Solutions of Differential Equations on Time Scales

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A Dissertation

Submitted in Partial Fulfillment of the

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IN
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Supervised By

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Rawalpindi, Pakistan
2007

Dedicated To

My Parents

And Sisters

CERTIFICATE

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We accept this dissertation as conforming to the required standard.

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2007

IN THE NAME OF

ALLAH

THE MOST BENEFICIENT

THE MOST MERCIFUL

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Faizullah

Solutions of differential equations on time scales

Faizullah

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ABSTRACT

Time scales calculus unifies two disparate notions, namely, difference and differential calculus. The theory of time scales calculus is really still in its infancy. In this dissertation we discuss the solution of differential equations on time scales.

Chapter one of this dissertation discusses the calculus on time scales that involves the notions and definitions of some particular type of derivatives (delta and nabla-derivatives), some related theorems with proofs, relations between delta and nabla derivatives, the notion of integration and related theorems. Some examples are included in order to explain these concepts.

Chapter two introduces dynamic equations on time scales. Initial value problems for first order dynamic equations on time scales are presented and some examples are included. The notion of the generalized exponential function is introduced and some of its properties are discussed. The method of solution of a second order linear differential equation is also presented.

Chapter three is concerned with the existence and uniqueness of solutions of linear boundary value problems (BVPs).

In chapter four, we derive new results on the existence and approximation of solutions of nonlinear BVPs. We use the method of lower and upper solutions [1] to establish a comparison result and an existence theorem. Finally, we introduce a generalized approximation technique and show that under some conditions there exists a monotone sequence $\{w_n\}$ of linear problems converging uniformly to a unique solution of the nonlinear BVP.

CONTENTS

1. **PRELIMINARIES**

1.1 Introduction

The theory of time scales was introduced by a German mathematician Stefen Hilger [2] (student of Bernd Aulbach) in his Ph.D thesis in 1988, in order to unify the theory of continuous and discrete calculus. He reasoned that there should be a way to connect the classical cases of the reals and integers, and interpret everything in between these cases. Any non-empty closed subset (in the usual topology) of the set of real numbers, $\mathbb{R}$, is called a time scale and is denoted by $\mathbb{T}$. For example:

$$\begin{aligned}
&\mathbb{R},\ \text{the set of real numbers;}\\
&\mathbb{Z},\ \text{the set of integers ;}\\
&\mathbb{N},\ \text{the set of natural numbers ;}\\
&\mathbb{N}_0,\ \text{the set of non-negative integers ;}\\
&[0,1]\cup[2,3], [0,1]\cup\mathbb{N};
\end{aligned}$$

etc are time scales; while the following are *not* time scales:

$$\begin{aligned}
&\mathbb{Q},\ \text{the set of rational numbers ;}\\
&\mathbb{Q}',\ \text{the set of irrational numbers ;}\\
&(0,1),\ \text{the open interval between 0 and 1;}\\
&\text{and } \mathbb{C}, \text{the set of complex numbers;}
\end{aligned}$$

because the first three are not closed in $\mathbb{R}$ and the fourth one is not a subset of $\mathbb{R}$.

There are many applications of time scales, since hybrid models are very common in reality. Many populations show population explosion in some time periods while in others growth is stunted. This type of phenomenon is not easily modeled by using just differential or difference equations alone. Moreover, the theory of time scales is applicable to any field in which dynamic processes can be described with discrete or continuous models. Since many economic models are dynamic models with continuous or discrete time therefore, the results of time scales calculus are directly applicable to economics as well. For example, money is invested and accounted at specific (discrete) times but the growth due to the investment is continuous in time. The time scales analysis is so useful that it allows one to create the time scale on a fly and analyze its dynamics.

This chapter discusses some basic definitions and notions of the calculus on time scales. In section 2 some basic concepts of the calculus on time scales are presented. Section 3 is devoted to nabla and delta derivatives, while in section 4 the notion of integration on time scales is discussed. Some examples are also included. Good references to the material of this chapter include the book [3] and papers $[4, 5, 6, 7]$.

1.2 Basic definitions and concepts

The following definitions and concepts will be used throughout this work. Since an arbitrary time scale will not be connected, therefore the jump operators are defined to describe moving forward and backward on a time scale.

Definition 1.2.1. *Forward jump operator*: Let $\mathbb{T}$ be a time scale. For $t \in \mathbb{T}$, the *forward jump operator* $\delta : \mathbb{T} \to \mathbb{T}$ is defined by $\delta(t) = inf\{s \in \mathbb{T} : s > t\}$.

Definition 1.2.2. *Backward jump operator*: For $t \in \mathbb{T}$, the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) = sup\{s \in \mathbb{T} : s < t\}$.

The following four definitions describe the behavior for an arbitrary time scale $\mathbb{T}$ at each point $t \in \mathbb{T}$, using the jump operators.

Definition 1.2.3. *Right, Left-scattered*: A point $t \in \mathbb{T}$ is called *right-scattered*, if $\delta(t) > t$ and it is called *left-scattered* if $\rho(t) < t$.

Definition 1.2.4. *Isolated point*: A point $t \in \mathbb{T}$ is called an *isolated point* if t is right-scattered and left-scattered simultaneously.

Definition 1.2.5. *Right, Left-dense*: A point is called a *right-dense* point if $\delta(t) = t$ and $t < \sup \mathbb{T}$, and a *left-dense* point if $\rho(t) = t$ and $t > \inf \mathbb{T}$.

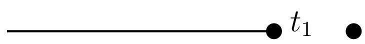
Definition 1.2.6. *Dense point*: A point $t \in \mathbb{T}$ is called *dense point* if t is right-dense and left-dense simultaneously.

Table 1 shows the classification of points:

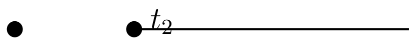
Table 1. Classification of points.

t right-scattered	$t < \delta(t)$ or $\delta(t) > t$
t left-scattered	$t > \rho(t)$ or $\rho(t) < t$
t isolated	$\rho(t) < t < \delta(t)$
t right-dense	$t = \delta(t)$
t left-dense	$t = \rho(t)$
t dense	$\rho(t) = t = \delta(t)$.

We can also represent these points by the following diagram:

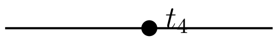


t_1 is right-scattered and left-dense;



t_2 is left-scattered and right-dense;

t_3 is left-scattered and right-scattered at the same time;



t_4 is left-dense and right-dense at the same time, while, t_3 is isolated point and t_4 is dense point.

A function that indicates the distance from one point to the next is defined as

Definition 1.2.7. *Graininess function*: The function $\mu : \mathbb{T} \to [0 \quad \infty)$ defined by $\mu(t) = \delta(t) - t$ is called the *graininess function*.

This function along with jump operators, plays an important role in explaining differences in behavior between the discrete and continuous parts of a time scale.

In the following example, the forward jump operator $\delta(t)$, the backward jump operator $\rho(t)$, right-scattered points, left-scattered points, isolated points, right-dense, left-dense, dense-points and the graininess function for the sets $\mathbb{R}$ and $\mathbb{Z}$ are illustrated.

Example 1.2.8. For $\mathbb{T} = \mathbb{R}$ and $t \in \mathbb{R}$,

$$\delta(t) = \inf\{s \in \mathbb{R} : s > t\} = \inf(t, \infty) = t,$$

$$\rho(t) = \sup\{s \in \mathbb{R} : s < t\} = \sup(-\infty, t) = t.$$

Clearly, $\delta(t) = t$ and $t < \sup \mathbb{T} = \sup \mathbb{R} = \infty$, therefore t is right-dense.

Also, $\rho(t) = t$ and $t > \inf \mathbb{T} = \inf \mathbb{R} = -\infty$, therefore t is left-dense. Hence, t is dense. Further,

$$\mu(t) = \delta(t) - t = t - t = 0.$$

Thus the graininess function of $\mathbb{R}$ is zero.

For $\mathbb{T} = \mathbb{Z}$ and $t \in \mathbb{Z}$,

$$\delta(t) = \inf\{s \in \mathbb{Z} : s > t\} = \inf\{t + 1, t + 2, t + 3, ...\} = t + 1$$

and for all $t \in \mathbb{Z}$,

$$\rho(t) = sup\{s \in \mathbb{Z} : s < t\} = sup\{..., t-3, t-2, t-1\} = t-1.$$

Since $\delta(t) = t+1 > t$, therefore t is right-scattered. Also, $\rho(t) = t-1 < t$, implies t is left-scattered. Hence t is an isolated point. Moreover,

$$\mu(t) = \delta(t) - t = t + 1 - t = 1.$$

Hence the graininess function of $\mathbb{Z}$ is 1.

The above observations may be presented in Table 2:

Table2. Jump operators and graininess functions for $\mathbb{R}$ and $\mathbb{Z}$.

time scale $\mathbb{T}$	$\mathbb{R}$	$\mathbb{Z}$
backward jump operator $[\rho(t)]$	t	$t-1$
forward jump operator $[\delta(t)]$	t	$t+1$
graininess function $[\mu(t)]$	0	1

1.3 Calculus on time scales

The following notations and definitions are useful for our work:

(**1**) For the function $f : \mathbb{T} \to \mathbb{R}$, the composition $f^\delta : \mathbb{T} \to \mathbb{R}$ is defined by

$$f^\delta(t) = (fo\delta)(t) = f(\delta(t));$$

(**2**) The set $\mathbb{T}^k$ is defined by

$$\mathbb{T}^k = \begin{cases} \mathbb{T} \setminus \{t_1\}, & \text{if } \mathbb{T} \text{ has a left-scattered maximum } t_1; \\ \mathbb{T}, & \text{otherwise;} \end{cases}$$

(**3**) The set $\mathbb{T}_k$ is defined by

$$\mathbb{T}_k = \begin{cases} \mathbb{T} \setminus \{t_2\}, & \text{if } \mathbb{T} \text{ has a right-scattered minimum } t_2; \\ \mathbb{T}, & \text{otherwise;} \end{cases}$$

(**4**) While $\mathbb{T}^* = \mathbb{T}^k \cap \mathbb{T}_k$;

(**5**) A neighborhood $\mathbb{N}$ of a point $t \in \mathbb{T}$ (i.e., $\mathbb{N} = (t-\delta, t+\delta) \cap \mathbb{T}$ for some $\delta > 0$) is the set of points in $\mathbb{T}$, indicates a small distance from t.

1.3.1 Δ-Differentiation

Definition 1.3.1. Suppose $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^k$. Then the delta derivative, $f^{\Delta}(t)$ of f at t is defined to be the number (if exists) with the property that for any given $\varepsilon > 0$ there is a neighborhood U of t such that

$$\mid [f(\delta(t)) - f(s)] - f^{\Delta}(t)[\delta(t) - s] \mid \leq \epsilon \mid \delta(t) - s \mid,$$

for all $s \in U$.

The function f is said to be delta (or Hilger) differentiable (or simply differentiable) on $\mathbb{T}^k$ if $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}^k$.

The following simple examples illustrate the definition of delta derivatives.

Example 1.3.2.

(*i*) If $f : \mathbb{T} \to \mathbb{R}$ is defined by $f(t) = \alpha$ for all $t \in \mathbb{T}$, where $\alpha \in \mathbb{R}$ is constant, then $f^{\Delta}(t) = 0$. Because for any $\epsilon > 0$,

$$|f(\delta(t)) - f(s) - 0(\delta(t) - s)| = |\alpha - \alpha| = 0 \leq \epsilon|\delta(t) - s|,$$

holds for all $s \in \mathbb{T}$.

(ii) If $f : \mathbb{T} \to \mathbb{R}$ is defined by $f(t) = t$ for all $t \in \mathbb{T}$, then $f^{\Delta}(t) = 1$. Again, it is because of the relation

$$|f(\delta(t)) - f(s) - 1.(\delta(t) - s)| = |\delta(t) - s - (\delta(t) - s)| = 0 \leq \epsilon|\delta(t) - s|,$$

holds for all $s \in \mathbb{T}$ and for any $\epsilon > 0$.

Some basic relationships of the delta derivative are summarized in the following theorem.

Theorem 1.3.3. *Suppose $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}$. Then:*

(**1**) *If f is differentiable at t, then f is continuous at t.*

(**2**) *If f is continuous at t and t is right-scattered, then f is differentiable at t and*

$$f^{\Delta}(t) = \frac{f(\delta(t)) - f(t)}{\mu(t)}.$$

(**3**) *If t is right-dense, then f is differentiable at t if and only if the limit*

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s},$$

exists as a finite number. In this case

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

(**4**) *If f is differentiable at t, then*

$$f(\delta(t)) = f(t) + \mu(t) f^{\Delta}(t).$$

Proof.

(1) Assume that f is differentiable at t. Choose $\epsilon \in (0, 1)$ and define

$$\epsilon^* = \frac{\epsilon}{1 + |f^{\Delta}(t)| + 2\mu(t)}.$$

Then $\epsilon^* \in (0,1)$ and by definition of delta derivative, there exists a neighborhood U of t such that

$$|f(\delta(t)) - f(s) - (\delta(t) - s)f^{\Delta}(t)| \leq \epsilon^*|\delta(t) - s|, \tag{1.3.1}$$

for all $s \in U$. Now, for all $s \in U \cap (t - \epsilon^*, t + \epsilon^*)$, using (1.3.1), it follows that

$$\begin{aligned} |f(t) - f(s)| &= |\{f(\delta(t)) - f(s) - f^{\Delta}(t)(\delta(t) - s)\} - \\ &\quad \{f(\delta(t)) - f(t) - \mu(t)f^{\Delta}(t)\} + (t - s)f^{\Delta}(t)| \\ &\leq \epsilon^*|\delta(t) - s| + \epsilon^*\mu(t) + |t - s||f^{\Delta}(t)| \\ &\leq \epsilon^*|\delta(t) - t + t - s| + \epsilon^*\mu(t) + |t - s||f^{\Delta}(t)| \\ &\leq \epsilon^*[\mu(t) + |t - s| + \mu(t) + |f^{\Delta}(t)|] \\ &< \epsilon^*[1 + |f^{\Delta}(t)| + 2\mu(t)] = \epsilon, \end{aligned}$$

Consequently, f is continuous at t.

