Existence and Uniqueness of solutions for Fractional Difference Equations

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Existence and Uniqueness of solutions for Fractional Difference Equations

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Abstract

This dissertation deals with the study of existence and uniqueness of solutions for discrete fractional calculus. We discuss some basic concepts of difference calculus and discrete fractional calculus.

We establish new results of existence, uniqueness and global uniqueness for fractional difference second order initial value problem, using Caputo difference operator. We develop these results with the help of Banach contraction principle and Schauder fixed point theorem.

We establish some conditions for the existence and uniqueness of the solutions of boundary value problem by Caputo approach. We construct existence and uniqueness results by using contraction principle and Brouwer theorem.

Finally, we construct some useful properties satisfied by the Green's function for a certain fractional difference boundary value problem. We also establish some new conditions to ensure the existence and non-existence of positive solutions for the boundary value problem. For construction of these results we use Guo-Krasnoselskii fixed point theorem. Some examples are also added to demonstrate the applicability of our results.

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To My Parents and Siblings

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Chapter 1

Introduction

Fractional calculus is a field of mathematics that deals with the derivatives and integrals of non-integer order. The origins of fractional calculus can be traced back to the end of the seventeenth century. In beginning, many mathematicians put their interest and efforts and also developed their own definitions for derivatives and integrals of fractional order. The most famous of these definitions in fractional calculus are the Riemann-Liouville, Grunwald-Letnikov and Caputo definition. On the other hand, the Weyl fractional integral of order α was introduced by Weyl in 1917. The major difference between Weyl and Riemann-Liouville fractional integral definition are the limits of integration with kernel.

Fractional calculus has a long history. To the author's knowledge, significant work did not appear in this area until the mid-1950's. A renewed interest shown within the past thirty years in both the study of fractional calculus and fractional order differential equations. But the idea of fractional order difference equations is very recent. In 1956 Kuttner [24] was first mentioned the difference of fractional order for sequence of any complex number.

Diaz and Olsen in 1974 [10] introduced a discrete fractional difference operator, and generalize the binomial formula for the n^{th} order difference operator Δ^n. Fractional order sum and difference of functions defined by Miller and Rose [26] in 1989. In 1991 Miller and Ross [27] published their paper on fractional Green's functions. Atici and Eloe in [1-3] have developed a transform method in discrete fractional calculus, solved the initial value problems in discrete fractional calculus and introduced a two point boundary value problem for a finite fractional difference equation. Abdeljawad [4] discusses Riemann and Caputo fractional difference and relate their properties. In Riemann-Liouville fractional difference the derivative of constant function is not zero, while in Caputo fractional difference the derivative of constant function is zero. In this dissertation we establish existence and uniqueness results of solutions for the nonlinear fractional difference problem.

This chapter is concerned with basic definitions and preliminary concepts related to fractional difference calculus. In first section we define the gamma function, used in further concepts. Second section is about basic concept of difference operator. In section three we define factorial function and their properties. Section four is related to infinite sum and in section five we discuss fractional sum and differences definitions and properties. In last section we gave some definitions and results about functional analysis.

1.1 The gamma function

The gamma function ia an extended form of factorial function. Swiss mathematician Leonhard Euler (1707-1783) was first to introduce the gamma function to generalize the factorial to non integer value. In 19^{th} century many mathematicians made significant progress in studies of gamma function.

The gamma function denoted by $\Gamma(\alpha)$ is define as:

$$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y} dy, \qquad \alpha > 0. \tag{1.1}$$

The improper integral (1.1) is convergent for $\alpha > 0$.

We list here some properties of gamma function:

(i) $\Gamma(\alpha) > 0, for\ \alpha > 0$,

(ii) $\Gamma(n+1) = n!$,

(iii) $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$.

The defined gamma function can be extended for all $\alpha \in \mathbb{R} \setminus \{0, -1, -2, \cdots\}$.

Lemma 1.1.1.

$$\frac{\Gamma(q+1)}{\Gamma(q-k+1)} = \frac{1}{k+1}\left[\frac{\Gamma(q+2)}{\Gamma(q-k+1)} - \frac{\Gamma(q+1)}{\Gamma(q-k)}\right]. \tag{1.2}$$

Proof. For $q, k \in \mathbb{R}, q > k > -1$, then

$$\begin{aligned}
\frac{\Gamma(q+2)}{\Gamma(k+2)\Gamma(q-k+1)} - \frac{\Gamma(q+1)}{\Gamma(k+2)\Gamma(q-k)} &= \frac{(q+1)\Gamma(q+1)}{\Gamma(k+2)\Gamma(q-k+1)} - \frac{(q-k)\Gamma(q+1)}{\Gamma(k+2)\Gamma(q-k+1)} \\
&= \frac{\Gamma(q+1)}{\Gamma(k+2)\Gamma(q-k+1)}\left(q+1-(q-k)\right) \\
&= \frac{(k+1)\Gamma(q+1)}{\Gamma(k+2)\Gamma(q-k+1)} \\
&= \frac{\Gamma(q+1)}{\Gamma(k+1)\Gamma(q-k+1)}.
\end{aligned}$$

We can write it as

$$\frac{\Gamma(q+1)}{\Gamma(k+1)\Gamma(q-k+1)} = \frac{\Gamma(q+2)}{\Gamma(k+2)\Gamma(q-k+1)} - \frac{\Gamma(q+1)}{\Gamma(k+2)\Gamma(q-k)}$$

$$\frac{\Gamma(q+1)}{\Gamma(q-k+1)} = \frac{1}{k+1}\left[\frac{\Gamma(q+2)}{\Gamma(q-k+1)} - \frac{\Gamma(q+1)}{\Gamma(q-k)}\right].$$

□

1.2 The difference operator

Mathematical computations are based on equations and we can find solutions of these functions recursively from given values, such equations are called difference equations. We can simplify the calculations that involves in solving or analyzing the difference equations by using difference calculus.

In this section, we give basic definitions and results of forward difference operator from [23]. As in differential calculus the differential operator is the basic component, similarly, the difference operator has the main role in difference calculus. Finite difference is the discrete form of derivatives.

Definition 1.2.1. Let $y(t)$ be a function of variable t where t belongs to real or complex number. Then the forward difference operator Δ is defined as:

$$\Delta y(t) = y(t+1) - y(t). \tag{1.3}$$

For domain of y we will take the set of consecutive integers such as the set of natural numbers. Sometimes it is useful to take the complex number or a continuous set such as the interval $[0, \infty)$ as the domain of y.
The difference operator may be applied on function of two or more variables. For example,

$$\Delta_y y e^m = (y+1)e^m - ye^m = e^m$$

and

$$\Delta_m y e^m = ye^{m+1} - ye^m = ye^m(e-1).$$

The second order difference Δ^2 is defined as

$$\begin{aligned} \Delta^2 y(t) &= \Delta(\Delta y(t)) \\ &= \Delta[y(t+1) - y(t)] \\ &= y(t+2) - y(t+1) - y(t+1) + y(t) \\ &= y(t+2) - 2y(t+1) - y(t). \end{aligned}$$

The n^{th} order difference formula for all positive integers n is define as:

$$\Delta^n y(t) = y(t+n) - ny(t+n-1) + \frac{n(n-1)}{2!} y(t+n-2) + ... + (-1)^n y(t), \tag{1.4}$$

which can be verified by induction.
Using the definition for binomial coefficients equation (1.4) can be written as

$$\Delta^n y(t) = \sum_{i=0}^{n} \binom{n}{i} (-1)^i y(t+n-i). \tag{1.5}$$

Definition 1.2.2. Let $y(t)$ be a function. Then the shift operator E is defined as

$$Ey(t) = y(t+1).$$

If I is identity $y(t) = I$, then $\Delta = E - I$.

Theorem 1.2.1. *[14] Let x, y be functions of t. Then*
(a) $\Delta\beta = 0$, *(β is constant),*
(b) $\Delta^m(\Delta^n y(t)) = \Delta^{m+n} y(t)$ *m, n represent all positive integers,*
(c) $\Delta(x(t) + y(t)) = \Delta x(t) + \Delta y(t)$,

(*d*) $\Delta(Cy(t)) = C\Delta y(t)$ *where* C *is a constant,*
(*e*) $\Delta(x(t)y(t)) = x(t)\Delta y(t) + Ey(t)\Delta x(t)$,
(*f*) $\Delta\left(\frac{x(t)}{y(t)}\right) = \frac{y(t)\Delta x(t) - x(t)\Delta y(t)}{y(t)Ey(t)}$.

Proof. Proof of parts $(a - d)$ are simple.

$$
\begin{aligned}
(e) \qquad \Delta x(t)y(t) &= x(t+1)y(t+1) - x(t)y(t) \\
&= x(t+1)y(t+1) - x(t)y(t+1) + x(t)y(t+1) - x(t)y(t) \\
&= y(t+1)(x(t+1) - x(t)) + x(t)(y(t+1) - y(t)) \\
&= x(t)\Delta y(t) + Ey(t)\Delta x(t).
\end{aligned}
$$

$$
\begin{aligned}
(f) \qquad \Delta\left(\frac{x(t)}{y(t)}\right) &= \frac{x(t+1)}{y(t+1)} - \frac{x(t)}{y(t)} \\
&= \frac{y(t)x(t+1) - x(t)y(t+1)}{y(t)y(t+1)} \\
&= \frac{y(t)x(t+1) - x(t)y(t) + x(t)y(t) - x(t)y(t+1)}{y(t)Ey(t)} \\
&= \frac{y(t)\Delta x(t) - x(t)\Delta y(t)}{y(t)Ey(t)}.
\end{aligned}
$$

□

Theorem 1.2.2. *[23] Let* β *be any constant. Then,*
(*a*) $\Delta\beta^t = (\beta - 1)\beta^t$,
(*b*) $\Delta \sin\beta t = 2\sin\frac{\beta}{2}\cos\beta(t + \frac{1}{2})$,
(*c*) $\Delta \cos\beta t = -2\sin\frac{\beta}{2}\sin\beta(t + \frac{1}{2})$,
(*d*) $\Delta \log\beta t = \log(1 + \frac{1}{t})$,
(*e*) $\Delta \log\Gamma(t) = \log t$.
Here $\log t$ *is logarithm of the positive number* t.

Proof. (a)

$$
\begin{aligned}
\Delta\beta^t &= \beta^{t+1} - \beta^t \\
&= \beta^t(\beta - 1).
\end{aligned}
$$

(b)

$$\begin{aligned}
\Delta \sin \beta t &= \sin(\beta t+\beta) - \sin \beta t \\
\textit{since} \qquad \sin\alpha - \sin\theta &= 2\cos\left(\frac{\alpha+\theta}{2}\right)\sin\left(\frac{\alpha-\theta}{2}\right) \\
&= 2\cos\left(\frac{\beta t+\beta+\beta t}{2}\right)\sin\left(\frac{\beta t+\beta-\beta t}{2}\right) \\
&= 2\cos\left(\beta t+\frac{\beta}{2}\right)\sin\left(\frac{\beta}{2}\right) \\
&= 2\cos\beta\left(t+\frac{1}{2}\right)\sin\beta\left(\frac{1}{2}\right).
\end{aligned}$$

(c) Proof of part (c) is similar to part (b).

(d)

$$\begin{aligned}
\Delta \log \beta t &= \log\beta(t+1) - \log\beta t \\
&= \log\beta\left(\frac{t+1}{t}\right) \\
&= \log\beta\left(1+\frac{1}{t}\right).
\end{aligned}$$

(e)

$$\begin{aligned}
\Delta \log\Gamma(t) &= \log\Gamma(t+1) - \log\Gamma(t) \\
&= \log\frac{\Gamma(t+1)}{\Gamma(t)} \\
&= \log\frac{t\Gamma(t)}{\Gamma(t)} = \log t.
\end{aligned}$$

□

1.3 Falling function

This section contains the basic definition of falling function and some useful results, which are used in the definitions of discrete fractional calculus.

Definition 1.3.1. [23] The falling factorial power is denoted by $t^{(v)}$ and read as "t to the v falling". The $t^{(v)}$ is defined as follows:

(a) If $v = 1, 2, 3, ...$, then $t^{(v)} = t(t-1)(t-2)...(t-v+1)$.

(b) If $v = 0$, then $t^{(v)} = 1$.

(c) If $v = -1, -2, -3, ...$, then $t^{(v)} = \frac{1}{(t+1)(t+2)...(t-v)}$.

(d) If v is not an integer then

$$t^{(v)} = \frac{\Gamma(t+1)}{\Gamma(t-v+1)}. \tag{1.6}$$

The expression in (d) is also valid for v is an integer, except those values that make the gamma function undefined. This is the generalized form of falling function.

Since the binomial coefficient is defined as:

$$\begin{pmatrix} n \\ i \end{pmatrix} = \frac{n(n-1)...(n-i+1)}{i!}.$$

If n and i are positive integers and $n \geq v$, then by above definition we get

$$\begin{aligned} \begin{pmatrix} n \\ i \end{pmatrix} &= \frac{n(n-1)...(n-i+1)(n-i)!}{i!(n-i)!} \\ &= \frac{n!}{i!(n-i)!} \\ &= \frac{\Gamma(n+1)}{\Gamma(i+1)\Gamma(n-i+1)} \\ &= \frac{n^{(i)}}{\Gamma(i+1)}. \end{aligned}$$

This is the relation between binomial coefficient and the falling factorial power. That suggest a definition for extended binomial coefficient, which is given as:

Definition 1.3.2. [23] The binomial coefficient denoted by $\begin{pmatrix} t \\ v \end{pmatrix}$ is defined as

$$\begin{pmatrix} t \\ v \end{pmatrix} = \frac{t^{(v)}}{\Gamma(v+1)}.$$

There are many useful identities of binomial coefficients. By using the definition of falling factorial and binomial coefficient we can easily verify these identities. Such as

(i)

$$\begin{pmatrix} t \\ v \end{pmatrix} = \begin{pmatrix} t \\ t-v \end{pmatrix} \quad \text{(symmetry)}. \tag{1.7}$$

Proof. Taking right hand side of equation (1.7)

$$\begin{aligned}\begin{pmatrix} t \\ t-v \end{pmatrix} &= \frac{t^{(t-v)}}{\Gamma(t-v+1)} \\ &= \frac{\Gamma(t+1)}{\Gamma(t-v+1)\Gamma(t-(t-v)+1)} \\ &= \frac{\Gamma(t+1)}{\Gamma(t-v+1)\Gamma(v+1)} \\ &= \frac{t^{(v)}}{\Gamma(v+1)} = \begin{pmatrix} t \\ v \end{pmatrix}.\end{aligned}$$

Hence right hand side equals to left hand side. □

(ii)

$$\begin{pmatrix} t \\ v \end{pmatrix} = \frac{t}{v}\begin{pmatrix} t-1 \\ v-1 \end{pmatrix} \qquad \text{(moving out of parentheses).} \tag{1.8}$$

Proof. Taking right hand side of equation (1.8)

$$\begin{aligned}\frac{t}{v}\begin{pmatrix} t-1 \\ v-1 \end{pmatrix} &= \frac{t}{v}\frac{(t-1)^{(v-1)}}{\Gamma(v-1+1)} \\ &= \frac{t}{v}\frac{\Gamma(t-1+1)\Gamma}{\Gamma(v)\Gamma(t-1-(v-1)+1)} \\ &= \frac{t\Gamma(t)}{v\Gamma(v)\Gamma(t-v+1)} \\ &= \frac{\Gamma(t+1)}{\Gamma(v+1)\Gamma(t-v+1)} = \begin{pmatrix} t \\ v \end{pmatrix}.\end{aligned}$$

Hence right hand side equals to left hand side. □

(iii)

$$\begin{pmatrix} t \\ v \end{pmatrix} = \begin{pmatrix} t-1 \\ v \end{pmatrix} + \begin{pmatrix} t-1 \\ v-1 \end{pmatrix} \qquad \text{(addition formula).} \tag{1.9}$$

Proof. Taking right hand side of equation (1.9)

$$\begin{aligned}
\begin{pmatrix} t-1 \\ v \end{pmatrix} + \begin{pmatrix} t-1 \\ v-1 \end{pmatrix} &= \frac{(t-1)^{(v)}}{\Gamma(v+1)} + \frac{(t-1)^{(v-1)}}{\Gamma(v-1+1)} \\
&= \frac{\Gamma(t-1+1)}{\Gamma(v+1)\Gamma(t-1-v+1)} + \frac{\Gamma(t-1+1)}{\Gamma(v)\Gamma(t-1-v+2)} \\
&= \frac{\Gamma(t)}{\Gamma(v+1)\Gamma(t-v)} + \frac{\Gamma(t)}{\Gamma(v)\Gamma(t-v+1)} \\
&= \frac{(t-v)\Gamma(t)}{\Gamma(v+1)\Gamma(t-v+1)} + \frac{v\Gamma(t)}{\Gamma(v+1)\Gamma(t-v+1)} \\
&= \frac{\Gamma(t)}{\Gamma(v+1)\Gamma(t-v+1)}(t-v+v) \\
&= \frac{t\Gamma(t)}{\Gamma(v+1)\Gamma(t-v+1)} \\
&= \frac{\Gamma(t+1)}{\Gamma(v+1)\Gamma(t-v+1)} = \begin{pmatrix} t \\ v \end{pmatrix}.
\end{aligned}$$

Hence right hand side equals to left hand side. □

Theorem 1.3.1. *[23] Let Δ be the function of t and $v > 0$. Then we have:*
(a) $\Delta_t t^{(v)} = vt^{(v-1)}$,
(b) $v^{(v)} = \Gamma(v+1)$,
(c) $\Delta_t \begin{pmatrix} t \\ v \end{pmatrix} = \begin{pmatrix} t \\ v-1 \end{pmatrix}, \qquad (v \neq 0)$,
(d) $\Delta_t \begin{pmatrix} v+t \\ t \end{pmatrix} = \begin{pmatrix} v+t \\ t+1 \end{pmatrix}$,
(e) $\Delta_t \Gamma(t) = (t-1)\Gamma(t)$.

Proof. (a) Let r be arbitrary, then

$$\begin{aligned}
\Delta_t t^{(v)} &= \Delta_t \frac{\Gamma(t+1)}{\Gamma(t-v+1)} \\
&= \frac{\Gamma(t+2)}{\Gamma(t-v+2)} - \frac{\Gamma(t+1)}{\Gamma(t-v+1)} \\
&= \frac{(t+1)\Gamma(t+1)}{\Gamma(t-v+2)} - \frac{(t-v+1)\Gamma(t+1)}{\Gamma(t-v+2)}
\end{aligned}$$

$$= \frac{\Gamma(t+1)}{\Gamma(t-v+2)}\left(t+1-t+v-1\right)$$
$$= \frac{v\Gamma(t+1)}{\Gamma(t+1-(v+1))} = vt^{(v-1)}.$$

(b)

$$\begin{aligned} v^{(v)} &= \frac{\Gamma(v+1)}{\Gamma(v+1-v)} \\ &= \frac{\Gamma(v+1)}{\Gamma(1)} = \Gamma(v+1). \end{aligned}$$

(c)

$$\begin{aligned} \Delta_t \begin{pmatrix} t \\ v \end{pmatrix} &= \Delta_t \frac{t^{(v)}}{\Gamma(v+1)} \\ &= \frac{vt^{(v-1)}}{\Gamma(v+1)} = \frac{vt^{(v-1)}}{v\Gamma(v)} \\ &= \frac{t^{(v-1)}}{\Gamma(v)} = \begin{pmatrix} t \\ v-1 \end{pmatrix}. \end{aligned}$$

(d)

$$\begin{aligned} \Delta_t \begin{pmatrix} v+t \\ t \end{pmatrix} &= \Delta_t \frac{(v+t)^{(t)}}{\Gamma(t+1)} \\ &= \Delta_t \frac{\Gamma(v+t+1)}{\Gamma(t+1)\Gamma(v+t+1-t)} \\ &= \Delta_t \frac{\Gamma(v+t+1)}{\Gamma(v+1)\Gamma(t+1)} \\ &= \frac{\Gamma(v+t+2)}{\Gamma(v+1)\Gamma(t+2)} - \frac{\Gamma(v+t+1)}{\Gamma(v+1)\Gamma(t+1)} \\ &= \frac{(v+t+1)\Gamma(v+t+1)}{\Gamma(v+1)\Gamma(t+2)} - \frac{(t+1)\Gamma(v+t+1)}{\Gamma(v+1)\Gamma(t+2)} \\ &= \frac{\Gamma(v+t+1)}{\Gamma(v+1)\Gamma(t+2)}\left(v+t+1-t-1\right) \\ &= \frac{v\Gamma(v+t+1)}{v\Gamma(v)\Gamma(t+2)} \end{aligned}$$

$$= \frac{\Gamma(t+v+1)}{\Gamma((t+1)+1)\Gamma((t+v+1)-(t+1))}$$
$$= \frac{(t+v)^{(t+1)}}{\Gamma(t+2)} = \begin{pmatrix} v+t \\ t+1 \end{pmatrix}.$$

(e)

$$\begin{aligned} \Delta_t \Gamma(t) &= \Gamma(t+1) - \Gamma(t) \\ &= t\Gamma(t) - \Gamma(t) \\ &= (t-1)\Gamma(t). \end{aligned}$$

□

1.4 Summation

As in differential calculus we have antiderivative of a function. Similarly in discrete calculus we introduce the parallel concept of antiderivative. The inverse operator of difference operator is called the indefinite sum.

Definition 1.4.1. For any real valued function $y(t)$. The indefinite sum or anti-difference of $y(t)$ denoted by $\sum y(t)$, or $\Delta^{-1}y(t)$ is define as

$$\Delta\left(\sum y(t)\right) = y(t).$$

Since we know that the indefinite integral is not unique for example,

$$\int 6t\,dt = 3t^2 + C \qquad (C\,is\,any\,constant).$$

Similarly, indefinite sum is not unique.

Theorem 1.4.1. *[23] Let $y(t)$ is a real valued function. If $Y(t)$ is an indefinite sum of $y(t)$, then the indefinite sum of $y(t)$ is given as*

$$\sum y(t) = Y(t) + C,$$

where C is constant, and $\Delta C = 0$.

The following theorem provide some useful properties of indefinite sum.

Theorem 1.4.2. *[23] Let β be an arbitrary constant and $\Delta C = 0$. Then*

(a) $\sum \beta^t = \frac{\beta^t}{\beta-1} + C, \qquad (\beta \neq 1),$

(b) $\sum \sin\beta t = -\frac{\cos\beta(t-\frac{1}{2})}{2\sin\frac{\beta}{2}} + C, \qquad (\beta \neq 2n\pi),$

(c) $\sum \cos\beta t = \frac{\sin\beta(t-\frac{1}{2})}{2\sin\frac{\beta}{2}} + C, \qquad (\beta \neq 2n\pi),$

(d) $\sum \log t = \log\Gamma(t) + C, \qquad (t > 0),$

(e) $\sum t^{(\beta)} = \frac{t^{(\beta+1)}}{\beta+1} + C, \qquad (\beta \neq -1),$

(f) $\sum \begin{pmatrix} t \\ \beta \end{pmatrix} = \begin{pmatrix} t \\ \beta+1 \end{pmatrix} + C,$ *and*

(g) $\sum \begin{pmatrix} \beta+t \\ t \end{pmatrix} = \begin{pmatrix} \beta+t \\ t-1 \end{pmatrix} + C.$

Proof. Proof of part (a) is immediate from Theorem 1.2.2 (a).
(b) According to Theorem 1.2.2(c) we have,

$$\begin{aligned}
\Delta\cos\beta t &= -2\sin\frac{\beta}{2}\sin\beta\left(t+\frac{1}{2}\right) \\
\Delta\cos\beta\left(t-\frac{1}{2}\right) &= -2\sin\frac{\beta}{2}\sin\beta\left(t+\frac{1}{2}-\frac{1}{2}\right) \\
\Delta\cos\beta\left(t-\frac{1}{2}\right) &= -2\sin\frac{\beta}{2}\sin\beta t \\
\sin\beta t &= \frac{-\Delta\cos\beta\left(t-\frac{1}{2}\right)}{2\sin\frac{\beta}{2}} \\
\sum\sin\beta t &= \sum\frac{-\Delta\cos\beta\left(t-\frac{1}{2}\right)}{2\sin\frac{\beta}{2}} = \frac{-\cos\beta\left(t-\frac{1}{2}\right)}{2\sin\frac{\beta}{2}} + C.
\end{aligned}$$

The proof of other parts are similar. □

Theorem 1.4.3. *[23] Some general properties of indefinite sum are as follows:*

(a) $\sum(x(t)+y(t)) = \sum x(t) + \sum y(t),$

(b) $\sum \beta y(t) = \beta\sum y(t), \qquad \beta$ *is any constant,*

(c) $\sum(x(t)\Delta y(t)) = x(t)y(t) - \sum Ey(t)\Delta x(t),$ *and*

(d) $\sum E(x(t)\Delta y(t)) = x(t)y(t) - \sum y(t)\Delta x(t).$

Proof. Part (a) and part (b) follows from Theorem 1.2.2. (c) To prove this we consider the part (d) of Theorem 1.2.1,

$$\begin{aligned}
\Delta(x(t)y(t)) &= x(t)\Delta y(t) + Ey(t)\Delta x(t) \\
x(t)\Delta y(t) &= \Delta(x(t)y(t)) - Ey(t)\Delta x(t) \\
\sum x(t)\Delta y(t) &= \sum [\Delta(x(t)y(t)) - Ey(t)\Delta x(t)] \\
\sum x(t)\Delta y(t) &= (x(t)y(t)) - \sum Ey(t)\Delta x(t) + c.
\end{aligned}$$

(d) Part (d) can be proved by rearrangement of part (c). □

Remark 1.4.1. *Part (c) and (d) of Theorem1.4.3 are known as summation by parts formula for indefinite sum.*

1.5 Fractional sum and difference

This section contains some basic definitions and results of discrete fractional sum and difference. Using these concepts throughout this dissertation we find some sufficient conditions for existence and uniqueness of solutions for discrete fractional calculus.

