon\eta'(x,y,y') + ... = y' + \epsilon\mathbf{X}y' + ...,\\
&\vdots\\
&\tilde{y}^{(n)} = y^{(n)} + \epsilon\eta^{(n)}(x,y,y',...,y^{(n)}) + ... = y^{(n)} + \epsilon\mathbf{X}y^{(n)} + ...,
\end{aligned}\right\} \tag{2.1.8}$$

where $\eta, \eta', ..., \eta^{(n)}$ are defined by

$$\eta' = \left.\frac{\partial\tilde{y}'}{\partial\epsilon}\right|_{\epsilon=0} = 0, ..., \eta^{(n)} = \left.\frac{\partial\tilde{y}^{(n)}}{\partial\epsilon}\right|_{\epsilon=0} = 0. \tag{2.1.9}$$

Substituting the results in eq.(2.1.8), we get

$$\begin{aligned}
\tilde{y}' &= y' + \epsilon\eta' + ... = \frac{d\tilde{y}}{d\tilde{x}} = \frac{dy + \epsilon d\eta + ...}{dx + \epsilon d\xi + ...} = \frac{y' + \epsilon(\frac{d\eta}{dx}) + ...}{1 + \epsilon(\frac{d\xi}{dx}) + ...} = y' + \epsilon(\frac{d\eta}{dx} - y'\frac{d\xi}{dx}) + ...\\
\tilde{y}^{(n)} &= y^{(n)} + \epsilon\eta^{(n)} + ... = \frac{d\tilde{y}^{(n-1)}}{d\tilde{x}} = y^{(n)} + \epsilon(\frac{d\eta^{(n-1)}}{dx} - y^{(n)}\frac{d\xi}{dx}) + ...,
\end{aligned}$$

from which we have

$$\eta^{(n)} = \frac{d\eta^{(n-1)}}{dx} - y^{(n)}\frac{d\xi}{dx}, \tag{2.1.10}$$

where $\frac{d}{dx}$ is given by

$$\mathbf{D} = \frac{d}{dx} = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + y'''\frac{\partial}{\partial y''} + ... \quad . \tag{2.1.11}$$

Therefore, the extension (or prolongation) up to the nth derivative of an infinitesimal generator $\mathbf{X}$ is given by

$$\mathbf{X}^{[n]} = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \eta'\frac{\partial}{\partial y'} + ... + \eta^{(n)}\frac{\partial}{\partial y^{(n)}}. \tag{2.1.12}$$

(We can use the same symbol $\mathbf{X}$ for both generator and its prolongation.)

2.1.3 Lie Point Symmetries of Differential Equations

The following theorem states the property of a symmetry that has to be true for every solution of a ODE.

Theorem 2.1.1. *[10]*

The nth-order differential eq.(2.1.1) admits a Lie point symmetry with generator eq.(2.1.12) if and only if

$$\mathbf{X}^{[n]}F = 0, \qquad (mod\, F = 0).$$

Finding the Lie point symmetries of a differential eq.(2.1.1) means finding the general solution $\xi(x, y)$, $\eta(x, y)$ of the symmetry condition eq.(2.1.12).

Symmetries of DE's

Now we illustrate the method to find symmetries of ODEs. In order to do so we would enlist a general scheme for the process which is further illustrated by an example. Consider the following example [11].

Consider the 2nd order homogenous DE given by

$$y'' = 0. \tag{2.1.13}$$

Step 1: *Finding the linearize symmetry condition of the DE.*

The linearize symmetry condition for this ODE is

$$\eta^{(2)} = 0,$$

i.e.

$$\eta_{,xx} + (2\eta_{,xy} - \xi_{,xx})y' + (\eta_{,yy} - 2\xi_{,xy})y'^2 - \xi_{,yy}y'^3 = 0.$$

Since ξ and η are independent of y', the linearize symmetry condition splits into the following system of equations

$$\left.\begin{aligned} &\eta_{,xx} = 0, \\ &2\eta_{,xy} - \xi_{,xx} = 0, \\ &\eta_{,yy} - 2\xi_{,xy} = 0, \\ &\xi_{,yy} = 0. \end{aligned}\right\} \tag{2.1.14}$$

Step 2: *Solving the system of DEs given by eq.(2.1.14).*

From the fourth equation of the above set of equations, we have

$$\xi(x, y) = A(x)y + B(x),$$

for some arbitrary functions A and B. The 3rd equation in the system eq.(2.1.14) gives

$$\eta(x, y) = A'(x)y^2 + C(x)y + D(x),$$

where C and D are also arbitrary functions. Then the remaining equations in eq.(2.1.14) give

$$\left.\begin{aligned} &A'''(x)y^2 + C''(x)y + D''(x) = 0, \\ &3A''(x)y + 2C'(x) - B''(x) = 0. \end{aligned}\right\} \tag{2.1.15}$$

Equating the powers of y in system eq.(2.1.15), we get the following system of ODEs

$$\left.\begin{aligned} &A''(x) = 0, \\ &C''(x) = 0, \\ &D''(x) = 0, \\ &B''(x) = 2C'(x). \end{aligned}\right\} \tag{2.1.16}$$

This system is easily solvable. So for every one parameter Lie group of symmetries of eq.(2.1.13), the function ξ and η are of the form

$$\begin{aligned} \xi(x, y) &= c_1 + c_3x + c_5y + c_7x^2 + c_8xy, \\ \eta(x, y) &= c_2 + c_4y + c_6x + c_7xy + c_8y^2, \end{aligned}$$

where $c_1, \cdots, c_8$ are arbitrary constants. Therefore, the most general infinitesimal generator is

$$\mathbf{X} = \sum_{i=1}^{8} c_i \mathbf{X_i}$$

where,

$$\left.\begin{array}{l} \mathbf{X_1} = \frac{\partial}{\partial x}, \quad \mathbf{X_2} = \frac{\partial}{\partial y}, \quad \mathbf{X_3} = x\frac{\partial}{\partial x}, \quad \mathbf{X_4} = y\frac{\partial}{\partial y}, \quad \mathbf{X_5} = y\frac{\partial}{\partial x} \\ \mathbf{X_6} = x\frac{\partial}{\partial y}, \quad \mathbf{X_7} = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}, \quad \mathbf{X_8} = xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}. \end{array}\right\} \tag{2.1.17}$$

2.1.4 Lie Bracket

Let $\mathbf{X_m}$ and $\mathbf{X_n}$ be two infinitesimal generators (or symmetries) of a DE. The *Lie bracket* [10] of the two generators $\mathbf{X_m}$ and $\mathbf{X_n}$ is defined by

$$[\mathbf{X_m}, \mathbf{X_n}] = \mathbf{X_m}\mathbf{X_n} - \mathbf{X_n}\mathbf{X_m}, \tag{2.1.18}$$

which is again a symmetry of the DE.

2.1.5 Lie Algebra

A *Lie algebra* [12] is simply a vector space over some field K together with a Lie bracket satisfying the following conditions.

(i). $[\alpha\mathbf{X_1} + \beta\mathbf{X_2}, \mathbf{X_3}] = \alpha[\mathbf{X_1}, \mathbf{X_3}] + \beta[\mathbf{X_2}, \mathbf{X_3}]$,

$[\mathbf{X_1}, \alpha\mathbf{X_2} + \beta\mathbf{X_3}] = \alpha[\mathbf{X_1}, \mathbf{X_2}] + \beta[\mathbf{X_1}, \mathbf{X_3}]$,

(ii). $[\mathbf{X_1}, \mathbf{X_2}] = -[\mathbf{X_2}, \mathbf{X_1}]$,

(iii). $[\mathbf{X_1}, [\mathbf{X_2}, \mathbf{X_3}]] + [\mathbf{X_2}, [\mathbf{X_3}, \mathbf{X_1}]] + [\mathbf{X_3}, [\mathbf{X_1}, \mathbf{X_2}]] = 0$.

A Lie algebra is called *complex* if K is a field of complex numbers and is called *real* if K is a field of reals. The vectors $\mathbf{X_1}$, $\mathbf{X_2}$, and $\mathbf{X_3}$ are called the generators of Lie algebra. If number of basic generators are finite then the Lie algebra is finite dimensional otherwise infinite dimensional.

2.1.6 Noether Symmetry

A symmetry is a *Noether symmetry* [10] if it satisfies

$$\mathbf{X}^{[1]}L + (\mathbf{D}\xi)L = \mathbf{D}A, \tag{2.1.19}$$

where $A(s, q^a)$ $(q^0 = t, q^1 = x, q^2 = y, q^3 = z)$ is some gauge term, $\mathbf{D}$ is an operator given by

$$\mathbf{D} = \frac{\partial}{\partial s} + \dot{q}^a \frac{\partial}{\partial q^a}, \tag{2.1.20}$$

and L is the Lagrangian

$$L = L(s, q^a, \dot{q}^a), \tag{2.1.21}$$

where dot '$\cdot$' is derivative with respect to geodetic parameter s, and

$$\mathbf{X}^{[1]} = \xi\left(s, q^i\right)\frac{\partial}{\partial s} + \eta^a\left(s, q^i\right)\frac{\partial}{\partial q^a} + \dot{\eta}^a\left(s, q^i, \dot{q}^i\right)\frac{\partial}{\partial \dot{q}^a}, \quad a, i = 1, 2, ..., n. \tag{2.1.22}$$

Here $\mathbf{X}^{[1]}$ is the first-order prolonged generator and the formula for $\dot{\eta}^a$ is as follows:

$$\dot{\eta}^a = \frac{d}{ds}\eta^a - \dot{q}^a\frac{d}{ds}\xi$$
$$\textit{or} \qquad \dot{\eta}^a = \eta^a_{,s} + \dot{q}^b\eta^a_{,b} + \dot{q}^a\xi_{,s} - \dot{q}^a\dot{q}^b\xi_{,b}. \tag{2.1.23}$$

Example

Consider the 2nd order DE given by

$$y''(x) + y = 0. \tag{2.1.24}$$

The corresponding Lagrangian of the differential eq.(2.1.24) is the difference of the kinetic and potential energy given as

$$L = \frac{1}{2}\left(y'^2 - y^2\right). \tag{2.1.25}$$

Step 1: *Applying the Noether symmetry condition to the DE.*

The Noether symmetry condition given by eq.(2.1.19) implies that

$$-y\eta+\left[\eta_{,x}+(\eta_{,y}-\xi_{,x})y'-\xi_{,y}y'^2\right]y'+\frac{1}{2}\left(y'^2-y^2\right)\left(\xi_{,x}+y'\xi_{,y}\right)=A_{,x}+y'A_{,y}. \quad (2.1.26)$$

Since ξ, η, and A are independent of y' (i.e., functions of x and y only), the Noether symmetry condition splits into the following system of equations

$$\left.\begin{array}{l}\eta_{,x}-\frac{1}{2}y^2\xi_{,y}=A_{,y},\\ 2\eta_{,y}-\xi_{,x}=0,\\ \xi_{,y}=0,\\ -y\eta-\frac{1}{2}y^2\xi_{,x}=A_{,x}.\end{array}\right\} \quad (2.1.27)$$

Step 2: *Solving the system of DEs given by eq.(2.1.27).*

From the third equation of the above set of equations, we have

$$\xi=\xi(x).$$

The second equation in the system eq.(2.1.27) gives

$$\eta(x,y)=\frac{1}{2}\xi_{,x}y+\alpha_1(x),$$

where $\alpha_1(x)$ is an arbitrary function. The first equation in the system given by eq.(2.1.27) leads to

$$A(x,y)=\frac{1}{4}\xi_{,xx}y^2+\alpha_1'(x)y+\alpha_2(x),$$

where $\alpha_2(x)$ is an arbitrary function. Then the remaining equation in eq.(2.1.27) give

$$-y^2\xi_{,x}-\alpha_1(x)y=\frac{1}{4}\xi_{,xxx}y^2+\alpha_1''(x)y+\alpha_2'(x). \quad (2.1.28)$$

Equating the powers of y in eq.(2.1.28), we get the following system of ODEs

$$\left.\begin{array}{l}\frac{1}{4}\xi_{,xxx}+\xi_{,x}=0,\\ \alpha_1''(x)+\alpha_1(x)=0,\\ \alpha_2'=0.\end{array}\right\} \quad (2.1.29)$$

This system is easily solvable. So the Noether symmetries of eq.(2.1.24), the function ξ and η are of the form

$$\begin{aligned}
\xi(x,y) &= c_4 + c_5\cos 2x + c_6 \sin 2x,\\
\eta(x,y) &= \big(-c_5 \sin 2x + c_6\cos 2x\big)y + c_2\cos x + c_3 \sin x,\\
A(x,y) &= -\big(c_5\cos 2x + c_6 \sin 2x\big)y^2 + \big(-c_2 \sin x + c_3\cos x\big)y + c_1,
\end{aligned}$$

where $c_1, \cdots, c_6$ are arbitrary constants. Therefore, the most general infinitesimal generator is

$$\mathbf{X} = \sum_{i=1}^{5} c_i \mathbf{X_i}$$

where,

$$\begin{aligned}
\mathbf{X_0} &= \frac{\partial}{\partial x},\\
\mathbf{X_1} &= \cos 2x\frac{\partial}{\partial x} - \sin 2x\frac{\partial}{\partial y}; \qquad A_1 = -y^2\cos 2x,\\
\mathbf{X_2} &= \sin 2x\frac{\partial}{\partial x} + \cos 2x\frac{\partial}{\partial y}; \qquad A_2 = -y^2\sin 2x,\\
\mathbf{X_3} &= \cos x\frac{\partial}{\partial y}; \qquad A_3 = -y\sin x,\\
\mathbf{X_4} &= \sin x\frac{\partial}{\partial y}; \qquad A_4 = y\cos x.
\end{aligned}$$

In general, the Noether symmetries have vast applications in physics because they can be used to simplify a given system of differential equations as well as to determine the integrability of the system.

2.2 The Classification of Plane Symmetric Spacetimes by Isometries

In General Relativity spacetime symmetries play an important role in finding solutions of the EFEs. These solutions are called metric/spacetimes and are then classified using different schemes [2]. One classification scheme is according to their spacetime symmetries (isometries, homotheties, conformal symmetries, Ricci collineation, curvature collineations, matter collineations etc.) [13]-[30]. The importance of this scheme is that it provides the complete classification of spacetimes according to their respective spacetime symmetries as well as new solutions of the EFEs. In this chapter, we review the work done by Feroze [7, 8] in which the classification of the plane symmetric spacetimes has been carried out according to the *isometries*. The method adopted for the classification of the plane symmetric spacetimes was developed by Qadir and Ziad [14, 31].

The most general form of the plane symmetric spacetime is, [2]

$$ds^2 = e^{2\nu(t,x)}dt^2 - e^{2\lambda(t,x)}dx^2 - e^{2\mu(t,x)}(dy^2 + dz^2). \tag{2.2.1}$$

The Killing equations for the above given metric were solved to give a complete classification of these spacetimes according to their isometries and metrics [7]. The

Killing equations eq.(2.0.3) for the metric given by eq.(2.2.1) reduce to

$$\dot{\nu}k^0 + \nu^{'}k^1 + k^0{}_{,0} = 0, \quad (2.2.2)$$

$$e^{2\upsilon}k^0{}_{,1} - e^{2\lambda}k^1{}_{,0} = 0, \quad (2.2.3)$$

$$e^{2\upsilon}k^0{}_{,2} - e^{2\mu}k^2{}_{,0} = 0, \quad (2.2.4)$$

$$e^{2\upsilon}k^0{}_{,3} - e^{2\mu}k^3{}_{,0} = 0, \quad (2.2.5)$$

$$\dot{\lambda}k^0 + \lambda^{'}k^1 + k^1{}_{,1} = 0, \quad (2.2.6)$$

$$e^{2\lambda}k^1{}_{,2} + e^{2\mu}k^2{}_{,1} = 0, \quad (2.2.7)$$

$$e^{2\lambda}k^1{}_{,3} + e^{2\mu}k^3{}_{,1} = 0, \quad (2.2.8)$$

$$\dot{\mu}k^0 + \mu^{'}k^1 + k^2{}_{,2} = 0, \quad (2.2.9)$$

$$k^2{}_{,3} + k^3{}_{,2} = 0, \quad (2.2.10)$$

$$\dot{\mu}k^0 + \mu^{'}k^1 + k^3{}_{,3} = 0. \quad (2.2.11)$$

where '.' and '′' represent partial derivatives w.r.t. t and x respectively. Eqs.(2.2.2)-(2.2.11) represent a system of ten coupled PDEs to be solved to give seven unknown functions, $\nu(t,x)$, $\lambda(t,x)$, $\mu(t,x)$ and $k^a = k^a\left(x^i\right)(a, i = 0,1,2,3)$. Thus, the above system was solved to get the Killing vector field $\mathbf{K}$ given by

$$\left.\begin{array}{l} \mathbf{K} = [e^{2(\mu-\nu)}\{\frac{1}{2}\dot{A}_1(t,x)(z^2+y^2) + \dot{A}_2(t,x)z + \dot{A}_3(t,x)y\} \\ \quad + A_0(t,x)]\partial/\partial t - [\frac{1}{2}e^{2(\mu-\lambda)}\{A_1^{'}(t,x)(z^2+y^2) + A_2^{'}(t,x)z \\ \quad + yA_3^{'}(t,x)\} - A_4(t,x)]\partial/\partial x + [\frac{1}{2}c_3(z^2-y^2) + yzc_2 \\ \quad + c_1 z + A_1(t,x)y + A_3(t,x)]\partial/\partial y + [-yzc_3 + \frac{1}{2}c_2(z^2-y^2) \\ \quad - c_1 y + A_1(t,x)z + A_2(t,x)]\partial/\partial z. \end{array}\right\} \quad (2.2.12)$$

subject to the conditions,

$$\dot{\mu} e^{-2\nu} \dot{A}_1 - \mu^{'} e^{-2\lambda} A_1^{'} = 0, \tag{2.2.13}$$

$$\dot{\mu} e^{-2\nu} \dot{A}_2 - \mu^{'} e^{-2\lambda} A_2^{'} = -c_2, \tag{2.2.14}$$

$$\dot{\mu} e^{-2\nu} \dot{A}_3 - \mu^{'} e^{-2\lambda} A_3^{'} = c_3, \tag{2.2.15}$$

$$\dot{\mu} A_0 + \mu^{'} A_4 = -A_1, \tag{2.2.16}$$

$$\ddot{A}_k + (2\dot{\mu} - \dot{\nu})\dot{A}_k - \nu^{'} e^{2(\nu-\lambda)} A_k^{'} = 0, \tag{2.2.17}$$

$$\dot{A}_k^{'} + (\mu^{'} - \nu^{'})\dot{A}_k + (\dot{\mu} - \dot{\lambda}) A_k^{'} = 0, \tag{2.2.18}$$

$$A_k^{''} + (2\mu^{'} - \lambda^{'}) A_k^{'} - \dot{\lambda} e^{2(\lambda-\nu)} \dot{A}_k = 0 \ (k = 1, 2, 3), \tag{2.2.19}$$

$$\dot{\nu} A_0 + \nu^{'} A_4 + \dot{A}_0 = 0\,, \tag{2.2.20}$$

$$\dot{\lambda} A_0 + \lambda^{'} A_4 + A_4^{'} = 0, \tag{2.2.21}$$

$$e^{2\nu} A_0^{'} - e^{2\lambda} \dot{A}_4 = 0. \tag{2.2.22}$$

After simplification, the problem has been reduced to sixteen coupled non-linear second order PDEs given by eqs.(2.2.13)-(2.2.22). These PDEs were solved to get the eight unknown functions $\nu, \lambda, \mu, A_{0,} A_{4,} A_k$ of two variables and $A_{0,} A_{4,} A_k \,(k = 1, 2, 3)$ are the functions of integration. The arbitrary constants that appeared in the Killing vector field, determines the number of generators of the Lie algebra.

The above system of equations can be reduced to simpler form by substituting the suitable transformation. Thus,

$$\dot{\mu}F_1 - \mu^{'}G_1 = 0\,, \tag{2.2.23}$$

$$\dot{\mu}F_2 - \mu^{'}G_2 = -c_2, \tag{2.2.24}$$

$$\dot{\mu}F_3 - \mu^{'}G_3 = c_3, \tag{2.2.25}$$

$$\dot{\mu}F_4 - \mu^{'}G_4 = -A_1, \tag{2.2.26}$$

$$(e^{\nu}F_i)^{\cdot} - \nu^{'}e^{\nu}G_i = 0, \tag{2.2.27}$$

$$\left(e^{\lambda}G_i\right)^{'} - \dot{\lambda}e^{\lambda}F_i = 0, \tag{2.2.28}$$

$$e^{2\nu}F_i^{'} + e^{2\lambda}\dot{G}_i = 0, \tag{2.2.29}$$

where (i=1,2,3,4). Now, one has the following cases:

(**I**) $\dot{\mu} = 0,\ \mu' = 0$,

(**II**) $\dot{\mu} = 0,\ \mu' \neq 0$,

(**III**) $\dot{\mu} \neq 0,\ \mu' = 0$,

(**IV**) $\dot{\mu} \neq 0,\ \mu' \neq 0$.

The metric is static if it has the time like vector $\frac{\partial}{\partial t}$, so we are only discussing the first two cases (i.e., static) because the other two cases are for the non-static part of the given metric.

2.2.1 $\dot{\mu} = 0,\ \mu' = 0$

As $\dot{\mu} = 0,\ \mu' = 0$, this implies $\mu = 2\ln a$ (where a is arbitrary constant). Then the metric given by eq. (2.2.1) is reduced to

$$ds^2 = ds_1^2 - a^2(dy^2 + dz^2), \tag{2.2.30}$$

where,

$$ds_1^2 = e^{2\nu(t,x)}dt^2 - e^{2\lambda(t,x)}dx^2. \tag{2.2.31}$$

is a metric of two dimensional V_2 space. The constant 'a' can be absorbed in the definitions of y and z, therefore, the metric given by eq.(2.2.1) takes the form

$$ds^2 = ds_1^2 - (dy^2 + dz^2). \tag{2.2.32}$$

and ds_1^2 admitting a group G_1 of motion. This can be done by using the theorem given by Eisenhart [32], which states that *"a necessary and sufficient condition that a V_n admits a group G_1 of motion is that there exists a coordinate system for which the coefficients g_{ij} do not involve one coordinate say x; then the curves of parameter x are the trajectories of motion"*. Therefore, the metric given by eq.(2.2.32) can either be written as

$$\mathbf{I}\,(\mathbf{A}) \qquad ds_1^2 = e^{2\nu(x)}dt^2 - e^{2\lambda(x)}dx^2, \tag{2.2.33}$$

or

$$\mathbf{I}(\mathbf{B}) \qquad ds_1^2 = e^{2\nu(t)}dt^2 - e^{2\lambda(t)}dx^2. \tag{2.2.34}$$

Here, we are reviewing only the static part i.e. $\mathbf{I}\,(\mathbf{A})$. Redefining x in eq.(2.2.33) s.t. $\lambda(x) = 0$, then the metric becomes

$$ds^2 = e^{2\nu(x)}dt^2 - dx^2 - dy^2 - dz^2. \tag{2.2.35}$$

The following cases can be obtained by solving the system of equations given by eqs.(2.2.23)-(2.2.29) for this metric.

$\mathbf{I}\,[\mathbf{A}(\mathbf{a})]$: $\nu = \ln\cosh kx$. The metric eq.(2.2.1) reduces to

$$ds^2 = \cosh^2\left(\frac{x}{a}\right) dt^2 - dx^2 - dy^2 - dz^2, \tag{2.2.36}$$

The corresponding Killing vector field is

$$\left.\begin{aligned}
&\mathbf{K} = [c_0 - \tanh\left(\tfrac{x}{a}\right)\left(c_4 \sin\left(\tfrac{t}{a}\right) - c_5 \cos\left(\tfrac{t}{a}\right)\right)]\partial/\partial t \\
&+ \left[\, c_4 \cos\left(\tfrac{t}{a}\right) + c_5 \sin\left(\tfrac{t}{a}\right)\right] \partial/\partial x \\
&+ [c_1 z + c_3]\,\partial/\partial y + [-c_1 y + c_2]\,\partial/\partial z.
\end{aligned}\right\} \tag{2.2.37}$$

There are six arbitrary constants appearing in $\mathbf{K}$, hence the metric admits a G_6 as the maximal group of motion. The generators of G_6 are $\mathbf{X_j}\,(j=0,1,....,5)$ corresponding to each c_k $\;(k=0,1,...,5)$ are

$$\left.\begin{array}{ll} \mathbf{X_0}=\partial/\partial t\,, & \mathbf{X_1}=\; z\partial/\partial y-y\partial/\partial z, \\ \mathbf{X_2}=\partial/\partial z, & \mathbf{X_3}=\partial/\partial y, \\ \multicolumn{2}{l}{\mathbf{X_4}=-\tanh\left(\tfrac{x}{a}\right)\sin\left(\tfrac{t}{a}\right)\partial/\partial t+\cos\left(\tfrac{t}{a}\right)\partial/\partial x,} \\ \multicolumn{2}{l}{\mathbf{X_5}=\tanh\left(\tfrac{x}{a}\right)\cos\left(\tfrac{t}{a}\right)\partial/\partial t+\sin\left(\tfrac{t}{a}\right)\partial/\partial x.} \end{array}\right\} \tag{2.2.38}$$

Here $\mathbf{X_0}$ is a timelike vector, therefore, the metric is static. The structure constants of the corresponding Lie algebra for this metric are

$$C^5_{04}=-\frac{1}{a},\qquad C^4_{05}=C^0_{45}=\frac{1}{a},\qquad C^3_{12}=-1,\qquad C^2_{13}=1.$$

$\mathbf{I}[\mathbf{A}(\mathbf{b})]$ $\;:\nu=\ln\left(\frac{x}{a}\right)$. Therefore, the metric eq.(2.2.1) reduces to

$$ds^2=\Big(\frac{x}{a}\Big)^2dt^2-dx^2-dy^2-dz^2. \tag{2.2.39}$$

This metric admits G_{10} as the maximal symmetry group. Therefore, the corresponding Killing vector field is

$$\left.\begin{array}{l} \mathbf{K}=[c_0+x^{-1}\{(c_6y+c_4z-c_8)\sinh\left(\tfrac{t}{a}\right)+(c_7y+c_5z-c_9) \\ \cosh\left(\tfrac{t}{a}\right)\}]\partial/\partial t+[-(c_6y+c_4z-c_8)\cosh\left(\tfrac{t}{a}\right) \\ -(c_7y+c_5z-c_9)\sinh\left(\tfrac{t}{a}\right)]\partial/\partial x+[c_1z+c_3+x(c_6\cosh\left(\tfrac{t}{a}\right) \\ +c_7\sinh\left(\tfrac{t}{a}\right))]\partial/\partial y+[-c_1y+c_2+x(c_4\cosh\left(\tfrac{t}{a}\right)+c_5\sinh\left(\tfrac{t}{a}\right))\partial/\partial z. \end{array}\right\} \tag{2.2.40}$$

The first four generators are same as given by eq.(2.2.38) for the metric eq.(2.2.36). The remaining generators in this case are

$$\left.\begin{array}{l} \mathbf{X_4}=-\tfrac{az}{x}\sinh\left(\tfrac{t}{a}\right)\partial/\partial t+z\cosh\left(\tfrac{t}{a}\right)\partial/\partial x-x\cosh\left(\tfrac{t}{a}\right)\partial/\partial z, \\ \mathbf{X_5}=-\tfrac{az}{x}\cosh\left(\tfrac{t}{a}\right)\partial/\partial t+z\sinh\left(\tfrac{t}{a}\right)\partial/\partial x-x\sinh\left(\tfrac{t}{a}\right)\partial/\partial z, \\ \mathbf{X_6}=-\tfrac{ay}{x}\sinh\left(\tfrac{t}{a}\right)\partial/\partial t+y\cosh\left(\tfrac{t}{a}\right)\partial/\partial x-x\cosh\left(\tfrac{t}{a}\right)\partial/\partial y, \\ \mathbf{X_7}=-\tfrac{ay}{x}\cosh\left(\tfrac{t}{a}\right)\partial/\partial t+y\sinh\left(\tfrac{t}{a}\right)\partial/\partial x-x\sinh\left(\tfrac{t}{a}\right)\partial/\partial y, \\ \mathbf{X_8}=\tfrac{1}{x}\sinh\left(\tfrac{t}{a}\right)\partial/\partial t-\tfrac{1}{a}\cosh\left(\tfrac{t}{a}\right)\partial/\partial x, \\ \mathbf{X_9}=\tfrac{1}{x}\cosh\left(\tfrac{t}{a}\right)\partial/\partial t-\tfrac{1}{a}\sinh\left(\tfrac{t}{a}\right)\partial/\partial x. \end{array}\right\} \tag{2.2.41}$$