(2) Assume that f is continuous at t and t is right-scattered. Then by continuity

$$\lim_{s \to t} \frac{f(\delta(t)) - f(s)}{\delta(t) - s} = \frac{f(\delta(t)) - f(t)}{\delta(t) - t} = \frac{f(\delta(t)) - f(t)}{\mu(t)}.$$

Hence for given $\epsilon > 0$, there is a neighborhood U of t such that

$$\left|\frac{f(\delta(t)) - f(s)}{\delta(t) - s} - \frac{f(\delta(t)) - f(t)}{\mu(t)}\right| \leq \epsilon \text{ for all } s \in U,$$

which implies that

$$|[f(\delta(t)) - f(s)] - \frac{f(\delta(t)) - f(t)}{\mu(t)}[\delta(t) - s]| \leq \epsilon|\delta(t) - s| \text{ for all } s \in U.$$

Hence we get the desired result

$$f^{\Delta}(t) = \frac{f(\delta(t)) - f(t)}{\mu(t)}.$$

(3) Assume f is differentiable at t and t is right-dense. By the definition of differentiability, there is a neighborhood U of t such that

$$|[f(\delta(t)) - f(s)] - f^{\Delta}(t)[\delta(t) - s]| \leq \epsilon|\delta(t) - s| \text{ for all } s \in U,$$

which implies that

$$|f(t) - f(s) - f^{\Delta}(t)[t - s]| \leq \epsilon|t - s| \text{ for all } s \in U, \text{ as } \delta(t) = t.$$

It follows that

$$|\frac{f(t) - f(s)}{t - s} - f^{\Delta}(t)| \leq \epsilon \text{ for all } s \in U, \text{ and } s \neq t.$$

Hence,

$$f^{\Delta}(t) = lim_{s \to t}\frac{f(t) - f(s)}{t - s},$$

as required.

Conversely, suppose that t is right-dense and $lim_{s \to t}\frac{f(t) - f(s)}{t - s}$ exists as a finite number. In this case $f^{\Delta}(t) = lim_{s \to t}\frac{f(t) - f(s)}{t - s}$ and for all $s \in U$ with $s \neq t$ there exists $\epsilon > 0$ such that

$$|\frac{f(t) - f(s)}{t - s} - f^{\Delta}(t)| \leq \epsilon,$$

it implies that

$$|f(t) - f(s) - f^{\Delta}(t)(t - s)| \leq \epsilon|t - s|.$$

Since $\delta(t) = t$, it follows that

$$|f(\delta(t)) - f(s) - f^{\Delta}(t)(\delta(t) - s)| \leq \epsilon|\delta(t) - s| \text{ for all } s \in U.$$

Hence f is differentiable at t.

(4) If t is right-dense, then $\delta(t) = t$ and $\mu(t) = 0$. Hence

$$f(\delta(t)) = f(t) = f(t) + \mu(t) f^{\Delta}(t).$$

If t is right-scattered, that is, $\delta(t) > t$, then by (2) we have

$$f(\delta(t)) = f(t) + \mu(t) \frac{f(\delta(t)) - f(t)}{\mu(t)} = f(t) + \mu(t) f^{\Delta}(t).$$

□

The delta derivative gives the standard definitions of difference and differentiation for $\mathbb{T} = \mathbb{Z}$ and $\mathbb{T} = \mathbb{R}$ as shown in the following example.

Example 1.3.4.

(*i*) $\mathbb{T} = \mathbb{Z}$: In this case, $\mu = 1$ and for every $t \in \mathbb{Z}$,

$$f^{\Delta}(t) = \frac{f(\delta(t)) - f(t)}{\mu(t)} = \frac{f(t+1) - f(t)}{1} = \Delta f(t) = f(t+1) - f(t).$$

where Δ is the forward difference operator.

(*ii*) $\mathbb{T} = \mathbb{R}$: In this case $\mu = 0$ and for every $t \in \mathbb{R}$,

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s} = f'(t),$$

which is the usual derivative.

Theorem 1.3.5. *Suppose f, $g : \mathbb{T} \to \mathbb{R}$ are differentiable at $t \in \mathbb{T}$. Then:*

(1) *The sum $f + g : \mathbb{T} \to \mathbb{R}$ is differentiable at t and*

$$(f + g)^{\Delta}(t) = f^{\Delta}(t) + g^{\Delta}(t).$$

(2) *The product $fg : \mathbb{T} \to \mathbb{R}$ is differentiable at t and*

$$\begin{aligned}(fg)^{\Delta}(t) &= f^{\Delta}(t) g(t) + f(\delta(t)) g^{\Delta}(t) \\ &= f(t) g^{\Delta}(t) + f^{\Delta}(t) g(\delta(t)).\end{aligned}$$

(**3**) *For any constant* α, *the product* αf *is differentiable at* t *and*

$$(\alpha f)^{\Delta}(t) = \alpha (f)^{\Delta}(t).$$

(**4**) *If* $f(t)f(\delta(t)) \neq 0$, *then* $\frac{1}{f}$ *is differentiable at* t *and*

$$(\frac{1}{f})^{\Delta}(t) = -\frac{f^{\Delta}(t)}{f(t)f(\delta(t))}.$$

(**5**) *The quotient* $\frac{f}{g} : \mathbb{T} \to \mathbb{R}$ *is differentiable, if* $g(t)g(\delta(t)) \neq 0$ *and*

$$(\frac{f}{g})^{\Delta}(t) = \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g(\delta(t))}.$$

Proof.

(1) Assume that f and g are Δ-differentiable at $t \in \mathbb{T}$. Then, for given $\epsilon > 0$, there exist neighborhoods U_1 and U_2 of t such that

$$|f(\delta(t)) - f(s) - f^{\Delta}(t)(\delta(t) - s)| \leq \frac{\epsilon}{2}|\delta(t) - s| \text{ for all } s \in U_1,$$

and

$$|g(\delta(t)) - g(s) - g^{\Delta}(t)(\delta(t) - s)| \leq \frac{\epsilon}{2}|\delta(t) - s| \text{ for all } s \in U_2.$$

Then,

$$\begin{aligned}
&|(f+g)(\delta(t)) - (f+g)(s) - [f^{\Delta}(t) + g^{\Delta}(t)](\delta(t) - s)| \\
&\qquad = |f(\delta(t)) - f(s) - f^{\Delta}(t)(\delta(t) - s) + g(\delta(t)) - g(s) - g^{\Delta}(t)(\delta(t) - s)| \\
&\qquad \leq |f(\delta(t)) - f(s) - f^{\Delta}(t)(\delta(t) - s)| + |g(\delta(t)) - g(s) - g^{\Delta}(t)(\delta(t) - s)| \\
&\qquad \leq \frac{\epsilon}{2}|\delta(t) - s| + \frac{\epsilon}{2}|\delta(t) - s| = \epsilon|\delta(t) - s|,
\end{aligned}$$

for all $s \in U = U_1 \cap U_2$. Hence, $f + g$ is differentiable at t and

$$(f+g)^{\Delta} = f^{\Delta} + g^{\Delta}.$$

(2) Choose $\epsilon \in (0,1)$ and define

$$\epsilon^* = \frac{\epsilon}{1 + |f(t)| + |g(\delta(t))| + |g^\Delta(t)|}.$$

Then $\epsilon^* \in (0,1)$ and hence there exist neighborhoods U_1, U_2 and U_3 of t such that

$$|f(\delta(t)) - f(s) - f^\Delta(t)(\delta(t) - s)| \leq \epsilon^* |\delta(t) - s| \text{ for all } s \in U_1,$$

$$|g(\delta(t)) - g(s) - g^\Delta(t)(\delta(t) - s)| \leq \epsilon^* |\delta(t) - s| \text{ for all } s \in U_2,$$

and

$$|f(t) - f(s)| \leq \epsilon^* \text{ for all } s \in U_3.$$

Then,

$$\begin{aligned}
&|(fg)(\delta(t)) - (fg)(s) - [f^\Delta(t)g(\delta(t)) + f(t)g^\Delta(t)](\delta(t) - s)| \\
&\quad = |[f(\delta(t)) - f(s) - f^\Delta(t)(\delta(t) - s)]g(\delta(t)) \\
&\quad + [g(\delta(t)) - g(s) - g^\Delta(t)(\delta(t) - s)]f(t) \\
&\quad + [g(\delta(t)) - g(s) - g^\Delta(t)(\delta(t) - s)][f(s) - f(t)] + (\delta(t) - s)g^\Delta(t)[f(s) - f(t)]| \\
&\quad \leq \epsilon^*|\delta(t) - s||g(\delta(t))| + \epsilon^*|\delta(t) - s||f(t)| + \epsilon^*\epsilon^*|\delta(t) - s| + \epsilon^*|\delta(t) - s||g^\Delta(t)| \\
&\quad = \epsilon^*|\delta(t) - s|[|g(\delta(t))| + |f(t)| + \epsilon^* + |g^\Delta(t)|] \\
&\quad \leq \epsilon^*|\delta(t) - s|[1 + |f(t)| + |g(\delta(t))| + |g^\Delta(t)|] = \epsilon|\delta(t) - s|,
\end{aligned}$$

for all $s \in U = U_1 \cap U_2 \cap U_3$. Thus the product fg is differentiable and $(fg)^\Delta = f^\Delta g^\delta + fg^\Delta$ at t. The other part can be proved similarly. Notice that

$$(fgh)^\Delta = f^\Delta gh + f^\delta g^\Delta h + f^\delta g^\delta h^\Delta.$$

(3) Define $\epsilon^* = \epsilon/|\alpha|$, $|\alpha| \neq 0$, $\epsilon > 0$, then by the definition of differentiability of f at t, it follows that

$$|[f(\delta(t)) - f(s)] - f^\Delta(t)(\delta(t) - s)| \leq \epsilon^* |\delta(t) - s| \text{ for any } s \in U,$$

where U is a neighborhood of t. Multiplying both sides by $|\alpha|$, we have

$$|[\alpha f(\delta(t)) - \alpha f(s)] - \alpha f^{\Delta}(t)(\delta(t) - s)| \leq |\alpha|\epsilon^{*}|\delta(t) - s| \text{ for any } s \in U,$$

which implies that

$$|[(\alpha f)(\delta(t)) - (\alpha f)(s)] - (\alpha f^{\Delta}(t))(\delta(t) - s)| \leq \epsilon|\delta(t) - s| \text{ for any } s \in U.$$

Hence (αf) is delta differentiable at t and

$$(\alpha f)^{\Delta}(t) = \alpha f^{\Delta}(t).$$

(4) For any $\epsilon > 0$, define $\epsilon^{*} = \epsilon|f(\delta(t))f(s)|$. Since f is differentiable at t, hence,

$$|[f(\delta(t)) - f(s)] - f^{\Delta}(t)(\delta(t) - s)| \leq \epsilon^{*}|\delta(t) - s| \text{ for any } s \in U.$$

Assume that $f(t)f(\delta(t)) \neq 0$. Dividing both sides by $f(t)f(\delta(t))$, we have

$$|\frac{1}{f(s)} - \frac{1}{f(\delta(t))} + f^{\Delta}(t)\frac{\delta(t) - s}{f(\delta(t))f(s)}| \leq \frac{\epsilon^{*}}{|f(\delta(t))f(s)|}|\delta(t) - s|,$$

which implies that

$$|\frac{1}{f(\delta(t))} - \frac{1}{f(s)} + f^{\Delta}(t)\frac{\delta(t) - s}{f(\delta(t))f(s)}| \leq \epsilon|\delta(t) - s|,$$

for any $s \in U$. Hence the delta derivative of $\frac{1}{f}$ exists and

$$(\frac{1}{f})^{\Delta}(t) = -\frac{f^{\Delta}(t)}{f(t)f(\delta(t))}.$$

Proof of the quotient formula (5) can be obtained by using (2) and (4). □

The following theorem gives general rule for the power formula of Δ differentiation.

Theorem 1.3.6. *Let α be constant and $m \in \mathbb{N}$.*

(1) *If* $f(t) = (t-\alpha)^m$, *then*

$$f^{\Delta}(t) = \sum_{n=0}^{m-1} (\delta(t)-\alpha)^n (t-\alpha)^{m-1-n}. \tag{1.3.2}$$

(2) *If* $g(t) = \dfrac{1}{(t-\alpha)^m}$, *then*

$$g^{\Delta}(t) = -\sum_{n=0}^{m-1} \frac{1}{(\delta(t)-\alpha)^{m-n}(t-\alpha)^{n+1}}, \ \textit{provided } (t-\alpha)(\delta(t)-\alpha) \neq 0. \tag{1.3.3}$$

Proof.

(1) Let $h_n(t) = (t-\alpha)^n$, by the product rule,

$$\begin{aligned} h_n^{\Delta}(t) = [(t-\alpha)(t-\alpha)^{n-1}]^{\Delta} &= 1.(t-\alpha)^{n-1} + (\delta(t)-\alpha)[(t-\alpha)^{n-1}]^{\Delta} \\ &= (t-\alpha)^{n-1} + (\delta(t)-\alpha)h_{n-1}^{\Delta}(t). \end{aligned}$$

Similarly,

$$\begin{aligned} h_{n-1}^{\Delta}(t) = [(t-\alpha)(t-\alpha)^{n-2}]^{\Delta} &= 1.(t-\alpha)^{n-2} + (\delta(t)-\alpha)[(t-\alpha)^{n-2}]^{\Delta} \\ &= 1.(t-\alpha)^{n-2} + (\delta(t)-\alpha)h_{n-2}^{\Delta}(t). \end{aligned}$$

Therefore,

$$\begin{aligned} h_n^{\Delta}(t) &= (t-\alpha)^{n-1} + (\delta(t)-\alpha)[(t-\alpha)^{n-2} + (\delta(t)-\alpha)h_{n-2}^{\Delta}(t)] \\ &= (t-\alpha)^{n-1} + (\delta(t)-\alpha)(t-\alpha)^{n-2} + (\delta(t)-\alpha)^2 h_{n-2}^{\Delta}(t) \\ &= (t-\alpha)^{n-1} + (\delta(t)-\alpha)(t-\alpha)^{n-2} + (\delta(t)-\alpha)^2(t-\alpha)^{n-3} \\ &+ (\delta(t)-\alpha)^3(t-\alpha)^{n-4} + ... + (\delta(t)-\alpha)^{n-1}h_{n-(n-1)}^{\Delta}(t) \\ &= (t-\alpha)^{n-1} + (\delta(t)-\alpha)(t-\alpha)^{n-2} + (\delta(t)-\alpha)^2(t-\alpha)^{n-3} \\ &+ (\delta(t)-\alpha)^3(t-\alpha)^{n-4} + ... + (\delta(t)-\alpha)^{n-1}.1, \end{aligned}$$

since $h_1^{\Delta}(t) = (t-\alpha)^{\Delta} = 1$. Hence

$$f^{\Delta}(t) = h_m^{\Delta}(t) = \sum_{n=0}^{m-1} (\delta(t)-\alpha)^n (t-\alpha)^{m-1-n}.$$

Part (2) can be proved in a similar way. □

As a verification of the above theorem, take $f(t) = t^2$, $f(t) = \frac{1}{t}$ and $f(t) = \frac{1}{t^2}$ in the following example.