1.5.1 Fractional sum

As we know that the n repeated definite integrals of continuous function f is define as:

$$\begin{aligned}
y(t) &= \int_a^t \int_a^s \int_a^{\tau_1} \cdots \int_a^{\tau_{n-2}} f(\tau_{n-1})(d\tau_{n-1} \cdots d\tau_2 d\tau_1 ds) \\
&= \int_a^t \frac{(t-s)^{n-1}}{(n-1)!} f(s) ds, \qquad t \in [a, \infty).
\end{aligned} \tag{1.10}$$

Similarly, the n repeated definite sums of discrete function f is written as:

$$\begin{aligned}
y(t) &= \sum_{s=a}^{t-1} \sum_{\tau_1=a}^{s-1} \cdots \sum_{\tau n-1=a}^{\tau_{n-2}-1} f(\tau_{n-1}) \\
&= \sum_{s=a}^{t-n} \frac{\prod_{j=0}^{n-1}(t-s-1-j)}{(n-1)!} f(s), \qquad t \in \mathbb{N}_a,
\end{aligned} \tag{1.11}$$

where $(t-s-1)^{(n-1)} = \prod_{j=0}^{n-1}(t-s-1-j)$.
From equation (1.11) the n^{th} integer sum of f is defined as:

$$y(t) = (\Delta_a^{-n} f)(t) = \sum_{s=a}^{t-n} \frac{(t-s-1)^{(n-1)}}{(n-1)!} f(s), \qquad t \in \mathbb{N}_a.$$

This expression motivates the definition of the v^{th} fractional sum.

Definition 1.5.1. [2] If $v > 0$ and $\sigma(s) = s+1$. Then the v^{th} order fractional sum of f is defined by

$$\Delta^{-v} f(t) = \frac{1}{\Gamma(v)} \sum_{s=a}^{t-v} (t-\sigma(s))^{(v-1)} f(s). \tag{1.12}$$

Note that Δ^{-v} maps functions defined on $\mathbb{N}_a$ to function defined on $\mathbb{N}_{a+v}$ where
$\mathbb{N}_a = \{a, a+1, a+2, ...\}$. Also note that $t^{(v)} = \frac{\Gamma(t+1)}{\Gamma(t+1-v)}$.

Example 1.5.1. We find the value of $\Delta^{-\frac{3}{2}}1$ for $t \in \{\frac{3}{2}, \frac{3}{2}+1, \frac{3}{2}+2, \cdots\}$ by using the definition of the fractional sum.

$$\begin{aligned}
\Delta_0^{-\frac{3}{2}} 1 &= \frac{1}{\Gamma(\frac{3}{2})} \sum_{s=0}^{t-\frac{3}{2}} (t-\sigma(s))^{(\frac{3}{2}-1)} 1 \\
&= \frac{1}{\Gamma(\frac{3}{2})} \sum_{s=0}^{t-\frac{3}{2}} (t-\sigma(s))^{(\frac{1}{2})} \\
&= \frac{1}{\Gamma(\frac{3}{2})} \sum_{s=0}^{t-\frac{3}{2}} \left(\frac{\Gamma(t-s)}{\Gamma(t-s-\frac{1}{2})} \right) \\
&= \frac{1}{\Gamma(\frac{3}{2})} \left[\sum_{s=0}^{t-\frac{3}{2}-1} \left(\frac{\Gamma(t-s)}{\Gamma(t-s-\frac{1}{2})} \right) + \Gamma\left(\frac{3}{2}\right) \right].
\end{aligned}$$

Using Lemma 1.1.1, we get

$$\begin{aligned}
&= \frac{1}{\Gamma(\frac{3}{2})} \left[\frac{1}{\frac{3}{2}} \sum_{s=0}^{t-\frac{5}{2}} \left(\frac{\Gamma(t-s+1)}{\Gamma(t-s-\frac{1}{2})} - \frac{\Gamma(t-s)}{\Gamma(t-s-\frac{1}{2}-1)} \right) + \Gamma\left(\frac{3}{2}\right) \right] \\
&= \frac{1}{\Gamma(\frac{3}{2})} \left[\frac{1}{\frac{3}{2}} \left(\frac{\Gamma(t+1)}{\Gamma(t-\frac{1}{2})} - \frac{\Gamma(\frac{5}{2})}{\Gamma(\frac{5}{2}-\frac{3}{2})} \right) + \Gamma\left(\frac{3}{2}\right) \right] \\
&= \frac{1}{\Gamma(\frac{3}{2})} \left[\frac{1}{\frac{3}{2}} \left(t^{(\frac{3}{2})} - \frac{3}{2}\Gamma(\frac{3}{2}) \right) + \Gamma\left(\frac{3}{2}\right) \right] = \frac{2}{3\Gamma(\frac{3}{2})} t^{(\frac{3}{2})} = \frac{4}{3\sqrt{\pi}} t^{(\frac{3}{2})}.
\end{aligned}$$

Lemma 1.5.1. *[2]*(**Fractional sum power rule**)

Let $v \neq 1$ and assume $\mu + v + 1$ is not a non-positive integer. Then

$$\Delta^{-v} t^{(\mu)} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+v+1)} t^{(\mu+v)}. \tag{1.13}$$

Proof. Define

$$h_1(t) := \frac{\Gamma(\mu+1)}{\Gamma(\mu+v+1)} t^{(\mu+v)}$$
$$h_2(t) := \Delta^{-v} t^{(\mu)} = \frac{1}{\Gamma(v)} \sum_{s=\mu}^{t-v} (t-s-1)^{(v-1)} s^{(\mu)}.$$

We show that h_1, h_2 each satisfy the following well-posed initial value problem

$$(t-(\mu+v)+1)\Delta h(t) = (\mu+v)h(t), \tag{1.14}$$
$$h(\mu+v) = \Gamma(\mu+1). \tag{1.15}$$

First we consider h_1 and prove that h_1 satisfies (1.14) then

$$\begin{aligned}
(t-(\mu+v)+1)\Delta h_1(t) &= \frac{(t-(\mu+v)+1)\Gamma(\mu+1)}{\Gamma(\mu+v+1)} \Delta t^{(\mu+v)} \\
&= \frac{(\mu+v)(t-(\mu+v)+1)\Gamma(\mu+1)}{\Gamma(\mu+v+1)} t^{(\mu+v-1)} \\
&= \frac{(\mu+v)\Gamma(\mu+1)}{\Gamma(\mu+v+1)} t^{(\mu+v)} = (\mu+v)h_1(t).
\end{aligned}$$

Now, we prove h_1 satisfies (1.15) that is

$$\begin{aligned}
h_1(\mu+v) &= \frac{\Gamma(\mu+1)}{\Gamma(\mu+v+1)} (\mu+v)^{(\mu+v)} \\
&= \frac{\Gamma(\mu+1)\Gamma(\mu+v+1)}{\Gamma(\mu+v+1)} = \Gamma(\mu+1).
\end{aligned}$$

Thus, h_1 satisfied both conditions.

Now, we consider h_2. To satisfying condition (1.14) we proceed as follows,

$$\begin{aligned}
h_2(t) &= \frac{1}{\Gamma(v)} \sum_{s=\mu}^{t-v} (t-s-1)^{(v-1)} s^{(\mu)} \\
&= \frac{1}{\Gamma(v)} \sum_{s=\mu}^{t-v} (t-s-1-(v-2))(t-s-1)^{(v-2)} s^{(\mu)} \\
&= \frac{1}{\Gamma(v)} \sum_{s=\mu}^{t-v} (t-s-1-v-\mu+2+\mu)(t-s-1)^{(v-2)} s^{(\mu)}
\end{aligned}$$

$$= \frac{1}{\Gamma(v)} \sum_{s=\mu}^{t-v} (t-s-v-\mu+1+\mu)(t-s-1)^{(v-2)} s^{(\mu)}$$

$$= \frac{(t-(v+\mu)+1)}{\Gamma(v)} \sum_{s=\mu}^{t-v} (t-s-1)^{(v-2)} s^{(\mu)} - \frac{1}{\Gamma(v)} \sum_{s=\mu}^{t-v} (s-\mu)(t-s-1)^{(v-2)} s^{(\mu)}. \qquad (1.16)$$

Consider first term of above equation(1.16)

$$\frac{(t-(v+\mu)+1)}{\Gamma(v)} \sum_{s=\mu}^{t-v} (t-s-1)^{(v-2)} s^{(\mu)}.$$

As we know that

$$\Delta h_2(t) = \frac{(v-1)}{\Gamma(v)} \sum_{s=\mu}^{t-v} (t-s-1)^{(v-2)} s^{(\mu)} + (t+1-v)^{\mu}.$$

So we can write above equation as

$$= (t-(v+\mu)+1) \left(\frac{\Delta h_2 - (t+1-v)^{\mu}}{(v-1)} \right)$$

$$= \frac{(t-(v+\mu)+1)\Delta h_2}{(v-1)} - \frac{(t+1-v)^{(\mu+1)}}{(v-1)}.$$

Consider second term of (1.16)

$$\frac{1}{\Gamma(v)} \sum_{s=\mu}^{t-v} (s-\mu)(t-s-1)^{(v-2)} s^{(\mu)}.$$

Using summation by parts rule, we have

$$\Delta((t-s)^{(v-1)} s^{(\mu+1)}) = (t-s-1)^{(v-1)}(\mu+1)s^{(\mu)} - (v-1)(t-s-1)^{(v-2)} s^{(\mu+1)}. \qquad (1.17)$$

We know that

$$\sum_{s=\mu}^{t-v} (t-\sigma(s))^{(v-2)} s^{(\mu+1)} = \sum_{s=\mu}^{t-v} (t-\sigma(s))^{(v-2)} (s-\mu) s^{(\mu)}.$$

Equation (1.17) can be written as

$$\sum_{s=\mu}^{t-v} (t-\sigma(s))^{(v-2)} s^{(\mu+1)} = \frac{1}{(v-1)} \left[(\mu+1) \sum_{s=\mu}^{t-v} (t-\sigma(s))^{(v-1)} s^{(\mu)} - \sum_{s=\mu}^{t-v} \Delta \left((t-s)^{(v-1)} s^{(\mu+1)} \right) \right]$$

$$= \frac{1}{(v-1)} \left[(\mu+1) \sum_{s=\mu}^{t-v} (t-\sigma(s))^{(v-1)} s^{(\mu)} - (v-1)^{(v-1)} (t+1-v)^{(\mu+1)} \right]$$

$$\frac{1}{\Gamma(v)} \sum_{s=\mu}^{t-v} (t-\sigma(s))^{(v-2)} s^{(\mu+1)} = \frac{(\mu+1)}{(v-1)} h_2(t) + \frac{(t+1-v)^{(\mu+1)}}{(v-1)}.$$

Substitute these values in (1.16)

$$h_2(t) = \frac{(t-(v+\mu)+1)\Delta h_2}{(v-1)} - \frac{(t+1-v)^{(\mu+1)}}{(v-1)} - \frac{(\mu+1)}{(v-1)}h_2(t) + \frac{(t+1-v)^{(\mu+1)}}{(v-1)}$$

$$\begin{aligned}(t-(v+\mu)+1)\Delta h_2 &= (v-1+\mu+1)h_2(t)\\ &= (v+\mu)h_2(t).\end{aligned}$$

Now to prove h_2 satisfies (1.15) we consider

$$h_2(t) = \Delta^{-v}t^{(\mu)} = \frac{1}{\Gamma(v)}\sum_{s=\mu}^{t-v}(t-s-1)^{(v-1)}s^{(\mu)}$$

$$h_2(\mu+v) = \frac{1}{\Gamma(v)}\sum_{s=\mu}^{\mu+v-v}(\mu+v-s-1)^{(v-1)}s^{(\mu)}$$

$$\begin{aligned}&= \frac{1}{\Gamma(v)}(\mu+v-\mu-1)^{(v-1)}\mu^{(\mu)}\\ &= \frac{\Gamma(\mu+1)}{\Gamma(v)}(v-1)^{(v-1)}\\ &= \frac{\Gamma(\mu+1)}{\Gamma(v)}\Gamma(v) = \Gamma(\mu+1).\end{aligned}$$

Thus we conclude that $h_1 = h_2$ by uniqueness of solutions to the well-posed initial value problem (1.14) and (1.15). □

Theorem 1.5.1. *[2] Let $v, u > 0$. Let f be a real valued function defined on $\mathbb{N}_a$. Then*

$$\Delta_{a+v}^{-v}[\Delta_a^{-u}f(t)] = \Delta^{-(u+v)}f(t) = \Delta_{a+v}^{-u}[\Delta_a^{-v}f(t)].$$

Proof.

$$\begin{aligned}\Delta_{a+v}^{-v}[\Delta_a^{-u}f(t)] &= \frac{1}{\Gamma(v)}\sum_{s=a+\mu}^{t-v}(t-\sigma(s))^{(v-1)}\left(\frac{1}{\Gamma(\mu)}\sum_{r=a}^{s-\mu}(t-\sigma(r))^{(\mu-1)}f(r)\right)\\ &= \frac{1}{\Gamma(\mu)\Gamma(v)}\sum_{s=a+\mu}^{t-v}\sum_{r=a}^{s-\mu}((t-\sigma(s))^{(v-1)}(t-\sigma(r))^{(\mu-1)}f(r)\\ &= \frac{1}{\Gamma(\mu)\Gamma(v)}\sum_{r=a}^{t-(v+\mu)}\sum_{s=r+\mu}^{t-v}((t-\sigma(s))^{(v-1)}(s-\sigma(r))^{(\mu-1)}f(r).\end{aligned}$$

Let $y = (s - \sigma(r))$. Then

$$= \frac{1}{\Gamma(\mu)\Gamma(v)} \sum_{r=a}^{t-(v+\mu)} \left[\sum_{y=\mu-1}^{(t-r-1)-v} ((t-r-1) - \sigma(Y))^{(v-1)} (y)^{(\mu-1)} \right] f(r)$$

$$= \frac{1}{\Gamma(\mu)} \sum_{r=a}^{t-(v+\mu)} \left[\frac{1}{\Gamma(v)} \sum_{y=\mu-1}^{t-r-1-v} (t-y-r-2)^{(v-1)} (y)^{(\mu-1)} \right] f(r)$$

$$= \frac{1}{\Gamma(\mu)} \sum_{r=a}^{t-(v+\mu)} \left[\Delta_{\mu-1}^{-v} (t)^{(\mu-1)} \right]_{t-r-1} f(r). Since \qquad \Delta_{a+\mu}^{-v} (t-a)^{(\mu)} \qquad = \mu^{-(v)} (t-a)^{(\mu+v)}$$

$$= \frac{1}{\Gamma(\mu)} \sum_{r=a}^{t-(v+\mu)} (\mu-1)^{-(v)} (t-r-1)^{(\mu-1+v)} f(r)$$

$$= \frac{1}{\Gamma(\mu)} \sum_{r=a}^{t-(v+\mu)} \frac{\Gamma(\mu)}{\Gamma(\mu+v)} (t-\sigma(r))^{((\mu+v)-1)} f(r)$$

$$= \Delta^{-(u+v)} f(t).$$

□

1.5.2 Riemann-Liouville fractional difference

In this section we define Riemann- Liouville fractional difference operator and some basic results on Riemann- Liouville fractional difference operator.

Definition 1.5.2. [5] If $v > 0$ and $n - 1 < v < n$, where n denotes a positive integer. Then the v^{th} order Riemann-Liouville fractional difference of f is defined as

$$\Delta^v f(t) = \Delta^n (\Delta^{-(n-v)} f(t)), \qquad \forall t \in \mathbb{N}_{a+n-v}, \tag{1.18}$$

where Δ^n is the n^{th} order forward difference operator. Also it is clear that Δ^v maps functions defined on $\mathbb{N}_a$ to function defined on $\mathbb{N}_{a+n-v}$.

Example 1.5.2. Find $\Delta_0^{\frac{3}{2}} 1$.

Let $n = 2$ then by using Definition 1.5.2 we get

$$\Delta_0^{\frac{3}{2}} 1 = \Delta^2 (\Delta^{-(2-\frac{3}{2})} 1) = \Delta^2 (\Delta^{-\frac{1}{2}} 1)$$
$$= \Delta^2 \frac{2}{\sqrt{\pi}} t^{(\frac{1}{2})} \quad \text{by Definition 1.5.1}$$

$$= \frac{2}{\pi}\left(\frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{-1}{2})}t^{(\frac{3}{2})}\right) \quad \text{by power rule}$$
$$= \frac{-1}{2\sqrt{\pi}}t^{(\frac{3}{2})}.$$

Lemma 1.5.2. *[2]***(Fractional difference power rule)**
Let $v \neq 1$ and assume $\mu - v + 1$ is not a non-positive integer. Then

$$\Delta^v t^{(\mu)} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-v+1)}t^{(\mu-v)}. \tag{1.19}$$

Proof. Taking left hand side and applying the definition of fractional difference, then we have

$$\begin{aligned}\Delta^v t^{(\mu)} &= \Delta^N \Delta^{-(N-v)} t^{(\mu)} \\ &= \Delta^N \frac{\Gamma(\mu+1)}{\Gamma(\mu+N+1-v)} t^{(\mu+N-v)}\end{aligned}$$

$$\begin{aligned}&= \frac{\Gamma(\mu+1)}{\Gamma(\mu+N+1-v)}\Delta^N t^{(\mu+N-v)} \\ &= \frac{\Gamma(\mu+1)}{\Gamma(\mu+N+1-v)}(\mu+N-v)^{(N)} t^{(\mu+N-v-N)} \\ &= \frac{\Gamma(\mu+1)\Gamma(\mu+N-v+1)}{\Gamma(\mu+N+1-v)\Gamma(\mu+N-v+1-N)} t^{(\mu-v)} \\ &= \frac{\Gamma(\mu+1)}{\Gamma(\mu-v+1)} t^{(\mu-v)}.\end{aligned}$$

□

Theorem 1.5.2. *Let $v, \mu > 0$ and f be a real valued function defined on $\mathbb{N}_a$. Where $N-1 < v \leq N$. Then*

$$\Delta^v_{a+\mu}[\Delta^{-\mu}_a f(t)] = \Delta^{v-\mu}_a f(t).$$

Proof. We have given that $f : \mathbb{N}_a \to \mathbb{R}$ and $v, \mu > 0$. Let $t \in \mathbb{N}_{a+\mu+N-v}$ then,

$$\begin{aligned}\Delta^v_{a+\mu}[\Delta^{-\mu}_a f(t)] &= \Delta^N \Delta^{-(N-v)}_{a+\mu}[\Delta^{-\mu}_a f(t)] \\ &= \Delta^N \Delta^{-(N-v+\mu)} f(t) \quad \text{by Theorem 1.5.1} \\ &= \Delta^{N-(N-v+\mu)}_a f(t) \\ &= \Delta^{v-\mu}_a f(t).\end{aligned}$$

□

Theorem 1.5.3. *Let $v > 0$ and $f : \mathbb{N}_a \to \mathbb{R}$ be given, suppose that $k \in \mathbb{N}_0$. Then for $t \in \mathbb{N}_{a+v}$ we have*

$$\Delta_a^{-v}\Delta^k f(t) = \Delta_a^{k-v} f(t) - \sum_{j=0}^{k-1} \frac{\Delta^j f(a)}{\Gamma(v-k+j+1)}(t-a)^{(v-k+j)}. \tag{1.20}$$

Moreover if $\mu > 0$ with $M-1 < \mu \le M$, then for $t \in \mathbb{N}_{a+M-\mu+v}$ we have

$$\Delta_{a+M-\mu}^{-v}\Delta_a^{\mu} f(t) = \Delta_a^{\mu-v} f(t) - \sum_{j=0}^{M-1} \frac{\Delta_a^{j-(M-\mu)} f(a+M-\mu)}{\Gamma(v-M+j+1)}(t-a-M+\mu)^{(v-M+j)}. \tag{1.21}$$

Proof. $k \in \mathbb{N}_0$ and $t \in \mathbb{N}_{a+v}$ be given, assume that $v > 0$ with $v \notin (1, 2, ..., k-1)$, then

$$\begin{aligned}
\Delta_a^{-v}\Delta^k f(t) &= \frac{1}{\Gamma(v)} \sum_{s=a}^{t-v} ((t-\sigma(s))^{(v-1)}[\Delta^k f(s)] \\
&= \frac{1}{\Gamma(v)} \sum_{s=a}^{(t-v+1)-1} ((t-\sigma(s))^{(v-1)}\Delta[\Delta^{k-1} f(s)].
\end{aligned}$$

Using sum by parts rule we obtain

$$\begin{aligned}
&= \frac{1}{\Gamma(v)}\left[(t-s)^{(v-1)}\Delta^{k-1} f(s)\right]_{s=a}^{t-v+1} + \frac{1}{\Gamma(v)}\sum_{s=a}^{t-v}(v-1)(t-\sigma(s))^{(v-2)}\Delta^{k-1} f(s) \\
&= \Delta^{k-1} f(t-v+1) - \frac{(t-a)^{(v-1)}}{\Gamma(v)}\Delta^{k-1} f(a) + \frac{1}{\Gamma(v-1)}\sum_{s=a}^{t-v}(t-\sigma(s))^{(v-2)}\Delta^{k-1} f(s) \\
&= \frac{1}{\Gamma(v-1)}\sum_{s=a}^{t-v+1}(t-\sigma(s))^{(v-2)}\Delta^{k-1} f(s) - \frac{\Delta^{k-1} f(a)(t-a)^{(v-1)}}{\Gamma(v)} \\
&= \frac{1}{\Gamma(v-1)}\sum_{s=a}^{t-(v-1)}(t-\sigma(s))^{((v-1)-1)}\Delta^{k-1} f(s) - \frac{\Delta^{k-1} f(a)}{\Gamma(v)}(t-a)^{(v-1)} \\
&= \Delta_a^{-(v-1)}\Delta^{k-1} f(t) - \frac{\Delta^{k-1} f(a)}{\Gamma(v)}(t-a)^{(v-1)} \\
&= \Delta_a^{1-v}\Delta^{k-1} f(t) - \frac{\Delta^{k-1} f(a)}{\Gamma(v)}(t-a)^{(v-1)}.
\end{aligned}$$

Summing by parts $(k-1)$ more times we obtain

$$\begin{aligned}
\Delta_a^{-v}\Delta^k f(t) &= \Delta_a^{1-v}\Delta^{k-1} f(t) - \frac{\Delta^{k-1} f(a)}{\Gamma(v)}(t-a)^{(v-1)} \\
&= \Delta_a^{2-v}\Delta^{k-2} f(t) - \frac{\Delta^{k-1} f(a)}{\Gamma(v)}(t-a)^{(v-1)} - \frac{\Delta^{k-2} f(a)}{\Gamma(v-1)}(t-a)^{(v-2)} \\
&\vdots
\end{aligned}$$

$$= \Delta^{k-v} f(t) - \sum_{i=1}^{k} \frac{\Delta^{k-i} f(a)}{\Gamma(v-i+1)} (t-a)^{(v-i)}$$

$$= \Delta^{k-v} f(t) - \sum_{j=0}^{k-1} \frac{\Delta^{j} f(a)}{\Gamma(v-k+j+1)} (t-a)^{(v-k+j)} \qquad \text{for}(t \in \mathbb{N}_a + v).$$

Suppose $v \in (1, 2, ..., k-1)$ then $k - v \in \mathbb{N}$ for $t \in \mathbb{N}_{a+v}$

$$\begin{aligned}
\Delta_a^{-v} \Delta^k f(t) &= \Delta^{k-v} \Delta^{-(k-v)} \Delta^{-v} \Delta^k f(t) \\
&= \Delta^{k-v} [\Delta^{-k} \Delta^k f(t)] \\
&= \Delta^{k-v} \left[f(t) - \sum_{j=0}^{k-1} \frac{\Delta^j f(a)}{\Gamma(k-k+j+1)} (t-a)^{(k-k+j)} \right] \\
&= \Delta^{k-v} \left[f(t) - \sum_{j=0}^{k-1} \frac{\Delta^j f(a)}{\Gamma(j+1)} (t-a)^{(j)} \right]
\end{aligned}$$

$$\begin{aligned}
&= \Delta^{k-v} f(t) - \sum_{j=0}^{k-1} \frac{\Delta^j f(a)}{\Gamma(j+1)} \Delta^{k-v} (t-a)^{(j)} \\
&= \Delta^{k-v} f(t) - \sum_{j=0}^{k-1} \frac{\Delta^j f(a) \Gamma(j+1)}{\Gamma(j+1) \Gamma(j+1-k+v)} (t-a)^{(j-k+v)} \\
&= \Delta^{k-v} f(t) - \sum_{j=0}^{k-1} \frac{\Delta^j f(a)}{\Gamma(j+1-k+v)} (t-a)^{(j-k+v)}.
\end{aligned}$$

The relation (4.12) is proved. For proof of (1.21) we suppose that $v, \mu > 0$ with $M-1 < \mu \leq M$. Define $h(t) = \Delta_a^{-(M-\mu)} f(t)$ and $b = a + M - \mu$ be the first point, then

$$\begin{aligned}
\Delta_{a+M-\mu}^{-v} \Delta_a^{\mu} f(t) &= \Delta^{-v} \Delta^{\mu)} \Delta^M \Delta^{-M} f(t) = \Delta^{-v} \Delta^M \Delta^{-(M-\mu)} f(t) \\
&= \Delta^{-v} \Delta^M h(t) \\
&= \Delta_{a+M-\mu}^{M-v} h(t) - \sum_{j=0}^{M-1} \frac{\Delta^j h(b)}{\Gamma(j+1-M+v)} (t-b)^{(j-M+v)} \\
&= \Delta_{a+M-\mu}^{M-v} \Delta^{-(M-\mu)} f(t) - \sum_{j=0}^{M-1} \frac{\Delta^j \Delta^{-(M-\mu)} f(b)}{\Gamma(j+1-M+v)} (t-b)^{(j-M+v)} \\
&= \Delta_a^{\mu-v} f(t) - \sum_{j=0}^{M-1} \frac{\Delta^{j-M+\mu} f(a+M-\mu)}{\Gamma(j+1-M+v)} (t-a-M+\mu)^{(j-M+v)}.
\end{aligned}$$

□

Theorem 1.5.4. *Let $v > 0$ and $f : \mathbb{N}_a \to \mathbb{R}$ be a given function. Then*

$$\Delta^{-v}\Delta f(t) = \Delta\Delta^{-v} f(t) - \frac{(t-a)^{(v-1)}}{\Gamma(v)} f(a).$$

Lemma 1.5.3. *[3] Suppose* $0 \leq N - 1 < v \leq N$. *Then*

$$\Delta^{-v}\Delta^{v} y(t) = y(t) + C_1 t^{(v-1)} + C_2 t^{(t-2)} + ... + C_N t^{(v-N)},$$

for some $C_k \in \mathbb{R},\ k = 1, 2, ..., N.$

1.5.3 Caputo fractional difference

In this section, we study the definition and some results of discrete fractional calculus using Caputo difference operator.

