Iota energy of some subclasses of digraphs and signed digraphs

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To
Mama and Baba

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Abstract

Gutman introduced the notion of energy of a graph in 1978 which turned into one of the fundamental concepts of spectral graph theory. An extension of the notion of graph energy to digraphs and sidigraphs was made in 2008 and 2014, respectively.

The adjacency matrix of both digraphs and sidigraphs are not necessarily symmetric, therefore their eigenvalues may lie in the complex plane. The iota energy of digraphs is defined as the sum of absolute values of the imaginary part of the eigenvalues of its adjacency matrix. The iota energy of sidigraphs is defined analogously. The extremal iota energy of unicyclic digraphs of fixed order is known. In this thesis, we take the set of those digraphs (respectively, sidigraphs) of fixed order which have two linear subdigraphs of equal length and find the digraphs (respectively, sidigraphs) in this class with smallest and largest iota energy. Moreover, we define a new class of tricyclic digraphs of fixed order which have five linear subdigraphs such that one of the directed cycles does not share any vertex with the other two directed cycles and the remaining two directed cycles are of same length sharing at least one vertex. We find the digraphs in this class with smallest and largest values of iota energy. Furthermore, we consider a similar class of n-vertex tricyclic sidigraphs and find sidigraphs in this class with extremal iota energy.

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Chapter 1

Introduction to graph theory

This chapter gives some basic concepts of graph theory accompanied by few examples in order to make these concepts comprehensible. The objective is to make the reader familiar with the fundamental properties of graphs and their matrices.

1.1 Introduction of the theory of graphs

In 1736, the concept of graph theory was originated from the Königsberg's bridge problem. The city of Königsberg, situated in Prussia, consisted of four regions linked by seven bridges. The problem involved the traversal of all seven bridges exactly once in a single round trip. Euler came up with a model of the city and showed that no such route exist.

A graph G is an ordered triple $(\mathcal{V}, \mathcal{E}, \psi_G)$ containing a finite set of vertices $\mathcal{V}$, a set of edges $\mathcal{E}$ and an incidence function ψ_G that associates with each edge two vertices (not necessarily distinct). For simplicity of notation, we denote a graph by $G = (\mathcal{V}, \mathcal{E})$. The order and size of a graph G are the cardinalities $|\mathcal{V}|$ and $|\mathcal{E}|$, respectively. An edge whose one end point is v and other end point is w is denoted by vw or wv. If v and w are the endpoints of an edge, then they are adjacent. The vertex $v \in \mathcal{V}$ and edge $e \in \mathcal{E}$ are incident if v is an endpoint of e. An isolated vertex is the vertex which has no edge incident on it and a vertex is called a leaf if it is adjacent to exactly one vertex.

An edge is said to be a loop if its endpoints are same and multiple edges are those two edges which have same pair of endpoints. A graph that has at least one loop is called a pseudograph, otherwise multigraph. If a graph has neither loops nor multiple edges, then it is called a simple graph. The set of all vertices of $G = (\mathcal{V}, \mathcal{E})$ adjacent to $v \in \mathcal{V}$ is called the neighbourhood $N(v)$ of the vertex v. The cardinality $|N(v)|$ is known as the degree $d(v)$ of a vertex v. A loop adds two to the degree of any vertex. A subgraph of a graph $G = (\mathcal{V}, \mathcal{E})$ is a graph $H = (\mathcal{C}, \mathcal{D})$ such that $\mathcal{C} \subseteq \mathcal{V}$ and $\mathcal{D} \subseteq \mathcal{E}$.

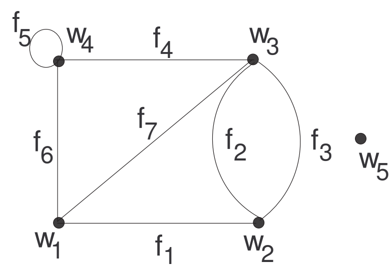


Figure 1.1: A pseudograph $G = (\mathcal{V}, \mathcal{E})$

Example 1.1. The graph $G = (\mathcal{V}, \mathcal{E})$ shown in Figure 1.1 has vertex set $\mathcal{V} = \{w_1, w_2, w_3, w_4, w_5\}$ and edge set $\mathcal{E} = \{f_1, f_2, f_3, f_4, f_5, f_6, f_7\}$. The vertices w_2 and w_3 have a multiple edge between them and the vertex w_4 has a loop. The vertex w_5 is an isolated vertex since there is no edge incident on it. The neighbourhood of w_1 in this graph is $N(w_1) = \{w_2, w_3, w_4\}$.

A walk is a finite alternating list $v_0 e_1 v_1 e_2 v_2 \ldots e_n v_n$ of vertices and edges such that, for $1 \leq i \leq n$, the endpoints of e_i are v_{i-1} and v_i. A v, w-walk is a walk with v and w as its first and last vertices. The edges and vertices in a walk are not necessarily distinct. If the edges $e_1, \ldots, e_n$ of a walk are distinct, then the walk is said to be a trail and a trail without the repetition of its vertices is called a path P_n. A v, w-path can be defined analogously. A trail whose first and last vertices are same is a closed trail and is called a circuit. Moreover, a cycle C_n is a circuit with no repeated vertex, except the first and last vertex. The length of a walk, trail, path or cycle is its number of edges. A graph G is connected if it has a v, w-path whenever $v, w \in \mathcal{V}$. A maximal connected subgraph of G is called a component of a graph G. A complete graph is a simple graph having pairwise adjacent vertices, denoted by K_n. A graph which is either K_2 or C_n $(n \geq 3)$ is called an elementary figure. A graph whose each component is an elementary figure is said to be a basic figure.

The adjacency matrix of a simple graph $G = (\mathcal{V}, \mathcal{E})$ of order n is a square and symmetric matrix $A(G) = [a_{ij}]$, where

$$a_{ij} = \begin{cases} 1 & \text{if } v_i v_j \in \mathcal{E} \\ 0 & \text{otherwise.} \end{cases}$$

The characteristic polynomial of a graph is the characteristic polynomial of its adjacency matrix, that is, $\Phi_G(x) = \det(xI - A(G))$, where I is the identity matrix of order $n \times n$. The zeros $x_1, \ldots, x_n$ of $\Phi_G(x)$ are called the eigenvalues of the graph G. The spectrum of a graph G, denoted by $\mathrm{Spec}(G)$, is the set of distinct eigenvalues of a graph G together with their multiplicities.

1.2 Spectral graph theory and energy of graphs

Spectral graph theory can be considered to be an endeavor to make use of linear algebra, specifically, the well-developed theory of matrices for the aim of graph theory and its applications. Two major graph theoretic researches were going on when spectral graph theory came forward in 1950s. One was Hückel molecular orbital (HMO) method, put forward by Erich Hückel in 1930, and the other was the study on relationship between structural and spectral properties of graphs. The monograph Spectra of Graphs by Cvetković et al. [7] discovered the links between these two lines in 1980s. This monograph was updated in 1988 under the title Recent Results in the Theory of Graph Spectra [5]. The third edition of this monograph came out in 1995 [6]. From that point forward the theory has progressed to a more noteworthy expand and many research papers have been published.

Energy of a graph is among the richest concepts in spectral graph theory. The notion of graph energy was put forward by Gutman [13] in 1978. Let $x_1, \ldots, x_n$ be the eigenvalues of a graph G, then the energy of G is defined by:

$$E(G) = \sum_{k=1}^{n} |x_k|.$$

Recall that a simple graph G of order n has a symmetric adjacency matrix $A(G)$ of order $n \times n$. The zeros $x_1, \ldots, x_n$ of the characteristic polynomial $\Phi_G(x)$ of G are the eigenvalues of G. The information on the structure of the graph can be obtained by the coefficients of the characteristic polynomial, as should be obvious in the following theorem called the coefficient theorem for graphs.

Theorem 1.2 (Cvetković et al. [7]). *Let $\Phi_G(x) = x^n + \sum_{k=1}^{n} a_k x^{n-k}$ be the characteristic polynomial of a graph G of order n. Then*

$$a_k = \sum_{L \in \mathcal{L}_k} (-1)^{p(L)} 2^{|c(L)|},$$

for all $k = 1, \ldots, n$, where $\mathcal{L}_k$ is the set of all basic figures L of G of order k, $p(L)$ denotes number of components of L and $c(L)$ denotes the set of all cycles of L.

1.3 Signed graphs

An ordered pair $S = (G, \sigma)$ is called a signed graph (or briefly sigraph), where $G = (\mathcal{V}, \mathcal{E})$ is the underlying graph of S and $\sigma : \mathcal{E} \to \{-1, 1\}$ is the signing function. An edge of S with a $+1$ sign (respectively, -1 sign) is called a positve edge (respectively, negative edge). Denote by $\mathcal{E}^+$ and $\mathcal{E}^-$ respectively the sets of positive and negative edges of S. Thus, $\mathcal{E} = \mathcal{E}^+ \cup \mathcal{E}^-$ is the set of signed edges of S. Throughout this dissertation, positive edges (respectively, negative edges) are denoted by solid lines (respectively, dotted lines).

The product of the signs of the edges of a sigraph is called the sign of a sigraph. If the sign of a sigraph is positive (respectively, negative), then the sigraph is said to be positive(respectively, negative). If each cycle of a sigraph is positive, then the sigraph is said to be balanced, otherwise unbalanced. The sigraph obtained by negating each edge of S is denoted by $-S$ and is called the negative of S. The sigraph S in Figure 1.2 is unbalanced having two negative cycles.

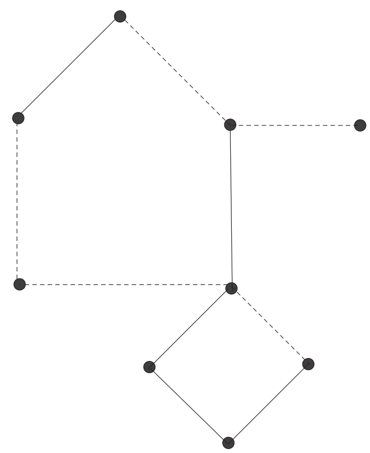


Figure 1.2: Unbalanced sigraph with two negative cycles

The adjacency matrix of an n-vertex sigraph $S = (G, \sigma)$, where $G = (\mathcal{V}, \mathcal{E})$, is a square and symmetric matrix $A(S) = [s_{ij}]$ such that

$$s_{ij} = \begin{cases} \sigma(v_i v_j) & \text{if } v_i v_j \in \mathcal{E} \\ 0 & \text{otherwise.} \end{cases}$$

The characteristic polynomial $\Phi_S(x) = \det(xI - A(S))$ of a sigraph S is the characteristic polynomial of its adjacency matrix $A(S)$. The zeros of $\Phi_S(x)$ are called the eigenvalues of S. The spectrum of S is the list of distinct eigenvalues of S with their multiplicities.

Energy in sigraphs was introduced by Germina et al. [12] and is defined to be the sum of absolute values of eigenvalues of S. The coefficient theorem for sigraphs is as follows:

Theorem 1.3 (Acharya [1]). *Let $\Phi_S(x) = x^n + \sum_{k=1}^{n} c_k(S)x^{n-k}$ be the characteristic polynomial of a sigraph S. Then*

$$c_k(S) = \sum_{L \in \mathcal{L}_k} (-1)^{p(L)} 2^{|c(L)|} \prod_{Z \in c(L)} s(Z),$$

for all $k = 1, \ldots, n$, where $\mathcal{L}_k$ is the set of all basic figures L of S of order k, $p(L)$ denotes the number of components of L, $c(L)$ denotes the set of all cycles of L and $s(Z)$ the sign of cycle Z.

1.4 Directed graphs

A pair $D = (\mathcal{V}, \mathcal{A})$ is called a directed graph (or briefly digraph), where $\mathcal{V}$ denotes the set of vertices and $\mathcal{A}$ is the set of arcs. An arc from a vertex v to a vertex w is represented by vw and we say that the vertices v and w are adjacent. A loop is an arc whose endpoints are same. Throughout this thesis, the digraphs are assumed to be without loops.

The number of arcs entering (respectively, leaving) a vertex v in a digraph D is called the indegree $d^-(v)$ (respectively, outdegree $d^+(v)$) of the vertex v. A digraph is said to be a linear digraph if every vertex has both indegree and outdegree equal to one. A directed path of length $n-1$ $(n \geq 2)$, denoted by P_n, is a digraph with n vertices $\{v_k \mid k = 1, \ldots, n\}$ and $n-1$ arcs $\{v_k v_{k+1} \mid k = 1, \ldots, n-1\}$. A directed cycle of length n $(n \geq 2)$, denoted by C_n, is a digraph with the vertex set $\{v_k \mid k = 1, \ldots, n\}$ and arc set $\{v_k v_{k+1} \mid k = 1, \ldots, n-1\} \cup \{v_n v_1\}$.

A digraph is said to be a connected digraph if its underlying graph is connected. A digraph $D = (\mathcal{V}, \mathcal{A})$ is called a strongly connected digraph if for every pair $v, w \in \mathcal{V}$, there is a directed path from v to w and one from w to v. The maximal connected subdigraphs of a digraph are known as the components of a digraph. The maximal strongly connected subdigraphs of a digraph are the strong components of the digraph. A digraph containing no directed cycle is called an acyclic digraph. A digraph $D = (\mathcal{V}, \mathcal{A})$ is unicyclic if $|\mathcal{A}| = |\mathcal{V}|$ and there is a unique directed cycle. A digraph $D = (\mathcal{V}, \mathcal{A})$ is bicyclic if $|\mathcal{A}| = |\mathcal{V}| + 1$ and there are two directed cycles. A digraph $D = (\mathcal{V}, \mathcal{A})$ is tricyclic if $|\mathcal{A}| = |\mathcal{V}| + 2$ and there are three directed cycles. Two directed cycles that share at least one vertex are said to be joined directed cycles, otherwise disjoint directed cycles.

The adjacency matrix of an n-vertex digraph $D = (\mathcal{V}, \mathcal{A})$ is an $n \times n$ matrix $A(D) =$

$[a_{jk}]$ defined by:

$$a_{jk} = \begin{cases} 1 & \text{if } v_j v_k \in \mathcal{A} \\ 0 & \text{otherwise.} \end{cases}$$

The characteristic polynomial $\det(xI - A(D))$ of the digraph D is denoted by $\Phi_D(x)$. The zeros $x_1, \ldots, x_n$ of $\Phi_D(x)$ are the eigenvalues of digraph D. The eigenvalues of digraphs may lie in the complex plane because the adjacency matrix of a digraph is not necessarily symmetric. The multiset of eigenvalues of D is called the spectrum of D and is denoted by $\mathrm{Spec}(D)$.

1.5 Signed digraphs

An ordered pair $S = (D, \sigma)$ is said to be a signed digraph (or briefly sidigraph), where $D = (\mathcal{V}, \mathcal{A})$ is a digraph called the underlying digraph of S and $\sigma : \mathcal{A} \to \{-1, 1\}$ is known as the signing function. $\mathcal{A}^+$ and $\mathcal{A}^-$ respectively denote the sets of positive and negative arcs of S. Thus, $\mathcal{A} = \mathcal{A}^+ \cup \mathcal{A}^-$ is the set of signed arcs of S. Throughout this thesis, positive arcs (respectively, negative arcs) are denoted by solid lines (respectively, dotted lines).

