Compactification and Foliation of Some Spacetimes

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Dedicated
To My Parents,
Buddy,
Siblings
and
Abid

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Abstract

In general, foliation of a manifold is its partitioning into submanifolds. In particular, from relativistic point of view, it is used to split the *spacetime* into *space* and *time*. Foliation is often useful to understand the clear concept of *time*. In this thesis, we have first reviewed the Brill, Cavallo and Isenberg procedure to obtain the foliation by spacelike hypersurfaces of constant mean extrinsic curvature (K-slicing) for spherically symmetric static spacetimes. After this, we used that procedure to review the foliation of the *Reissner Nordstorm* and the *Schwarzschild* spacetimes and then we review the flat foliation for the *Schwarzschild* metric. At the end, we have discussed coordinates (non-singular) for the Taub-Nut spacetime.

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Introduction

In general, foliation is the splitting of the space into subspaces such that the union of all subspaces is the whole space while the intersection of every subspace with every other subspace is empty. More precisely, a foliation of the space is its partition into subspaces. After the unification of the *space* and *time* into *spacetime* one may think why do we need foliation? Why are we reversing the procedure of unification? So as to comprehend the appropriate answer of these questions, consider the notion of a *moment of time* in *Newtonian* theory. Here in this theory, there is no vagueness in the idea of a given *moment of time*. In Special Relativity (SR), there is some vagueness because simultaneity is not universal, but, after specification of an inertial frame, the concept becomes precise. Globally we don't have inertial frames[1] in GR unless the *spacetime* is flat and hence, the notion of *moment of time* is completely unclear. Therefore this notion translates in GR as 'a time slice' or 'a spacelike hypersurface'. Foliation of *spacetime* into *space* and *time* help us to understand that how *time* evolves in space. In cosmology, we also need the clear notion of time because we sometimes assume that the universe is same for all observers (homogeneity) in all directions (isotropy) at a given *moment of time*.

York [2] defined a time parameter proportional to the *Mean Extrinsic* Curvature (MEC), due to this, foliation of a metric may be expected by constant MEC, K, hypersurfaces (hereafter called K-surfaces). Brill, Cavallo and Isenberg (hereafter called BCI) [3] provide a thorough discussion on foliation by K-surfaces of spherically symmetric static spacetimes. They show a foliation of the *Schwarzschild* metric but did not provide complete foliation of it. Eardly and Smarr [4] indicated that there is a K-slicing of the *Schwarzschild* metric but did not provide explicit method for its con-

struction. A complete foliation of the *Schwarzschild*, *Reisnner Nordstorm* (RN) and extreme *Reisnner Nordstorm* (eRN) metrics by means of K-surfaces is provided in [5], [6] and [7] respectively while foliation of the RN and the *Schwarzschild* metrics by flat hyprsurfaces is given in [8] and for eRN metric in [9].
The layout of the thesis is as follows:

In Chapter 1, some basic concepts of differential geometry which are useful in GR are discussed. Then, after defining the energy-momentum tensor, Einstein Field Equations (hereafter called EFEs) are derived. Brief introductions of blackholes and Carter Penrose (CP) diagrams are given at the end of this chapter.

In Chapter 2, we, first, explain some exact solutions of the the EFEs and then, classify these solutions into two categories based on zero and non-zero cosmological constant $\wedge$. In the former category, we, first, discuss the singularities, non-singular compactified coordinates and CP diagrams of Minkowski, *Schwarzschild*, RN and eRN metrics. Then, a brief introduction of *Kerr* metric is given. While in the later category, we, first discuss de-Sitter and *Anti* de-Sitter metrics and their singularities, after this, we derive the *Schwarzschild* de-Sitter (SdS) and *Schwarzschild Anti* de-Sitter (SAdS) metrics.

In Chapter 3, foliation is defined in general and some examples are given. Then, after reviewing the BCI procedure, we reviewed the K-slicing of the *Schwarzschild* metric (SM) and RN metric and rederive the generalized K-surface equation. In the end, we discuss flat hypersurfaces for the *Schwarzschild* metric.

In Chapter 4, we have discussed the CP diagram and non-singular coordinates for Taub-Nut metric.