Definition 1.5.3. [5] If $v > 0$ and $n - 1 < v < n$, where n denotes a positive integer. Then the v^{th} order Caputo fractional difference of f is defined as

$$\Delta_*^v f(t) = \Delta^{-(n-v)}(\Delta^n f(t)), \qquad \forall t \in \mathbb{N}_{a+v}, \tag{1.22}$$

where Δ^n is the n^{th} order forward difference operator. Also it is clear that Δ_*^v maps functions defined on $\mathbb{N}_a$ to function defined on $\mathbb{N}_{a+n-v}$.

We are taking the same Example 1.5.2 and compute the Caputo fractional difference.

Example 1.5.3. Use the Definition 1.5.3 and find $\Delta_0^{\frac{3}{2}} 1$.
Let $n = 2$ then by using Definition 1.5.3 we get

$$\begin{aligned}\Delta_0^{\frac{3}{2}} 1 &= \Delta^{-(2-\frac{3}{2}}\Delta^2 1) \\ &= 0.\end{aligned}$$

The above example shows the basic difference in Riemann and Caputo fractional difference. In Riemann fractional difference the difference of constant is not equal to zero while in Caputo fractional difference the difference of constant is zero.

Theorem 1.5.5. *[5] For non integer $v > 0$, $n = \lceil v \rceil, \mu = n - v$, it holds*

$$f(t) = \sum_{k=0}^{n-1} \frac{(t-a)^{(k)}}{k!} \Delta^k f(a) + \frac{1}{\Gamma(v)} \sum_{s=a+\mu}^{t-v} (t-s-1)^{v-1} \Delta_*^v f(s), \tag{1.23}$$

where f is defined on $\mathbb{N}_a$ with $a \in \mathbb{R}$.

Lemma 1.5.4. *Let $v > 0$ and f be a function defined on $\mathbb{N}_a$. Then*

$$\Delta^{-v}\Delta_*^v f(t) = f(t) - \sum_{k=0}^{n-1} \frac{(t-a)^{(k)}}{k!}\Delta^k f(a).$$

The proof can be achieved by applying the definition of fractional sum and Caputo fractional difference.

Lemma 1.5.5. *Suppose $0 < v$. Then*

$$\Delta^{-v}\Delta_*^v y(t) = y(t) + C_0 + C_1 t + C_2 t^2 + ... + C_N t^N,$$

for some $C_k \in \mathbb{R}, k = 0, 1, 2, ..., N$.

1.6 Some results from functional analysis

In this section, some basic definitions and results of analysis from [18, 22] are given. These results are used throughout this dissertation to find the existence and uniqueness of solutions for initial value and boundary value problems of discrete fractional calculus.

Definition 1.6.1. Let $\mathbf{X}$ be a set and $\mathbf{X} \subset C[c, d]$ is said to be equicontinuous if and only if for each $\epsilon > 0$ there exist $\delta > o$ such that

$$|f(y_1) - f(y_2)| < \epsilon \quad \text{for all} \;\; y_1, y_2 \in [c, d],$$

whenever $|y_1 - y_2| < \delta$ for all $f \in \mathbf{X}$.

Definition 1.6.2. A subset $\mathbf{X}$ of a vector space V is said to be convex iff $\lambda y_1 + (1-\lambda)y_2$ lies in $\mathbf{X}$ for each $y_1, y_2 \in \mathbf{X}$, and each λ with $0 \leq \lambda \leq 1$.

Definition 1.6.3. A Banach space $\mathcal{B}$ is a complete normed vector space.

Definition 1.6.4. Let V and W be two vector spaces. Then Φ is called an operator from V into W, such that $\Phi : V \to W$. Where V is domain and W is range.

Definition 1.6.5. Let $\mathbf{X}$ be closed subset of Banach space $\mathcal{B}$. Then a mapping $\Phi : X \to X$ is called contraction, if for any $y_1, y_2 \in \mathbf{X}$ and there exist a non-negative real number $0 < r < 1$ such that

$$\|\Phi(y_1) - \Phi(y_2)\| \leq r\|y_1 - y_2\|.$$

Definition 1.6.6. Let $\mathcal{B}$ be the Banach space and $\mathbf{X}$ be a closed subset of $\mathcal{B}$. Φ is said to satisfy Lipschitz condition on $\mathbf{X}$ with Lipschitz constant $L < \infty$, if and only if for any $y_1, y_2 \in \mathbf{X}$

$$\|\Phi(y_1) - \Phi(y_2)\| \leq L\|y_1 - y_2\|.$$

Definition 1.6.7. A linear operator Φ mapping a Banach space $\mathcal{B}$ into itself it is said to be completely continuous operator if it maps every bounded set into a relatively compact set.

Definition 1.6.8. A point y is said to be a fixed point of the operator Φ if and only if $y = \Phi y$.

Definition 1.6.9. Let $\mathbf{E}$ be a subset of the Banach space $\mathcal{B}$. $\mathbf{E}$ is said to be a cone if and only if it satisfy the following conditions:
(i)E is closed and convex.
(ii) If $f \in E,\ yf \in E$ for every $y \geq 0$.
(iii) If $f, -f \in E$ then $f = 0$.

Definition 1.6.10. A class of sets in a real Banach space $\mathcal{B}$ is said to cover a given set $\mathbf{X}$ iff each point of $\mathbf{X}$ lies at least one sets.

Definition 1.6.11. Let $\mathcal{B}$ be a real Banach space. A set $\mathbf{X} \subset \mathcal{B}$ is said to be compact iff every class of open sets which cover $\mathbf{X}$ has a finite subclass which also cover $\mathbf{X}$.

Definition 1.6.12. Let $\mathbf{X}$ be a subset of the real Banach space $\mathcal{B}$. Then $\mathbf{X}$ is relatively compact iff its closure $\bar{\mathbf{X}}$ is compact.

Theorem 1.6.1. *(Leray-Schauder alternative theorem [17]. Let $\mathcal{B}$ be a Banach space with $C \subseteq \mathcal{B}$ closed and convex. Assume U is a relatively open subset of C with $0 \in U$ and $\Phi : \overline{U} \to C$ is a continuous, compact map. Then either*
(1)Φ has a fixed point in $\overline{U}$, or
(2) there exist $u \in \partial U$ and $\lambda \in (0, 1)$ with $u = \lambda u$.

Theorem 1.6.2. *(Schauder fixed point theorem ([19]. If U is a closed, bounded convex subset of a Banach space $\mathcal{B}$ and $\Phi : U \to U$ is completely continuous, then Φ has a fixed point in U.*

Theorem 1.6.3. *(Ascoli-Arzela theorem [25]). Let $\mathcal{B}$ be a Banach space, and $S = \{s(t)\}$ is a family of continuous mappings $s : [a, b] \to X$. If S is uniformly bounded and equicontinuous, and for any $t^* \in [a, b]$, the set $\{s(t^*)\}$ is relatively compact, then there exists a uniformly convergent sequence $\{s_n(t)\}(n = 1, 2, 3, ..., t \in [a, b]$ in S.*

Lemma 1.6.1. *(Mazur's Lemma [29]). If $\widetilde{S}$ is a compact subset of Banach space $\mathcal{B}$, then its convex closure $\overline{conv}\widetilde{S}$ is compact.*

Theorem 1.6.4. *[12] Let $\mathcal{B}$ be a Banach space and $E \in \mathcal{B}$ be a cone. Assume Ω_1 and Ω_2 are open discs contained in $\mathcal{B}$ with $0 \in \Omega_1$, $\bar{\Omega}_1 \subset \Omega_2$ and let*

$$\Phi : E \cap (\Omega_2 \backslash \bar{\Omega}_1) \to E,$$

be a completely continuous operator such that, either

(i) $\|\Phi y\| \leq \|y\|$ *for* $y \in E \cap \partial\Omega_1$ *and* $\|\Phi y\| \geq \|y\|$ *for* $y \in E \cap \partial\Omega_2$*, or*

(ii) $\|\Phi y\| \geq \|y\|$ *for* $y \in E \cap \partial\Omega_1$ *and* $\|\Phi y\| \leq \|y\|$ *for* $y \in E \cap \partial\Omega_2$*.*

Then Φ has at least one fixed point in $E \cap (\bar{\Omega}_2 \backslash \Omega_1)$.

Chapter 2

Initial value problem in discrete fractional calculus

In this chapter, we will discuss some methods of solutions and the existence and uniqueness of solutions for certain initial value problems in discrete fractional calculus. The interest in the study of existence and uniqueness of solutions for initial value problems (IVP) and boundary value problems (BVP) has increased in last decade. For convenience of numerical calculations the research about fractional order difference equations becomes important.

Many authors have done quality of work in this field of study. Atici and Eloe [1] in 2009 published their paper. The particular goal of their study was to solved well-defined discrete fractional difference equations and they also solved the half-order linear problem with constant coefficients using the method of undetermined coefficients and with a transform method. F. Chen [8] also stuided the existence and uniqueness of solution for IVP. His work deals with the initial value problem of a nonlinear fractional difference equation of order $0 < \alpha \leq 1$, with the Caputo difference operator. J. Jagan and Deekshitulu [21] proved some existence and uniqueness results for solutions of various classes of fractional order difference equations.

During the course of this chapter, in section 2.1 we deal with the initial value problem by the method of successive approximation. In section 2.2, we study the half order linear problem with coefficients solved by undetermined coefficients method. In section 2.3, we discuss the discrete transform method for the solution of initial value problems. Section 2.4, contains the extension of the work in [8]. Here the difference is we take second order initial value problem rather than order one, also we have taken different initial conditions. We developed the existence and uniqueness results of solutions for second order Caputo fractional difference equation.

2.1 Method of successive approximation

This section is concerned with a nonlinear fractional difference equation of $v-th$ order ($0 < v \leq 1$) with an initial condition [1]. By mean of some commutativity properties we will construct a summation equation for fractional difference initial value problem.
Consider the following initial value problem to a nonlinear fractional difference equation:

$$\Delta^{v}y(t) = f(t+v-1, y(t+v-1)), \qquad t = 0, 1, 2, \cdots, \qquad 0 < v \leq 1, \tag{2.1}$$

$$\Delta^{v-1}y(t)|_{t=0} = \alpha_0, \tag{2.2}$$

where f is a real valued function and α_0 is a real number. The solution $y(t)$ is defined on $\mathbb{N}_{v-1}$, if exists.
First, we construct a summation equation of initial value problem (2.1) and (2.2). Applying the fractional sum operator Δ^{-v} on equation (2.1), then we have

$$\Delta^{-v}\Delta^{v}y(t) = \Delta^{-v}f(t+v-1, y(t+v-1)), \qquad t = v, v+1, v+2, \cdots.$$

Now, we apply Theorem 1.5.1 to the right side of the above equation to obtain,

$$\Delta^{-v}\Delta^{v}y(t) = \Delta^{-v}\Delta\Delta^{-(1-v)}y(t) = \Delta^{-v}f(t+v-1, y(t+v-1)). \tag{2.3}$$

Using Theorem 1.5.4 to the right side of equation (2.3)

$$\Delta\Delta^{-v}\Delta^{v-1}y(t) - \frac{(t-0)^{(v-1)}}{\Gamma(v)}\Delta^{v-1}y(t)|_{t=0} = \Delta^{-v}f(t+v-1, y(t+v-1))$$

$$\Delta\Delta^{-v}\Delta^{v}\Delta^{-1}y(t) = \frac{t^{(v-1)}}{\Gamma(v)}y(t)|_{t=0} + \Delta^{-v}f(t+v-1, y(t+v-1)).$$

Using initial condition, we obtain for $t \in \mathbb{N}_{v-1}$

$$y(t) = \frac{t^{(v-1)}}{\Gamma(v)}\alpha_0 + \frac{1}{\Gamma(v)}\sum_{s=0}^{t-v}(t-\sigma(s))^{(v-1)}f(s+v-1, y(s+v-1)). \tag{2.4}$$

The above equation represents the summation equation of the initial value problem (2.1) and (2.2).

Example 2.1.1. [1] Consider the initial value problem with constant coefficient.

$$\Delta^v y(t) = \lambda y(t+v-1), \qquad t = 0, 1, 2, \cdots, \qquad 0 < v \le 1, \tag{2.5}$$

$$\Delta^{v-1} y(t)|_{t=0} = \alpha_0. \tag{2.6}$$

The function is defined on $\mathbb{N}_{v-1}$, and $\Delta^{v-1}y(t)|_{t=0} = y(v-1)$. Since $(-t)^{(-t)} = \Gamma(1-t)$. Then the above initial value problem is equivalent to,

$$\begin{aligned} \Delta^v y(t) &= \lambda y(t+v-1), \qquad t = 0, 1, 2, \cdots, \\ y(v-1) &= \alpha_0. \end{aligned}$$

By equation (2.4) the solution of the initial value problem (2.5) and (2.6) is given as,

$$y(t) = \frac{t^{(v-1)}}{\Gamma(v)}\alpha_0 + \frac{\lambda}{\Gamma(v)} \sum_{s=0}^{t-v} (t-\sigma(s))^{(v-1)} y(s+v-1).$$

We use the method of successive approximations to obtain an explicit solution. Set

$$y_0(t) = \frac{t^{(v-1)}}{\Gamma(v)}\alpha_0,$$

$$\text{and} \qquad \begin{aligned} y_n(t) &= y_0(t) + \frac{\lambda}{\Gamma(v)} \sum_{s=0}^{t-v} (t-\sigma(s))^{(v-1)} y_{n-1}(s+v-1) \\ &= y_0(t) + \lambda \Delta^{-v} y_{n-1}(t+v-1), \qquad n = 1, 2, 3, \cdots. \end{aligned}$$

For $n = 1$, we use the power rule (Lemma 1.5.1),

$$\begin{aligned} y_1(t) &= y_0(t) + \lambda \Delta^{-v} y_0(t+v-1) \\ &= \frac{t^{(v-1)}}{\Gamma(v)}\alpha_0 + \lambda \Delta^{-v} \frac{(t+v-1)^{(v-1)}}{\Gamma(v)}\alpha_0 \\ &= \frac{t^{(v-1)}}{\Gamma(v)}\alpha_0 + \alpha_0 \lambda \frac{\Gamma(v-1+1)}{\Gamma(v)\Gamma(v-1+v+1)}(t+v-1)^{(v-1+v)} \\ &= \frac{t^{(v-1)}}{\Gamma(v)}\alpha_0 + \alpha_0 \lambda \frac{\Gamma(v)}{\Gamma(v)\Gamma(2v)}(t+v-1)^{(2v-1)} \\ &= \alpha_0 \left(\frac{t^{(v-1)}}{\Gamma(v)} + \lambda \frac{(t+v-1)^{(2v-1)}}{\Gamma(2v)} \right). \end{aligned}$$

After repeated application of power rule, we have

$$y_n(t) = \alpha_0 \sum_{k=0}^{n} \frac{\lambda^k}{\Gamma((k+1)v)} (t+(k-1)(v-1))^{(kv+v-1)}, \qquad n = 0, 1, 2, 3, \cdots.$$

Let $n \to \infty$, we obtain the solution

$$y(t) = \alpha_0 \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma((k+1)v)} (t + (k-1)(v-1))^{(kv+v-1)}. \tag{2.7}$$

Set $v = 1$, then

$$y(t) = \alpha_0 \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} (t)^{(k)} = a_0(1+\lambda)^t.$$

This is the unique solution of initial value problem (2.5) and (2.6).

2.2 Undetermined coefficients method

This section contains the half-order difference equation with constant coefficients and we apply method of undetermined coefficients, as Miller and Ross used in [28] to find the solution.

Example 2.2.1. [1] Consider the following half-order difference equation,

$$\Delta^{1/2} y(t) + a\Delta^0 y(t - 1/2) = 0, \quad t = 0, 1, 2, \cdots.$$

Suppose that a solution $y(t)$ has the form

$$y(t) = CF(t, 0, \lambda) + F(t, -1/2, \lambda).$$

Since y is defined on $\mathbb{N}_{-1/2}$. Substituting the value of $y(t)$ in above equation,

$$\begin{aligned} \Delta^{1/2} y(t) + a\Delta^0 y(t - 1/2) &= \Delta^{1/2}\{CF(t, 0, \lambda) + F(t, -1/2, \lambda)\} \\ &\quad + a\Delta^0\{CF(t - 1/2, 0, \lambda) + F(t - 1/2, -1/2, \lambda)\}. \end{aligned} \tag{2.8}$$

Also we know that $F(t, v, \alpha) = \Delta^{-v}(1+\alpha)^t$, where v is any real number. We can write the equation (2.8) as

$$\begin{aligned} \Delta^{1/2} y(t) + a\Delta^0 y(t - 1/2) = C\Delta^{1/2}(1+\lambda)^t + \Delta^{1/2}\Delta^{1/2}(1+\lambda)^t + aC(1+\lambda)^{-1/2}(1+\lambda)^t \\ + a(1+\lambda)^{-1/2}\Delta^{1/2}(1+\lambda)^t. \end{aligned} \tag{2.9}$$

We simplify the $\Delta^{1/2}\Delta^{1/2}(1+\lambda)^t$ part of equation (2.9) by using the power rule and Theorem 1.5.4, so we have

$$\begin{aligned} \Delta^{1/2}\Delta^{1/2}(1+\lambda)^t &= \Delta^{1/2}\Delta\Delta^{-1/2}(1+\lambda)^t \\ &= \Delta^{1/2}\left(\Delta^{-1/2}\Delta(1+\lambda)^t + \frac{(t+1/2)^{(-1/2)}}{\Gamma(1/2)}(1+\lambda)^{-1/2}\right) \end{aligned}$$

$$
\begin{aligned}
&= \Delta^{1/2}\Delta^{-1/2}\Delta(1+\lambda)^t + \Delta^{1/2}\left(\frac{(t+1/2)^{(-1/2)}}{\Gamma(1/2)}(1+\lambda)^{-1/2}\right)\\
&= \Delta(1+\lambda)^t + \Delta\Delta^{-1/2}\left(\frac{(t+1/2)^{(-1/2)}}{\Gamma(1/2)}(1+\lambda)^{-1/2}\right)\\
&= \left((1+\lambda)^{t+1} - (1+\lambda)^t\right) + \Delta\left(\frac{\Gamma(1/2)(t+1/2)^{(-1/2+1/2)}}{\Gamma(1/2)\Gamma(-1/2+1/2+1)}(1+\lambda)^{-1/2}\right)\\
&= \lambda(1+\lambda)^t + \Delta(1+\lambda)^{-1/2}\\
&= \lambda(1+\lambda)^t.
\end{aligned}
$$

By substituting this value in equation (2.9), we get

$$
\begin{aligned}
\Delta^{1/2}y(t) + a\Delta^0 y(t-1/2) &= C\Delta^{1/2}(1+\lambda)^t + \lambda(1+\lambda)^t + aC(1+\lambda)^{-1/2}(1+\lambda)^t\\
&+ a(1+\lambda)^{-1/2}\Delta^{1/2}(1+\lambda)^t\\
&= \Delta^{1/2}(1+\lambda)^t\left(C + a(1+\lambda)^{-1/2}\right) + (1+\lambda)^t\left(\lambda + Ca(1+\lambda)^{-1/2}\right)\\
&= F(t,-1/2,\lambda)\left(C + a(1+\lambda)^{-1/2}\right) + F(t,0,\lambda)\left(\lambda + Ca(1+\lambda)^{-1/2}\right)\\
&= (1+\lambda)^{-1/2}F(t,-1/2,\lambda)\left(C(1+\lambda)^{1/2} + a\right)\\
&+ (1+\lambda)^{-1/2}F(t,0,\lambda)\left(\lambda(1+\lambda)^{1/2} + Ca\right)\\
&= (1+\lambda)^{-1/2}F(t,-1/2,\lambda)\left(C(1+\lambda)^{1/2} + a\right)\\
&+ C(1+\lambda)^{-1/2}F(t,0,\lambda)\left(C^{-1}\lambda(1+\lambda)^{1/2} + a\right).
\end{aligned}
$$

Set $P(\lambda) = \sqrt{\lambda(1+\lambda)} + a$, and $\lambda = C^2$, then we can write above equation as

$$
\begin{aligned}
\Delta^{1/2}u(t) + a\Delta^0 u(t-1/2) &= (1+\lambda)^{-1/2}F(t,-1/2,\lambda)\left(\sqrt{\lambda(1+\lambda)} + a\right)\\
&+ \sqrt{\lambda}(1+\lambda)^{-1/2}F(t,0,\lambda)\left(\frac{\lambda}{\sqrt{\lambda}}\sqrt{1+\lambda} + a\right)\\
&= (1+\lambda)^{-1/2}P(\lambda)\left(F(t,-1/2,\lambda) + \sqrt{\lambda}F(t,0,\lambda)\right).
\end{aligned}
$$

Thus,

$$
y(t) = \sqrt{\lambda}F(t,0,\lambda) + F(t,-1/2,\lambda).
$$

This represents the solution of half-order difference equation for $t \in \{-1/2, 1/2, 3/2, \cdots\}$.

2.3 Discrete transform method

In [2] authors extended the discrete Laplace transform method to develop a discrete transform method for fractional difference equations.
Discrete transform (R-transform) for $f \in \mathbb{N}_a$ is defined as

$$R_{t_0}(f(t))(s) = \sum_{t=t_0}^{\infty} \left(\frac{1}{s+1}\right)^{t+1} f(t). \tag{2.10}$$

Discrete transform is not the same as the more common z-transform. This discrete transform is the Laplace transform on the time scale of integers.

Lemma 2.3.1. *[2] Let for any $v \in \mathbb{R}\backslash\{\cdots,-2,-1,0\}$, then*

$$\begin{aligned} &(i) \quad R_{v-1}(t^{(v-1)})(s) = \frac{\Gamma(v)}{s^v}, \\ &(ii) \quad R_{v-1}(t^{(v-1)}\alpha^t)(s) = \frac{\alpha^{v-1}\Gamma(v)}{(s+1-\alpha)^v}. \end{aligned}$$

Proof. (i)If $v = 1$, then we have straight forward calculations with geometric series. For $v = 2, 3, \cdots$ the problem is solved by induction method. First, we apply Theorem 1.3.1 (i) and then use summation by parts. For $0 < v < 1$, we have

$$R_{v-1}(t^{(v-1)}) = \sum_{t=v-1}^{\infty} \left(\frac{1}{s+1}\right)^{t+1} t^{(v-1)}.$$

Substitute $t = \mu + v - 1$, we get

$$\begin{aligned} R_{v-1}((\mu+v-1)^{(v-1)}) &= \sum_{\mu=0}^{\infty} \left(\frac{1}{s+1}\right)^{\mu+v} (\mu+v-1)^{(v-1)} \\ &= \left(\frac{1}{s+1}\right)^{v} \sum_{\mu=0}^{\infty} \left(\frac{1}{s+1}\right)^{\mu} \frac{\Gamma(\mu+v-1+1)}{\Gamma(\mu+v-v+1)} \\ &= \left(\frac{1}{s+1}\right)^{v} \sum_{\mu=0}^{\infty} \left(\frac{1}{s+1}\right)^{\mu} \frac{\Gamma(\mu+v)}{\Gamma(\mu+1)}. \end{aligned}$$

We use the following equality [[6], Theorem 2.2.1] of hypergeometric function ${}_2F_1$ to the right side of above equation,

$$\frac{\Gamma(c)}{\Gamma(b)} \sum_{n=0}^{\infty} \frac{(a)^{(n)}\Gamma(n+b)}{n!\Gamma(n+c)} X^n = {}_2F_1(a,b;c;x).$$

And if $c, b > 0$ then

$$ {}_2F_1(a,b;c;x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{n+b-1}(1-t)^{c-b-1}dt. $$

So we have

$$ \begin{aligned} \sum_{t=v-1}^{\infty}\left(\frac{1}{s+1}\right)^{t+1} t^{(v-1)} &= \frac{1}{(s+1)^v}\Gamma(v)_2F_1\left(1,v;1;\frac{1}{s+1}\right) \\ &= \frac{1}{(s+1)^v}\frac{\Gamma(v)\Gamma(1)}{\Gamma(v)\Gamma(1-v)}\int_0^1 t^{v-1}(1-t)^{-v}\left(1-\frac{t}{s+1}\right)^{-1}dt \\ &= \frac{1}{(s+1)^v}\frac{1}{\Gamma(1-v)}\int_0^1 t^{v-1}(1-t)^{-v}\left(\frac{s+1-t}{s+1}\right)^{-1}dt \\ &= \frac{1}{(s+1)^{v-1}}\frac{1}{\Gamma(1-v)}\int_0^1 (1-\mu)^{v-1}\mu^{-v}(s+u)^{-1}d\mu \qquad \because t = 1-\mu \\ &= \frac{1}{(s+1)^{v-1}}\frac{1}{\Gamma(1-v)}\int_0^1 \frac{(1-\mu)^{v-1}\mu^{-v}}{(s+u)}d\mu. \end{aligned} $$

Here we use a property of beta function to get

$$ \begin{aligned} \sum_{t=v-1}^{\infty}\left(\frac{1}{s+1}\right)^{t+1} t^{(v-1)} &= \frac{1}{(s+1)^{v-1}}\frac{1}{\Gamma(1-v)}\int_0^1 B(v,1-v)(1+s)^{v-1}s^{-v} \\ &= \frac{\Gamma(v)}{s^v}. \end{aligned} $$

For $0 < v < 1$, the proof is complete.
The proof of (ii) follows from the proof of (i). □

We shall use the convolution theorem by Bohner and Peterson [11] in which they have developed the convolution on the time scale of integers.
Assume $h(t) = t^{(v-1)}$ and $g(t) = a^t$ then convolution is defined as

$$ (h * g)(t) = \sum_{s=0}^{t-v}(t-\sigma(s))^{(v-1)}g(s), $$

where $v \in \mathbb{R}/\{\cdots,-2,-1,0\}$.