An arc of a sidigraph S from v to w is represented by vw and we say that the vertices v and w are adjacent. A signed directed path of length $n-1$ $(n \geq 2)$, denoted by P_n, is a sidigraph on n vertices $\{v_k \mid k = 1, \ldots, n\}$ with $n-1$ signed arcs $\{v_k v_{k+1} \mid k = 1, \ldots, n-1\}$. A signed directed cycle of length n $(n \geq 2)$, denoted by C_n, is a sidigraph having vertices $\{v_k \mid k = 1, \ldots, n\}$ and signed arcs $\{v_k v_{k+1} \mid k = 1, \ldots, n-1\} \cup \{v_n v_1\}$. If each vertex of a sidigraph has both indegree and outdegree equal to one, then it is called a linear sidigraph. The product of the signs of the arcs of a sidigraph is called the sign of a sidigraph. If the sign of a sidigraph is positive (respectively, negative), then the sidigraph is said to be positive (respectively, negative). If each signed directed cycle of a sidigraph is positive, then sidigraph is called cycle-balanced, otherwise non cycle-balanced. A sidigraph $S = (D, \sigma)$ is called a strongly connected sidigraph if D is strongly connected. A sidigraph $S = (D, \sigma)$ is unicyclic (respectively, bicyclic) if D is unicyclic (respectively, bicyclic). Similarly, a sidigraph $S = (D, \sigma)$ is tricyclic if D is tricyclic. Two signed directed cycles that share at least one vertex are said to be joined signed directed cycles, otherwise disjoint signed directed cycles.

The adjacency matrix of a sidigraph $S = (D, \sigma)$, where $D = (\mathcal{V}, \mathcal{A})$ and $\mathcal{V} = \{v_1, \dots, v_n\}$, is the $n \times n$ matrix $A(S) = [s_{ij}]$, where

$$s_{ij} = \begin{cases} \sigma(v_i v_j) & \text{if } v_i v_j \in \mathcal{A} \\ 0 & \text{otherwise.} \end{cases}$$

The characteristic polynomial $\det(xI - A(S))$ of a sidigraph S is denoted by $\phi_S(x)$. The zeros of $\phi_S(x)$ are called the eigenvalues of S. Note that the adjacency matrix $A(S)$ is not necessarily a symmetric matrix. Hence, the eigenvalues of S may lie in the complex plane. The multiset of the eigenvalues of S form the spectrum $\mathrm{spec}(S)$ of S.

1.6 Overview

The arrangement of this exposition is as follows:

In Chapter 2, we survey some fundamental results associated with energy of digraphs and sidigraphs. These results have been obtained from [1, 3, 7, 10, 20, 21].

In Chapter 3, we find smallest and largest iota energy among all n-vertex digraphs containing two linear subdigraphs of equal length. We define a similar class for sidigraphs and find sidigraphs in this class with extremal iota energy.

In Chapter 4, we define a class of n-vertex tricyclic digraphs containing five linear subdigraphs such that one of the directed cycles does not share any vertex with the other two directed cycles and the remaining two directed cycles are of same length sharing at least one vertex. We find the digraphs in this class with smallest and largest iota energy. Moreover, we introduce a similar class of n-vertex tricyclic sidigraphs and find extremal iota energy among the sidigraphs in this class.

Chapter 2

Energy of digraphs and sidigraphs

The energy of a simple graph G of order n was introduced quite some time ago by Gutman [13] as $E(G) = \sum\limits_{k=1}^{n} \left|\lambda_k\right|$, where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of G. This concept of graph energy was extended to digraphs by Peña and Rada [20]. The energy of a sigraph was introduced by Germina et al. [12]. A generalization of the digraph energy to sidigraphs was proposed by Pirzada and Bhat [21]. This chapter includes some known results associated with the energy of digraphs and sidigraphs.

2.1 Energy of digraphs

The following theorem gives the coefficients of the characteristic polynomial of digraphs.

Theorem 2.1 (Cvetković et al. [7])**.** *Let $\Phi_D(x) = x^n + \sum\limits_{k=1}^{n} b_k x^{n-k}$ be the characteristic polynomial of a digraph D. Then*

$$b_k = \sum_{L \in \mathcal{L}_k} (-1)^{comp(L)},$$

for every $k = 1, \ldots, n$, where $\mathcal{L}_k$ denotes the set of all linear subdigraphs L of D with exactly k vertices and $comp(L)$ is the number of components of L.

Recall that the adjacency matrix $A(D)$ of a digraph D is not necessarily symmetric, therefore its eigenvalues may lie in the complex plane. Thus, the energy of a digraph D

of order n is defined by:

$$E(D) = \sum_{k=1}^{n} \big|\mathrm{Re}(y_k)\big|,$$

where $y_1, \ldots, y_n$ are eigenvalues of D and $\mathrm{Re}(y_k)$ is the real part of the eigenvalue y_k [20].

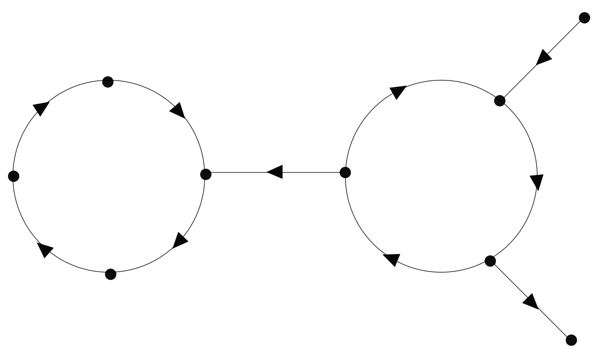


Figure 2.1: A digraph with two directed cycles

Example 2.2. Consider a digraph D shown in Figure 2.1. From Theorem 2.1, we obtain

$$\begin{aligned}\Phi_D(x) &= x^9 - x^6 - x^5 + x^2\\ &= x^2(x+1)(x-1)^2(x^2+1)(x^2+x+1).\end{aligned}$$

The spectrum of D is as follows:

$$\mathrm{Spec}(D) = \left\{0^2, -1, 1^2, \pm i, \frac{-1 \pm i\sqrt{3}}{2}\right\}$$

Hence, $E(D) = 4$.

One can easily find the energy of a digraph without calculating the zeros of its characteristic polynomial with the help of Coulson's integral formula which is stated as follows:

Theorem 2.3 (Peña and Rada [20])**.** *Let D be a digraph with characteristic polynomial $\Phi_D(x)$ and eigenvalues $y_1, \ldots, y_n$, then*

$$E(D) = \sum_{k=1}^{n} |\mathrm{Re}(y_k)| = \frac{1}{\pi}\ p.v \int_{-\infty}^{+\infty} \left(n - \frac{ix\Phi'_D(ix)}{\Phi_D(ix)}\right) dx,$$

where $\mathrm{Re}(y_k)$ denotes the real part of the eigenvalue y_k.

A direct consequence of Theorem 2.3 is the following corollary.

Corollary 2.4 (Peña and Rada [20]). *Let Φ be a monic polynomial of degree n with zeros $y_1, \ldots, y_n$ and define $\gamma(t) = t^n \Phi(\frac{i}{t})$. Then*

$$\sum_{k=1}^{n} |\mathrm{Re}(y_k)| = \frac{1}{\pi} \; p.v \; \int_{-\infty}^{+\infty} \log|\gamma(t)| \frac{dt}{t^2},$$

where $\mathrm{Re}(y_k)$ is the real part of y_k.

Recall that a digraph $D = (\mathcal{V}, \mathcal{A})$ is said to be a strongly connected digraph if there exists a directed path from v to w and one from w to v for every pair $v, w \in \mathcal{V}$. The maximal connected subdigraphs of a digraph are the components of the digraph. The maximal strongly connected subdigraphs of a digraph are the strong components of the digraph. The energy of a digraph is related to the energy of its strong components in the next theorem.

Theorem 2.5 (Pena and Rada [20]). *If a digraph D has strong components $D_1, \ldots, D_r$. Then*

$$E(D) = \sum_{i=1}^{r} E(D_i).$$

In the theory of digraph energy, one of the basic problems is to find smallest and largest energy over a particular set of digraphs. The extremal energy in the class of unicyclic digraphs of fixed order are found by Peña and Rada [20] as follows:

Theorem 2.6 (Pena and Rada [20]). *A digraph containing C_2, C_3 or C_4 has the smallest energy among all n-vertex unicyclic digraphs. The largest energy is obtained in the cycle C_n of length n.*

Farooq et al. [10] considered the set of those digraphs which contain two linear subdigraphs of equal length, say p, and denoted it by $\mathcal{D}_n$. The authors gave the following energy formulas for a digraph $H \in \mathcal{D}_n$:

$$E(H) = \begin{cases} 2\sqrt[p]{2}\cot\frac{\pi}{p} & \text{if } p \equiv 0(\mathrm{mod}4) \\ 2\sqrt[p]{2}\csc\frac{\pi}{p} & \text{if } p \equiv 2(\mathrm{mod}4) \\ \sqrt[p]{2}\csc\frac{\pi}{2p} & \text{if } p \equiv 1(\mathrm{mod}2). \end{cases}$$

The lower bound for the energy of digraphs in $\mathcal{D}_n$ is mentioned below:

Lemma 2.7 (Farooq et al. [10])**.** *Let $n \geq 3$. Take a digraph $H \in \mathcal{D}_n$ whose both linear subdigraphs are of length p such that $2 \leq p \leq n-1$. Then*

$$E(H) \geq \begin{cases} \sqrt[p]{2}\left(\frac{2p}{\pi} - \frac{\pi}{p}\right) & \text{if } p \equiv 0(\text{mod}4) \\ \sqrt[p]{2}\left(\frac{2p}{\pi}\right) & \text{otherwise.} \end{cases}$$

The upper bound for the energy of digraphs in $\mathcal{D}_n$ is as follows:

Lemma 2.8 (Farooq et al. [10])**.** *Let $n \geq 3$. Take a digraph $H \in \mathcal{D}_n$ whose both linear subdigraphs are of length p such that $2 \leq p \leq n-1$. Then*

$$E(H) \leq \begin{cases} \sqrt[p]{2}\left(\frac{2p}{\pi}\right) & \text{if } p \equiv 0(\text{mod}4) \\ \sqrt[p]{2}\left(\frac{2p}{\pi} + \frac{2p\pi}{6p^2-\pi^2}\right) & \text{if } p \equiv 2(\text{mod}4) \\ \sqrt[p]{2}\left(\frac{2p}{\pi} + \frac{2p\pi}{24p^2-\pi^2}\right) & \text{if } p \equiv 1(\text{mod}2). \end{cases}$$

Next theorem gives smallest and largest energy in $\mathcal{D}_n$.

Theorem 2.9 (Farooq et al. [10])**.** *Let $n \geq 6$. Take a digraph $H \in \mathcal{D}_n$ whose both linear subdigraphs are of length p such that $2 \leq p \leq n-1$. Then, $H \in \mathcal{D}_n$ attains smallest energy when $p = 4$ and largest energy when $p = n-1$ among all digraphs of $\mathcal{D}_n$.*

2.2 Energy of sidigraphs

The coefficient theorem for sidigraphs is as follows:

Theorem 2.10 (Acharya et al. [3])**.** *Let $\phi_S(x) = x^n + \sum\limits_{i=1}^{n} c_i x^{n-i}$ be the characteristic polynomial of a sidigraph S. Then*

$$c_i = \sum_{L \in \mathcal{L}_i} (-1)^{p(L)} \prod_{Z \in c(L)} s(Z),$$

for all $i = 1, \ldots, n$, where $\mathcal{L}_i$ denotes the set of all linear subdigraphs L of S with exactly i vertices, $p(L)$ is the number of components of L, $c(L)$ is the set of all cycles of L and $s(Z)$ denotes the sign of cycle Z.

The spectral criterion for cycle-balanced sidigraphs is given by the following result.

Theorem 2.11 (Acharya [1]). *A sidigraph $S = (D, \sigma)$ is cycle-balanced if and only if S and D are cospectral.*

The adjacency matrix of a sidigraph is not necessarily symmetric, therefore its eigenvalues lie in the complex plane. For a sidigraph S of order n, the energy of S is defined by:

$$E(S) = \sum_{k=1}^{n} \left|\mathrm{Re}(y_k)\right|,$$

where $y_1, \ldots, y_n$ are eigenvalues of S and $\mathrm{Re}(y_k)$ is the real part of the eigenvalue y_k [21].

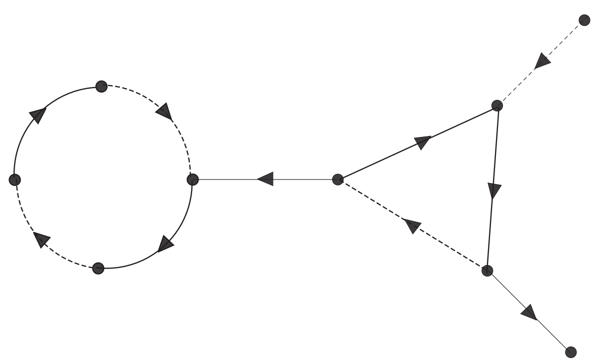


Figure 2.2: A sidigraph with two signed directed cycles

Example 2.12. Let S be the sidigraph shown in Figure 2.2. From Theorem 2.10, we obtain

$$\begin{aligned} \phi_S(x) &= x^9 + x^6 - x^5 - x^2 \\ &= x^2(x-1)(x+1)^2(x^2+1)(x^2-x+1). \end{aligned}$$

The spectrum of S is given by

$$\mathrm{Spec}(S) = \left\{0^2, 1, -1^2, \pm i, \frac{1 \pm i\sqrt{3}}{2}\right\}$$

Therefore, $E(S) = 4$.

Analogous to Theorem 2.5, Pirzada and Bhat [21] related the energy of a sidigraph with the energy of its strong components.

Theorem 2.13 (Pirzada and Bhat [21]). *Let $S_1, \dots, S_k$ be the strong components of a sidigraph S on n vertices. Then*

$$E(S) = \sum_{i=1}^{k} E(S_i).$$

Let C_k^+ and C_k^- respectively denote the positive and negative directed cycle of order k, $k \geq 2$. Pirzada and Bhat [21] calculated the following energy formulas for positive directed cycles.

$$E(C_k^+) = \begin{cases} 2\cot\frac{\pi}{k} & \text{if } k \equiv 0(\text{mod}4) \\ 2\csc\frac{\pi}{k} & \text{if } k \equiv 2(\text{mod}4) \\ \csc\frac{\pi}{2k} & \text{if } k \equiv 1(\text{mod}2). \end{cases}$$

Pirzada and Bhat [21] also derived the formulas for the energy of negative directed cycles of length k.

$$E(C_k^-) = \begin{cases} 2\csc\frac{\pi}{k} & \text{if } k \equiv 0(\text{mod}4) \\ 2\cot\frac{\pi}{k} & \text{if } k \equiv 2(\text{mod}4) \\ \csc\frac{\pi}{2k} & \text{if } k \equiv 1(\text{mod}2). \end{cases}$$

The following two theorems are the immediate consequence of the energy formulas for positive and negative sidirected cycles.

Theorem 2.14 (Pirzada and Bhat [21]). *Among all non cycle-balanced n-vertex unicyclic sidigraphs, the negative directed cycle of order 2 has minimal energy. Further, the negative directed cycle on n vertices has the maximal energy.*

Theorem 2.15 (Pirzada and Bhat [21]). *Energy of positive and negative directed cycles satisfies the following:*

(1) *Energy of the positive directed cycle of odd order equals energy of the negative directed cycle of same order.*

(2) Energy of the negative directed cycle of even order is greater than energy of the positive directed cycle of same order if and only if $n \equiv 0(\text{mod}4)$.