Therefore, one has the following structure constants:

$$C_{79}^{3} = C_{67}^{0} = C_{59}^{2} = C_{04}^{5} = C_{05}^{4} = C_{06}^{7} = C_{07}^{6} = C_{08}^{9} = C_{09}^{8} = 1,$$

$$C_{12}^{3} = C_{14}^{6} = C_{15}^{7} = C_{24}^{8} = C_{25}^{9} = C_{36}^{8} = C_{37}^{9} = C_{48}^{2} = C_{68}^{3} = -1,$$

$$C_{13}^{2} = C_{16}^{4} = C_{17}^{5} = C_{45}^{0} = C_{46}^{1} = C_{57}^{1} = 1.$$

$\mathbf{I}[\mathbf{A}(\mathbf{c})]$: $\nu = \ln\cos\left(\frac{x}{a}\right)$. Therefore, the corresponding metric is

$$ds^2 = \cos^2\left(\frac{x}{a}\right)dt^2 - dx^2 - dy^2 - dz^2. \tag{2.2.42}$$

The corresponding Killing vector field is given by

$$\left.\begin{aligned} &\mathbf{K} = c_0 + \tan\left(\tfrac{x}{a}\right)\left(c_4\sinh\left(\tfrac{t}{a}\right) + c_5\cosh\left(\tfrac{t}{a}\right)\right)]\partial/\partial t + [c_4\cosh\left(\tfrac{t}{a}\right) \\ &+c_5\sinh\left(\tfrac{t}{a}\right)]\partial/\partial x + [c_1 z + c_3]\partial/\partial y + [-c_1 y + c_2]\partial/\partial z. \end{aligned}\right\} \tag{2.2.43}$$

This metric eq.(2.2.42) admits six isometries. The first four generators are same as for eq.(2.2.36) whereas the remaining generators are

$$\left.\begin{aligned} \mathbf{X_4} &= \tan\left(\tfrac{x}{a}\right)\sinh\left(\tfrac{t}{a}\right)\partial/\partial t + \cosh\left(\tfrac{t}{a}\right)\partial/\partial x, \\ \mathbf{X_5} &= \tan\left(\tfrac{x}{a}\right)\cosh\left(\tfrac{t}{a}\right)\partial/\partial t + \sinh\left(\tfrac{t}{a}\right)\partial/\partial x. \end{aligned}\right\} \tag{2.2.44}$$

Therefore,

$$C_{04}^{5} = C_{45}^{0} = C_{05}^{4} = \frac{1}{a}, \quad C_{12}^{3} = -1, \quad C_{13}^{2} = 1.$$

$\mathbf{I}[\mathbf{A}(\mathbf{d})]$: In this case $\nu = \frac{x}{a}$. Hence, the metric eq.(2.2.1) becomes

$$ds^2 = e^{2x/a}dt^2 - dx^2 - dy^2 - dz^2. \tag{2.2.45}$$

The Killing vectors field for this metric is

$$\left.\begin{aligned} \mathbf{K} = &\left[c_0 - \tfrac{c_4}{2}(t^2/a + ae^{-2x/a}) - c_5 t/a\right]\partial/\partial t + (c_4 t + c_5)\,\partial/\partial x \\ &+ (c_1 z + c_3)\,\partial/\partial y + (-c_1 y + c_2)\,\partial/\partial z. \end{aligned}\right\} \tag{2.2.46}$$

The Killing vectors other than the first four given as in eq.(2.2.38) are

$$\left.\begin{aligned} \mathbf{X_4} &= -\tfrac{1}{2}(t^2/a + ae^{-2x/a})\partial/\partial t + t\partial/\partial x, \\ \mathbf{X_5} &= -(t/a)\partial/\partial t + \partial/\partial x. \end{aligned}\right\} \tag{2.2.47}$$

The structure constants of the corresponding Lie algebra for this metric are

$$C^5_{04} = C^2_{13} = 1, \quad C^4_{45} = \frac{1}{a}, \quad C^3_{12} = -1, \quad C^1_{05} = -\frac{1}{a}.$$

$\mathbf{I[A(e)]}$: Here $\nu = 0$. Thus, the metric eq.(2.2.1) reduces to the Minkowski space given by

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2. \tag{2.2.48}$$

The Killing vector field is

$$\left.\begin{array}{l} \mathbf{K} = [c_7 z + c_9 y + c_4 x + c_0]\partial/\partial t + [-c_6 z - c_8 y + c_4 t + c_5]\partial/\partial x \\ +[c_1 z + c_8 x + c_9 t + c_3]\partial/\partial y + [-c_1 y + c_6 x + c_7 t + c_2]\partial/\partial z. \end{array}\right\} \tag{2.2.49}$$

The metric eq.(2.2.48) admits G_{10} as a maximal group of motion. The generators $\mathbf{X_j}$ $(j = 4, 5, ..., 10)$ of G_{10} are

$$\left.\begin{array}{ll} \mathbf{X_4} = x\partial/\partial t + t\partial/\partial x, & \mathbf{X_5} = \partial/\partial x, \\ \mathbf{X_6} = -z\partial/\partial x + x\partial/\partial z, & \mathbf{X_7} = z\partial/\partial t + t\partial/\partial z, \\ \mathbf{X_8} = -y\partial/\partial x + x\partial/\partial y, & \mathbf{X_9} = y\partial/\partial t + t\partial/\partial y. \end{array}\right\} \tag{2.2.50}$$

Then, one has

$$C^5_{04} = C^2_{07} = C^3_{09} = C^2_{13} = C^6_{18} = C^7_{19} = C^0_{27} = C^4_{89} = C^0_{39} = 1,$$

$$C^3_{12} = C^8_{16} = C^9_{17} = C^5_{26} = C^5_{38} = C^0_{45} = C^6_{47} = C^8_{49} = C^1_{68} = -1,$$

$$C^7_{46} = C^9_{48} = C^2_{56} = C^3_{58} = C^4_{67} = C^1_{79} = 1.$$

2.2.2 $\dot{\mu} = 0, \ \mu' \neq 0$

For this case, the metrics are as follows:

$\mathbf{II(a)}$: Here $\nu = \mu$, then the metric eq.(2.2.1) takes the form

$$ds^2 = e^{2\nu(x)}(dt^2 - dy^2 - dz^2) - dx^2. \tag{2.2.51}$$

The Killing vector field for this metric is

$$\left.\begin{aligned}\mathbf{K} = (c_0 + c_4 z + c_5 y)\partial/\partial t + (c_1 z + c_4 t + c_2)\partial/\partial y \\ +(-c_1 y + c_5 t + c_3)\partial/\partial z.\end{aligned}\right\} \tag{2.2.52}$$

i.e. the metric admits six isometries. $\mathbf{X_j}(j = 0, 1, 2, 3)$ are same as for the metric eq.(2.2.54). The remaining are

$$\left.\begin{aligned}\mathbf{X_4} = z\partial/\partial t + t\partial/\partial z, \\ \mathbf{X_5} = y\partial/\partial t + t\partial/\partial y,\end{aligned}\right\} \tag{2.2.53}$$

and the structure constants of the corresponding Lie algebra are

$$C^2_{04} = C^3_{05} = C^2_{13} = C^4_{15} = C^0_{24} = C^0_{45} = 1, \qquad C^3_{12} = C^5_{14} = -1.$$

II(b) : The metric is given by

$$ds^2 = e^{2\nu(x)}dt^2 - dx^2 - e^{2\mu(x)}(dy^2 + dz^2). \tag{2.2.54}$$

For this metric, eq.(2.2.12) becomes

$$\mathbf{K} = c_0\partial/\partial t + (c_1 z + c_3)\partial/\partial y + (-c_1 y + c_2)\partial/\partial z. \tag{2.2.55}$$

Hence, eq.(2.2.54) admits symmetry group G_4 whose generators are

$$\left.\begin{aligned}\mathbf{X_0} = \partial/\partial t, \quad \mathbf{X_1} = z\partial/\partial y - y\partial/\partial z, \\ \mathbf{X_2} = \partial/\partial z, \quad \mathbf{X_3} = \partial/\partial y,\end{aligned}\right\} \tag{2.2.56}$$

and the structure constants of the Lie algebra are

$$C^3_{12} = -1, \qquad C^2_{13} = 1.$$

II(c) : Here $\mu = cx$ and $\mu = \nu$. Then the metric is

$$ds^2 = e^{2\nu(x)}dt^2 - dx^2 - e^{2cx}(dy^2 + dz^2). \tag{2.2.57}$$

The metric eq.(2.2.57) admits four isometries. The Killing vector field, generators of G_4, and the structure constants are same as for eq.(2.2.54) .

II(d) : Here $\nu = ax, (a \neq 0)$ and $\mu = \nu$. The metric is

$$ds^2 = e^{2ax}(dt^2 - dy^2 - dz^2) - dx^2, \tag{2.2.58}$$

admits $G_{10} =< \mathbf{X_0}, \mathbf{X_1}, ..., \mathbf{X_9} >$ as maximal group of motion. The Killing vector field is

$$\left.\begin{array}{l} \mathbf{K} = [c_0 - ac_5 t - a\frac{c_4}{2}(z^2 + y^2 + t^2 + \frac{1}{a^2}e^{-2ax}) + (c_8 t + c_6)z \\ +(-c_9 t + c_7)y]\partial/\partial t \ + [-\frac{1}{a}(c_8 z - c_9 y) + c_4 t + c_5]\partial/\partial x \\ +[\frac{c_9}{2}(z^2 - y^2 - t^2 + \frac{1}{a^2}e^{-2ax}) \ \ + c_8 yz + c_1 z - a(c_4 t + c_5)y \\ +c_7 t + c_3]\partial/\partial y + [\frac{c_8}{2}(z^2 - y^2 + t^2 - \frac{1}{a^2}e^{-2ax}) \\ -c_9 yz - c_1 y - a(c_4 t + c_5)z + c_6 t + c_2]\partial/\partial z. \end{array}\right\} \tag{2.2.59}$$

The generators $\mathbf{X_j}(j = 4, 5, ..., 9)$ are

$$\left.\begin{array}{l} \mathbf{X_4} = -\frac{1}{2}a(z^2 + y^2 + t^2 + \frac{1}{a^2}\ e^{-2ax})\partial/\partial t \ + t\partial/\partial x \\ \qquad - aty\partial/\partial y - atz\partial/\partial z, \\ \mathbf{X_5} = -at\partial/\partial t + \partial/\partial x - ay\partial/\partial y - az\partial/\partial z, \\ \mathbf{X_6} = z\partial/\partial t + t\partial/\partial z, \\ \mathbf{X_7} = \ y\partial/\partial t + t\partial/\partial y, \\ \mathbf{X_8} = tz\partial/\partial t - \frac{1}{a}z\partial/\partial x + yz\partial/\partial y \ + \frac{1}{2}(z^2 - y^2 + t^2 - \frac{1}{a^2}\, e^{-2ax})\partial/\partial z, \\ \mathbf{X_9} = -ty\partial/\partial t + \frac{1}{a}y\partial/\partial x + \frac{1}{2}(z^2 - y^2 - t^2 + \frac{1}{a^2}\ e^{-2ax})\partial/\partial y - yz\partial/\partial z. \end{array}\right\} \tag{2.2.60}$$

The structure constants of the corresponding Lie algebra are

$C^5_{04} = C^2_{06} = C^3_{07} = C^2_{13} = C^6_{08} = C^6_{17} = C^9_{18} = C^0_{26} = C^1_{29} = C^0_{37} = C^1_{38} = C^1_{67} = 1,$

$C^7_{09} = C^3_{12} = C^7_{16} = C^8_{19} = -1, \qquad C^0_{05} = C^6_{24} = C^2_{25} = C^7_{34} = C^3_{35} = C^8_{58} = C^9_{59} = -a,$

$C^5_{28} = C^4_{68} = -\frac{1}{a}, \qquad C^4_{45} = C^8_{46} = C^9_{47} = a, \qquad C^5_{39} = C^4_{79} = \frac{1}{a}.$

II(e) : $\nu = ax, (a \neq 0)$ and $\mu = cx$. Thus, the metric is given by

$$ds^2 = e^{2ax}dt^2 - dx^2 - e^{2cx}(dy^2 + dz^2). \tag{2.2.61}$$

The Killing vector field for this metric is

$$\left.\begin{array}{l}\mathbf{K}=(c_0-ac_5t)\partial/\partial t-c_5\partial/\partial x+(c_1z+c_3-cc_5y)\partial/\partial y\\ +(-c_1y+c_2-cc_5z)\partial/\partial z.\end{array}\right\}\tag{2.2.62}$$

The metric admits five isometries, four generators are same as for metric eq.(2.2.55) the remaining one is

$$\mathbf{X_5}=-at\partial/\partial t-\partial/\partial x-cy\partial/\partial y-cz\partial/\partial z.\tag{2.2.63}$$

The structure constants of the corresponding Lie algebra are

$$C^0_{05}=-a,\quad C^2_{25}=C^3_{35}=-c,\quad C^2_{13}=1,\quad C^3_{12}=-1.$$

II(f) : Here $\mu=cx$ and $a=0$. Hence, the metric is

$$ds^2=dt^2-dx^2-e^{2cx}(dy^2+dz^2),\tag{2.2.64}$$

admits seven isometries. The Killing vector field is

$$\left.\begin{array}{l}\mathbf{K}=c_0\partial/\partial t+[\frac{1}{c}(c_6y-c_5z)+c_4]\partial/\partial x\\ +[\frac{c_6}{2}(z^2-y^2+\frac{1}{c^2}e^{-2cx})+c_5yz-cc_4y+c_1z+c_3]\partial/\partial y\\ +[\frac{c_5}{2}(z^2-y^2-\frac{1}{c^2}e^{-2cx})-c_6yz-cc_4z-c_1y+c_2]\partial/\partial z,\end{array}\right\}\tag{2.2.65}$$

The generators $\mathbf{X_j}(j=0,1,2,3)$ of $G_7=<\mathbf{X_0},\mathbf{X_1},...,\mathbf{X_6}>$ are same as for the previous cases, the remaining generators $\mathbf{X_j}$ $(j=4,5,6)$ are

$$\left.\begin{array}{l}\mathbf{X_4}=c(\frac{1}{c}\partial/\partial x-y\partial/\partial y-z\partial/\partial z),\\ \mathbf{X_5}=-\frac{z}{c}\partial/\partial x+yz\partial/\partial y+\frac{1}{2}(z^2-y^2-\frac{1}{c^2}\ e^{-2cx})\partial/\partial z,\\ \mathbf{X_6}=\frac{y}{c}\partial/\partial x+\frac{1}{2}(z^2-y^2+\frac{1}{c^2}\ e^{-2cx})\partial/\partial y-yz\partial/\partial z.\end{array}\right\}\tag{2.2.66}$$

Then the structure constants of the corresponding Lie algebra are

$C^3_{12}=C^5_{16}=-1,\quad C^2_{13}=C^6_{15}=C^1_{26}=C^1_{35}=1,\quad C^4_{36}=\frac{1}{c},$

$C^6_{46}=-\frac{1}{c},\quad C^2_{24}=C^4_{25}=C^3_{34}=C^5_{45}=-c.$

Chapter 3

The Classification of Plane Symmetric Static Spacetimes by Noether Symmetries

To start the classification according to the Noether symmetries, we consider the corresponding Lagrangian of the metric eq.(2.2.35) given as

$$L = e^{2\nu(x)}\dot{t}^2 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2. \tag{3.0.1}$$

This Lagrangian will admit the Noether symmetry $\mathbf{X}$ if it satisfies eq.(2.1.19). This implies

$$\begin{aligned}
&2\nu'(x)e^{2\nu(x)}\eta^1\dot{t}^2 + 2e^{2\nu(x)}\dot{t}[\eta^0_{,s} + \dot{t}(\eta^0_{,t} - \xi_{,s}) + \dot{x}\eta^0_{,x} + \dot{y}\eta^0_{,y} + \dot{z}\eta^0_{,z}\\
&-\dot{t}^2\xi_{,t} - \dot{t}\dot{x}\xi_{,x} - \dot{t}\dot{y}\xi_{,y} - \dot{t}\dot{z}\xi_{,z}] - 2\dot{x}[\eta_{,s} + \dot{t}\eta^1_{,t} + \dot{x}(\eta^1_{,x} - \xi_{,s}) + \dot{y}\eta^1_{,y} + \dot{z}\eta^1_{,z}\\
&-\dot{t}\dot{x}\xi_{,t} - \dot{x}^2\xi_{,x} - \dot{x}\dot{y}\xi_{,y} - \dot{x}\dot{z}\xi_{,z}] - 2\dot{y}[\eta^2_{,s} + \dot{t}\eta^2_{,t} + \dot{x}\eta^2_{,x} + \dot{y}(\eta^2_{,y} - \xi_{,s}) + \dot{z}\eta^2_{,z}\\
&-\dot{t}\dot{y}\xi_{,t} - \dot{x}\dot{y}\xi_{,x} - \dot{y}^2\xi_{,y} - \dot{y}\dot{z}\xi_{,z}] - 2\dot{z}[\eta^3_{,s} + \dot{t}\eta^3_{,t} + \dot{x}\eta^3_{,x} + \dot{y}\eta^3_{,y}\\
&+\dot{z}(\eta^3_{,z} - \xi_{,s}) - \dot{t}\dot{z}\xi_{,t} - \dot{x}\dot{z}\xi_{,x} - \dot{y}\dot{z}\xi_{,y} - \dot{z}^2\xi_{,z}] + (\xi_{,s} + \dot{t}\xi_{,t} + \dot{x}\xi_{,x}\\
&+\dot{y}\xi_{,y} + \dot{z}\xi_{,z})(e^{2\nu(x)}\dot{t}^2 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2)\\
&= A_s + \dot{t}A_{,t} + \dot{x}A_{,x} + \dot{y}A_{,y} + \dot{z}A_{,z}
\end{aligned} \tag{3.0.2}$$

Eq.(3.0.2) provides a set of nineteen independent PDEs with seven unknown functions $\nu(x), \xi, A$, and $\eta^a (a = 0, 1, 2, 3)$.

$$\xi_{,t} = 0, \tag{3.0.3}$$

$$\xi_{,x} = 0, \tag{3.0.4}$$

$$\xi_{,y} = 0, \tag{3.0.5}$$

$$\xi_{,z} = 0, \tag{3.0.6}$$

$$A_{,s} = 0, \tag{3.0.7}$$

$$2e^{2\nu(x)}\eta^0_{,s} = A_{,t}, \tag{3.0.8}$$

$$-2\eta^1_{,s} = A_{,x}, \tag{3.0.9}$$

$$-2\eta^2_{,s} = A_{,y}, \tag{3.0.10}$$

$$-2\eta^3_{,s} = A_{,z}, \tag{3.0.11}$$

$$e^{2\nu(x)}\eta^0_{,x} = \eta^1_{,t}, \tag{3.0.12}$$

$$e^{2\nu(x)}\eta^0_{,y} = \eta^2_{,t}, \tag{3.0.13}$$

$$e^{2\nu(x)}\eta^0_{,z} = \eta^3_{,t}, \tag{3.0.14}$$

$$2\eta^1_{,x} = \xi_{,s}, \tag{3.0.15}$$

$$2\eta^2_{,y} = \xi_{,s}, \tag{3.0.16}$$

$$2\eta^3_{,z} = \xi_{,s}, \tag{3.0.17}$$

$$\eta^1_{,y} + \eta^2_{,x} = 0, \tag{3.0.18}$$

$$\eta^1_{,z} + \eta^3_{,x} = 0, \tag{3.0.19}$$

$$\eta^2_{,z} + \eta^3_{,y} = 0, \tag{3.0.20}$$

$$2\nu'(x)\eta^1 + 2\eta^0_{,t} = \xi_{,s}. \tag{3.0.21}$$

Here ξ, A, and η^a $(a = 0, 1, 2, 3)$ are the functions of s and q^a, where q^a represents t, x, y, z. Eqs.(3.0.3)-(3.0.6) give $\xi = \xi(s)$ and eq.(3.0.7) leads to $A = A(t, x, y, z)$. Differentiating eqs.(3.0.8)-(3.0.11) w.r.t. s, we have the following equations:

$$\eta^0_{,ss} = 0 \;\Rightarrow\; \eta^0 = a_1(t, x, y, z)s + a_2(t, x, y, z),$$

$$\eta^1_{,ss} = 0 \;\Rightarrow\; \eta^1 = a_3(t,x,y,z)s + a_4(t,x,y,z),$$

$$\eta^2_{,ss} = 0 \;\Rightarrow\; \eta^2 = a_5(t,x,y,z)s + a_6(t,x,y,z),$$

$$\eta^3_{,ss} = 0 \;\Rightarrow\; \eta^3 = a_7(t,x,y,z)s + a_8(t,x,y,z).$$

Similarly, differentiating eq.(3.0.15) twice w.r.t. s, we get

$$\xi_{sss} = 0 \Rightarrow \xi = c_0\frac{s^2}{2} + c_1 s + c_2. \tag{3.0.22}$$

Therefore, we have

$$\left.\begin{array}{l} \xi = c_0\frac{s^2}{2} + c_1 s + c_2, \\ \eta^0 = a_1(t,x,y,z)s + a_2(t,x,y,z), \\ \eta^1 = a_3(t,x,y,z)s + a_4(t,x,y,z), \\ \eta^2 = a_5(t,x,y,z)s + a_6(t,x,y,z), \\ \eta^3 = a_7(t,x,y,z)s + a_8(t,x,y,z), \\ A = A(t,x,y,z). \end{array}\right\} \tag{3.0.23}$$

Now, differentiate eq.(3.0.18) w.r.t. t, x, y, and z respectively results in

$$\eta^1_{,ty} + \eta^2_{,tx} = 0, \tag{3.0.24}$$

$$\eta^1_{,xy} + \eta^2_{,xx} = 0.$$

If we consider eq.(3.0.15) we can see that $\eta^1_{,xy} = 0$. Therefore, the above equation reduces to

$$\eta^2_{,xx} = 0, \tag{3.0.25}$$

$$\eta^1_{,yy} + \eta^2_{xy} = 0.$$

Similarly, eq.(3.0.16) implies that $\eta^2_{,xy} = 0$, which leads to

$$\eta^1_{,yy} = 0, \tag{3.0.26}$$

$$\eta^1_{,yz} + \eta^2_{,xz} = 0. \tag{3.0.27}$$

Considering eq.(3.0.19) and differentiating w.r.t. t, x, y, and z respectively, we have

$$\eta^1_{,tz} + \eta^3_{,tx} = 0, \tag{3.0.28}$$

$$\eta^1_{xz} + \eta^3_{,xx} = 0.$$

Recalling eq.(3.0.15), we can see that $\eta^1_{,xz} = 0$. Therefore, the above equation becomes

$$\eta^3_{,xx} = 0, \tag{3.0.29}$$

$$\eta^1_{,yz} + \eta^3_{,xy} = 0, \tag{3.0.30}$$

$$\eta^1_{,zz} + \eta^3_{xz} = 0.$$

Similarly, if we differentiate eq.(3.0.17) w.r.t. x, we get $\eta^3_{,xz} = 0$. Therefore, the above equation leads to

$$\eta^1_{,zz} = 0. \tag{3.0.31}$$

Consider eq.(3.0.20), after differentiating w.r.t. t, x, y, and z respectively, we have

$$\eta^2_{,tz} + \eta^3_{,ty} = 0, \tag{3.0.32}$$

$$\eta^2_{,xz} + \eta^3_{,xy} = 0, \tag{3.0.33}$$

$$\eta^2_{,yz} + \eta^3_{,yy} = 0.$$

This equation reduces to

$$\eta^3_{,yy} = 0, \tag{3.0.34}$$

when we differentiate eq.(3.0.16) w.r.t z i.e. $\eta^2_{,yz} = 0$.

$$\eta^2_{,zz} + \eta^3_{,yz} = 0.$$

Using eq.(3.0.17), differentiate it w.r.t. y then the above equation implies that

$$\eta^2_{,zz} = 0. \tag{3.0.35}$$

All the eqs.(3.0.30), (3.0.27), and (3.0.33) together implies that

$$\eta^2_{,xz} = 0.$$

Therefore,

$$\eta^3_{,xy} = 0, \qquad \eta^1_{,yz} = 0. \tag{3.0.36}$$

Now differentiating eq.(3.0.21) w.r.t. t, x, y, and z respectively, we have

$$\nu'(x)\eta^1_{,t} + \eta^0_{,tt} = 0, \tag{3.0.37}$$

$$\nu''(x)\eta^1 + \nu'(x)\eta^1_{,x} + \eta^0_{,tx} = 0, \tag{3.0.38}$$

$$\nu'(x)\eta^1_{,y} + \eta^0_{,ty} = 0, \tag{3.0.39}$$

$$\nu'(x)\eta^1_{,z} + \eta^0_{,tz} = 0. \tag{3.0.40}$$

Similarly, differentiating eq.(3.0.12) w.r.t. t, x, y, and z respectively, we get

$$e^{2\nu(x)}\eta^0_{,tx} - \eta^1_{,tt} = 0, \tag{3.0.41}$$

$$\eta^0_{,xx} + 2\nu'(x)\eta^0_{,x} = 0, \tag{3.0.42}$$

as we see that from eq.(3.0.15), $\eta^1_{,tx} = 0$.

$$e^{2\nu(x)}\eta^0_{,xy} - \eta^1_{,ty} = 0, \tag{3.0.43}$$

$$e^{2\nu(x)}\eta^0_{,xz} - \eta^1_{,tz} = 0. \tag{3.0.44}$$

After differentiating eq.(3.0.13) w.r.t. t, x, y, and z respectively, we obtain

$$e^{2\nu(x)}\eta^0_{,ty} - \eta^2_{,tt} = 0, \tag{3.0.45}$$

$$2\nu'(x)e^{2\nu(x)}\eta^0_{,y} + e^{2\nu(x)}\eta^0_{,xy} - \eta^2_{,tx} = 0, \tag{3.0.46}$$

$$\eta^0_{,yy} = 0, \tag{3.0.47}$$

as we see that $\eta^2_{,ty} = 0$.

$$e^{2\nu(x)}\eta^0_{,yz} - \eta^2_{,tz} = 0. \tag{3.0.48}$$

Consider eq.(3.0.14) and differentiating it w.r.t. t, x, y, and z respectively, we have

$$e^{2\nu(x)}\eta^0_{,tz} - \eta^3_{,tt} = 0, \tag{3.0.49}$$

$$2\nu'(x)e^{2\nu(x)}\eta^0_{,z} + e^{2\nu(x)}\eta^0_{,xz} - \eta^3_{,tx} = 0, \tag{3.0.50}$$

$$e^{2\nu(x)}\eta^0_{,yz} - \eta^3_{,ty} = 0, \tag{3.0.51}$$

$$\eta^0_{,zz} = 0, \tag{3.0.52}$$

as we see that $\eta^3_{,tz} = 0$.

If we solve eqs.(3.0.48), (3.0.51), and (3.0.32) simultaneously, we get

$$\eta^2_{,tz} = 0, \tag{3.0.53}$$

which implies

$$\eta^3_{,ty} = 0, \qquad \eta^0_{,yz} = 0. \tag{3.0.54}$$

Eqs.(3.0.24), (3.0.43), and (3.0.46) together implies that

$$\eta^0_{,xy} + \nu'(x)\eta^0_{,y} = 0. \tag{3.0.55}$$

Solving eqs.(3.0.28), (3.0.44), and (3.0.50) simultaneously, we obtain

$$\eta^0_{,xz} + \nu'(x)\eta^0_{,z} = 0. \tag{3.0.56}$$

Now, differentiating eq.(3.0.42) w.r.t. y and eq.(3.0.55) w.r.t. x and subtracting them gives

$$\nu'(x)\eta^0_{,xy} + \nu''(x)\eta^0_{,y} = 0. \tag{3.0.57}$$

Similarly, we take eqs.(3.0.42) and (3.0.56) and differentiate them w.r.t. z and x respectively. Now, we subtract the resulting equations and one gets

$$\nu'(x)\eta^0_{,xz} + \nu''(x)\eta^0_{,z} = 0. \tag{3.0.58}$$

Now, solving all the partial derivatives of the form $\eta^a_{,bc} = 0$, where $a = 0, 1, 2, 3$ and b, c represents t, x, y, and z to find the functions ξ, η^a.