Lemma 2.3.2. *[2] Let for any $v \in \mathbb{R}/\{\cdots,-2,-1,0\}$ we have*

$$ R_v((h * g)(t))(s) = R_{v-1}(h(t))(s)R_0(g(t))(s). $$

Proof. We have

$$\begin{aligned}
R_v((h*g)(t))(s) &= \sum_{t=v}^{\infty}\left(\frac{1}{s+1}\right)^{t+1}(h*g)(s)\\
&= \sum_{t=v}^{\infty}\left(\frac{1}{s+1}\right)^{t+1}\sum_{r=0}^{t-v}(t-\sigma(r))^{(v-1)}g(r)\\
&= \sum_{r=0}^{\infty}\sum_{t=r+v}^{\infty}\left(\frac{1}{s+1}\right)^{t+1}(t-\sigma(r))^{(v-1)}g(r)\\
&= \sum_{r=0}^{\infty}\sum_{\tau=v-1}^{\infty}\left(\frac{1}{s+1}\right)^{\tau+r+2}\tau^{(v-1)}g(r) \qquad \because \tau = t-r-1\\
&= \left(\sum_{\tau=v-1}^{\infty}\left(\frac{1}{s+1}\right)^{\tau+1}\tau^{(v-1)}\right)\left(\sum_{r=0}^{\infty}\left(\frac{1}{s+1}\right)^{r+1}g(r)\right)\\
&= R_{v-1}(h(t))(s)R_0(g(t))(s).
\end{aligned}$$

□

We have another property of R-transform that is used to solve fractional difference equation. That is

$$R_{a+v}(\Delta^{-v}f(t))(s) = s^{-v}R_a(f(t))(s). \tag{2.11}$$

Lemma 2.3.3. *[2] Let $0 < v < 1$ and the function f defined for $\{v-1, v, v+1, \cdots\}$*

$$R_0(\Delta^v f(t))(s) = s^v R_{v-1}(f(t)) - f(v-1).$$

Proof. In [7, 11] it is shown that, if m is a positive integer, then

$$R_0(\Delta^m f(t))(s) = s^m R_0(f)(s) - \sum_{k=0}^{m-1} s^{m-k-1}\Delta^k f|_{t=0}.$$

If we take $m = 1$, then we have

$$\begin{aligned}
R_0(\Delta^m f(t)) = R_0(\Delta\Delta^{-(1-v)}f(t)) &= \sum_{t=0}^{\infty}\left(\frac{1}{s+1}\right)^{t+1}\Delta(\Delta^{-(1-v)})f(t)\\
= s\sum_{t=0}^{\infty}\left(\frac{1}{s+1}\right)^{t+1}&\Delta^{-(1-v)}f(t) - \sum_{k=0}^{1-1}s^{1-k-1}\Delta^k\Delta^{-(1-v)}f|_{t=0}\\
= s\sum_{t=0}^{\infty}\left(\frac{1}{s+1}\right)^{t+1}&\Delta^{-(1-v)}f(t) - \Delta^{-(1-v)}f|_{t=0}
\end{aligned}$$

$$
\begin{aligned}
&= s\sum_{t=0}^{\infty}\left(\frac{1}{s+1}\right)^{t+1}\Delta^{-(1-v)}f(t) - \frac{1}{\Gamma(v-1)}\sum_{r=v-1}^{v-1}(-r-1)^{(-v)}f(r)\\
&= s\sum_{t=0}^{\infty}\left(\frac{1}{s+1}\right)^{t+1}\Delta^{-(1-v)}f(t) - f(v-1)\\
&= s\sum_{t=0}^{\infty}\left(\frac{1}{s+1}\right)^{t+1}\frac{1}{\Gamma(v-1)}\sum_{r=v-1}^{t+v-1}(t-r-1)^{(-v)}f(r) - f(v-1)\\
&= \frac{s}{\Gamma(v-1)}\sum_{t=r-v+1}^{\infty}\left(\frac{1}{s+1}\right)^{t+1}\sum_{r=v-1}^{\infty}(t-r-1)^{(-v)}f(r) - f(v-1)\\
&= \frac{s}{\Gamma(v-1)}\sum_{\tau=-v}^{\infty}\left(\frac{1}{s+1}\right)^{\tau+r+2}\sum_{r=v-1}^{\infty}(\tau)^{(-v)}f(r) - f(v-1)\\
&= \frac{s}{\Gamma(v-1)}\sum_{\tau=-v}^{\infty}\left(\frac{1}{s+1}\right)^{\tau+1}(\tau)^{(-v)}\sum_{r=v-1}^{\infty}\left(\frac{1}{s+1}\right)^{r+1}f(r) - f(v-1)\\
&= \frac{s}{\Gamma(v-1)}R_{-v}\tau^{(-v)}(s)R_{v-1}(f(t))(s) - f(v-1)\\
&= \frac{s}{\Gamma(v-1)}\frac{\Gamma(1-v)}{s^{1-v}}R_{v-1}(f(t)) - f(v-1) \quad \text{by Lemma 2.3.1}\\
&= s^{v}R_{v-1}(f(t)) - f(v-1).
\end{aligned}
$$

□

Example 2.3.1. [2] We take $\Delta^{1/2}y(t) = \lambda$ for $t = 0, 1, 2, \cdots$. We shall find the solution of $y(t)$ define on $\{\frac{-1}{2}, \frac{1}{2}, \frac{3}{2}, \cdots\}$.

Apply R-transform on both side of given problem,

$$
\begin{aligned}
R_0(\Delta^{1/2}y(t)) &= R_0(\lambda)\\
s^{1/2}R_{-1/2}(y(t)) - y\left(-\frac{1}{2}\right) &= \frac{\lambda}{s} \quad \text{by Lemma2.3.3}\\
R_{-1/2} &= \frac{y(-1/2)}{s^{1/2}} + \frac{\lambda}{s^{3/2}}.
\end{aligned}
$$

Since $R_{-1/2}(t^{(-1/2)}) = \frac{\Gamma(1/2)}{s^{1/2}}$ and $R_{1/2}(t^{(1/2)}) = \frac{\Gamma(3/2)}{s^{3/2}}$, then we can write above equation as

$$
R_{-1/2} = \frac{y(-1/2)}{\Gamma(1/2)}R_{-1/2}(t^{(-1/2)}) + \frac{\lambda}{\Gamma(3/2)}R_{1/2}(t^{(1/2)}).
$$

Note that $R_{1/2}(t^{(1/2)}) = R_{-1/2}(t^{(1/2)})$, thus we have the solution for $t \in \{\frac{-1}{2}, \frac{1}{2}, \frac{3}{2}, \cdots\}$

$$
y(t) = \frac{y(-1/2)}{\Gamma(1/2)}(t^{(-1/2)}) + \frac{\lambda}{\Gamma(3/2)}(t^{(1/2)}).
$$

2.4 Existence and uniqueness results for an initial value problem

In this study, we will establish necessary and sufficient conditions for the existence and uniqueness of solutions for a class of fractional difference equations. We deal with following initial value problem to nonlinear fractional difference equation of order $1 < v \leq 2$, with the Caputo difference operator.

$$\Delta_*^v y(t) = f(t+v-1, y(t+v-1)), \qquad t \in \mathbb{N}_{v-1}, \qquad 1 < v \leq 2, \tag{2.12}$$

$$y(v-2) = y_0, \qquad \Delta y(v-2) = y_1. \tag{2.13}$$

Here Δ_*^v is discrete Caputo fractional difference, $f : [0, \infty) \times Y \to Y$ is continuous in t and Y is a real Banach space with the norm $\|y\| = \sup\{|y(t)|, t \in \mathbb{N}\}, \mathbb{N}_v = \{v, v+1, \cdots .\}$. This work is the extension of [8]. The difference here is we consider second order fractional difference equation rather than order one and we used different initial conditions. By using Caputo difference operator we construct equivalence summation equation of problem (2.12) and (2.13) which is different from as given in [8]. We will use fixed point theorems, Leray-Schauder alternative theorem and Schauder fixed point theorem to obtain the existence and uniqueness of a solutions for ((2.12)-(2.13)).

First, we construct a summation equation of the problem (2.12) by using the Theorem 1.5.5, when $1 < v \leq 2$ and $a = v-2$.

$$\begin{aligned} y(t) &= \sum_{k=0}^{1} \frac{(t-(v-2))^{(k)}}{k!} \Delta^k y(v-2) + \frac{1}{\Gamma(v)} \sum_{s=v-2+2-v}^{t-v} (t-s-1)^{(v-1)} \Delta_*^v y(s) \\ &= y(v-2) + (t-v+2)\Delta y(v-2) + \frac{1}{\Gamma(v)} \sum_{s=0}^{t-v} (t-s-1)^{(v-1)} \Delta_*^v y(s) \\ &= y(v-2) + (t-v+2)\Delta y(v-2) + \frac{1}{\Gamma(v)} \sum_{s=0}^{t-v} (t-s-1)^{(v-1)} f(s+v-1, y(s+v-1)) \\ &= y_0 + y_1(t-v+2) + \frac{1}{\Gamma(v)} \sum_{s=0}^{t-v} (t-s-1)^{(v-1)} f(s+v-1, y(s+v-1)). \end{aligned}$$

It represents the summation form of problem ((2.12)-(2.13)).

Lemma 2.4.1. *A function $y(t) : N \to Y$ is a solution of the IVP ((2.12)-(2.13)) if and only if $y(t)$ is a solution of the following fractional sum equation:*

$$y(t) = y_0 + y_1(t - v + 2) + \frac{1}{\Gamma(v)} \sum_{s=0}^{t-v} (t - s - 1)^{(v-1)} f(s + v - 1, y(s + v - 1)). \tag{2.14}$$

Proof. Suppose that $y(t)$ is a solution of IVP (2.12), that is $\Delta_*^v y(t) = f(t + v - 1, y(t + v - 1))$ for $t \in \mathbb{N}_{v-1}$, an approach of Theorem 1.5.5 implies equation (2.14).
Conversely, let $y(t)$ is a solution of equation (2.14), then

$$y(t) = y_0 + y_1(t - v + 2) + \frac{1}{\Gamma(v)} \sum_{s=0}^{t-v} (t - s - 1)^{(v-1)} f(s + v - 1, y(s + v - 1)). \tag{2.15}$$

Consequently Theorem 1.5.5 yields that,

$$y(t) = y_0 + y_1(t - v + 2) + \frac{1}{\Gamma(v)} \sum_{s=0}^{t-v} (t - s - 1)^{(v-1)} \Delta_*^v y(s). \tag{2.16}$$

By comparing equations (2.15) and (2.16),we get

$$\frac{1}{\Gamma(v)} \sum_{s=0}^{t-v} (t - s - 1)^{(v-1)} [\Delta_*^v y(s) - f(s + v - 1, y(s + v - 1))] = 0.$$

We have that $\Delta_*^v y(t) = f(t + v - 1, y(t + v - 1))$ for $t \in \mathbb{N}_{v-1}$, which implies that $y(t)$ is a solution of problem (2.12). □

Lemma 2.4.2. *We have*

$$\sum_{s=0}^{t-v} (t - s - 1)^{(v-1)} = \frac{\Gamma(t + 1)}{v\Gamma(t - v + 1)}. \tag{2.17}$$

Proof. We know

$$\begin{aligned} \sum_{s=0}^{t-v} (t - s - 1)^{(v-1)} &= \sum_{s=0}^{t-v} \frac{\Gamma(t - s)}{\Gamma(t - s - v + 1)} \\ &= \sum_{s=0}^{t-v-1} \frac{\Gamma(t - s)}{\Gamma(t - s - v + 1)} + \Gamma(v). \end{aligned}$$

By using Lemma 1.1.1, we have

$$\begin{aligned}
&= \sum_{s=0}^{t-v-1} \frac{1}{v}\left[\frac{\Gamma(t-s+1)}{\Gamma(t-s-v+1)} - \frac{\Gamma(t-s)}{\Gamma(t-s-v)}\right] + \Gamma(v) \\
&= \frac{1}{v}\left[\frac{\Gamma(t+1)}{\Gamma(t-v+1)} - \Gamma(v+1)\right] + \Gamma(v) \\
&= \frac{\Gamma(t+1)}{v\Gamma(t-v+1)}.
\end{aligned}$$

□

2.4.1 Uniqueness result

In this section, we establish uniqueness result of problem (2.12) by applying the Banach contraction principle. Set $\mathbb{N}_k = \{0, 1, 2, \cdots, k\}$, where $k \in \mathbb{N}$.

Theorem 2.4.1. *Let $f : [0, k] \times Y \to Y$ is locally Lipschitz continuous (with constant L) on Y, then the IVP (2.12) has unique solution $y(t)$ on $t \in \mathbb{N}$ provided that,*

$$\frac{L\Gamma(k+1)}{\Gamma(v+1)\Gamma(k-v+1)} < 1. \tag{2.18}$$

Proof. A mapping $\Phi : Y|_{t\in\mathbb{N}_k} \to Y|_{t\in\mathbb{N}_k}$ is defined as:

$$(\Phi y)(t) = y_0 + y_1(t-v+2) + \frac{1}{\Gamma(v)}\sum_{s=0}^{t-v}(t-s-1)^{(v-1)} f(s+v-1, y(s+v-1)), \tag{2.19}$$

for $t \in \mathbb{N}_k$. We show that Φ is contraction, for $x, y \in Y|_{t\in\mathbb{N}_k}$ it follows that
$\|(\Phi x)(t) - (\Phi y)(t)\|$

$$\begin{aligned}
&\le \frac{1}{\Gamma(v)}\sum_{s=0}^{t-v}(t-s-1)^{(v-1)}\|f(s+v-1, x(s+v-1)) - f(s+v-1, y(s+v-1))\| \\
&\le \frac{L}{\Gamma(v)}\sum_{s=0}^{t-v}(t-s-1)^{(v-1)}\|x-y\| \\
&\le \frac{L}{v\Gamma(v)}\frac{\Gamma(t+1)}{\Gamma(t-v+1)}\|x-y\| \\
&\le \frac{L(k+1)(k)\cdots(t+1)\Gamma(t+1)}{v\Gamma(v)(k-v)\cdots(t-v+1)\Gamma(t-v+1)}\|x-y\| \\
&\le \frac{L\Gamma(k+1)}{\Gamma(v+1)\Gamma(k-v+1)}\|x-y\|.
\end{aligned}$$

By applying Banach contraction principle, Φ has unique fixed point $y^*(t)$ which is the unique solution of the IVP (2.12) and (2.13). □

2.4.2 Existence result

This section contains the existence result of problem (2.12). For this we show that the operator Φ is completely continuous and has at least one fixed point.

Theorem 2.4.2. *Let there exist $L_1, L_2 > 0$ such that $\|f(t,y)\| \leq L_1\|y\| + L_2$ for $y \in Y$, and the set $\widetilde{S} = \{(t-s-1)^{(v-1)} f(s+v-1, y(s+v-1)) : y \in Y, s \in \{v-1, \cdots, v+t\}\}$ is relatively compact for every $t \in \mathbb{N}_k$, then for IVP (2.12) there exist at least one solution $y(t)$ on $t \in \mathbb{N}_k$ provided that,*

$$\frac{L_1\Gamma(k+1)}{\Gamma(v+1)\Gamma(k-v+1)} < 1. \tag{2.20}$$

Proof. Let Φ be the operator defined by equation (2.19), The set Λ is defined as follows:

$$\Lambda = \{y(t) : \|y(t)\| \leq M+1, t \in \mathbb{N}_k\}, \tag{2.21}$$

where

$$M = \frac{\Gamma(v+1)\Gamma(k-v+1)(\|y_0\| + (k-v+2)\|y_1\|) + L_2\Gamma(k+1)}{\Gamma(v+1)\Gamma(k-v+1) - L_1\Gamma(k+1)}. \tag{2.22}$$

Suppose that there exist $y \in \Lambda$ and $\lambda \in (0,1)$ such that $y = \lambda\Phi y$. We claim that $\|y\| \neq M+1$. Since

$$y(t) = \lambda y_0 + \lambda y_1(t-v+2) + \frac{\lambda}{\Gamma(v)}\sum_{s=0}^{t-v}(t-s-1)^{(v-1)} f(s+v-1, y(s+v-1)),$$

then,

$$\begin{aligned}
\|y(t)\| &\leq \|y_0\| + \|y_1\|(t-v+2) + \frac{1}{\Gamma(v)}\sum_{s=0}^{t-v}(t-s-1)^{(v-1)}\|f(s+v-1, y(s+v-1))\| \\
&\leq \|y_0\| + \|y_1\|(t-v+2) + \frac{1}{\Gamma(v)}\sum_{s=0}^{t-v}(t-s-1)^{(v-1)}(L_1\|y\| + L_2) \\
&\leq \|y_0\| + \|y_1\|(k-v+2) + \frac{L_1\Gamma(k+1)}{\Gamma(v+1)\Gamma(k-v+1)}\|y\| + \frac{L_2\Gamma(k+1)}{\Gamma(v+1)\Gamma(k-v+1)}.
\end{aligned}$$

Taking $\|y\|$ common,

$$\|y\|\left(1-\frac{L_1\Gamma(k+1)}{\Gamma(v+1)\Gamma(k-v+1)}\right) \leq \|y_0\|+\|y_1\|(k-v+2)+\frac{L_2\Gamma(k+1)}{\Gamma(v+1)\Gamma(k-v+1)}$$

$$\|y\| \leq \frac{\Gamma(v+1)\Gamma(k-v+1)(\|y_0\|+(k-v+2)\|y_1\|)+L_2\Gamma(k+1)}{\Gamma(v+1)\Gamma(k-v+1)-L_1\Gamma(k+1)}.$$

Which implies that $\|y\| \neq M+1$.
Since f is continuous therefore the operator Φ is continuous. Now we prove that the operator Φ is also completely continuous in Λ. For any $\epsilon > 0$, there exists $t_1, t_2 \in \mathbb{N}_k, (t_1 > t_2)$ where $\mathbb{N}_k = \{0, 1, 2, \cdots, k\}$ such that,

$$\left|\frac{(t_1-2)(t_1-3)\cdots(t_2-1)}{(t_1-v)(t_1-v-1)\cdots(t_2-v+1)}-1\right| < \frac{\Gamma(v)\Gamma(k-v+1)}{(L_1M+L_2\Gamma(k-1))}\epsilon. \tag{2.23}$$

Then we have

$$\|(\Phi y)(t_1)-(\Phi y)(t_2)\|$$

$$= \|y_1\{(t_1-v+2)-(t_2-v+2)\}\|+\frac{1}{\Gamma(v)}\|\sum_{s=0}^{t_1-v}(t_1-s-1)^{(v-1)}f(s+v-1,y(s+v-1))$$

$$-\sum_{s=0}^{t_2-v}(t_2-s-1)^{(v-1)}f(s+v-1,y(s+v-1))\|$$

$$\leq \|y_1(t_1-t_2)\|+\frac{1}{\Gamma(v)}\|\sum_{s=0}^{t_2-v}\left[(t_1-s-1)^{(v-1)}-(t_2-s-1)^{(v-1)}\right]f(s+v-1,y(s+v-1))\|$$

$$+\frac{1}{\Gamma(v)}\|\sum_{s=t_2-v+1}^{t_1-v}(t_1-s-1)^{(v-1)}f(s+v-1,y(s+v-1))\|$$

$$\leq \|y_1\|(t_1-t_2)+\frac{L_1M+L_2}{\Gamma(v)}\left[\sum_{s=0}^{t_2-v}(t_1-s-1)^{(v-1)}-\sum_{s=0}^{t_2-v}(t_2-s-1)^{(v-1)}\right]$$

$$+\frac{L_1M+L_2}{\Gamma(v)}\sum_{s=t_2-v+1}^{t_1-v}(t_1-s-1)^{(v-1)}$$

$$= \|y_1\|(t_1-t_2)+\frac{L_1M+L_2}{v\Gamma(v)}\left[\frac{\Gamma(t_1-1)}{\Gamma(t_1-v+1)}-\frac{\Gamma(t_1-t_2+v)}{\Gamma(t_1-t_2)}-\frac{\Gamma(t_2-1)}{\Gamma(t_2-v+1)}+\frac{\Gamma(t_1-t_2+v)}{\Gamma(t_1-t_2)}\right]$$

$$= \|y_1\|(t_1-t_2)+\frac{L_1M+L_2}{v\Gamma(v)}\left[\frac{\Gamma(t_1-1)}{\Gamma(t_1-v+1)}-\frac{\Gamma(t_2-1)}{\Gamma(t_2-v+1)}\right]$$

$$= \|y_1\|(t_1-t_2)+\frac{L_1M+L_2}{v\Gamma(v)}\frac{\Gamma(t_2-1)\Gamma(t_2-v+1)}{\Gamma(t_2-1)\Gamma(t_2-v+1)}\left[\frac{\Gamma(t_1-1)}{\Gamma(t_1-v+1)}-\frac{\Gamma(t_2-1)}{\Gamma(t_2-v+1)}\right]$$

$$
\begin{aligned}
&= \|y_1\|(t_1-t_2) + \frac{L_1M+L_2}{v\Gamma(v)}\frac{\Gamma(t_2-1)}{\Gamma(t_2-v+1)}\left[\frac{\Gamma(t_1-1)\Gamma(t_2-v+1)}{\Gamma(t_1-v+1)\Gamma(t_2-1)} - 1\right] \\
&\le \|y_1\|k + \frac{L_1M+L_2}{v\Gamma(v)}\frac{\Gamma(k-1)}{\Gamma(k-v+1)}\left[\frac{(t_1-2)(t_1-3)\cdots(t_2-1)}{(t_1-v)(t_1-v-1)\cdots(t_2-v+1)} - 1\right] \\
&< \|y_1\|k + \frac{L_1M+L_2}{v\Gamma(v)}\frac{\Gamma(k-1)}{\Gamma(k-v+1)}\frac{v\Gamma(v)\Gamma(k-v+1)}{(L_1M+L_2)\Gamma(k-1)}\epsilon \\
&< \|y_1\|k + \epsilon.
\end{aligned}
$$

This shows that the set $\Phi\Lambda$ is an equicontinuous set. The condition that $\widetilde{S}$ is relatively compact and by Lemma 1.6.1, we know that for any $t^* \in \mathbb{N}_k$, $\overline{conv}\widetilde{S}$ is compact.

$$
\begin{aligned}
(\Phi y_n)(t^*) &= y_0 + y_1(t^*-v+2) + \frac{1}{\Gamma(v)}\sum_{s=0}^{t^*-v}(t^*-s-1)^{(v-1)}f(s+v-1, y_n(s+v-1)) \\
(\Phi y_n)(t^*) &= y_0 + y_1(t^*-v+2) + \frac{1}{\Gamma(v)}\zeta_n,
\end{aligned}
$$

where

$$
\zeta_n : \sum_{s=0}^{t^*-v}(t^*-s-1)^{(v-1)}f(s+v-1, y_n(s+v-1)). \tag{2.24}
$$

We know that, $\zeta_n \in \overline{conv}\widetilde{S}$, and $\overline{conv}\widetilde{S}$ is convex and compact. Then for any $t^* \in \mathbb{N}_k$, the set $\{(\Phi y_n)(t^*)\}(n = 1, 2, \cdots)$ is relatively compact. According to Theorem 1.6.3, every $\{(\Phi y_n)(t)\}$ contains a uniform convergent subsequence $\{(\Phi y_{n_k})(t)\}(k = 1, 2, \cdots)\}$ on $\mathbb{N}_k$, which shows that the set $\Phi\Lambda$ is relatively compact. Since $\Phi\Lambda$ is bounded, equicontinuous and relatively compact set, which means that Φ is completely continuous.

Therefore, the Leray-schauder fixed point theorem guarantees that Φ has a fixed point, which means that for $t \in \mathbb{N}_k$, there exist at least one solution of the IVP (2.12). □

Corollary 2.4.1. *Suppose that there exists $M > 0$ such that $\|f(t,y)\| \le M$ for any $t \in [0,k]$ and $y \in Y$, and the set $\widetilde{S} = \{(t-s-1)^{(v-1)}f(s+v-1, y(s+v-1)) : y \in Y, s \in \{v-1, \cdots, v+t\}\}$ is relatively compact for every $t \in \mathbb{N}_k$, then for $t \in \mathbb{N}_k$, there exists at least one solution of the IVP* (2.12).

Proof. Suppose $L_1 = 0, L_2 = M$, and by applying Theorem 2.4.2 we directly obtain the result. □

Corollary 2.4.2. *Let the function f satisfies $lim_{\|y\|\to 0}\|f(t,y)\|/\|y\| = 0$,and for every $t \in \mathbb{N}_k$, the set $\widetilde{S} = \{(t-s-1)^{(v-1)}f(s+v-1, y(s+v-1)) : y \in Y, s \in \{v-1, \cdots, v+t\}\}$ is relatively compact, then for $t \in \mathbb{N}_k$, there exists at least one solution of the IVP* (2.12).