Chapter 1

Basics of Relativity

Sir Isaac Newton compared the acceleration of different objects using gravitational force. He observed that how the gravity varies with the distance and discovered his universal law of gravitation, in which every object attracts every other object in the universe, with the force which varies directly with the product of their masses and inversely to the square of the distance between their centers. Thus in Newtonian mechanics, every object experiences same gravitational field irrespective of its speed and mass.

In 1905, Einstein published SR, which relies on two assumptions:

1) The laws of physics are same for all inertial observers. This means that there is no physical difference for any two observers in an inertial frame.

2) The speed of light in vacuum is same for all inertial observers.

With these assumptions Galilean transformations of classical mechanics are replaced by Lorentz transformations. SR is limited to the study of uniform, unaccelerated motion. In other words it deals with the motion in straight line with constant speed. But regardless of its restriction it made tremendous predictions e.g. the events that happen simultaneously for one observer does not happen at the same time for another observer that is moving relative to the first one. Which shows that the idea of absolute time and must be discarded and in SR time and space are infact the part of the single 4-dimensional continuum known as spacetime.

In 1915, Einstein published his GR, which generalizes SR for arbitrary motion. GR modifies the framework of SR by allowing the manifold to be curved. The basis of GR are:

1) The principle of equivalence which gives the equivalence of gravitational and inertial masses in local inertial coordinate systems. It states that in a local inertial coordinate system, all laws of physics are same as in SR.

2) Principle of general covariance which states that if the equation remains valid in SR (absence of gravity) and it preserves its form under coordinate transformation (tensor equation) then the equation remains true in all coordinate systems.

In this chapter we review some basics of differential geometry which are used in GR. We start with the basic definitions of manifolds and differential operators which are useful in GR. Further, after defining the vectors and tensors for curved spaces, we introduce the notion of metric tensor and the classification of hypersurfaces based on it. We discuss the *mean* and *Gaussian* curvatures and their generalizations in higher dimensions. After that we define energy momentum tensor and then derive the EFEs. At the end, a brief introduction of black holes and CP diagrams is given.

1.1 Manifolds

Manifolds are the spaces which are locally same as the *Euclidean* space so that coordinate patches covered them. This framework allows differentiation to be defined, but it does not differentiate intrinsically between charts. Therefore manifold structure defines only those concepts which are independent of the choice of coordinates [10]. For a precise definition of manifold, we need some basic definitions.

Let $\Re^n$ be the *Euclidean* space of dimension n, *i.e*, the set of all n-tuples $(x^1, x^2, ..., x^n)$ with usual topology. A map $\psi : U \subset \Re^n \to U' \subset \Re^m$, where U and U' are open subsets, is said to be of class $C^k (k \geq 1)$ if the coordinates $\{x'^1, x'^2, ..., x'^n\}$ of the image point $\psi(p)$ in U' are k-times continuously differentiable (k^{th} derivative exists and is continuous) of the coordinates $\{x^1, x^2, ..., x^n\}$ of p in U. If a map is C^k $\forall$ $k \geq 0$ then it is said to be C^∞.

A C^k n-dimensional manifold M is a set together with a collection of subsets $\{O_\beta\}$

satisfying the following proprieties:
1) the set O_β cover M $i.e$ $M = \bigcup O_\beta$.
2) For each β, there is bijective (*injective* and *surjective*) map $\gamma : O_\beta \to U_\beta$, where U_β is an open subset of $\Re^n$.
3) If $O_\alpha \bigcup O_\beta$ is non-empty, then the map

$$\gamma_\alpha o \gamma_\beta^{-1} : \gamma_\beta(O_\alpha \bigcup O_\beta) \to \gamma_\alpha(O_\alpha \bigcup O_\beta)$$

is a C^k map of an open subset of $\Re^n$ to an open subset of $\Re^n$. Each γ_β is called a *chart*. We can define a topology on a manifold M and consider it as topological space. The definition of manifold mentioned above is very general, we impose two further conditions of *Housdorff* and *second countable* on the definition to make it more precise.
A topological space (X, T) is said to be *Housdorff* if for each $(p, q) \varepsilon X$ and $p \neq q$, we can find open sets O_p, $O_q \varepsilon T$ such that $p \varepsilon O_p$, $q \varepsilon O_q$ and both O_p, O_q are distinct open sets. It is second countable if it has a countable basis and is locally *Euclidean* if every point p in M has a neighborhood V such that there is a homeomorphism ϕ from V onto an open subset of $\Re^n$. A topological manifold is a *Housdorff*, *second countable*, locally *Euclidean* space [11].