From eq.(3.0.47), we have

$$a_{1,yy}(t, x, y, z)s + a_{2,yy}(t, x, y, z) = 0.$$

Comparing coefficients of s and s^0;

$$\begin{aligned} s &: \quad a_{1,yy}(t, x, y, z) = 0 \quad \Rightarrow a_1(t, x, y, z) = b_1(t, x, z)y + b_2(t, x, z), \\ s^0 &: \quad a_{2,yy}(t, x, y, z) = 0 \quad \Rightarrow a_1(t, x, y, z) = b_3(t, x, z)y + b_4(t, x, z), \end{aligned}$$

$$\eta^0 = [b_1(t, x, z)y + b_2(t, x, z)]s + b_3(t, x, z)y + b_4(t, x, z). \tag{3.0.59}$$

Using eq.(3.0.54), we have

$$b_{1,z}(t, x, z)s + b_{3,z}(t, x, z) = 0.$$

Comparing coefficients of s and s^0;

$$\begin{aligned} s &: \quad b_{1,z}(t, x, z) = 0 \quad \Rightarrow b_1(t, x, z) = d_1(t, x), \\ s^0 &: \quad b_{3,z}(t, x, z) = 0 \quad \Rightarrow b_3(t, x, z) = d_3(t, x), \end{aligned}$$

$$\eta^0 = [d_1(t, x))y + b_2(t, x, z)]s + d_3(t, x)y + b_4(t, x, z). \tag{3.0.60}$$

Taking into account eq.(3.0.52), we have

$$b_{2,zz}(t, x, z)s + b_{4,zz}(t, x, z) = 0.$$

Comparing coefficients of s and s^0;

$$\begin{aligned} s &: \quad b_{2,zz}(t, x, z) = 0 \quad \Rightarrow b_2(t, x, z) = d_2(t, x)z + d_0(t, x), \\ s^0 &: \quad b_{4,zz}(t, x, z) = 0 \quad \Rightarrow b_4(t, x, z) = d_4(t, x)z + d_5(t, x), \end{aligned}$$

$$\eta^0 = [d_1(t,x))y + d_2(t,x)z + d_0(t,x)]s + d_3(t,x)y + d_4(t,x)z + d_5(t,x). \tag{3.0.61}$$

Now, differentiating eq.(3.0.15) w.r.t. x gives

$$\eta^1_{,xx} = 0 \qquad \Rightarrow a_{3,xx}(t,x,y,z)s + a_{4,xx}(t,x,y,z) = 0.$$

Comparing coefficients of s and s^0;

$$s \quad : \qquad a_{3,xx}(t,x,y,z) = 0 \qquad \Rightarrow a_3(t,x,y,z) = b_5(t,y,z)x + b_6(t,y,z),$$

$$s^0 \quad : \qquad a_{4,xx}(t,x,y,z) = 0 \qquad \Rightarrow a_4(t,x,y,z) = b_7(t,y,z)x + b_8(t,y,z),$$

$$\eta^1 = [b_5(t,y,z)x + b_6(t,y,z)]s + b_7(t,y,z)x + b_8(t,y,z). \tag{3.0.62}$$

From eq.(3.0.26), we have

$$\eta^1_{,yy} = 0 \qquad \Rightarrow [b_{5,yy}(t,y,z)x + b_{6,yy}(t,y,z)]s + b_{7,yy}(t,y,z)x + b_{8,yy}(t,y,z) = 0.$$

Comparing coefficients of xs, s, x, constant;

$$xs \quad : \qquad b_{5,yy}(t,y,z) = 0 \qquad \Rightarrow b_5(t,y,z) = d_6(t,z)y + d_7(t,z),$$

$$s \quad : \qquad b_{6,yy}(t,y,z) = 0 \qquad \Rightarrow b_6(t,y,z) = d_8(t,z)y + d_9(t,z),$$

$$x \quad : \qquad b_{7,yy}(t,y,z) = 0 \qquad \Rightarrow b_7(t,y,z) = d_{10}(t,z)y + d_{11}(t,z),$$

$$\text{constant term} \quad : \qquad b_{8,yy}(t,y,z) = 0 \qquad \Rightarrow b_8(t,y,z) = d_{12}(t,z)y + d_{13}(t,z),$$

$$\begin{aligned} \eta^1 = {} & [(d_6(t,z)y + d_7(t,z))x + d_8(t,z)y + d_9(t,z)]s \\ & + d_{10}(t,z)y + d_{11}(t,z))x + d_{12}(t,z)y + d_{13}(t,z). \end{aligned} \tag{3.0.63}$$

Moreover, differentiate eq.(3.0.15) w.r.t. t, we get

$$\eta^1_{,xy} = 0 \qquad \Rightarrow d_6(t,z)s + d_{10}(t,z) = 0.$$

Comparing coefficients of s and s^0;

$$s \quad : \qquad d_6(t,z) = 0,$$

$$s^0 \quad : \qquad d_{10}(t,z) = 0,$$

$$\eta^1 = [d_7(t,z)x + d_8(t,z)y + d_9(t,z)]s + d_{11}(t,z)x + d_{12}(t,z)y + d_{13}(t,z). \quad (3.0.64)$$

Also, differentiate eq.(3.0.15) w.r.t. z, we get

$$\eta^1_{,xz} = 0 \qquad \Rightarrow d_{7,z}(t,z)s + d_{11,z}(t,z) = 0.$$

Comparing coefficients of s and s^0;

$$s \quad : \qquad d_{7,z}(t,z) = 0 \Rightarrow d_7(t,z) = d_7(t),$$
$$s^0 \quad : \qquad d_{11,z}(t,z) = 0 \Rightarrow d_{11}(t,z) = d_{11}(t),$$

$$\eta^1 = [d_7(t)x + d_8(t,z)y + d_9(t,z)]s + d_{11}(t)x + d_{12}(t,z)y + d_{13}(t,z). \quad (3.0.65)$$

Using eq.(3.0.36), we have

$$\eta^1_{,yz} = 0 \qquad \Rightarrow d_{8,z}(t,z)s + d_{12,z}(t,z) = 0.$$

Comparing coefficients of s and s^0;

$$s \quad : \qquad d_{8,z}(t,z) = 0 \qquad \Rightarrow d_8(t,z) = d_8(t),$$
$$s^0 \quad : \qquad d_{12,z}(t,z) = 0 \qquad \Rightarrow d_{12}(t,z) = d_{12}(t),$$

$$\eta^1 = [d_7(t)x + d_8(t)y + d_9(t,z)]s + d_{11}(t)x + d_{12}(t)y + d_{13}(t,z). \quad (3.0.66)$$

Taking into account eq.(3.0.31), one obtain

$$\eta^1_{,zz} = 0 \qquad \Rightarrow d_{9,zz}(t,z)s + d_{13,zz}(t,z) = 0.$$

Comparing coefficients of s and s^0;

$$s \quad : \qquad d_{9,zz}(t,z) = 0 \qquad \Rightarrow d_9(t,z) = d_9(t)z + d_{14}(t),$$
$$s^0 \quad : \qquad d_{13,zz}(t,z) = 0 \qquad \Rightarrow d_{13}(t,z) = d_{13}(t)z + d_{15}(t),$$

$$\eta^1 = [d_7(t)x + d_8(t)y + d_9(t)z + d_{14}(t)]s + d_{11}(t)x + d_{12}(t)y + d_{13}(t)z + d_{15}(t). \quad (3.0.67)$$

In addition, differentiating eq.(3.0.15) w.r.t. t, we get

$$\eta^1_{,tx} = 0 \qquad \Rightarrow d_7'(t)s + d_{11}'(t) = 0.$$

Comparing coefficients of s and s^0;

$$s \quad : \qquad d_7'(t) = 0 \qquad \Rightarrow d_7(t) = c_7,$$
$$s^0 \quad : \qquad d_{11}'(t) = 0 \qquad \Rightarrow d_{11}(t) = c_{11},$$

where c_7 and c_{11} are constants of integration. Therefore,

$$\eta^1 = [c_7x + d_8(t)y + d_9(t)z + d_{14}(t)]s + c_{11}x + d_{12}(t)y + d_{13}(t)z + d_{15}(t). \quad (3.0.68)$$

Differentiate eq.(3.0.16) w.r.t. y, we get

$$\eta^2_{,yy} = 0 \qquad \Rightarrow a_{5,yy}(t,x,y,z)s + a_{6,yy}(t,x,y,z) = 0.$$

Comparing coefficients of s and s^0;

$$s \quad : \qquad a_{5,yy}(t,x,y,z) = 0 \qquad \Rightarrow a_5(t,x,y,z) = b_9(t,x,z)y + b_{10}(t,x,z),$$
$$s^0 \quad : \qquad a_{6,yy}(t,x,y,z) = 0 \qquad \Rightarrow a_6(t,x,y,z) = b_{11}(t,x,z)y + b_{12}(t,x,z),$$

$$\eta^2 = [b_9(t,x,z)y + b_{10}(t,x,z)]s + b_{11}(t,x,z)y + b_{12}(t,x,z). \quad (3.0.69)$$

From eq.(3.0.25),

$$\eta^2_{,xx} = 0 \qquad \Rightarrow [b_{9,xx}(t,x,z)y + b_{10,xx}(t,x,z)]s + b_{11,xx}(t,x,z)y + b_{12,xx}(t,x,z) = 0.$$

Comparing coefficients of ys, s, y, constant;

$$ys \quad : \qquad b_{9,xx}(t,x,z) = 0 \qquad \Rightarrow b_9(t,x,z) = d_{16}(t,z)x + d_{17}(t,z),$$
$$s \quad : \qquad b_{10,xx}(t,x,z) = 0 \qquad \Rightarrow b_{10}(t,x,z) = d_{18}(t,z)x + d_{19}(t,z),$$
$$y \quad : \qquad b_{11,xx}(t,x,z) = 0 \qquad \Rightarrow b_{11}(t,x,z) = d_{20}(t,z)x + d_{21}(t,z),$$
$$\text{constant term} \quad : \qquad b_{12,xx}(t,x,z) = 0 \qquad \Rightarrow b_{12}(t,x,z) = d_{22}(t,z)x + d_{23}(t,z),$$

$$\eta^2 = [(d_{16}(t,z)x + d_{17}(t,z))y + d_{18}(t,z)x + d_{19}(t,z)]s + (d_{20}(t,z)x + d_{21}(t,z))y + d_{22}(t,z)x + d_{23}(t,z). \tag{3.0.70}$$

Likewise, differentiate eq.(3.0.16) w.r.t. x, one gets

$$\eta^2_{,xy} = 0 \qquad \Rightarrow d_{16}(t,z)s + d_{20}(t,z) = 0.$$

Comparing coefficients of s and s^0;

$$s \quad : \qquad d_{16}(t,z) = 0,$$

$$s^0 \quad : \qquad d_{20}(t,z) = 0,$$

$$\eta^2 = [d_{17}(t,z)y + d_{18}(t,z)x + d_{19}(t,z)]s + d_{21}(t,z)y + d_{22}(t,z)x + d_{23}(t,z). \tag{3.0.71}$$

Now differentiate eq.(3.0.16) w.r.t. z, we obtain

$$\eta^2_{,yz} = 0 \qquad \Rightarrow d_{17,z}(t,z)s + d_{21,z}(t,z) = 0.$$

Comparing coefficients of s and s^0;

$$s \quad : \qquad d_{17,z}(t,z) = 0 \qquad \Rightarrow d_{17}(t,z) = d_{17}(t),$$

$$s^0 \quad : \qquad d_{21,z}(t,z) = 0 \qquad \Rightarrow d_{21}(t,z) = d_{21}(t),$$

$$\eta^2 = [d_{17}(t)y + d_{18}(t,z)x + d_{19}(t,z)]s + d_{21}(t)y + d_{22}(t,z)x + d_{23}(t,z). \tag{3.0.72}$$

Using eq.(3.0.36), we have

$$\eta^2_{,xz} = 0 \qquad \Rightarrow d_{18,z}(t,z)s + d_{22,z}(t,z) = 0.$$

Comparing coefficients of s and s^0;

$$s \quad : \qquad d_{18,z}(t,z) = 0 \qquad \Rightarrow d_{18}(t,z) = d_{18}(t),$$

$$s^0 \quad : \qquad d_{22,z}(t,z) = 0 \qquad \Rightarrow d_{22}(t,z) = d_{22}(t),$$

$$\eta^2 = [d_{17}(t)y + d_{18}(t)x + d_{19}(t,z)]s + d_{21}(t)y + d_{22}(t)x + d_{23}(t,z). \tag{3.0.73}$$

Taking into account eq.(3.0.35), we have

$$\eta^2_{,zz} = 0 \qquad \Rightarrow d_{19,zz}(t,z)s + d_{23,zz}(t,z) = 0.$$

Comparing coefficients of s and s^0;

$$\begin{aligned} s &: \qquad d_{19,zz}(t,z) = 0 \qquad \Rightarrow d_{19}(t,z) = d_{19}(t)z + d_{24}(t), \\ s^0 &: \qquad d_{23,zz}(t,z) = 0 \qquad \Rightarrow d_{23}(t,z) = d_{23}(t)z + d_{25}(t), \end{aligned}$$

$$\begin{aligned} \eta^2 = {} & [d_{17}(t)y + d_{18}(t)x + d_{19}(t)z + d_{24}(t)]s \\ & + d_{21}(t)y + d_{22}(t)x + d_{23}(t)z + d_{25}(t). \end{aligned} \tag{3.0.74}$$

Also, differentiate eq.(3.0.16) w.r.t. t, we get

$$\eta^2_{,ty} = 0 \qquad \Rightarrow d'_{17}(t)s + d'_{21}(t) = 0.$$

Comparing coefficients of s and s^0;

$$\begin{aligned} s &: \qquad d'_{17}(t) = 0 \qquad \Rightarrow d_{17}(t) = c_{17}, \\ s^0 &: \qquad d'_{21}(t) = 0 \qquad \Rightarrow d_{21}(t) = c_{21}, \end{aligned}$$

where c_{17} and c_{21} are the constants of integration. Thus,

$$\eta^2 = [c_{17}y + d_{18}(t)x + d_{19}(t)z + d_{24}(t)]s + c_{21}y + d_{22}(t)x + d_{23}(t)z + d_{25}(t). \tag{3.0.75}$$

From eq.(3.0.54),

$$\eta^2_{,tz} = 0 \qquad \Rightarrow d'_{19}(t)s + d'_{23}(t) = 0.$$

Comparing coefficients of s and s^0;

$$\begin{aligned} s &: \qquad d'_{19}(t) = 0 \qquad \Rightarrow d_{19}(t) = c_{19}, \\ s^0 &: \qquad d'_{23}(t) = 0 \qquad \Rightarrow d_{23}(t) = c_{23}, \end{aligned}$$

where c_{19} and c_{23} are constants of integration. Hence,

$$\eta^2 = [c_{17}y + d_{18}(t)x + c_{19}z + d_{24}(t)]s + c_{21}y + d_{22}(t)x + c_{23}z + d_{25}(t). \tag{3.0.76}$$

Moreover, differentiate eq.(3.0.17) w.r.t. z, we get

$$\eta^3_{,zz} = 0 \qquad \Rightarrow a_{7,zz}(t,x,y,z)s + a_{8,zz}(t,x,y,z) = 0,$$

Comparing coefficients of s and s^0;

$$\begin{aligned} s \quad &: \qquad a_{7,zz}(t,x,y,z) = 0 \qquad \Rightarrow a_7(t,x,y,z) = b_{13}(t,x,y)z + b_{14}(t,x,y), \\ s^0 \quad &: \qquad a_{8,zz}(t,x,y,z) = 0 \qquad \Rightarrow a_8(t,x,y,z) = b_{15}(t,x,y)z + b_{16}(t,x,y), \end{aligned}$$

$$\eta^3 = [b_{13}(t,x,y)z + b_{14}(t,x,y)]s + b_{15}(t,x,y)z + b_{16}(t,x,y). \tag{3.0.77}$$

Using eq.(3.0.29), we have

$$\eta^3_{,xx} = 0 \qquad \Rightarrow [b_{13,xx}(t,x,y)z + b_{14,xx}(t,x,y)]s + b_{15,xx}(t,x,y)z + b_{16,xx}(t,x,y) = 0.$$

Comparing coefficients of zs, s, z, constant;

$$\begin{aligned} zs \quad &: \quad b_{13,xx}(t,x,y) = 0 \qquad \Rightarrow b_{13}(t,x,y) = d_{26}(t,y)x + d_{27}(t,y), \\ s \quad &: \quad b_{14,xx}(t,x,y) = 0 \qquad \Rightarrow b_{14}(t,x,y) = d_{28}(t,y)x + d_{29}(t,y), \\ z \quad &: \quad b_{15,xx}(t,x,y) = 0 \qquad \Rightarrow b_{15}(t,x,y) = d_{30}(t,y)x + d_{31}(t,y), \\ \text{constant term} \quad &: \quad b_{16,xx}(t,x,y) = 0 \qquad \Rightarrow b_{16}(t,x,y) = d_{32}(t,y)x + d_{33}(t,y), \end{aligned}$$

$$\begin{aligned} \eta^3 = {} & [(d_{26}(t,y)x + d_{27}(t,y))z + d_{28}(t,y)x + d_{29}(t,y)]s \\ & + (d_{30}(t,y)x + d_{31}(t,y))z + d_{32}(t,y)x + d_{33}(t,y). \end{aligned} \tag{3.0.78}$$

Now, differentiating eq.(3.0.17) w.r.t. x, we get

$$\eta^3_{,xz} = 0 \qquad \Rightarrow d_{26}(t,y)s + d_{30}(t,y) = 0.$$

Comparing coefficients of s and s^0;

$$\begin{aligned} s : \qquad & d_{26}(t,y) = 0, \\ s^0 : \qquad & d_{30}(t,y) = 0, \end{aligned}$$

$$\eta^3 = [d_{27}(t,y)z + d_{28}(t,y)x + d_{29}(t,y)]s + d_{31}(t,y)z + d_{32}(t,y)x + d_{33}(t,y). \quad (3.0.79)$$

Furthermore, we differentiate eq.(3.0.17) w.r.t. y, we get

$$\eta^3_{,yz} = 0 \qquad \Rightarrow d_{27,y}(t,y)s + d_{31,y}(t,y) = 0.$$

Comparing coefficients of s and s^0;

$$\begin{aligned} s &: \qquad d_{27,y}(t,y) = 0 \qquad \Rightarrow d_{27}(t,y) = d_{27}(t), \\ s^0 &: \qquad d_{31,y}(t,y) = 0 \qquad \Rightarrow d_{31}(t,y) = d_{31}(t), \end{aligned}$$

$$\eta^3 = [d_{27}(t)z + d_{28}(t,y)x + d_{29}(t,y)]s + d_{31}(t)z + d_{32}(t,y)x + d_{33}(t,y). \quad (3.0.80)$$

Taking into account eq.(3.0.36), we get

$$\eta^3_{,xy} = 0 \qquad \Rightarrow d_{28,y}(t,y)s + d_{32,y}(t,y) = 0.$$

Comparing coefficients of s and s^0;

$$\begin{aligned} s &: \qquad d_{28,y}(t,y) = 0 \qquad \Rightarrow d_{28}(t,y) = d_{28}(t), \\ s^0 &: \qquad d_{32,y}(t,y) = 0 \qquad \Rightarrow d_{32}(t,y) = d_{32}(t), \end{aligned}$$

$$\eta^3 = [d_{27}(t)z + d_{28}(t)x + d_{29}(t,y)]s + d_{31}(t)z + d_{32}(t)x + d_{33}(t,y). \quad (3.0.81)$$

From eq.(3.0.34),

$$\eta^3_{,yy} = 0 \qquad \Rightarrow d_{29,yy}(t,y)s + d_{33,yy}(t,y) = 0.$$

Comparing coefficients of s and s^0;

$$\begin{aligned} s &: \qquad d_{29,yy}(t,y) = 0 \qquad \Rightarrow d_{29}(t,y) = d_{29}(t)y + d_{34}(t), \\ s^0 &: \qquad d_{33,yy}(t,y) = 0 \qquad \Rightarrow d_{33}(t,y) = d_{33}(t)y + d_{35}(t), \end{aligned}$$

$$\begin{aligned} \eta^3 = {} & [d_{27}(t)z + d_{28}(t)x + d_{29}(t)y + d_{34}(t)]s \\ & + d_{31}(t)z + d_{32}(t)x + d_{33}(t)y + d_{35}(t). \end{aligned} \quad (3.0.82)$$

Using eq.(3.0.54), we have

$$\eta^3_{,ty} = 0 \qquad \Rightarrow d'_{29}(t)s + d'_{33}(t) = 0.$$

Comparing coefficients of s and s^0;

$$\begin{aligned} s \quad &: \qquad d'_{29}(t) = 0 \qquad \Rightarrow d_{29}(t) = c_{29}, \\ s^0 \quad &: \qquad d'_{33}(t) = 0 \qquad \Rightarrow d_{33}(t) = c_{33}, \end{aligned}$$

where c_{29} and c_{33} are constants of integration. Therefore,

$$\begin{aligned} \eta^3 = {} & [d_{27}(t)z + d_{28}(t)x + c_{29}y + d_{34}(t)]s \\ & + d_{31}(t)z + d_{32}(t)x + c_{33}y + d_{35}(t). \end{aligned} \tag{3.0.83}$$

Now, we differentiate eq.(3.0.17) w.r.t t, we get

$$\eta^3_{,tz} = 0 \qquad \Rightarrow d'_{27}(t)s + d'_{31}(t) = 0.$$

Comparing coefficients of s and s^0;

$$\begin{aligned} s \quad &: \qquad d'_{27}(t) = 0 \qquad \Rightarrow d_{27}(t) = c_{27}, \\ s^0 \quad &: \qquad d'_{31}(t) = 0 \qquad \Rightarrow d_{31}(t) = c_{31}, \end{aligned}$$

where c_{27} and c_{31} are constants of integration. Thus,

$$\eta^3 = [c_{27}z + d_{28}(t)x + c_{29}y + d_{34}(t)]s + c_{31}z + d_{32}(t)x + c_{33}y + d_{35}(t). \tag{3.0.84}$$

Therefore, eq.(3.0.23) reduces to

$$\left.\begin{aligned}
&\xi = c_0\tfrac{s^2}{2} + c_1 s + c_2,\\
&\eta^0 = [d_1(t,x))y + d_2(t,x)z + d_0(t,x)]s\\
&\qquad + d_3(t,x)y + d_4(t,x)z + d_5(t,x),\\
&\eta^1 = [c_7 x + d_8(t)y + d_9(t)z + d_{14}(t)]s\\
&\qquad + c_{11}x + d_{12}(t)y + d_{13}(t)z + d_{15}(t),\\
&\eta^2 = [c_{17}y + d_{18}(t)x + c_{19}z + d_{24}(t)]s\\
&\qquad + c_{21}y + d_{22}(t)x + c_{23}z + d_{25}(t),\\
&\eta^3 = [c_{27}z + d_{28}(t)x + c_{29}y + d_{34}(t)]s\\
&\qquad + c_{31}z + d_{32}(t)x + c_{33}y + d_{35}(t),\\
&A = A(t,x,y,z).
\end{aligned}\right\} \tag{3.0.85}$$

Taking into account eqs.(3.0.15)–(3.0.17), we get $c_7 = c_{17} = c_{27} = \frac{1}{2}c_0$ and $c_{11} = c_{21} = c_{31} = \frac{1}{2}c_1$.

From eq.(3.0.18), we have $d_{18}(t) = -d_8(t)$ and $d_{22}(t) = -d_{12}(t)$ and from eq.(3.0.19), we have $d_{28}(t) = -d_9(t)$ and $d_{32}(t) = -d_{13}(t)$. Also by eq.(3.0.20), we have $c_{29} = -c_{19}$ and $c_{33} = -c_{23}$. Therefore, eq.(3.0.85) becomes

$$\left.\begin{aligned}
&\xi = c_0\tfrac{s^2}{2} + c_1 s + c_2,\\
&\eta^0 = [d_1(t,x))y + d_2(t,x)z + d_0(t,x)]s\\
&\qquad + d_3(t,x)y + d_4(t,x)z + d_5(t,x),\\
&\eta^1 = [c_0\tfrac{x}{2} + d_8(t)y + d_9(t)z + d_{14}(t)]s\\
&\qquad + c_1\tfrac{x}{2} + d_{12}(t)y + d_{13}(t)z + d_{15}(t),\\
&\eta^2 = [c_0\tfrac{y}{2} - d_8(t)x + c_{19}z + d_{24}(t)]s\\
&\qquad + c_1\tfrac{y}{2} - d_{12}(t)x + c_{23}z + d_{25}(t),\\
&\eta^3 = [c_0\tfrac{z}{2} - d_9(t)x - c_{19}y + d_{34}(t)]s\\
&\qquad + c_1\tfrac{z}{2} - d_{13}(t)x - c_{23}y + d_{35}(t),\\
&A = A(t,x,y,z).
\end{aligned}\right\} \tag{3.0.86}$$

In the Lagrangian given by eq.(3.0.1), $\nu(x)$ is an arbitrary function. For this arbitrary function $\nu(x)$, there are two possibilities:

(A). $\nu(x) = 0$,

(B). $\nu(x) \neq 0$.

3.1 Case of $\nu(x) = 0$

If $\nu(x) = 0$, then eq.(3.0.1) reduces to

$$L = \dot{t}^2 - \dot{x}^2 - (\dot{y}^2 + \dot{z}^2). \tag{3.1.1}$$

Now solving all the partial derivatives of the form $\eta^a_{,bc} = 0$, where $(a = 0, 1, 2, 3)$ and b, c represents t, x, y, and z.