Proof. According to $lim_{\|y\|\to 0}\|f(t,y)\|/\|y\| = 0$, there exists $P > 0$, for any $\epsilon > 0$ such that $\|f(t,y)\| \leq \epsilon P$ for any $\|y\| \leq P$. Suppose $M = \epsilon E$, then corollary 2.4.1 yields results of corollary 2.4.2. □

Corollary 2.4.3. *Let the function $F_1 : R^+ \to R^+$ is a non decreasing continuous function and there exists $L_3, L_4 > 0$ such that*

$$\|f(t,y)\| \leq L_3 F_1(\|y\|) + L_4, \qquad t \in [0,k], \tag{2.25}$$

$$\lim_{\mu\to+\infty} F_1(\mu) < +\infty, \tag{2.26}$$

and for every $t \in \mathbb{N}_k$, the set $\widetilde{S} = \{(t-s-1)^{(v-1)} f(s+v-1, y(s+v-1)) : y \in Y, s \in \{v-1, \cdots, v+t\}\}$ is relatively compact, then for $t \in \mathbb{N}_k$, there exists at least one solution of the IVP (2.12)*.*

Proof. We know that there exists positive constant D_1, d_2, by inequality (2.25) $F_1(\mu) \leq D_1$, for all $\mu \geq d_2$. Then we have $F_1(\mu) \leq D_1 + D_2$, for all $\mu \geq 0$. Suppose $M = L_3(D_1 + D_2) + L_4$, then corollary 2.4.1 yields results of corollary 2.4.3. □

2.4.3 Global uniqueness

Theorem 2.4.3. *Let $f : [0, +\infty) \times Y \to Y$ is globally Lipschits continuous (with constant L) on Y, then the IVP* (2.12) *has a unique solution $y(t)$ provided that $0 < L < \frac{1}{2}(\Gamma(v))(\Gamma(3-v))$.*

Proof. Let $\Phi : Y \to Y$, for $t \in \{0, 1\}$, be the operator defined by equation (2.19). For any $x, y \in Y$ it follows that,

$$\begin{aligned}
&\|(\Phi x)(t) - (\Phi y)(t)\| \\
&\leq \frac{1}{\Gamma(v)}\sum_{s=0}^{t-v}(t-s-1)^{(v-1)}\|f(s+v-1, x(s+v-1)) - f(s+v-1, y(s+v-1))\| \\
&\leq \frac{L}{\Gamma(v)}\sum_{s=0}^{t-v}(t-s-1)^{(v-1)}\|x-y\| \\
&\leq \frac{L}{\Gamma(v)}(v-1)^{(v-1)}\|x-y\| \\
&= \frac{L}{\Gamma(v)}\Gamma(v)\|x-y\| \\
&= L\|x-y\|.
\end{aligned}$$

Since

$$\begin{aligned}\Gamma(v)\Gamma(3-v) &= \Gamma(v)(2-v)\Gamma(2-v) \\ &= \Gamma(v)(2-v)(1-v)\Gamma(1-v)\end{aligned}$$

$$\begin{aligned}&= (1-v)(2-v)\Gamma(v)\Gamma(1-v) \\ &= (1-v)(2-v)\frac{\pi}{\sin\pi v} \qquad \because (\Gamma(v)\Gamma(1-v) = \frac{\pi}{\sin\pi v}).\end{aligned}$$

Note that $\pi(1-v) \leq \sin\pi v$ and $(2-v) \leq 1$ for $1 \leq v \leq 2$,

$$\begin{aligned}\Rightarrow \pi(2-v)(1-v) &\leq \sin\pi v \\ (1-v)(2-v)\frac{\pi}{\sin\pi v} &\leq 1 \\ \Rightarrow \Gamma(v)\Gamma(3-v) &\leq 1 \\ 1+\Gamma(v)\Gamma(3-v) &\leq 2\end{aligned}$$

So $L < \frac{1}{2}(\Gamma(v)\Gamma(3-v)) \leq 1$, Φ has a fixed point by applying Banach contraction principle, which is a unique solution of the IVP (2.12) on $t \in \{0,1\}$.

Define a mapping $\Phi_1 : Y \to Y$, for $t \in \{1,2\}$. Since we have y_2 and y_3 exist

$$(\Phi_1 y)(t) = y_2 + y_3(t-v+2) + \frac{1}{\Gamma(v)}\sum_{s=0}^{t-v}(t-s-1)^{(v-1)}f(s+v-1, y(s+v-1)).$$

For any $x, y \in Y, t \in \{1,2\}$,we have

$\|(\Phi_1 x)(t) - (\Phi_1 y)(t)\|$

$$\begin{aligned}&\leq \frac{1}{\Gamma(v)}\sum_{s=0}^{t-v}(t-s-1)^{(v-1)}\|f(s+v-1, x(s+v-1)) - f(s+v-1, y(s+v-1))\| \\ &\leq \frac{L}{\Gamma(v)}\sum_{s=0}^{t-v}(t-s-1)^{(v-1)}\|x-y\| \\ &\leq \frac{L}{\Gamma(v)}\sum_{s=0}^{2-v}(2-s-1)^{(v-1)}\|x-y\| \\ &= \frac{L}{\Gamma(v)}[(1)^{v-1} + (v-1)^{(v-1)}]\|x-y\| \\ &= \frac{L}{\Gamma(v)}[\frac{\Gamma(2)}{\Gamma(3-v)} + \Gamma(v)]\|x-y\|\end{aligned}$$

$$
\begin{aligned}
&= L\left\{\frac{1}{\Gamma(v)\Gamma(3-v)} + 1\right\}\|x-y\| \\
&= L\left\{\frac{1+\Gamma(v)\Gamma(3-v)}{\Gamma(v)\Gamma(3-v)}\right\}\|x-y\| \\
&\le \frac{2L}{\Gamma(v)\Gamma(3-v)}.
\end{aligned}
$$

Since $L/\Gamma(v)\Gamma(3-v) < 1$, Φ has a fixed point by applying Banach contraction principle, which is a unique solution of the IVP (2.12) on $t \in \{1,2\}$. In general, we may define the operator Φ_m for $t \in \{m, m+1\}$ as follows:

$$
(\Phi_m y)(t) = y_m + y_{m+1}(t-v+2) + \frac{1}{\Gamma(v)}\sum_{s=0}^{t-v}(t-s-1)^{(v-1)} f(s+v-1, y(s+v-1)).
$$

Similarly, by applying Banach contraction principle, we may obtain the IVP (2.12) has a unique solution on $t \in \{m, m+1\}$. Then $y(t)$, defined as follows:

$$
y(t) = \begin{cases} y_0, y_1 & t = 0 \\ y_2, y_3 & t = 1 \\ \vdots & \\ y_m, y_{m+1} & t = m \\ \vdots & \end{cases} \tag{2.27}
$$

Then $y(t)$ is the unique solution of IVP (2.12) for, $t \in \mathbb{N}$. □

Chapter 3

Boundary value problems in discrete fractional calculus

This chapter focuses on a class of boundary value problems in discrete fractional calculus. There are some papers that deal with the existence of fractional difference boundary value problems. Ferhan and Paul [3] discussed the two point boundary value problem for a finite fractional difference equations using Riemann Liouville difference operator and obtained some sufficient conditions for the existence of positive solutions. In [15] C. S. Goodrich established some existence results for a discrete fractional three point boundary value problem. M. Holm in [20] showed the existence of a positive solution to the nonlinear discrete, (N-1,1) fractional order boundary value problem. C. S. Goodrich [13] considered a discrete fractional boundary value problem in which he used the Riemann Liouville difference operator. He proved the existence and uniqueness of solutions using various tools from nonlinear functional analysis.

In the first section of this chapter, we reviewed the results for existence of positive solution for a two point boundary value problem by using the Krasnoselskii fixed point theorem. In the second section, we establish some conditions for the existence of the solutions of boundary value problem. We discuss the same boundary value problem as given in [13]. Here the difference from Goodrich's work is that the boundary conditions are different and we use Caputo difference operator instead of Riemann Liouville difference operator.

3.1 Two point boundary value problem

In this section, we discuss the following two point boundary value problem for fractional difference equations.

$$-\Delta^v y(t) = f(t+v-1, y(t+v-1)), \quad t = 1, 2, 3, \cdots, b+1, \tag{3.1}$$

$$y(v-2) = 0, \quad y(v+b+1) = 0, \tag{3.2}$$

where the function f, defined as $f : [v, v+b]_{\mathbb{N}_{v-1}} \times \mathbb{R} \to \mathbb{R}$ is continuous and $1 < v \leq 2$ is a real number and $b \geq 2$ an integer [3].
We can write above boundary value problem as:

$$-\Delta^v y(t) = h(t+v-1), \quad t = 1, 2, 3, \cdots, b+1, \tag{3.3}$$

with boundary condition (3.2).

3.1.1 Green's function

We convert the fractional boundary value problem (FBVP),(3.3) and (3.2) into summation form by using Lemma 1.5.3, and then we construct the Green's function.

Theorem 3.1.1. *[3] Suppose $1 < v \leq 2$, then the unique solution of FBVP (3.3) and (3.2) is*

$$y(t) = \sum_{s=1}^{b+1} G(t,s) h(s+v-1),$$

where

$$G(t,s) = \frac{1}{\Gamma(v)} \begin{cases} \frac{t^{(v-1)} - (v+b+1-\sigma(s))^{(v-1)}}{(v+b+1)^{(v-1)}} - (t-\sigma(s))^{(v-1)}, & s < t-v+1 \leq b+1; \\ \\ \frac{t^{(v-1)}(v+b+1-\sigma(s))^{(v-1)}}{(v+b+1)^{(v-1)}}, & t-v+1 \leq s \leq b+1. \end{cases}$$

Proof. First we use Lemma 1.5.3 to convert equation (3.3) and (3.2) into summation form. Since we know that $y(t)$ is defined on $[v-2, v+b+1] \subset \mathbb{N}_{v-2}$, we obtain

$$y(t) = -\Delta^{-v} h(t+v-1) + c_1 t^{(v-1)} + c_2 t^{(v-2)}, \tag{3.4}$$

where $c_1, c_2 \in \mathbb{R}$. Now, we use boundary conditions to find the value of c_1 and c_2.
For $y(v-2) = 0$, we get

$$\begin{aligned} y(v-2) = 0 &= -\Delta^{-v} h(t+v-1)|_{t=v-2} + c_1(v-2)^{(v-1)} + c_2(v-2)^{(v-2)} \\ &= -\frac{1}{\Gamma(v)} \sum_{s=1}^{t-v} (t-\sigma(s))^{(v-1)} h(s+v-1)|_{t=v-2} + c_1 \frac{\Gamma(v-1)}{\Gamma(v-1-v+1)} + c_2(v-2)^{(v-2)} \\ &= \frac{1}{\Gamma(v)} \sum_{s=1}^{v-2-v} (v-2-\sigma(s))^{(v-1)} h(s+v-1) + c_1 \frac{\Gamma(v-1)}{\Gamma(0)} + c_2(v-2)^{(v-2)} \end{aligned}$$

$\Rightarrow c_2(v-2)^{(v-2)} = 0 \Rightarrow c_2 = 0.$

For $y(v+b+1) = 0$,

$$\begin{aligned} y(v+b+1) = 0 &= -\Delta^{-v} h(t+v-1)|_{t=v+b+1} + c_1(v+b+1)^{(v-1)} \\ &= -\frac{1}{\Gamma(v)} \sum_{s=1}^{t-v} (t-\sigma(s))^{(v-1)} h(s+v-1)|_{t=v+b+1} + c_1(v+b+1)^{(v-1)} \\ c_1 &= \frac{1}{\Gamma(v)(v+b+1)^{(v-1)}} \sum_{s=1}^{b+1} (v+b+1-\sigma(s))^{(v-1)} h(s+v-1). \end{aligned}$$

Substituting these values of c_1 and c_2 in equation (3.4), then we have

$$y(t) = -\frac{1}{\Gamma(v)} \sum_{s=1}^{t-v} (t-\sigma(s))^{(v-1)} h(s+v-1) + \frac{t^{(v-1)}}{\Gamma(v)(v+b+1)^{(v-1)}} \sum_{s=1}^{b+1} (v+b+1-\sigma(s))^{(v-1)} h(s+v-1)$$

Hence,

$$\begin{aligned} y(t) = &-\frac{1}{\Gamma(v)} \sum_{s=1}^{t-v} \left[(t-\sigma(s))^{(v-1)} - \frac{t^{(v-1)}(v+b+1-\sigma(s))^{(v-1)}}{(v+b+1)^{(v-1)}} \right] h(s+v-1) \\ &+ \frac{t^{(v-1)}}{\Gamma(v)(v+b+1)^{(v-1)}} \sum_{s=t-v+1}^{b+1} (v+b+1-\sigma(s))^{(v-1)} h(s+v-1). \end{aligned}$$

□

Lemma 3.1.1. *[3] Suppose that a and b be any two nonnegative real numbers and v be any positive real number such that $v < a \leq b$. Then the following hold:*

(i) $1/u^{(v)}$ is a decreasing function for $u \in (0, \infty)_{\mathbb{N}}$,

(ii) $(a-u)^{(v)}/(b-u)^{(v)}$ *is a decreasing function for* $u \in [0, a-v)_{\mathbb{N}}$.

Proof. Proof of (i) is simple.
(ii) Since,

$$\frac{(a-u)(b-u-v)}{(a-u-v)(b-u)} \geq 1.$$

It follows that

$$\frac{(a-u)\Gamma(a-u)}{(a-u-v)\Gamma(a-u-v)} \frac{(b-u-v)\Gamma(b-u-v)}{(b-u)\Gamma(b-u)} \geq \frac{\Gamma(a-u)\Gamma(b-u-v)}{\Gamma(a-u-v)\Gamma(b-u)}.$$

As we know that $\Gamma(u+1) = u\Gamma(u)$, then we obtain

$$\frac{\Gamma(1+a-u)}{\Gamma(1+a-u-v)} \frac{\Gamma(1+b-u-v)}{\Gamma(1+b-u)} \geq \frac{\Gamma(a-u)\Gamma(b-u-v)}{\Gamma(a-u-v)\Gamma(b-u)}$$
$$\frac{(a-u)^{(v)}}{(b-u)^{(v)}} \geq \frac{(a-u-1)^{(v)}}{(b-u-1)^{(v)}}.$$

□

Theorem 3.1.2. *[3] The Green's function satisfies the following properties:*

(i) $G(t,s) > 0$ *for* $t \in [v-1, v+b]_{\mathbb{N}_{v-1}}$ *and* $s \in [1, b+1]_{\mathbb{N}}$,

(ii) $\max_{t \in [v-1, v+b]_{\mathbb{N}_{v-1}}} G(t,s) = G(s+v-1, s)$ *for* $s \in [1, b+1]_{\mathbb{N}}$,

(iii) $\min_{(b+v)/4 \leq t \leq 3(b+v)/4} G(t,s) \geq \gamma \max_{t \in [v-1, v+b]_{\mathbb{N}_{v-1}}} G(t,s) = \gamma G(s+v-1, s)$ *for some* $\gamma \in (0,1), s \in [0, N]_{\mathbb{N}}$.

Proof. (i) We can easily see that $\Delta_t G(t,s) > 0$ for $t - v + 1 \leq s < b + 1$,

$$G(t,s) = \frac{t^{(v-1)}}{\Gamma(v)(v+b+1)^{(v-1)}} \sum_{s=t-v+1}^{b+1} (v+b+1-\sigma(s))^{(v-1)}$$
$$\Delta_t G(t,s) = \frac{(v-1)t^{(v-2)}}{\Gamma(v)(v+b+1)^{(v-1)}} \sum_{s=t-v+1}^{b+1} (v+b+1-\sigma(s))^{(v-1)} > 0.$$

For $s < t - v + 1 \leq b + 1$, we have

$$G(t,s) = \left[(t-\sigma(s))^{(v-1)} - \frac{t^{(v-1)}(v+b+1-\sigma(s))^{(v-1)}}{(v+b+1)^{(v-1)}}\right]$$
$$\Delta_t G(t,s) = (v-1)(t-\sigma(s))^{(v-2)} - \frac{(v-1)t^{(v-2)}(v+b+1-\sigma(s))^{(v-1)}}{(v+b+1)^{(v-1)}}$$

$$= \frac{(v-1)(t-\sigma(s))^{(v-2)}(v+b+1)^{(v-1)} - (v-1)t^{(v-2)}(v+b+1-\sigma(s))^{(v-1)}}{(v+b+1)^{(v-1)}}$$
$$= \frac{(v-1)}{(v+b+1)^{(v-1)}}[(t-\sigma(s))^{(v-2)}(v+b+1)^{(v-1)} - t^{(v-2)}(v+b+1-\sigma(s))^{(v-1)}].$$

Thus $\Delta_t G(t,s) < 0$ means $\Delta_t G(t,s)$ is decreasing with respect to t if and only if

$$\frac{t^{(v-2)}(v+b+1-\sigma(s))^{(v-1)}}{(t-\sigma(s))^{(v-2)}(v+b+1)^{(v-1)}} < 1.$$

It follows from the Lemma 4.1.1 (i) that $t^{(\alpha)}$ is increasing and $t^{(-\alpha)}$ is decreasing if $0 < \alpha \leq 1$. Since $G(v-2,s) = G(v+b+1,s) = 0$ and

$$G(s+v-1,s) = \frac{(s+v-1)^{(v-1)}(v+b-s)^{(v-1)}}{(v+b+1)^{(v-1)}} > 0, \qquad s \in [1, b+1]_{\mathbb{N}}.$$

This proves (i).

(iii) Since

$$G(s+v-1,s) = \frac{1}{\Gamma(v)} \begin{cases} \frac{(s+v-1)^{(v-1)}(v+b+1-\sigma(s))^{(v-1)}}{(v+b+1)^{(v-1)}}, & s < t-v+1 \leq b+1; \\ \frac{(s+v-1)^{(v-1)}(v+b+1-\sigma(s))^{(v-1)}}{(v+b+1)^{(v-1)}}, & t-v+1 \leq s \leq b+1. \end{cases}$$

$$\frac{G(t,s)}{G(s+v-1,s)} = \begin{cases} \frac{(t-\sigma(s))^{(v-1)}(v+b+1)^{(v-1)} + t^{(v-1)}(v+b+1-\sigma(s))^{(v-1)}}{(s+v-1)^{(v-1)}(v+b+1-\sigma(s))^{(v-1)}}, & s < t-v+1; \\ \frac{t^{(v-1)}}{(s+v-1)^{(v-1)}}, & t-v+1 \leq s. \end{cases}$$

For $t-v+1 < s$ and $(b+v)/4 \leq t \leq 3(b+v)/4$, we obtain

$$\frac{G(t,s)}{G(s+v-1,s)} = \frac{t^{(v-1)}}{(s+v-1)^{(v-1)}} \geq \frac{((b+v)/4)^{(v-1)}}{(b+v)^{(v-1)}}. \tag{3.5}$$

Since $G(t,s)$ is decreasing with respect to t, then for $s \leq t-v+1$, we have

$$\frac{G(t,s)}{G(s+v-1,s)} = \frac{-(t-\sigma(s))^{(v-1)}(v+b+1)^{(v-1)} + t^{(v-1)}(v+b+1-\sigma(s))^{(v-1)}}{(s+v-1)^{(v-1)}(v+b+1-\sigma(s))^{(v-1)}}$$
$$\geq \frac{(3(b+v)/4-\sigma(s))^{(v-1)}(v+b+1)^{(v-1)} + (3(b+v)/4)^{(v-1)}(v+b+1-\sigma(s))^{(v-1)}}{(s+v-1)^{(v-1)}(v+b+1-\sigma(s))^{(v-1)}}.$$

Define

$$m(s) = \frac{1}{(s+v-1)^{(v-1)}}\left[\left(\frac{3(b+v)}{4}\right)^{(v-1)} - \frac{(3(b+v)/4-\sigma(s))^{(v-1)}(v+b+1)^{(v-1)}}{(v+b+1-\sigma(s))^{(v-1)}}\right].$$

Using Lemma 4.1.1(ii), $\frac{(3(b+v)/4-\sigma(s))^{(v-1)}}{(v+b+1-\sigma(s))^{(v-1)}}$, is decreasing for $1 \leq s < (3(b+v)/4) - v + 1$. Hence

$$\begin{aligned} m(s) &\geq \frac{1}{(s+v-1)^{(v-1)}}\left[\left(\frac{3(b+v)}{4}\right)^{(v-1)} - \frac{(3(b+v)/4-\sigma(1))^{(v-1)}(v+b+1)^{(v-1)}}{(v+b+1-\sigma(1))^{(v-1)}}\right] \\ &\geq \frac{1}{(3(v+b)/4)^{(v-1)}}\left[\left(\frac{3(b+v)}{4}\right)^{(v-1)} - \frac{(3(b+v)/4-\sigma(1))^{(v-1)}(v+b+1)^{(v-1)}}{(v+b+1-\sigma(1))^{(v-1)}}\right]. \end{aligned}$$

By applying Lemma 4.1.1, we get

$$\frac{G(t,s)}{G(s+v-1,s)} \geq \frac{1}{(3(v+b)/4)^{(v-1)}}\left[\left(\frac{3(b+v)}{4}\right)^{(v-1)} - \frac{(3(b+v)/4-\sigma(1))^{(v-1)}(v+b+1)^{(v-1)}}{(v+b+1-\sigma(1))^{(v-1)}}\right]. \tag{3.6}$$

For our convenes define

$$\tilde{\gamma} := \left[\left(\frac{3(b+v)}{4}\right)^{(v-1)} - \frac{(3(b+v)/4-\sigma(1))^{(v-1)}(v+b+1)^{(v-1)}}{(v+b+1-\sigma(1))^{(v-1)}}\right].$$

From equation (3.5) and (3.6) it follows that for $s \in [1, b+1]_{\mathbb{N}}$,

$$\min_{(b+v)/4 \leq t \leq 3(b+v)/4} G(t,s) \geq \gamma \max_{t \in [v-1, v+b]_{\mathbb{N}_{v-1}}} G(t,s) = \gamma G(s+v-1,s),$$

where

$$\gamma = \min\left\{\frac{\tilde{\gamma}}{(3(v+b)/4)^{(v-1)}}, \frac{((b+v)/4)^{(v-1)}}{(b+V)^{(v-1)}}\right\}. \tag{3.7}$$

□

3.1.2 Existence of positive solution

This section contains some results for existence of solutions for the problem (3.1)-(3.2). We use the Guo-Krasnoselskii cone theoretic fixed point Theorem 1.6.4 and obtain some sufficient conditions for existence of positive solutions. To find the solution $y(t)$, we use equivalent summation equation for the problem (3.1)-(3.2) is

$$y(t) = \sum_{s=1}^{b+1} G(t,s) f(s+v-1, y(s+v-1)), \quad t \in [v-2, v+b+1]_{\mathbb{N}_{v-2}}. \tag{3.8}$$

We define Banach space $\mathcal{B}$ as $\mathcal{B} = \{y : [v-2, v+b+1]_{\mathbb{N}_{v-2}} \to \mathbb{R} : y(v-2) = y(v+b+1) = 0\}$, with norm $\|y\| = \max |y(t)|$, where $t \in [v-2, v+b+1]_{\mathbb{N}_{v-1}}$.

Now define cones E and E_0 in $\mathcal{B}$ as:

$$E = \left\{ y \in \mathcal{B} : y(t) \geq 0 \ \text{ for } \ t \in [v-1, v+b]_{\mathbb{N}_{v-2}} \right\}$$

$$E_0 = \left\{ y \in E : \min_{t \in \left[\frac{v+b}{4}, \frac{3(v+b)}{4}\right]_{\mathbb{N}_{v-1}}} y(t) \geq \gamma \|y\| \right\}.$$

The operator $\Phi : \mathcal{B} \to \mathcal{B}$, is defined as

$$\Phi y(t) = \sum_{s=1}^{b+1} G(t,s) f(s+v-1, y(s+v-1)), \quad t \in [v-2, v+b+1]_{\mathbb{N}_{v-2}}.$$

Thus, $y(t)$ is a solution of the problem (3.1) and (3.2) if and only if y is a fixed point of the operator Φ.

Note that Φ is a summation operator on a finite set, so Φ is completely continuous. We state three conditions that will be used in following theorems:

(H_1) : $f(t,\xi) \geq 0, \quad (t,\xi) \in [v, v+b]_{\mathbb{N}_{v-1}} \times [0, \infty)$;

(H_2) : $f(t,y) = h(t)g(y)$, where h is a positive function, g is a nonnegative function and $\lim_{y \to 0^+} \frac{g(y)}{y} = 0, \quad \lim_{y \to \infty} \frac{g(y)}{y} = \infty.$

(H_3) : $f(t,y) = h(t)g(y)$, where h is a positive function, g is a nonnegative function and $\lim_{y \to 0^+} \frac{g(y)}{y} = \infty, \quad \lim_{y \to \infty} \frac{g(y)}{y} = 0.$

Lemma 3.1.2. *[3] Suppose that condition (H_1) holds. Then $\Phi y \in E_0$ for all $y \in E$. In particular, the operator Φ leaves the cone E invariant.*

Proof. For all $y \in E$, using Theorem 3.1.2 and condition (H_1), we get $\Phi(y) \geq 0$ for all $t \in [v-1, v+b]_{\mathbb{N}_{v-1}}$. Further, from Theorem 3.1.2 (iii) it follows that

$$\min_{t \in \left[\frac{v+b}{4}, \frac{3(v+b)}{4}\right]_{\mathbb{N}_{v-1}}} \Phi(y) \geq \gamma \sum_{s=1}^{b+1} \max G(t,s) f(s+v-1, y(s+v-1)) \geq \gamma \|\Phi\|.$$

Thus, $\Phi(y) \in E_0$. □

Theorem 3.1.3. *[3] Suppose that condition* (H_1) *and* (H_2) *are satisfied. Then, the problem* (3.1) *and* (3.2) *has at least one solution* $y \neq 0 \in E_0$.

Proof. Let

$d = \lfloor \frac{3(v+b)}{4} - v + 1 \rfloor + 1 - \lceil \frac{v+b}{4} - v + 1 \rceil,$

$M = \max G(t,s), \quad m = \min G(t,s)$ for $(t,s) \in [v-1, v+b]_{\mathbb{N}_{v-2}} \times [1, b+1]_{\mathbb{N}}$, also let $h = \min h(t), \quad H(t) = \max h(t)$ for $t \in [v, v+b]_{\mathbb{N}_{v-2}}$.