(3) Energy of the negative directed cycle of even order is less than energy of the positive directed cycle of same order if and only if $n \equiv 2(\text{mod}4)$.

In the theory of graph energy, the Coulson's integral formula plays a fundamental role. The Coulson's integral formula for energy of sidigraphs is given as follows:

Theorem 2.16 (Pirzada and Bhat [21]). *Let S be an n-vertex sidigraph with characteristic polynomial $\phi_S(x)$ and eigenvalues $y_1, \ldots, y_n$. Then*

$$E(S) = \sum_{k=1}^{n} |\text{Re}(y_k)| = \frac{1}{\pi}\ p.v \int_{-\infty}^{+\infty} \left(n - \frac{ix\phi_S'(ix)}{\phi_S(ix)} \right) dx,$$

where $\text{Re}(y_k)$ *denotes the real part of the eigenvalue* y_k.

A consequence of Theorem 2.16 is the following corollary.

Corollary 2.17 (Pirzada and Bhat [21]). *Let ϕ be a monic polynomial of degree n with zeros $y_1, \ldots, y_n$ and define $\gamma(t) = t^n \phi(\frac{i}{t})$. Then*

$$\sum_{k=1}^{n} |\text{Re}(y_k)| = \frac{1}{\pi}\ p.v \int_{-\infty}^{+\infty} \log|\gamma(t)| \frac{dt}{t^2},$$

where $\text{Re}(y_k)$ *is the real part of* y_k.

Recall that $S_{n,h}$ denotes the set of sidigraphs with n vertices containing cycles of length h. The following theorem gives the characteristic polynomial of a sidigraph $S \in S_{n,h}$.

Theorem 2.18 (Pirzada and Bhat [21]). *If $S \in S_{n,h}$, then*

$$\phi_S(x) = x^n + \sum_{k=1}^{\lfloor \frac{n}{h} \rfloor} (-1)^k c^*(S, kh) x^{n-kh},$$

where $c^*(S, kh)$ =number of positive linear sidigraphs of order kh − number of negative linear sidigraphs of order kh, for every $k = 1, \ldots, \lfloor \frac{n}{h} \rfloor$.

Let $S^*_{n,h} = \{S \in S_{n,h} : c^*(S, kh) \geq 0, k = 1, \ldots, \lfloor \frac{n}{h} \rfloor\}$. A reflexive and transitive relation, that is, a quasi-order relation over $S^*_{n,h}$ was defined by Pirzada and Bhat [21] as follows:

Definition 2.19 (Pirzada and Bhat [21])**.** Let S_1 and S_2 be two elements of $S^*_{n,h}$. Then $S_1 \preceq S_2$ if the following condition holds:

$$c^*(S_1, kh) \leq c^*(S_2, kh) \text{ for every } k = 1, \ldots, \lfloor \tfrac{n}{h} \rfloor.$$

If $S_1 \preceq S_2$ and there exists k such that $c^*(S_1, kh) < c^*(S_2, kh)$ then $S_1 \prec S_2$.

Clearly, $\preceq$ is a quasi-order relation over $S^*_{n,h}$. In the following theorem, the energy increases over the set $S^*_{n,h}$ with respect to quasi-order relation.

Theorem 2.20 (Pirzada and Bhat [21])**.** *If $h \equiv 2 (\mathrm{mod}\ 4)$, then the energy increases over $S^*_{n,h}$ with respect to the quasi-order relation, that is, if $S_1, S_2 \in S^*_{n,h}$ then $S_1 \prec S_2$ implies that $E(S_1) < E(S_2)$.*

For more details, see [2, 4, 11, 15, 17–19, 22, 23].

Chapter 3

Extremal iota energy of a subclass of bicyclic digraphs and sidigraphs

Khan et al. [16] introduced the notion of iota energy of digraphs and defined it as $E_c(D) = \sum_{k=1}^{n} \big|\mathrm{Im}(\lambda_k)\big|$, where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of D and $\mathrm{Im}(\lambda_k)$ is the imaginary part of the eigenvalue λ_k. The authors found the smallest and largest iota energy among all unicyclic digraphs of fixed order, $n \geq 2$. This concept of iota energy is extended to sidigraphs by Farooq et al. [9] recently. The definition of iota energy of a sidigraph is analogous to the definition of the iota energy of a digraph. In this chapter, we take a class of digraphs containing two linear subdigraphs of equal length and find digraphs in this class with extremal iota energy. Furthermore, we define a similar class of sidigraphs and find smallest and largest iota energy among the sidigraphs in this class.

3.1 Iota energy of digraphs

We begin with the following lemma useful to prove a few results in this chapter.

Lemma 3.1. *The function f defined by $f(x) = \sqrt[x]{2}\cot\frac{\pi}{2x}$ is strictly increasing on $[2, \infty)$.*

Proof. It suffices to show that $f'(x) > 0$ for all $x \in [2, \infty)$ to prove that f is strictly increasing on $[2,\infty)$. We have

$$\begin{aligned} f'(x) &= \frac{\sqrt[x]{2}\pi}{2x^2}\left(\csc^2\frac{\pi}{2x}\right)+\left(-\frac{\sqrt[x]{2}\ln 2}{x^2}\right)\cot\frac{\pi}{2x} \\ &= \frac{\sqrt[x]{2}}{x^2}\left(\frac{\pi}{2}\csc^2\frac{\pi}{2x}-\ln 2\cot\frac{\pi}{2x}\right). \end{aligned}$$

We know that $\csc\frac{\pi}{2x}>1$ and $\csc\frac{\pi}{2x}>\cot\frac{\pi}{2x}$ for each $x\in[2,\infty)$. This gives

$$\csc^2\frac{\pi}{2x}>\cot\frac{\pi}{2x},$$

for each $x\in[2,\infty)$. As $\frac{\pi}{2}>\ln 2$, we obtain

$$\frac{\pi}{2}\csc^2\frac{\pi}{2x}>\ln 2\cot\frac{\pi}{2x},$$

for each $x\in[2,\infty)$. This proves that $f'(x)>0$ for each $x\in[2,\infty)$. Thus, f is strictly increasing on $[2,\infty)$. $\square$

For $n\geq 2$, let $\mathcal{K}_n$ be a set of digraphs on n vertices such that each digraph in $\mathcal{K}_n$ has two linear subdigraphs of equal length. We denote a digraph in $\mathcal{K}_n$ by G_r, where subscript r denotes the length of each linear subdigraph such that $2\leq r\leq n$. We note that $r=n$ if and only if $n=2$. Using Theorem 2.1, we obtain

$$\Phi_{G_r}(x)=x^{n-r}(x^r-2).$$

The eigenvalues of G_r are 0 and $\sqrt[r]{2}\exp\left(\frac{2j\pi i}{r}\right)$, where $j=0,1,\ldots,r-1$ and the eigenvalue 0 has multiplicity $n-r$. Thus, the iota energy of G_r is given by

$$E_c(G_r)=\sqrt[r]{2}\sum_{j=0}^{r-1}\left|\sin\frac{2j\pi}{r}\right|. \tag{3.1}$$

It is shown in Khan et al. [16] that

$$\sum_{j=0}^{r-1}\left|\sin\frac{2j\pi}{r}\right|=\begin{cases} 2\cot\frac{\pi}{r} & \text{if } r\equiv 0(\text{mod}2) \\ \cot\frac{\pi}{2r} & \text{if } r\equiv 1(\text{mod}2). \end{cases} \tag{3.2}$$

Using (3.1) and (3.2), we get the following iota energy formulas for a digraph $G_r\in\mathcal{K}_n$:

$$E_c(G_r)=\begin{cases} 2\sqrt[r]{2}\cot\frac{\pi}{r} & \text{if } r\equiv 0(\text{mod}2) \\ \sqrt[r]{2}\cot\frac{\pi}{2r} & \text{if } r\equiv 1(\text{mod}2). \end{cases} \tag{3.3}$$

Next, we characterize the digraphs in $\mathcal{K}_n$ with smallest and largest iota energy. When $r \equiv 0(\text{mod}2)$, the following lemma demonstrates that the iota energy of digraphs in $\mathcal{K}_n$ increases as the length of their linear subdigraphs increases.

Lemma 3.2. *Let $n \geq 6$, $4 \leq r < n-1$ and $r \equiv 0(\text{mod}2)$. For $G_r, G_{r+1} \in \mathcal{K}_n$, we have*

$$E_c(G_{r+1}) > E_c(G_r).$$

Proof. As $r \equiv 0(\text{mod}2)$, we have $r+1 \equiv 1(\text{mod}2)$. Using formula (3.3), we obtain

$$\begin{aligned}
E_c(G_{r+1}) - E_c(G_r) &= \sqrt[r+1]{2}\cot\frac{\pi}{2(r+1)} - 2\sqrt[r]{2}\cot\frac{\pi}{r} \\
&= \sqrt[r+1]{2}\cot\frac{\pi}{2(r+1)} - 2\sqrt[r]{2}\left(\frac{\cos^2\frac{\pi}{2r} - \sin^2\frac{\pi}{2r}}{2\sin\frac{\pi}{2r}\cos\frac{\pi}{2r}}\right) \\
&= \sqrt[r+1]{2}\cot\frac{\pi}{2(r+1)} - \sqrt[r]{2}\left(\cot\frac{\pi}{2r} - \tan\frac{\pi}{2r}\right) \\
&= \sqrt[r+1]{2}\cot\frac{\pi}{2(r+1)} - \sqrt[r]{2}\cot\frac{\pi}{2r} + \sqrt[r]{2}\tan\frac{\pi}{2r}.
\end{aligned}$$

By using Lemma 3.1, we have

$$\sqrt[r+1]{2}\cot\frac{\pi}{2(r+1)} - \sqrt[r]{2}\cot\frac{\pi}{2r} > 0,$$

where $4 \leq r < n-1$. Also $\sqrt[r]{2}\tan\frac{\pi}{2r} > 0$, where $4 \leq r < n-1$. Thus

$$E_c(G_{r+1}) - E_c(G_r) > 0.$$

This gives the required result. $\square$

When $r \equiv 1(\text{mod}2)$, next lemma illustrates that the iota energy of digraphs in $\mathcal{K}_n$ increases as the length of their linear subdigraphs increases.

Lemma 3.3. *Let $n \geq 6$, $4 \leq r \leq n-1$ and $r \equiv 1(\text{mod}2)$. For $G_{r-1}, G_r \in \mathcal{K}_n$, we have*

$$E_c(G_r) > E_c(G_{r-1}).$$

Proof. As $r \equiv 1(\mathrm{mod}2)$, it holds that $r-1 \equiv 0(\mathrm{mod}2)$. Using formula (3.3), we obtain

$$\begin{aligned}
E_c(G_r) - E_c(G_{r-1}) &= \sqrt[r]{2}\cot\frac{\pi}{2r} - 2\sqrt[r-1]{2}\cot\frac{\pi}{r-1} \\
&= \sqrt[r]{2}\cot\frac{\pi}{2r} - 2\sqrt[r-1]{2}\left(\frac{\cos^2\frac{\pi}{2(r-1)} - \sin^2\frac{\pi}{2(r-1)}}{2\sin\frac{\pi}{2(r-1)}\cos\frac{\pi}{2(r-1)}}\right) \\
&= \sqrt[r]{2}\cot\frac{\pi}{2r} - \sqrt[r-1]{2}\left(\cot\frac{\pi}{2(r-1)} - \tan\frac{\pi}{2(r-1)}\right) \\
&= \sqrt[r]{2}\cot\frac{\pi}{2r} - \sqrt[r-1]{2}\cot\frac{\pi}{2(r-1)} + \sqrt[r-1]{2}\tan\frac{\pi}{2(r-1)}.
\end{aligned}$$

By Lemma 3.1, we have

$$\sqrt[r]{2}\cot\frac{\pi}{2r} - \sqrt[r-1]{2}\cot\frac{\pi}{2(r-1)} > 0,$$

where $4 \le r \le n-1$. Also, we know that $\sqrt[r-1]{2}\tan\frac{\pi}{2(r-1)} > 0$ for $4 \le r \le n-1$. Hence

$$E_c(G_r) - E_c(G_{r-1}) > 0.$$

This proves the assertion. □

A direct consequence of Lemma 3.2 and Lemma 3.3 is as follows:

Corollary 3.4. *Let $n \ge 6$ and $4 \le r_2 \le r_1 \le n-1$. For $G_{r_1}, G_{r_2} \in \mathcal{K}_n$, we have*

$$E_c(G_{r_1}) \ge E_c(G_{r_2}).$$

Using Corollary 3.4, we prove the following theorem.

Theorem 3.5. *Let $n \ge 2$ and $G_r \in \mathcal{K}_n$, where $2 \le r \le n$. Then G_r has smallest iota energy when $r = 2$ and largest iota energy when $r = n-1$ among all digraphs of $\mathcal{K}_n$.*

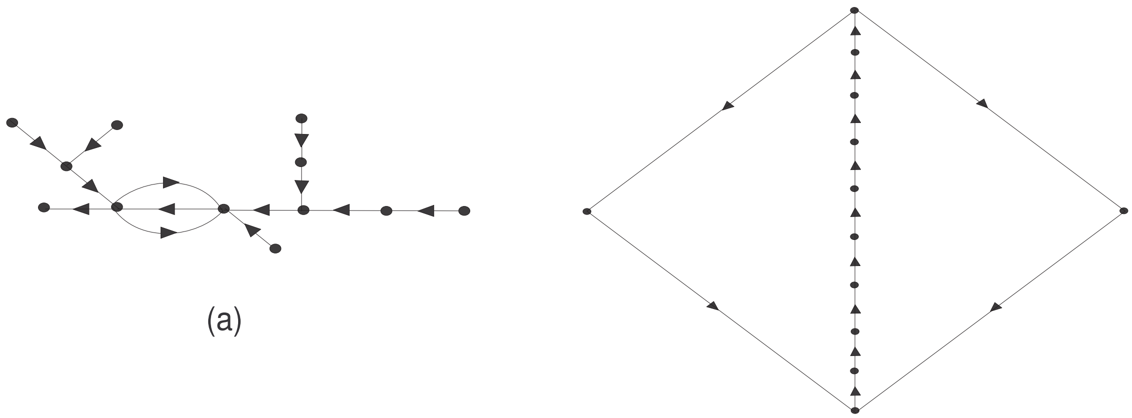
Proof. It can easily be seen from formula (3.3) that

$$E_c(G_r) = \begin{cases} 0 & \text{if } r = 2 \\ \sqrt{3}\sqrt[3]{2} & \text{if } r = 3. \end{cases} \tag{3.4}$$

Let $r \ge 4$ and $G_q \in \mathcal{K}_n$, where $q \ge r$. Then from formula (3.3) and Corollary 3.4, we obtain

$$\sqrt{3}\sqrt[3]{2} < 2\sqrt[4]{2} \le E_c(G_r) \le E_c(G_q). \tag{3.5}$$

It is easy to comprehend from (3.4) and (3.5) that $E_c(G_r)$ is smallest when $r = 2$. The case when $n = 2$ is obvious. Assume $n \geq 3$. Then, $\mathcal{K}_n$ contains a digraph with two linear subdigraphs each of length $n-1$. Therefore, from (3.5) it is clear that $E_c(G_r)$ is largest when $r = n-1$. $\square$



(a)

(b)

Figure 3.1: (a) The digraph with smallest iota energy in $\mathcal{K}_{12}$. (b) The digraph with largest iota energy in $\mathcal{K}_{12}$.