From eq.(3.0.55), we have

$$\eta^0_{,xy} = 0 \qquad \Rightarrow d_{1,x}(t,x)s + d_{3,x}(t,x) = 0.$$

Comparing coefficients of s and s^0;

$$\begin{aligned} s \quad &: \qquad d_{1,x}(t,x) = 0 \qquad \Rightarrow d_1(t,x) = d_1(t), \\ s^0 \quad &: \qquad d_{3,x}(t,x) = 0 \qquad \Rightarrow d_3(t,x) = d_3(t). \end{aligned}$$

Using eq.(3.0.56), we have

$$\eta^0_{,xz} = 0 \qquad \Rightarrow d_{2,x}(t,x)s + d_{4,x}(t,x) = 0.$$

Comparing coefficients of s and s^0;

$$\begin{aligned} s \quad &: \qquad d_{2,x}(t,x) = 0 \qquad \Rightarrow d_2(t,x) = d_2(t), \\ s^0 \quad &: \qquad d_{4,x}(t,x) = 0 \qquad \Rightarrow d_4(t,x) = d_4(t). \end{aligned}$$

Therefore, eq.(3.0.86) reduces to

$$\eta^0 = [d_1(t)y + d_2(t)z + d_0(t,x)]s + d_3(t)y + d_4(t)z + d_5(t,x). \tag{3.1.2}$$

Taking into account eq.(3.0.42), one have

$$\eta^0_{,xx} = 0 \qquad \Rightarrow d_{0,xx}(t,x)s + d_{5,xx}(t,x) = 0.$$

Comparing coefficients of s and s^0;

$$\begin{aligned} s \quad &: \qquad d_{0,xx}(t,x) = 0 \qquad \Rightarrow d_0(t,x) = d_0(t)x + d_{36}(t), \\ s^0 \quad &: \qquad d_{5,xx}(t,x) = 0 \qquad \Rightarrow d_5(t,x) = d_5(t)x + d_{37}(t). \end{aligned}$$

Therefore, eq.(3.1.2) becomes

$$\begin{aligned} \eta^0 = {} & [d_1(t)y + d_2(t)z + d_0(t)x + d_{36}(t)]s + d_3(t)y \\ & + d_4(t)z + d_5(t)x + d_{37}(t). \end{aligned} \tag{3.1.3}$$

From eq.(3.0.39), one obtain

$$\eta^0_{,ty} = 0 \qquad \Rightarrow d_1'(t)s + d_3'(t) = 0.$$

Comparing coefficients of s and s^0;

$$\begin{aligned} s \quad &: \qquad d_1'(t) = 0 \qquad \Rightarrow d_1(t) = c_{38}, \\ s^0 \quad &: \qquad d_3'(t) = 0 \qquad \Rightarrow d_3(t) = c_3. \end{aligned}$$

where c_{38} and c_3 are constants of integration. Now let us consider eq.(3.0.40),

$$\eta^0_{,tz} = 0 \qquad \Rightarrow d_2'(t)s + d_4'(t) = 0.$$

Comparing coefficients of s and s^0;

$$\begin{aligned} s \quad &: \qquad d_2'(t) = 0 \qquad \Rightarrow d_2(t) = c_{39}, \\ s^0 \quad &: \qquad d_4'(t) = 0 \qquad \Rightarrow d_4(t) = c_4. \end{aligned}$$

where c_{39} and c_4 are constants of integration. From eq.(3.0.38), we have

$$\eta^0_{,tx} = 0 \qquad \Rightarrow d_0'(t)s + d_5'(t) = 0.$$

Comparing coefficients of s and s^0;

$$s \;:\quad d_0'(t) = 0 \quad \Rightarrow d_0(t) = c_{40},$$
$$s^0 \;:\quad d_5'(t) = 0 \quad \Rightarrow d_5(t) = c_5.$$

where c_5 and c_{40} are constants of integration. Therefore, eq.(3.1.3) reduces to

$$\eta^0 = [c_{38}y + c_{39}z + c_{40}x + d_{36}(t)]s + c_3y + c_4z + c_5x + d_{37}(t). \tag{3.1.4}$$

Taking into account eq.(3.0.37), we obtain

$$\eta^0_{,tt} = 0 \quad \Rightarrow d_{36}''(t)s + d_{37}''(t) = 0.$$

Comparing coefficients of s and s^0;

$$s \;:\quad d_{36}''(t) = 0 \quad \Rightarrow d_{36}(t) = c_{36}t + c_{41},$$
$$s^0 \;:\quad d_{37}''(t) = 0 \quad \Rightarrow d_{37}(t) = c_{37}t + c_{42}.$$

where c_{36}, c_{37}, c_{41}, and c_{42} are constants of integration. Hence, eq.(3.1.4) becomes

$$\eta^0 = [c_{38}y + c_{39}z + c_{40}x + c_{36}t + c_{41}]s + c_3y + c_4z + c_5x + c_{37}t + c_{42}. \tag{3.1.5}$$

From eq.(3.0.24),

$$\eta^1_{,ty} = 0 \quad \Rightarrow d_8'(t)s + d_{12}'(t) = 0.$$

Comparing coefficients of s and s^0;

$$s \;:\quad d_8'(t) = 0 \quad \Rightarrow d_8(t) = c_8,$$
$$s^0 \;:\quad d_{12}'(t) = 0 \quad \Rightarrow d_{12}(t) = c_{12}.$$

where c_8 and c_{12} are constants of integration. Using eq.(3.0.28),

$$\eta^1_{,tz} = 0 \quad \Rightarrow d_9'(t)s + d_{13}'(t) = 0.$$

Comparing coefficients of s and s^0;

$$s \;:\qquad d_9'(t) = 0 \qquad \Rightarrow d_9(t) = c_9,$$
$$s^0 \;:\qquad d_{13}'(t) = 0 \qquad \Rightarrow d_{13}(t) = c_{13}.$$

where c_9 and c_{13} are constants of integration. Taking into account eq.(3.0.41), we have

$$\eta^1_{,tt} = 0 \qquad \Rightarrow d_{14}''(t)s + d_{15}''(t) = 0.$$

Comparing coefficients of s and s^0;

$$s \;:\qquad d_{14}''(t) = 0 \qquad \Rightarrow d_{14}(t) = c_{14}t + c_{43},$$
$$s^0 \;:\qquad d_{15}''(t) = 0 \qquad \Rightarrow d_{15}(t) = c_{15}t + c_{44},$$

where c_{14}, c_{15}, c_{43}, and c_{44} are constants of integration. Therefore,

$$\eta^1 = [\frac{c_0}{2}x + c_8 y + c_9 z + c_{14}t + c_{43}]s + \frac{c_1}{2}x + c_{12}y + c_{13}z + c_{15}t + c_{44}. \tag{3.1.6}$$

Thus, eq.(3.0.86) becomes

$$\left.\begin{array}{l}
\xi = \frac{c_0}{2}s^2 + c_1 s + c_2, \qquad A = A(t,x,y,z),\\
\eta^0 = [c_{38}y + c_{39}z + c_{40}x + c_{36}t + c_{41}]s + c_3 y + c_4 z + c_5 x + c_{37}t + c_{42},\\
\eta^1 = [\frac{c_0}{2}x + c_8 y + c_9 z + c_{14}t + c_{43}]s + \frac{c_1}{2}x + c_{12}y + c_{13}z + c_{15}t + c_{44},\\
\eta^2 = [\frac{c_0}{2}y - c_8 x + c_{19}z + d_{24}(t)]s + [\frac{c_1}{2}y - c_{12}x + c_{23}z + d_{25}(t)],\\
\eta^3 = [\frac{c_0}{2}z - c_9 x - c_{19}y + d_{34}(t)]s + [\frac{c_1}{2}z - c_{13}x - c_{23}y + c_{35}(t)].
\end{array}\right\} \tag{3.1.7}$$

From eq.(3.0.45),

$$\eta^2_{,tt} = 0 \qquad \Rightarrow d_{24}''(t)s + d_{25}''(t) = 0.$$

Comparing coefficients of s and s^0;

$$s \;:\qquad d_{24}''(t) = 0 \qquad \Rightarrow d_{24}(t) = c_{24}t + c_{45},$$
$$s^0 \;:\qquad d_{25}''(t) = 0 \qquad \Rightarrow d_{25}(t) = c_{25}t + c_{46},$$

where c_{24}, c_{25}, c_{45}, and c_{46} are constants of integration. Hence,

$$\eta^2 = [\frac{c_0}{2}y - c_8x + +c_{19}z + c_{24}t + c_{45}]s + [\frac{c_1}{2}y - c_{12}x + c_{23}z + c_{25}t + c_{46}]. \quad (3.1.8)$$

Using eq.(3.0.49),

$$\eta^3_{,tt} = 0 \qquad \Rightarrow d''_{34}(t)s + d''_{35}(t) = 0.$$

Comparing coefficients of s and s^0;

$$\begin{aligned} s \quad &: \qquad d''_{34}(t) = 0 \qquad \Rightarrow d_{34}(t) = c_{34}t + c_{47}, \\ s^0 \quad &: \qquad d''_{35}(t) = 0 \qquad \Rightarrow d_{35}(t) = c_{35}t + c_{48}, \end{aligned}$$

where c_{34}, c_{35}, c_{47}, and c_{48} are constants of integration. Therefore,

$$\begin{aligned} \eta^3 = [&\tfrac{c_0}{2}z - c_9x - c_{19}y + c_{34}t + c_{47}]s \\ &+ [\tfrac{c_1}{2}z - c_{13}x - c_{23}y + c_{35}t + c_{48}]. \end{aligned} \quad (3.1.9)$$

Hence, eq.(3.1.7) reduces to

$$\left.\begin{aligned} &\xi = \tfrac{c_0}{2}s^2 + c_1s + c_2, \qquad A = A(t,x,y,z), \\ &\eta^0 = [c_{38}y + c_{39}z + c_{40}x + c_{36}t + c_{41}]s + c_3y + c_4z + c_5x + c_{37}t + c_{42}, \\ &\eta^1 = [\tfrac{c_0}{2}x + c_8y + c_9z + c_{14}t + c_{43}]s + \tfrac{c_1}{2}x + c_{12}y + c_{13}z + c_{15}t + c_{44}, \\ &\eta^2 = [\tfrac{c_0}{2}y - c_8x + c_{19}z + c_{24}t + c_{45}]s + [\tfrac{c_1}{2}y - c_{12}x + c_{23}z + c_{25}t + c_{46}], \\ &\eta^3 = [\tfrac{c_0}{2}z - c_9x - c_{19}y + c_{34}t + c_{47}]s + [\tfrac{c_1}{2}z - c_{13}x - c_{23}y + c_{35}t + c_{48}]. \end{aligned}\right\} \quad (3.1.10)$$

From eqs.(3.0.12)–(3.0.14) and eq.(3.0.21) we have the following implications:

$c_{40}s + c_5 = c_{14}s + c_{15} \qquad \Rightarrow c_{40} = c_{14},\ c_{15} = c_5.$

$c_{38}s + c_3y = c_{24}s + c_{25} \qquad \Rightarrow c_{38} = c_{24},\ c_{25} = c_3.$

$c_{39}s + c_4 = c_{34}s + c_{35} \qquad \Rightarrow c_{39} = c_{34},\ c_{35} = c_4.$

$2(c_{36}s + c_{37}) = c_0s + c_1 \qquad \Rightarrow c_{36} = \frac{c_0}{2},\ c_{37} = \frac{c_1}{2}.$

Therefore, eq.(3.1.10) becomes

$$\left.\begin{array}{l}
\xi = \frac{c_0}{2}s^2 + c_1 s + c_2, \qquad A = A(t,x,y,z), \\
\eta^0 = [c_{24}y + c_{34}z + c_{14}x + \frac{c_0}{2}t + c_{41}]s + c_3 y + c_4 z + c_5 x + \frac{c_1}{2}t + c_{42}, \\
\eta^1 = [\frac{c_0}{2}x + c_8 y + c_9 z + c_{14}t + c_{43}]s + \frac{c_1}{2}x + c_{12}y + c_{13}z + c_5 t + c_{44}, \\
\eta^2 = [\frac{c_0}{2}y - c_8 x + c_{19}z + c_{24}t + c_{45}]s + [\frac{c_1}{2}y - c_{12}x + c_{23}z + c_3 t + c_{46}], \\
\eta^3 = [\frac{c_0}{2}z - c_9 x - c_{19}y + c_{34}t + c_{47}]s + [\frac{c_1}{2}z - c_{13}x - c_{23}y + c_4 t + c_{48}].
\end{array}\right\} \qquad (3.1.11)$$

Now, eq.(3.0.8) implies that

$$A(t,x,y,z) = c_0\frac{t^2}{2} + 2c_{14}xt + c_{24}yt + 2c_{34}tz + 2c_{41}t + h_1(x,y,z). \qquad (3.1.12)$$

Using eq.(3.0.9), we have

$$-2(c_0\frac{x}{2} + c_8 y + c_9 z + c_{14}t + c_{43}) = 2c_{14}t + h_{1,x}(x,y,z).$$

By comparing coefficient of t : $c_{14} = 0$,

and $\qquad h_1(x,y,z) = -c_0\frac{x^2}{2} - 2c_8 xy - 2c_9 xz - 2c_{43}x + h_2(y,z).$

Therefore, eq.(3.1.12) reduces to

$$\begin{aligned} A(t,x,y,z) = {} & \tfrac{c_0}{2}(t^2 - x^2) + 2c_{24}ty + 2c_{34}tz + 2c_{41}t \\ & - 2c_8 xy - 2c_9 xz - 2c_{43}x + h_2(y,z) \end{aligned} \qquad (3.1.13)$$

Taking into account eq.(3.0.10), one obtains

$$-2[c_0\frac{y}{2} - c_8 x + c_{19}z + c_{24}t + c_{45}] = 2c_{24}t - 2c_8 x + h_{2,y}(y,z).$$

By comparing coefficients of x and t, we have $c_8 = 0$, $c_{24} = 0$,

and $\qquad h_2(y,z) = -\frac{c_0}{2}y^2 - 2c_{19}yz - 2c_{45}y + h_3(z).$

Therefore, eq.(3.1.13) becomes

$$\begin{aligned} A(t,x,y,z) = {} & \tfrac{c_0}{2}(t^2 - x^2 - y^2) + 2c_{34}tz + 2c_{41}t \\ & - 2c_9 xz - 2c_{43}x - 2c_{19}yz - 2c_{45}y + h_3(z) \end{aligned} \qquad (3.1.14)$$

From eq.(3.0.11),

$$-2[\frac{c_0}{2}z - c_9x - c_{19}y + c_{34}t + c_{47}] = 2c_{34}t - 2c_9x - 2c_{19}y + h_3'(z).$$

By comparing coefficient of t, x, and y: $c_9 = 0 = c_{19} = c_{34}$,

and $\qquad h_3(z) = -\frac{c_0}{2}z^2 - 2c_{47}z + c.$

Therefore, eq.(3.1.14) gives

$$A(t,x,y,z) = \frac{c_0}{2}(t^2 - x^2 - y^2 - z^2) + 2c_{41}t - 2c_{43}x - 2c_{45}y - 2c_{47}z + c. \quad (3.1.15)$$

Therefore, eq.(3.1.11) reduce to

$$\left.\begin{aligned}
&\xi = \tfrac{c_0}{2}s^2 + c_1 s + c_2,\\
&\eta^0 = [\tfrac{c_0}{2}t + c_{41}]s + c_3y + c_4z + c_5x + \tfrac{c_1}{2}t + c_{42},\\
&\eta^1 = [\tfrac{c_0}{2}x + c_{43}]s + \tfrac{c_1}{2}x + c_{12}y + c_{13}z + c_5t + c_{44},\\
&\eta^2 = [\tfrac{c_0}{2}y + c_{45}]s + \tfrac{c_1}{2}y - c_{12}x + c_{23}z + c_3t + c_{46},\\
&\eta^3 = [\tfrac{c_0}{2}z + c_{47}]s + \tfrac{c_1}{2}z - c_{13}x - c_{23}y + c_4t + c_{48},\\
&A = \tfrac{c_0}{2}(t^2 - x^2 - y^2 - z^2) + 2c_{41}t - 2c_{43}x\\
&\qquad - 2c_{45}y - 2c_{47}z + c.
\end{aligned}\right\} \quad (3.1.16)$$

Substituting eq.(3.1.16) in generator $\mathbf{X}$, such that

$$\mathbf{X} = \xi\frac{\partial}{\partial s} + \eta^a\frac{\partial}{\partial q^a}, \quad (3.1.17)$$

where $a = 0, 1, 2, 3$ and q represents t, x, y, z. The arbitrary constants in $\mathbf{X}$ determine the number of generators of the Lie algebra. This implies

$$\begin{aligned}
\mathbf{X} = &\left(\tfrac{c_0}{2}s^2 + c_1s + c_2\right)\tfrac{\partial}{\partial s} + \left([\tfrac{c_0}{2}t + c_{41}]s + c_3y + c_4z + c_5x + \tfrac{c_1}{2}t\right.\\
&\left.+ c_{42}\right)\tfrac{\partial}{\partial t} + \left([\tfrac{c_0}{2}x + c_{43}]s + \tfrac{c_1}{2}x + c_{12}y + c_{13}z + c_5t + c_{44}\right)\tfrac{\partial}{\partial x}\\
&+ \left([\tfrac{c_0}{2}y + c_{45}]s + \tfrac{c_1}{2}y - c_{12}x + c_{23}z + c_3t + c_{46}\right)\tfrac{\partial}{\partial y} + \left([\tfrac{c_0}{2}z\right.\\
&\left.+ c_{47}]s + \tfrac{c_1}{2}z - c_{13}x - c_{23}y + c_4t + c_{48}\right)\tfrac{\partial}{\partial z}.
\end{aligned} \quad (3.1.18)$$

There are seventeen arbitrary constants c_i's appearing in eq.(3.2.18). Hence, the metric given by eq.(2.2.48) admits 17 Noether symmetries. The metric is, therefore, known as the Minkowski space. The generators (or symmetries) are as follows:

$$
\begin{aligned}
\mathbf{X_0} &= \frac{\partial}{\partial t}, \\
\mathbf{X_1} &= z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, \\
\mathbf{X_2} &= \frac{\partial}{\partial z} \\
\mathbf{X_3} &= \frac{\partial}{\partial y}, \\
\mathbf{X_4} &= x\frac{\partial}{\partial t} + t\frac{\partial}{\partial x}, \\
\mathbf{X_5} &= \frac{\partial}{\partial x}, \\
\mathbf{X_6} &= z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}, \\
\mathbf{X_7} &= z\frac{\partial}{\partial t} + t\frac{\partial}{\partial z}, \\
\mathbf{X_8} &= y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, \\
\mathbf{X_9} &= y\frac{\partial}{\partial t} + t\frac{\partial}{\partial y}, \\
\mathbf{Y_0} &= \frac{\partial}{\partial s}, \\
\mathbf{Y_1} &= s\frac{\partial}{\partial y}; \qquad A_1 = -2y, \\
\mathbf{Y_2} &= s\frac{\partial}{\partial z}; \qquad A_2 = -2z, \\
\mathbf{Y_3} &= \frac{1}{2}\big[s^2\frac{\partial}{\partial s} + st\frac{\partial}{\partial t} + sx\frac{\partial}{\partial x} + sy\frac{\partial}{\partial y} + sz\frac{\partial}{\partial z}\big]; \qquad A_3 = \frac{1}{2}(t^2 - x^2 - y^2 - z^2), \\
\mathbf{Y_4} &= s\frac{\partial}{\partial t}; \qquad A_4 = 2t, \\
\mathbf{Y_5} &= s\frac{\partial}{\partial x}; \qquad A_5 = -2x, \\
\mathbf{Y_6} &= s\frac{\partial}{\partial s} + \frac{1}{2}\big(t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\big).
\end{aligned}
$$

The structure constants of the corresponding Lie algebra are

$$C^5_{04} = C^2_{07} = C^3_{09} = C^2_{13} = C^6_{18} = C^7_{19} = C^0_{27} = C^4_{89} = C^0_{39} = C^{\hat{4}}_{\hat{0}\hat{3}} = 1,$$
$$C^7_{46} = C^9_{48} = C^2_{56} = C^3_{58} = C^4_{67} = C^1_{79} = C^0_{27} = C^0_{39} = C^3_{\hat{0}\hat{1}} = C^2_{\hat{0}\hat{2}} = 1,$$
$$C^{\hat{6}}_{8\hat{1}} = C^{\hat{5}}_{9\hat{1}} = C^{\hat{1}}_{\hat{1}\hat{2}} = C^{\hat{6}}_{6\hat{2}} = C^{\hat{5}}_{7\hat{2}} = C^{\hat{3}}_{\hat{3}\hat{4}} = C^{\hat{6}}_{4\hat{5}} = C^{\hat{2}}_{7\hat{5}} = C^{\hat{1}}_{9\hat{5}} = C^{\hat{5}}_{4\hat{6}} = -1,$$
$$C^3_{12} = C^8_{16} = C^9_{17} = C^5_{26} = C^5_{38} = C^0_{45} = C^6_{47} = C^8_{49} = C^1_{68} = -1,$$
$$C^{\hat{0}}_{\hat{0}\hat{4}} = C^0_{\hat{0}\hat{5}} = C^5_{\hat{0}\hat{6}} = C^{\hat{2}}_{1\hat{1}} = C^{\hat{2}}_{6\hat{6}} = C^{\hat{1}}_{8\hat{6}} = 1, \quad C^{\hat{1}}_{\hat{1}\hat{4}} = C^{\hat{2}}_{2\hat{3}} = -\frac{1}{2},$$
$$C^{\hat{5}}_{0\hat{3}} = C^{\hat{6}}_{5\hat{3}} = C^0_{0\hat{4}} = C^2_{2\hat{4}} = C^3_{3\hat{4}} = C^5_{5\hat{4}} = C^0_{\hat{4}\hat{5}} = C^5_{\hat{4}\hat{6}} = \frac{1}{2}.$$

Here, the superscript (or subscript) $\hat{i}$ is the representation of the symmetries $\mathbf{Y_i}$.

3.2 Case of $\nu(x) \neq 0$

Substitute eq.(3.0.55) into eq.(3.0.57) and eq.(3.0.56) into eq.(3.0.58), we get

$$\left.\begin{aligned} [\nu''(x) + \nu'^2(x)]\eta^0_{,y} &= 0, \\ [\nu''(x) + \nu'^2(x)]\eta^0_{,z} &= 0. \end{aligned}\right\} \tag{3.2.1}$$

In eq.(3.2.1), we have two possibilities:

$$\nu''(x) + \nu'^2(x) = 0,$$

or

$$\nu''(x) + \nu'^2(x) \neq 0 \Rightarrow \eta^0_{,y} = \eta^0_{,z} = 0.$$

3.2.1 Case of $[\nu''(x) + \nu'^2(x)] = 0$

This implies $\nu(x) = \ln\left(\frac{x}{a}\right)$ (a=constant of integration).

Therefore, eq.(3.0.1) becomes

$$L = \left(\frac{x}{a}\right)^2 \dot{t}^2 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2. \tag{3.2.2}$$

From eq.(3.0.14),

$$\left(\frac{x}{a}\right)^2 \left[d_2(t,x)s + d_4(t,x)\right] = [-d'_9(t)x + d'_{34}(t)]s + [-d'_{13}(t)x + d'_{35}(t)].$$

Comparing coefficients of s and s^0;

$$\begin{aligned} s \quad &: \qquad d_2(t,x) = \frac{-a^2}{x} d_9'(t) + \Big(\frac{a}{x}\Big)^2 d_{34}'(t), \\ s^0 \quad &: \qquad d_4(t,x) = \frac{-a^2}{x} d_{13}'(t) + \Big(\frac{a}{x}\Big)^2 d_{35}'(t). \end{aligned}$$

Using eq.(3.0.13), we have

$$\Big(\frac{x}{a}\Big)^2 [d_1(t,x)s + d_3(t,x)] = [-d_8'(t)x + d_{34}'(t)x]s + [-d_{12}'(t)x + d_{25}'(t)x]$$

Comparing coefficients of s and s^0;

$$\begin{aligned} s \quad &: \qquad d_1(t,x) = \frac{-a^2}{x} d_8'(t) + \Big(\frac{a}{x}\Big)^2 d_{24}'(t), \\ s^0 \quad &: \qquad d_3(t,x) = \frac{-a^2}{x} d_{12}'(t) + \Big(\frac{a}{x}\Big)^2 d_{25}'(t), \end{aligned}$$

$$\begin{aligned} \eta^0 = &\big[\{\tfrac{-a^2}{x} d_8'(t) + \big(\tfrac{a}{x}\big)^2 d_{24}'(t)\}y + \{\tfrac{-a^2}{x} d_9'(t) + \big(\tfrac{a}{x}\big)^2 d_{34}'(t)\}z + d_0(t,x)\big]s \\ &+ \big[\{\tfrac{-a^2}{x} d_{12}'(t) + \big(\tfrac{a}{x}\big)^2 d_{25}'(t)\}y + \{\tfrac{-a^2}{x} d_{13}'(t) + \big(\tfrac{a}{x}\big)^2 d_{35}'(t)\}z + d_5(t,x)\big]. \end{aligned} \tag{3.2.3}$$

Taking into account eq.(3.0.39),

$$\frac{1}{x}[d_8(t)s + d_{12}(t)] = -\big[\{\frac{-a^2}{x} d_8''(t) + \Big(\frac{a}{x}\Big)^2 d_{24}''(t)\}s + \{\frac{-a^2}{x} d_{12}''(t) + \Big(\frac{a}{x}\Big)^2 d_{25}''(t)\}\big]$$

Comparing coefficients of s and s^0;

$$\begin{aligned} s \quad &: \qquad \frac{1}{x} d_8(t) = \frac{a^2}{x} d_8''(t) - \Big(\frac{a}{x}\Big)^2 d_{24}''(t), \\ s^0 \quad &: \qquad \frac{1}{x} d_{12}(t) = \frac{a^2}{x} d_{12}''(t) - \Big(\frac{a}{x}\Big)^2 d_{25}''(t). \end{aligned}$$

Comparing coefficients of $\frac{1}{x}$ and $\frac{1}{x^2}$;

$$\begin{aligned} \frac{1}{x} \quad &: \qquad d_8''(t) - \frac{1}{a^2} d_8(t) = 0 \qquad \Rightarrow d_8(t) = k_1 \cosh\Big(\frac{t}{a}\Big) + k_2 \sinh\Big(\frac{t}{a}\Big), \\ \frac{1}{x^2} \quad &: \qquad d_{24}''(t) = 0 \qquad \Rightarrow d_{24}(t) = k_3 t + k_4. \end{aligned}$$

Comparing coefficients of $\frac{1}{x}$ and $\frac{1}{x^2}$;

$$\frac{1}{x} \quad : \qquad d''_{12}(t) - \frac{1}{a^2} d_{12}(t) = 0 \qquad \Rightarrow d_{12}(t) = k_5 \cosh\big(\frac{t}{a}\big) + k_6 \sinh\big(\frac{t}{a}\big),$$
$$\frac{1}{x^2} \quad : \qquad d''_{25}(t) = 0 \qquad \Rightarrow d_{25}(t) = k_7 t + k_8.$$

where k_i $(i = 1, ..., 8)$ are constants of integration. Now from eq.(3.0.40), we have

$$\frac{1}{x}[d_9(t)s + d_{13}(t)] = -\big[\{\frac{-a^2}{x} d''_9(t) + \big(\frac{a}{x}\big)^2 d''_{34}(t)\}s + \{\frac{-a^2}{x} d''_{13}(t) + \big(\frac{a}{x}\big)^2 d''_{35}(t)\}\big].$$

Comparing coefficients of s and s^0;

s : $\quad \frac{1}{x} d_9(t) = \frac{a^2}{x} d''_9(t) - \left(\frac{a}{x}\right)^2 d''_{34}(t).$

Comparing coefficients of $\frac{1}{x}$ and $\frac{1}{x^2}$;

$$\frac{1}{x} \quad : \qquad d''_{19}(t) - \frac{1}{a^2} d_9(t) = 0 \qquad \Rightarrow d_9(t) = k_9 \cosh\left(\frac{t}{a}\right) + k_{10} \sinh\left(\frac{t}{a}\right),$$
$$\frac{1}{x^2} \quad : \qquad d''_{34}(t) = 0 \qquad \Rightarrow d_{34}(t) = k_{11} t + k_{12}.$$

s^0 : $\quad \frac{1}{x} d_{13}(t) = \frac{a^2}{x} d''_{13}(t) - \frac{1}{a^2} d_{13}(t) = 0.$

Comparing coefficient of $\frac{1}{x}$ and $\frac{1}{x^2}$;

$$\frac{1}{x} \quad : \qquad d''_{13}(t) - \frac{1}{a^2} d_{13}(t) = 0 \Rightarrow d_{13}(t) = k_{13} \cosh\big(\frac{t}{a}\big) + k_{14} \sinh\big(\frac{t}{a}\big),$$
$$\frac{1}{x^2} \quad : \qquad d''_{35}(t) = 0 \Rightarrow d_{35}(t) = k_{15} t + k_{16}.$$

where k_9 $(i = 9, ..., 16)$ are constants of integration. Therefore, eq.(3.0.86) reduces to

$$\left.\begin{aligned}
&\xi = \tfrac{c_0}{2}s^2 + c_1 s + c_2, \qquad A = A(t,x,y,z),\\
&\eta^0 = [\{\tfrac{-a}{x}\big(k_1\sinh(\tfrac{t}{a}) + k_2\cosh(\tfrac{t}{a})\big) + \big(\tfrac{a}{x^2}\big)k_3\}y +\\
&\quad \{\tfrac{-a}{x}\big(k_9\sinh(\tfrac{t}{a}) + k_{10}\cosh(\tfrac{t}{a})\big) + \big(\tfrac{a}{x^2}\big)k_{11}\}z + d_0(t,x)]s +\\
&\quad \{\tfrac{-a}{x}\big(k_5\sinh(\tfrac{t}{a}) + k_6\cosh(\tfrac{t}{a})\big) + \big(\tfrac{a}{x^2}\big)k_7\}y\\
&\quad + \{\tfrac{-a}{x}\big(k_{13}\sinh(\tfrac{t}{a}) + k_{14}\cosh(\tfrac{t}{a})\big) + \big(\tfrac{a}{x^2}\big)k_{15}\}z + d_5(t,x),\\
&\eta^1 = [\tfrac{c_0}{2}x + \big(k_1\cosh(\tfrac{t}{a}) + k_2\sinh(\tfrac{t}{a})\big)y + \big(k_9\cosh(\tfrac{t}{a}) + k_{10}\sinh(\tfrac{t}{a})\big)z\\
&\quad + d_{14}(t)]s + \tfrac{c_1}{2}x + \big(k_5\cosh(\tfrac{t}{a}) + k_6\sinh(\tfrac{t}{a})\big)y\\
&\quad + \big(k_{13}\cosh(\tfrac{t}{a}) + k_{14}\sinh(\tfrac{t}{a})\big)z + d_{15}(t),\\
&\eta^2 = [\tfrac{c_0}{2}y - \big(k_1\cosh(\tfrac{t}{a}) + k_2\sinh(\tfrac{t}{a})\big)x + c_{19}z + k_3 t + k_4]s +\\
&\quad + \tfrac{c_1}{2}y - \big(k_5\cosh(\tfrac{t}{a}) + k_6\sinh(\tfrac{t}{a})\big)x + c_{23}z + k_7 t + k_8,\\
&\eta^3 = [\tfrac{c_0}{2}z - \big(k_9\cosh(\tfrac{t}{a}) + k_{10}\sinh(\tfrac{t}{a})\big)x - c_{19}y + k_{11}t + k_{12}]s +\\
&\quad + \tfrac{c_1}{2}z - \big(k_{13}\cosh(\tfrac{t}{a}) + k_{14}\sinh(\tfrac{t}{a})\big)x - c_{23}y + k_{15}t + k_{16}.
\end{aligned}\right\} \quad (3.2.4)$$