We find $0 < n$ such that if $0 \leq y \leq n$, then

$$g(y) \leq \frac{n}{(b+1)MH}.$$

When $0 < n < N$ such that if $N \leq y$, then

$$g(y) \geq \frac{N}{a\gamma mh}.$$

For $y \in E_0$ with $\|y\| = n$, then for all $t \in [v-1, v+b]_{\mathbb{N}_{v-1}}$,

$$\Phi y(t) \leq M \sum_{s=1}^{b+1} h(s+v-1) g(y(s+v-1)) \leq MH \frac{n}{(b+1)MH} \sum_{s=1}^{b+1} = n = \|y\|. \tag{3.9}$$

Suppose $\Omega_1 = \{y \in \mathcal{B} : \|y\| < n\}$. Then by equation (3.9) we get $\|\Phi y\| \leq \|y\|, y \in E_0 \cap \partial\Omega_1$. Now we further set $N_1 = N/\gamma > N$ and $\Omega_1 = \{y \in \mathcal{B} : \|y\| < N_1\}$. Then, $y \in E_0$ and $\|y\| = N_1$

$$\min_{t \in \left[\frac{v+b}{4}, \frac{3(v+b)}{4}\right]_{\mathbb{N}_{v-2}}} y(t) \geq \gamma \|y\| = \gamma N_1 = N;$$

Thus, $y(t) \geq N$ for all $t \in \left[\frac{v+b}{4}, \frac{3(v+b)}{4}\right]_{\mathbb{N}_{v-2}}$. Therefore, for all $t \in [v-1, v+b]_{\mathbb{N}_{v-1}}$

$$\begin{aligned}\Phi y(t) &\geq mh \sum_{s=1}^{b+1} g(y(s+v-1)) \geq mh \sum_{\lceil \frac{v+b}{4} - v + 1 \rceil}^{\lfloor \frac{3(v+b)}{4} - v + 1 \rfloor} g(y(s+v-1)) \\ &\geq mh \frac{N}{a\gamma mh} \sum_{\lceil \frac{v+b}{4} - v + 1 \rceil}^{\lfloor \frac{3(v+b)}{4} - v + 1 \rfloor} 1 = \frac{N}{\gamma} = N_1 = \|y\|.\end{aligned}$$

In particular, $\|\Phi y\| \geq \|y\|, y \in E_0 \cap \partial\Omega_2$. By using Theorem 1.6.4 (i), we get that Φ has a fixed point y in $E_0 \cap (\bar{\Omega}_2 \backslash \Omega_1)$. □

Theorem 3.1.4. *[3] Suppose that condition* (H_1) *and* (H_3) *are satisfied. Then, the problem* (3.1) *and* (3.2) *has at least one solution* $y \neq 0 \in E_0$.

3.2 Existence and uniqueness results for boundary value problems

In this section, we consider the following two point fractional difference boundary value problem.

$$\Delta_*^v y(t) = f(t+v-1, y(t+v-1)), \tag{3.10}$$

$$y(v-1) = 0, \quad y(v+n+1) = 0, \tag{3.11}$$

where $1 < v \leq 2$ and $n \geq 2$ an integer. The nonlinear function $f : [v, v+n]_{\mathbb{N}_{v-1}} \times \mathbb{R} \to \mathbb{R}$ is supposed to be continuous. This problem is similar to the problem discussed in [13]. Here the difference is we use Caputo difference operator with different boundary conditions. First, we construct the equivalence summation form of the problem (3.10) and (3.11), by using the Caputo fractional difference operator instead of Riemann fractional difference operator. Then, we establish some uniqueness and existence results by using contraction principal and Brouwer's theorem.

Now, we state and prove an important theorem, that represents the summation form of problem (3.10) and (3.11). We will use this representation in finding the results on existence and uniqueness.

Theorem 3.2.1. *Let $1 < v \leq 2$ and $g : [v, v+n]_{\mathbb{N}_{v-1}} \to \mathbb{R}$, then unique solution of problem*

$$\begin{cases} \Delta_*^v y(t) = g(t+v-1), \\ y(v-1) = 0, \quad y(v+n+1) = 0, \end{cases}$$

is

$$y(t) = \frac{1}{\Gamma(v)} \sum_{s=a}^{t-v} (t-\sigma(s))^{(v-1)} g(s+v-1) + \frac{(v-1-t)}{(n+2)\Gamma(v)} \sum_{s=a}^{n+1} (v+n+1-\sigma(s))^{(v-1)} g(s+v-1). \tag{3.12}$$

Proof. Applying Lemma 1.5.5, we convert the difference equation $\Delta_*^v y(t) = g(t+v-1)$ into a summation equation.

$$y(t) = \Delta^{-v} g(t+v-1) + c_1 + c_2 t, \tag{3.13}$$

where $c_1, c_2 \in \mathbb{R}$. Now we use boundary conditions to find the values of constant c_1 and c_2. First, we apply the boundary condition $y(v-1) = 0$,

$$\begin{aligned} y(v-1) = 0 &= \Delta^{-v} g(t+v-1)|_{t=v-1} + c_1 + c_2(v-1) \\ &= \frac{1}{\Gamma(v)} \sum_{s=a}^{v-1-v} (v-1-\sigma(s))^{(v-1)} g(s+v-1) + c_1 + c_2(v-1) = c_1 + c_2(v-1). \end{aligned} \tag{3.14}$$

Apply second boundary condition $y(v+n+1) = 0$,

$$y(v+n+1) = 0 = \Delta^{-v} g(t+v-1)|_{t=v+n+1} + c_1 + c_2(v+n+1). \tag{3.15}$$

By subtracting equations (3.14) from (3.15), we obtain

$$\begin{aligned} c_2(v+n+1) - c_2(v-1) &= -\Delta^{-v} g(t+v-1)|_{t=v+n+1} \\ c_2(v+n+1-v+1) &= -\frac{1}{\Gamma(v)} \sum_{s=a}^{v+n+1-v} (v+n+1-\sigma(s))^{(v-1)} g(s+v-1) \\ c_2 &= -\frac{1}{(n+2)\Gamma(v)} \sum_{s=a}^{n+1} (v+n+1-\sigma(s))^{(v-1)} g(s+v-1). \end{aligned}$$

By adding equations (3.14) and (3.15), we get

$$\begin{aligned} 2c_1 &= -\Delta^{-v} g(t+v-1)|_{t=v+n+1} - c_2(v+n+1+v-1) \\ &= -\frac{1}{\Gamma(v)} \sum_{s=a}^{v+n+1-v} (v+n+1-\sigma(s))^{(v-1)} g(s+v-1) - c_2(2v+n) \\ &= -\frac{1}{\Gamma(v)} \sum_{s=a}^{n+1} (v+n+1-\sigma(s))^{(v-1)} g(s+v-1) + \frac{(2v+n)}{(n+2)\Gamma(v)} \sum_{s=a}^{n+1} (v+n+1-\sigma(s))^{(v-1)} g(s+v-1) \\ &= \frac{1}{\Gamma(v)} \sum_{s=a}^{n+1} (v+n+1-\sigma(s))^{(v-1)} g(s+v-1) \left(-1 + \frac{(2v+n)}{(n+2)} \right) \\ c_1 &= \frac{(v-1)}{(n+2)\Gamma(v)} \sum_{s=a}^{n+1} (v+n+1-\sigma(s))^{(v-1)} g(s+v-1). \end{aligned}$$

Substituting values of c_1 and c_2 in equation (3.13),

$$= \frac{1}{\Gamma(v)} \sum_{s=a}^{t-v} (t-\sigma(s))^{(v-1)} g(s+v-1) + \frac{(v-1)}{(n+2)\Gamma(v)} \sum_{s=a}^{n+1} (v+n+1-\sigma(s))^{(v-1)} g(s+v-1)$$

$$- \frac{t}{(n+2)\Gamma(v)} \sum_{s=a}^{n+1} (v+n+1-\sigma(s))^{(v-1)} g(s+v-1)$$

$$= \frac{1}{\Gamma(v)} \sum_{s=a}^{t-v} (t-\sigma(s))^{(v-1)} g(s+v-1) + \frac{(v-1-t)}{(n+2)\Gamma(v)} \sum_{s=a}^{n+1} (v+n+1-\sigma(s))^{(v-1)} g(s+v-1).$$

□

3.3 Existence and uniqueness of solution

Now, we establish some sufficient conditions for existence and uniqueness of solution of boundary value problem (3.10) and (3.11). For this we define the operator Φ as:
$\Phi : \mathbb{R} \to \mathbb{R}$ for $t \in [v-1, v+n+1]_{\mathbb{N}_{v-2}}$, where

$$\begin{aligned} (\Phi y)(t) &= \frac{1}{\Gamma(v)} \sum_{s=a}^{t-v} (t-\sigma(s))^{(v-1)} f(s+v-1, y(s+v-1)) \\ &+ \frac{(v-1-t)}{(n+2)\Gamma(v)} \sum_{s=a}^{n+1} (v+n+1-\sigma(s))^{(v-1)} f(s+v-1, y(s+v-1)). \end{aligned} \tag{3.16}$$

Now we use this operator to establish our first theorem of uniqueness.

Theorem 3.3.1. *Assume that $f(t, y)$ is Lipschitz with constant L, in y. Then the boundary value problem* (3.10) *has a unique solution provided*

$$2L \prod_{i=1}^{n} \left(\frac{v+i}{i} \right) < 1. \tag{3.17}$$

Proof. We prove this theorem by showing that Φ is a contraction mapping. For any $y_1, y_2 \in \mathbb{R}$, such that $\|\Phi y_1 - \Phi y_2\|$

$$\leq \max_{t \in [v-1, v+n+1]_{\mathbb{N}_{v-2}}} \left[\frac{1}{\Gamma(v)} \sum_{s=a}^{t-v} (t-\sigma(s))^{(v-1)} \right] \|f(s+v-1, y_1(s+v-1)) - f(s+v-1, y_2(s+v-1))\|$$

$$
\begin{aligned}
&+ \max_{t\in[v-1,v+n+1]_{\mathbb{N}_{v-2}}} \left[\frac{(v-1-t)}{(n+2)\Gamma(v)}\sum_{s=a}^{n+1}(v+n+1-\sigma(s))^{(v-1)}\right]\|f(s+v-1,y_1(s+v-1))\\
&- f(s+v-1,y_2(s+v-1))\|\\
&\leq \max_{t\in[v-1,v+n+1]_{\mathbb{N}_{v-2}}} \frac{L}{\Gamma(v)}\sum_{s=a}^{t-v}(t-\sigma(s))^{(v-1)}\|y_1-y_2\|\\
&+ \max_{t\in[v-1,v+n+1]_{\mathbb{N}_{v-2}}} \frac{L(v-1-t)}{(n+2)\Gamma(v)}\sum_{s=a}^{n+1}(v+n+1-\sigma(s))^{(v-1)}\|y_1-y_2\|\\
&\leq L\|y_1-y_2\| \max_{t\in[v-1,v+n+1]_{\mathbb{N}_{v-2}}} \left[\frac{1}{\Gamma(v)}\sum_{s=a}^{t-v}(t-\sigma(s))^{(v-1)}\right]\\
&+ L\|y_1-y_2\| \max_{t\in[v-1,v+n+1]_{\mathbb{N}_{v-2}}} \left[\frac{(v-1-t)}{(n+2)\Gamma(v)}\sum_{s=a}^{n+1}(v+n+1-\sigma(s))^{(v-1)}\right]. \qquad (3.18)
\end{aligned}
$$

We consider the first term on the right hand side of equation (3.18),

$$
\begin{aligned}
&L\|y_1-y_2\|\left[\frac{1}{\Gamma(v)}\sum_{s=a}^{t-v}(t-s-1)^{(v-1)}\right]\\
&= L\|y_1-y_2\|\left[\frac{1}{\Gamma(v)}\sum_{s=a}^{t-v}\frac{\Gamma(t-s)}{\Gamma(t-s-v+1)}\right]\\
&= L\|y_1-y_2\|\left[\frac{1}{\Gamma(v)}\sum_{s=a}^{t-v-1}\frac{\Gamma(t-s)}{\Gamma(t-s-v+1)}+1\right]\\
&= L\|y_1-y_2\|\left[\frac{1}{\Gamma(v)}\sum_{s=a}^{t-v-1}\frac{1}{v}\left(\frac{\Gamma(t-s+1)}{\Gamma(t-s-v+1)}-\frac{\Gamma(t-s)}{\Gamma(t-s-v)}\right)+1\right]\\
&= L\|y_1-y_2\|\left[\frac{1}{\Gamma(v)}\frac{1}{v}\left(\frac{\Gamma(t-1+1)}{\Gamma(t-1-v+1)}-\frac{\Gamma(t-t+v+1)}{\Gamma(t-t+v+1-v)}\right)+1\right]\\
&= L\|y_1-y_2\|\left[\frac{1}{\Gamma(v+1)}\left(\frac{\Gamma(t)}{\Gamma(t-v)}-\frac{\Gamma(v+1)}{\Gamma(1)}\right)+1\right]\\
&= L\|y_1-y_2\|\left[\frac{\Gamma(t)}{\Gamma(v+1)\Gamma(t-v)}-1+1\right]\\
&= L\|y_1-y_2\|\left[\frac{\Gamma(t)}{\Gamma(v+1)\Gamma(t-v)}\right]\\
&\leq L\|y_1-y_2\|\left[\frac{\Gamma(v+n+1)}{\Gamma(v+1)\Gamma(v+n+1-v)}\right]
\end{aligned}
$$

$$\leq L\|y_1 - y_2\| \left[\frac{\Gamma(v+n+1)}{\Gamma(v+1)\Gamma(n+1)} \right]$$
$$\leq L\|y_1 - y_2\| \left[\frac{(v+n)!}{v!n!} \right]$$
$$\leq L\|y_1 - y_2\| \left[\frac{(v+n)(v+n-1)(v+n-2)\cdots(v+n-(n-1))(v+n-n)}{v!n!} \right]$$
$$\leq L\|y_1 - y_2\| \left[\frac{(v+n)(v+n-1)(v+n-2)\cdots(v+n-(n-1))}{n!} \right]$$
$$\leq L\|y_1 - y_2\| \prod_{i=1}^{n} \left(\frac{v+i}{i} \right).$$

Now we consider the second term on the right hand side of equation(3.18)

$$L\|y_1 - y_2\| \left[\frac{(v-1-t)}{(n+2)\Gamma(v)} \sum_{s=a}^{n+1} (v+n+1-\sigma(s))^{(v-1)} \right]$$
$$\leq L\|y_1 - y_2\| \left[\frac{1}{\Gamma(v)} \sum_{s=a}^{n+1} (v+n-s)^{(v-1)} \right]$$
$$\leq L\|y_1 - y_2\| \left[\frac{1}{\Gamma(v)} \sum_{s=a}^{n+1} \frac{\Gamma(v+n-s+1)}{\Gamma(n+2-s)} \right]$$
$$\leq L\|y_1 - y_2\| \left[\frac{1}{\Gamma(v)} \sum_{s=a}^{n} \frac{\Gamma(v+n-s+1)}{\Gamma(n+2-s)} + 1 \right]$$
$$\leq L\|y_1 - y_2\| \left[\frac{1}{\Gamma(v)} \sum_{s=a}^{n} \frac{1}{v} \left(\frac{\Gamma(v+n+2-s)}{\Gamma(n+2-s)} - \frac{\Gamma(v+n+1-s)}{\Gamma(n+1-s)} \right) + 1 \right]$$
$$\leq L\|y_1 - y_2\| \left[\frac{1}{\Gamma(v)} \frac{1}{v} \left(\frac{\Gamma(v+n+2-1)}{\Gamma(n+2-1)} - \frac{\Gamma(v+n+1-n)}{\Gamma(n+1-n)} \right) + 1 \right]$$
$$\leq L\|y_1 - y_2\| \left[\frac{1}{\Gamma(v+1)} \left(\frac{\Gamma(v+n+1)}{\Gamma(n+1)} - \frac{\Gamma(v+1)}{\Gamma(1)} \right) + 1 \right]$$
$$\leq L\|y_1 - y_2\| \left[\frac{1}{\Gamma(v+1)} \frac{\Gamma(v+n+1)}{\Gamma(n+1)} - 1 + 1 \right]$$
$$\leq L\|y_1 - y_2\| \left[\frac{\Gamma(v+n+1)}{\Gamma(v+1)\Gamma(n+1)} \right]$$
$$\leq L\|y_1 - y_2\| \left[\frac{(v+n)!}{v!n!} \right]$$
$$\leq L\|y_1 - y_2\| \prod_{i=1}^{n} \left(\frac{v+i}{i} \right).$$

Substitute these value in equation (3.18), we obtain

$$\|\Phi y_1 - \Phi y_2\| \leq 2L \prod_{i=1}^{n} \left(\frac{v+i}{i}\right) \|y_1 - y_2\|.$$

Hence this proves the uniqueness of solution for the problem (3.10) and (3.11). □

Now we establish an existence result by using Brouwer theorem.

Theorem 3.3.2. *Assume that there exists a constant $M > 0$ such that the function $f(t,y)$ satisfies the following condition*

$$\max_{(t,y)\in[v-1,v+n]_{\mathbb{N}_{v-1}}\times[-M,M]} |f(t,y)| \leq \frac{M}{\frac{2\Gamma(v+n+1)}{\Gamma(v+1)\Gamma(n+1)}}. \tag{3.19}$$

Then the problem (3.10) *and* (3.11) *has at least one solution say y_0 and $|y_0(t)| \leq M$, for all $t \in [v-1, v+n]_{\mathbb{N}_{v-2}}$.*

Proof. We define the Banach space $\mathcal{B}$ as:

$$\mathcal{B} = \{y \in \mathbb{R} : \|y\| \leq M\}.$$

Let the operator Φ defined in equation (3.16) and Φ is continuous operator. Now we have to show that $\Phi : \mathcal{B} \to \mathcal{B}$, whenever $\|y\| \leq M$, it follows that $\|\Phi y\| \leq M$.
Suppose that the inequality (3.19) holds for $f(t,y)$. For our convenience put

$$\xi_0 = \frac{M}{\frac{2\Gamma(v+n+1)}{\Gamma(v+1)\Gamma(n+1)}},$$

where ξ_0 is completely positive constant. By using this notation we have

$$\begin{aligned}
\|\Phi y\| \leq & \max_{t\in[v-1,v+n]_{\mathbb{N}_{v-2}}} \frac{1}{\Gamma(v)} \sum_{s=1}^{t-v} (t-s-1)^{(v-1)} |f(s+v-1, y(s+v-1))| \\
& + \max_{t\in[v-1,v+n]_{\mathbb{N}_{v-2}}} \frac{(v-1-t)}{(n+2)\Gamma(v)} \sum_{s=1}^{n+1} (v+n-s)^{(v-1)} |f(s+v-1, y(s+v-1))| \\
& \leq \xi_0 \max_{t\in[v-1,v+n]_{\mathbb{N}_{v-2}}} \left[\frac{1}{\Gamma(v)} \sum_{s=1}^{t-v} (t-s-1)^{(v-1)} + \sum_{s=1}^{n+1} \frac{(v-1-t)(v+n-s)^{(v-1)}}{(n+2)\Gamma(v)}\right]
\end{aligned}$$

$$\leq \xi_0 \max_{t\in[v-1,v+n]_{\mathbb{N}_{v-2}}} \left[\frac{1}{\Gamma(v)}\sum_{s=1}^{t-v}(t-s-1)^{(v-1)} + \frac{1}{\Gamma(v)}\sum_{s=1}^{n+1}(v+n-s)^{(v-1)}\right]$$
$$\leq \xi_0 \left[\frac{1}{\Gamma(v)}\sum_{s=1}^{v+n+1-v}(v+n+1-s-1)^{(v-1)} + \frac{1}{\Gamma(v)}\sum_{s=1}^{n+1}(v+n-s)^{(v-1)}\right]$$
$$\leq \xi_0 \left[\frac{1}{\Gamma(v)}\sum_{s=1}^{n+1}(v+n-s)^{(v-1)} + \frac{1}{\Gamma(v)}\sum_{s=1}^{n+1}(v+n-s)^{(v-1)}\right]$$
$$\leq \xi_0 \left[\frac{2}{\Gamma(v)}\sum_{s=1}^{n+1}\frac{\Gamma(v+n-s+1)}{\Gamma(v+n-s+1-v+1)}\right]$$
$$\leq \xi_0 \left[\frac{2}{\Gamma(v)}\sum_{s=a}^{n}\frac{\Gamma(v+n-s+1)}{\Gamma(n+2-s)}+2\right]$$
$$\leq \xi_0 \left[\frac{2}{\Gamma(v)}\sum_{s=a}^{n}\frac{1}{v}\left(\frac{\Gamma(v+n+2-s)}{\Gamma(n+2-s)} - \frac{\Gamma(v+n+1-s)}{\Gamma(n+1-s)}\right)+2\right]$$
$$\leq \xi_0 \left[\frac{2}{\Gamma(v)}\frac{1}{v}\left(\frac{\Gamma(v+n+2-1)}{\Gamma(n+2-1)} - \frac{\Gamma(v+n+1-n)}{\Gamma(n+1-n)}\right)+2\right]$$
$$\leq \xi_0 \left[\frac{2}{\Gamma(v+1)}\left(\frac{\Gamma(v+n+1)}{\Gamma(n+1)} - \frac{\Gamma(v+1)}{\Gamma(1)}\right)+2\right]$$
$$\leq \xi_0 \left[\frac{2}{\Gamma(v+1)}\frac{\Gamma(v+n+1)}{\Gamma(n+1)} - 2+2\right]$$
$$\leq \xi_0 \left[\frac{2\Gamma(v+n+1)}{\Gamma(v+1)\Gamma(n+1)}\right] = M.$$

Hence we get $\Phi : \mathcal{B} \to \mathcal{B}$, as required. Consequently by the Brouwer theorem it follows that there exists a fixed point of the map, say $\Phi y_0 = y_0$ with $y_0 \in \mathcal{B}$. The function y_0 is a solution of the boundary value problem (3.10) and (3.11). Also y_0 satisfies $|y_0(t)| \leq M$ for all $t \in [v-1, v+n+1]_{\mathbb{N}_{v-2}}$. This complete the proof. □

Chapter 4

Existence of positive solution for discrete fractional boundary value problem

Motivated by some recent study on existence of positive solutions for fractional difference equations, in this chapter we consider the following class of two-point boundary value problem for fractional difference equations

$$-\Delta^{\alpha}_{\alpha-2}y(t) = f(t+\alpha-1, y(t+\alpha-1)), \tag{4.1}$$

$$y(\alpha-2) = 0, \quad \Delta^{\alpha-1}_{\alpha-1}y(\alpha+k) = 0, \tag{4.2}$$

where $1 < \alpha \leq 2$, $t \in [0, k+1]_{\mathbb{N}_0}$ and $k \geq 2$ an integer. The nonlinear function $f : [\alpha-1, \alpha+k-1]_{\mathbb{N}_{\alpha-1}} \times \mathbb{R} \to \mathbb{R}^+$ is suppose to be continuous. A similar boundary value problem was discussed in [16] by Goodrich. Here the boundary condition at $(\alpha+k)$ has a difference of order $\alpha-1$ instead of order one. Also in this study we establish various new existence and non-existence results for positive solution which are absent in [16]. This work is accepted for publication [30]. We shall obtain an equivalent summation equation of problem (4.1) and derive a different Green's function as compared to in [16]. Some useful inequality will be established for the Green's function that allow us to used Krasnoselskii's theorem to obtain existence results for the problem. Finally we establish various existence and non-existence results for positive solutions and interval for parameter are obtained for which there exist positive solutions.

4.1 Green's function and its properties

We reduce the fractional difference boundary value problem (FDBVP) in to an equivalent summation equation to establish existence theorem. In the following lemma we construct the Green's function of boundary value problem (4.1).

Lemma 4.1.1. *Let* $1 < \alpha \leq 2$ *and* $h \in C([\alpha-1, \alpha+k]_{\mathbb{N}_{\alpha-1}})$. *Then unique solution of discrete fractional boundary value problem*

$$\begin{cases} \Delta^{\alpha}_{\alpha-2} y(t) + h(t+\alpha-1) = 0, \\ y(\alpha-2) = 0, \quad \Delta^{\alpha-1}_{\alpha-1} y(\alpha+k) = 0, \end{cases}$$

is

$$y(t) = \sum_{s=0}^{k+1} G(t,s) h(s+\alpha-1),$$

where

$$G(t,s) = \begin{cases} \frac{t^{(\alpha-1)} - (t-s-1)^{(\alpha-1)}}{\Gamma(\alpha)}, & s \leq t-\alpha \leq k+1; \\ \frac{t^{(\alpha-1)}}{\Gamma(\alpha)}, & t-\alpha < s \leq k+1. \end{cases} \tag{4.3}$$

Proof. Using Lemma 1.5.3 we may reduce the difference equation $\Delta^{\alpha}_{\alpha-2} y(t) + h(t+\alpha-1) = 0$ to an equivalent summation equation

$$y(t) = -\Delta_0^{-\alpha} h(t+\alpha-1) + c_1 t^{(\alpha-1)} + c_2 t^{(\alpha-2)} \tag{4.4}$$

where $t \in [\alpha-1, \alpha+k]_{\mathbb{N}_{\alpha-1}}$ and $c_1, c_2 \in \mathbb{R}$. Now applying the first boundary condition $y(\alpha-2) = 0$ we immediately get $c_2 = 0$. For second boundary condition, applying $\Delta^{\alpha-1}_{\alpha-1}$ on both sides of equation (4.4) and using Theorem 1.3.1 and Theorem 1.5.1, we get

$$\Delta^{\alpha-1}_{\alpha-1} y(t) = -\Delta_0^{-1} h(t+\alpha-1) + c_1 \Gamma(\alpha). \tag{4.5}$$

Now using the boundary condition $\Delta^{\alpha-1}_{\alpha-1} y(\alpha+k) = 0$ in equation (4.5), we have

$$c_1 \Gamma(\alpha) = \sum_{s=0}^{\lceil(\alpha-1)+k\rceil} h(s+\alpha-1). \tag{4.6}$$

Since $v - 1 \leq 1$. Then

$$c_1 = \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{k+1} h(s+\alpha-1). \tag{4.7}$$

Substituting values of c_1 and c_2 in equation (4.4) we get

$$\begin{aligned} y(t) &= -\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{t-\alpha}(t-s-1)^{(\alpha-1)}h(s+\alpha-1) + \frac{t^{(\alpha-1)}}{\Gamma(\alpha)}\sum_{s=0}^{k+1}h(s+\alpha-1) \\ &= \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{t-\alpha}[t^{(\alpha-1)} - (t-s-1)^{(\alpha-1)}]h(s+\alpha-1) + \frac{t^{(\alpha-1)}}{\Gamma(\alpha)}\sum_{s=t-\alpha+1}^{k+1}h(s+\alpha-1) \\ &= \sum_{s=0}^{k+1}G(t,s)h(s+\alpha-1). \end{aligned}$$

□

Lemma 4.1.2. *The Green function $G(t,s)$ satisfies following properties:*

(i) $G(t,s) > 0$ *for* $t \in [\alpha-1, \alpha+k]_{\mathbb{N}_{\alpha-1}}$ *and* $s \in [0, k+1]_{\mathbb{N}_0}$,

(ii) $\max_{t\in[\alpha-1,\alpha+k]_{\mathbb{N}_{\alpha-1}}} G(t,s) = G(s+\alpha, s)$, *and*

(iii) $\min_{t\in\left[\frac{(k+\alpha)}{4},\frac{3(k+\alpha)}{4}\right]_{\mathbb{N}_{\alpha-1}}} G(t,s) \geq \tilde{\gamma}\max_{t\in[\alpha-1,\alpha+k]_{\mathbb{N}_{\alpha-1}}} G(t,s)$ *for some* $\tilde{\gamma} \in (0,1)$, $s \in [0, k+1]_{\mathbb{N}_0}$.