Example 3.6. The digraphs with smallest and largest iota energy in $\mathcal{K}_{12}$ are shown in Figure 3.1.

3.2 Iota energy of sidigraphs

For $n \geq 2$, let $\mathcal{Q}_n$ be a set of sidigraphs on n vertices such that each sidigraph in $\mathcal{Q}_n$ has two linear subdigraphs of equal length. Let $S_r \in \mathcal{Q}_n$, where r is the length of each linear subdigraph such that $2 \leq r \leq n$. We note that $r = n$ if and only if $n = 2$. Also for any $S_r \in \mathcal{Q}_n$, a linear subdigraph of S_r is a cycle. Let $\mathcal{Q}_n^{++} \subset \mathcal{Q}_n$ be the subset of all those sidigraphs whose both linear subdigraphs are positive. Also, let $\mathcal{Q}_n^{+-} \subset \mathcal{Q}_n$ be the subset of all those sidigraphs whose one linear subdigraph is positive and other is negative. Similarly, let $\mathcal{Q}_n^{--} \subset \mathcal{Q}_n$ be the subset of all those sidigraphs whose both linear subdigraphs are negative.

If $S_r \in \mathcal{Q}_n^{++}$ then by Theorem 2.11, we have the following result which is analogue of Theorem 3.5.

Theorem 3.7. *Let $n \geq 2$ and $S_r \in \mathcal{Q}_n^{++}$, where $2 \leq r \leq n$. Then S_r has smallest iota energy when $r = 2$ and largest iota energy when $r = n-1$ among all sidigraphs of $\mathcal{Q}_n^{++}$.*

If $S_r \in \mathcal{Q}_n^{+-}$ then from Theorem 2.10, the characteristic polynomial of S_r is given by $\phi_{S_r}(x) = x^n$. Thus

$$E_c(S_r) = 0. \tag{3.6}$$

If $S_r \in \mathcal{Q}_n^{--}$ then from Theorem 2.10, we obtain

$$\phi_{S_r}(x) = x^{n-r}(x^r + 2).$$

The eigenvalues of S_r are 0 and $\sqrt[r]{2}\exp\left(\frac{(2j+1)\pi i}{r}\right)$, where $j = 0, 1, \ldots, r-1$ and the eigenvalue 0 has multiplicity $n - r$. Thus, the iota energy of $S_r \in \mathcal{Q}_n^{--}$ is given by

$$E_c(S_r) = \sqrt[r]{2}\sum_{j=0}^{r-1} \big|\sin\frac{(2j+1)\pi}{r}\big|. \tag{3.7}$$

It is shown in Farooq et al. [9] that

$$\sum_{j=0}^{r-1} |\sin\frac{(2j+1)\pi}{r}| = \begin{cases} 2\csc\frac{\pi}{r} & \text{if } r \equiv 0(\text{mod}2) \\ \cot\frac{\pi}{2r} & \text{if } r \equiv 1(\text{mod}2). \end{cases} \tag{3.8}$$

Using (3.7) and (3.8), the iota energy formulas for $S_r \in \mathcal{Q}_n^{--}$ are given by

$$E_c(S_r) = \begin{cases} 2\sqrt[r]{2}\csc\frac{\pi}{r} & \text{if } r \equiv 0(\text{mod}2) \\ \sqrt[r]{2}\cot\frac{\pi}{2r} & \text{if } r \equiv 1(\text{mod}2). \end{cases} \tag{3.9}$$

We find the sidigraphs with extremal iota energy among the sidigraphs in $\mathcal{Q}_n^{--}$. When $r \equiv 0(\text{mod}2)$, the following lemma demonstrates that the iota energy of sidigraphs in $\mathcal{Q}_n^{--}$ increases as the length of their linear subdigraphs increases.

Lemma 3.8. *Let $n \geq 6$, $4 \leq r \leq n-1$ and $r \equiv 0(\mathrm{mod}2)$. For $S_{r-1}, S_r \in \mathcal{Q}_n^{--}$, we have*

$$E_c(S_r) > E_c(S_{r-1}).$$

Proof. As $r \equiv 0(\mathrm{mod}2)$, we have $r-1 \equiv 1(\mathrm{mod}2)$. Using formula (3.9), we obtain

$$\begin{aligned}
E_c(S_r) - E_c(S_{r-1}) &= 2\sqrt[r]{2}\csc\frac{\pi}{r} - \sqrt[r-1]{2}\cot\frac{\pi}{2(r-1)} \\
&= 2\sqrt[r]{2}\left(\frac{1}{2\sin\frac{\pi}{2r}\cos\frac{\pi}{2r}}\right) - \sqrt[r-1]{2}\cot\frac{\pi}{2(r-1)} \\
&= \sqrt[r]{2}\left(\frac{\sin^2\frac{\pi}{2r}+\cos^2\frac{\pi}{2r}}{\sin\frac{\pi}{2r}\cos\frac{\pi}{2r}}\right) - \sqrt[r-1]{2}\cot\frac{\pi}{2(r-1)} \\
&= \sqrt[r]{2}\left(\tan\frac{\pi}{2r}+\cot\frac{\pi}{2r}\right) - \sqrt[r-1]{2}\cot\frac{\pi}{2(r-1)} \\
&= \sqrt[r]{2}\tan\frac{\pi}{2r} + \sqrt[r]{2}\cot\frac{\pi}{2r} - \sqrt[r-1]{2}\cot\frac{\pi}{2(r-1)}.
\end{aligned}$$

By Lemma 3.1, we have

$$\sqrt[r]{2}\cot\frac{\pi}{2r} - \sqrt[r-1]{2}\cot\frac{\pi}{2(r-1)} > 0,$$

where $4 \leq r \leq n-1$. Also, we know that $\sqrt[r]{2}\tan\frac{\pi}{2r} > 0$ for $4 \leq r \leq n-1$. Thus

$$E_c(S_r) - E_c(S_{r-1}) > 0.$$

This gives the required result. $\square$

When $r \equiv 1(\mathrm{mod}\,2)$, next lemma illustrates that the iota energy of sidigraphs in $\mathcal{Q}_n^{--}$ increases as the length of their linear subdigraphs increases.

Lemma 3.9. *Let $n \geq 6$, $4 \leq r < n-1$ and $r \equiv 1(\mathrm{mod}2)$. For $S_r, S_{r+1} \in \mathcal{Q}_n^{--}$, we have*

$$E_c(S_{r+1}) > E_c(S_r).$$

Proof. As $r \equiv 1(\text{mod}2)$, it holds that $r+1 \equiv 0(\text{mod}2)$. Using formula (3.9), we obtain

$$\begin{aligned}
E_c(S_{r+1}) - E_c(S_r) &= 2\sqrt[r+1]{2}\csc\frac{\pi}{r+1} - \sqrt[r]{2}\cot\frac{\pi}{2r} \\
&= 2\sqrt[r+1]{2}\left(\frac{1}{2\sin\frac{\pi}{2(r+1)}\cos\frac{\pi}{2(r+1)}}\right) - \sqrt[r]{2}\cot\frac{\pi}{2r} \\
&= \sqrt[r+1]{2}\left(\frac{\sin^2\frac{\pi}{2(r+1)} + \cos^2\frac{\pi}{2(r+1)}}{\sin\frac{\pi}{2(r+1)}\cos\frac{\pi}{2(r+1)}}\right) - \sqrt[r]{2}\cot\frac{\pi}{2r} \\
&= \sqrt[r+1]{2}\left(\tan\frac{\pi}{2(r+1)} + \cot\frac{\pi}{2(r+1)}\right) - \sqrt[r]{2}\cot\frac{\pi}{2r} \\
&= \sqrt[r+1]{2}\tan\frac{\pi}{2(r+1)} + \sqrt[r+1]{2}\cot\frac{\pi}{2(r+1)} - \sqrt[r]{2}\cot\frac{\pi}{2r}.
\end{aligned}$$

By Lemma 3.1, we have

$$\sqrt[r+1]{2}\cot\frac{\pi}{2(r+1)} - \sqrt[r]{2}\cot\frac{\pi}{2r} > 0,$$

where $4 \leq r < n-1$. Also $\sqrt[r+1]{2}\tan\frac{\pi}{2(r+1)} > 0$, where $4 \leq r < n-1$. Thus

$$E_c(S_{r+1}) - E_c(S_r) > 0.$$

This proves the assertion. □

A direct consequence of Lemma 3.8 and Lemma 3.9 is as follows:

Corollary 3.10. *Let $n \geq 6$ and $4 \leq r_2 \leq r_1 \leq n-1$. For $S_{r_1}, S_{r_2} \in \mathcal{Q}_n^{--}$, we have*

$$E_c(S_{r_1}) \geq E_c(S_{r_2}).$$

Using Corollary 3.10, we prove the following theorem.

Theorem 3.11. *Let $n \geq 2$ and $S_r \in \mathcal{Q}_n^{--}$, where $2 \leq r \leq n$. Then S_r has smallest iota energy when $r = 3$ and largest iota energy when $r = n-1$ among all sidigraphs of $\mathcal{Q}_n^{--}$.*

Proof. Let $S_r \in \mathcal{Q}_n^{--}$, where $2 \leq r \leq n$. Then from formula (3.9), we have

$$E_c(S_r) = \begin{cases} 2\sqrt[2]{2} & \text{if } r = 2 \\ \sqrt{3}\sqrt[3]{2} & \text{if } r = 3. \end{cases} \tag{3.10}$$

Let $r \geq 4$ and $S_q \in \mathcal{Q}_n^{--}$, where $q \geq r$. Then from formula (3.9) and Corollary 3.10, we obtain

$$2\sqrt[2]{2} < 2^{\frac{3}{2}}\sqrt[4]{2} \leq E_c(S_r) \leq E_c(S_q). \tag{3.11}$$

It is easy to understand from (3.10) and (3.11) that $E_c(S_r)$ is smallest when $r = 3$. The case when $n = 2$ is obvious. Assume $n \geq 3$. Then, $\mathcal{Q}_n^{--}$ contains a sidigraph with two linear subdigraphs each of length $n - 1$. Therefore, from (3.11) it is clear that $E_c(S_r)$ is largest when $r = n - 1$. □

We know that $\sin y < y < \tan y$ for each $y \in (0, \frac{\pi}{2})$. This gives

$$\csc y > \cot y, \tag{3.12}$$

for each $y \in (0, \frac{\pi}{2})$. The next theorem follows from (3.6), formula (3.9), inequality (3.12), Theorem 3.7 and Theorem 3.11.

Theorem 3.12. *Let $n \geq 2$ and $S_r \in \mathcal{Q}_n$, where $2 \leq r \leq n$. Then*

(1) S_r has smallest iota energy when

(i) $S_r \in \mathcal{Q}_n^{++}$ and $r = 2$, or

(ii) $S_r \in \mathcal{Q}_n^{+-}$.

(2) S_r has largest iota energy when both linear subdigraphs are of length $r = n - 1$ such that

(i) $S_r \in \mathcal{Q}_n^{--}$ and $r \equiv 0(\mathrm{mod} 2)$, or

(ii) $S_r \in \mathcal{Q}_n^{++}$ or $S_r \in \mathcal{Q}_n^{--}$ and $r \equiv 1(\mathrm{mod} 2)$.

Example 3.13. The sidigraphs with smallest and largest iota energy in $\mathcal{Q}_{12}$ are shown in Figure 3.2.

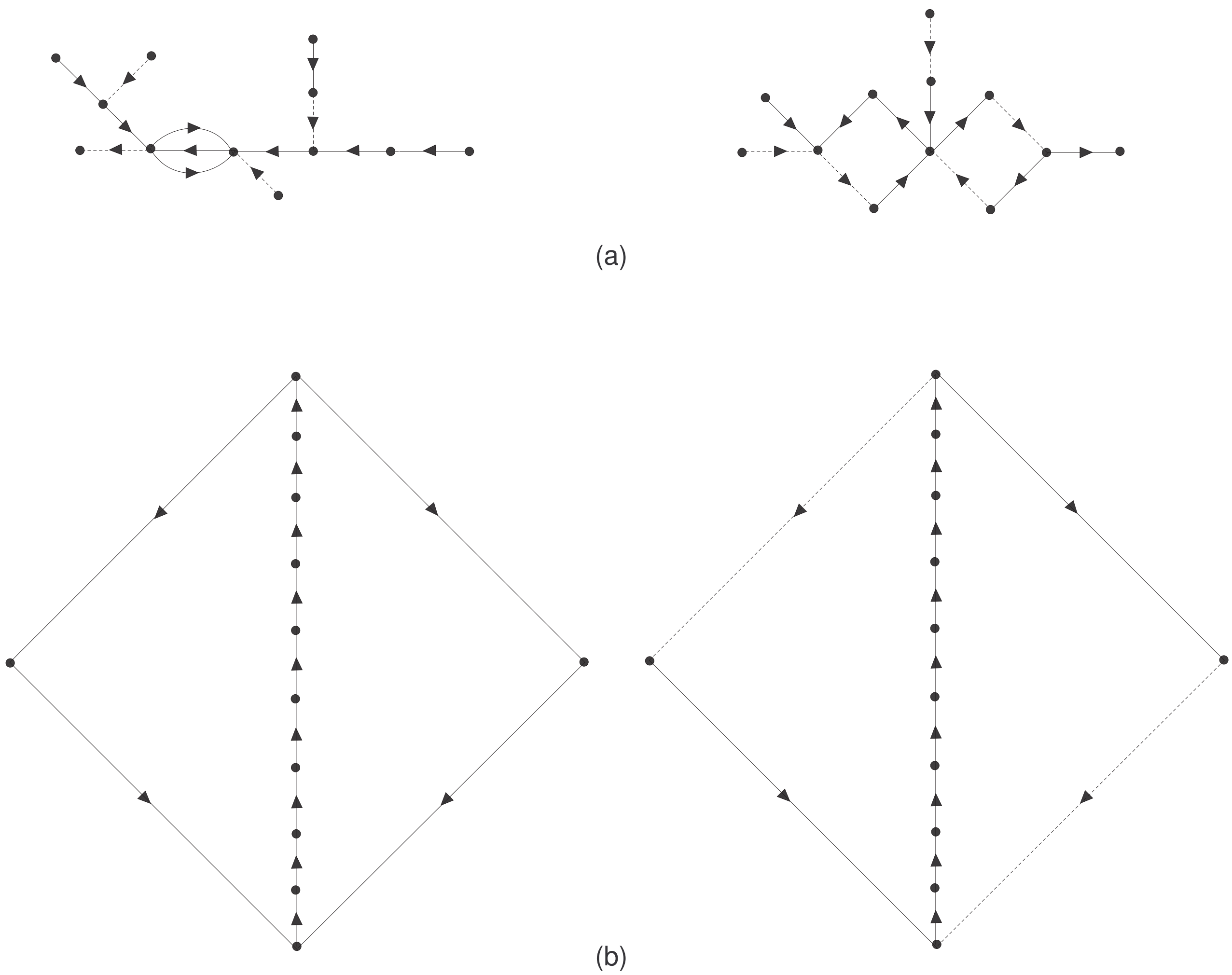


Figure 3.2: (a) The sidigraphs with smallest iota energy in $\mathcal{Q}_{12}$. (b) The sidigraphs with largest iota energy in $\mathcal{Q}_{12}$.