Using eq.(3.0.43), one obtain

$$\big(\tfrac{x}{a}\big)^2\big[\{\tfrac{a}{x^2}\big(k_1\sinh(\tfrac{t}{a}) + k_2\cosh(\tfrac{t}{a})\big) - 2\tfrac{a}{x^3}k_3\}s + \{\tfrac{a}{x^2}\big(k_5\sinh(\tfrac{t}{a}) + k_6\cosh(\tfrac{t}{a})\big) - 2\tfrac{a}{x^3}k_7\}\big] = \tfrac{1}{a}[\big(k_1\sinh(\tfrac{t}{a}) + k_2\cosh(\tfrac{t}{a})\big)s + \big(k_5\sinh(\tfrac{t}{a}) + k_6\cosh(\tfrac{t}{a})\big)].$$

Comparing coefficients of s and s^0;

$$s \;:\quad k_3 = 0,$$
$$s^0 \;:\quad k_7 = 0.$$

Taking into account eq.(3.0.44), we get

$$\big(\tfrac{x}{a}\big)^2\big[\{\tfrac{a}{x^2}\big(k_9\sinh(\tfrac{t}{a}) + k_{10}\cosh(\tfrac{t}{a})\big) - 2\tfrac{a}{x^3}k_{11}\}s + \{\tfrac{a}{x^2}\big(k_{13}\sinh(\tfrac{t}{a}) + k_{14}\cosh(\tfrac{t}{a})\big) - 2\tfrac{a}{x^3}k_{15}\}\big] = \tfrac{1}{a}[\big(k_9\sinh(\tfrac{t}{a}) + k_{10}\cosh(\tfrac{t}{a})\big)s + \big(k_{13}\sinh(\tfrac{t}{a}) + k_{14}\cosh(\tfrac{t}{a})\big)].$$

Comparing coefficients of s and s^0;

$$s \;:\quad k_{11} = 0,$$
$$s^0 \;:\quad k_{15} = 0.$$

Hence, eq.(3.2.4) becomes

$$\left.\begin{aligned}
&\xi = \tfrac{c_0}{2}s^2 + c_1 s + c_2, \qquad A = A(t,x,y,z),\\
&\eta^0 = [\tfrac{-a}{x}\big(k_1\sinh(\tfrac{t}{a}) + k_2\cosh(\tfrac{t}{a})\big)y - \tfrac{a}{x}\big(k_9\sinh(\tfrac{t}{a})\\
&\quad + k_{10}\cosh(\tfrac{t}{a})\big)z + d_0(t,x)]s - \tfrac{a}{x}\big(k_5\sinh(\tfrac{t}{a}) + k_6\cosh(\tfrac{t}{a})\big)y\\
&\quad - \tfrac{a}{x}\big(k_{13}\sinh(\tfrac{t}{a}) + k_{14}\cosh(\tfrac{t}{a})\big)z + d_5(t,x),\\
&\eta^1 = [\tfrac{c_0}{2}x + \big(k_1\cosh(\tfrac{t}{a}) + k_2\sinh(\tfrac{t}{a})\big)y + \big(k_9\cosh(\tfrac{t}{a})\\
&\quad + k_{10}\sinh(\tfrac{t}{a})\big)z + d_{14}(t)]s + \tfrac{c_1}{2}x + \big(k_5\cosh(\tfrac{t}{a}) + k_6\sinh(\tfrac{t}{a})\big)y\\
&\quad + \big(k_{13}\cosh(\tfrac{t}{a}) + k_{14}\sinh(\tfrac{t}{a})\big)z + d_{15}(t),\\
&\eta^2 = [\tfrac{c_0}{2}y - \big(k_1\cosh(\tfrac{t}{a}) + k_2\sinh(\tfrac{t}{a})\big)x + c_{19}z + k_4]s+\\
&\quad + c_1 y - \big(k_5\cosh(\tfrac{t}{a}) + k_6\sinh(\tfrac{t}{a})\big)x + c_{23}z + k_8,\\
&\eta^3 = [\tfrac{c_0}{2}z - \big(k_9\cosh(\tfrac{t}{a}) + k_{10}\sinh(\tfrac{t}{a})\big)x - c_{19}y + k_{12}]s+\\
&\quad + \tfrac{c_1}{2}z - \big(k_{13}\cosh(\tfrac{t}{a}) + k_{14}\sinh(\tfrac{t}{a})\big)x - c_{23}y + k_{16}.
\end{aligned}\right\} \tag{3.2.5}$$

From eq.(3.0.42),

$x^2\big[\{\frac{-2a}{x^3}(k_1\sinh(\frac{t}{a})+k_2\cosh(\frac{t}{a}))y-\frac{2a}{x^3}(k_9\sinh(\frac{t}{a})+k_{10}\cosh(\frac{t}{a}))z+d_{0,xx}(t,x)\}s-\frac{2a}{x^3}(k_5\sinh(\frac{t}{a})+k_6\cosh(\frac{t}{a}))y-\frac{2a}{x^3}(k_{13}\sinh(\frac{t}{a})+k_{14}\cosh(\frac{t}{a}))z+d_{5,xx}(t,x)\big]+2x\big[\{\frac{a}{x^2}(k_1\sinh(\frac{t}{a}(+k_2\cosh(\frac{t}{a}))y+\frac{a}{x^2}(k_9\sinh(\frac{t}{a})+k_{10}\cosh(\frac{t}{a}))z+d_0(t,x)\}s+\frac{a}{x^2}(k_5\sinh(\frac{t}{a})+k_6\cosh(\frac{t}{a}))y+\frac{a}{x^2}(k_{13}\sinh(\frac{t}{a})+k_{14}\cosh(\frac{t}{a}))z+d_{5,x}(t,x)\big]=0.$

Comparing coefficients of s and s^0;

$$s: \qquad x^2 d_{0,xx}(t,x) + 2x d_{0,x}(t,x) = 0,$$

$$d_0(t,x) = B_1(t) + B_2(t)\frac{1}{x}, \tag{3.2.6}$$

$$s^0: \qquad x^2 d_{5,xx}(t,x) + 2x d_5(t,x),$$

$$d_5(t,x) = B_3(t) + B_4(t)\frac{1}{x}. \tag{3.2.7}$$

Thus, we have

$$\begin{aligned}
\eta^0 = &\big[-\tfrac{ay}{x}(k_1\sinh(\tfrac{t}{a}) + k_2\cosh(\tfrac{t}{a})) - \tfrac{az}{x}(k_9\sinh(\tfrac{t}{a}) + k_{10}\cosh(\tfrac{t}{a}))\\
&+ B_1(t) + \tfrac{B_2(t)}{x}\big]s + \big[-\tfrac{ay}{x}(k_5\sinh(\tfrac{t}{a}) + k_6\cosh(\tfrac{t}{a}))\\
&- \tfrac{az}{x}(k_{13}\sinh(\tfrac{t}{a}) + k_{14}\cosh(\tfrac{t}{a})) + B_3(t) + \tfrac{B_4(t)}{x}\big].
\end{aligned} \tag{3.2.8}$$

From eq.(3.0.41), we get

$\left(\frac{x}{a}\right)^2\big[\{\frac{y}{x^2}(k_1\cosh(\frac{t}{a})+k_2\sinh(\frac{t}{a}))+\frac{z}{x^2}(k_9\cosh(\frac{t}{a})+k_{10}\sinh(\frac{t}{a}))-\frac{B_2'(t)}{x^2}\}s+\{\frac{y}{x^2}(k_5\cosh(\frac{t}{a})+k_6\sinh(\frac{t}{a}))+\frac{z}{x^2}(k_{13}\cosh(\frac{t}{a})+k_{14}\sinh(\frac{t}{a}))-\frac{B_4'(t)}{x^2}\}\big] = \big[\{\frac{1}{a^2}(k_1\cosh(\frac{t}{a})+k_2\sinh(\frac{t}{a}))y+\frac{1}{a^2}(k_9\cosh(\frac{t}{a})+k_{10}\sinh(\frac{t}{a}))z+d_{14}''(t)\}s+\frac{1}{a^2}(k_5\cosh(\frac{t}{a})+k_6\sinh(\frac{t}{a}))y+(k_{13}\cosh(\frac{t}{a})+k_{14}\sinh(\frac{t}{a}))z+d_{15}''(t)\big]$.

Comparing coefficient of s and s^0;

$$s: \quad d_{14}''(t) = \tfrac{-1}{a^2}B_2'(t),$$

$$d_{14}'(t) = \frac{-1}{a^2}B_2(t) + k_{17}, \tag{3.2.9}$$

$$s^0: \quad d_{15}''(t) = \tfrac{-1}{a^2}B_4'(t),$$

$$d_{15}'(t) = \frac{-1}{a^2}B_4(t) + k_{18}. \tag{3.2.10}$$

where k_{17} and k_{18} are constants of integration. Now using eq.(3.0.37), we obtain

$\frac{1}{x}\big[\{\frac{1}{a}(k_1\sinh(\frac{t}{a})+k_2\cosh(\frac{t}{a}))y+\frac{1}{a}(k_9\sinh(\frac{t}{a})+k_{10}\cosh(\frac{t}{a}))z+d_{14}'(t)\}s+\frac{1}{a}(k_5\sinh(\frac{t}{a})+k_6\cosh(\frac{t}{a}))y+\frac{1}{a}(k_{13}\sinh(\frac{t}{a})+k_{14}\cosh(\frac{t}{a}))z+d_{15}'(t)\big] = -\big[\{\frac{-y}{ax}(k_1\sinh(\frac{t}{a})+k_2\cosh(\frac{t}{a}))-\frac{z}{ax}\frac{1}{a}(k_9\sinh(\frac{t}{a})+k_{10}\cosh(\frac{t}{a}))+B_1''(t)+\frac{B_2''(t)}{x}\}s-\frac{y}{ax}(k_5\sin(\frac{t}{a})+k_6\cosh(\frac{t}{a}))-\frac{z}{ax}(k_{13}\sinh(\frac{t}{a})+k_{14}\cosh(\frac{t}{a}))+B_3''(t)+\frac{B_4''(t)}{x}\big]$.

Comparing coefficients of s and s^0;

$s: \quad \frac{1}{x}d_{14}'(t) = -B_1''(t) - \frac{B_2''(t)}{x}$.

Comparing coefficients of $\frac{1}{x}$ and $\left(\frac{1}{x}\right)^0$;

$$\frac{1}{x} \quad : \quad B_2(t) = k_{19}\cosh(\frac{t}{a}) + k_{20}\sinh(\frac{t}{a}) + k_{17},$$

$$\text{const} \quad : \quad B_1''(t) = 0 \quad \Rightarrow A_1(t) = k_{21}t + k_{22},$$

$s^0: \quad \frac{1}{x}d_{15}'(t) = -B_3''(t) - \frac{B_4''(t)}{x}$

Comparing coefficients of $\frac{1}{x}$ and $\left(\frac{1}{x}\right)^0$;

$$\frac{1}{x} \quad : \quad B_4(t) = k_{23}\cosh(\frac{t}{a}) + k_{24}\sinh(\frac{t}{a}) + k_{18},$$

$$\text{const} \quad : \quad B_3''(t) = 0 \quad \Rightarrow B_3(t) = k_{25}t + k_{26}.$$

where k_i $(i = 19, ..., 25)$ are constants of integration. Therefore, eq.(3.2.9) implies

$$d_{14}(t) = -\frac{1}{a}\big(k_{19}\sinh(\frac{t}{a}) + k_{20}\cosh(\frac{t}{a})\big) + k_{27}t + k_{28}.$$

and eq.(3.2.10) gives

$$d_{15}(t) = -\frac{1}{a}\big(k_{23}\sinh(\frac{t}{a}) + k_{24}\cosh(\frac{t}{a})\big) + k_{29}t + k_{30}$$

where k_{27}, k_{28}, k_{29}, and k_{30} are constants of integration. Therefore, eqs.(3.2.6)-(3.2.7) implies

$$\begin{aligned} d_0(t,x) &= k_{21}t + k_{22} + \frac{1}{x}\big(k_{19}\cosh(\frac{t}{a}) + k_{20}\sinh(\frac{t}{a}) + k_{17}\big), \\ d_5(t,x) &= k_{25}t + k_{26} + \frac{1}{x}\big(k_{23}\cosh(\frac{t}{a}) + k_{24}\sinh(\frac{t}{a}) + k_{18}\big). \end{aligned}$$

Therefore, eq.(3.2.5) becomes

$$\left.\begin{aligned} &\xi = \tfrac{c_0}{2}s^2 + c_1 s + c_2, \qquad A = A(t,x,y,z), \\ &\eta^0 = [\tfrac{-a}{x}\big(k_1\sinh(\tfrac{t}{a}) + k_2\cosh(\tfrac{t}{a})\big)y - \tfrac{a}{x}\big(k_9\sinh(\tfrac{t}{a}) + k_{10}\cosh(\tfrac{t}{a})\big)z \\ &\quad + k_{21}t + k_{22} + \tfrac{1}{x}(k_{19}\cosh(\tfrac{t}{a}) + k_{20}\sinh(\tfrac{t}{a}) + k_{17})]s \\ &\quad - \tfrac{a}{x}\big(k_5\sinh(\tfrac{t}{a}) + k_6\cosh(\tfrac{t}{a})\big)y - \tfrac{a}{x}\big(k_{13}\sinh(\tfrac{t}{a}) + k_{14}\cosh(\tfrac{t}{a})\big)z \\ &\quad + k_{25}t + k_{26} + \tfrac{1}{x}(k_{23}\cosh(\tfrac{t}{a}) + k_{24}\sinh(\tfrac{t}{a}) + k_{18}), \\ &\eta^1 = [\tfrac{c_0}{2}x + \big(k_1\cosh(\tfrac{t}{a}) + k_2\sinh(\tfrac{t}{a})\big)y + \big(k_9\cosh(\tfrac{t}{a}) + k_{10}\sinh(\tfrac{t}{a})\big)z \\ &\quad - \tfrac{1}{a}\big(k_{19}\sinh(\tfrac{t}{a}) + k_{20}\cosh(\tfrac{t}{a})\big) + k_{27}t + k_{28}]s \\ &\quad + \tfrac{c_1}{2}x + \big(k_5\cosh(\tfrac{t}{a}) + k_6\sinh(\tfrac{t}{a})\big)y + \big(k_{13}\cosh(\tfrac{t}{a}) + k_{14}\sinh(\tfrac{t}{a})\big)z \\ &\quad - \tfrac{1}{a}\big(k_{23}\sinh(\tfrac{t}{a}) + k_{24}\cosh(\tfrac{t}{a})\big) + k_{29}t + k_{30}, \\ &\eta^2 = [\tfrac{c_0}{2}y - \big(k_1\cosh(\tfrac{t}{a}) + k_2\sinh(\tfrac{t}{a})\big)x + c_{19}z + k_4]s+ \\ &\quad + \tfrac{c_1}{2}y - \big(k_5\cosh(\tfrac{t}{a}) + k_6\sinh(\tfrac{t}{a})\big)x + c_{23}z + k_8, \\ &\eta^3 = [\tfrac{c_0}{2}z - \big(k_9\cosh(\tfrac{t}{a}) + k_{10}\sinh(\tfrac{t}{a})\big)x - c_{19}y + k_{12}]s+ \\ &\quad + \tfrac{c_1}{2}z - \big(k_{13}\cosh(\tfrac{t}{a}) + k_{14}\sinh(\tfrac{t}{a})\big)x - c_{23}y + k_{16}. \end{aligned}\right\} \quad (3.2.11)$$

From eq.(3.0.38),

$\frac{1}{x^2}\big[\{\frac{c_0 x}{2} + (k_1\cosh(\frac{t}{a}) + k_2\sinh(\frac{t}{a}))y + (k_9\cosh(\frac{t}{a}) + k_{10}\sinh(\frac{t}{a}))z - \frac{1}{a}\big(k_{19}\sinh(\frac{t}{a}) + k_{20}\cosh(\frac{t}{a})\big) + k_{27}t + k_{28}\}s + \frac{c_1 x}{2} + (k_5\cosh(\frac{t}{a}) + k_6\sinh(\frac{t}{a}))y + (k_{13}\cosh(\frac{t}{a}) + k_{14}\sinh(\frac{t}{a}))z -$

$\frac{1}{a}\big(k_{23}\sinh(\frac{t}{a})+k_{24}\cosh(\frac{t}{a})\big)+k_{29}t+k_{30}\big]+\frac{1}{x}(\frac{c_0 s}{2}+\frac{c_1}{2})+\big[\{\frac{y}{x^2}(k_1\cosh(\frac{t}{a})+k_2\sinh(\frac{t}{a}))+$
$\frac{z}{x}(k_9\cosh(\frac{t}{a})+k_{10}\sinh(\frac{t}{a}))-\frac{1}{ax^2}(k_{19}\sinh(\frac{t}{a})+k_{20}\cosh(\frac{t}{a}))\}s+\frac{y}{x^2}(k_5\cosh(\frac{t}{a})+k_6\sinh(\frac{t}{a}))+$
$\frac{z}{x}(k_{13}\cosh(\frac{t}{a})+k_{14}\sinh(\frac{t}{a}))-\frac{1}{ax^2}(k_{23}\cosh(\frac{t}{a})+k_{24}\sinh(\frac{t}{a}))\big]=0.$

By Comparing coefficients of s and s^0;

$$s: \qquad k_{27}t+k_{28}=0 \qquad \Rightarrow k_{27}=0=k_{28},$$

$$s^0: \qquad k_{29}t+k_{30}=0 \qquad \Rightarrow k_{29}=0=k_{30}.$$

$$\begin{aligned}\eta^1 = [&\tfrac{c_0}{2}x+\big(k_1\cosh(\tfrac{t}{a})+k_2\sinh(\tfrac{t}{a})\big)y+\big(k_9\cosh(\tfrac{t}{a})+k_{10}\sinh(\tfrac{t}{a})\big)z\\ &-\tfrac{1}{a}\big(k_{19}\sinh(\tfrac{t}{a})+k_{20}\cosh(\tfrac{t}{a})\big)]s+\tfrac{c_1}{2}x+\big(k_5\cosh(\tfrac{t}{a})+k_6\sinh(\tfrac{t}{a})\big)y\\ &+\big(k_{13}\cosh(\tfrac{t}{a})+k_{14}\sinh(\tfrac{t}{a})\big)z-\tfrac{1}{a}\big(k_{23}\sinh(\tfrac{t}{a})+k_{24}\cosh(\tfrac{t}{a})\big).\end{aligned} \tag{3.2.12}$$

Using eq.(3.0.9), we have

$$\begin{aligned}A(t,x,y,z) = &-\tfrac{c_0x^2}{2}-2xy(k_1\cosh(\tfrac{t}{a})+k_2\sinh(\tfrac{t}{a}))-2xz(k_9\cosh(\tfrac{t}{a})+k_{10}\sinh(\tfrac{t}{a}))\\ &+2\tfrac{x}{a}\big(k_{19}\sinh(\tfrac{t}{a})+k_{20}\cosh(\tfrac{t}{a})\big)+h_1(t,y,z).\end{aligned} \tag{3.2.13}$$

From eq.(3.0.10),

$$-2\big[\frac{c_0y}{2}-\big(k_1\cosh(\frac{t}{a})+k_2\sinh(\frac{t}{a})\big)x+c_{19}z+k_4\big]=-2x\big(k_1\cosh(\frac{t}{a})+k_2\sinh(\frac{t}{a})\big)+h_{1,y}(t,y,z).$$

By comparing coefficient of x;

$$k_1=0=k_2,$$

and

$$h_1(t,y,z)=-\frac{c_0y^2}{2}-2c_{19}yz-2k_4y+h_2(t,z).$$

Therefore, eq.(3.2.13) reduces to

$$\begin{aligned}A(t,x,y,z) = &-\tfrac{c_0}{2}(x^2+y^2)-2xz(k_9\cosh(\tfrac{t}{a})+k_{10}\sinh(\tfrac{t}{a}))\\ &+2\tfrac{x}{a}(k_{19}\sinh(\tfrac{t}{a})+k_{20}\cosh(\tfrac{t}{a}))-2c_{19}yz-2k_4y+h_2(t,z).\end{aligned} \tag{3.2.14}$$

Taking into account eq.(3.0.11), we have

$-2[\frac{c_0z}{2}-(k_9\cosh(\frac{t}{a})+k_{10}\sinh(\frac{t}{a}))x-c_{19}y+k_{12}]=-2x(k_9\cosh(\frac{t}{a})+k_{10}\sinh(\frac{t}{a}))-$

$2c_{19}y + h_{2,z}(t,z)$.

By Comparing coefficients x and y;

$$x: \qquad k_9 = k_{10} = 0,$$

$$y: \qquad c_{19} = 0.$$

Therefore, $\qquad h_2(t,z) = -\frac{c_0 z^2}{2} - 2k_{12}z + h_3(t).$

Hence, eq.(3.2.14) reduces to

$$\begin{aligned} A(t,x,y,z) = &-\tfrac{c_0}{2}(x^2+y^2+z^2) + 2\tfrac{x}{a}(k_{19}\sinh(\tfrac{t}{a}) + k_{20}\cosh(\tfrac{t}{a})) - 2k_4 y \\ &- 2k_{12}z + h_3(t). \end{aligned} \tag{3.2.15}$$

From eq.(3.0.8),

$$2\tfrac{x}{a}^2\big[k_{21}t + k_{22} + \tfrac{1}{x}\big(k_{19}\cosh(\tfrac{t}{a}) + k_{20}\sinh(\tfrac{t}{a}()\big] = 2\tfrac{x}{a^2}\big(k_{19}\sinh(\tfrac{t}{a}) + k_{20}\cosh(\tfrac{t}{a})\big) + h_3'(t),$$

$$h_3(t) = 2\frac{x}{a}^2\big(k_{21}\frac{t^2}{2} + k_{22}t + k_{31}\big), \qquad (k_{31} = const.\,of\,integration)$$

Therefore,

$$\begin{aligned} A(t,x,y,z) = &-\tfrac{c_0}{2}(x^2+y^2+z^2) + 2\tfrac{x}{a}(k_{19}\sinh(\tfrac{t}{a}) + k_{20}\cosh(\tfrac{t}{a})) \\ &- 2k_4 y - 2k_{12}z + 2\tfrac{x}{a}^2\big(k_{21}\tfrac{t^2}{2} + k_{22}t\big) + k_{31}. \end{aligned} \tag{3.2.16}$$

Taking into account eqs.(3.0.8), (3.0.9), (3.0.12), and (3.0.21), we have

$$k_{17} = k_{21} = k_{22} = k_{18} = k_{25} = 0.$$

Therefore, eq.(3.2.11) reduce to

$$\left.\begin{aligned}
&\xi = \tfrac{c_0}{2}s^2 + c_1 s + c_2,\\
&\eta^0 = [\tfrac{1}{x}(k_{19}\cosh(\tfrac{t}{a}) + k_{20}\sinh(\tfrac{t}{a}))]s - \tfrac{a}{x}\big(k_5\sinh(\tfrac{t}{a}) + k_6\cosh(\tfrac{t}{a})\big)y\\
&\qquad - \tfrac{a}{x}\big(k_{13}\sinh(\tfrac{t}{a}) + k_{14}\cosh(\tfrac{t}{a})\big)z + k_{26}\\
&\qquad + \tfrac{1}{x}(k_{23}\cosh(\tfrac{t}{a}) + k_{24}\sinh(\tfrac{t}{a})),\\
&\eta^1 = [\tfrac{c_0}{2}x - \tfrac{1}{a}\big(k_{19}\sinh(\tfrac{t}{a}) + k_{20}\cosh(\tfrac{t}{a})\big)]s + \tfrac{c_1}{2}x\\
&\qquad + \big(k_5\cosh(\tfrac{t}{a}) + k_6\sinh(\tfrac{t}{a})\big)y + \big(k_{13}\cosh(\tfrac{t}{a}) + k_{14}\sinh(\tfrac{t}{a})\big)z\\
&\qquad - \tfrac{1}{a}\big(k_{23}\sinh(\tfrac{t}{a}) + k_{24}\cosh(\tfrac{t}{a})\big),\\
&\eta^2 = [\tfrac{c_0}{2}y + k_4]s + \tfrac{c_1}{2}y - \big(k_5\cosh(\tfrac{t}{a}) + k_6\sinh(\tfrac{t}{a})\big)x\\
&\qquad + c_{23}z + k_8,\\
&\eta^3 = [\tfrac{c_0}{2}z + k_{12}]s + \tfrac{c_1}{2}z - \big(k_{13}\cosh(\tfrac{t}{a}) + k_{14}\sinh(\tfrac{t}{a})\big)x\\
&\qquad - c_{23}y + k_{16},\\
&A = -\tfrac{c_0}{2}(x^2 + y^2 + z^2) + 2\tfrac{x}{a}(k_{19}\sinh(\tfrac{t}{a}) + k_{20}\cosh(\tfrac{t}{a}))\\
&\qquad - 2k_4 y - 2k_{12}z + k_{31}.
\end{aligned}\right\} \tag{3.2.17}$$

Substituting eq.(3.2.17) into eq.(3.1.17), one gets

$$\begin{aligned}
\mathbf{X} = {}& \big(\tfrac{c_0}{2}s^2 + c_1 s + c_2\big)\tfrac{\partial}{\partial s} + \Bigg([\tfrac{1}{x}(k_{19}\cosh\tfrac{t}{a} + k_{20}\sinh\tfrac{t}{a})]s + k_{26}\\
& - \tfrac{a}{x}\big(k_5\sinh\tfrac{t}{a} + k_6\cosh\tfrac{t}{a}\big)y - \tfrac{a}{x}\big(k_{13}\sinh\tfrac{t}{a} + k_{14}\cosh\tfrac{t}{a}\big)z\\
& + \tfrac{1}{x}(k_{23}\cosh\tfrac{t}{a} + k_{24}\sinh\tfrac{t}{a})\Bigg)\tfrac{\partial}{\partial t} + \Bigg([\tfrac{c_0}{2}x - \tfrac{1}{a}\big(k_{19}\sinh\tfrac{t}{a}\\
& + k_{20}\cosh\tfrac{t}{a}\big)]s + \tfrac{c_1}{2}x + \big(k_5\cosh\tfrac{t}{a} + k_6\sinh\tfrac{t}{a}\big)y + \big(k_{13}\cosh\tfrac{t}{a}\\
& + k_{14}\sinh\tfrac{t}{a}\big)z - \tfrac{1}{a}\big(k_{23}\sinh\tfrac{t}{a} + k_{24}\cosh\tfrac{t}{a}\big)\Bigg)\tfrac{\partial}{\partial x} + \Bigg([\tfrac{c_0}{2}y + k_4]s\\
& + \tfrac{c_1}{2}y - \big(k_5\cosh\tfrac{t}{a} + k_6\sinh\tfrac{t}{a}\big)x + c_{23}z + k_8\Bigg)\tfrac{\partial}{\partial y} + \Bigg([\tfrac{c_0}{2}z\\
& + k_{12}]s + \tfrac{c_1}{2}z - \big(k_{13}\cosh\tfrac{t}{a} + k_{14}\sinh\tfrac{t}{a}\big)x - c_{23}y + k_{16}\Bigg)\tfrac{\partial}{\partial z}
\end{aligned} \tag{3.2.18}$$