Example 1.3.7. (i) For $f(t) = t^2$, using formula (1.3.2) with $m = 2$ and $\alpha = 0$,

$$\begin{aligned} f^{\Delta}(t) &= \sum_{n=0}^{1}(\delta(t) - 0)^n (t - 0)^{2-1-n} = \sum_{n=0}^{1}[\delta(t)]^n t^{1-n} \\ &= [\delta(t)]^0 t^{1-0} + [\delta(t)]^1 t^{1-1} f^{\Delta}(t) = t + \delta(t). \end{aligned}$$

(ii) For $f(t) = \frac{1}{t}$, using (1.3.3) with $m = 1$ and $\alpha = 0$,

$$\begin{aligned} f^{\Delta}(t) = -\sum_{n=0}^{m-1} \frac{1}{(\delta(t) - \alpha)^{m-n}(t - \alpha)^{n+1}} &= -\frac{1}{(\delta(t) - 0)^{1-0}(t - 0)^{0+1}} \\ &= -\frac{1}{\delta(t)(t)} = -\frac{1}{t\delta(t)}. \end{aligned}$$

(iii) For $f(t) = \frac{1}{t^2}$, we have,

$$\begin{aligned} f^{\Delta}(t) &= -\sum_{n=0}^{1} \frac{1}{(\delta(t) - 0)^{2-n}(t - 0)^{n+1}} \\ &= \frac{1}{(\delta(t) - 0)^{2-0}(t - 0)^{0+1}} - \frac{1}{(\delta(t) - 0)^{2-1}(t - 0)^{1+1}} \\ &= -\frac{1}{t[\delta(t)]^2} - \frac{1}{t^2\delta(t)} = -\frac{t + \delta(t)}{t^2\delta^2(t)}, \text{ where } t^2\delta^2(t) \neq 0. \end{aligned}$$

1.3.2 ∇-Differentiation and relation of ∇ to Δ.

Definition 1.3.8. Suppose $f : \mathbb{T} \to \mathbb{R}$ be a function and let $t \in \mathbb{T}_k$. Then the nabla derivative of f at t is defined to be the number $f^{\nabla}(t)$ (if exists) with the property that for each $\varepsilon > 0$ there is a neighborhood $U = (t-\delta, t+\delta) \cap \mathbb{T}$ (for some $\delta > 0$) of t such that

$$|[f(\rho(t)) - f(s)] - f^{\nabla}(t)(\rho(t) - s)| \leq \epsilon|\rho(t) - s| \text{ for all } s \in U.$$

Then f is said to be nabla differentiable on $\mathbb{T}_k$ if $f^{\nabla}(t)$ exists for all $t \in \mathbb{T}_k$.

Notice that in the case $\mathbb{T} = \mathbb{R}$, $f^{\nabla}(t) = f'(t)$ and for $\mathbb{T} = \mathbb{Z}$,

$$f^{\nabla}(t) = \nabla f(t) = f(t) - f(t-1),$$

the usual backward operator.

Definition 1.3.9. The *backward graininess function* $\nu : \mathbb{T}_k \to [0, \infty)$ is defined by

$$\nu(t) = t - \rho(t)$$

and the function $f^{\rho} : \mathbb{T}_k \to \mathbb{R}$ is defined by $f^{\rho}(t) = f(\rho(t))$ for all $t \in \mathbb{T}_k$.

Definition 1.3.10. A function $F : \mathbb{T} \to \mathbb{R}$ is called a ∇-antiderivative of $f : \mathbb{T} \to \mathbb{R}$ if $F^{\nabla}(t) = f(t)$ holds for all $t \in \mathbb{T}_k$.

Definition 1.3.11. A function $f : \mathbb{T} \to \mathbb{R}$ is said to be *left-dense continuous* (or *ld-continuous*) if:

(**1**) it is continuous at left-dense points; and

(**2**) its right-sided limits exist (finite) at right-dense points.

Definition 1.3.12. A function $p : \mathbb{T} \to \mathbb{R}$ is called *ν-regressive* if $1 + \nu(t)p(t) \neq 0$ for all $t \in \mathbb{T}_k$.

The following are some basic properties of the nabla derivative. The proofs can be found in [2].

Theorem 1.3.13. (**1**) *If f is ∇-differentiable at t, then f is continuous at t.*

(**2**) *If f is continuous at t and t is left-scattered, then f is ∇ -differentiable at t with*

$$f^{\nabla}(t) = \frac{f(\rho(t)) - f(t)}{\rho(t) - t}.$$

(**3**) *If t is left-dense, then f is ∇-differentiable at t if and only if*

$$\lim_{s\to t}\frac{f(t)-f(s)}{t-s},$$

exists as a finite number. In this case

$$f^{\nabla}(t)=\lim_{s\to t}\frac{f(t)-f(s)}{t-s}.$$

(**4**) *If f is ∇-differentiable at t, then*

$$f(\rho(t))=f(t)+[\rho(t)-t]f^{\nabla}(t).$$

Theorem 1.3.14. *Suppose $f,\ g:\mathbb{T}\to\mathbb{R}$ are ∇-differentiable at $t\in\mathbb{T}$. Then:*

(**1**) *The sum $f+g:\mathbb{T}\to\mathbb{R}$ is ∇-differentiable at t and*

$$(f+g)^{\nabla}(t)=f^{\nabla}(t)+g^{\nabla}(t).$$

(**2**) *The product $fg:\mathbb{T}\to\mathbb{R}$ is ∇-differentiable at t and*

$$(fg)^{\nabla}(t)=f^{\nabla}(t)g(t)+f(\rho(t))g^{\nabla}(t)=f(t)g^{\nabla}(t)+f^{\nabla}(t)g(\rho(t)).$$

(**3**) *The quotient $\dfrac{f}{g}:\mathbb{T}\to\mathbb{R}$ is ∇-differentiable if $g(t)g(\rho(t))\neq 0$ and*

$$(\frac{f}{g})^{\nabla}(t)=\frac{f^{\nabla}(t)g(t)-f(t)g^{\nabla}(t)}{g(t)g(\rho(t))}.$$

(**4**) *For any constant $\alpha, \alpha f:\mathbb{T}\to\mathbb{R}$ is ∇-differentiable at t and*

$$(\alpha f)^{\nabla}(t)=\alpha(f)^{\nabla}(t).$$

(**5**) *If $f(t)f(\rho(t))\neq 0$, then $\dfrac{1}{f}$ is ∇-differentiable at t and*

$$(\frac{1}{f})^{\nabla}(t)=-\frac{f^{\nabla}(t)}{f(t)f(\rho(t))}.$$

The following mean value result on time scales will be used in the proof of the next theorem.

Theorem 1.3.15. *Let f be a continuous function on $[\alpha, \beta] \subset \mathbb{T}$ that is Δ-differentiable on $[\alpha, \beta)$. Then there exist $\xi, \tau \in [\alpha, \beta)$ such that*

$$f^{\Delta}(\tau) \leq \frac{f(\beta) - f(\alpha)}{\beta - \alpha} \leq f^{\Delta}(\xi). \tag{1.3.4}$$

For the proof see [8, 9].

The following theorems establish the relationship between Δ and ∇ derivatives of a function on an arbitrary time scale.

Theorem 1.3.16. *If $f : \mathbb{T} \to \mathbb{C}$ is Δ-differentiable on $\mathbb{T}^k$ and if f^{Δ} is continuous on $\mathbb{T}^k$, then f is $\bigtriangledown$-differentiable on $\mathbb{T}_k$ and $f^{\bigtriangledown}(t) = f^{\Delta}(\rho(t))$ for all $t \in \mathbb{T}$.*

Proof. Let $t \in \mathbb{T}$ be fixed. The proof is split into several cases:

Case 1, t is left scattered: Since f is Δ- differentiable, it is continuous. Therefore, f is ∇ -differentiable at t and

$$f^{\bigtriangledown}(t) = \frac{f(\rho(t)) - f(t)}{\rho(t) - t}. \tag{1.3.5}$$

As $\rho(t)$ is right-scattered, hence

$$f^{\Delta}(\rho(t)) = \frac{f(\delta(\rho(t))) - f(\rho(t))}{\delta(\rho(t)) - \rho(t)} = \frac{f(t) - f(\rho(t))}{t - \rho(t)} = \frac{f(\rho(t)) - f(t)}{\rho(t) - t}. \tag{1.3.6}$$

From (1.3.5) and (1.3.6), it follows that

$$f^{\Delta}(\rho(t)) = f^{\bigtriangledown}(t).$$

Case 2, t is dense: Since t is left-dense, therefore f is ∇-differentiable at t and

$$f^{\bigtriangledown}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}. \tag{1.3.7}$$

Also since f is right-dense, therefore f is Δ-differentiable at t and

$$f^{\Delta}(t) = \lim_{s\to t} \frac{f(t)-f(s)}{t-s}. \tag{1.3.8}$$

From (1.3.7) and (1.3.8)

$$f^{\Delta}(t) = f^{\nabla}(t).$$

Case 3, t is left-dense and right-scattered: Since f is a continuous function on $[s,t] \subset \mathbb{T}$, that is Δ-differentiable on $[s,t[$. Therefore by Theorem (1.3.15) there exist $\xi, \tau \in [s,t[$ such that

$$f^{\Delta}(\tau) \leq \frac{f(t)-f(s)}{t-s} \leq f^{\Delta}(\xi),$$

where ξ, τ are between s and t, since $\xi \to t$, $\tau \to t$ as $s \to t$ and f^{Δ} is continuous. Hence by Theorem (1.3.3),

$$f^{\Delta}(t) = \lim_{s\to t} \frac{f(t)-f(s)}{t-s}. \tag{1.3.9}$$

On the other hand since t is left-dense, therefore

$$f^{\nabla}(t) = \lim_{t\to s} \frac{f(t)-f(s)}{t-s}. \tag{1.3.10}$$

From (1.3.9) and (1.3.10)

$$f^{\Delta}(t) = f^{\nabla}(t).$$

□

Theorem 1.3.17. *If $f : \mathbb{T} \to \mathbb{C}$ is ∇-differentiable on $\mathbb{T}_k$ and if f^{∇} is continuous on $\mathbb{T}_k$ then f is Δ-differentiable on $\mathbb{T}^k$ and $f^{\Delta}(t) = f^{\nabla}(\delta(t))$.*

1.4 Integration

Definition 1.4.1. (*Regulated function*): A function $f : \mathbb{T} \to \mathbb{R}$ is called *regulated* if:

(**1**) its right-sided limits exist (finite) at all right-dense points; and

(**2**) its left-sided limits exist (finite) at all left-dense points.

Definition 1.4.2. (*rd-continuous function*): A function $f : \mathbb{T} \to \mathbb{R}$ is said to be *rd-continuous* if:

(**1**) it is continuous at right-dense points; and

(**2**) its left-sided limits exist (finite) at left-dense points.

Definition 1.4.3. A function $F : \mathbb{T} \to \mathbb{R}$ is called an antiderivative of $f : \mathbb{T} \to \mathbb{R}$ if $F^{\Delta}(t) = f(t)$ holds for all $t \in \mathbb{T}^k$. Let $f : \mathbb{T} \to \mathbb{R}$ be a regulated function. The indefinite integral of f is defined by

$$\int f(t)\Delta t = F(t) + C,$$

where C is an arbitrary constant and F is a pre-antiderivative of f. The definite integral is defined by

$$\int_a^b f(t)\Delta t = F(s) - F(a).$$

Note that $a, b \in \mathbb{T}$ and $a \leq b$, the time scale integral

$$\int_a^b f(t)\Delta t = \int_a^b f(t)dt, \text{ if } \mathbb{T} = \mathbb{R},$$

is the ordinary integral and

$$\int_a^b f(t)\Delta t = \sum_{s=a}^{b-1} f(s), \text{ if } \mathbb{T} = \mathbb{Z}.$$

The following theorem summarizes some basic results of the delta integrals. Let $C_{rd} = \{f : \mathbb{T} \to \mathbb{R} : f \text{ is rd-continuous}\}$.

Theorem 1.4.4. *If $a, b, c \in \mathbb{T}$, $\alpha \in \mathbb{R}$, and $f, g \in C_{rd}$, then the delta integrals have the following properties:*

(1) $\int_a^b [f(t)+g(t)]\Delta t = \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t;$

(2) $\int_a^b (\alpha f)(t)\Delta t = \alpha \int_a^b f(t)\Delta t;$

(3) $\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t;$

(4) $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t;$

(5) $\int_a^a f(t)\Delta t = 0;$

(6) $\int_a^b f(\delta(t))g^{\Delta}(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)g(t)\Delta t;$

(7) $\int_a^b f(t)g^{\Delta}(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)g(\delta(t))\Delta t;$

(8) *If* $f(t) \geq 0$ *for all* $a \leq t < b$, *then* $\int_a^b f(t)\Delta t \geq 0$;

(9) *If* $|f(t)| \leq g(t)$ *on* $[a,b)$, *then*

$$|\int_a^b f(t)\Delta t| \leq \int_a^b g(t)\Delta t.$$

Proof. Since f and g are rd-continuous, therefore they posses anti-delta derivatives and hence the definite integral properties can easily be proved by using the properties of delta-derivatives. □

Example 1.4.5. Consider two basic integrals of $f(s) = 1$ and $f(s) = s$ on an arbitrary time scale $\mathbb{T}$.

(i) Let $a, t \in \mathbb{T}$. Then, since

$$(s)^{\Delta} = 1,$$

therefore

$$\int_a^t 1\Delta s = t - a,$$

for any $t \in \mathbb{T}$.

(ii) Let $a, t \in \mathbb{T}$. Then, since

$$(s^2)^\Delta = s + \delta(s),$$

therefore

$$\int_a^t (s^2)^\Delta \Delta s = \int_a^t s\Delta s + \int_a^t \delta(s)\Delta s,$$

it implies that,

$$\int_a^t s\Delta s = \int_a^t (s^2)^\Delta \Delta s - \int_a^t \delta(s)\Delta s.$$

Hence,

$$\int_a^t s\Delta s = t^2 - a^2 - \int_a^t \delta(s)\Delta s.$$

Example 1.4.6. To find the integral $\int_a^t s\Delta s$, when $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$ with $a \in \mathbb{N}$ and $t \in \mathbb{T}$, we have

(i) If $\mathbb{T} = \mathbb{R}$, then for any $t \in \mathbb{R}$, $\delta(t) = t$ and from the above example

$$\begin{aligned}\int_a^t s\Delta s &= t^2 - a^2 - \int_a^t s ds \\ &= t^2 - a^2 - \frac{1}{2}(t^2 - a^2) \\ &= \frac{1}{2}(t^2 - a^2).\end{aligned}$$

(ii) If $\mathbb{T} = \mathbb{Z}$, then for any $t \in \mathbb{Z}$, $\delta(t) = t + 1$ and from the example

$$\begin{aligned}
\int_a^t s\Delta s &= t^2 - a^2 - \int_a^t (s+1)\Delta s \\
&= t^2 - a^2 - \sum_{s=a}^{t-1}(s+1) \\
&= t^2 - a^2 - \sum_{s=0}^{t-1}(s+1) + \sum_{s=0}^{a-1}(s+1) \\
&= \frac{1}{2}(t^2 - t - a^2 + a).
\end{aligned}$$

The following theorem and lemma will be used in our work.