1.2 Vectors and Tensors

In $\Re^n$, it is assumed that space has natural structure of n-dimensional vector space and has a point to serve as the origin. The natural rules of addition and scalar multiplication satisfy the vector space axioms. In SR, spacetime similarly has the natural framework of 4-dimensional vector space. When we consider the curved geometries as we do in GR, the $\Re^n$ framework of vector space is vanished. For example, there is no natural way of how to add two points on a sphere and get a third point on it. Due to the presence of curvature we lose the ability to draw the preferred curves from one point to another, or to move them uniquely around the manifold.
We can recover the framework of vector space for curved manifolds in the limit of

"infinitesimal displacements" about a point. For manifolds like the sphere, which naturally arise as surface embedded in $\Re^n$, the concept of tangent vector can be made more precise but in many situations, especially in GR, one is given a manifold without an embedding of it in $\Re^n$. Thus it is valuable to define the tangent vector in a way that refers only to the intrinsic structure of the manifold, not to its possible embeddings in $\Re^n$. Such a definition is provided by the idea of a tangent vector as a directional derivative [12]. In $\Re^n$, vectors and directional derivatives have bijection between them. A vector $\underline{x} = (x^1, x^2, .., x^n)$ defines the operator $x^i \frac{\partial}{\partial x^i}$ (called directional derivative operator) and vice versa [12]. Directional derivatives obey the *Leibnitz* rule and linearity when act on the functions. Therefore on a manifold M let $\wp$ be the collection of C^∞ functions from M into $\Re$. The tangent vector u at point $p \varepsilon M$ is defined to be the map $u : \wp \rightarrow \Re$ which obeys both linearity and *Leibnitz* rule. The tangent vector can be expressed as the linear combination of its basis vectors $\{\frac{\partial}{\partial x^i}\}$ such that

$$\underline{u} = u^i X_i = u^i \frac{\partial}{\partial x^i}. \tag{1.1}$$

If we take $\{\frac{\partial}{\partial x'^i}\}$ be the different coordinate bases, then we can express $\{\frac{\partial}{\partial x^i}\}$ in terms of $\{\frac{\partial}{\partial x'^i}\}$ as

$$\frac{\partial}{\partial x^i} = \frac{\partial x'^j}{\partial x^i}|_{\psi(p)} \frac{\partial}{\partial x'^j}. \tag{1.2}$$

From eqs.(1.1) and (1.2), we have

$$u'^i = u^j \frac{\partial x'^i}{\partial x^j}. \tag{1.3}$$

Eq.(1.3) is the vector transformation law.

At each point $p \varepsilon M$ lying on the C^∞ curve $D(t)$, we can relate a tangent vector v to the curve D by setting $v(f)$, where $f \varepsilon \wp$, equal the function's derivative $foD : \Re \rightarrow \Re$ at p,

$$v(f) = \frac{d}{dt}(foD) = \frac{dx^i}{dt}\frac{\partial f}{\partial x^i} = \frac{dx^i}{dt} X_i(f). \tag{1.4}$$

This shows that the components v^i of the tangent vector to the curve D are

$$v^i = \frac{dx^i}{dt}. \tag{1.5}$$

Tensors are the generalization of vectors. Vector and scalar are the tensors of rank one and zero respectively. Let V be the finite dimensional vector space and V^* be its dual space. A tensor T of type (r,s) is a multilinear map

$$T : V^* \text{ x } V^* \text{ x } \ldots \text{ x } V^* \text{ x } V \text{ x } V \text{ x } \ldots \text{ x } V \ \rightarrow \Re.$$

In general, the components of a tensor T of type (r,s) transform as

$$T'^{\alpha_1\alpha_2......\alpha_s}{}_{\gamma_1\gamma_2......\gamma_r} = \frac{\partial x'^{\alpha_1}}{\partial x^{\beta_1}}\frac{\partial x'^{\alpha_2}}{\partial x^{\beta_2}}\cdot\cdot\cdot\frac{\partial x'^{\alpha_s}}{\partial x^{\beta_s}}\frac{\partial x^{\rho_1}}{\partial x'^{\gamma_1}}\frac{\partial x^{\rho_2}}{\partial x'^{\gamma_2}}\cdot\cdot\cdot\frac{\partial x^{\rho_r}}{\partial x'^{\gamma_r}}T^{\beta_1\beta_2......\beta_s}{}_{\rho_1\rho_2......\rho_r}. \tag{1.6}$$