There are 17 arbitrary constants c_i's appearing in $\mathbf{X}$. Therefore, the metric given by eq.(2.2.39) admits 17 Noether symmetries. The generators are as follows:

$$
\begin{aligned}
\mathbf{X_0} &= \frac{\partial}{\partial t},\\
\mathbf{X_1} &= z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z},\\
\mathbf{X_2} &= \frac{\partial}{\partial z},\\
\mathbf{X_3} &= \frac{\partial}{\partial y},\\
\mathbf{X_4} &= \frac{-az}{x}\sinh\left(\frac{t}{a}\right)\frac{\partial}{\partial t} + z\cosh\left(\frac{t}{a}\right)\frac{\partial}{\partial x} - x\cosh\left(\frac{t}{a}\right)\frac{\partial}{\partial z},\\
\mathbf{X_5} &= \frac{-az}{x}\cosh\left(\frac{t}{a}\right)\frac{\partial}{\partial t} + z\sinh\left(\frac{t}{a}\right)\frac{\partial}{\partial x} - x\sinh\left(\frac{t}{a}\right)\frac{\partial}{\partial y},\\
\mathbf{X_6} &= \frac{-ay}{x}\sinh\left(\frac{t}{a}\right)\frac{\partial}{\partial t} + y\cosh\left(\frac{t}{a}\right)\frac{\partial}{\partial x} - x\cosh\left(\frac{t}{a}\right)\frac{\partial}{\partial y},\\
\mathbf{X_7} &= \frac{-ay}{x}\cosh\left(\frac{t}{a}\right)\frac{\partial}{\partial t} + y\sinh\left(\frac{t}{a}\right)\frac{\partial}{\partial x} - x\sinh\left(\frac{t}{a}\right)\frac{\partial}{\partial y},\\
\mathbf{X_8} &= \frac{1}{x}\sinh\left(\frac{t}{a}\right)\frac{\partial}{\partial t} - \frac{1}{a}\cosh\left(\frac{t}{a}\right)\frac{\partial}{\partial x},\\
\mathbf{X_9} &= \frac{1}{x}\cosh\left(\frac{t}{a}\right)\frac{\partial}{\partial t} - \frac{1}{a}\sinh\left(\frac{t}{a}\right)\frac{\partial}{\partial x},\\
\mathbf{Y_0} &= \frac{\partial}{\partial s},\\
\mathbf{Y_1} &= s\frac{\partial}{\partial y} \qquad ; A_1 = -2y,\\
\mathbf{Y_2} &= s\frac{\partial}{\partial z} \qquad ; A_2 = -2z,\\
\mathbf{Y_3} &= \frac{1}{2}\big[s^2\frac{\partial}{\partial s} + sx\frac{\partial}{\partial x} + sy\frac{\partial}{\partial y} + sz\frac{\partial}{\partial z}\big] \qquad ; A_3 = \frac{1}{2}(-x^2 - y^2 - z^2),\\
\mathbf{Y_4} &= \frac{s}{x}\sinh\left(\frac{t}{a}\right)\frac{\partial}{\partial t} - \frac{s}{a}\cosh\left(\frac{t}{a}\right)\frac{\partial}{\partial x} \qquad ; A_4 = \frac{2x}{a}\cosh(\frac{t}{a}),\\
\mathbf{Y_5} &= \frac{s}{x}\cosh\left(\frac{t}{a}\right)\frac{\partial}{\partial t} - \frac{s}{a}\sinh\left(\frac{t}{a}\right)\frac{\partial}{\partial x} \qquad ; A_5 = \frac{2x}{a}\sinh\left(\frac{t}{a}\right),\\
\mathbf{Y_6} &= s\frac{\partial}{\partial s} + \frac{1}{2}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right).
\end{aligned}
$$

The structure constants of the corresponding Lie algebra are

$$C^5_{04} = C^4_{05} = C^7_{06} = C^6_{07} = C^9_{08} = C^8_{09} = C^2_{13} = C^4_{16} = C^5_{17} = C^0_{45} = C^{\hat{2}}_{1\hat{1}} = 1,$$
$$C^1_{46} = C^1_{57} = C^2_{59} = C^0_{67} = C^3_{79} = C^3_{\hat{0}\hat{1}} = C^2_{\hat{0}\hat{2}} = C^{\hat{4}}_{\hat{0}\hat{3}} = C^{\hat{0}}_{\hat{0}\hat{4}} = C^8_{\hat{0}\hat{5}} = C^9_{\hat{0}\hat{6}} = 1,$$
$$C^3_{12} = C^6_{14} = C^7_{15} = C^8_{24} = C^9_{25} = C^8_{36} = C^9_{37} = C^2_{48} = C^3_{68} = C^{\hat{1}}_{1\hat{2}} = C^{\hat{3}}_{\hat{3}\hat{4}} = -1,$$
$$C^{\hat{1}}_{2\hat{3}} = C^{\hat{5}}_{8\hat{3}} = C^{\hat{6}}_{9\hat{3}} = C^2_{2\hat{4}} = C^3_{3\hat{4}} = C^8_{8\hat{4}} = C^9_{9\hat{4}} = C^{\hat{5}}_{\hat{4}\hat{5}} = C^{\hat{6}}_{\hat{4}\hat{6}} = \frac{1}{2},$$
$$C^{\hat{5}}_{0\hat{5}} = C^{\hat{6}}_{0\hat{6}} = C^{\hat{2}}_{4\hat{5}} = C^{\hat{1}}_{7\hat{6}} = \frac{1}{2}, \qquad C^{\hat{1}}_{\hat{1}\hat{4}} = C^{\hat{2}}_{\hat{2}\hat{4}} = C^{\hat{1}}_{3\hat{3}} = -\frac{1}{2},$$
$$C^{\hat{4}}_{6\hat{1}} = C^{\hat{6}}_{7\hat{1}} = C^{\hat{4}}_{4\hat{2}} = C^{\hat{5}}_{5\hat{2}} = a.$$

3.2.2 Case of $[\nu''(x) + \nu'^2(x)] \neq 0$

When $[\nu''(x) + \nu'^2(x)] \neq 0 \Rightarrow \eta^0_y = 0$, $\eta^0_z = 0$.

There are three possibilities:

(a). $\nu''(x) + \nu'^2(x) = k$, (k is positive)

(b). $\nu''(x) + \nu'^2(x) = -k$,

(c). $\nu''(x) + \nu'^2(x) = f(x)$.

(a) Case of $\nu''(x) + \nu'^2(x) = k$, **(**$k > 0$**)**

This implies $\nu(x) = \ln\big(\cosh\big(\frac{x}{a}\big)\big)$, (taking $k = 1/a^2$).

Therefore, eq.(3.0.1) reduces to

$$L = \cosh^2\big(\frac{x}{a}\big)\dot{t}^2 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2. \tag{3.2.19}$$

From eq.(3.0.55),

$$\eta^0_{,xy} = 0 \qquad \Rightarrow d_{1,x}(t,x)s + d_{3,x}(t,x) = 0.$$

Comparing coefficients of s and s^0;

$$s \quad : \qquad d_{1,x}(t,x) = 0 \qquad \Rightarrow d_1(t,x) = d_1(t),$$
$$s^0 \quad : \qquad d_{3,x}(t,x) = 0 \qquad \Rightarrow d_3(t,x) = d_3(t),$$

$$\eta^0 = [d_1(t)y + d_2(t,x)z + d_0(t,x)]s + d_3(t)y + d_4(t,x)z + d_5(t,x). \tag{3.2.20}$$

Using eq.(3.0.39), we have

$$\eta^0_{,ty} = 0 \qquad \Rightarrow d_1'(t)s + d_3'(t) = 0.$$

Comparing coefficients of s and s^0;

$$\begin{aligned} s \quad &: \qquad d_1'(t) = 0 \qquad \Rightarrow d_1(t) = p_1, \\ s^0 \quad &: \qquad d_3'(t) = 0 \qquad \Rightarrow d_3(t) = p_3, \end{aligned}$$

where p_1 and p_3 are the constants of integration. Therefore,

$$\eta^0 = [p_1 y + d_2(t,x)z + d_0(t,x)]s + p_3 y + d_4(t,x)z + d_5(t,x). \tag{3.2.21}$$

Taking into account eq.(3.0.56),

$$\eta^0_{,xz} = 0 \qquad \Rightarrow d_{2,x}(t,x)s + d_{4,x}(t,x) = 0.$$

Comparing coefficients of s and s^0;

$$\begin{aligned} s \quad &: \qquad d_{2,x}(t,x) = 0 \qquad \Rightarrow d_2(t,x) = d_2(t), \\ s^0 \quad &: \qquad d_{4,x}(t,x) = 0 \qquad \Rightarrow d_4(t,x) = d_4(t). \end{aligned}$$

From eq.(3.0.40),

$$\eta^0_{,tz} = 0 \qquad \Rightarrow d_2'(t)s + d_4'(t) = 0.$$

Comparing coefficients of s and s^0;

$$\begin{aligned} s \quad &: \qquad d_2'(t) = 0 \qquad \Rightarrow d_2(t) = p_2, \\ s^0 \quad &: \qquad d_4'(t) = 0 \qquad \Rightarrow d_4(t) = p_4, \end{aligned}$$

where p_2 and p_4 are the constants of integration. Hence,

$$\eta^0 = [p_1 y + p_2 z + d_0(t,x)]s + p_3 y + p_4 z + d_5(t,x). \tag{3.2.22}$$

Using eq.(3.0.43), we have

$$\eta^1_{,ty} = 0 \qquad \Rightarrow d_8'(t)s + d_{12}'(t) = 0.$$

Comparing coefficients of s and s^0;

$$\begin{aligned} s &: \qquad d_8'(t) = 0 \qquad \Rightarrow d_8(t) = p_{11}, \\ s^0 &: \qquad d_{12}'(t) = 0 \qquad \Rightarrow d_{12}(t) = p_{12}. \end{aligned}$$

where p_{11} and p_{12} are the constants of integration. Taking into account eq.(3.0.44), we have

$$\eta^1_{,tz} = 0 \qquad \Rightarrow d_9'(t)s + d_{13}'(t) = 0.$$

Comparing coefficients of s and s^0;

$$\begin{aligned} s &: \qquad d_9'(t) = 0 \qquad \Rightarrow d_9(t) = p_{13}, \\ s^0 &: \qquad d_{13}'(t) = 0 \qquad \Rightarrow d_{13}(t) = p_{14}, \end{aligned}$$

where p_{13} and p_{14} are the constants of integration. Thus,

$$\eta^1 = \left[\frac{c_0}{2}x + p_{11}y + p_{13}z + d_{14}(t)\right]s + \frac{c_1}{2}x + p_{12}y + p_{14}z + d_{15}(t). \tag{3.2.23}$$

From eq.(3.0.45),

$$\eta^2_{,tt} = 0 \qquad \Rightarrow d_{24}''(t)s + d_{25}''(t) = 0.$$

Comparing coefficients of s and s^0;

$$\begin{aligned} s &: \qquad d_{24}''(t) = 0 \qquad \Rightarrow d_{24}(t) = p_{19}t + p_{20}, \\ s^0 &: \qquad d_{25}''(t) = 0 \qquad \Rightarrow d_{25}(t) = p_{21}t + p_{22}, \end{aligned}$$

where p_{19}, p_{20}, p_{21}, and p_{22} are the constants of integration. Therefore,

$$\eta^2 = \left[\frac{c_0}{2}y - p_{11}x + c_{19}z + p_{19}t + p_{20}\right]s + \frac{c_1}{2}y - p_{12}x + c_{23}z + p_{21}t + p_{22}. \tag{3.2.24}$$

From eq.(3.0.49),

$$\eta^3_{,tt} = 0 \qquad \Rightarrow d''_{34}(t)s + d''_{35}(t) = 0.$$

Comparing coefficients of s and s^0;

$$\begin{aligned} s \quad &: \qquad d''_{34}(t) = 0 \qquad \Rightarrow d_{34}(t) = p_{23}t + p_{24}, \\ s^0 \quad &: \qquad d''_{35}(t) = 0 \qquad \Rightarrow d_{35}(t) = p_{25}t + p_{26}, \end{aligned}$$

where p_{23}, p_{24}, p_{25}, and p_{26} are the constants of integration. Hence,

$$\eta^3 = \left[\frac{c_0}{2}z - p_{13}x - c_{19}y + p_{23}t + p_{24}\right]s + \frac{c_1}{2}z - p_{14}x - c_{23}y + p_{25}t + p_{26}. \tag{3.2.25}$$

Therefore, eq.(3.0.86) reduces to

$$\left.\begin{aligned} &\xi = \tfrac{c_0}{2}s^2 + c_1 s + c_2, \qquad A = A(t,x,y,z), \\ &\eta^0 = \big[p_1 y + p_2 z + d_0(t,x)\big]s + \big[p_3 y + p_4 z + d_5(t,x)\big], \\ &\eta^1 = \big[\tfrac{c_0}{2}x + p_{11}y + p_{13}z + d_{14}(t)\big]s + \tfrac{c_1}{2}x + p_{12}y \\ &\qquad + p_{14}z + d_{15}(t), \\ &\eta^2 = \big[\tfrac{c_0}{2}y - p_{11}x + c_{19}z + p_{19}t + p_{20}\big]s + \tfrac{c_1}{2}y - p_{12}x \\ &\qquad + c_{23}z + p_{21}t + p_{22}, \\ &\eta^3 = \big[\tfrac{c_0}{2}z - p_{13}x - c_{19}y + p_{23}t + p_{24}\big]s + \tfrac{c_1}{2}z - p_{14}x \\ &\qquad - c_{23}y + p_{25}t + p_{26}. \end{aligned}\right\} \tag{3.2.26}$$

Using (3.0.14), we have

$$\cosh^2\big(\frac{x}{a}\big)\big(p_2 s + p_4\big) = p_{23}s + p_{25}.$$

Comparing coefficients of s and s^0;

$$\begin{aligned} s : \qquad & p_{23} = p_2 \cosh^2\big(\frac{x}{a}\big), \\ s^0 : \qquad & p_{25} = p_4 \cosh^2\big(\frac{x}{a}\big). \end{aligned}$$

From eq.(3.0.13), we have

$$\cosh^2\big(\frac{x}{a}\big)(p_1 s + p_3) = p_{19}s + p_{21},$$

$$s: \qquad p_{19} = p_1\cosh^2\big(\frac{x}{a}\big),$$
$$s^0: \qquad p_{21} = p_3\cosh^2\big(\frac{x}{a}\big).$$

Taking into account eq.(3.0.12), we have

$$\cosh^2\big(\frac{x}{a}\big)\big[d_{0,x}(t,x)s + d_{s,x}(t,x)\big] = d'_{14}(t)s + d'_{15}(t),$$

$$s: \qquad d_0(t,x) = a\tanh\big(\frac{x}{a}\big)d'_{14}(t) + f_1(t),$$
$$s^0: \qquad d_5(t,x) = a\tanh\big(\frac{x}{a}\big)d'_{15}(t) + f_2(t).$$

Therefore, eq.(3.2.19) reduces to

$$\left.\begin{aligned}
&\xi = \tfrac{c_0}{2}s^2 + c_1 s + c_2, \qquad A = A(t,x,y,z),\\
&\eta^0 = \big[p_1 y + p_2 z + a\tanh\big(\tfrac{x}{a}\big)d'_{14}(t) + f_1(t)\big]s + p_3 y\\
&\quad + p_4 z + a\tanh\big(\tfrac{x}{a}\big)d'_{15}(t) + f_2(t),\\
&\eta^1 = \big[\tfrac{c_0}{2}x + p_{11}y + p_{13}z + d_{14}(t)\big]s + \tfrac{c_1}{2}x + p_{12}y\\
&\quad + p_{14}z + d_{15}(t),\\
&\eta^2 = \big[\tfrac{c_0}{2}y - p_{11}x + c_{19}z + p_1\cosh^2\big(\tfrac{x}{a}\big)t + p_{20}\big]s + \tfrac{c_1}{2}y\\
&\quad - p_{12}x + c_{23}z + p_3\cosh^2\big(\tfrac{x}{a}\big)t + p_{22},\\
&\eta^3 = \big[\tfrac{c_0}{2}z - p_{13}x - c_{19}y + p_2\cosh^2\big(\tfrac{x}{a}\big)t + p_{24}\big]s + \tfrac{c_1}{2}z - p_{14}x\\
&\quad - c_{23}y + p_4\cosh^2\big(\tfrac{x}{a}\big)t + p_{26}.
\end{aligned}\right\} \tag{3.2.27}$$

Eqs.(3.0.39) and (3.0.40) together gives

$$p_{11} = 0 = p_{12},$$

$$p_{13} = 0 = p_{14}.$$

Therefore, eq.(3.2.27) becomes

$$\left.\begin{aligned}
&\xi = \tfrac{c_0}{2}s^2 + c_1 s + c_2, \qquad A = A(t,x,y,z),\\
&\eta^0 = \big[p_1 y + p_2 z + a\tanh\left(\tfrac{x}{a}\right)d'_{14}(t) + f_1(t)\big]s\\
&\qquad + \big[p_3 y + p_4 z + a\tanh\left(\tfrac{x}{a}\right)d'_{15}(t) + f_2(t)\big],\\
&\eta^1 = \big[\tfrac{c_0}{2}x + d_{14}(t)\big]s + \tfrac{c_1}{2}x + d_{15}(t),\\
&\eta^2 = \big[\tfrac{c_0}{2}y + c_{19}z + p_1\cosh^2\left(\tfrac{x}{a}\right)t + p_{20}\big]s\\
&\qquad + \tfrac{c_1}{2}y - p_{12}x + c_{23}z + p_3\cosh^2\left(\tfrac{x}{a}\right)t + p_{22},\\
&\eta^3 = \big[\tfrac{c_0}{2}z - c_{19}y + p_2\cosh^2\left(\tfrac{x}{a}\right)t + p_{24}\big]s\\
&\qquad + \tfrac{c_1}{2}z - c_{23}y + p_4\cosh^2\left(\tfrac{x}{a}\right)t + p_{26}.
\end{aligned}\right\} \tag{3.2.28}$$

From eq.(3.0.37),

$\frac{1}{a}\tanh\left(\frac{x}{a}\right)\big[d'_{14}(t)s + d'_{15}(t)\big] = -\big[\{a\tanh\left(\frac{x}{a}\right)d'''_{14}(t) + f''_1(t)\}s + \{a\tanh\left(\frac{x}{a}\right)d'''_{15}(t) + f''_2(t)\}\big].$

Comparing coefficients of s and s^0;

$s: \quad \frac{1}{a}\tanh\left(\frac{x}{a}\right)d'_{14}(t) + a\tanh\left(\frac{x}{a}\right)d'''_{14}(t) = -f''_1(t).$

Comparing coefficients of $\tanh\left(\frac{x}{a}\right)$ and const;

$$\begin{aligned}
\tanh\left(\frac{x}{a}\right) &: \qquad d_{14}(t) = p_{27} + p_{28}\cos\left(\frac{t}{a}\right) + p_{29}\sin\left(\frac{t}{a}\right),\\
const &: \qquad f''_1(t) = 0 \qquad \Rightarrow f_1(t) = p_{33}t + p_{34},
\end{aligned}$$

$s^0: \quad \frac{1}{a}\tanh\left(\frac{x}{a}\right)d'_{15}(t) + a\tanh\left(\frac{x}{a}\right)d'''_{15}(t) = -f'''_2(t).$

Comparing coefficient of $\tanh\left(\frac{x}{a}\right)$ and const;

$$\begin{aligned}
tanh\left(\frac{x}{a}\right) &: \qquad d_{15}(t) = p_{30} + p_{31}cos\left(\frac{t}{a}\right) + p_{32}sin\left(\frac{t}{a}\right),\\
const &: \qquad f''_2(t) = 0 \Rightarrow f_2(t) = p_{35}t + p_{36}.
\end{aligned}$$

where p_i $(i = 27, ..., 36)$ are the constants of integration. Therefore, eq.(3.2.28) reduces to

$$
\left.\begin{aligned}
&\xi = \tfrac{c_0}{2}s^2 + c_1 s + c_2, \qquad A = A(t,x,y,z),\\
&\eta^0 = \left[p_1 y + p_2 z + \tanh\left(\tfrac{x}{a}\right)\{-p_{28}\sin\left(\tfrac{t}{a}\right) + p_{29}\cos\left(\tfrac{t}{a}\right)\} + p_{33}t + p_{34}\right]s\\
&\quad + \left[p_3 y + p_4 z + \tanh\left(\tfrac{x}{a}\right)\{-p_{31}\sin\left(\tfrac{t}{a}\right) + p_{32}\cos\left(\tfrac{t}{a}\right)\} + p_{35}t + p_{36}\right],\\
&\eta^1 = \left[\tfrac{c_0}{2}x + p_{27} + p_{28}\cos\left(\tfrac{t}{a}\right) + p_{29}\sin\left(\tfrac{t}{a}\right)\right]s + \tfrac{c_1}{2}x\\
&\quad + p_{30} + p_{31}\cos\left(\tfrac{t}{a}\right) + p_{32}\sin\left(\tfrac{t}{a}\right),\\
&\eta^2 = \left[\tfrac{c_0}{2}y + c_{19}z + p_1\cosh^2\left(\tfrac{x}{a}\right)t + p_{20}\right]s + \tfrac{c_1}{2}y\\
&\quad + c_{23}z + p_3\cosh^2\left(\tfrac{x}{a}\right)t + p_{22},\\
&\eta^3 = \left[\tfrac{c_0}{2}z - c_{19}y + p_2\cosh^2\left(\tfrac{x}{a}\right)t + p_{24}\right]s + \tfrac{c_1}{2}z\\
&\quad - c_{23}y + p_4\cosh^2\left(\tfrac{x}{a}\right)t + p_{26}.
\end{aligned}\right\} \qquad (3.2.29)
$$

From eq.(3.0.21),

$\frac{2}{a}\tanh\left(\frac{x}{a}\right)\left[\{c_0\frac{x}{2}+p_{27}+p_{28}\cos\left(\frac{t}{a}\right)+p_{29}\sin\left(\frac{t}{a}\right)\}s+c_1\frac{x}{2}+p_{30}+p_{31}\cos\left(\frac{t}{a}\right)+p_{32}\sin\left(\frac{t}{a}\right)\right]+$ $2\left[\frac{1}{a}\tanh\left(\frac{x}{a}\right)\left(-p_{28}\cos\left(\frac{t}{a}\right)-p_{29}\sin\left(\frac{t}{a}\right)+p_{33}\right)s+\frac{1}{a}\tanh\left(\frac{x}{a}\right)\left(-p_{31}\cos\left(\frac{t}{a}\right)-p_{32}\sin\left(\frac{t}{a}\right)+\right.\right.$ $\left.\left.p_{35}\right)\right] - (c_0 s + c_1) = 0.$

Comparing coefficients of s and s^0;

$s : \frac{2}{a}\tanh\left(\frac{x}{a}\right)\left[c_0\frac{x}{2} + p_{27}\right] - c_0 + 2p_{33} = 0.$

Comparing coefficients of $x\tanh\left(\frac{x}{a}\right)$, $\tanh\left(\frac{x}{a}\right)$ and const;

$$
\begin{aligned}
x\tanh\left(\frac{x}{a}\right) &: \qquad c_0 = 0,\\
\tanh\left(\frac{x}{a}\right) &: \qquad p_{27} = 0,\\
const &: \qquad p_{33} = 0,
\end{aligned}
$$

$s^0 : \frac{2}{a}\tanh\left(\frac{x}{a}\right)\left[c_1\frac{x}{2} + p_{30}\right] + p_{35} - c_1 = 0.$

Comparing coefficients of $x\tanh\left(\frac{x}{a}\right)$, $\tanh\left(\frac{x}{a}\right)$ and const;

$$
\begin{aligned}
x\tanh\left(\frac{x}{a}\right) &: \qquad c_1 = 0,\\
\tanh\left(\frac{x}{a}\right) &: \qquad p_{30} = 0,\\
const &: \qquad p_{35} = 0.
\end{aligned}
$$

Therefore, eq.(3.2.29) becomes

$$\left.\begin{aligned}
&\xi = \tfrac{c_0}{2}s^2 + c_1 s + c_2, \qquad\qquad A = A(t,x,y,z)\\
&\eta^0 = \big[p_1 y + p_2 z + \tanh\big(\tfrac{x}{a}\big)\{-p_{28}\sin\big(\tfrac{t}{a}\big) + p_{29}\cos\big(\tfrac{t}{a}\big)\} + p_{34}\big]s\\
&\qquad + \big[p_3 y + p_4 z + \tanh\big(\tfrac{x}{a}\big)\{-p_{31}\sin\big(\tfrac{t}{a}\big) + p_{32}\cos\big(\tfrac{t}{a}\big)\} + p_{36}\big],\\
&\eta^1 = \big[p_{28}\cos\big(\tfrac{t}{a}\big) + p_{29}\sin\big(\tfrac{t}{a}\big)\big]s + p_{31}\cos\big(\tfrac{t}{a}\big) + p_{32}\sin\big(\tfrac{t}{a}\big),\\
&\eta^2 = \big[c_{19}z + p_1\cosh^2\big(\tfrac{x}{a}\big)t + p_{20}\big]s + c_{23}z + p_3\cosh^2\big(\tfrac{x}{a}\big)t + p_{22},\\
&\eta^3 = \big[-c_{19}y + p_2\cosh^2\big(\tfrac{x}{a}\big)t + p_{24}\big]s - c_{23}y + p_4\cosh^2\big(\tfrac{x}{a}\big)t + p_{26}.
\end{aligned}\right\} \qquad (3.2.30)$$

Taking into account eq.(3.0.9), we have

$$A(t,x,y,z) = -2x\big[p_{28}\cos\big(\frac{t}{a}\big) + p_{29}\sin\big(\frac{t}{a}\big)\big] + r_1(t,y,z). \qquad (3.2.31)$$

Using eq.(3.0.10), we have

$$r_1(t,y,z) = -2\big[p_1\cosh^2\big(\frac{x}{a}\big)t + c_{19}z + p_{20}\big] + r_2(t,z).$$

Therefore, eq.(3.2.31) becomes

$$A(t,x,y,z) = -2x\big[p_{28}\cos\big(\frac{t}{a}\big) + p_{29}\sin\big(\frac{t}{a}\big)\big] - 2\big[p_1\cosh^2\big(\frac{x}{a}\big)t + c_{19}z + p_{20}\big] + r_2(t,z). \qquad (3.2.32)$$

From eq.(3.0.11),

$$-2\big[p_2\cosh^2\big(\frac{x}{a}\big)t - c_{19}y + p_{24}\big] = -2c_{19}y + r_{2,z}(t,z).$$

Comparing coefficient of y : $c_{19} = 0$.