Theorem 1.4.7.

$$(\int_a^t f(t,s)\Delta s)^{\Delta} = f(\delta(t), t) + \int_a^t f^{\Delta}(t,s)\Delta s.$$

Proof. Let $g(t) = \int_a^t f(t,s)\Delta s$, $t \in T^k$.
Case 1: If t is right-scattered, then

$$\begin{aligned}
g^{\Delta}(t) &= \frac{g(\delta(t)) - g(t)}{\delta(t) - t} \\
&= \frac{1}{\delta(t) - t}[\int_a^{\delta(t)} f(\delta(t), s)\Delta s - \int_a^t f(t,s)\Delta s] \\
&= \frac{1}{\delta(t) - t}[\int_a^t f(\delta(t), s)\Delta s + \int_t^{\delta(t)} f(\delta(t), s)\Delta s - \int_a^t f(t,s)\Delta s] \\
&= \int_a^t \frac{f(\delta(t), s) - f(t,s)}{\delta(t) - t}\Delta s + \frac{1}{\delta(t) - t}\int_t^{\delta(t)} f(\delta(t), s)\Delta s \\
&= \int_a^t f^{\Delta}(t,s)\Delta s + \frac{f(\delta(t), t)}{\delta(t) - t}\int_t^{\delta(t)} \Delta s \\
&= \int_a^t f^{\Delta}(t,s)\Delta s + \frac{f(\delta(t), t)}{\delta(t) - t}(\delta(t) - t) \\
&= \int_a^t f^{\Delta}(t,s)\Delta s + f(\delta(t), t).
\end{aligned}$$

Case 2: If t is right-dense, then

$$\begin{aligned} g^{\Delta}(t) &= \lim_{r\to t}\frac{g(t)-g(r)}{t-r} = \lim_{r\to t}\frac{1}{t-r}[\int_a^t f(t,s)\Delta s - \int_a^r f(r,s)\Delta s] \\ &= \lim_{r\to t}\frac{1}{t-r}[\int_a^t f(t,s)\Delta s - \int_a^t f(r,s)\Delta s - \int_t^r f(r,s)\Delta s] \qquad (1.4.1) \\ &= \lim_{r\to t}\int_a^t \frac{f(t,s)-f(r,s)}{t-r}\Delta s + lim_{r\to t}\frac{1}{t-r}\int_r^t f(r,s)\Delta s. \end{aligned}$$

Now we have to show that

$$\lim_{r\to t}\frac{1}{t-r}\int_r^t f(r,s)\Delta s = f(t,t), \qquad (1.4.2)$$

$$\lim_{r\to t}\int_a^t \frac{f(t,s)-f(r,s)}{t-s}\Delta s = \int_a^t f^{\Delta}(t,s)\Delta s. \qquad (1.4.3)$$

To show (1.4.2) since f is continuous, it will be uniformly continuous on any compact subset of $\mathbb{T}\times\mathbb{T}$. Therefore for arbitrary $\epsilon > 0$, there exists a neighborhood U of t such that

$$|f(r,s)-f(t,t)| < \epsilon \text{ for all } r,s\in U.$$

Consequently,

$$\begin{aligned} |\frac{1}{t-r}\int_r^t f(r,s)\Delta s - f(t,t)| &= |\frac{1}{t-r}\int_r^t [f(r,s)-f(t,t)]\Delta s| \\ &\le \frac{1}{|t-r|}\int_r^t |f(r,s)-f(t,t)|\Delta s \\ &\le \frac{\epsilon}{|t-r|}|\int_r^t \Delta s| = \epsilon, \end{aligned}$$

which implies that

$$\lim_{r\to t}\frac{1}{t-r}\int_r^t f(r,s)\Delta s = f(t,t).$$

By the mean value theorem,

$$f^{\Delta}(\tau,s) \le \frac{f(t,s)-f(r,s)}{t-r} \le f^{\Delta}(\xi,s),$$

where τ, ξ are between r, t and $\xi \to t$, $\tau \to t$ as $r \to t$. On the other hand, since f^{Δ} is uniformly continuous on any compact subset of $\mathbb{T} \times \mathbb{T}$. Therefore

$$f^{\Delta}(t,s) = \lim_{r \to t} \frac{f(t,s) - f(r,s)}{t-r}.$$

Hence,

$$g^{\Delta}(t) = \int_a^t f^{\Delta}(t,s)\Delta s + f(t,t).$$

□

Lemma 1.4.8. *If a function $f : \mathbb{T} \to \mathbb{R}$ has a local maximum at a point $t_0 \in \mathbb{T}^{k^2}$, then $f^{\Delta\Delta}(\rho(t_0)) \leq 0$ provided that t_0 is not simultaneously ld and rs and that $f^{\Delta\Delta}(\rho(t_0))$ exists.*

For the proof of this lemma see [10].

2. INITIAL VALUE PROBLEMS

In this chapter, the concept of dynamic equations on time scales is introduced. In section 1 initial value problems for first order equations on time scales and some properties of the generalized exponential function are discussed. Some examples are also included. In section 2 solutions of homogeneous and non-homogeneous equations of the type $-[p(t)y^{\triangle}(t)]^{\nabla} + q(t)y(t) = 0$ and $-[p(t)y^{\triangle}(t)]^{\nabla} + q(t)y(t) = h(t)$ are presented.

For much of the work in this chapter, we refer to [11, 12].

2.1 *First order differential equations on time scales*

Definition 2.1.1. Let $f : \mathbb{T} \times \mathbb{R}^2 \to \mathbb{R}$. Then the equation

$$y^{\Delta}(t) = f(t, y(t), y(\delta(t))),\ t \in \mathbb{T}^k, \tag{2.1.1}$$

is called a first order delta dynamic equation on a time scale $\mathbb{T}$. A function $y : \mathbb{T} \to \mathbb{R}$ that satisfies (2.1.1) for each $t \in \mathbb{T}^k$ is called its solution. Similarly, the equation

$$y^{\nabla}(t) = f(t, y(t), y(\rho(t))),\ t \in \mathbb{T}_k, \tag{2.1.2}$$

is called a first order Nabla dynamic equation.

Definition 2.1.2. For given $t \in \mathbb{T}$ and $y_0 \in \mathbb{R}$, the problem

$$y^{\Delta}(t) = f(t, y(t), y(\delta(t))),\ y(t_0) = y_0, \tag{2.1.3}$$

is called an initial value problem for a first order dynamic equation on a time scale $\mathbb{T}$ and a function $y(t)$ that satisfies (2.1.3) for each $t \in \mathbb{T}^k$ with

$y(t_0) = y_0$ is called its solution. Similarly, for given $t \in \mathbb{T}$ and $y_0 \in \mathbb{R}$, the problem

$$y^{\nabla}(t) = f(t, y(t), y(\rho(t))),\ y(t_0) = y_0, \tag{2.1.4}$$

is called an initial value problem for first order Nabla dynamic equation and a function $y(t)$ that satisfies (2.1.4) for each $t \in \mathbb{T}_k$ with $y(t_0) = y_0$ is called a solution of this IVP.

In the following example we discuss solution of an initial value problem for two special cases of $\mathbb{T}$, that is, $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{N}_0$.

Example 2.1.3. Consider the IVP

$$y^{\Delta}(t) = t^2 y(t),\ y(0) = 1, \text{ where } 0 \in \mathbb{T}. \tag{2.1.5}$$

(i) If $\mathbb{T} = \mathbb{R}$, the IVP takes the form

$$y'(t) = t^2 y(t),\ y(t_0) = y_0,$$

that has a solution of the form

$$y = exp(\frac{t^3}{3}).$$

(ii) For $\mathbb{T} = \mathbb{N}_0$, the IVP takes the form

$$\triangle y(t) = t^2 y(t),\ y(t_0) = y_0$$

or $y(t+1) = (t^2+1)y(t)$ for $t = 0, 1, 2, ...$ with $y(0) = 1$. Solving this equation we have

$$y(1) = (0^2 + 1)y(0),$$

$$y(2) = (1^2 + 1)y(1),$$

$$y(3) = (2^2 + 1)y(2),$$

$$y(4) = (3^2 + 1)y(3),$$

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$$y(t) = \prod_{s=1}^{t-1}[(s-1)^2+1].$$

Hence the type of solution of an IVP depends on the nature of the time scale. Next, we discuss the general solutions of the initial value problems of the type

$$y^{\Delta} = p(t)y, \quad y(t_0) = 1,\ t \in \mathbb{T}. \tag{2.1.6}$$

where $p : \mathbb{T} \to \mathbb{R}$.

Definition 2.1.4. (*Generalized exponential function*): The *generalized exponential function*, see [12, 13] on $\mathbb{T}$ is denoted by $e_p(t, t_0)$ and is defined as

$$e_p(t,s) = \begin{cases} exp[\int_{t_0}^{t} \frac{1}{\mu(t)} ln(1+\mu(s)p(s))\Delta s], & \mu(t) > 0; \\ exp[\int_{t_0}^{t} p(s)\Delta s], & \mu(t) = 0, \end{cases}$$

where ln is the principal logarithm function and $t_0 \in \mathbb{T}$.

We state the following theorems in the sequel, which establish existence and uniqueness of solution of the IVP (2.1.6). For the proofs, see [13, 14].

Theorem 2.1.5. *If p is rd-continuous and regressive, then the IVP (2.1.6) has a unique solution.*

Theorem 2.1.6. *If p is regressive and $t_0 \in \mathbb{T}$, then the unique solution to the IVP (2.1.6) is the generalized exponential function $e_p(t, t_0)$ on $\mathbb{T}$.*

Example 2.1.7. The IVP

$$y^{\Delta}(t) = t^2 y(t),\ y(0) = 1 \text{ with } 0 \in \mathbb{T},$$

has a unique solution $e_{t^2}(t, 0)$. Because:

(i) For $\mathbb{T} = \mathbb{R}$, $\mu(t) = 0$, we have

$$e_{t^2}(t,0) = exp \int_0^t s^2 \Delta s = exp \int_0^t s^2 ds = exp[\frac{t^3}{3}].$$

(ii) For $\mathbb{T} = \mathbb{N}_0$, $\mu(t) = 1$, therefore

$$e_{t^2}(t,0) = exp \int_0^t ln(1 + 1.s^2)\Delta s = exp \sum_{s=0}^{t-1} ln(1 + 1.s^2) = \prod_{s=1}^{t-1}((s-1)^2 + 1).$$

2.1.1 Properties of the generalized exponential function

The following properties of exponential function are of interest.

(1) If p and q are regressive and rd-continuous, then $e_p \,.\, e_q = e_{p \odot q}$, where the addition $\odot$ is defined by $p \odot q = p + q + \mu pq$.

Proof. By the definition of Δ-derivative, it follows that

$$e_p(\delta(t), s) = e_p(t,s) + \mu(t)e_p^{\Delta}(t,s) = e_p(t,s) + \mu(t)p(t)e_p(t,s) = [1 + \mu(t)p(t)]e_p(t,s), \tag{2.1.7}$$

where $e_p^{\Delta}(t,s)$ denotes the derivatives of $e_p(t,s)$ with respect to t. Let $y = e_p(.,t_0)e_q(.,t_0)$ where p and q are regressive . By (2.1.7), it follows that

$$\begin{aligned} y^{\Delta}(t) &= e_p^{\Delta}(t,t_0)e_q(t,t_0) + e_p(\delta(t),t_0)e_q^{\Delta}(t,t_0) \\ &= p(t)e_p(t,t_0)e_q(t,t_0) + [1 + \mu(t)p(t)]e_p(t,t_0)q(t)e_q(t,t_0) \\ &= [p(t) + q(t) + \mu(t)p(t)q(t)]e_p(t,t_0)e_q(t,t_0) \\ &= [p(t) + q(t) + \mu(t)p(t)q(t)]y(t) = (p \odot q)(t)y. \end{aligned}$$

Hence that y solves the initial value problem

$$y^{\Delta} = (p \odot q)(t)y, \quad y(t_0) = 1$$

and consequently, by theorem (2.1.6) $y = e_{p \odot q}(.,t_0)$. Thus

$$e_p.e_q = e_{p \odot q}.$$

□

(2) $p \odot q$ is regressive if and only if p and q are regressive.

Proof. Suppose p and q are regressive, then

$$1 + \mu(t)p(t) \neq 0 \text{ and } 1 + \mu(t)q(t) \neq 0. \tag{2.1.8}$$

Now in view of (2.1.8), we have

$$\begin{aligned} 1 + \mu(t)(p \odot q) &= 1 + \mu(p + q + \mu pq) = 1 + \mu p + \mu q + \mu^2 pq \\ &= 1(1 + \mu p) + \mu q(1 + \mu p) = (1 + \mu p)(1 + \mu q) \neq 0. \end{aligned}$$

Hence $p \odot q$ is regressive.

Conversely, let $p \odot q$ is regressive, then

$$1 + \mu(p \odot q) \neq 0, \text{ that is, } (1 + \mu p)(1 + \mu q) \neq 0,$$

which implies that $1+\mu p \neq 0$ and $1+\mu q \neq 0$. Hence p and q are regressive. □

(3) If p and q are regressive and rd-continuous, then $\dfrac{e_p}{e_q} = e_{p \oslash q}$, where $p \oslash q = \dfrac{p - q}{1 + \mu q}$.

Proof. Let

$$y = \frac{e_p(., t_0)}{e_q(., t_0)}, \tag{2.1.9}$$

using (2.1.7), we have

$$\begin{aligned} y^{\triangle}(t) &= \frac{e_p^{\triangle}(t, t_0)e_q(t, t_0) - e_p(t, t_0)e_q^{\triangle}(t, t_0)}{e_q(t, t_0)e_q(\delta(t), t_0)} \\ &= \frac{p(t)e_p(t, t_0)e_q(t, t_0) - e_p(t, t_0)q(t)e_q(t, t_0)}{e_q(t, t_0)[1 + \mu(t)q(t)]e_q(t, t_0)}, \end{aligned}$$

which in view of eq.(2.1.9) implies that

$$\begin{aligned} y^{\triangle}(t) &= \frac{[p(t)+q(t)]e_p(t,t_0)e_q(t,t_0)}{[1+\mu(t)q(t)]e_q(t,t_0)e_q(t,t_0)} \\ &= \frac{[p(t)+q(t)]e_p(t,t_0)}{[1+\mu(t)q(t)]e_q(t,t_0)} \\ &= \frac{p(t)+q(t)}{1+\mu(t)q(t)}y(t) \\ &= (p \oslash q)y(t). \end{aligned}$$

This implies that y solve the initial value problem

$$y^{\Delta} = (p \oslash q)y, \quad y(t_0) = 1,$$

and consequently, $y = e_{p \oslash q}(., t_0)$. Hence,

$$\frac{e_p}{e_q} = e_{p \oslash q}.$$

□

(4) $p \oslash q$ is regressive if and only if both p and q are regressive.

Proof. Suppose p and q are regressive. Then,

$$1 + \mu p \neq 0 \text{ and } 1 + \mu q \neq 0. \tag{2.1.10}$$

It follows that

$$1 + \mu(p \oslash q) = 1 + \frac{p-q}{1+\mu q}\mu = \frac{1+\mu q + \mu p - \mu q}{1+\mu q} = \frac{1+\mu p}{1+\mu q} \neq 0.$$

Hence $p \oslash q$ is regressive.
Conversely, suppose that $p \oslash q$ is regressive, then $1 + \mu(p \oslash q) \neq 0$, which implies that

$$\frac{1+\mu p}{1+\mu q} \neq 0, \text{ that is, } 1 + \mu p \neq 0 \text{ and } 1 + \mu q \neq 0.$$

Hence p and q are regressive. □

For further properties of the generalized exponential function see [15, 16].