3.3 Conclusion

In this chapter, a class $\mathcal{K}_n$ of n-vertex digraphs containing two linear subdigraphs of equal length is defined. We showed that a digraph $G_r \in \mathcal{K}_n$ attains smallest and largest iota energy among the digraphs in $\mathcal{K}_n$ when the length of both linear subdigraphs is 2 and $n-1$, respectively. We have also introduced a class of n-vertex sidigraphs $\mathcal{Q}_n$ containing two linear subdigraphs of equal length. We found sidigraphs with smallest and largest iota energy in $\mathcal{Q}_n$. It will be a challenging problem to find digraphs (respectively, sidigraphs) with extremal iota energy among a more extensive set of bicyclic digraphs (respectively, sidigraphs) where the length of both linear subdigraphs is not necessarily equal.

Chapter 4

Extremal iota energy of a subclass of tricyclic digraphs and sidigraphs

Farooq et al. [8] found the smallest and largest iota energy among all n-vertex digraphs containing two vertex-disjoint directed cycles. In chapter 3, we computed the extremal iota energy over the class of all n-vertex digraphs which contain two linear subdigraphs of equal length. We also computed the smallest and largest iota energy among all n-vertex sidigraphs containing two linear subdigraphs of equal length. In this chapter, we define a class $\mathcal{F}_n$ of n-vertex tricyclic digraphs containing five linear subdigraphs such that one of the directed cycles does not share any vertex with the other two directed cycles and the remaining two directed cycles are of same length sharing at least one vertex. We find the digraphs in $\mathcal{F}_n$ with smallest and largest iota energy. We also consider a similar class of tricyclic sidigraphs of fixed order and find extremal values of iota energy among the sidigraphs in this class.

4.1 Known results

Following lemmas are helpful in proving a few results.

Lemma 4.1 (Hafeez and Khan [14]). *Consider the sequences* $< a_k >$ *and* $< b_k >$, *where* a_k *and* b_k *are given by*

$$a_k = \begin{cases} 2\cot\frac{\pi}{k} & \text{if } k \equiv 0(\bmod 2) \\ \cot\frac{\pi}{2k} & \text{if } k \equiv 1(\bmod 2). \end{cases}$$

and

$$b_k = \begin{cases} 2\csc\frac{\pi}{k} & \text{if } k \equiv 0(\bmod 2) \\ \cot\frac{\pi}{2k} & \text{if } k \equiv 1(\bmod 2). \end{cases}$$

Then $\{a_k\}$ *is strictly increasing for* $k \geq 4$ *and* $\{b_k\}$ *is strictly increasing for* $k \geq 3$.

Lemma 4.2 (Farooq et al. [8]). *For* $y \in (0, \frac{\pi}{2}]$, *the following inequality holds:*

$$\frac{1}{y} - 0.429y \leq \cot y \leq \frac{1}{y} - \frac{y}{3}.$$

Lemma 4.3 (Khan et al. [17]). *Let* y, p, q *be real numbers such that* $y \geq p > 0$ *and* $q > 0$. *Then*

$$\frac{\pi y}{qy^2 - \pi^2} \leq \frac{\pi p}{qp^2 - \pi^2}.$$

It is known that for any real number y with $0 \leq y \leq \frac{\pi}{2}$, the following inequalities hold:

$$\sin y \leq y, \qquad \sin y \geq y - \frac{y^3}{3!}. \tag{4.1}$$

4.2 Iota energy of digraphs

For $n \geq 4$, let $\mathcal{F}_n$ be a set of n-vertex tricyclic digraphs containing five linear subdigraphs such that one of the directed cycles does not share any vertex with the other two directed cycles and the remaining two directed cycles are of same length sharing at least one vertex. Let $D_p^m \in \mathcal{F}_n$, where superscript m denotes the length of each joined directed cycle and subscript p denotes the length of disjoint directed cycle such that $2 \leq m \leq n-2$ and $2 \leq p \leq n-2$. Using Theorem 2.1, we obtain

$$\Phi_{D_p^m}(x) = x^{n-m-p}(x^m - 2)(x^p - 1).$$

The eigenvalues of D_p^m are 0, $\sqrt[m]{2}\exp\left(\frac{2k\pi i}{m}\right)$ and $\exp\left(\frac{2j\pi i}{p}\right)$, where $k = 0, 1, \ldots, m-1$ and $j = 0, 1, \ldots, p-1$ and the eigenvalue 0 has multiplicity $n - m - p$. Thus, the iota energy of D_p^m is given by

$$E_c(D_p^m) = \sqrt[m]{2}\sum_{k=0}^{m-1}\left|\sin\frac{2k\pi}{m}\right| + \sum_{j=0}^{p-1}\left|\sin\frac{2j\pi}{p}\right|. \tag{4.2}$$

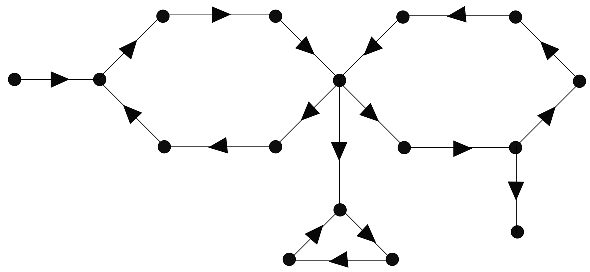


Figure 4.1: $D_3^6 \in \mathcal{F}_{16}$

Example 4.4. Consider $D_3^6 \in \mathcal{F}_{16}$ shown in Figure 4.1. By (4.2), the iota energy of D_3^6 is given by:

$$E_c(D_3^6) = \sqrt[6]{2}\sum_{k=0}^{5}\left|\sin\frac{2k\pi}{6}\right| + \sum_{j=0}^{2}\left|\sin\frac{2j\pi}{3}\right|.$$

It is shown in Khan et al. [16] that

$$\sum_{j=0}^{m-1}\left|\sin\frac{2j\pi}{m}\right| = \begin{cases} 2\cot\frac{\pi}{m} & \text{if } m \equiv 0(\text{mod}2) \\ \cot\frac{\pi}{2m} & \text{if } m \equiv 1(\text{mod}2). \end{cases} \tag{4.3}$$

Using (4.2) and (4.3), we get the following iota energy formulas for a digraph $D_p^m \in \mathcal{F}_n$:

$$E_c(D_p^m) = \begin{cases} 2\left(\sqrt[m]{2}\cot\frac{\pi}{m} + \cot\frac{\pi}{p}\right) & \text{if } m \equiv 0(\text{mod}2), p \equiv 0(\text{mod}2) \\ 2\sqrt[m]{2}\cot\frac{\pi}{m} + \cot\frac{\pi}{2p} & \text{if } m \equiv 0(\text{mod}2), p \equiv 1(\text{mod}2) \\ \sqrt[m]{2}\cot\frac{\pi}{2m} + 2\cot\frac{\pi}{p} & \text{if } m \equiv 1(\text{mod}2), p \equiv 0(\text{mod}2) \\ \sqrt[m]{2}\cot\frac{\pi}{2m} + \cot\frac{\pi}{2p} & \text{if } m \equiv 1(\text{mod}2), p \equiv 1(\text{mod}2). \end{cases} \tag{4.4}$$

Next, we characterize the digraphs in $\mathcal{F}_n$ with smallest and largest iota energy. The following lemma demonstrates an increase in the iota energy of digraphs in $\mathcal{F}_n$ with the increase in length of their disjoint directed cycles.

Lemma 4.5. *Let $n \geq 7$, $m = 2$ and $4 \leq p < n-2$. Take $D_p^2, D_{p+1}^2 \in \mathcal{F}_n$. Then*

$$E_c(D_{p+1}^2) > E_c(D_p^2).$$

Proof. Let $p \equiv 0(\text{mod}2)$, that is, $p+1 \equiv 1(\text{mod}2)$. Then by formula (4.4) and Lemma 4.1, we find

$$\begin{aligned} E_c(D_{p+1}^2) - E_c(D_p^2) &= \cot\frac{\pi}{2(p+1)} - 2\cot\frac{\pi}{p} \\ &> 0. \end{aligned} \tag{4.5}$$

Next, let $p \equiv 1(\text{mod}2)$. In this case, $p \geq 5$ and $p+1 \equiv 0(\text{mod}2)$. Using formula (4.4) and Lemma 4.1, we obtain

$$\begin{aligned} E_c(D_{p+1}^2) - E_c(D_p^2) &= 2\cot\frac{\pi}{p+1} - \cot\frac{\pi}{2p} \\ &> 0. \end{aligned} \tag{4.6}$$

From (4.5) and (4.6), we have

$$E_c(D_{p+1}^2) > E_c(D_p^2).$$

This gives the required result. □

An easy consequence of Lemma 4.5 is given below:

Corollary 4.6. *Let $n \geq 7$, $m = 2$ and $4 \leq p_2 \leq p_1 \leq n-2$. Take $D_{p_1}^2, D_{p_2}^2 \in \mathcal{F}_n$. Then*

$$E_c(D_{p_1}^2) \geq E_c(D_{p_2}^2).$$

Next lemma illustrates an increase in the iota energy of digraphs in $\mathcal{F}_n$ with the increase in length of their joined directed cycles when $m \equiv 0(\text{mod}2)$.

Lemma 4.7. *Let $n \geq 7$, $p = 2$, $4 \leq m < n-2$ and $m \equiv 0(\bmod 2)$. Take $D_2^m, D_2^{m+1} \in \mathcal{F}_n$. Then*

$$E_c(D_2^{m+1}) > E_c(D_2^m).$$

Proof. As $m \equiv 0(\mathrm{mod} 2)$, we have $m+1 \equiv 1(\mathrm{mod} 2)$. Then by formula (4.4), we obtain

$$E_c(D_2^{m+1}) - E_c(D_2^m) = \sqrt[m+1]{2}\cot\frac{\pi}{2(m+1)} - 2\sqrt[m]{2}\cot\frac{\pi}{m}.$$

The rest of the proof follows from the proof of Lemma 3.2. □

The following lemma indicates an increase in the iota energy of digraphs in $\mathcal{F}_n$ with the increase in length of their joined directed cycles when $m \equiv 1(\mathrm{mod} 2)$.

Lemma 4.8. *Let $n \geq 7$, $p = 2$, $4 \leq m \leq n-2$ and $m \equiv 1(\bmod 2)$. Take $D_2^{m-1}, D_2^m \in \mathcal{F}_n$, where $m \geq 5$. Then*

$$E_c(D_2^m) > E_c(D_2^{m-1}).$$

Proof. As $m \equiv 1(\bmod 2)$, it holds that $m-1 \equiv 0(\bmod 2)$. Using formula (4.4), we obtain

$$E_c(D_2^m) - E_c(D_2^{m-1}) = \sqrt[m]{2}\cot\frac{\pi}{2m} - 2\sqrt[m-1]{2}\cot\frac{\pi}{m-1}.$$

Rest of the proof follows from the proof of Lemma 3.3. □

Combining Lemmas 4.7 and 4.8, we have the following result:

Corollary 4.9. *Let $n \geq 7$, $p = 2$ and $4 \leq m_2 \leq m_1 \leq n-2$. Take $D_2^{m_1}, D_2^{m_2} \in \mathcal{F}_n$. Then*

$$E_c(D_2^{m_1}) \geq E_c(D_2^{m_2}).$$

The following lemma is useful in proving our main result.

Lemma 4.10. *Let $n \geq 7$ and $D_2^k, D_k^2 \in \mathcal{F}_n$, where $4 \leq k \leq n-2$. Then we have the following:*

$$E_c(D_2^k) > E_c(D_k^2).$$

Proof. By formula (4.4), we obtain

$$E_c(D_2^k) = \begin{cases} 2\sqrt[k]{2}\cot\frac{\pi}{k} & \text{if } k \equiv 0(\mathrm{mod} 2) \\ \sqrt[k]{2}\cot\frac{\pi}{2k} & \text{if } k \equiv 1(\mathrm{mod} 2). \end{cases} \tag{4.7}$$

Also, formula (4.4) gives

$$E_c(D_k^2) = \begin{cases} 2\cot\frac{\pi}{k} & \text{if } k \equiv 0(\text{mod}2) \\ \cot\frac{\pi}{2k} & \text{if } k \equiv 1(\text{mod}2). \end{cases} \tag{4.8}$$

As $\sqrt[k]{2} > 1$, formulas (4.7) and (4.8) clearly show that the following inequality holds:

$$E_c(D_2^k) > E_c(D_k^2).$$

This gives the required result. □

Using Corollary 4.6, one can easily obtain the inequality mentioned below:

$$E_c(D_{n-2}^2) \geq E_c(D_p^2), \tag{4.9}$$

where $4 \leq p \leq n-2$. The next inequality can be obtained using Corollary 4.9.

$$E_c(D_2^{n-2}) \geq E_c(D_2^m), \tag{4.10}$$

where $4 \leq m \leq n-2$. By (4.9), (4.10) and Lemma 4.10, we find

$$E_c(D_2^{n-2}) > E_c(D_{n-2}^2). \tag{4.11}$$

The following theorem gives the digraphs in $\mathcal{F}_n$ with smallest and largest iota energy.

Theorem 4.11. *Let $n \geq 4$ and $D_p^m \in \mathcal{F}_n$, where $2 \leq m \leq n-2$ and $2 \leq p \leq n-2$. Then D_p^m has smallest iota energy when $m = p = 2$ and largest iota energy when $m = n-2$ and $p = 2$ among all digraphs of $\mathcal{F}_n$.*

Proof. It can easily be seen from formula (4.4) that

$$E_c(D_p^m) = \begin{cases} 0 & \text{if } m = 2 \text{ and } p = 2 \\ \sqrt{3} & \text{if } m = 2 \text{ and } p = 3 \\ 2 & \text{if } m = 2 \text{ and } p = 4 \\ \sqrt{3}\sqrt[3]{2} & \text{if } m = 3 \text{ and } p = 2. \end{cases} \tag{4.12}$$

Let $[m = 2$ and $p \geq 5]$ or $[p = 2$ and $m \geq 4]$. Then from formula (4.4) and inequalities (4.9) – (4.11), we obtain

$$\sqrt{3}\sqrt[3]{2} < 2\sqrt[4]{2} \leq E_c(D_p^m) < E_c(D_2^{n-2}). \tag{4.13}$$

It is easy to understand from (4.12) and (4.13) that $E_c(D_p^m)$ is smallest when $m = 2$ and $p = 2$. Furthermore, from (4.13) it is clear that $E_c(D_p^m)$ is largest when $m = n - 2$ and $p = 2$. □

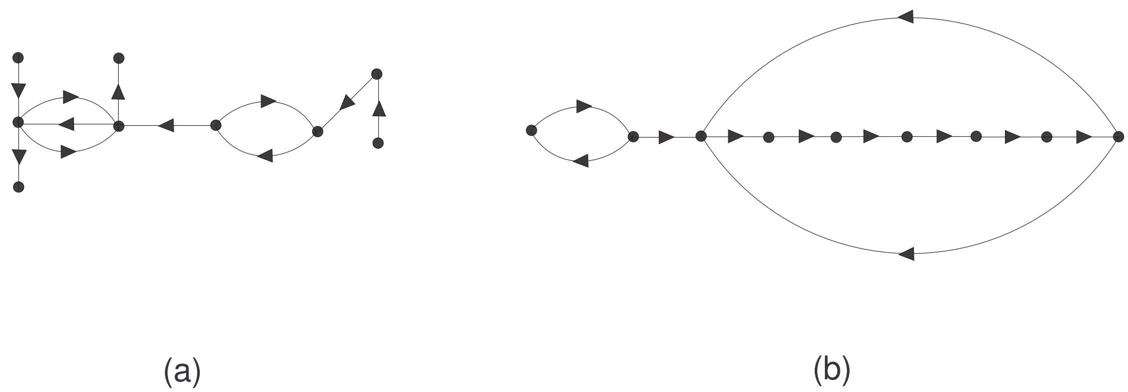


Figure 4.2: (a) The digraph with smallest iota energy in $\mathcal{F}_9$. (b) The digraph with largest iota energy in $\mathcal{F}_9$.