Therefore, $\qquad r_2(t,z) = -2z\big[p_2\cosh^2\big(\tfrac{x}{a}\big)t + p_{24}\big] + r_3(t).$

$$\begin{aligned}
A(t,x,y,z) = &-2x\big[p_{28}\cos\big(\tfrac{t}{a}\big) + p_{29}\sin\big(\tfrac{t}{a}\big)\big] - 2\big[p_1\cosh^2\big(\tfrac{x}{a}\big)t + c_{19}z + p_{20}\big]\\
&- 2z\big[p_2\cosh^2\big(\tfrac{x}{a}\big)t + p_{24}\big] + r_3(t).
\end{aligned} \qquad (3.2.33)$$

From eq.(3.0.8),

$2\cosh^2\big(\tfrac{x}{a}\big)\big[p_1 y + p_2 z + \tanh\big(\tfrac{x}{a}\big)\{-p_{28}\sin\big(\tfrac{t}{a}\big) + p_{29}\cos\big(\tfrac{t}{a}\big)\} + p_{34}\big] = -2\tfrac{x}{a}\big[-p_{28}\sin\big(\tfrac{t}{a}\big) +$

$p_{29}\cos\left(\frac{t}{a}\right)\big] - 2y\big[p_1\cosh^2\left(\frac{x}{a}\right)\big] - 2z\big[p_2\cosh^2\left(\frac{x}{a}\right)\big] + r_3'(t).$

Comparing the coefficient of x, y, z and $\sinh 2\left(\frac{x}{a}\right)$;

$$y \quad : \qquad p_1 = 0,$$

$$z \quad : \qquad p_2 = 0,$$

$$x: \qquad p_{28}\sin\left(\frac{t}{a}\right) - p_{29}\cos\left(\frac{t}{a}\right) = 0, \tag{3.2.34}$$

$$\sinh 2\left(\frac{x}{a}\right): \qquad -p_{28}\sin\left(\frac{t}{a}\right) + p_{29}\cos\left(\frac{t}{a}\right) = 0. \tag{3.2.35}$$

Solving eqs.(3.2.34) and (3.2.35), we get

$$p_{28} = 0 = p_{29},$$

$$r_3'(t) = 0,$$

$$r_3(t) = p_{37}. \qquad (p_{37} = const.\, of\ integration)$$

Therefore,

$$A(t, x, y, z) = -2p_{20}y - 2p_{24}z + p_{37}. \tag{3.2.36}$$

and thus eq.(3.2.30) reduces to

$$\left.\begin{aligned} &\xi = c_2, \\ &\eta^0 = p_{34}s + p_3y + p_4z + \tanh\left(\tfrac{x}{a}\right)\left[-p_{31}\sin\left(\tfrac{t}{a}\right) + p_{32}\cos\left(\tfrac{t}{a}\right)\right] + p_{36}, \\ &\eta^1 = p_{31}\cos\left(\tfrac{t}{a}\right) + p_{32}\sin\left(\tfrac{t}{a}\right), \\ &\eta^2 = p_{20}s + c_{23}z + p_{22}, \\ &\eta^3 = p_{24}s - c_{23}y + p_{26}, \\ &A = -2p_{20}y - 2p_{24}z + p_{37}. \end{aligned}\right\} \tag{3.2.37}$$

From eqs.(3.0.13) and (3.0.14) we have $p_3 = 0$ and $p_4 = 0$. Further, eq.(3.0.8) gives $p_{34} = 0$.

Therefore, eq.(3.2.37) becomes

$$\left.\begin{aligned}
&\xi = c_2,\\
&\eta^0 = \tanh\left(\tfrac{x}{a}\right)\left[-p_{31}\sin\left(\tfrac{t}{a}\right)+p_{32}\cos\left(\tfrac{t}{a}\right)\right]+p_{36},\\
&\eta^1 = p_{31}\cos\left(\tfrac{t}{a}\right)+p_{32}\sin\left(\tfrac{t}{a}\right),\\
&\eta^2 = p_{20}s+c_{23}z+p_{22},\\
&\eta^3 = p_{24}s-c_{23}y+p_{26},\\
&A = -2p_{20}y-2p_{24}z+p_{37}.
\end{aligned}\right\} \tag{3.2.38}$$

Thus, substituting eq.(3.2.38) in $\mathbf{X}$ gives

$$\begin{aligned}
\mathbf{X} = {} & c_2\tfrac{\partial}{\partial s}+\left(\tanh\left(\tfrac{x}{a}\right)\left[-p_{31}\sin\left(\tfrac{t}{a}\right)+p_{32}\cos\left(\tfrac{t}{a}\right)\right]+p_{36}\right)\tfrac{\partial}{\partial t}\\
& +\left(p_{31}\cos\left(\tfrac{t}{a}\right)+p_{32}\sin\left(\tfrac{t}{a}\right)\right)\tfrac{\partial}{\partial x}+\left(p_{20}s+c_{23}z+p_{22}\right)\tfrac{\partial}{\partial y}\\
& +\left(p_{24}s-c_{23}y+p_{26}\right)\tfrac{\partial}{\partial z}
\end{aligned} \tag{3.2.39}$$

There are 9 arbitrary constants appearing in $\mathbf{X}$. Hence, the metric given by eq.(2.2.36) admits 9 Noether symmetries. Therefore, eq.(3.2.39) leads to the following set of symmetries.

$$\begin{aligned}
\mathbf{X_0} &= \frac{\partial}{\partial t},\\
\mathbf{X_1} &= z\frac{\partial}{\partial y}-y\frac{\partial}{\partial z},\\
\mathbf{X_2} &= \frac{\partial}{\partial z},\\
\mathbf{X_3} &= \frac{\partial}{\partial y},\\
\mathbf{X_4} &= -\tanh\left(\frac{x}{a}\right)\sin\left(\frac{t}{a}\right)\frac{\partial}{\partial t}+\cos\left(\frac{t}{a}\right)\frac{\partial}{\partial x},\\
\mathbf{X_5} &= \tanh\left(\frac{x}{a}\right)\cos\left(\frac{t}{a}\right)\frac{\partial}{\partial t}+\sin\left(\frac{t}{a}\right)\frac{\partial}{\partial x},\\
\mathbf{Y_0} &= \frac{\partial}{\partial s},\\
\mathbf{Y_1} &= s\frac{\partial}{\partial y}; \qquad A_1 = -2y,\\
\mathbf{Y_2} &= s\frac{\partial}{\partial z}; \qquad A_2 = -2z.
\end{aligned}$$

The structure constants of the corresponding Lie algebra are

$$C^2_{13} = C^3_{\hat{0}\hat{1}} = C^2_{\hat{0}\hat{2}} = C^{\hat{2}}_{1\hat{1}} = 1, \qquad C^5_{04} = -\frac{1}{a}, \qquad C^4_{05} = C^0_{45} = \frac{1}{a}, \qquad C^3_{12} = C^{\hat{1}}_{1\hat{2}} = -1.$$

(b) Case of $\nu''(x) + \nu'^2(x) = -k, \quad (k > 0)$

This implies $\nu(x) = \ln\big(\cos(\frac{x}{a})\big)$, (taking $k = 1/a^2$)

Therefore, eq.(3.0.1) gives

$$L = \cos^2\big(\frac{x}{a}\big)\dot{t}^2 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2, \tag{3.2.40}$$

Using eq.(3.2.26) and substitute values into eq.(3.0.14),

$$\cos^2\big(\frac{x}{a}\big)\big[p_2 s + p_4\big] = p_{23}s + p_{25}.$$

Comparing coefficients of s and s^0;

$$s: \qquad p_{23} = p_2\cos^2\big(\frac{x}{a}\big),$$

$$s^0: \qquad p_{25} = p_4\cos^2\big(\frac{x}{a}\big).$$

From eq.(3.0.13),

$$cos^2\big(\frac{x}{a}\big)\big[p_1 s + p_3\big] = p_{19}s + p_{21}.$$

Comparing coefficients of s and s^0;

$$s: \qquad p_{19} = p_1\cos^2\big(\frac{x}{a}\big),$$

$$s^0: \qquad p_{21} = p_1\cos^2\big(\frac{x}{a}\big).$$

Using eq.(3.0.12), we have

$$\cos^2\big(\frac{x}{a}\big)\big[d_{0,x}(t,x)s + d_{5,x}(t,x)\big] = d'_{14}(t) + d'_{15}(t),$$

$$s: \qquad d_{0,x}(t,x) = \sec^2\big(\frac{x}{a}\big)d'_{14}(t),$$

$$d_0(t,x) = a\tan\left(\frac{x}{a}\right)d'_{14}(t) + f_3(t), \tag{3.2.41}$$

$$s^0: \qquad d_{5,x}(t,x) = \sec^2\left(\frac{x}{a}\right)d'_{15}(t),$$

$$d_0(t,x) = a\tan\left(\frac{x}{a}\right)d'_{15}(t) + f_4(t). \tag{3.2.42}$$

Taking into account eqs.(3.0.39) and (3.0.40), we have

$$p_{11} = p_{12} = 0,$$

$$p_{13} = p_{14} = 0.$$

Therefore, eq.(3.2.26) reduces to

$$\left.\begin{aligned}
&\xi = \tfrac{c_0}{2}s^2 + c_1 s + c_2, \qquad A = A(t,x,y,z),\\
&\eta^0 = \left[p_1 y + p_2 z + a\tan\left(\tfrac{x}{a}\right)d'_{14}(t) + f_3(t)\right]s\\
&\qquad + \left[p_3 y + p_4 z + a\tan\left(\tfrac{x}{a}\right)d'_{15}(t) + f_4(t)\right],\\
&\eta^1 = \left[\tfrac{c_0}{2}x + d_{14}(t)\right]s + \tfrac{c_1}{2}x + d_{15}(t),\\
&\eta^2 = \left[\tfrac{c_0}{2}y + c_{19}z + p_1 cos^2\left(\tfrac{x}{a}\right)t + p_{20}\right]s + \tfrac{c_1}{2}y\\
&\qquad + c_{23}z + p_3\cos^2\left(\tfrac{x}{a}\right)t + p_{22},\\
&\eta^3 = \left[\tfrac{c_0}{2}z - c_{19}y + p_2\cos^2\left(\tfrac{x}{a}\right)t + p_{24}\right]s + \tfrac{c_1}{2}z - c_{23}y\\
&\qquad + p_4\cos^2\left(\tfrac{x}{a}\right)t + p_{26}.
\end{aligned}\right\} \tag{3.2.43}$$

From eq.(3.0.37),

$$-\frac{1}{a}\tan\left(\frac{x}{a}\right)\left[d'_{14}(t)s + d'_{15}(t)\right] = -\left[\{a\tan\left(\frac{x}{a}\right)d'''_{14}(t) + f''_3(t)\}s + \{a\tan\left(\frac{x}{a}\right)d'''_{15}(t) + f''_4(t)\}\right].$$

Comparing coefficients of s and s^0;

$s: \qquad \frac{-1}{a}\tan(\frac{x}{a})d'_{14}(t) + a\tan(\frac{x}{a})d'''_{14}(t) = -f''_3(t).$

Comparing coefficients of $\tan(\frac{x}{a})$ and *const.*;

$$\tan\left(\frac{x}{a}\right) \quad : \qquad d_{14}(t) = p_{38} + p_{39}\cosh\left(\frac{t}{a}\right) + p_{40}\sinh\left(\frac{t}{a}\right),$$

$$const. \quad : \qquad f_3(t) = p_{41}t + p_{42},$$

$s^0: \qquad -\frac{1}{a}\tan(\frac{x}{a})d'_{15}(t) + a\tan(\frac{x}{a})d'''_{15}(t) = f''_4(t).$

Comparing coefficient of $\tan(\frac{x}{a})$ and *const.*;

$$\tan\Big(\frac{x}{a}\Big) \quad : \qquad d_{15}(t) = p_{43} + p_{44}\cosh\Big(\frac{t}{a}\Big) + p_{45}\sinh\Big(\frac{t}{a}\Big),$$
$$const. \quad : \qquad f_4(t) = p_{46}t + p_{47}.$$

where p_i $(i = 38, ..., 47)$ are the constants of integration. Therefore, eq.(3.2.43) becomes

$$\left.\begin{aligned}
&\xi = \tfrac{c_0}{2}s^2 + c_1 s + c_2, \qquad A = A(t,x,y,z)\\
&\eta^0 = \big[p_1 y + p_2 z + \tan\big(\tfrac{x}{a}\big)\{p_{39}\sinh(\tfrac{t}{a}) + p_{40}\cosh(\tfrac{t}{a})\}\\
&\qquad + p_{41}t + p_{42}\big]s + \big[p_3 y + p_4 z + \tan\big(\tfrac{x}{a}\big)\{p_{44}\sinh(\tfrac{t}{a})\\
&\qquad + p_{45}\cosh(\tfrac{t}{a})\} + p_{46}t + p_{47}\big],\\
&\eta^1 = \big[\tfrac{c_0}{2}x + p_{38} + p_{39}\cosh(\tfrac{t}{a}) + p_{40}\sinh(\tfrac{t}{a})\big]s\\
&\qquad + \tfrac{c_1}{2}x + p_{43} + p_{44}\cosh(\tfrac{t}{a}) + p_{45}\sinh(\tfrac{t}{a}),\\
&\eta^2 = \big[\tfrac{c_0}{2}y + c_{19}z + p_1\cos^2\big(\tfrac{x}{a}\big)t + p_{20}\big]s + \tfrac{c_1}{2}y\\
&\qquad + c_{23}z + p_3\cos^2\big(\tfrac{x}{a}\big)t + p_{22},\\
&\eta^3 = \big[\tfrac{c_0}{2}z - c_{19}y + p_2\cos^2\big(\tfrac{x}{a}\big)t + p_{24}\big]s + \tfrac{c_1}{2}z\\
&\qquad - c_{23}y + p_4\cos^2\big(\tfrac{x}{a}\big)t + p_{26}.
\end{aligned}\right\} \tag{3.2.44}$$

From eq.(3.0.21),

$\frac{-2}{a}\tan\big(\frac{x}{a}\big)\big[\{c_0\frac{x}{2} + p_{38} + p_{39}\cosh\big(\frac{t}{a}\big) + p_{40}\sinh\big(\frac{t}{a}\big)\}s + \{c_1\frac{x}{2} + p_{43} + p_{44}\cosh\big(\frac{t}{a}\big) + p_{45}\sinh\big(\frac{t}{a}\big)\}\big] + 2\big[\{\frac{1}{a}\tan\big(\frac{x}{a}\big)\big(p_{40}\sinh\big(\frac{t}{a}\big) + p_{39}\cosh\big(\frac{t}{a}\big)\big) + p_{41}\}s + \frac{1}{a}\tan\big(\frac{x}{a}\big)\{p_{44}\cosh\big(\frac{t}{a}\big) + p_{45}\sinh\big(\frac{t}{a}\big)\} + p_{46}\big] - (c_0 s + c_1) = 0.$

Comparing coefficients of s and s^0;

$$s: \qquad \frac{-2}{a}\tan\Big(\frac{x}{a}\Big)\Big[c_0\frac{x}{2} + p_{38} + p_{39}\cosh\Big(\frac{t}{a}\Big) + p_{40}\sinh\Big(\frac{t}{a}\Big)\Big] + 2p_{41}$$
$$-c_0 + \frac{2}{a}\tan\Big(\frac{x}{a}\Big)\Big[p_{39}\cosh\Big(\frac{t}{a}\Big) + p_{40}\sinh\Big(\frac{t}{a}\Big)\Big] = 0,$$

Comparing coefficients of $x\tan\big(\frac{x}{a}\big), \tan\big(\frac{x}{a}\big)$ and *const.*;

$$x\tan\Big(\frac{x}{a}\Big): \qquad c_0 = 0,$$

$$\tan\left(\frac{x}{a}\right): \qquad p_{38}=0,$$

$$const.: \qquad p_{41}=0.$$

$$s^0: \qquad \frac{-2}{a}\tan\left(\frac{x}{a}\right)\left[c_1\frac{x}{2}+p_{43}+p_{44}\cosh\left(\frac{t}{a}\right)+p_{45}\sinh\left(\frac{t}{a}\right)\right] \\ +\frac{2}{a}\tan\left(\frac{x}{a}\right)\left[p_{44}\cosh\left(\frac{t}{a}\right)+p_{45}\sinh\left(\frac{t}{a}\right)\right]+2p_{46}-c_1=0.$$

Comparing coefficients of $x\tan\left(\frac{x}{a}\right)$, $\tan\left(\frac{x}{a}\right)$ and $const.$;

$$x\tan\left(\frac{x}{a}\right) \; : \qquad c_1=0,$$
$$\tan\left(\frac{x}{a}\right) \; : \qquad p_{43}=0,$$
$$const. \; : \qquad p_{46}=0.$$

Therefore, eq.(3.2.44) becomes

$$\left.\begin{aligned} &\xi=c_2, \qquad A=A(t,x,y,z),\\ &\eta^0=\left[p_1y+p_2z+\tan\left(\tfrac{x}{a}\right)\{p_{39}\sinh(\tfrac{t}{a})+p_{40}\cosh(\tfrac{t}{a})\}+p_{42}\right]s\\ &\qquad+\left[p_3y+p_4z+\tan\left(\tfrac{x}{a}\right)\{p_{44}\sinh(\tfrac{t}{a})+p_{45}\cosh(\tfrac{t}{a})\}+p_{47}\right],\\ &\eta^1=\left[p_{39}\cosh(\tfrac{t}{a})+p_{40}\sinh(\tfrac{t}{a})\right]s+p_{44}\cosh(\tfrac{t}{a})+p_{45}\sinh(\tfrac{t}{a}),\\ &\eta^2=\left[p_1\cos^2\left(\tfrac{x}{a}\right)t+c_{19}z+p_{20}\right]s+c_{23}z+p_3\cos^2\left(\tfrac{x}{a}\right)t+p_{22},\\ &\eta^3=\left[p_2\cos^2\left(\tfrac{x}{a}\right)t-c_{19}y+p_{24}\right]s-c_{23}y+p_4\cos^2\left(\tfrac{x}{a}\right)t+p_{26}. \end{aligned}\right\} \tag{3.2.45}$$

From eq.(3.0.9), we have

$$A(t,x,y,z)=-2[p_{39}\cosh\left(\frac{t}{a}\right)+p_{40}\sinh\left(\frac{t}{a}\right)]+w_1(t,y,z). \tag{3.2.46}$$

Using eq.(3.0.10), we get

$$w_1(t,y,z)=-2y[p_1\cos^2\left(\frac{x}{a}\right)t+c_{19}z+p_{20}]+w_2(t,z).,$$

Therefore,

$$A(t,x,y,z)=-2[p_{39}\cosh\left(\frac{t}{a}\right)+p_{40}\sinh\left(\frac{t}{a}\right)]-2y[p_1\cos^2\left(\frac{x}{a}\right)t+c_{19}z+p_{20}]+w_2(t,z). \tag{3.2.47}$$

From eq.(3.0.11),

$$-2[p_2 \cos^2\big(\frac{x}{a}\big)t - c_{19}y + p_{24}] = -2c_{19}y + w_{2,z}(t,z).$$

Comparing coefficients of y;

$$y: \qquad c_{19} = 0,$$

and $$w_2(t,z) = -2z[p_2 \cos^2\big(\tfrac{x}{a}\big)t + p_{24}] + w_3(t),$$

Hence,

$$A(t,x,y,z) = -2[p_{39}\cosh\big(\tfrac{t}{a}\big) + p_{40}\sinh\big(\tfrac{t}{a}\big)] - 2y[p_1\cos^2\big(\tfrac{x}{a}\big)t + p_{20}] \\ - 2z[p_2\cos^2\big(\tfrac{x}{a}\big)t + p_{24}] + w_3(t). \qquad (3.2.48)$$

From eq.(3.0.8),

$2\cos^2\big(\frac{x}{a}\big)[p_1y + p_2z + \tan\big(\frac{x}{a}\big)\{p_{39}\sinh\big(\frac{t}{a}\big) + p_{40}\cosh\big(\frac{t}{a}\big)\} + p_{42}] = w_3'(t) - \frac{2x}{a}[p_{39}\sinh\big(\frac{t}{a}\big) + p_{40}\cosh\big(\frac{t}{a}\big)] - 2yp_1\cos^2\big(\frac{x}{a}\big) - 2zp_2\cos^2\big(\frac{x}{a}\big).$

Comparing coefficients of x, y, z and $\sin 2\big(\frac{x}{a}\big)$;

$$y: \qquad p_1 = 0,$$

$$z: \qquad p_2 = 0,$$

$$x: \qquad p_{39}\sinh\big(\frac{t}{a}\big) + p_{40}\cosh\big(\frac{t}{a}\big) = 0, \qquad (3.2.49)$$

$$\sin 2\big(\frac{x}{a}\big): \qquad p_{39}\sinh\big(\frac{t}{a}\big) + p_{40}\cosh\big(\frac{t}{a}\big) = 0. \qquad (3.2.50)$$

By solving eqs.(3.2.49) and (3.2.50) simultaneously, we obtain

$$p_{39} = p_{40} = 0.$$

Therefore, we obtain $w_3(t) = p_{48}$. $\quad (p_{48} = const.\, of\ integration)$

Thus, eq.(3.2.48) reduces to

$$A(t,x,y,z) = -2p_{20}y - 2p_{24}z + p_{48}. \qquad (3.2.51)$$

Eqs.(3.0.13), (3.0.14), and (3.0.8) together implies that

$$p_3 = p_4 = p_{42} = 0.$$

Therefore, eq.(3.2.45) reduces to

$$\left.\begin{aligned}
&\xi = c_2,\\
&\eta^0 = \tan\left(\tfrac{x}{a}\right)\left[p_{44}\sinh(\tfrac{t}{a}) + p_{45}\cosh(\tfrac{t}{a})\right] + p_{47},\\
&\eta^1 = p_{44}\cosh(\tfrac{t}{a}) + p_{45}\sinh(\tfrac{t}{a}),\\
&\eta^2 = p_{20}s + c_{23}z + p_{22},\\
&\eta^3 = p_{24}s - c_{23}y + p_{26},\\
&A = -2p_{20}y - 2p_{24}z + p_{48}.
\end{aligned}\right\} \tag{3.2.52}$$

Thus, from eq.(3.2.52) the generator $\mathbf{X}$ leads to

$$\begin{aligned}
\mathbf{X} = {} & c_2\tfrac{\partial}{\partial s} + \left(\tan\left(\tfrac{x}{a}\right)\left[p_{44}\sinh(\tfrac{t}{a}) + p_{45}\cosh(\tfrac{t}{a})\right] + p_{47}\right)\tfrac{\partial}{\partial t}\\
& + \left(p_{44}\cosh(\tfrac{t}{a}) + p_{45}\sinh(\tfrac{t}{a})\right)\tfrac{\partial}{\partial x} + \left(p_{20}s + c_{23}z + p_{22}\right)\tfrac{\partial}{\partial y}\\
& + \left(p_{24}s - c_{23}y + p_{26}\right)\tfrac{\partial}{\partial t}
\end{aligned} \tag{3.2.53}$$

Hence, the metric given by eq.(2.2.42) admits 9 Noether symmetries. Hence, eq.(3.2.53) gives the following symmetries:

$$\begin{aligned}
\mathbf{X_0} &= \frac{\partial}{\partial t},\\
\mathbf{X_1} &= z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z},\\
\mathbf{X_2} &= \frac{\partial}{\partial z},\\
\mathbf{X_3} &= \frac{\partial}{\partial y},\\
\mathbf{X_4} &= \tan\left(\frac{x}{a}\right)\sinh\left(\frac{t}{a}\right)\frac{\partial}{\partial t} + \cosh\left(\frac{t}{a}\right)\frac{\partial}{\partial x},\\
\mathbf{X_5} &= \tan\left(\frac{x}{a}\right)\cosh\left(\frac{t}{a}\right)\frac{\partial}{\partial t} + \sinh\left(\frac{t}{a}\right)\frac{\partial}{\partial x},\\
\mathbf{Y_0} &= \frac{\partial}{\partial s},\\
\mathbf{Y_1} &= s\frac{\partial}{\partial y}; \qquad A_1 = -2y,\\
\mathbf{Y_2} &= s\frac{\partial}{\partial z}; \qquad A_2 = -2z.
\end{aligned}$$

The structure constants of the corresponding Lie Algebra for this metric are

$$C^5_{04} = C^4_{05} = C^0_{45} = \frac{1}{a}, \qquad C^3_{12} = C^{\hat{1}}_{1\hat{2}} = -1, \qquad C^2_{13} = C^3_{\hat{0}\hat{1}} = C^2_{\hat{0}\hat{2}} = C^{\hat{2}}_{1\hat{1}} = 1.$$

(c) Case of $\nu''(x) + \nu'^2(x) = f(x)$

In this case,

$$\eta^0_{,y} = \eta^0_{,z} = 0.$$

By eq.(3.2.26), we get

$$p_1 = p_2 = p_3 = p_4 = 0.$$

From eqs.(3.0.13) and (3.0.14), we get

$$p_{19}s + p_{21} = 0,$$

$$p_{23}s + p_{25} = 0.$$

Comparing coefficients of s and s^0;

$$s: \qquad p_{19} = p_{21} = 0,$$

$$s^0: \qquad p_{23} = p_{25} = 0.$$

Therefore, eq.(3.2.26) reduces to

$$\left.\begin{array}{l} \xi = \frac{c_0 s^2}{2} + c_1 s + c_2, \qquad A = A(t,x,y,z), \\ \eta^0 = d_0(t,x)s + d_5(t,x), \\ \eta^1 = [\frac{c_0 x}{2} + p_{11}y + p_{13}z + d_{14}(t)]s + \frac{c_1 x}{2} + p_{12}y + p_{14}z + d_{15}(t), \\ \eta^2 = [\frac{c_0 y}{2} - p_{11}x + c_{19}z + p_{20}]s + \frac{c_1 y}{2} - p_{12}x + c_{23}z + p_{22}, \\ \eta^3 = [\frac{c_0 z}{2} - p_{13}x - c_{19}y + p_{24}]s + \frac{c_1 z}{2} - p_{14}x - c_{23}y + p_{26}. \end{array}\right\} \qquad (3.2.54)$$

Eqs.(3.0.39) and (3.0.40) implies that

$$p_{11} = p_{12} = 0,$$

$$p_{13} = p_{14} = 0.$$

Thus, eq.(3.2.54) reduces to

$$\left.\begin{array}{l} \xi = \frac{c_0 s^2}{2} + c_1 s + c_2, \qquad A = A(t,x,y,z), \\ \eta^0 = d_0(t,x)s + d_5(t,x), \\ \eta^1 = [\frac{c_0 x}{2} + d_{14}(t)]s + \frac{c_1 x}{2} + d_{15}(t), \\ \eta^2 = [\frac{c_0 y}{2} + c_{19}z + p_{20}]s + \frac{c_1 y}{2} + c_{23}z + p_{22}, \\ \eta^3 = [\frac{c_0 z}{2} - c_{19}y + p_{24}]s + \frac{c_1 z}{2} - c_{23}y + p_{26}. \end{array}\right\} \tag{3.2.55}$$

From eq.(3.0.12),

$$e^{\nu(x)}[d_{0,x}(t,x)s + d_{5,x}(t,x)] = d'_{14}(t)s + d'_{15}(t).$$

Comparing coefficients of s and s^0;

$$s: \qquad e^{\nu(x)}d_{0,x}(t,x) = d'_{14}(t), \tag{3.2.56}$$

$$s^0: \qquad e^{\nu(x)}d_{5,x}(t,x) = d'_{15}(t). \tag{3.2.57}$$

Taking into account eq.(3.0.37), we have

$$\nu(x)[d'_{14}(t)s + d'_{15}(t)] = -[d_{0,tt}(t,x)s + d_{5,t}(t,x)],$$

Comparing coefficients of s and s^0;

$$s: \qquad \nu(x)d'_{14}(t) = -d_{0,tt}(t,x), \tag{3.2.58}$$

$$s^0: \qquad \nu(x)d'_{15}(t) = -d_{5,tt}(t,x). \tag{3.2.59}$$

Now, differentiate eqs.(3.2.56), (3.2.57) w.r.t. t twice and eqs.(3.2.58), (3.2.59) w.r.t. x then subtract the resulting equations respectively, we get

$$d'''_{14}(t) - \nu''(x)e^{2\nu(x)}d'_{14}(t) = 0, \tag{3.2.60}$$

$$d'''_{15}(t) - \nu''(x)e^{2\nu(x)}d'_{15}(t) = 0. \tag{3.2.61}$$

The Trivial Solution

For trivial solution, in eqs.(3.2.60) and (3.2.61) we have

$$d'_{14}(t) = 0 \quad \Rightarrow d_{14}(t) = c_{50},$$

and

$$d'_{15}(t) = 0 \quad \Rightarrow d_{15}(t) = c_{51}.$$

where c_{50} and c_{51} are constants of integration. Taking into account eqs.(3.2.56) and (3.2.57), we get

$$d_0(t,x) = 0 \quad \Rightarrow d_0(t,x) = \beta_1(t),$$

and

$$d_5(t,x) = 0 \quad \Rightarrow d_5(t,x) = \beta_2(t).$$

From eqs.(3.2.58) and (3.2.59), one obtain

$$\beta''_1(t) = 0 \quad \Rightarrow \beta_1(t) = c_{52}t + c_{53},$$

and

$$\beta''_2(t) = 0 \quad \Rightarrow \beta_2(t) = c_{54}t + c_{55}.$$

where c_{52}, c_{53}, c_{54}, and c_{55} are constants of integration. Hence, eq.(3.2.55) reduces to

$$\left.\begin{array}{l} \xi = \frac{c_0 s^2}{2} + c_1 s + c_2, \qquad A = A(t,x,y,z), \\ \eta^0 = [c_{52}t + c_{53}]s + c_{54}t + c_{55}, \\ \eta^1 = [\frac{c_0 x}{2} + c_{50}]s + \frac{c_1 x}{2} + c_{51}, \\ \eta^2 = [\frac{c_0 y}{2} + c_{19}z + p_{20}]s + \frac{c_1 y}{2} + c_{23}z + p_{22}, \\ \eta^3 = [\frac{c_0 z}{2} - c_{19}y + p_{24}]s + \frac{c_1 z}{2} - c_{23}y + p_{26}. \end{array}\right\} \tag{3.2.62}$$