Now, we are in position to solve the initial value problem of the type

$$x^{\Delta} = -p(t)x^{\delta}, \quad x(t_0) = 1, \tag{2.1.11}$$

where p is rd-continuous and regressive. By theorem (2.1.5), the IVP has a unique solution x. It follows that

$$x^{\Delta}(t) = -p(t)x(\delta(t)) = -p(t)[x(t) + \mu(t)x^{\Delta}(t)],$$

which implies that

$$\begin{aligned} &x^{\Delta}(t) + p(t)\mu(t)x^{\Delta}(t) = -p(t)x(t), \text{ that is,} \\ &[1 + p(t)\mu(t)]x^{\Delta}(t) = -p(t)x(t), \\ &x^{\Delta}(t) = \frac{-p(t)}{1 + \mu(t)p(t)}x(t) = (\oslash p(t))x(t). \end{aligned}$$

Hence, x is a solution of the IVP

$$y^{\triangle}(t) = (\oslash p(t))y(t), \quad y(t_0) = 1.$$

Theorem 2.1.8. *If p is rd-continuous and regressive then $e_p(., t_0)$ and $e_{\oslash p}(., t_0)$ are the unique solutions of (2.1.6) and (2.1.11) respectively.*

The exponential functions can be used to find solutions of higher order dynamic equations on time scale as shown in the following examples.

Example 2.1.9. Consider a linear third order equation of the type

$$y^{\triangle^3} - 2y^{\triangle^2} - y^{\triangle} + 2y = 0. \tag{2.1.12}$$

Choose $y = e_{\lambda}(t, t_0)$ to be a solution of (2.1.12), we have

$$\begin{aligned} &\lambda^3 e_{\lambda}(t, t_0) - 2\lambda^2 e_{\lambda}(t, t_0) - \lambda e\lambda_(t, t_0) + 2e_{\lambda}(t, t_0) = 0, \text{ that is,} \\ &(\lambda^3 - 2\lambda^2 - \lambda + 2)e_{\lambda}(t, t_0) = 0, \text{ that is,} \\ &(\lambda + 1)(\lambda - 1)(\lambda - 2)e_{\lambda}(t, t_0) = 0, \end{aligned}$$

known as the auxiliary equation. Its roots are:

$$\lambda = -1, \lambda = 1, \lambda = 2.$$

Hence

$$e_{-1}(., t_0),\ e_1(., t_0),\ e_2(., t_0)$$

are solutions of (2.1.12). The general solution of (2.1.12) is therefore,

$$y(t) = C_1 e_{-1}(t, t_0) + C_2 e_1(t, t_0) + C_3 e_2(t, t_0).$$

Example 2.1.10. Consider the initial value problem

$$y^{\Delta^2} = a^2 y, \quad y(t_0) = 1,\ y^{\Delta}(t_0) = 0, \tag{2.1.13}$$

where a is a regressive constant. Choose $y = e_\lambda(t, t_0)$ to be a solution of (2.1.13), we have

$$\lambda^2 e_\lambda(t, t_0) - a^2 e_\lambda(t, t_0) = 0,$$

it implies that

$$(\lambda + a)(\lambda - a) e_\lambda(t, t_0) = 0.$$

Hence, $\lambda = -a$ and $\lambda = a$. Therefore

$$e_{-a}(t, t_0) \text{ and } e_a(t, t_0),$$

are the solutions of (2.1.13). Hence the general solution of (2.1.13) is

$$y = C_1 e_{-a}(t, t_0) + C_2 e_a(t, t_0). \tag{2.1.14}$$

Taking the delta derivative,

$$y^{\Delta} = -aC_1 e_{-a}(t, t_0) + aC_2 e_a(t, t_0). \tag{2.1.15}$$

Using the initial conditions, we have

$$C_1 + C_2 = 1,\ C_1 - C_2 = 0. \tag{2.1.16}$$

Solving (2.1.16), we obtain $C_1 = \frac{1}{2}$ and $C_2 = \frac{1}{2}$. Hence,

$$y = \frac{1}{2}[e_{-a}(t,t_0) + e_a(t,t_0)] = cosh_a(t,t_0),$$

is a particular solution.

2.2 *Second order linear differential equations on time scales*

In this section solutions of second ordered homogeneous differential equation on $\mathbb{T}^* = T^k \bigcap \mathbb{T}_k$ of the type

$$-[p(t)y^{\triangle}(t)]^{\nabla} + q(t)y(t) = 0, \tag{2.2.1}$$

are discussed. Assume that $q : \mathbb{T} \to \mathbb{C}$ is continuous, $p : \mathbb{T} \to \mathbb{C}$ is ∇-differentiable on $\mathbb{T}_k$ such that $p^{\nabla} : \mathbb{T}_k \to \mathbb{C}$ is continuous and $p(t) \neq 0$ for all $t \in \mathbb{T}$.

Definition 2.2.1. A function $y : \mathbb{T} \to \mathbb{C}$ is said to be a solution of (2.2.1) if and only if the following hold:

1. y is Δ-differentiable;
2. $y^{\Delta} : \mathbb{T}^k \to \mathbb{C}$ is ∇-differentiable on $\mathbb{T}_k$;
3. $(y^{\Delta})^{\nabla} : \mathbb{T}^k \to \mathbb{C}$ is continuous;
4. (2.2.1) holds for all $t \in \mathbb{T}$.

Definition 2.2.2. *Quasi* Δ*-derivative*

The *quasi* Δ*-derivative* of y at t is defined by $y^{[\Delta]}(t) = p(t)y^{\Delta}(t)$.

Theorem 2.2.3. *Let* $t_0 \in \mathbb{T}^k$ *be fixed and* C_0, C_1 *be given constants. Then equation* (2.2.1) *has a unique solution* y *such that*

$$y(t_0) = C_0,\ y^{[\triangle]}(t_0) = C_1.$$

Proof. Let $y(t)$ be a solution of equation (2.2.1) and set $p(t)y^{\triangle}(t) = u(t)$ then, (2.2.1) can be written as an equivalent first-ordered system

$$\begin{aligned} y^{\triangle}(t) &= \frac{u(t)}{p(t)}, \\ u^{\nabla}(t) &= q(t)y(t). \end{aligned} \tag{2.2.2}$$

Now by theorems (1.3.13) and (1.3.16), we have

$$\begin{aligned} y^{\nabla}(t) &= y^{\triangle}(\rho(t)) \\ &= \frac{u(\rho(t))}{p(\rho(t))} \\ &= \frac{1}{p(\rho(t))}[u(t) + (\rho(t) - t)u^{\nabla}(t)] \\ &= \frac{1}{p(\rho(t))}[u(t) + (\rho(t) - t)q(t)y(t)], \end{aligned} \tag{2.2.3}$$

and by using theorems (1.3.3) and (1.3.16)

$$\begin{aligned} u^{\triangle}(t) &= u^{\nabla}(\delta(t)) \\ &= q(\delta(t))y(\delta(t)) \\ &= q(\delta(t))[y(t) + (\delta(t) - t)y^{\triangle}(t)] \\ &= q(\delta(t))[y(t) + \frac{\delta(t) - t}{p(t)}u(t)]. \end{aligned} \tag{2.2.4}$$

Now from equations (2.2.2) and (2.2.3) we have

$$\begin{pmatrix} y^{\nabla}(t) \\ u^{\nabla}(t) \end{pmatrix} = \begin{pmatrix} (\rho(t) - t)\dfrac{q(t)}{p(\rho(t))} & \dfrac{1}{p(\rho(t))} \\ q(t) & 0 \end{pmatrix} \begin{pmatrix} y(t) \\ u(t) \end{pmatrix},$$

or

$$x^{\nabla}(t) = B_1(t)x(t).$$

Also from equations (2.2.2) and (2.2.4) we have

$$\begin{pmatrix} y^{\triangle}(t) \\ u^{\triangle}(t) \end{pmatrix} = \begin{pmatrix} 0 & \dfrac{1}{p(t)} \\ q(\delta(t)) & (\delta(t)-t)\dfrac{q(\delta(t))}{p(t)} \end{pmatrix} \begin{pmatrix} y(t) \\ u(t) \end{pmatrix},$$

or

$$x^{\triangle}(t) = B_2(t)x(t).$$

□

Definition 2.2.4. (*Wronskian*)

If y, $z : \mathbb{T} \to \mathbb{C}$ are Δ-differentiable on $\mathbb{T}^k$, then the *Wronskian* of y and z is defined by

$$W_t(y,z) = \begin{pmatrix} y(t) & z(t) \\ y^{[\triangle]}(t) & z^{[\triangle]}(t) \end{pmatrix} = y(t)z^{[\triangle]}(t) - y^{[\triangle]}z(t),$$

or

$$W_t(y,z) = p(t)[y(t)z^{\triangle}(t) - y^{\triangle}(t)z(t)].$$

In the following theorems, properties of the Wronskian are discussed.

Theorem 2.2.5. *The Wronskian of any two solutions of the equation (2.2.1) is independent of t.*

Proof. Let y and z be two solutions of equation (2.2.1), then for $t \in \mathbb{T}_k$

$$W_t(y,z) = y(t)z^{[\triangle]}(t) - y^{[\triangle]}(t)z(t).$$

Taking the nabla-derivative,

$$\begin{aligned}(W_t(y,z))^{\nabla} &= [y(t)z^{[\triangle]}(t) - y^{[\triangle]}(t)z(t)]^{\nabla} \\ &= \{y(t)z^{[\triangle]}(t)\}^{\nabla} - \{y^{[\triangle]}(t)z(t)\}\nabla \\ &= y(t)[z^{[\triangle]}(t)]^{\nabla} + y^{\nabla}(t)z^{[\triangle]}(\rho(t)) - [y^{[\triangle]}(t)]^{\nabla}z(t) - y^{[\triangle]}(\rho(t))z^{\nabla}(t) \\ &= y(t)[p(t)z^{\triangle}(t)]^{\nabla} + y^{\nabla}(t)p(\rho(t))z^{\triangle}(\rho(t) - [p(t)y^{\triangle}(t)]^{\nabla}z(t) - p(\rho(t))y^{\triangle}(t)z^{\nabla}(t) \\ &= y(t)q(t)z(t) + y^{\nabla}(t)p(\rho(t))z^{\triangle}(\rho(t)) - q(t)y(t)z(t) - p(\rho(t))y^{\triangle}(\rho(t))z^{\nabla}(t) \\ &= p(\rho(t))[y^{\nabla}(t)z^{\triangle}(\rho(t)) - y^{\triangle}(\rho(t))z^{\nabla}(t)] \\ &= p(\rho(t))[y^{\nabla}(t)z^{\nabla}(t) - y^{\nabla}(t)z^{\nabla}(t)] = 0.\end{aligned}$$

Hence $W_t(y,z) = constant$. □

Corollary 2.2.6. *If y and z are both solutions of equation (2.2.1) then either $W_t(y,z) = 0$ for all $t \in \mathbb{T}$ or $W_t(y,z) \neq 0$ for all $t \in \mathbb{T}$.*

The proofs of the following two theorems are the same as in the case of usual differential equations.

Theorem 2.2.7. *Any two solutions of equation (2.2.1) are linearly independent if and only if their wronskian is non-zero.*

Theorem 2.2.8. *Equation (2.2.1) has two linearly independent solutions and every solution of equation (2.2.1) is a linear combination of these solutions.*

Let us now consider the non-homogeneous equation

$$-[p(t)y^{\triangle}(t)]^{\nabla} + q(t)y(t) = h(t), \tag{2.2.5}$$

where $h : \mathbb{T} \to \mathbb{C}$ is a continuous function.

Theorem 2.2.9. *Suppose that y_1 and y_2 form the fundamental set of solutions of the homogeneous equation (2.2.1) and $\omega = W_t(y_1, y_2)$. Then the general solution of the non-homogeneous equation (2.2.5) is given by*

$$y(t) = C_1 y_1(t) + C_2 y_2(t) + \frac{1}{\omega}\int_{\rho(t_0)}^{t} [y_1(t)y_2(s) - y_1(s)y_2(t)]h(s)\nabla s,$$

where t_0 is a fixed point in $\mathbb{T}_k$, C_1 and C_2 are arbitrary constants.

Proof. It is sufficient to show that the function

$$z(t) = \frac{1}{\omega}\int_{\rho(t_0)}^{t} [y_1(t)y_2(s) - y_1(s)y_2(t)]h(s)\nabla s \tag{2.2.6}$$

is a particular solution of equation (2.2.5)

$$z^{\triangle}(t) = \frac{1}{\omega}\int_{\rho(t_0)}^{t} [y_1^{\triangle}(t)y_2(s) - y_1(s)y_2^{\triangle}(t)]h(s)\nabla s,$$

that is,

$$\begin{aligned} p(t)z^{\triangle}(t) &= \frac{1}{\omega}\int_{\rho(t_0)}^{t} [p(t)y_1^{\triangle}(t)y_2(s) - y_1(s)p(t)y_2^{\triangle}(t)]h(s)\nabla s \\ &= \frac{1}{\omega}\int_{\rho(t_0)}^{t} [y_1^{[\triangle]}(t)y_2(s) - y_1(s)y_2^{[\triangle]}(t)]h(s)\nabla s. \end{aligned} \tag{2.2.7}$$

Taking the ∇-derivative and using Theorem (1.4.7), it follows that

$$\begin{aligned} [p(t)z^{\triangle}(t)]^{\nabla} &= \frac{1}{\omega}(\int_{\rho(t_0)}^{t} [y_1^{[\triangle]}(t)y_2(s) - y_1(s)y_2^{[\triangle]}(t)]h(s)\nabla s)^{\nabla} \\ &= \frac{1}{\omega}[y_1^{[\triangle]}(\rho(t))y_2(t) - y_1(t)y_2^{[\triangle]}(\rho(t))]h(t) \\ &+ \frac{1}{\omega}\int_{\rho(t_0)}^{t} ([y_1^{[\triangle]}(t)]^{\nabla}y_2(s) - y_1(s)[y_2^{[\triangle]}(t)]^{\nabla})h(s)\nabla s \\ &= \frac{1}{\omega}[y_1^{[\triangle]}(\rho(t))y_2(t) - y_1(t)y_2^{[\triangle]}(\rho(t))]h(t) \\ &+ \frac{1}{\omega}\int_{\rho(t_0)}^{t} ([p(t)y_1^{\triangle}(t)]^{\nabla}y_2(s) - y_1(s)[p(t)y_2^{\triangle}(t)]^{\nabla})h(s)\nabla s. \end{aligned} \tag{2.2.8}$$

Now

$$y_1^{[\Delta]}(\rho(t))y_2(t) - y_1(t)y_2^{[\triangle]}(\rho(t)) = -W_t(y_1, y_2) = -\omega. \tag{2.2.9}$$