Example 4.12. The digraphs with smallest and largest iota energy in $\mathcal{F}_9$ are shown in Figure 4.2.

4.3 Iota energy of sidigraphs

For $n \geq 4$, let $\mathcal{J}_n$ be a set of n-vertex tricyclic sidigraphs containing five linear subdigraphs such that one of the signed directed cycles does not share any vertex with the other two signed directed cycles and the remaining two signed directed cycles are of same length sharing at least one vertex. We denote a sidigraph in $\mathcal{J}_n$ by S_p^m, where m is the length of those signed directed cycles which share at least one vertex and p is the length of the signed directed cycle which does not share any vertex with the other two signed directed cycles such that $2 \leq m \leq n-2$ and $2 \leq p \leq n-2$. Let $\mathcal{J}_n^{+++} \subset \mathcal{J}_n$ be the subset of all those sidigraphs whose all three signed directed cycles are positive. Also, let $\mathcal{J}_n^{++-} \subset \mathcal{J}_n$ be the subset of all those sidigraphs whose joined signed directed cycles are positive and disjoint signed directed cycle is negative. Let $\mathcal{J}_n^{--+} \subset \mathcal{J}_n$ be the subset of all those sidigraphs whose joined signed directed cycles are negative and disjoint signed directed cycle is positive. Similarly, let $\mathcal{J}_n^{+-+} \subset \mathcal{J}_n$ be the subset of all those sidigraphs whose one of the joined signed directed cycles is positive and the other is negative whereas the disjoint signed directed cycle is positive. Also, let $\mathcal{J}_n^{+--} \subset \mathcal{J}_n$ be the subset of all those sidigraphs whose one of the joined signed directed cycles is positive and the other is negative whereas the disjoint signed directed cycle is negative. Lastly, let $\mathcal{J}_n^{---} \subset \mathcal{J}_n$ be the subset of all those sidigraphs whose all three signed directed cycles are negative. There are following six possible cases:

Case 1

If $S_p^m \in \mathcal{J}_n^{+++}$ then by Theorem 2.11, we have the following result which is analogue of Theorem 4.11.

Theorem 4.13. *Let $n \geq 4$ and $S_p^m \in \mathcal{J}_n^{+++}$, where $2 \leq m \leq n-2$ and $2 \leq p \leq n-2$. Then S_p^m has smallest iota energy when $m = p = 2$ and largest iota energy when $m = n-2$ and $p = 2$ among all sidigraphs of $\mathcal{J}_n^{+++}$.*

Case 2

If $S_p^m \in \mathcal{J}_n^{++-}$ then from Theorem 2.10, we obtain

$$\phi_{S_p^m}(x) = x^{n-m-p}(x^m - 2)(x^p + 1).$$

The eigenvalues of S_p^m are 0, $\sqrt[m]{2}\exp\left(\frac{2k\pi i}{m}\right)$ and $\exp\left(\frac{(2j+1)\pi i}{p}\right)$, where $k = 0, 1, \ldots, m-1$ and $j = 0, 1, \ldots, p-1$ and the eigenvalue 0 has multiplicity $n-m-p$. Thus, the iota energy of S_p^m is given by

$$E_c(S_p^m) = \sqrt[m]{2}\sum_{k=0}^{m-1} \left| \sin \frac{2k\pi}{m} \right| + \sum_{j=0}^{p-1} \left| \sin \frac{(2j+1)\pi}{p} \right|. \tag{4.14}$$

It is shown in Farooq et al. [9] that

$$\sum_{j=0}^{p-1} \left| \sin \frac{(2j+1)\pi}{p} \right| = \begin{cases} 2\csc\frac{\pi}{p} & \text{if } p \equiv 0(\text{mod}2) \\ \cot\frac{\pi}{2p} & \text{if } p \equiv 1(\text{mod}2). \end{cases} \tag{4.15}$$

Using (4.3), (4.14) and (4.15), the iota energy formulas for $S_p^m \in \mathcal{J}_n^{++-}$ are given by

$$E_c(S_p^m) = \begin{cases} 2\left(\sqrt[m]{2}\cot\frac{\pi}{m} + \csc\frac{\pi}{p}\right) & \text{if } m \equiv 0(\text{mod}2), p \equiv 0(\text{mod}2) \\ 2\sqrt[m]{2}\cot\frac{\pi}{m} + \cot\frac{\pi}{2p} & \text{if } m \equiv 0(\text{mod}2), p \equiv 1(\text{mod}2) \\ \sqrt[m]{2}\cot\frac{\pi}{2m} + 2\csc\frac{\pi}{p} & \text{if } m \equiv 1(\text{mod}2), p \equiv 0(\text{mod}2) \\ \sqrt[m]{2}\cot\frac{\pi}{2m} + \cot\frac{\pi}{2p} & \text{if } m \equiv 1(\text{mod}2), p \equiv 1(\text{mod}2). \end{cases} \tag{4.16}$$

We find the smallest and largest iota energy among the sidigraphs in $\mathcal{J}_n^{++-}$. The following lemma demonstrates an increase in the iota energy of sidigraphs in $\mathcal{J}_n^{++-}$ with the increase in length of their disjoint signed directed cycles.

Lemma 4.14. *Let $n \geq 7$, $m = 2$ and $4 \leq p < n-2$. Take $S_p^2, S_{p+1}^2 \in \mathcal{J}_n^{++-}$. Then*

$$E_c(S_{p+1}^2) > E_c(S_p^2).$$

Proof. Let $p \equiv 0(\text{mod}2)$, that is, $p+1 \equiv 1(\text{mod}2)$. Then by formula (4.16) and Lemma 4.1, we find

$$\begin{aligned} E_c(S_{p+1}^2) - E_c(S_p^2) &= \cot\frac{\pi}{2(p+1)} - 2\csc\frac{\pi}{p} \\ &> 0. \end{aligned} \tag{4.17}$$

Next, let $p \equiv 1(\text{mod}2)$. In this case, $p \geq 5$ and $p+1 \equiv 0(\text{mod}2)$. Using formula (4.16) and Lemma 4.1, we obtain

$$\begin{aligned} E_c(S_{p+1}^2) - E_c(S_p^2) &= 2\csc\frac{\pi}{p+1} - \cot\frac{\pi}{2p} \\ &> 0. \end{aligned} \tag{4.18}$$

From (4.17) and (4.18), we have

$$E_c(S_{p+1}^2) > E_c(S_p^2).$$

This gives the required result. $\square$

An direct result of Lemma 4.14 is as follows:

Corollary 4.15. *Let $n \geq 7$, $m = 2$ and $4 \leq p_2 \leq p_1 \leq n-2$. Take $S_{p_1}^2, S_{p_2}^2 \in \mathcal{J}_n^{++-}$. Then*

$$E_c(S_{p_1}^2) \geq E_c(S_{p_2}^2).$$

Next lemma illustrates an increase in the iota energy of sidigraphs in $\mathcal{J}_n^{++-}$ with the increase in length of their joined signed directed cycles when $m \equiv 0(\text{mod}2)$.

Lemma 4.16. *Let $n \geq 7$, $p = 2$, $4 \leq m < n-2$ and $m \equiv 0(\text{mod}2)$. Take $S_2^m, S_2^{m+1} \in \mathcal{J}_n^{++-}$. Then*

$$E_c(S_2^{m+1}) > E_c(S_2^m).$$

Proof. As $m \equiv 0(\text{mod}2)$, we have $m+1 \equiv 1(\text{mod}2)$. Then by formula (4.16), we obtain

$$E_c(S_2^{m+1}) - E_c(S_2^m) = \sqrt[m+1]{2}\cot\frac{\pi}{2(m+1)} + 2 - 2\sqrt[m]{2}\cot\frac{\pi}{m} - 2.$$

Rest of the proof follows from the proof of Lemma 3.2. $\square$

The following lemma indicates an increase in the iota energy of sidigraphs in $\mathcal{J}_n^{++-}$ with the increase in length of their joined signed directed cycles when $m \equiv 1(\text{mod}2)$.

Lemma 4.17. *Let $n \geq 7$, $p = 2$, $4 \leq m \leq n-2$ and $m \equiv 1(\text{mod}2)$. Take $S_2^{m-1}, S_2^m \in \mathcal{J}_n^{++-}$, where $m \geq 5$. Then*

$$E_c(S_2^m) > E_c(S_2^{m-1}).$$

Proof. As $m \equiv 1(\text{mod}2)$, it holds that $m-1 \equiv 0(\text{mod}2)$. Using formula (4.16), we obtain

$$E_c(S_2^m) - E_c(S_2^{m-1}) = \sqrt[m]{2}\cot\frac{\pi}{2m} + 2 - 2\sqrt[m-1]{2}\cot\frac{\pi}{m-1} - 2.$$

The rest of the proof follows from the proof of Lemma 3.3. $\square$

A direct outcome of Lemmas 4.16 and 4.17 is as follows:

Corollary 4.18. *Let $n \geq 7$, $p = 2$ and $4 \leq m_2 \leq m_1 \leq n-2$. Take $S_2^{m_1}, S_2^{m_2} \in \mathcal{J}_n^{++-}$. Then*

$$E_c(S_2^{m_1}) \geq E_c(S_2^{m_2}).$$

The next lemma is helpful in proving our main result.

Lemma 4.19. *Let $n \geq 7$ and $S_2^k, S_k^2 \in \mathcal{J}_n^{++-}$, where $4 \leq k \leq n-2$. Then we have the following:*

$$E_c(S_2^k) > E_c(S_k^2).$$

Proof. By formula (4.16), we obtain

$$E_c(S_2^k) = \begin{cases} 2\sqrt[k]{2}\cot\frac{\pi}{k} + 2 & \text{if } k \equiv 0(\text{mod}2) \\ \sqrt[k]{2}\cot\frac{\pi}{2k} + 2 & \text{if } k \equiv 1(\text{mod}2). \end{cases} \tag{4.19}$$

Also, formula (4.16) gives

$$E_c(S_k^2) = \begin{cases} 2\csc\frac{\pi}{k} & \text{if } k \equiv 0(\text{mod}2) \\ \cot\frac{\pi}{2k} & \text{if } k \equiv 1(\text{mod}2). \end{cases} \tag{4.20}$$

Let $k \equiv 0(\text{mod}2)$. Note that $\sqrt[k]{2} > 1$. This along with (4.19) and (4.20) imply

$$\begin{aligned} E_c(S_2^k) - E_c(S_k^2) &= 2\left(\sqrt[k]{2}\cot\frac{\pi}{k} + 1 - \csc\frac{\pi}{k}\right) \\ &> \cot\frac{\pi}{k} + 1 - \csc\frac{\pi}{k}. \end{aligned} \tag{4.21}$$

Let $a_k = \cot\frac{\pi}{k} + 1$ and $b_k = \csc\frac{\pi}{k}$. Then by Lemma 4.2, we find

$$\begin{aligned} a_k &= \cot\frac{\pi}{k} + 1 \\ &\geq \frac{k}{\pi} - 0.429\frac{\pi}{k} + 1. \end{aligned} \tag{4.22}$$

As $k \geq 4$, we have $-0.429\frac{\pi}{k} \geq -0.429\frac{\pi}{4}$. Thus

$$\begin{aligned} a_k &\geq \frac{k}{\pi} - 0.429\frac{\pi}{4} + 1 \\ &\geq \frac{k}{\pi} + 0.663. \end{aligned} \tag{4.23}$$

Also, using inequality (4.1) and Lemma 4.3, we obtain

$$\begin{aligned}
b_k &= \csc\frac{\pi}{k} \\
&\leq \frac{1}{\frac{\pi}{k}(1-\frac{\pi^2}{6k^2})} \\
&= \frac{k}{\pi}\left(\frac{6k^2}{6k^2-\pi^2}\right) \\
&= \frac{k}{\pi}\left(1+\frac{\pi^2}{6k^2-\pi^2}\right) \\
&= \frac{k}{\pi}+\frac{k\pi}{6k^2-\pi^2} \\
&\leq \frac{k}{\pi}+\frac{4\pi}{6\times 4^2-\pi^2} \\
&\leq \frac{k}{\pi}+0.146.
\end{aligned} \tag{4.24}$$

From (4.22) – (4.24), we obtain

$$\begin{aligned}
a_k - b_k &= \cot\frac{\pi}{k}+1-\csc\frac{\pi}{k} \\
&\geq \frac{k}{\pi}+0.663-\frac{k}{\pi}-0.146 \\
&> 0.
\end{aligned} \tag{4.25}$$

Using (4.21) – (4.25), we obtain

$$\begin{aligned}
E_c(S_2^k)-E_c(S_k^2) &> \cot\frac{\pi}{k}+1-\csc\frac{\pi}{k} \\
&> 0.
\end{aligned} \tag{4.26}$$

Next, let $k \equiv 1(\mathrm{mod}2)$. From (4.19) and (4.20), the following inequality can be easily obtained.

$$E_c(S_2^k)-E_c(S_k^2) > 0. \tag{4.27}$$

From (4.26) and (4.27), we obtain the required result. □

Using Corollary 4.15, one can easily obtain the inequality mentioned below:

$$E_c(S_{n-2}^2) \geq E_c(S_p^2), \tag{4.28}$$

where $4 \leq p \leq n-2$. The next inequality can be obtained using Corollary 4.18.

$$E_c(S_2^{n-2}) \geq E_c(S_2^m), \tag{4.29}$$

where $4 \leq m \leq n-2$. By (4.28), (4.29) and Lemma 4.19, we find

$$E_c(S_2^{n-2}) > E_c(S_{n-2}^2). \tag{4.30}$$

The following result gives the sidigraphs in $\mathcal{J}_n^{++-}$ with smallest and largest iota energy.