Proof. (i) For $t-\alpha < s \leq k+1$, obviously $G(t,s) > 0$.
We know that $t^{(x)}$ is increasing function for $0 < x < 1$, then $(t-s-1)^{(\alpha-1)} < t^{(\alpha-1)}$. Thus $G(t,s) > 0$ for $0 < s < t-\alpha \leq k+1$.
(ii) For $t-\alpha < s \leq k+1$, clearly $\Gamma(\alpha)\Delta_t G(t,s) = (\alpha-1)t^{(\alpha-2)} > 0$ which implies that G is increasing for $0 \leq t \leq s+\alpha$. Since $t^{(x)}$ is a decreasing function for $x \in (-1,0)$. Thus $\Delta_t G(t,s) = (\alpha-1)[t^{(\alpha-2)} - (t-s-1)^{(\alpha-2)}] < 0$, which implies that G is decreasing in t, whenever $s+\alpha < t \leq \alpha+k$. Hence $\max_{t\in[\alpha-1,\alpha+k]_{\mathbb{N}_{\alpha-2}}} G(t,s) = G(s+\alpha, s)$.

Now

$$\frac{G(t,s)}{G(s+\alpha,s)} = \begin{cases} \frac{t^{(\alpha-1)}-(t-s-1)^{(\alpha-1)}}{(s+\alpha)^{(\alpha-1)}}, & s \leq t-\alpha \leq k+1; \\ \frac{t^{(\alpha-1)}}{(s+\alpha)^{(\alpha)}}, & t-\alpha < s \leq k+1. \end{cases} \tag{4.8}$$

Observe that, for $\frac{(k+\alpha)}{4} \leq t \leq \frac{3(k+\alpha)}{4}$ and $t-\alpha < s \leq k$, we have $t^{(\alpha-1)} \geq \left(\frac{k+\alpha}{4}\right)^{(\alpha-1)}$ and $(s+\alpha)^{(\alpha-1)} \leq (k+\alpha)^{(\alpha-1)}$. Then

$$\frac{G(t,s)}{G(s+\alpha,s)} > \frac{\left(\frac{1}{4}(k+\alpha)\right)^{(\alpha-1)}}{(k+\alpha)^{(\alpha-1)}}.$$

For $\frac{(k+\alpha)}{4} \leq t \leq \frac{3(k+\alpha)}{4}$ and $s \leq t-\alpha$, we have $(t-s-1)^{\alpha-1} \leq \left(\frac{3}{4}(k+\alpha)-s-1\right)^{\alpha-1} < \left(\frac{3}{4}(k+\alpha)-1\right)^{\alpha-1}$ and $t^{(\alpha-1)} \geq (s+\alpha)^{(\alpha-1)}$. Consequently

$$\frac{G(t,s)}{G(s+\alpha-1,s)} \geq 1 - \frac{\left(\frac{3}{4}(k+\alpha)-1\right)^{(\alpha-1)}}{(\frac{3}{4}(k+\alpha))^{(\alpha-1)}}.$$

Hence,

$$\min_{t\in\left[\frac{(k+\alpha)}{4},\frac{3(k+\alpha)}{4}\right]_{\mathbb{N}_{\alpha-1}}} G(t,s) \geq \tilde{\gamma} \max_{t\in[\alpha-1,\alpha+k]_{\mathbb{N}_{\alpha-1}}} G(t,s),$$

for $\tilde{\gamma} := \min\left\{\frac{\left(\frac{1}{4}(k+\alpha)\right)^{(\alpha-1)}}{(k+\alpha)^{(\alpha-1)}},\ 1-\frac{\left(\frac{3}{4}(k+\alpha)-1\right)^{(\alpha-1)}}{(\frac{3}{4}(k+\alpha))^{(\alpha-1)}}\right\}$.

Note that $\frac{\left(\frac{1}{4}(k+\alpha)\right)^{(\alpha-1)}}{(k+\alpha)^{(\alpha-1)}} < 1$ and also $\frac{\left(\frac{3}{4}(k+\alpha)-1\right)^{(\alpha-1)}}{(\frac{3}{4}(k+\alpha))^{(\alpha-1)}} < 1$. Therefore $0 < \tilde{\gamma} < 1$. □

4.2 Existence of positive solutions

In the study of boundary value problem (4.1) and (4.2) to establish the existence of positive solutions we shall use the well-known Guo-Krasnoselskii fixed point theorem.

Definition 4.2.1. We call a function $y(t)$ a positive solution of problem (4.1), if $y(t) \in C[\alpha-2,\alpha+k]_{\mathbb{N}_{\alpha-2}}$ and $y(t) \geq 0$ for $t \in [\alpha-2,\alpha+k]_{\mathbb{N}_{\alpha-2}}$ and satisfies (4.1).

We define Banach space $\mathcal{B}$ by

$$\mathcal{B} = \{y \in C([\alpha-2,\ \alpha+k]_{\mathbb{N}_{\alpha-2}}) : \Delta_{\alpha-1}^{\alpha-2} y \in C([\alpha-1,\ \alpha+k]_{\mathbb{N}_{\alpha-1}}),\ \alpha \in (1,2]\}$$

with the norm $\|y\| = \max|y(t)|,\ t \in [\alpha-2,\ \alpha+k]_{\mathbb{N}_{\alpha-2}}$. Further, for some $\tilde{\gamma} \in (0,1)$ we define cone

$$E := \left\{y \in \mathcal{B} : y(t) \geq 0, \min_{t\in[\frac{k+\alpha}{4},\ \frac{3(k+\alpha)}{4}]_{\mathbb{N}_{\alpha-2}}} y(t) \geq \tilde{\gamma}\|y\|\right\}. \tag{4.9}$$

Let the operator $\Phi : \mathcal{B} \to \mathcal{B}$, defined as

$$\Phi y(t) = \sum_{s=0}^{k+1} G(t,s) f(s+\alpha-1, y(s+\alpha-1)), \tag{4.10}$$

then $y \in \mathcal{B}$ is a solution of (4.10) if and only of $y \in \mathcal{B}$ is a solution of (4.1) and (4.2).

Lemma 4.2.1. *Let Φ be defined as in (4.10) and E in (4.9). Then $\Phi : E \to E$ and Φ is completely continuous.*

Proof. Let $y \in E$. Then by Lemma 4.1.2, we have

$$\begin{aligned}
\min_{t\in[\frac{(k+\alpha)}{4},\frac{3(k+\alpha)}{4}]_{\mathbb{N}_{\alpha-2}}} \Phi y(t) &\geq \min_{t\in[\frac{(k+\alpha)}{4},\frac{3(k+\alpha)}{4}]_{\mathbb{N}_{\alpha-2}}} \sum_{s=0}^{k+1} G(t,s) f(s+\alpha-1, y(s+\alpha-1)) \\
&\geq \tilde{\gamma} \sum_{s=0}^{k+1} \max_{t\in[\alpha-1,\ \alpha+k]_{\mathbb{N}_{\alpha-1}}} G(t,s) f(s+\alpha-1, y(s+\alpha-1)) \\
&= \tilde{\gamma} \max_{t\in[\alpha-2,\ \alpha+k]_{\mathbb{N}_{\alpha-1}}} \Phi y(t) = \tilde{\gamma}\|\Phi y\|.
\end{aligned}$$

Hence $\Phi : E \to E$. Since f is continuous, therefore the operator Φ is also continuous. It remains to prove that the operator Φ is completely continuous in E. For fixed $l > 0$, consider bounded subset A of E defined as $A = \{y \in E : \|y\| \leq l\}$. Define
$M_0 = \max_{0\leq y\leq R,\ t\in[\alpha-2,\alpha+k]_{\mathbb{N}_{\alpha-1}}} f(t+\alpha-1, y(t+\alpha-1))$.

$$\begin{aligned}
|\Phi y(t)| &\leq \sum_{s=0}^{k+1} G(t,s)|f(s+\alpha-1, y(s+\alpha-1))| \\
&\leq M_0 \sum_{s=0}^{k+1} G(s+\alpha, s) \\
&\leq \frac{M_0}{\Gamma(\alpha)} \frac{\Gamma(k+\alpha+1)}{\Gamma(k+2)}.
\end{aligned}$$

Hence $\Phi(A)$ is uniformly bounded. Now for any $\varepsilon > 0$, there exists $t_1,\ t_2 \in [\alpha-1, \alpha+k]_{\mathbb{N}_{\alpha-1}}$ with $t_1 < t_2$ such that

$$(t_2 - t_1) < \frac{\varepsilon}{2M_0(k+1)},$$

then we have

$$\begin{aligned}
\|\Phi y(t_1) - \Phi y(t_2)\| \leq & \frac{1}{\Gamma(\alpha)} \Big\| \Big(\sum_{s=0}^{t_1-\alpha} (t_1-s-1)^{(\alpha-1)} - \sum_{s=0}^{t_2-\alpha} (t_2-s-1)^{(\alpha-1)} \Big) f(s+\alpha-1, y(s+\alpha-1)) \Big\| \\
& + \frac{(t_2^{(\alpha-1)} - t_1^{(\alpha-1)})}{\Gamma(\alpha)} \sum_{s=0}^{k+1} f(s+\alpha-1, y(s+\alpha-1))
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{M_0}{\Gamma(\alpha)}\Big\|\Big(\sum_{s=0}^{t_1-\alpha}[(t_1-s-1)^{(\alpha-1)}-(t_2-s-1)^{(\alpha-1)}]\Big)-\sum_{s=t_1-\alpha+1}^{t_2-\alpha}(t_2-s-1)^{(\alpha-1)}\Big\| \\
&\quad+\frac{M_0(k+1)}{\Gamma(\alpha)}(t_2^{(\alpha-1)}-t_1^{(\alpha-1)}) \\
&=\frac{M_0}{\alpha\Gamma(\alpha)}\Big\|(t_2-t_1+\alpha-1)^{(\alpha)}-(t_2-1)^{(\alpha)}+(t_1-1)^{(\alpha)}-(t_2-t_1+\alpha-1)^{(\alpha)}\Big\| \\
&\quad+\frac{M_0(k+1)}{\Gamma(\alpha)}(t_2^{(\alpha-1)}-t_1^{(\alpha-1)}) \\
&=\frac{M_0(k+1)}{\alpha\Gamma(\alpha)}[(t_2-1)^{(\alpha)}-(t_1-1)^{(\alpha)}]+\frac{M_0(k+1)}{\Gamma(\alpha)}(t_2^{(\alpha-1)}-t_1^{(\alpha-1)}) \\
&\le\frac{M_0(k+1)}{\alpha\Gamma(\alpha)}[(t_2-1)^{(\alpha-1)}-(t_1-1)^{(\alpha-1)}]+\frac{M_0(k+1)}{\Gamma(\alpha)}(t_2^{(\alpha-1)}-t_1^{(\alpha-1)}) \\
&\le\frac{2M_0(k+1)}{\Gamma(\alpha)}(t_2^{(\alpha-1)}-t_1^{(\alpha-1)}).
\end{aligned}
$$

Since $t^{(\alpha-1)}$ is a decreasing function, therefore $(t-1)^{(\alpha-1)} \le t^{(\alpha-1)}$ and

$$
\begin{aligned}
t_2^{(\alpha-1)}-t_1^{(\alpha-1)} &\le \Delta(t)^{(\alpha-1)}|_{t=t_2-1}(t_2-t_1)=(\alpha-1)(t_2-1)^{(\alpha-2)}(t_2-t_1) \\
&\le(\alpha-1)(\alpha-2)^{(\alpha-2)}(t_2-t_1)=(\alpha-1)\Gamma(\alpha-1)(t_2-t_1) \\
&=\Gamma(\alpha)(t_2-t_1).
\end{aligned}
$$

Hence $\|\Phi y(t_1)-\Phi y(t_2)\| \le 2M_0(k+1)(t_2-t_1) < \varepsilon$, which implies that $\Phi(A)$ is an equicontinuous set. Consequently $\Phi(A)$ relatively compact in $\mathcal{B}$ by Arzelà-Ascoli theorem. We have proved that Φ is completely continuous operator. □

We state some conditions for the function f which will be used to establish some existence results of positive solutions.

(H_1): There exists $\omega > 0$ such that $f(t,y) \le \frac{1}{2}\mu\omega$ whenever $0 \le y \le \omega$,

(H_2): There exists $\omega > 0$ such that $f(t,y) \ge \zeta\omega$ whenever $\tilde{\gamma}\omega \le y \le \omega$.

Now we prove the following existence result.

Theorem 4.2.1. *Suppose that there are distinct $\omega_1, \omega_2 > 0$ such that condition (H_1) holds at $\omega = \omega_1$ and condition (H_2) holds at $\omega = \omega_2$. Suppose also that $f(t,y) \ge 0$ and continuous. Then the problem (4.1) has at least one positive solution, say y_0, such that $|y_0|$ lies between ω_1 and ω_2.*

Proof. Suppose that $0 < \omega_1 < \omega_2$. For $y \in E$, by Lemma 4.2.1 $\Phi y \in E$ and Φ is a completely continuous operator. Now substitute $\Omega_1 = \{y \in E : ||y|| < \omega_1\}$ and let $\mu := \frac{2}{\sum_{s=0}^{k+1} G(s+\alpha,s)}$. Note that for $y \in \partial\Omega_1$, we have $||y|| = \omega_1$, so that condition (H_1) holds for all $y \in \partial\Omega_1$. Then for $y \in E \cap \partial\Omega_1$, we find that

$$
\begin{aligned}
||\Phi y|| &= \max_{t \in [\alpha-1,\alpha+k]_{\mathbb{N}_{\alpha-1}}} \sum_{s=0}^{k+1} G(t,s) f(s+\alpha-1, y(s+\alpha-1)) \\
&\leq \sum_{s=0}^{k+1} G(s+\alpha,s) f(s+\alpha-1, y(s+\alpha-1)) \\
&\leq \frac{1}{2}\mu\omega_1 \sum_{s=0}^{k+1} G(s+\alpha,s) = \omega_1 = ||y||.
\end{aligned}
$$

Therefore we find that $||\Phi y|| \leq ||y||$, whenever $y \in E \cap \partial\Omega_1$. Hence we get that the operator Φ is a cone compression on $E \cap \Omega_1$.

On the other hand, substitute $\Omega_2 = \{y \in E : ||y|| < \omega_2\}$ and let $\zeta = 1/\vartheta$, where

$$
\vartheta := \sum_{s=\lceil \frac{\alpha+k}{4} - \alpha \rceil}^{\lfloor \frac{3(\alpha+k)}{4} - \alpha \rfloor} \tilde{\gamma} G\left(\lfloor \frac{k+1}{2} \rfloor + \alpha, s \right).
$$

Note that for $y \in \partial\Omega_2$, we have $||y|| = \omega_2$, then the condition (H_2) holds for all $y \in \partial\Omega_2$. Also note that $\left\{ \lfloor \frac{k+1}{2} \rfloor + \alpha \right\} \subset \left[\frac{k+\alpha}{4}, \frac{3(k+\alpha)}{4} \right]$. Then, for $y \in E \cap \partial\Omega_2$, we find that

$$
\begin{aligned}
\Phi y \left(\lfloor \frac{k+1}{2} \rfloor + \alpha \right) &= \sum_{s=0}^{k+1} G\left(\lfloor \frac{k+1}{2} \rfloor + \alpha, s \right) f(s+\alpha-1, y(s+\alpha-1)) \\
&\geq \zeta\omega_2 \sum_{s=\lceil \frac{\alpha+k}{4} - \alpha \rceil}^{\lfloor \frac{3(\alpha+k)}{4} - \alpha \rfloor} \tilde{\gamma} G\left(\lfloor \frac{k+1}{2} \rfloor + \alpha, s \right) = \omega_2.
\end{aligned}
$$

Therefore $||\Phi y|| \geq ||y||$, whenever $y \in E \cap \partial\Omega_2$. Hence we get that the operator Φ is a cone expansion on $E \cap \partial\Omega_2$. Then, from Theorem 1.6.4 it follows that the operator Φ has a fixed point. This means that (4.1) has a positive solution, say y_0 with $\omega_1 \leq ||y_0|| \leq \omega_2$, as claimed. □

We now provide some new results that yield the existence or non existence of positive solutions. We shall assume that $f(t,y)$ has the special form $f(t,y) \equiv \lambda f_1(t) f_2(y)$.

(H_3) Suppose that $\lambda > 0$ and f_1, f_2 are nonnegative.
Let

$$f_0 = \lim_{y\to 0} \frac{f_2(y)}{y}, \qquad f_\infty = \lim_{y\to\infty} \frac{f_2(y)}{y}.$$

Theorem 4.2.2. *Assume that condition (H_3) holds.*
(H_4) If $f_0 = 0$ and $f_\infty = \infty$, then for all $\lambda > 0$ problem (4.1) has a positive solution.
(H_5) If $f_0 = \infty$ and $f_\infty = 0$, then for all $\lambda > 0$ problem (4.1) has a positive solution.

Proof. We prove that the problem (4.1) has a positive solution, it is sufficient to find that operator $\Phi : E \to E$ has a fixed point defined by equation (4.10). From Lemma 4.2.1 it follows that operator $\Phi : E \to E$ is completely continuous. Moreover,if we fix $\omega > 0$, let $E_\omega = \{y \in E : \|y\| < \omega\}$, and $\partial E_\omega = \{y \in E : \|y\| = \omega\}$, then the operator $\Phi : E_\omega \to E$ defined as $\Phi y(t) := \lambda \sum_{s=0}^{k+1} G(t,s) f_1(s) f_2(y(s))$, is completely continuous, $\Phi y \neq 0$ for $y \in \partial E_\omega$.

(H_4) By $f_0 = 0$, for $0 < \epsilon < \frac{1}{\lambda \sum_{s=0}^{k+1} G(s+\alpha,s) f_1(s)}$, there exists positive $\omega_1 > 0$ such that $f_2(y) < \epsilon|y|$ whenever $0 < |y| < \omega_1$.
Let $\Omega_{\omega_1} = \{y \in E : \|y\| < \omega_1\}$. For $y \in \partial\Omega_{\omega_1} \cap E$, we have

$$\begin{aligned} |\Phi y(t)| &\le \lambda\epsilon \sum_{s=0}^{k+1} G(s+\alpha, s) f_1(s) |y(s)| \\ &\le \omega_1 \lambda\epsilon \sum_{s=0}^{k+1} G(s+\alpha, s) f_1(s) < \omega_1, \end{aligned}$$

which shows that $\|\Phi y(t)\| \le \omega_1 = \|y\|$ for $y \in \partial\Omega_{\omega_1} \cap E$.

By $f_\infty = \infty$, for $M > \frac{1}{\lambda\tilde{\gamma}^2 \sum_{s=\lceil \frac{k+\alpha}{4} \rceil}^{\lfloor \frac{3(k+\alpha)}{4} \rfloor} G(s+\alpha,s) f_1(s)}$, there exists $k_0 > 0$ such that $f_2(y) > M|y|$, whenever $|y| > k_0$. Take $\omega_2 > \left\{\omega_1, \frac{k_0}{\tilde{\gamma}}\right\}$, and that let $\Omega_{\omega_2} = \{y \in E : \|y\| < \omega_2\}$. Then for $y \in \partial\Omega_{\omega_2} \cap E, y(t) \ge \tilde{\gamma}\|y\| = \tilde{\gamma}\omega_2 > k_0$, for $\frac{k+\alpha}{4} \le t \le \frac{3(k+\alpha)}{4}$. Whence

$$\Phi y(t) \ge \lambda \sum_{s=\lceil \frac{k+\alpha}{4} \rceil}^{\lfloor \frac{3(k+\alpha)}{4} \rfloor} G(t,s) f_1(s) f_2(y(s))$$

$$\geq \lambda\tilde{\gamma}M \sum_{s=\lceil\frac{k+\alpha}{4}\rceil}^{\lfloor\frac{3(k+\alpha)}{4}\rfloor} G(s+\alpha,s)f_1(s)|y(s)|$$

$$\geq \lambda\tilde{\gamma}^2 M \sum_{s=\lceil\frac{k+\alpha}{4}\rceil}^{\lfloor\frac{3(k+\alpha)}{4}\rfloor} G(s+\alpha,s)f_1(s)||y|| > ||y||,$$

which implies that $||\Phi y|| \geq ||y||$ for $y \in \partial\Omega_{\omega_2} \cap E$. It follows from Theorem 1.6.4 (i) that Φ has at least one fixed point in $E \cap (\bar{\Omega}_{\omega_2}/\Omega_{\omega_1})$. The fixed point $y \in E \cap (\bar{\Omega}_{\omega_2}/\Omega_{\omega_1})$ is the positive solution of (4.1) by Lemma 4.1.1.

(H_5) The assumption $f_0 = \infty$, for $M > \frac{1}{\lambda\tilde{\gamma}^2 \sum_{s=\lceil\frac{k+\alpha}{4}\rceil}^{\lfloor\frac{3(k+\alpha)}{4}\rfloor} G(s+\alpha,s)f_1(s)}$, there exists a positive $\omega_1 > 0$ such that $f_2(y) > M|y|$, whenever $0 < |y| < \omega_1$. Let $\Omega_{\omega_1} = \{y \in E : \|y\| < \omega_1\}$, for $y \in \partial\Omega_{\omega_1} \cap E$, we have

$$\Phi y(t) \geq \lambda \sum_{s=\lceil\frac{k+\alpha}{4}\rceil}^{\lfloor\frac{3(k+\alpha)}{4}\rfloor} G(t,s)f_1(s)f_2(y(s))$$

$$\geq \lambda\tilde{\gamma}M \sum_{s=\lceil\frac{k+\alpha}{4}\rceil}^{\lfloor\frac{3(k+\alpha)}{4}\rfloor} G(s+\alpha,s)f_1(s)|y(s)|$$

$$\geq \lambda\tilde{\gamma}^2 M \sum_{s=\lceil\frac{k+\alpha}{4}\rceil}^{\lfloor\frac{3(k+\alpha)}{4}\rfloor} G(s+\alpha,s)f_1(s)||y|| > ||y||,$$

which implies that $||\Phi y|| \geq ||y||$ for $y \in \partial\Omega_{\omega_1} \cap E$.

By $f_\infty = 0$, for $0 < \epsilon < \frac{1}{2\lambda\sum_{s=0}^{k+1} G(s+\alpha,s)f_1(s)}$, there exists positive $k_0 > 0$ such that $f_2(y) < \epsilon|y|$, whenever $|y| > k_0$.
Hence, $f_2(y) \leq \epsilon|y| + C$, for $y \in [0,+\infty)$, where $C = \max_{0\leq y\leq k_0} f_2(y) + 1$. Let $\Omega_{\omega_2} = \{y \in E : ||y|| < \omega_2\}$, where $\omega_2 > \{\omega_1, 2C\lambda\sum_{s=0}^{k+1} G(s+\alpha,s)f_1(s)\}$, then, for $y \in \partial\Omega_{\omega_2} \cap E$, we have

$$|\Phi y(t)| \leq \lambda\epsilon \sum_{s=0}^{k+1} G(s+\alpha,s)f_1(s)|y(s)| + \lambda C \sum_{s=0}^{k+1} G(s+\alpha,s)f_1(s)$$

$$\leq \omega_2\lambda\epsilon \sum_{s=0}^{k+1} G(s+\alpha,s)f_1(s) + \lambda C \sum_{s=0}^{k+1} G(s+\alpha,s)f_1(s)$$

$$\leq \frac{\omega_2}{2} + \frac{\omega_2}{2} = \omega_2,$$

which implies $||\Phi y|| \leq ||y||$ for $y \in \partial\Omega_{\omega_2} \cap E$.

From Theorem 1.6.4 (ii), it follows that Φ has at least one fixed point y in $E \cap (\bar{\Omega}_{\omega_2}/\Omega_{\omega_1})$. From Lemma 4.1.1, the fixed point $y \in E \cap (\bar{\Omega}_{\omega_2}/\Omega_{\omega_1})$ is the positive solution of (4.1). □

Theorem 4.2.3. *Assume that condition* (H_3) *holds and* $f_0 = 0$ *or* $f_\infty = 0$*. Then there exists* $\lambda_0 > 0$ *such that for all* $\lambda > \lambda_0$ *problem* (4.1) *has a positive solution.*

Proof. In view of Lemma 4.2.1 and assumption H_3, if we fix $\omega > 0$, define $E_\omega = \{y \in E : ||y|| < \omega\}$, and $\partial E_\omega = \{y \in E : ||y|| = \omega\}$, then, the operator $\Phi : E_\omega \to E$ is completely continuous, $\Phi y \neq 0$ for $y \in \partial E_\omega$.