As, if $\rho(t) = t$ then (2.2.9) is obvious. If $\rho(t) < t$, then in view of the formula

$$y_i(t) = y_i(\rho(t)) + (t - \rho(t))y_i^{\triangle}(\rho(t)),\ i = 1, 2,$$

it follows that

$$\begin{aligned} y_1^{[\triangle]}(\rho(t))y_2(t) - y_1(t)y_2^{[\triangle]}(\rho(t)) &= p(\rho(t))[y_1^{\triangle}(\rho(t))y_2(t) - y_1(t)y_2^{\triangle}(\rho(t))] \\ &= p(\rho(t))[y_1^{\Delta}(\rho(t))y_2(\rho(t)) + (t-\rho(t))y_2^{\Delta}(\rho(t))y_1^{\Delta}(\rho(t)) \\ &- y_1(\rho(t))y_2^{\Delta}(\rho(t)) - (t-\rho(t))y_1^{\Delta}(\rho(t))y_2^{\Delta}(\rho(t))] \\ &= p(\rho(t))[y_1^{\Delta}(\rho(t))y_2(\rho(t)) - y_1(\rho(t))y_2^{\Delta}(\rho(t))] \\ &= -W_{\rho(t)}(y_1, y_2) = -\omega. \end{aligned} \tag{2.2.10}$$

Now

$$[p(t)y_i^{\Delta}(t)]^{\nabla} = q(t)y_i(t), i = 1, 2. \tag{2.2.11}$$

Since the Wronskian of any two solutions of equation (2.2.1) is constant, therefore using eq.(2.2.9) and eq.(2.2.11) in eq.(2.2.8) , we get

$$\begin{aligned} [p(t)z^{\triangle}(t)]^{\nabla} &= \frac{1}{\omega}(-\omega)h(t) + \frac{1}{\omega}\int_{\rho(t_0)}^{t}[q(t)y_1(t)y_2(s) - y_1(s)q(t)y_2(t)]h(s)\nabla s \\ &= -h(t) + q(t)\frac{1}{\omega}\int_{\rho(t_0)}^{t}[y_1(t)y_2(s) - y_1(s)y_2(t)]h(s)\nabla s. \end{aligned}$$

Hence by using (2.2.6).

$$-[p(t)z^{\triangle}(t)]^{\nabla} + q(t)z(t) = h(t),$$

which implies that $z(t)$ satisfies equation (2.2.5), so $z(t)$ is a particular solution of (2.2.5). □

3. LINEAR BOUNDARY VALUE PROBLEMS AND THE GREEN'S FUNCTION

In this chapter the existence and uniqueness of solutions of linear boundary value problems (BVPs) of the type

$$\begin{aligned} -[p(t)y^{\triangle}(t)]^{\nabla} + q(t)y(t) = h(t),\ t \in [a,b]_{\mathbb{T}}, \\ \alpha y(\rho(a)) - \beta y^{[\Delta]}(\rho(a)) = 0,\ \gamma y(b) + \delta y^{[\Delta]}(b) = 0, \end{aligned} \tag{3.0.1}$$

are discussed, where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ and $|\alpha| + |\beta| \neq 0$, $|\gamma| + |\delta| \neq 0$, $a \leq b$ are fixed in $\mathbb{T}$ and $a \in \mathbb{T}_k$, $b \in \mathbb{T}^k$. In section 1, the solution of the homogeneous linear boundary value problem is presented, while in section 2, uniqueness of the solution and the Green's function corresponding to (3.0.1) is investigated. Almost all of the materials of this chapter are taken from [9].

Note that, if $\beta = 0$ and $\delta p(b) = \gamma[\delta(b) - b]$, then the boundary conditions in (3.0.1) in view of the relation

$$\gamma[y(\delta(b)) - (\delta(b) - b)y^{\Delta}(b)] + \delta y^{[\Delta]}(b) = 0,$$

take the form

$$y(\rho(a)) = 0, y(\delta(b)) = 0,$$

known as the conjugate (or Dirichlet) boundary conditions. This means that the boundary conditions in (3.0.1) include the Dirichlet boundary conditions as a special case.

3.1 Homogeneous linear boundary value problem

Let $\phi(t)$ and $\psi(t)$ be two solutions of the homogeneous equation

$$-[p(t)y^{\Delta}(t)]^{\nabla} + q(t)y(t) = 0,\ t \in [a,b]_{\mathbb{T}}, \tag{3.1.1}$$

such that

$$\phi(\rho(a)) = \beta,\ \ \phi^{[\Delta]}(\rho(a)) = \alpha, \tag{3.1.2}$$

$$\psi(b) = \delta,\ \psi^{[\Delta]}(b) = -\gamma. \tag{3.1.3}$$

The Wronskian is given by

$$D = -W_t(\phi,\psi) = \phi^{[\Delta]}(t)\psi(t) - \phi(t)\psi^{[\Delta]}(t). \tag{3.1.4}$$

We know that the Wronskian of any solution of (3.0.1) is independent of $t \in [\rho(a), b]_{\mathbb{T}}$, take $t = \rho(a)$ in (3.1.4) and using (3.1.2), it follows that

$$D = \phi^{[\Delta]}(\rho(a))\psi(\rho(a)) - \phi(\rho(a))\psi^{[\Delta]}(\rho(a)) = \alpha\psi(\rho(a)) - \beta\psi^{[\Delta]}(\rho(a)). \tag{3.1.5}$$

Similarly for $t = b$ and in view of (3.1.3), it follows that

$$D = \phi^{[\Delta]}(b)\psi(b) - \phi(b)\psi^{[\Delta]}(b) = \delta\phi^{[\Delta]}(b) + \gamma\phi(b). \tag{3.1.6}$$

From (3.1.5) and (3.1.6),

$$D = \alpha\psi(\rho(a)) - \beta\psi^{[\Delta]}(\rho(a)) = \delta\phi^{[\Delta]}(b) + \gamma\phi(b).$$

Theorem (2.2.7), demonstrates that $D \neq 0$ if and only if ϕ and ψ are linearly independent.

Theorem 3.1.1. *$D \neq 0$ if and only if the homogeneous equation* (3.1.1) *has only the trivial solution, satisfying the boundary conditions presented in* (3.1.1).

Proof. If $D \neq 0$, then by theorem (2.2.7), ϕ and ψ form a fundamental set of solutions. Hence, the general solution of the BVP (3.1.1) is given by

$$y(t) = C_1\phi(t) + C_2\psi(t), \tag{3.1.7}$$

where C_1 and C_2 are constants. The boundary condition in (3.0.1) implies that

$$\alpha[C_1\phi(\rho(a)) + C_2\psi(\rho(a))] - \beta[C_1\phi^{[\Delta]}(\rho(a)) + C_2\psi^{[\Delta]}(\rho(a))] = 0,$$

which in view of (3.1.2) gives

$$\alpha[C_1\beta + C_2\psi(\rho(a))] - \beta[C_1\alpha + C_2\psi^{[\Delta]}(\rho(a))] = 0.$$

Consequently,

$$C_2[\alpha\psi(\rho(a)) - \beta\psi^{[\Delta]}(\rho(a))] = 0,$$

which yields $C_2 = 0$ as $\alpha\psi(\rho(a)) - \beta\psi^{[\Delta]}(\rho(a)) = D \neq 0$. On the other hand, using (3.1.3), the boundary condition in (3.0.1) implies that

$$\gamma[C_1\phi(b) + C_2\psi(b)] + \delta[C_1\phi^{[\Delta]}(b) + C_2\psi^{[\Delta]}(b)] = 0,$$

that is,

$$\gamma[C_1\phi(b) + C_2\delta] + \delta[C_1\phi^{[\Delta]}(b) + C_2(-\gamma)] = C_1[\gamma\phi(b) + \delta\phi^{[\Delta]}(b)] = 0,$$

which implies that $C_1 = 0$. Hence $y = 0$. □

3.2 Non-homogeneous linear boundary value problem

Theorem 3.2.1. *If $D \neq 0$, then the non-homogeneous BVP (3.0.1) has a unique solution $y(t)$ given by*

$$y(t) = \int_{\rho(a)}^{b} G(t,s)h(s)\nabla s,$$

where

$$G(t,s) = \frac{1}{D}\begin{cases} \phi(t)\psi(s), & \rho(a) \le t \le s \le \delta(b), \\ \phi(s)\psi(t), & \rho(a) \le s \le t \le \delta(b), \end{cases}$$

called green function of BVP (3.0.1).

Proof. If $D \neq 0$, then by theorem (2.2.9), the general solution of the non-homogeneous problem (3.0.1) is given by

$$y(t) = C_1\phi(t) + C_2\psi(t) + \frac{1}{D}\int_{\rho(a)}^{t} [\phi(s)\psi(t) - \phi(t)\psi(s)]h(s)\nabla s, \tag{3.2.1}$$

where C_1 and C_2 are constants. Taking the Δ- derivative,

$$y^{[\triangle]}(t) = C_1\phi^{[\Delta]}(t) + C_2\psi^{[\Delta]}(t) + \frac{1}{D}\int_{\rho(a)}^{t} [\phi(s)\psi^{[\Delta]}(t) - \phi^{[\Delta]}(t)\psi(s)]h(s)\nabla s. \tag{3.2.2}$$

Now, at $t = \rho(a)$,

$$y(\rho(a)) = C_1\phi(\rho(t)) + C_2\psi(\rho(a)) = C_1\beta + C_2\psi(\rho(a)),$$

$$y^{[\Delta]}(\rho(a)) = C_1\phi^{[\Delta]}(\rho(a)) + C_2\psi^{[\Delta]}(\rho(a)) = C_1\alpha + C_2\psi^{[\Delta]}(\rho(a)).$$

Using the values $y(\rho(a))$ and $y^{[\Delta]}(\rho(a))$ in the boundary conditions, we have

$$\alpha[C_1\beta + C_2\psi(\rho(a))] - \beta[C_1\alpha + C_2\psi^{[\Delta]}(\rho(a))] = 0,$$

which leads to

$$C_2[\alpha\psi(\rho(a)) - \beta\psi^{[\Delta]}(\rho(a))] = 0.$$

Hence $C_2 = 0$ as $\alpha\psi(\rho(a)) - \beta\psi^{[\Delta]}(\rho(a)) = D \neq 0$. Thus, (3.2.1) and (3.2.2) take the forms

$$y(t) = C_1\phi(t) + \frac{1}{D}\int_{\rho(a)}^{t} [\phi(s)\psi(t) - \phi(t)\psi(s)]h(s)\nabla s, \tag{3.2.3}$$

$$y^{[\triangle]}(t) = C_1\phi^{[\Delta]}(t) + \frac{1}{D}\int_{\rho(a)}^{t} [\phi(s)\psi^{[\Delta]}(t) - \phi^{[\Delta]}(t)\psi(s)]h(s)\nabla s. \tag{3.2.4}$$

In view of (3.1.3), at $t = b$,

$$\begin{aligned} y(b) &= C_1\phi(b) + \frac{1}{D}\int_{\rho(a)}^{b} [\phi(s)\psi(b) - \phi(b)\psi(s)]h(s)\nabla s \\ &= C_1\phi(b) + \frac{1}{D}\int_{\rho(a)}^{b} [\delta\phi(s) - \phi(b)\psi(s)]h(s)\nabla s, \end{aligned} \tag{3.2.5}$$

$$\begin{aligned} y^{[\Delta]}(b) &= C_1\phi^{[\Delta]}(b) + \frac{1}{D}\int_{\rho(a)}^{b} [\phi(s)\psi^{[\Delta]}(b) - \phi^{[\Delta]}(b)\psi(s)]h(s)\nabla s \\ &= C_1\phi^{[\Delta]}(b) + \frac{1}{D}\int_{\rho(a)}^{b} [-\gamma\phi(s) - \phi^{[\Delta]}(b)\psi(s)]h(s)\nabla s. \end{aligned} \tag{3.2.6}$$

Substituting (3.2.5) and (3.2.6) in the boundary condition, implies that

$$\begin{aligned} &\gamma[C_1\phi(b) + \frac{1}{D}\int_{\rho(a)}^{b} [\delta\phi(s) - \phi(b)\psi(s)]h(s)\nabla s] \\ &\quad + \delta[C_1\phi^{[\Delta]}(b) + \frac{1}{D}\int_{\rho(a)}^{b} [-\gamma\phi(s) - \phi^{[\Delta]}(b)\psi(s)]h(s)\nabla s] = 0, \\ &C_1[\gamma\phi(b) + \delta\phi^{[\Delta]}(b)] - \frac{\gamma\phi(b) + \delta\phi^{[\Delta]}(b)}{D}\int_{\rho(a)}^{b} \psi(s)h(s)\nabla s = 0. \end{aligned} \tag{3.2.7}$$

Solving for C_1, we have

$$C_1 = \frac{1}{D}\int_{\rho(a)}^{b} \psi(s)h(s)\nabla s.$$

Using C_1 in (3.2.3) we have

$$y(t) = \frac{1}{D}\int_{\rho(a)}^{b} \phi(t)\psi(s)h(s)\nabla s + \frac{1}{D}\int_{\rho(a)}^{t} [\phi(s)\psi(t) - \phi(t)\psi(s)]h(s)\nabla s,$$

which implies that

$$y(t) = \frac{1}{D}\int_{\rho(a)}^{t} \phi(s)\psi(t)h(s)\nabla s + \frac{1}{D}\int_{t}^{b} \phi(t)\psi(s)h(s)\nabla s = \int_{\rho(a)}^{b} G(t,s)h(s)\nabla s,$$

where

$$G(t,s) = \frac{1}{D}\begin{cases} \phi(s)\psi(t), & \rho(a) \le s \le t \le \delta(b); \\ \phi(t)\psi(s), & \rho(a) \le t \le s \le \delta(b). \end{cases}$$

□

4. EXISTENCE AND APPROXIMATION OF SOLUTIONS OF NON-LINEAR BOUNDARY VALUE PROBLEMS

In this chapter, we derive new results dealing with the existence and approximations of solutions of the BVPs of the form

$$\begin{aligned} & -[p(t)y^{\Delta}(t)]^{\nabla} + q(t)y(t) = f(t, y(t)),\ t \in [a, b]_{\mathbb{T}}, \\ & c_1 y(\rho(a)) - c_2 y^{[\Delta]}(\rho(a)) = 0,\ d_1 y(b) + d_2 y^{[\Delta]}(b) = 0, \end{aligned} \tag{4.0.1}$$

where $c_1, c_2, d_1, d_2 \in \mathbb{C}$ and $|c_1| + |c_2| \neq 0$, $|d_1| + |d_2| \neq 0$, $p(t) > 0$, $a \leq b$ are fixed in $\mathbb{T}$ and $a \in \mathbb{T}_k$, $b \in \mathbb{T}^k$.

We introduce the concept of lower and upper solutions for the BVP (4.0.1) and develop a comparison result. Then we investigate the existence of solution in the presence of well ordered lower and upper solutions [1] for the BVP (4.0.1). Finally, we develop a generalized approximation technique and prove that there exists a monotone sequence $\{w_n\}$ of solutions of linear problems converging uniformly to a unique solution of the BVP (4.0.1).