Theorem 4.20. *Let $n \geq 4$ and $S_p^m \in \mathcal{J}_n^{++-}$, where $2 \leq m \leq n-2$ and $2 \leq p \leq n-2$. Then S_p^m has smallest iota energy when $m = 2$ and $p = 3$ and largest iota energy when $m = n-2$ and $p = 2$ among all sidigraphs of $\mathcal{J}_n^{++-}$.*

Proof. Let $S_p^m \in \mathcal{J}_n^{++-}$, where $2 \leq m \leq n-2$ and $2 \leq p \leq n-2$. Then, it can easily be seen from formula (4.16) that

$$E_c(S_p^m) = \begin{cases} 2 & \text{if } m = 2 \text{ and } p = 2 \\ \sqrt{3} & \text{if } m = 2 \text{ and } p = 3 \\ 2\sqrt{2} & \text{if } m = 2 \text{ and } p = 4 \\ \sqrt{5+2\sqrt{5}} & \text{if } m = 2 \text{ and } p = 5 \\ 4 & \text{if } m = 2 \text{ and } p = 6 \\ \sqrt{3}\sqrt[3]{2}+2 & \text{if } m = 3 \text{ and } p = 2. \end{cases} \tag{4.31}$$

Let [$m = 2$ and $p \geq 7$] or [$p = 2$ and $m \geq 4$]. Also, let $S_2^{n-2} \in \mathcal{J}_n^{++-}$. Then from formula (4.16) and inequalities (4.28) – (4.30), we obtain

$$\sqrt{3}\sqrt[3]{2}+2 < 2\sqrt[4]{2}+2 \leq E_c(S_p^m) < E_c(S_2^{n-2}). \tag{4.32}$$

It is clear from (4.31) and (4.32) that $E_c(S_p^m)$ is smallest when $m = 2$ and $p = 3$. Furthermore, from (4.32) it is easy to understand that $E_c(S_p^m)$ is largest when $m = n-2$ and $p = 2$. $\square$

Case 3

If $S_p^m \in \mathcal{J}_n^{--+}$ then from Theorem 2.10, we obtain

$$\phi_{S_p^m}(x) = x^{n-m-p}(x^m + 2)(x^p - 1).$$

The eigenvalues of S_p^m are 0, $\sqrt[m]{2}\exp\left(\frac{(2k+1)\pi i}{m}\right)$ and $\exp\left(\frac{2j\pi i}{p}\right)$, where $k = 0, 1, \ldots, m-1$ and $j = 0, 1, \ldots, p-1$ and the eigenvalue 0 has multiplicity $n - m - p$. Thus, the iota energy of S_p^m is given by

$$E_c(S_p^m) = \sqrt[m]{2}\sum_{k=0}^{m-1}\Big|\sin\frac{(2k+1)\pi}{m}\Big| + \sum_{j=0}^{p-1}\Big|\sin\frac{2j\pi}{p}\Big|. \tag{4.33}$$

Using (4.3), (4.15) and (4.33), the iota energy formulas for $S_p^m \in \mathcal{J}_n^{--+}$ are given by

$$E_c(S_p^m) = \begin{cases} 2\left(\sqrt[m]{2}\csc\frac{\pi}{m} + \cot\frac{\pi}{p}\right) & \text{if } m \equiv 0(\text{mod}2), p \equiv 0(\text{mod}2) \\ 2\sqrt[m]{2}\csc\frac{\pi}{m} + \cot\frac{\pi}{2p} & \text{if } m \equiv 0(\text{mod}2), p \equiv 1(\text{mod}2) \\ \sqrt[m]{2}\cot\frac{\pi}{2m} + 2\cot\frac{\pi}{p} & \text{if } m \equiv 1(\text{mod}2), p \equiv 0(\text{mod}2) \\ \sqrt[m]{2}\cot\frac{\pi}{2m} + \cot\frac{\pi}{2p} & \text{if } m \equiv 1(\text{mod}2), p \equiv 1(\text{mod}2). \end{cases} \tag{4.34}$$

Next, we find the smallest and largest iota energy among the sidigraphs in $\mathcal{J}_n^{--+}$. The following lemma demonstrates an increase in the iota energy of sidigraphs in $\mathcal{J}_n^{--+}$ with the increase in length of their disjoint signed directed cycles.

Lemma 4.21. *Let $n \geq 7$, $m = 2$ and $4 \leq p < n - 2$. Take $S_p^2, S_{p+1}^2 \in \mathcal{J}_n^{--+}$. Then*

$$E_c(S_{p+1}^2) > E_c(S_p^2).$$

Proof. Let $p \equiv 0(\text{mod}2)$, that is, $p + 1 \equiv 1(\text{mod}2)$. Then by formula (4.34) and Lemma 4.1, we find

$$\begin{aligned} E_c(S_{p+1}^2) - E_c(S_p^2) &= 2\sqrt[2]{2} + \cot\frac{\pi}{2(p+1)} - 2\sqrt[2]{2} - 2\cot\frac{\pi}{p} \\ &> 0. \end{aligned} \tag{4.35}$$

Next, let $p \equiv 1(\text{mod}2)$. In this case, $p \geq 5$ and $p+1 \equiv 0(\text{mod}2)$. Using formula (4.34) and Lemma 4.1, we obtain

$$\begin{aligned} E_c(S^2_{p+1}) - E_c(S^2_p) &= 2\sqrt[2]{2} + 2\cot\frac{\pi}{p+1} - 2\sqrt[2]{2} - \cot\frac{\pi}{2p} \\ &> 0. \end{aligned} \tag{4.36}$$

From (4.35) and (4.36), we have

$$E_c(S^2_{p+1}) > E_c(S^2_p).$$

This gives the required result. □

A direct result of Lemma 4.21 is as follows:

Corollary 4.22. *Let $n \geq 7$, $m = 2$ and $4 \leq p_2 \leq p_1 \leq n-2$. Take $S^2_{p_1}, S^2_{p_2} \in \mathcal{J}^{--+}_n$. Then*

$$E_c(S^2_{p_1}) \geq E_c(S^2_{p_2}).$$

Next lemma illustrates an increase in the iota energy of sidigraphs in $\mathcal{J}^{--+}_n$ with the increase in length of their joined signed directed cycles when $m \equiv 0(\text{mod}2)$.

Lemma 4.23. *Let $n \geq 7$, $p = 2$, $4 \leq m \leq n-2$ and $m \equiv 0(\text{mod}2)$. Take $S^{m-1}_2, S^m_2 \in \mathcal{J}^{--+}_n$, where $m \geq 5$. Then*

$$E_c(S^m_2) > E_c(S^{m-1}_2).$$

Proof. As $m \equiv 0(\text{mod}2)$, we have $m-1 \equiv 1(\text{mod}2)$. Then by formula (4.34), we obtain

$$E_c(S^m_2) - E_c(S^{m-1}_2) = 2\sqrt[m]{2}\csc\frac{\pi}{m} - \sqrt[m-1]{2}\cot\frac{\pi}{2(m-1)}.$$

Rest of the proof follows from the proof of Lemma 3.8. □

The following lemma indicates an increase in the iota energy of sidigraphs in $\mathcal{J}^{--+}_n$ with the increase in length of their joined signed directed cycles when $m \equiv 1(\text{mod}2)$.

Lemma 4.24. *Let $n \geq 7$, $p = 2$, $4 \leq m < n-2$ and $m \equiv 1(\text{mod}2)$. Take $S^m_2, S^{m+1}_2 \in \mathcal{J}^{--+}_n$. Then*

$$E_c(S^{m+1}_2) > E_c(S^m_2).$$

Proof. As $m \equiv 1(\text{mod}2)$, it holds that $m+1 \equiv 0(\text{mod}2)$. Using formula (4.34), we obtain

$$E_c(S_2^{m+1}) - E_c(S_2^m) = 2\sqrt[m+1]{2}\csc\frac{\pi}{m+1} - \sqrt[m]{2}\cot\frac{\pi}{2m}.$$

The rest of the proof follows from the proof of Lemma 3.9. □

A direct outcome of Lemmas 4.23 and 4.24 is as follows:

Corollary 4.25. *Let $n \geq 7$, $p = 2$ and $4 \leq m_2 \leq m_1 \leq n-2$. Take $S_2^{m_1}, S_2^{m_2} \in \mathcal{J}_n^{--+}$. Then*

$$E_c(S_2^{m_1}) \geq E_c(S_2^{m_2}).$$

The following lemma is helpful in proving our main result.

Lemma 4.26. *Let $n \geq 7$ and $S_2^k, S_k^2 \in \mathcal{J}_n^{--+}$, where $4 \leq k \leq n-2$. Then we have the following:*

$$E_c(S_k^2) > E_c(S_2^k).$$

Proof. By formula (4.34), we obtain

$$E_c(S_k^2) = \begin{cases} 2\left(\sqrt[2]{2} + \cot\frac{\pi}{k}\right) & \text{if } k \equiv 0(\text{mod}2) \\ 2\sqrt[2]{2} + \cot\frac{\pi}{2k} & \text{if } k \equiv 1(\text{mod}2). \end{cases} \tag{4.37}$$

Also, formula (4.34) gives

$$E_c(S_2^k) = \begin{cases} 2\sqrt[k]{2}\csc\frac{\pi}{k} & \text{if } k \equiv 0(\text{mod}2) \\ \sqrt[k]{2}\cot\frac{\pi}{2k} & \text{if } k \equiv 1(\text{mod}2). \end{cases} \tag{4.38}$$

One can easily see from formulas (4.37) and (4.38) that the following inequality holds:

$$E_c(S_k^2) > E_c(S_2^k).$$

This gives the required result. □

Using Corollary 4.22, one can easily obtain the inequality mentioned below:

$$E_c(S_{n-2}^2) \geq E_c(S_p^2), \tag{4.39}$$

where $4 \leq p \leq n-2$. The next inequality can be obtained using Corollary 4.25.

$$E_c(S_2^{n-2}) \geq E_c(S_2^m), \tag{4.40}$$

where $4 \leq m \leq n-2$. By (4.39), (4.40) and Lemma 4.26, we find

$$E_c(S_{n-2}^2) > E_c(S_2^{n-2}). \tag{4.41}$$

The next result gives the sidigraphs in $\mathcal{J}_n^{--+}$ with smallest and largest iota energy.

Theorem 4.27. *Let $n \geq 4$ and $S_p^m \in \mathcal{J}_n^{--+}$, where $2 \leq m \leq n-2$ and $2 \leq p \leq n-2$. Then S_p^m has smallest iota energy when $m = 3$ and $p = 2$ and largest iota energy when $m = 2$ and $p = n-2$ among all sidigraphs of $\mathcal{J}_n^{--+}$.*

Proof. It can easily be seen from formula (4.34) that

$$E_c(S_p^m) = \begin{cases} 2\sqrt[2]{2} & \text{if } p = 2 \text{ and } m = 2 \\ \sqrt{3}\sqrt[3]{2} & \text{if } p = 2 \text{ and } m = 3 \\ 2\sqrt{2}\sqrt[4]{2} & \text{if } p = 2 \text{ and } m = 4 \\ \sqrt[5]{2}\sqrt{5+2\sqrt{5}} & \text{if } p = 2 \text{ and } m = 5 \\ 4\sqrt[6]{2} & \text{if } p = 2 \text{ and } m = 6 \\ 2\sqrt[2]{2} + \sqrt{3} & \text{if } p = 3 \text{ and } m = 2. \end{cases} \tag{4.42}$$

Let $[p = 2$ and $m \geq 7]$ or $[m = 2$ and $p \geq 4]$. Also, let $S_{n-2}^2 \in \mathcal{J}_n^{--+}$. Then from formula (4.34) and inequalities (4.39) − (4.41), we obtain

$$2\sqrt[2]{2} + \sqrt{3} < 2\sqrt[2]{2} + 2 \leq E_c(S_p^m) < E_c(S_{n-2}^2). \tag{4.43}$$

It is clear from (4.42) and (4.43) that $E_c(S_p^m)$ is smallest when $m = 3$ and $p = 2$. Furthermore, from (4.43) it is easy to understand that $E_c(S_p^m)$ is largest when $m = 2$ and $p = n-2$. $\square$

Case 4

If $S_p^m \in \mathcal{J}_n^{+-+}$ then from Theorem 2.10, we obtain

$$\phi_{S_p^m}(x) = x^{n-p}(x^p - 1).$$

The eigenvalues of S_p^m are 0 and $\exp\left(\frac{2j\pi i}{p}\right)$, where $j = 0, 1, \ldots, p-1$ and the eigenvalue 0 has multiplicity $n-p$. Thus, the iota energy of S_p^m is given by

$$E_c(S_p^m) = \sum_{j=0}^{p-1} \left| \sin \frac{2j\pi}{p} \right|. \tag{4.44}$$

Using (4.3) and (4.44), the iota energy formulas for $S_p^m \in \mathcal{J}_n^{+-+}$ are given by

$$E_c(S_p^m) = \begin{cases} 2\cot\frac{\pi}{p} & \text{if } p \equiv 0(\text{mod} 2) \\ \cot\frac{\pi}{2p} & \text{if } p \equiv 1(\text{mod} 2). \end{cases} \tag{4.45}$$

The following result characterizes the sidigraphs in $\mathcal{J}_n^{+-+}$ with smallest and largest iota energy.

Theorem 4.28. *Let $n \geq 4$ and $S_p^m \in \mathcal{J}_n^{+-+}$, where $2 \leq m \leq n-2$ and $2 \leq p \leq n-2$. Then S_p^m has smallest iota energy when $p = 2$ and largest iota energy when $p = n-2$ among all sidigraphs of $\mathcal{J}_n^{+-+}$.*

Proof. It can easily be seen from formula (4.45) that

$$E_c(S_p^m) = \begin{cases} 0 & \text{if } p = 2 \\ \sqrt{3} & \text{if } p = 3. \end{cases} \tag{4.46}$$

Let $p \geq 4$ and $S_r^m \in \mathcal{J}_n^{+-+}$ is a sidigraph with length of the disjoint signed directed cycle r, where $r \geq p$. Then from formula (4.45) and Lemma 4.1, we obtain

$$\sqrt{3} < 2 \leq E_c(S_p^m) \leq E_c(S_r^m). \tag{4.47}$$

It is clear from (4.46) and (4.47) that $E_c(S_p^m)$ is smallest when $p = 2$. Furthermore, $\mathcal{J}_n^{+-+}$ contains a sidigraph with disjoint signed directed cycle of maximum length $n-2$. Therefore, from (4.47) it is easy to understand that $E_c(S_p^m)$ is largest when $p = n-2$. □

Case 5

If $S_p^m \in \mathcal{J}_n^{+--}$ then from Theorem 2.10, we obtain

$$\phi_{S_p^m}(x) = x^{n-p}(x^p + 1).$$

The eigenvalues of S_p^m are 0 and $\exp\left(\frac{(2j+1)\pi i}{p}\right)$, where $j = 0, 1, \ldots, p-1$ and the eigenvalue 0 has multiplicity $n - p$. Thus, the iota energy of S_p^m is given by

$$E_c(S_p^m) = \sum_{j=0}^{p-1} \Big| \sin \frac{(2j+1)\pi}{p} \Big|. \tag{4.48}$$

Using (4.15) and (4.48), the iota energy formulas for $S_p^m \in \mathcal{J}_n^{+--}$ are given by

$$E_c(S_p^m) = \begin{cases} 2\csc \frac{\pi}{p} & \text{if } p \equiv 0(\text{mod} 2) \\ \cot \frac{\pi}{2p} & \text{if } p \equiv 1(\text{mod} 2). \end{cases} \tag{4.49}$$

Next theorem gives the sidigraphs in $\mathcal{J}_n^{+--}$ with smallest and largest iota energy.