From eq.(3.0.10), we have

$$A(t,x,y,z) = -(\frac{c_0}{2})y^2 - 2c_{19}yz - 2p_{20}y + u_1(t,x,z). \tag{3.2.63}$$

Using eq.(3.0.11), we get

$$-2((\frac{c_0}{2})z - c_{19}y + p_{24}) = -2c_{19}y + u_{1,z}(t,x,z),$$

Comparing coefficient of y; $\quad c_{19} = 0,$

and $\quad u_1(t,x,z) = -(\frac{c_0}{2})z^2 - 2p_{24}z + u_2(t,x).$

Therefore, we have

$$A(t,x,y,z) = -\frac{c_0}{2}(y^2 + z^2) - 2p_{20}y - 2p_{24}z + u_2(t,x). \tag{3.2.64}$$

Also by eq.(3.0.9), we get

$$u_2(t,x) = -\frac{c_0}{2}x^2 - 2c_{50}x + u_3(t).$$

Thus,

$$A(t,x,y,z) = -\frac{c_0}{2}(x^2 + y^2 + z^2) - 2c_{50}x - 2p_{20}y - 2p_{24}z + u_3(t). \tag{3.2.65}$$

From eq.(3.0.8), one obtain

$$u_3(t) = e^{2\nu(x)}c_{52}t^2 + 2e^{2\nu(x)}c_{53}t + c_{56}.$$

Hence,

$$A(t,x,y,z) = -\frac{c_0}{2}(x^2 + y^2 + z^2) - 2c_{50}x - 2p_{20}y - 2p_{24}z + e^{2\nu(x)}(c_{52}t^2 + 2c_{53}t) + c_{56}. \tag{3.2.66}$$

Taking into account eq.(3.0.9), we have

$$e^{2\nu(x)}(c_{52}t^2 + 2c_{53}t) = 0.$$

Comparing coefficients of t^2 and t;

$$t^2 \quad : \quad c_{52} = 0,$$

$$t \quad : \quad c_{53} = 0.$$

Thus, eq.(3.2.62) becomes

$$\left.\begin{aligned}
&\xi = \tfrac{c_0 s^2}{2} + c_1 s + c_2,\\
&\eta^0 = c_{54} t + c_{55},\\
&\eta^1 = [\tfrac{c_0 x}{2} + c_{50}]s + \tfrac{c_1 x}{2} + c_{51},\\
&\eta^2 = [\tfrac{c_0 y}{2} + c_{19} z + p_{20}]s + \tfrac{c_1 y}{2} + c_{23} z + p_{22},\\
&\eta^3 = [\tfrac{c_0 z}{2} - c_{19} y + p_{24}]s + \tfrac{c_1 z}{2} - c_{23} y + p_{26},\\
&A = -\tfrac{c_0}{2}(x^2 + y^2 + z^2) - 2c_{50} x - 2p_{20} y - 2p_{24} z + c_{56}.
\end{aligned}\right\} \qquad (3.2.67)$$

From eq.(3.0.21), we have

$$2\nu'(x)\big([\frac{c_0 x}{2} + c_{50}]s + \frac{c_1 x}{2} + c_{51}\big) + 2c_{54} - (c_0 s + c_1),$$

Comparing coefficients of s and s^0;

$$s: \qquad \big(x\nu'(x) - 1\big)c_0 + 2\nu'(x)c_{50} = 0, \qquad (3.2.68)$$

$$s^0: \qquad \big(x\nu'(x) - 1\big)c_1 + 2\nu'(x)c_{51} + 2c_{54} = 0. \qquad (3.2.69)$$

Since $\nu'(x) \neq 0$, we can write eq.(3.2.68) as

$$\left(x - \frac{1}{\nu'(x)}\right)c_0 + 2c_{50} = 0,$$

When $\nu'(x) = \frac{1}{x}$, we have $c_{50} = 0$ and $\nu'(x) = \frac{1}{x}$ implies $\nu(x) = \ln\left(\frac{x}{a}\right)$, which is a Minkowski spacetime.

When $\nu'(x) \neq \frac{1}{x}$, c_0 and c_{50} are linearly independent, therefore, eq.(3.2.68) gives $c_0 = 0$ and $c_{50} = 0$. Similarly in eq.(3.2.69), c_1, c_{51}, and c_{54} are linearly independent. Thus, we have $c_1 = 0 = c_{51} = c_{54}$. Hence, eq.(3.2.67) reduces to

$$\left.\begin{aligned}
&\xi = c_2,\\
&\eta^0 = c_{55},\\
&\eta^1 = 0,\\
&\eta^2 = p_{20} s + c_{23} z + p_{22},\\
&\eta^3 = p_{24} s - c_{23} y + p_{26},\\
&A = -2p_{20} y - 2p_{24} z + c_{56}.
\end{aligned}\right\} \qquad (3.2.70)$$

After substituting eq.(3.2.70) into generator $\mathbf{X}$, one gets

$$\mathbf{X} = c_2\frac{\partial}{\partial s} + c_{63}\frac{\partial}{\partial t} + \left(p_{20}s + c_{23}z + p_{22}\right)\frac{\partial}{\partial y} + \left(p_{24}s - c_{23}y + p_{26}\right)\frac{\partial}{\partial z}. \tag{3.2.71}$$

The metric given by eq.(2.2.35) admits seven Noether symmetries. Thus, eq.(3.2.71) leads to the following Noether symmetries.

$$\left.\begin{aligned} &\mathbf{X_0} = \frac{\partial}{\partial t}, \\ &\mathbf{X_1} = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, \\ &\mathbf{X_2} = \frac{\partial}{\partial z}, \\ &\mathbf{X_3} = \frac{\partial}{\partial y}, \\ &\mathbf{Y_0} = \frac{\partial}{\partial s}, \\ &\mathbf{Y_1} = s\frac{\partial}{\partial y}; \qquad A_1 = -2y, \\ &\mathbf{Y_2} = s\frac{\partial}{\partial z}; \qquad A_2 = -2z. \end{aligned}\right\} \tag{3.2.72}$$

The structure constants in this case are

$$C^2_{13} = C^{\hat{2}}_{1\hat{1}} = C^3_{\hat{0}\hat{1}} = C^2_{\hat{0}\hat{2}} = 1, \qquad C^3_{12} = C^{\hat{3}}_{1\hat{2}} = -1.$$

Now, let us consider another case, when $\nu'(x) = \frac{c}{x}$ (where $c \neq 0, 1$) then

$$\nu(x) = c\,\ln\left(\frac{x}{a}\right).$$

Therefore, the Lagrangian given by eq.(3.0.1) becomes

$$L = \left(\frac{x}{a}\right)^{2c}\dot{t}^2 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2. \tag{3.2.73}$$

From eq.(3.0.21),

$$cc_1 + 2\frac{c}{x}c_{51} + 2c_{54} - c_1 = 0,$$

Comparing coefficients of $\frac{1}{x}$ and const;

$$\begin{aligned} \frac{1}{x} \quad &: \qquad c_{51} = 0, \\ \text{const} \quad &: \qquad c_1(c-1) + 2c_{54} = 0, \end{aligned}$$

or $$c_{54} = (1-c)\tfrac{c_1}{2}.$$

Therefore, eq.(3.2.67) reduces to

$$\left.\begin{aligned}
&\xi = c_1 s + c_2,\\
&\eta^0 = (1-c)\tfrac{c_1}{2}t + c_{55},\\
&\eta^1 = c_1\tfrac{x}{2},\\
&\eta^2 = p_{20}s + c_1\tfrac{y}{2} + c_{23}z + p_{22},\\
&\eta^3 = p_{24}s + c_1\tfrac{z}{2} - c_{23}y + p_{26},\\
&V = -2p_{20}y - 2p_{24}z + c_{56}.
\end{aligned}\right\} \tag{3.2.74}$$

Thus, eq.(3.2.74) gives the generator **X** as

$$\begin{aligned}\mathbf{X} = \big(c_1 s + c_2\big)\tfrac{\partial}{\partial s} + \big((1-c)\tfrac{c_1}{2}t + c_{63}\big)\tfrac{\partial}{\partial t} + +c_1\tfrac{x}{2}\tfrac{\partial}{\partial x} + \big(p_{20}s\\
+ c_1\tfrac{y}{2} + c_{23}z + p_{22}\big)\tfrac{\partial}{\partial y} + \big(p_{24}s + c_1\tfrac{z}{2} - c_{23}y + p_{26}\big)\tfrac{\partial}{\partial z}.\end{aligned} \tag{3.2.75}$$

Thus, the metric given by

$$ds^2 = \left(\frac{x}{a}\right)^{2c} dt^2 - dx^2 - dy^2 - dz^2, \tag{3.2.76}$$

admits 8 Noether symmetries. Hence, the generators are as follows:

$$\mathbf{X_0} = \frac{\partial}{\partial t}, \tag{3.2.77}$$

$$\mathbf{X_1} = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial t}, \tag{3.2.78}$$

$$\mathbf{X_2} = \frac{\partial}{\partial z}, \tag{3.2.79}$$

$$\mathbf{X_3} = \frac{\partial}{\partial y}, \tag{3.2.80}$$

$$\mathbf{Y_0} = \frac{\partial}{\partial s}, \tag{3.2.81}$$

$$\mathbf{Y_1} = s\frac{\partial}{\partial y}; \qquad A_1 = -2y, \tag{3.2.82}$$

$$\mathbf{Y_2} = s\frac{\partial}{\partial z}; \qquad A_2 = -2z, \tag{3.2.83}$$

$$\mathbf{Y_3} = s\frac{\partial}{\partial s} + (1-c)\frac{t}{2}\frac{\partial}{\partial t} + \frac{x}{2}\frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial y} + \frac{z}{2}\frac{\partial}{\partial z}. \tag{3.2.84}$$

The structure constants of the corresponding Lie algebra for this metric are

$$C^2_{13} = C^{\hat{2}}_{1\hat{1}} = C^{\hat{1}}_{1\hat{2}} = C^3_{\hat{0}\hat{1}} = C^2_{\hat{0}\hat{2}} = C^{\hat{0}}_{\hat{0}\hat{3}} = 1, \qquad C^2_{2\hat{3}} = C^3_{3\hat{3}} = \frac{1}{2}, \qquad C^3_{12} = -1,$$

$$C^{\hat{1}}_{\hat{1}\hat{3}} = C^{\hat{2}}_{\hat{2}\hat{3}} = -\frac{1}{2}, \qquad C^0_{0\hat{3}} = \frac{1-c}{2}.$$

The Non-Trivial Solution

For non-trivial solution, we have $d'_{14}(t) \neq 0$ and $d'_{15}(t) \neq 0$. Thus, we can rewrite eqs.(3.2.60) and (3.2.61) as

$$\frac{d'''_{14}(t)}{d'_{14}(t)} = \nu''(x)e^{2\nu(x)} = C, \tag{3.2.85}$$

$$\frac{d'''_{15}(t)}{d'_{15}(t)} = \nu''(x)e^{2\nu(x)} = C. \tag{3.2.86}$$

Here, we have two possibilities:

(1). $C \neq 0$,

(2). $C = 0$.

(1). $\mathbf{C \neq 0}$

Therefore, eq.(3.2.85) leads to

$$\frac{d'''_{14}(t)}{d'_{14}(t)} = C\,(= \frac{1}{a^2}),$$

and

$$\frac{d'''_{15}(t)}{d'_{15}(t)} = C\,(= \frac{1}{a^2}).$$

This implies that

$$d_{14}(t) = p_{30} + p_{31}\cosh(\frac{t}{a}) + p_{32}\sinh(\frac{t}{a}), \tag{3.2.87}$$

$$d_{15}(t) = p_{33} + p_{34}\cosh(\frac{t}{a}) + p_{35}\sinh(\frac{t}{a}). \tag{3.2.88}$$

and
$$\nu''(x)e^{2\nu(x)} = C.$$

After simplifying the above equation, we have

$$\ln \nu''(x) + 2\nu(x) = lnC.$$

Further, differentiating w.r.t. x, we obtain

$$\frac{\nu'''(x)}{\nu''(x)} + 2\nu'(x) = 0,$$

$$\Rightarrow [\nu''(x) + \nu'^2(x)]' = 0,$$

$$\Rightarrow [\nu''(x) + \nu'^2(x)] = constant.$$

Hence, we come to the conclusion that the possible values for $\nu(x)$ in this equation can only be

(i). $\nu(x) = \ln(x/a)$,

(ii). $\nu(x) = \ln[\cosh(x/a)]$,

and, (iii). $\nu(x) = \ln[\cos(x/a)]$.

All the above mentioned results have already been discussed in detail before.

(2). $\mathbf{C = 0}$

Thus, eq.(3.2.85) gives $\nu''(x) = 0$, which leads to $\nu(x) = \frac{x}{a} + b$ (where b is a constant of integration and can be absorbed in the definition of x). Therefore, eq.(3.0.1) becomes

$$L = e^{2x/a}\dot{t}^2 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2. \tag{3.2.89}$$

From eq.(3.2.85), we have

$$d'''_{14}(t) = 0 \qquad \Rightarrow d_{14}(t) = p_{36}\frac{t^2}{2} + p_{37}t + p_{38},$$

$$d'''_{15}(t) = 0 \qquad \Rightarrow d_{15}(t) = p_{39}\frac{t^2}{2} + p_{40}t + p_{41}.$$

where p_i $(i = 36, ..., 41)$ are the constants of integration. Therefore, eq.(3.2.62) reduces to

$$\left.\begin{array}{l} \xi = \frac{c_0 s^2}{2} + c_1 s + c_2, \qquad A = A(t,x,y,z), \\ \eta^0 = d_0(t,x)s + d_5(t,x), \\ \eta^1 = [\frac{c_0 x}{2} + p_{36}\frac{t^2}{2} + p_{37}t + p_{38}]s + \frac{c_1 x}{2} + p_{39}\frac{t^2}{2} + p_{40}t + p_{41}, \\ \eta^2 = [\frac{c_0 y}{2} + c_{19}z + p_{20}]s + \frac{c_1 y}{2} + c_{23}z + p_{22}, \\ \eta^3 = [\frac{c_0 z}{2} - c_{19}y + p_{24}]s + \frac{c_1 z}{2} - c_{23}y + p_{26}. \end{array}\right\} \tag{3.2.90}$$

Taking into account eq.(3.0.42), we get

$$d_0(t,x) = \alpha_1(t) + \alpha_2(t)e^{-2x/a}, \tag{3.2.91}$$

$$d_5(t,x) = \alpha_3(t) + \alpha_4(t)e^{-2x/a}. \tag{3.2.92}$$

Now, using eq.(3.0.38) this gives

$$d_{0,tx}(t,x) = -\frac{c_0}{2a}, \tag{3.2.93}$$

$$d_{5,tx}(t,x) = -\frac{c_1}{2a}. \tag{3.2.94}$$

Solving eqs.(3.2.91)-(3.2.94) simultaneously, we have

$$\alpha_2(t) = \frac{1}{4}c_0e^{2x/a}t + p_{42}.$$

and

$$\alpha_4(t) = \frac{1}{4}c_1e^{2x/a}t + c_{43}.$$

where p_{42} and p_{43} are the constants of integration. Therefore, eq.(3.2.90) reduces to

$$\left.\begin{aligned}
&\xi = \tfrac{c_0s^2}{2} + c_1s + c_2, \qquad A = A(t,x,y,z),\\
&\eta^0 = \left[\alpha_1(t) + \tfrac{1}{4}c_0t + p_{42}e^{-2x/a}\right]s + \alpha_3(t) + \tfrac{1}{4}c_1t + p_{43}e^{-2x/a},\\
&\eta^1 = [\tfrac{c_0x}{2} + p_{36}\tfrac{t^2}{2} + p_{37}t + p_{38}]s + \tfrac{c_1x}{2} + p_{39}\tfrac{t^2}{2} + p_{40}t + p_{41},\\
&\eta^2 = [\tfrac{c_0y}{2} + c_{19}z + p_{20}]s + \tfrac{c_1y}{2} + c_{23}z + p_{22},\\
&\eta^3 = [\tfrac{c_0z}{2} - c_{19}y + p_{24}]s + \tfrac{c_1z}{2} - c_{23}y + p_{26}.
\end{aligned}\right\} \tag{3.2.95}$$

From eq.(3.0.12), we have

$$e^{2x/a}\big(-\frac{2}{a}p_{42}e^{-2x/a}s - \frac{2}{a}p_{43}e^{-2x/a}\big) = (p_{36}t + p_{37})s + p_{39}t + p_{40},$$

Comparing coefficients of s and s^0;

$s:\ -\frac{2}{a}p_{42} = p_{36}t + p_{37}$,

Comparing coefficients of t and t^0;

$$\begin{aligned}
t \quad &: \qquad p_{36} = 0,\\
t^0 \quad &: \qquad p_{37} = -\frac{2}{a}p_{42}.
\end{aligned}$$

$s^0: \ -\frac{2}{a}p_{43} = p_{39}t + p_{40}$,

Comparing coefficients of t and t^0;

$$\begin{aligned} t &: \qquad p_{39} = 0, \\ t^0 &: \qquad p_{40} = -\frac{2}{a}p_{43}. \end{aligned}$$

Taking into account eq.(3.0.38), one obtain $c_0 = 0$ and $c_1 = 0$. Also, eqs.(3.2.58) and (3.2.59) leads to

$$\begin{aligned} \alpha_1(t) &= \frac{1}{a^2}p_{42}t^2 + p_{44}t + p_{45}, \\ \alpha_3(t) &= \frac{1}{a^2}p_{43}t^2 + p_{46}t + p_{47}. \end{aligned}$$

Thus, eq.(f16*) reduces to

$$\left.\begin{array}{l} \xi = c_2, \qquad\qquad A = A(t,x,y,z), \\ \eta^0 = \left[\frac{1}{a^2}p_{42}t^2 + p_{44}t + p_{45} + p_{42}e^{-2x/a}\right]s \\ +\frac{1}{a^2}p_{43}t^2 + p_{46}t + p_{47} + p_{43}e^{-2x/a}, \\ \eta^1 = [-\frac{2}{a}p_{42}t + p_{38}]s - \frac{2}{a}p_{43}t + p_{41}, \\ \eta^2 = [c_{19}z + p_{20}]s + c_{23}z + p_{22}, \\ \eta^3 = [-c_{19}y + p_{24}]s - c_{23}y + p_{26}. \end{array}\right\} \tag{3.2.96}$$

From eq.(3.0.10), we have

$$A(t,x,y,z) = -2c_{19}yz - 2p_{20}y + w_1(t,x,z). \tag{3.2.97}$$

Using eq.(3.0.11), we get

$$-2(-c_{19}y + p_{24}) = -2c_{19}y + w_{1,z}(t,x,z),$$

Comparing coefficient of y; $\quad c_{19} = 0$,

and $\qquad w_1(t,x,z) = -p_{24}z + w_2(t,x)$.

Therefore, we have

$$A(t,x,y,z) = -2p_{20}y - 2p_{24}z + w_2(t,x). \tag{3.2.98}$$

Also by eq.(3.0.9), we get

$$w_2(t,x) = \frac{4}{a}p_{42}tx - 2p_{38}x + w_3(t).$$

Therefore,

$$A(t,x,y,z) = -2p_{38}x - 2p_{20}y - 2p_{24}z + \frac{4}{a}p_{42}tx + w_3(t). \quad (3.2.99)$$

Taking into account eq.(3.0.8), we have

$$2e^{2x/a}\big(\frac{1}{a^2}p_{42}t^2 + p_{44}t + p_{45} + p_{42}e^{-2x/a}\big) = \frac{4}{a}p_{42}x + w_3'(t),$$

Comparing coefficients of x and $e^{2x/a}$;

$$\begin{aligned} x &\quad : \quad p_{42} = 0, \\ e^{2x/a} &\quad : \quad p_{45} = 0. \end{aligned}$$

and

$$w_3(t) = p_{44}e^{2x/a}t^2 + p_{48}.$$

Thus,

$$A(t,x,y,z) = -2p_{38}x - 2p_{20}y - 2p_{24}z + p_{44}e^{2x/a}t^2 + p_{48}. \quad (3.2.100)$$

From eq.(3.0.21), we have

$$\frac{2}{a}\big(p_{38}s - \frac{2}{a}p_{43}t + p_{41}\big) + 2\big(p_{44}s + \frac{2}{a^2}p_{43}t + p_{46}\big) = 0,$$

Comparing coefficients of s and s^0;

$$\begin{aligned} s: \ p_{44} &= -\frac{1}{a}p_{38}, \\ s^0: \ p_{46} &= -\frac{1}{a}p_{41}. \end{aligned}$$

Using eq.(3.0.8), we get $p_{38} = 0$. Thus,eq.(3.2.96) reduces to

$$\left.\begin{aligned}
&\xi = c_2,\\
&\eta^0 = \big(\tfrac{t^2}{a^2} + e^{-2x/a}\big)p_{43} - p_{41}\tfrac{t}{a} + p_{47},\\
&\eta^1 = -\tfrac{2}{a}p_{43}t + p_{41},\\
&\eta^2 = p_{20}s + c_{23}z + p_{22},\\
&\eta^3 = p_{24}s - c_{23}y + p_{26},\\
&A = -2p_{20}y - 2p_{24}z + p_{48}
\end{aligned}\right\} \tag{3.2.101}$$

Hence, from eq.(3.2.101) the generator $\mathbf{X}$ becomes

$$\begin{aligned}
\mathbf{X} = {}& c_2\tfrac{\partial}{\partial s} + \big((\tfrac{t^2}{a^2} + e^{-2x/a})p_{43} - p_{41}\tfrac{t}{a} + p_{47}\big)\tfrac{\partial}{\partial t}\\
&+ \big(-\tfrac{2}{a}p_{43}t + p_{41}\big)\tfrac{\partial}{\partial x} + \big(p_{20}s + c_{23}z + p_{22}\big)\tfrac{\partial}{\partial y}\\
&+ \big(p_{24}s - c_{23}y + p_{26}\big)\tfrac{\partial}{\partial z}
\end{aligned} \tag{3.2.102}$$

Therefore, the metric given by eq.(2.2.45) admits 9 Noether symmetries. Thus, the Noether symmetry generators are as follows:

$$\begin{aligned}
\mathbf{X_0} &= \frac{\partial}{\partial t},\\
\mathbf{X_1} &= z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z},\\
\mathbf{X_2} &= \frac{\partial}{\partial z},\\
\mathbf{X_3} &= \frac{\partial}{\partial y},\\
\mathbf{X_4} &= (a^2t^2 + e^{-2ax})\frac{\partial}{\partial t} - 2at\frac{\partial}{\partial x},\\
\mathbf{X_5} &= -at\frac{\partial}{\partial t} + \frac{\partial}{\partial x},\\
\mathbf{Y_0} &= \frac{\partial}{\partial s},\\
\mathbf{Y_1} &= s\frac{\partial}{\partial y}; \qquad A_1 = -2y,\\
\mathbf{Y_2} &= s\frac{\partial}{\partial z}; \qquad A_2 = -2z.
\end{aligned}$$

For this metric, the structure constants are

$$C^2_{13} = C^3_{\hat{0}\hat{1}} = C^2_{\hat{0}\hat{2}} = C^{\hat{2}}_{1\hat{1}} = 1, \quad C^5_{04} = -2a, \quad C^3_{12} = C^{\hat{1}}_{1\hat{2}} = -1, \quad C^0_{05} = -a, \quad C^4_{45} = a.$$

We summarized the obtained results (spacetimes) for the class under consideration in the table given below:

S. No.	$\nu(x)$	Spacetime	G_r	eq. ref	κT_{ab}
1.	0	Minkowski	17	(3.1.1)	0
2.	$(\frac{x}{a})^2$	Minkowski	17	(3.2.2)	0
3.	$\ln[\cosh(\frac{x}{a})]$	Bertotti-Robinson	9	(3.2.19)	$\kappa T_{22} = \frac{1}{a^2} = \kappa T_{33}$
4.	$\ln[\cos(\frac{x}{a})]$	Bertotti-Robinson	9	(3.2.40)	$\kappa T_{22} = -\frac{1}{a^2} = \kappa T_{33}$
5.	$\exp^{2x/a}$	Bertotti-Robinson	9	(3.2.89)	$\kappa T_{22} = \frac{1}{a^2} = \kappa T_{33}$
6.	$(\frac{x}{a})^{2c}$	-	8	(3.2.73)	$\kappa T_{22} = \frac{c(c-1)}{x^2} = \kappa T_{33}$
7.	arbitrary	-	7	(3.0.1)	$\kappa T_{22} = \nu''(x) + \nu'^2(x) = \kappa T_{33}$

Chapter 4

Conclusion

A class of plane symmetric static spacetimes have been classified according to their Noether symmetries and metrics. We have obtained a system of nineteen independent partial differential eqs.(3.0.3)-(3.0.21) from eq.(3.0.2). Solving this system we get some metrics. In each case, the corresponding Lagrangian of the metric admits either G_7 or G_9 or G_{17}. The corresponding Lagrangian of the metrics given by eqs.(3.1.1) and (3.2.2) admit G_{17} (Minkowski), the maximal number of Noether symmetries and eqs.(3.2.19), (3.2.40), and (3.2.89) admits G_9 (Bertotti-Robinson). Also, the metric given by eq.(2.2.35) admits the G_7 group which has the minimal number of Noether symmetries eq.(3.2.72) (in this case $\nu(x)$ is an arbitrary function) for the class under consideration i.e. **I(A)**. It is well known that the time translation $\mathbf{X_0}$ in eq.(3.2.72) corresponds to the conservation of energy, rotation $\mathbf{X_1}$ to the angular momentum, $\mathbf{Y_0}$ to the conservation of the Lagrangian, and translations in spatial directions $\mathbf{X_1}$, $\mathbf{X_2}$ corresponds to linear momentum in the respective directions. The interpretation of $\mathbf{Y_1}$ and $\mathbf{Y_2}$ is given in [33]. In the mentioned paper, it is shown that corresponding to $\mathbf{Y_1}$ and $\mathbf{Y_2}$, the conserved quantities are

$$y = -\frac{1}{2}sp_y + T_1, \tag{4.0.1}$$

$$z = -\frac{1}{2}sp_z + T_2, \tag{4.0.2}$$

where p_y and p_z are linear momentum in the $y-$ and $z-$direaction and T_1 and T_2 are constants. In eqs.(4.0.1) and (4.0.2), T_1 and T_2 are $y-$ and $z-$intercepts with the slopes $-\frac{1}{2}p_y$ and $-\frac{1}{2}p_z$ respectively. We obtained another case eq.(3.2.73) while classifying the plane symmetric static spacetimes according to their Noether symmetries and metrics. In this case, the nature of the metric depends on the value of a constant c. If $(c = 0, 1)$ then the metric given in eq.(3.2.73) is Minkowski, but we are not interested in discussing this case for these value of c. So, we take the case when $(c \neq 0, 1)$. Then the metric obtained in eq.(3.2.73) admits G_8 group of motion. In this case, the scaling symmetry $\mathbf{Y_3}$ which we do not obtained in any other case, contradicts with the theorem given in [34]. We also calculate the Lie algebra for each case.

Eq.(2.0.3) gives a system of ten Killing equations for the metric given by eq.(2.2.35) and the system of partial differential eqs.(3.0.3)–(3.0.21) can be reduced to the system of Killing eqs.(2.2.2)–(2.2.11) [35].

It would be of interest to classify the general plane symmetric static and non-static spacetimes according to their Noether symmetries. This work can be extended for the classification of the spherical and cylindrical spacetimes.

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