Definition 4.0.2. Let $C_{rd}([a, b]_{\mathbb{T}}, \mathbb{R})$ to be the set of all functions $f(t, y)$ such that $f(., y)$ is rd-continuous for each $y \in \mathbb{R}$ and $f(t, .)$ is continuous for each $t \in [a, b]_{\mathbb{T}}$.

Definition 4.0.3. Let $C^2_{rd}([a, b]_{\mathbb{T}})$ to be the set of all functions $y : \mathbb{T} \to \mathbb{R}$ such that

$$C^2_{rd}([a, b]_{\mathbb{T}}) = \{y : y,\ y^{\Delta} \in C([a, b]_{\mathbb{T}}) \text{ and } y^{\Delta\nabla} \in C_{rd}([a, b]_{\mathbb{T}})\}.$$

We recall the concept of lower and upper solutions [1].

Definition 4.0.4. : A function $\alpha \in C^2_{rd}([\rho(a), \delta(b)]_{\mathbb{T}})$ is called a *lower solution* of (4.0.1) if

$$-[p(t)\alpha^{\Delta}(t)]^{\nabla} + q(t)\alpha(t) \leq f(t, \alpha(t)),\ t \in [a, b]_{\mathbb{T}},$$
$$c_1\alpha(\rho(t)) - c_2\alpha^{[\Delta]}(\rho(t)) \leq 0,\ d_1\alpha(b) + d_2\alpha^{[\Delta]}(b) \leq 0.$$

Definition 4.0.5. : A function $\beta \in C^2_{rd}([\rho(a), \delta(b)]_{\mathbb{T}})$ is called an *upper solution* of (4.0.1) if

$$-[p(t)\beta^{\Delta}(t)]^{\nabla} + q(t)\beta(t) \geq f(t, \beta(t)),\ t \in [a, b]_{\mathbb{T}},$$
$$c_1\beta(\rho(a)) - c_2\beta^{[\Delta]}(\rho(a)) \geq 0,\ d_1\beta(b) + d_2\beta^{[\Delta]}(b) \geq 0.$$

Example 4.0.6. Consider the two point boundary value problem,

$$y^{\Delta\Delta}(t) = y^2 - t, \quad t \in [0, 1]_{\mathbb{T}}, \quad y(0) = 0, \quad y(1) = 0.$$

Take $\alpha = 0$ on $[0, 1]_{\mathbb{T}}$, then $\alpha^{\Delta} = 0$ and $\alpha^{\Delta\Delta} = 0$,

$$\alpha^{\Delta\Delta} - \alpha^2 = 0 \geq -t = f(t, \alpha),\ t \in [0, 1]_{\mathbb{T}}.$$

Moreover,

$$\alpha(0) = 0 = \alpha(1).$$

Hence $\alpha(t) = 0$ is a lower solution. Take $\beta(t) = t + 1$ on $[0, 1]_{\mathbb{T}}$, then

$$\beta^{\Delta}(t) = 1,\ \beta^{\Delta\Delta}(t) = 0 \text{ on } [0, 1]_{\mathbb{T}}.$$

Hence β satisfies $\beta(0) = 1 > 0$, $\beta(1) = 2 > 0$ and

$$\beta^{\Delta\Delta}(t) - f(t, \beta) = 0 - (t + 1)^2 + t = -(t^2 + t + 1) \leq 0.$$

Thus β is an upper solution.

4.1 Comparison result

Theorem 4.1.1. *Assume* α *and* β *are lower and upper solutions of the BVP* (4.0.1) *respectively. If* $f(t,y) \in C_{rd}[[\rho(a),\delta(b)]_{\mathbb{T}} \times \mathbb{R}]$ *is decreasing in* y *for each* $t \in [\rho(a),\delta(b)]_{\mathbb{T}}$*. Then* $\alpha(t) \le \beta(t)$ *on* $[\rho(a),\delta(b)]_{\mathbb{T}}$*.*

Proof. Let $v(t) = \alpha(t) - \beta(t)$. Then $v \in C^2_{rd}([\rho(a),\delta(b)]_{\mathbb{T}})$ and using the definitions of upper and lower solutions, we have

$$\begin{aligned}
-[p(t)v^{\Delta}(t)]^{\nabla} + q(t)v(t) &= -[p(t)(\alpha(t)-\beta(t))^{\Delta}]^{\nabla} + q(t)(\alpha(t)-\beta(t)) \\
&= -[p(t)\alpha^{\Delta}(t)]^{\nabla} + [p(t)\beta^{\Delta}(t)]^{\nabla} + q(t)\alpha(t) - q(t)\beta(t) \\
&= \{-[p(t)\alpha^{\Delta}(t)]^{\nabla} + q(t)\alpha(t)\} - \{-[p(t)\beta^{\Delta}(t)]^{\nabla} + q(t)\beta(t)\} \\
&\le f(t,\alpha(t)) - f(t,\beta(t)),\ t \in [a,b]_{\mathbb{T}}.
\end{aligned}$$

The boundary conditions imply that

$$\begin{aligned}
c_1 v(\rho(a)) - c_2 v^{[\Delta]}(\rho(a)) &= c_1[\alpha(\rho(a)) - \beta(\rho(a))] - c_2[\alpha(\rho(a)) - \beta(\rho(a))]^{[\Delta]} \\
&= [c_1\alpha(\rho(a)) - c_2\alpha^{[\Delta]}(\rho(a))] - [c_1\beta(\rho(a)) - c_2\beta^{[\Delta]}(\rho(a))] \\
&\le 0
\end{aligned}$$

and

$$\begin{aligned}
d_1 v(b) + d_2 v^{[\Delta]}(b) &= d_1[\alpha(b) - \beta(b)] + d_2[\alpha^{[\Delta]}(b) - \beta^{[\Delta]}(b)] \\
&= [d_1\alpha(b) + \alpha^{[\Delta]}(b)] - [d_1\beta(b) + d_2\beta^{[\Delta]}(b)] \\
&\le 0.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&-[p(t)v^{\Delta}(t)]^{\nabla} + q(t)v(t) \le f(t,\alpha(t)) - f(t,\beta(t)),\ t \in [a,b]_{\mathbb{T}} \\
&C_1 v(\rho(a)) - C_2 v^{[\Delta]}(\rho(a)) \le 0,\ d_1 v(b) + d_2 v^{[\Delta]}(b) \le 0.
\end{aligned} \tag{4.1.1}$$

Suppose that the conclusion of the theorem is not true, then, $v(t)$ has a positive maximum say M at some $t_0 \in [\rho(a),\delta(b)]_{\mathbb{T}}$. If $t_0 = \rho(a)$, then

$$v(\rho(a)) > 0 \text{ and } v^{\Delta}(\rho(a)) \le 0. \tag{4.1.2}$$

The boundary condition

$$c_1 v(\rho(a)) - c_2 v^{[\Delta]}(\rho(a)) \leq 0,$$

implies that

$$c_1 v(\rho(a)) \leq c_2 v^{[\Delta]}(\rho(a)) = c_2 p(\rho(a)) v^{\Delta}(\rho(a)) < 0,$$

which leads to $v(\rho(a)) < 0$, a contradiction.

If $t_0 = \delta(b)$, then

$$v(\delta(b)) > 0 \text{ and } v^{\Delta}(b) \geq 0. \tag{4.1.3}$$

The boundary condition

$$d_1 v(b) + d_2 v^{[\Delta]}(b) \leq 0,$$

implies that

$$d_1 v(b) \leq -d_2 v^{[\Delta]}(b) = -d_2 p(b) v^{\Delta}(b) < 0,$$

which contradict (4.1.3). Hence $t_0 \neq \delta(b)$. Therefore, $t_0 \in (\rho(a), \delta(b))$. Now, choose

$$t_0 = max\{e \in (\rho(a), \delta(b)) : v(e) = M\}, \text{ then } v(t) < v(t_0) \text{ for each } t \in (\rho(a), \delta(b)).$$

Firstly, we show that t_0 can not be ld and rs simultaneously. Suppose that t_0 is ld and rs, i.e. $\rho(t_0) = t_0 < \delta(t_0)$. Since,

$$v(\delta(t_0)) < v(t_0),$$

follows that

$$v^{\Delta}(t_0) < 0.$$

Also,

$$v^{\Delta}(t_0) = \lim_{t \to t_0^-} v^{\Delta}(t),$$

therefore,

$$\lim_{t \to t_0^-} v^{\Delta}(t) < 0,$$

which implies that, there exists a neighbourhood $N_\epsilon(t_0)$ such that

$$v^{\Delta}(t) < 0 \text{ for each } t \in N_\epsilon(t_0),$$

that is,

$$v(t) \geq v(t_0) \text{ for each } t \in N_\epsilon(t_0),$$

a contradiction. Hence, t_0 is not simultaneously left-dense and right-scattered. Therefore, by lemma (1.3.25), we have

$$v^{\Delta\Delta}(\rho(t_0)) \leq 0. \tag{4.1.4}$$

Let $v^{\Delta} = f$, then from (4.1.4), $(f^{\Delta})(\rho(t_0)) \leq 0$. Using the relation $f^{\nabla}(t_0) = f^{\Delta}(\rho(t_0))$, it follows that

$$f^{\nabla}(t_0) \leq 0.$$

Hence,

$$v^{\Delta\nabla}(t_0) \leq 0,$$

which implies that

$$\alpha^{\Delta\nabla}(t_0) \leq \beta^{\Delta\nabla}(t_0). \tag{4.1.5}$$

The relation

$$-\alpha^{[\Delta]\nabla}(t_0) \leq -q(t_0)\alpha(t_0) + f(t_0, \alpha(t_0)),$$

and the decreasing property of $f(t, y)$ in y, leads to

$$\begin{aligned} \alpha^{[\Delta]\nabla}(t_0) &\geq q(t_0)\alpha(t_0) - f(t_0, \alpha(t_0)) \\ &\geq q(t_0)\beta(t_0) - f(t_0, \alpha(t_0)) \\ &> q(t_0)\beta(t_0) - f(t_0, \beta(t_0)) \geq \beta^{[\Delta]\nabla}(t_0), \end{aligned}$$

a contradiction. Hence there does not exist $t_0 \in [\rho(a), \delta(b)]_{\mathbb{T}}$, such that $v(t_0) > 0$. Consequently, $v(t) \leq 0$, for every $t_0 \in [\rho(a), \delta(b)]_{\mathbb{T}}$. □

Corollary 4.1.2. *Under the hypothesis of the Theorem (4.1.1), the BVP (4.0.1) has a unique solution.*

Proof. Suppose that y_1 and y_2 are two solutions of the BVP (4.0.1). Then

$$-[p(t)y_1^{\Delta}(t)]^{\nabla} + q(t)y_1(t) = f(t, y_1(t)) \leq f(t, y_1(t)),\ t \in [a, b]_{\mathbb{T}},$$
$$c_1 y_1(\rho(a)) - c_2 y_1^{[\Delta]}(\rho(a)) = 0,\ d_1 y_1(b) + d_2 y_1^{[\Delta]}(b) = 0,$$

which implies that y_1 is a lower solution of (4.0.1). Also,

$$-[p(t)y_2^{\Delta}(t)]^{\nabla} + q(t)y_2(t) = f(t, y_2(t)) \geq f(t, y_2(t)),\ t \in [a, b]_{\mathbb{T}},$$
$$c_1 y_2(\rho(a)) - c_2 y_2^{[\Delta]}(\rho(a)) = 0,\ d_1 y_2(b) + d_2 y_2^{[\Delta]}(b) = 0,$$

which implies that y_2 is an upper solution of (4.0.1). Hence, by Theorem (4.1.1), it follows that

$$y_1(t) \leq y_2(t),\ t \in [\rho(a), \delta(b)]. \tag{4.1.6}$$

Now,

$$-[p(t)y_1^{\Delta}(t)]^{\nabla} + q(t)y_1(t) = f(t, y_1(t)) \geq f(t, y_1(t)),\ t \in [a, b]_{\mathbb{T}},$$
$$c_1 y_1(\rho(a)) - c_2 y_1^{[\Delta]}(\rho(a)) = 0,\ d_1 y_1(b) + d_2 y_1^{[\Delta]}(b) = 0,$$

which implies that y_1 is an upper solution of (4.0.1). Moreover,

$$-[p(t)y_2^{\Delta}(t)]^{\nabla} + q(t)y_2(t) = f(t, y_2(t)) \leq f(t, y_2(t)),\ t \in [a, b]_{\mathbb{T}},$$
$$c_1 y_2(\rho(a)) - c_2 y_2^{[\Delta]}(\rho(a)) = 0,\ d_1 y_2(b) + d_2 y_2^{[\Delta]}(b) = 0,$$

which implies that y_2 is a lower solution of (4.0.1). Hence, by Theorem (4.1.1), it follows that

$$y_2(t) \leq y_1(t),\ t \in [\rho(a), \delta(b)]. \tag{4.1.7}$$

From (4.1.6) and (4.1.7), we have $y_1(t) = y_2(t), t \in [\rho(a), \delta(b)]$. □

The following result is known. For the proof see [17].

Lemma 4.1.3. *If $f(t, y) \in C_{rd}[[\rho(a), \delta(b)]_{\mathbb{T}} \times \mathbb{R}]$ and is bounded uniformly on $[\rho(a), \delta(b)]_{\mathbb{T}} \times \mathbb{R}$, then the BVP (4.0.1) has a solution.*

4.2 Existence theorem (method of lower and upper solutions)

Theorem 4.2.1. *Assume that* α, β *are lower and upper solutions of the boundary value problem* (4.0.1) *such that* $\alpha \leq \beta$ *on* $[\rho(a), \delta(b)]_{\mathbb{T}}$. *If* $f(t, y) \in C_{rd}[[\rho(a), \delta(b)]_{\mathbb{T}} \times \mathbb{R}]$, *then the boundary value problem* (4.0.1) *has a solution* y *such that* $\alpha \leq y \leq \beta$ *on* $[\rho(a), \delta(b)]_{\mathbb{T}}$.