Theorem 4.29. *Let $n \geq 4$ and $S_p^m \in \mathcal{J}_n^{+--}$, where $2 \leq m \leq n-2$ and $2 \leq p \leq n-2$. Then S_p^m has smallest iota energy when $p = 3$ and largest iota energy when $p = n-2$ among all sidigraphs of $\mathcal{J}_n^{+--}$.*

Proof. Let $S_p^m \in \mathcal{J}_n^{+--}$, where $2 \leq m \leq n-2$ and $2 \leq p \leq n-2$. Then, it can easily be seen from formula (4.49) that

$$E_c(S_p^m) = \begin{cases} 2 & \text{if } p = 2 \\ \sqrt{3} & \text{if } p = 3. \end{cases} \tag{4.50}$$

Let $p \geq 4$ and $S_r^m \in \mathcal{J}_n^{+--}$, where $r \geq p$. Then from formula (4.49) and Lemma 4.1, we obtain

$$2 < 2\sqrt{2} \leq E_c(S_p^m) \leq E_c(S_r^m). \tag{4.51}$$

It is clear from (4.50) and (4.51) that $E_c(S_p^m)$ is smallest when $p = 3$. Furthermore, $\mathcal{J}_n^{+--}$ contains a sidigraph with disjoint signed directed cycle of maximum length $n-2$. Therefore, from (4.51) it is easy to understand that $E_c(S_p^m)$ is largest when $p = n-2$. □

Case 6

If $S_p^m \in \mathcal{J}_n^{---}$ then from Theorem 2.10, we obtain

$$\phi_{S_p^m}(x) = x^{n-m-p}(x^m+2)(x^p+1).$$

The eigenvalues of S_p^m are 0, $\sqrt[m]{2}\exp\left(\frac{(2k+1)\pi i}{m}\right)$ and $\exp\left(\frac{(2j+1)\pi i}{p}\right)$, where $k = 0, 1, \ldots, m-1$ and $j = 0, 1, \ldots, p-1$ and the eigenvalue 0 has multiplicity $n-m-p$. Thus, the iota energy of S_p^m is given by

$$E_c(S_p^m) = \sqrt[m]{2}\sum_{k=0}^{m-1}\left|\sin\frac{(2k+1)\pi}{m}\right| + \sum_{j=0}^{p-1}\left|\sin\frac{(2j+1)\pi}{p}\right|. \tag{4.52}$$

Using (4.15) and (4.52), the iota energy formulas for $S_p^m \in \mathcal{J}_n^{---}$ are given by

$$E_c(S_p^m) = \begin{cases} 2\left(\sqrt[m]{2}\csc\frac{\pi}{m} + \csc\frac{\pi}{p}\right) & \text{if } m \equiv 0(\text{mod}2), p \equiv 0(\text{mod}2) \\ 2\sqrt[m]{2}\csc\frac{\pi}{m} + \cot\frac{\pi}{2p} & \text{if } m \equiv 0(\text{mod}2), p \equiv 1(\text{mod}2) \\ \sqrt[m]{2}\cot\frac{\pi}{2m} + 2\csc\frac{\pi}{p} & \text{if } m \equiv 1(\text{mod}2), p \equiv 0(\text{mod}2) \\ \sqrt[m]{2}\cot\frac{\pi}{2m} + \cot\frac{\pi}{2p} & \text{if } m \equiv 1(\text{mod}2), p \equiv 1(\text{mod}2). \end{cases} \tag{4.53}$$

We find the smallest and largest iota energy among the sidigraphs in $\mathcal{J}_n^{---}$. The following lemma demonstrates an increase in the iota energy of sidigraphs in $\mathcal{J}_n^{---}$ with the increase in length of their disjoint signed directed cycles.

Lemma 4.30. *Let $n \geq 7$, $m = 2$ and $4 \leq p < n-2$. Take $S_p^2, S_{p+1}^2 \in \mathcal{J}_n^{---}$. Then*

$$E_c(S_{p+1}^2) > E_c(S_p^2).$$

Proof. Let $p \equiv 0(\text{mod}2)$, that is, $p+1 \equiv 1(\text{mod}2)$. Then by formula (4.53) and Lemma 4.1, we find

$$\begin{aligned} E_c(S_{p+1}^2) - E_c(S_p^2) &= 2\sqrt[2]{2} + \cot\frac{\pi}{2(p+1)} - 2\sqrt[2]{2} - 2\csc\frac{\pi}{p} \\ &> 0. \end{aligned} \tag{4.54}$$

Next, let $p \equiv 1(\text{mod}2)$. In this case, $p \geq 5$ and $p+1 \equiv 0(\text{mod}2)$. Using formula (4.53) and Lemma 4.1, we obtain

$$\begin{aligned} E_c(S^2_{p+1}) - E_c(S^2_p) &= 2\sqrt[2]{2} + 2\csc\frac{\pi}{p+1} - 2\sqrt[2]{2} - \cot\frac{\pi}{2p} \\ &> 0. \end{aligned} \tag{4.55}$$

From (4.54) and (4.55), we have

$$E_c(S^2_{p+1}) > E_c(S^2_p).$$

This gives the required result. $\square$

A direct result of Lemma 4.30 is as follows:

Corollary 4.31. *Let $n \geq 7$, $m = 2$ and $4 \leq p_2 \leq p_1 \leq n-2$. Take $S^2_{p_1}, S^2_{p_2} \in \mathcal{J}^{---}_n$. Then*

$$E_c(S^2_{p_1}) \geq E_c(S^2_{p_2}).$$

Next lemma illustrates an increase in the iota energy of sidigraphs in $\mathcal{J}^{---}_n$ with the increase in length of their joined signed directed cycles when $m \equiv 0(\text{mod}2)$.

Lemma 4.32. *Let $n \geq 7$, $p = 2$, $4 \leq m \leq n-2$ and $m \equiv 0(\text{mod}2)$. Take $S^{m-1}_2, S^m_2 \in \mathcal{J}^{---}_n$, where $m \geq 5$. Then*

$$E_c(S^m_2) > E_c(S^{m-1}_2).$$

Proof. As $m \equiv 0(\text{mod}2)$, we have $m-1 \equiv 1(\text{mod}2)$. Then by formula (4.53), we obtain

$$E_c(S^m_2) - E_c(S^{m-1}_2) = 2\sqrt[m]{2}\csc\frac{\pi}{m} + 2 - \sqrt[m-1]{2}\cot\frac{\pi}{2(m-1)} - 2.$$

The rest of the proof follows from the proof of Lemma 3.8. $\square$

The following lemma indicates an increase in the iota energy of sidigraphs in $\mathcal{J}^{---}_n$ with the increase in length of their joined signed directed cycles when $m \equiv 1(\text{mod}2)$.

Lemma 4.33. *Let $n \geq 7$, $p = 2$, $4 \leq m < n-2$ and $m \equiv 1(\text{mod}2)$. Take $S^m_2, S^{m+1}_2 \in \mathcal{J}^{---}_n$. Then*

$$E_c(S^{m+1}_2) > E_c(S^m_2).$$

Proof. As $m \equiv 1(\text{mod}2)$, it holds that $m+1 \equiv 0(\text{mod}2)$. Using formula (4.53), we obtain

$$E_c(S_2^{m+1}) - E_c(S_2^m) = 2\sqrt[m+1]{2}\csc\frac{\pi}{m+1} + 2 - \sqrt[m]{2}\cot\frac{\pi}{2m} - 2.$$

Rest of the proof follows from the proof of Lemma 3.9. $\square$

Combining Lemmas 4.32 and 4.33, we have the following result:

Corollary 4.34. *Let $n \geq 7$, $p = 2$ and $4 \leq m_2 \leq m_1 \leq n-2$. Take $S_2^{m_1}, S_2^{m_2} \in \mathcal{J}_n^{---}$. Then*

$$E_c(S_2^{m_1}) \geq E_c(S_2^{m_2}).$$

The following lemma is helpful in proving our main result.

Lemma 4.35. *Let $n \geq 7$ and $S_2^k, S_k^2 \in \mathcal{J}_n^{---}$, where $4 \leq k \leq n-2$. Then we have the following:*

$$E_c(S_k^2) > E_c(S_2^k).$$

Proof. By formula (4.53), we obtain

$$E_c(S_k^2) = \begin{cases} 2\left(\sqrt[2]{2} + \csc\frac{\pi}{k}\right) & \text{if } k \equiv 0(\text{mod}2) \\ 2\sqrt[2]{2} + \cot\frac{\pi}{2k} & \text{if } k \equiv 1(\text{mod}2). \end{cases} \tag{4.56}$$

Also, formula (4.53) gives

$$E_c(S_2^k) = \begin{cases} 2\sqrt[k]{2}\csc\frac{\pi}{k} + 2 & \text{if } k \equiv 0(\text{mod}2) \\ \sqrt[k]{2}\cot\frac{\pi}{2k} + 2 & \text{if } k \equiv 1(\text{mod}2). \end{cases} \tag{4.57}$$

One can easily see from formulas (4.56) and (4.57) that the following inequality holds:

$$E_c(S_k^2) > E_c(S_2^k).$$

This gives the required result. $\square$

Using Corollary 4.31, one can easily obtain the inequality mentioned below:

$$E_c(S_{n-2}^2) \geq E_c(S_p^2), \tag{4.58}$$

where $4 \leq p \leq n-2$. The next inequality can be obtained using Corollary 4.34.

$$E_c(S_2^{n-2}) \geq E_c(S_2^m), \tag{4.59}$$

where $4 \leq m \leq n-2$. By (4.58), (4.59) and Lemma 4.35, we find

$$E_c(S_{n-2}^2) > E_c(S_2^{n-2}). \tag{4.60}$$

The next theorem characterizes the sidigraphs in $\mathcal{J}_n^{---}$ with smallest and largest iota energy.

Theorem 4.36. *Let $n \geq 4$ and $S_p^m \in \mathcal{J}_n^{---}$, where $2 \leq m \leq n-2$ and $2 \leq p \leq n-2$. Then S_p^m has smallest iota energy when $m = 3$ and $p = 2$ and largest iota energy when $m = 2$ and $p = n-2$ among all sidigraphs of $\mathcal{J}_n^{---}$.*

Proof. It can easily be seen from formula (4.53) that

$$E_c(S_p^m) = \begin{cases} 2\sqrt[2]{2}+2 & \text{if } m=2 \text{ and } p=2 \\ 2\sqrt[2]{2}+\sqrt{3} & \text{if } m=2 \text{ and } p=3 \\ \sqrt{3}\sqrt[3]{2}+2 & \text{if } m=3 \text{ and } p=2. \end{cases} \tag{4.61}$$

Let $[m = 2$ and $p \geq 4]$ or $[p = 2$ and $m \geq 4]$. Also, let $S_{n-2}^2 \in \mathcal{J}_n^{---}$. Then from formula (4.53) and inequalities (4.58) – (4.60), we obtain

$$2\sqrt[2]{2}+2 < 2\sqrt{2}\sqrt[4]{2}+2 \leq E_c(S_p^m) < E_c(S_{n-2}^2). \tag{4.62}$$

It is clear from (4.61) and (4.62) that $E_c(S_p^m)$ is smallest when $m = 3$ and $p = 2$. Furthermore, from (4.62) it is easy to understand that $E_c(S_p^m)$ is largest when $m = 2$ and $p = n-2$. □

We know that $\sin y < y < \tan y$ for each $y \in (0, \frac{\pi}{2})$. This gives

$$\csc y > \cot y, \tag{4.63}$$

for each $y \in (0, \frac{\pi}{2})$. The next theorem follows from formulas (4.16), (4.34), (4.45), (4.49), (4.53), inequality (4.63), Theorems 4.13, 4.20, 4.27 – 4.29 and 4.36.

Theorem 4.37. *Let $n \geq 4$ and $S_p^m \in \mathcal{J}_n$, where $2 \leq m \leq n-2$ and $2 \leq p \leq n-2$. Then*

(1) S_p^m has smallest iota energy when

(i) $S_p^m \in \mathcal{J}_n^{+++}$ and $m = p = 2$, or

(ii) $S_p^m \in \mathcal{J}_n^{+-+}$ and $p = 2$.

(2) S_p^m has largest iota energy when $m = 2$ and $p = n-2$ such that

(i) $S_p^m \in \mathcal{J}_n^{---}$ and $p \equiv 0(\text{mod}2)$, or

(ii) $[S_p^m \in \mathcal{J}_n^{--+}$ or $S_p^m \in \mathcal{J}_n^{---}]$ and $p \equiv 1(\text{mod}2)$.

Example 4.38. The sidigraphs with smallest and largest iota energy in $\mathcal{J}_9$ are shown in Figure 4.3.

4.4 Conclusion

In this chapter, we defined a new class $\mathcal{F}_n$ of n-vertex tricyclic digraphs containing five linear subdigraphs such that one of the directed cycles does not share any vertex with the other two directed cycles and the remaining two directed cycles are of same length sharing at least one vertex. We showed that a digraph $D_p^m \in \mathcal{F}_n$ attains smallest iota energy among the digraphs in $\mathcal{F}_n$ when $m = p = 2$. Moreover, D_p^m has largest iota energy when $m = n-2$ and $p = 2$. We have also introduced a similar class $\mathcal{J}_n$ of n-vertex tricyclic sidigraphs. We found smallest and largest iota energy among the sidigraphs in $\mathcal{J}_n$. It will be interesting to find digraphs (respectively, sidigraphs) with extremal iota energy among a more general class of tricyclic digraphs (respectively, sidigraphs).

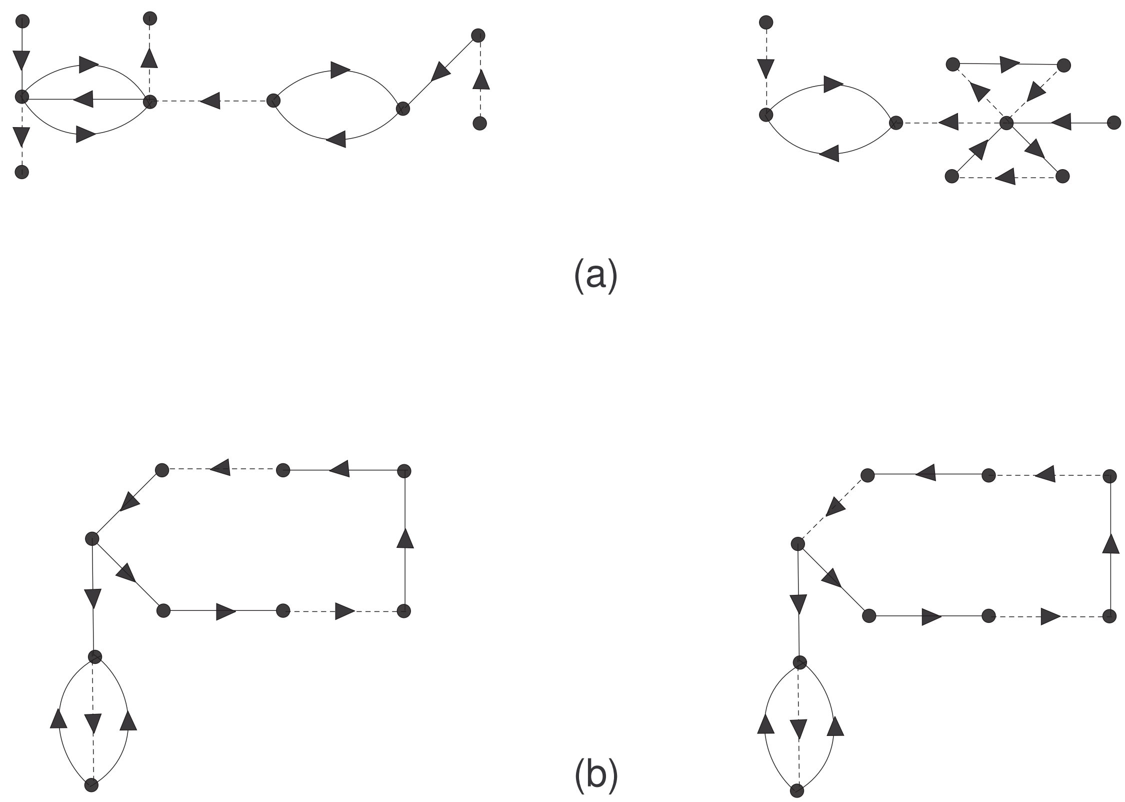


Figure 4.3: (a) The sidigraphs with smallest iota energy in $\mathcal{J}_9$. (b) The sidigraphs with largest iota energy in $\mathcal{J}_9$.

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