For fixed $\omega_1 > 0$, there exists $\lambda_0 > 0$, such that we define $\lambda_0 = \frac{\omega_1}{\tilde{\gamma} m_{\omega_1} \sum_{s=\lceil \frac{k+\alpha}{4} \rceil}^{\lfloor \frac{3(k+\alpha)}{4} \rfloor} G(s+\alpha,s) f_1(s)}$, and $\Omega_{\omega_1} = \{y \in E : ||y|| < \omega_1\}$, here $m_{\omega_1} = \min_{\tilde{\gamma}\omega_1 \leq y \leq \omega 1} f_2(y)$, by H_3 $, m_{\omega_1 > 0}$. Then for $y \in E \cap \partial\Omega_{\omega_1}$, by Lemma 4.1.2, we have

$$\min_{t \in [\frac{k+\alpha}{4}, \frac{3(k+\alpha)}{4}]} \Phi y(t) \geq \lambda\tilde{\gamma} \sum_{s=\lceil \frac{k+\alpha}{4} \rceil}^{\lfloor \frac{3(k+\alpha)}{4} \rfloor} G(s+\alpha,s) f_1(s) f_2(y(s))$$

$$> \lambda_0 \tilde{\gamma} m_{\omega_1} \sum_{s=\lceil \frac{k+\alpha}{4} \rceil}^{\lfloor \frac{3(k+\alpha)}{4} \rfloor} G(s+\alpha,s) f_1(s) = \omega_1 = ||y||,$$

which implies that

$$||\Phi y|| > ||y|| \;\; for \;\; y \in E \cap \partial\Omega_{\omega_1}, \lambda > \lambda_0. \tag{4.11}$$

Let $f_0 = 0$, then, for $0 < \epsilon < \frac{1}{\lambda \sum_{s=0}^{k+1} G(s+\alpha,s) f_1(s)}$, there exists positive $\bar{\omega}_2 > 0$ such that $f_2(y) < \epsilon|y|$, whenever $0 < |y| < \bar{\omega}_2$. Let $\Omega_{\omega_2} = \{y \in E : ||y|| < \omega_2\}, 0 < \omega_2 < \{\omega_1, \bar{\omega}_2\}$. For $y \in E \cap \partial\Omega_{\omega_2}$, we have

$$|\Phi y(t)| \leq \lambda\epsilon \sum_{s=0}^{k+1} G(s+\alpha,s) f_1(s)|y(s)| \leq \omega_2 \lambda\epsilon \sum_{s=0}^{k+1} G(s+\alpha,s) f_1(s) < \omega_2,$$

which imply that $||\Phi y|| \leq \omega_2 = ||y||$ for $y \in E \cap \partial\Omega_{\omega_2}$.

Now $f_\infty = 0$, then, $0 < \epsilon < \frac{1}{2\lambda \sum_{s=0}^{k+1} G(s+\alpha,s) f_1(s)}$, there exists positive $k_0 > 0$ such that $f_2(y) <$

$\epsilon|y|$, whenever $|y| > k_0$. Hence, $f_2(y) \leq \epsilon|y| + C$, for $y \in [0, +\infty)$, here $C = \max_{0\leq y\leq k_0} f_2(y) + 1$. Let $\Omega_{\omega_3} = \{y \in E : ||y|| < \omega_3\}$, where $\omega_3 > \left\{\omega_1, 2\lambda C \sum_{s=0}^{k+1} G(s+\alpha, s) f_1(s)\right\}$, then, for $y \in E \cap \partial\Omega_{\omega_3}$, we have

$$
\begin{aligned}
|\Phi y(t)| &\leq \omega_3 \lambda \epsilon \sum_{s=0}^{k+1} G(s+\alpha, s) f_1(s) + \lambda C \sum_{s=0}^{k+1} G(s+\alpha, s) f_1(s) \\
&< \frac{\omega_3}{2} + \frac{\omega_3}{2} = \omega_3,
\end{aligned}
$$

which imply that $||\Phi y|| \leq \omega_3 = ||y||$ for $y \in E \cap \partial\Omega_{\omega_3}$.
From Theorem 1.6.4, it follows that Φ has a fixed point in $y \in E \cap \bar{\Omega}_{\omega_2}/\Omega_{\omega_1}$ or $y \in E \cap \bar{\Omega}_{\omega_3}/\Omega_{\omega_1}$ according to $f_0 = 0$ or $f_\infty = 0$, respectively. Hence, problem (4.1) has a positive solution for $\lambda > \lambda_0$. □

Theorem 4.2.4. *Assume that condition* (H_3) *holds. If* $f_0 = f_\infty = 0$, *then there exists* $\lambda_0 > 0$ *such that for all* $\lambda > \lambda_0$ *problem* (4.1) *has two positive solutions.*

Proof. Let $0 < \omega_3 < \omega_4$, by the same argument as for inequality (4.11), there exists $\lambda_0 > 0$ such that

$$
||\Phi y|| > ||y|| \qquad \text{for} \qquad y \in E \cap \partial\Omega_{\omega_i} (i = 3, 4), \lambda > \lambda_0.
$$

Since we have given $f_0 = f_\infty = 0$, then, from the proof of Theorem 4.2.3, it follows that we can choose $0 < \omega_1 < \omega_3$ and $\omega_2 > \omega_4$ such that $||\Phi y|| < ||y||$ for $y \in E \cap \partial\Omega_{\omega_i} (i = 1, 2)$.
Thus, from Theorem 1.6.4 the operetor Φ has two fixed points $y_1 \in E \cap \bar{\Omega}_{\omega_3}/\Omega_{\omega_1}, y_2 \in E \cap \bar{\Omega}_{\omega_2}/\Omega_{\omega_4}$, which are the distinct positive solutions of problem (4.1). □

Theorem 4.2.5. *Assume that condition* H_3 *holds. If* $f_0 < \infty$ *and* $f_\infty < \infty$, *then there exists* $\lambda_0 > 0$ *such that for all* $0 < \lambda < \lambda_0$ *problem* (4.1) *has no positive solution.*

Proof. Since $f_0 < \infty$ and $f_\infty < \infty$, then for positive numbers $\epsilon_1, \epsilon_2 > 0$ there exist $0 < \omega_1 < \omega_2$ such that,

$$
f_2(y) \leq \epsilon_1 |y|, \qquad for \qquad |y| \leq \omega_1,
$$

and

$$
f_2(y) \leq \epsilon_2 |y|, \qquad for \qquad |y| \geq \omega_2.
$$

Let

$$\epsilon = \max\left\{\epsilon_1, \epsilon_2, \max\left\{\frac{f_2(y)}{y} : \omega_1 \le y \le \omega_2\right\}\right\}.$$

Therefore we have

$$f_2(y) \le \epsilon|y|, \qquad for \qquad \omega_1 \le |y| \le \omega_2.$$

On contrary, suppose that $v(t)$ is a positive solution of problem (4.1), then, for $0 < \lambda < \lambda_0$, we define $\lambda_0 = \frac{1}{\beta\epsilon\sum_{s=0}^{k} G(s+\alpha,s)f_1(s)}$, for some $0 < \beta < 1$. Then, we have

$$\begin{aligned}\|v\| = \|\Phi v\| &\le \lambda\epsilon\sum_{s=0}^{k+1} G(s+\alpha,s)f_1(s)\|v\| \\ &\le \|v\|\lambda_0\epsilon\sum_{s=0}^{k+1} G(s+\alpha,s)f_1(s) = \frac{\|v\|}{\beta}.\end{aligned}$$

That is, $\|v\| \le \frac{\|v\|}{\beta} < \|v\|$, which is a contradiction. Whence problem (4.1) has no positive solution. □

Theorem 4.2.6. *Assume that condition* (H_3) *holds. If* $f_0 = \infty$ *or* $f_\infty = \infty$, *then there exists* $\lambda_0 > 0$ *such that for all* $0 < \lambda < \lambda_0$ *problem* (4.1) *has a positive solution.*

Proof. Fix $\omega_1 > 0$, we define $\lambda_0 = \frac{\omega_1}{M_{\omega_1}\sum_{s=0}^{k+1} G(s+\alpha,s)f_1(s)}$, where $M_{\omega_1} = \max_{0\le y\le\omega_1} f_2(y) + 1$. Let $\Omega_{\omega_1} = \{y \in E : ||y|| < \omega_1\}$, for $y \in \partial\Omega_{\omega_1},\ 0 < \lambda < \lambda_0$, we have

$$\begin{aligned}|\Phi y(t)| &\le \lambda M_{\omega_1}\sum_{s=0}^{k+1} G(s+\alpha,s)f_1(s) \\ &< \lambda_0 M_{\omega_1}\sum_{s=0}^{k+1} G(s+\alpha,s)f_1(s) = \omega_1 = ||y||,\end{aligned}$$

which implies that

$$||\Phi y|| < ||y|| \qquad for \qquad y \in \partial\Omega_{\omega_1}, 0 < \lambda < \lambda_0. \tag{4.12}$$

Now $f_0 = \infty$, implies, for $M > \frac{1}{\lambda\tilde{\gamma}^2\sum_{\lceil\frac{k+\alpha}{4}\rceil}^{\lfloor\frac{3(k+\alpha)}{4}\rfloor} G(s+\alpha,s)f_1(s)}$, there exists positive $\bar{\omega} > 0$ such that $f_2(y) > M|y|$, whenever $0 < |y| < \bar{\omega}$.

Let $\Omega_{\omega_2} = \{y \in E : ||y|| < \omega_2\}, 0 < \omega_2 < \{\omega_1, \bar{\omega}\}$. For $y \in \partial\Omega_{\omega_2}$, we have

$$\begin{aligned}\Phi y(t) &\geq \lambda\tilde{\gamma}M \sum_{s=\lceil\frac{k+\alpha}{4}\rceil}^{\lfloor\frac{3(k+\alpha)}{4}\rfloor} G(s+\alpha, s)f_1(s)|y(s)| \\ &\geq \lambda\tilde{\gamma}^2 M \sum_{s=\lceil\frac{k+\alpha}{4}\rceil}^{\lfloor\frac{3(k+\alpha)}{4}\rfloor} G(s+\alpha, s)f_1(s)||y|| \\ &> ||y||,\end{aligned}$$

which implies that $||\Phi y|| \geq ||y||$ for $y \in \partial\Omega_{\omega_2}$.

The hypothesis $f_\infty = \infty$, for $M > \frac{1}{\lambda\tilde{\gamma}^2 \sum_{\lceil\frac{k+\alpha}{4}\rceil}^{\lfloor\frac{3(k+\alpha)}{4}\rfloor} G(s+\alpha,s)f_1(s)}$, there exists positive $k_0 > 0$ such that $f_2(y) > M|y|$, for $|y| > k_0$.

Let $\omega_3 > \{\omega_1, \frac{k_0}{\tilde{\gamma}}\}$, and that let $\Omega_{\omega_3} = \{y \in E : ||y|| < \omega_3\}$. Then, for $y \in \partial\Omega_{\omega_3}, y(t) \geq \tilde{\gamma}||y|| = \tilde{\gamma}\omega_3 > k_0$ for $\frac{k+\alpha}{4} \leq t \leq \frac{3(k+\alpha)}{4}$. Thus, we have

$$\begin{aligned}\Phi y(t) &\geq \lambda \sum_{\lceil\frac{k+\alpha}{4}\rceil}^{\lfloor\frac{3(k+\alpha)}{4}\rfloor} G(t,s)f_1(s)f_2(y(s)) \\ &\geq \lambda\tilde{\gamma}^2 M \sum_{\lceil\frac{k+\alpha}{4}\rceil}^{\lfloor\frac{3(k+\alpha)}{4}\rfloor} G(t,s)f_1(s)||y|| \\ &> ||y||,\end{aligned}$$

which implies that $||\Phi y|| \geq ||y||$ for $y \in \partial\Omega_{\omega_3}$.

From Theorem 1.6.4, it follows that Φ has a fixed point $y \in \Omega_{\omega_1}/\bar{\Omega}_{\omega_2}$ or $y \in \Omega_{\omega_3}/\bar{\Omega}_{\omega_1}$ according to $f_0 = \infty$ or $f_\infty = \infty$, respectively. Consequently, problem (4.1) has a positive solution for $0 < \lambda < \lambda_0$. □

Theorem 4.2.7. *Assume that condition* (H_3) *holds. If* $f_0 = f_\infty = \infty$, *then there exists* $\lambda_0 > 0$ *such that for all* $0 < \lambda < \lambda_0$ *problem* (4.1) *has two positive solutions.*

Proof. Let $0 < \omega_3 < \omega_4$, then the inequality (4.12) implies that there exists $\lambda_0 > 0$ such that

$$\|\Phi y\| < \|y\| \qquad \mathit{for} \qquad y \in \partial\Omega_{\omega_i}(i = 3, 4), 0 < \lambda < \lambda_0.$$

Since $f_0 = f_\infty = \infty$, then, it follows from the proof of Theorem 4.2.6 that we can choose $0 < \omega_1 < \omega_3$ and $\omega_2 > \omega_4$ such that

$$\|\Phi y\| > \|y\| \qquad for \qquad y \in \partial\Omega_{\omega_i}(i = 1, 2).$$

Hence, by Theorem 1.6.4 the operator Φ has two fixed points $y_1 \in \Omega_{\omega_3}\backslash\bar{\Omega}_{\omega_1}, y_2 \in \Omega_{\omega_2}\backslash\bar{\Omega}_{\omega_4}$ which are the distinct positive solutions of problem (4.1). □

Theorem 4.2.8. *Assume that condition* (H_3) *holds. If* $f_0 > 0$ *or* $f_\infty > 0$*, then there exists* $\lambda_0 > 0$ *such that for all* $\lambda > \lambda_0$ *problem* (4.1) *has no positive solution.*

Proof. Since $f_0 > 0$ and $f_\infty > 0$, then, for positive numbers $\eta_1, \eta_2 > 0$ there exists $0 < \omega_1 < \omega_2$ such that,

$$f_2(y) \geq \eta_1|y|, \qquad \text{for} \qquad |y| \leq \omega_1,$$

and

$$f_2(y) \geq \eta_2|y|, \qquad \text{for} \qquad |y| \geq \omega_2.$$

Let

$$\eta = \min\left\{\eta_1, \eta_2, \min\{\frac{f_2(y)}{y} : \tilde{\gamma}\omega_1 \leq y \leq \omega_2\}\right\}.$$

Then we can obtain that

$$f_2(y) \geq \eta|y|, \qquad \text{for} \qquad |y| \in R_+, |y| \leq \omega_1, \tag{4.13}$$

$$f_2(y) \geq \eta|y|, \qquad \text{for} \qquad |y| \in R_+, |y| \geq \tilde{\gamma}\omega_1. \tag{4.14}$$

On contrary, suppose that $\upsilon(t) \in E$ is a positive solution of problem (4.1), if $\|\upsilon\| \leq \omega_1$. Let $\lambda_0 = \frac{1}{\tilde{\gamma}^2\eta\sum_{s=\lceil\frac{(k+\alpha)}{4}\rceil}^{\lfloor\frac{3(k+\alpha)}{4}\rfloor} G(s+\alpha,s)f_1(s)}$, then, by equation (4.13), we have

$$f_2(\upsilon(t)) \geq \eta|\upsilon(t)|, \qquad t \in [\alpha - 1, \alpha + k].$$

On the other hand, if $\|\upsilon\| > \omega_1$, then,

$$\min_{\frac{(k+\alpha)}{4} \leq t \leq \frac{3(k+\alpha)}{4}} \upsilon(t) \geq \tilde{\gamma}\|\upsilon\| > \tilde{\gamma}\omega_1,$$

which, together with equation (4.14), implies that

$$f_2(\upsilon(t)) \geq \eta|\upsilon(t)|, \qquad t \in \left[\frac{(k+\alpha)}{4}, \frac{3(k+\alpha)}{4}\right].$$

Thus, for $\lambda > \lambda_0$, we have

$$\begin{aligned}
\upsilon(t) = \Phi\upsilon(t) &= \lambda \sum_{s=0}^{k+1} G(t,s) f_1(s) f_2(\upsilon(s)) \\
&\geq \lambda \sum_{s=\lceil \frac{(k+\alpha)}{4} \rceil}^{\lfloor \frac{3(k+\alpha)}{4} \rfloor} G(t,s) f_1(s) f_2(\upsilon(s)) \\
&\geq \tilde{\gamma}\eta\lambda \sum_{s=\lceil \frac{(k+\alpha)}{4} \rceil}^{\lfloor \frac{3(k+\alpha)}{4} \rfloor} G(s+\alpha,s) f_1(s) |\upsilon| \\
&\geq \tilde{\gamma}^2\eta\lambda \sum_{s=\lceil \frac{(k+\alpha)}{4} \rceil}^{\lfloor \frac{3(k+\alpha)}{4} \rfloor} G(s+\alpha,s) f_1(s) \|\upsilon\| \\
&\geq \tilde{\gamma}^2\eta\lambda_0 \sum_{s=\lceil \frac{(k+\alpha)}{4} \rceil}^{\lfloor \frac{3(k+\alpha)}{4} \rfloor} G(s+\alpha,s) f_1(s) \|\upsilon\| \\
&> \|\upsilon\|,
\end{aligned}$$

which is a contradiction $\|\upsilon\| > \|\upsilon\|$. Hence problem (4.1) has no positive solution. □

Example 4.2.1. Consider the following discrete boundary value problem

$$\begin{aligned}
-\Delta^{\alpha}_{\alpha-2} y(t) &= \frac{1 + (t+\alpha-1)^2 e^{(-1/2)(t+\alpha-1)}}{1+(t+\alpha-1)^2} + \frac{(t+\alpha-1)^2[1+\sin^2 y(t+\alpha-1)]}{1+y^2(t+\alpha-1)},\ t \in [0,k+1]_{\mathbb{N}_0}, \\
y(\alpha-2) &= 0,\ \Delta^{\alpha-1}_{\alpha-1} y(\alpha+k) = 0,
\end{aligned} \tag{4.15}$$

where $\alpha = \frac{3}{2}$ and $k = 5$. Let

$$f(t,y(t)) = \frac{1 + t^2 e^{(-1/2)t}}{1+t^2} + \frac{t^2[1+\sin^2 y(t)]}{1+y^2(t)}, \quad t \in [\frac{1}{2}, \frac{13}{2}]_{\mathbb{N}_{\frac{1}{2}}}.$$

First we find the lower bound and upper bound of a function $f(t,y(t))$.

$$f(t,y(t)) = \frac{t^2 e^{-1/2t} + 1}{1+t^2} + \frac{t^2[1+\sin^2 y(t)]}{1+y^2(t)}$$

$$\leq \frac{(13/2)^2 e^{-1/2(1/2)}+1}{1+(1/2)^2} + (13/2)^2(1+1) \qquad \because \max\sin(y)=1$$
$$= \frac{169e^{-1/4}+4}{5} + 84.5 \approx 111.623,$$

and

$$f(t,y(t)) \geq \frac{(1/2)^2 e^{-1/2(13/2)}+1}{1+(13/2)^2}$$
$$= \frac{e^{-13/4}+4}{173} \approx 0.0233455.$$

Now we find the value of $\tilde{\gamma}$ for the given function,

$$\tilde{\gamma} = \min\left\{ \frac{\left(\frac{1}{4}(k+\alpha)\right)^{(\alpha-1)}}{(k+\alpha)^{(\alpha-1)}},\ 1-\frac{\left(\frac{3}{4}(k+\alpha)-1\right)^{(\alpha-1)}}{\left(\frac{3}{4}(k+\alpha)\right)^{(\alpha-1)}} \right\}$$
$$= \min\left\{ \frac{\left(\frac{1}{4}(5+3/2)\right)^{(1/2)}}{(5+3/2)^{(1/2)}},\ 1-\frac{\left(\frac{3}{4}(5+3/2)-1\right)^{(1/2)}}{\left(\frac{3}{4}(5+3/2)\right)^{(1/2)}} \right\}$$
$$= \min\left\{ \frac{\left(\frac{13}{8}\right)^{(1/2)}}{\left(\frac{13}{2}\right)^{(1/2)}},\ 1-\frac{\left(\frac{31}{8}\right)^{(1/2)}}{\left(\frac{39}{8}\right)^{(1/2)}} \right\}$$
$$= \min\{0.52912, 0.102564\} \approx 0.102564.$$

The Green's function for corresponding problem is given as:

$$G(t,s) = \begin{cases} \frac{2\left(t^{(1/2)}-(t-s-1)^{(1/2)}\right)}{\sqrt{\pi}}, & s \leq t-3/2 \leq 6; \\ \frac{2t^{(1/2)}}{\sqrt{\pi}}, & t-3/2 < s \leq 6. \end{cases} \tag{4.16}$$

By using Green's function we find the value of μ and ζ. We note that

$$\mu = \frac{2}{\sum_{s=0}^{k+1} G(s+\alpha,s)} = \frac{2}{\sum_{s=0}^{6} G(s+3/2,s)}$$
$$= \frac{2}{G(3/2,0)+G(5/2,1)+G(7/2,2)+G(9/2,3)+G(11/2,4)+G(13/2,5)+G(15/2,6)}$$
$$= \frac{2}{9.80518} \approx 0.20397.$$

Since $\zeta = 1/\vartheta$, and

$$\vartheta = \sum_{s=\lceil \frac{\alpha+k}{4}-\alpha \rceil}^{\lfloor \frac{3(\alpha+k)}{4}-\alpha \rfloor} \tilde{\gamma} G\left(\lfloor \frac{k+1}{2} \rfloor + \alpha, s \right) = \sum_{s=\lceil \frac{1}{8} \rceil}^{\lfloor \frac{27}{2} \rfloor} \tilde{\gamma} G\left(9/2, s\right)$$

$$= (0.102564)\sum_{s=1}^{3} G\left(9/2, s\right)$$
$$= (0.102564)[G(9/2,1) + G(9/2,2) + G(9/2,3)] = (0.102564)(3.007816)$$
$$\approx 0.30849.$$

Then, $\zeta \approx 3.241557911$.
Now we are using the conditions of Theorem 4.2.1 that shows the existence of problem (4.15)

(H_1): We choose $\omega_1 = 1140$. Then

$$f(t, y(t)) \leq 111.623 < \frac{1}{2}\mu\omega_1 = \frac{1}{2}(0.20397)(1140) \approx 116.263.$$

(H_2): Now we choose $\omega_2 = 0.001$. Then we have

$$f(t, y(t)) \geq 0.0233455 > \zeta\omega_2 \approx 0.00324156.$$

Thus, this shows that problem (4.15) has at least one positive solution y_0, such that $0,001 \leq |y_0| \leq 1140$.

Example 4.2.2. Consider the discrete boundary value problem

$$\begin{aligned} -\Delta^{\alpha}_{\alpha-2} y(t) &= \lambda e^{-(t+\alpha-1)}\sqrt{(t+\alpha-1)^2+1}\left[\frac{(y(t+\alpha-1))^2}{\pi + (y(t+\alpha-1))^3}\right], \quad t \in \mathbb{N}_0, \\ y(\alpha-2) &= 0, \quad \Delta^{\alpha-1}_{\alpha-1} y(\alpha+k) = 0. \end{aligned} \tag{4.17}$$

where $1 < \alpha \leq 2$ and $k > 3$ is an integer. Let $\alpha = 3/2$ and $k = 4$. Taking $f(t, y(t)) = \lambda e^{-t}\sqrt{t^2+1}[\frac{y^2(t)}{\pi+y^3(t)}]$, for $t \in [\alpha-2, \alpha+k]_{\mathbb{N}_{\alpha-2}}$. Obviously $f(t, y(t)) = \lambda f_1(t) f_2(y(t))$, where $f_1(t) := e^{-t}\sqrt{t^2+1}$ and $f_2(y(t)) := \frac{y^2(t)}{\pi+y^3(t)}$ satisfy assumption (H_3).
For $\omega_1 = 8$, by computations we get $m_{\omega_1} = \min_{\tilde{\gamma}\omega_1 \leq y \leq \omega_1} f_2(y) = \min_{0.9696 \leq y \leq 8} (\frac{y^2(t)}{\pi+y^3(t)}) \approx 0.23195$. Furthermore

$$\begin{aligned} \lambda_0 &= \frac{\omega_1}{\tilde{\gamma} m_{\omega_1} \sum_{s=\lceil \frac{k+\alpha}{4} \rceil}^{\lfloor \frac{3(k+\alpha)}{4} \rfloor} G(s+\alpha, s) f_1(s)} \\ &= \frac{8}{(0.02811234)\sum_{s=2}^{4} \sqrt{s^2+1}e^{-s} G(s+\frac{3}{2}, s)} \approx 396.186. \end{aligned}$$

Moreover, $f_0 = 0$ and $f_\infty = 0$. All assumptions of the Theorem 4.2.4 are satisfied. Therefore, the boundary value problem (4.17) has one positive solution.

Chapter 5

Conclusion

We have studied the basic concept of difference calculus that were used for the development of our results. We discussed some fundamental concepts of discrete fractional calculus. Some basic properties about Riemann-Liouville and Caputo difference operator were included. We also state some fixed point theorem that used to find existence and uniqueness results.

We have discussed some well known methods to find the solution of initial value problems. We have taken the second order fractional difference initial value problem and used Caputo approach to establish the existence results of solutions. Further, we construct uniqueness and global uniqueness results for initial value problem by using Banach contraction principle.

The existence of positive solution for two point fractional difference boundary value problem have been discussed. We have established sufficient conditions for existence and uniqueness of solution for fractional difference boundary value problem. For construction of these results we used Caputo difference operator and also we used Banach contraction principle and Brouwer fixed point theorem.

We established various conditions for existence and non-existence of positive solutions for a class of nonlinear fractional difference boundary value problem, with the help of Krasnoselskii's theorem. For applicability of Krasnoselskii theorem we have constructed Green's function and some useful properties of Green's function for this boundary value problem by using Riemann-Liouville approach. Some examples were included for the appropriateness of our results.

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