Proof. Define the following modification F of f

$$F(t, y(t)) = \begin{cases} f(t, \beta(t)) + \dfrac{y(t) - \beta(t)}{1 + |y(t) - \beta(t)|}, & y(t) \geq \beta(t); \\ f(t, y(t)), & \alpha(t) \leq y(t) \leq \beta(t); \\ f(t, \alpha(t)) + \dfrac{\alpha(t) - y(t)}{1 + |\alpha(t) - y(t)|}, & y(t) \leq \alpha(t). \end{cases} \tag{4.2.1}$$

Clearly, $F \in C_{rd}[[\rho(a), \delta(b)]_{\mathbb{T}} \times \mathbb{R}]$ and is bounded on $[\rho(a), \delta(b)]_{\mathbb{T}} \times \mathbb{R}$. Hence by Lemma (4.1.3) the modified problem,

$$\begin{aligned} &- y^{[\Delta]\nabla}(t) + q(t)y(t) = F(t, y(t)), t \in [a, b]_{\mathbb{T}} \\ &c_1 y(\rho(a)) - c_2 y^{[\Delta]}(\rho(a)) = 0, d_1 y(b) + d_2 y^{[\Delta]}(b) = 0 \end{aligned} \tag{4.2.2}$$

has a solution. By definition of F, we have

$$F(t, \alpha(t)) = f(t, \alpha(t)) \geq -\alpha^{[\Delta]\nabla}(t) + q(t)\alpha(t), t \in [a, b]_{\mathbb{T}}$$

and

$$F(t, \beta(t)) = f(t, \beta(t)) \leq -\beta^{[\Delta]\nabla}(t) + q(t)\beta(t), t \in [a, b]_{\mathbb{T}}$$

which implies that α and β are lower and upper solutions of (4.2.2). Moreover, we note that any solution y of (4.2.2) such that

$$\alpha(t) \leq y(t) \leq \beta(t),\ t \in [\rho(a), \delta(b)]_{\mathbb{T}} \tag{4.2.3}$$

is a solution of (4.0.1). We only need to show that any solution y of (4.2.2) in fact does satisfy (4.2.3). Set

$$u(t) = \alpha(t) - y(t), t \in [\rho(a), \delta(b)]_{\mathbb{T}}$$

where y is a solution of (4.0.1), then $u \in C^2_{rd}[\rho(a), \delta(b)]_{\mathbb{T}}$ and the boundary conditions imply that

$$c_1 u(\rho(a)) - c_2 u^{[\Delta]}(\rho(a)) \leq 0, \; d_1 u(b) + d_2 u^{[\Delta]}(b) \leq 0.$$

Assume that

$$max\{u(t) : t \in [\rho(a), \delta(b)]_{\mathbb{T}}\} = u(t_0) > 0.$$

We can show as in the previous Theorem that $t_0 \neq \rho(a), \delta(b)$. Moreover, t_0 is not simultaneously ld and rs. Consequently,

$$u^{\triangle\nabla}(t_0) \leq 0. \tag{4.2.4}$$

On the other hand, using the definition of the lower solution and that of the modified function, we have

$$\begin{aligned}
-u^{[\triangle]\nabla}(t_0) &= -\alpha^{[\triangle]\nabla}(t_0) + y^{[\triangle]\nabla}(t_0) \\
&\leq f(t_0, \alpha(t_0)) - q(t_0)\alpha(t_0) + y^{[\triangle]\nabla}(t_0) \\
&= f(t_0, \alpha(t_0)) - q(t_0)\alpha(t_0) - F(t_0, y(t_0)) + q(t_0)y(t_0) \\
&= f(t_0, \alpha(t_0)) - q(t_0)\alpha(t_0) - f(t_0, \alpha(t_0)) - \frac{\alpha(t_0) - y(t_0)}{1 + |\alpha(t_0) - y(t_0)|} + q(t_0)y(t_0) \\
&= -q(t_0)u(t_0) - \frac{u(t_0)}{1 + u(t_0)} \\
&= -[q(t_0)u(t_0) + \frac{u(t_0)}{1 + u(t_0)}] < 0,
\end{aligned}$$

a contradiction. Hence

$$u(t) \leq 0, \text{ on } [\rho(a), \delta(b)]_{\mathbb{T}}.$$

Now, set

$$u(t) = y(t) - \beta(t), t \in [\rho(a), \delta(b)]_{\mathbb{T}},$$

where y is a solution of (4.0.1), then $u \in C^2_{rd}[\rho(a), \delta(b)]_{\mathbb{T}}$ and the boundary conditions imply that

$$c_1 u(\rho(a)) - c_2 u^{[\Delta]}(\rho(a)) \leq 0, \; d_1 u(b) + d_2 u^{[\Delta]}(b) \leq 0.$$

Assume that

$$max\{u(t) : t \in [\rho(a), \delta(b)]_{\mathbb{T}}\} = u(t_0) > 0.$$

We can show as in the previous Theorem that $t_0 \neq \rho(a), \delta(b)$. Moreover, t_0 is not simultaneously ld and rs. Consequently,

$$u^{\triangle\nabla}(t_0) \leq 0. \tag{4.2.5}$$

On the other hand, using the definition of the upper solution and that of the modified function, we have

$$\begin{aligned}
u^{[\triangle]\nabla}(t_0) &= y^{[\triangle]\nabla}(t_0) - \beta^{[\triangle]\nabla}(t_0) \\
&\geq y^{[\triangle]\nabla}(t_0) + f(t_0, \beta(t_0)) - q(t_0)\beta(t_0) \\
&= q(t_0)y(t_0) - F(t_0, y(t_0)) + f(t_0, \beta(t_0)) - q(t_0)\beta(t_0) \\
&= q(t_0)y(t_0) - q(t_0)\beta(t_0) - f(t_0, \beta(t_0)) - \frac{y(t_0) - \beta(t_0)}{1 + |y(t_0) - \beta(t_0)|} + f(t_0, \beta(t_0)) \\
&= q(t_0)(y(t_0) - \beta(t_0)) - \frac{y(t_0) - \beta(t_0)}{1 + |y(t_0) - \beta(t_0)|} \\
&= q(t_0)u(t_0) - \frac{u(t_0)}{1 + u(t_0)} > 0,
\end{aligned}$$

a contradiction. Hence

$$u(t) \leq 0,\ t \in [\rho(a), \delta(b)]_{\mathbb{T}}.$$

□

4.3 Generalized approximation technique

We develop the approximations scheme and show that under suitable conditions on f, there exists a bounded monotone sequence of solutions of linear problems that converges uniformly to a solution of the original problem. Let

$$\Omega = \{y \in C^2_{rd}[\rho(a), \delta(b)]_{\mathbb{T}} : \alpha(t) \leq y(t) \leq \beta(t), t \in [\rho(a), \delta(b)]_{\mathbb{T}}\}.$$

If $\frac{\partial^2 f(t,y)}{\partial y^2}$ is continuous and bounded on $[\rho(a), \delta(b)]_{\mathbb{T}} \times \Omega$, there exists a function ϕ such that

$$\frac{\partial^2}{\partial y^2}[f(t,y) + \phi(t,y)] \geq 0 \text{ on } [\rho(a), \delta(b)]_{\mathbb{T}} \times \Omega,$$

where $\phi \in C^2([\rho(a), \delta(b)]_{\mathbb{T}} \times \Omega)$ and is such that $\frac{\partial^2}{\partial y^2}\phi(t,y) \geq 0$ on $[\rho(a), \delta(b)]_{\mathbb{T}} \times \Omega$.

For example, let $M = max\{|f_{yy}(t,y)| : (t,y) \in [\rho(a), \delta(b)]_{\mathbb{T}} \times \Omega\}$, then we can choose $\phi = \frac{M}{2}y^2$. Clearly $\frac{\partial^2}{\partial y^2}[f(t,y) + \phi(t,y)] \geq 0$ on $[\rho(a), \delta(b)]_{\mathbb{T}} \times \Omega$.
Define $F : [\rho(a), \delta(b)]_T \times C[\rho(a), \delta(b)]_{\mathbb{T}} \to C[\rho(a), \delta(b)]_{\mathbb{T}}$ by $F(t,y) = f(t,y) + \phi(t,y)$, then $F \in C^2([\rho(a), \delta(b)]_{\mathbb{T}} \times \mathbb{R}$ and

$$\frac{\partial^2}{\partial y^2}F(t,y) \geq 0 \text{ on } [\rho(a), \delta(b)]_{\mathbb{T}} \times \Omega. \tag{4.3.1}$$

Theorem 4.3.1. *Assume that*

(1) *α, β are lower and upper solutions of the BVP* (4.0.1) *such that $\alpha \leq \beta$ on $[\rho(a), \delta(b)]_{\mathbb{T}}$.*

(2) *$f \in C^2([\rho(a), \delta(b)]_{\mathbb{T}} \times \mathbb{R})$ and f is decreasing in y for each $t \in [\rho(a), \delta(b)]_{\mathbb{T}}$. Then there exists a monotone sequence $\{w_n\}$ of linear solutions converging uniformly to a unique solution of the BVP* (4.0.1).

Proof. By the comparison and existence theorems (4.1.1) and (4.2.1), the conditions (1) and (2) ensure the existence of a unique solution y of the BVP (4.0.1) such that

$$\alpha(t) \leq y(t) \leq \beta(t), t \in [\rho(a), \delta(b)]_{\mathbb{T}}.$$

In view of (4.3.1), we obtain

$$F(t,y) \geq F(t,z) + F_y(t,z)(y - z) \text{ on } [\rho(a), \delta(b)]_{\mathbb{T}} \times \Omega,$$

which implies that

$$f(t,y) \geq f(t,z) + (y-z)F_y(t,z) - [\phi(t,y) - \phi(t,z)]. \tag{4.3.2}$$

The mean value theorem and the fact that ϕ_y is increasing in y on $[\rho(a),\delta(b)]_{\mathbb{T}} \times \Omega$, yield

$$\phi(t,y) - \phi(t,z) = \phi_y(t,c)(y-z) \leq \phi_y(t,\beta(t))(y-z), \tag{4.3.3}$$

where $z \leq c \leq y$. Using (4.3.3) in (4.3.2), we have

$$f(t,y) \geq f(t,z) + [F_y(t,z) - \phi_y(t,\beta(t))](y-z).$$

Define

$$g(t,y,z) = f(t,z) + (y-z)F_y(t,z) - [\phi(t,y) - \phi(t,z)].$$

We note that $g(.,y,z)$ is rd-continuous on $[\rho(a),\delta(b)]_{\mathbb{T}}$ for each $(y,z) \in \mathbb{R} \times \mathbb{R}$ and $g(t,.,.)$ is continuous for each $t \in [\rho(a),\delta(b)]_{\mathbb{T}}$. Moreover, g satisfies:

$$g_y(t,y,z) = F_y(t,z) - \phi_y(t,\beta(t)) \leq F_y(t,z) - \phi_y(t,z) = f_y(t,z) \leq 0$$

and

$$\begin{cases} f(t,y) \geq g(t,y,z), & y \geq z; \\ f(t,y) = g(t,y,y), \end{cases} \tag{4.3.4}$$

on $[\rho(a),\delta(b)]_{\mathbb{T}} \times \Omega$.

Now, we develop the iterative scheme to approximate the solution. As an initial approximation, we choose $w_0 = \alpha$ and consider the linear BVP

$$\begin{cases} -y^{[\triangle]\nabla}(t) + q(t)y(t) = g(t,y(t),w_0(t)), & t \in [a,b]_{\mathbb{T}}; \\ c_1 y(\rho(a)) - c_2 y^{[\Delta]}(\rho(a)) = 0, \\ d_1 y(b) + d_2 y^{[\Delta]}(b) = 0. \end{cases} \tag{4.3.5}$$

Using (4.3.4) and the definitions of lower and upper solutions, we get

$$g(t,w_0(t),w_0(t)) = f(t,w_0(t)) \geq -w_0^{[\triangle]\nabla}(t) + q(t)w_0(t),\ t \in [a,b]_{\mathbb{T}},$$

$$g(t,\beta(t),w_0(t)) \leq f(t,\beta(t)) \leq -\beta^{[\triangle]\nabla}(t)+q(t)\beta(t),\ t\in[a,b]_{\mathbb{T}},$$

which imply that w_0 and β are lower and upper solutions of (4.3.4) respectively. Hence by Theorems (4.1.1) and (4.2.1), there exists a unique solution $w_1 \in C^2_{rd}[\rho(a),\delta(b)]_{\mathbb{T}}$ of (4.3.4) such that

$$w_0(t) \leq w_1(t) \leq \beta(t),\ t\in[\rho(a),\delta(b)]_{\mathbb{T}}.$$

Using (4.3.4) and the fact that w_1 is a solution of (4.3.4), we obtain

$$\begin{cases} -w_1^{[\triangle]\nabla}(t)+q(t)w_1(t) = g(t,w_1(t),w_0(t)) \leq f(t,w_1(t)), \quad t\in[a,b]_{\mathbb{T}};\\ c_1w_1(\rho(a)) - c_2w_1^{[\Delta]}(\rho(a)) = 0,\\ d_1w_1(b)+d_2w_1^{[\Delta]}(b) = 0, \end{cases} \tag{4.3.6}$$

which implies that w_1 is a lower solution of (4.0.1). Using (4.3.4),(4.3.6) and the definition of upper solution, we have

$$g(t,w_1(t),w_1(t)) = f(t,w_1(t)) \geq -w_1^{[\Delta]\nabla}(t)+q(t)w_1(t), t\in[a,b]_{\mathbb{T}}$$

and

$$g(t,\beta(t),w_1(t)) \leq f(t,\beta(t)) \leq -\beta^{[\Delta]\nabla}(t)+q(t)\beta(t), t\in[a,b]_{\mathbb{T}},$$

which implies that w_1 and β are lower and upper solutions of

$$\begin{cases} -y^{[\triangle]\nabla}(t)+q(t)y(t) = g(t,y(t),w_1(t)), \quad t\in[a,b]_{\mathbb{T}};\\ c_1y(\rho(a)) - c_2y^{[\Delta]}(\rho(a)) = 0,\\ d_1y(b)+d_2y^{[\Delta]}(b) = 0. \end{cases} \tag{4.3.7}$$

Hence by Theorems (4.1.1) and (4.2.1), there exists a unique solution $w_2 \in C^2_{rd}[\rho(a),\delta(b)]_{\mathbb{T}}$ of (4.3.7) such that

$$w_1(t) \leq w_2(t) \leq \beta(t) \quad on \quad [\rho(a),\delta(b)]_{\mathbb{T}}.$$

Continuing in the above fashion, we obtain a bounded monotone sequence $\{w_n\}$ of solutions of linear problems satisfying

$$w_0(t) \leq w_1(t) \leq w_2(t) \leq w_3(t) \leq ... \leq w_n(t) \leq \beta(t),\ t\in[\rho(a),\delta(b)]_{\mathbb{T}},$$

where the element w_n of the sequence $\{w_n(t)\}$, is a solution of the linear problem

$$\begin{cases} -y^{[\triangle]\nabla}(t) + q(t)y(t) = g(t, y(t), w_{n-1}(t)), & t \in [a,b]_{\mathbb{T}}; \\ c_1 y(\rho(a)) - c_2 y^{[\Delta]}(\rho(a)) = 0, \\ d_1 y(b) + d_2 y^{[\Delta]}(b) = 0 \end{cases}$$

and is given by

$$w_n(t) = \int_{\rho(a)}^{b} G(t,s) g(s, w_n(s), w_{n-1}(s)) \Delta s.$$

Since $[\rho(a), \delta(b)]_{\mathbb{T}}$ is compact and the convergence is monotone and bounded, $\{w_n(t)\}$ converges uniformly to some function y, see [1,18]. Note that

$$g(t, w_n(t), w_{n-1}(t)) \to g(t, y(t), y(t)) = f(t, y(t)) \text{ as } n \to \infty.$$

Passing to the limit, we obtain

$$y(t) = \int_{\rho(a)}^{b} G(t,s) f(s, y(s)) \Delta s,$$

that is, y is a solution of (4.0.1). $\square$

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