Einstein's GR is the geometrical theory described in terms of tensors. The metric tensor which plays a key role in this theory because it gives the information of coordinates of a spacetime in terms of mass and gravity [13] . It is denoted by '**g**' and is defined as tensor of type $(0,2)$ [10] on a manifold M. It is the inner product on tangent space at every point. **g** has the components with respect to basis $\{e_i\}$ are

$$g_{ij} = g(e_i, e_j) = g(e_i, e_j). \tag{1.7}$$

If we use a coordinate basis $\left\{\frac{\partial}{\partial x^i}\right\}$, then g in terms of line element is

$$ds^2 = g_{ij}dx^i dx^j. \tag{1.8}$$

The inverse of g_{ij} is g^{ij} so that

$$g_{ij}g^{ik} = \delta^k_j, \tag{1.9}$$

where δ^k_j is the *Kronecker* delta (named after Leopold Kronecker) defined as

$$\delta^k_j = \begin{cases} 1 & \text{if} \quad j = k \ , \\ 0 & \text{if} \quad j \neq k \ . \end{cases}$$

A useful characterization of the metric is obtained by putting g_{ij} into its canonical form [14]. In this form the metric becomes

$$g_{ij} = diag(-1, -1, .., -1, +1, +1, ..., +1, 0, 0, ..., 0), \tag{1.10}$$

where '*diag*' means a diagonal matrix with the given elements. The signature of the metric depends on the signs of eigenvalues. If any of the eigenvalues are zero, the metric is degenerate, and its inverse metric does not exist. In GR, we deal with the continuous and non-degenerate metrics. If all of the signs of the eigenvalues are positive, the metric is called *Euclidean* or *Riemannian* while if there is a single minus or plus it is called *Lorentzian* or pseudo-*Riemannian*, and, if the metric have some positive and some negative eigenvalues, then it is called indefinite. In GR we deal with *Lorentzian* metrics.

1.3 Hypersurface

Let M be the m dimensional manifold and N be the $(m-1)-dimensional$ manifold. A hypersurface may be defined as an $(m-1)-dimensional$ manifold N embedded in an $m-dimensional$ manifold M [10]. It is therefore a surface in one higher dimension. If $\mathbf{g}$ is the metric on M, then there exists a corresponding induced metric $\mathbf{g}^*$ on N given by

$$ds^2 = g^*_{ab} n^a n^b. \tag{1.11}$$

The metric g^* classifies the hypersurface into three groups:

1) If $g^*_{ab} n^a n^b > 0$, the hypersurface will be Time like.

2) If $g^*_{ab} n^a n^b = 0$, the hypersurface will be Null like.

3) If $g^*_{ab} n^a n^b < 0$, the hypersurface will be Space like.

1.4 Differentiation on Manifolds

We normally study three types of differential operators on manifolds given by

1) Exterior Derivative,

2) *Lie* Derivative,

3) Covariant Derivative.

Exterior and Lie derivatives are purely defined by the manifold structure while covariant derivative is defined by placing an extra structure, which we call affine connection, on the manifold.

1.4.1 Exterior Derivative

In order to define the exterior differentiation, first, we need to define differential forms. A differential k-form w on an open subset U of a manifold M is a function that assigns to each point p in U as alternating k-linear function on the tangent space T_pM, *i.e* $w_p \varepsilon A_k(T_pM)$, where $A_k(T_pM)$ is the space of alternating k-tensors [11]. Mathematically a k-form is written as

$$w_p = b_I(p) dx_p^I \qquad 1 \leq i_1 \leq i_2 \leq ... \leq i_k \leq n, \tag{1.12}$$

where b_I's are the functions coefficients and

$$dx_p^I = dx_p^{i_1} \wedge dx_p^{i_2} \wedge ... \wedge dx_p^{i_k}, \tag{1.13}$$

where '$\wedge$' is called *wedge* and $dx^{i_1} \wedge dx^{i_2}$ is the wedge product.
Eq.(1.12) can simply be written as

$$w = b_I dx^I = b_{i_1 i_2 ... i_k} dx^{i_1} \wedge dx^{i_2} \wedge ... \wedge dx^{i_k}. \tag{1.14}$$

A 0-form is just a function while a 1-form is the differential of a function. The exterior derivative operator 'd' maps k-form linearly to $(k+1)$-form as

$$dw = db_{i_1 i_2 ... i_k} \wedge dx^{i_1} \wedge dx^{i_2} \wedge ... \wedge dx^{i_k}. \tag{1.15}$$

The exterior derivative of 0-form 'f' is

$$df = \frac{\partial f}{\partial x^j} dx^j. \tag{1.16}$$

The exterior derivative of a 1-form, $g = b_i dx^i$, is a 2-form given by

$$dg = db_i \wedge dx^i = \frac{\partial b_i}{\partial x^j} dx^i \wedge dx^j. \tag{1.17}$$

Also, by using anti-commutativity of *wedge* product, we have

$$d(df) = \frac{\partial^2 f}{\partial x^j \partial x^k} dx^j \wedge dx^k = -\frac{\partial^2 f}{\partial x^j \partial x^k} dx^j \wedge dx^k, \tag{1.18}$$

$$\implies d(df) = 0. \tag{1.19}$$

This holds true for any k-form.

1.4.2 Lie Differentiation

Consider any C^k $(k \geq 1)$ vector field $\underline{U}$ on manifold M. By fundamental theorem for systems of ODEs [15] there is a unique maximal curve $\underline{r}(t)$ through each point p of M such that $\underline{r}(0) = p$ and the tangent vector of that curve at the point $r(t)$ is the vector $\underline{X}|_{r(t)}$. If $\{U^\tau\}$ are local coordinates, so that the curve $\underline{r}(t)$ has the coordinates $x^\tau(t)$ and the vector $\underline{U}$ has the components U^τ, then this curve is locally a solution of a set of differential equations

$$\frac{dx^\tau}{dt} = U^\tau \left(x^1(t), x^2(t), .., x^n(t)\right). \tag{1.20}$$

This curve is called an integral curve of x with initial point p. For each point q of M, there is an open neighborhood ϕ of q and an $\epsilon > 0$ such that $\underline{U}$ defines a family of diffeomorphisms (maps which are *one-one* C^k and their inverses are also C^k) $\psi_t : \phi \rightarrow M$ whenever $|t| < \epsilon$, obtained by taking each point p in ϕ a parameter distance t along the integral curves of $\underline{U}$. This diffeomorphism maps each tensor field $\underline{V}$ at p of type (k, l) into $\psi_{t_*} V|_{\psi_{t(p)}}$.

The *Lie* derivative $L_{\underline{U}}\underline{V}$ (generalization of directional derivative) of a tensor field $\underline{V}$ with respect to $\underline{U}$ is defined as

$$L_{\underline{U}}\underline{V}|_p = \lim_{t\rightarrow 0} \frac{V|_p - \psi_{t_*} V|_p}{t}. \tag{1.21}$$

If U and V are both C^∞ vector fields, then

$$L_{\underline{U}}\underline{V} = [\underline{U}, \underline{V}], \tag{1.22}$$

where $[\underline{U}, \underline{V}]$ is the *Lie* bracket defined as

$$[\underline{U}, \underline{V}]f = \underline{U}(\underline{V}f) - \underline{V}(\underline{U}f) = -[\underline{V}, \underline{U}]f. \tag{1.23}$$

$$\Longrightarrow \; [\underline{U}, \underline{V}] = -[\underline{V}, \underline{U}] \tag{1.24}$$

The vector fields $\underline{U}$ and $\underline{V}$ will commute if $L_{\underline{U}}\underline{V} = 0$.

1.4.3 Covariant Derivative and Christoffel Symbols

For flat space, the operator ∂_ν is defined as

$$\partial_\nu : U^n_m \rightarrow U^n_{m+1}, \tag{1.25}$$

where U^n_m and U^n_{m+1} are tensor fields of rank $m+n$ and $m+n+1$ respectively. The map in the expression (1.25) obeys the following properties:

1) It acts linearly on each of its argument.

2) It obeys the Leibnitz rule on tensor product.

The action of ∂_ν depends upon the given choice of coordinate system. To define an operator $\triangledown$ (called covariant derivative operator) that should be independent of coordinate system being used. We require that $\triangledown$ be a map and it satisfies (1)-(2) also for the scalar function ϕ, it should reduce to partial derivative *i.e*

$$\triangledown_\mu \phi = \partial_\mu \phi. \tag{1.26}$$

The derivative (covariant) of a vector U^ν is given as

$$\triangledown_\mu U^\nu = \partial_\mu U^\nu + \Gamma^\nu_{\mu\beta} U^\beta. \tag{1.27}$$

Covariant derivative is also denoted by ";" [10]. In this notation above equation can be written as

$$U^\nu_{;\mu} = \partial_\mu U^\nu + \Gamma^\nu_{\mu\beta} U^\beta, \tag{1.28}$$

where $\Gamma^\nu_{\mu\beta}$ are the *Christoffel* symbols or connection symbols, defined as

$$\Gamma^\nu_{\mu\beta} = \frac{1}{2} g^{\nu a} \left(g_{a\beta,\mu} - g_{\mu\beta,a} + g_{a\mu,\beta} \right). \tag{1.29}$$

For mixed tensor the covariant derivative is

$$\begin{aligned} \triangledown_\beta T^{\alpha_1\alpha_2......\alpha_m}{}_{\gamma_1\gamma_2......\gamma_n} &= \partial_\beta T^{\alpha_1\alpha_2......\alpha_m}{}_{\gamma_1\gamma_2......\gamma_n} + \Gamma^{\alpha_1}_{\beta\rho} T^{\rho\alpha_2......\alpha_m}{}_{\gamma_1\gamma_2......\gamma_n} \\ &+ \Gamma^{\alpha_2}_{\beta\rho} T^{\alpha_1\rho......\alpha_m}{}_{\gamma_1\gamma_2......\gamma_n} + ... + \Gamma^{\alpha_m}_{\beta\rho} T^{\alpha_1\alpha_2......\rho}{}_{\gamma_1\gamma_2......\gamma_n} \\ &- \Gamma^{\rho}_{\beta\gamma_1} T^{\alpha_1\alpha_2......\alpha_m}{}_{\rho\gamma_2......\gamma_n} - \Gamma^{\rho}_{\beta\gamma_2} T^{\alpha_1\alpha_2......\alpha_m}{}_{\gamma_1\rho......\gamma_n} \\ &- ...... - \Gamma^{\rho}_{\beta\gamma_n} T^{\alpha_1\alpha_2......\alpha_m}{}_{\gamma_1\gamma_2......\rho}. \end{aligned} \tag{1.30}$$

1.5 Curvature

Let $\underline{\beta}(\varpi)$ be a curve given by

$$\underline{\beta}(\varpi) = \left(x^1(\varpi), x^2(\varpi), x^3(\varpi)\right). \tag{1.31}$$

We choose a rescaled parameter 's' such that $d\underline{\beta}/ds$ always has a unit magnitude. This parameter is called affine parameter. With this parameter the arc length of the curve is defined as

$$s = \int \left|\frac{d\underline{\beta}}{d\varpi}\right| d\varpi, \tag{1.32}$$

then

$$ds^2 = \left|\frac{d\underline{\beta}}{d\varpi}\right|^2 d\varpi^2 = \left(dx^1\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2. \tag{1.33}$$

The unit tangent vector $\underline{T}(\varpi)$ to the curve $\underline{\beta}(\varpi)$ is

$$\underline{T}(\varpi) = \frac{d\underline{\beta}/d\varpi}{\left|d\underline{\beta}/d\varpi\right|}, \tag{1.34}$$

and the unit normal $\underline{N}(\varpi)$ is

$$\underline{N}(\varpi) = \frac{\underline{T}'(\varpi)}{|\underline{T}'(\varpi)|}, \tag{1.35}$$

where $|\underline{T}'(\varpi)|$ gives the measure of curvature of the curve. The binormal vector $\underline{B}(\varpi)$ is

$$\underline{B}(\varpi) = \underline{T}(\varpi) \wedge \underline{N}(\varpi). \tag{1.36}$$

The quantity $|B'(s)|$ is called the second curvature.

A surface 'S' can be described as

$$S(v, \varpi) = \left(x^1(v, \varpi), x^2(v, \varpi), x^3(v, \varpi)\right). \tag{1.37}$$

Inserting eq.(1.37) in eq.(1.33), we get

$$ds^2 = Edv^2 + 2Fdvd\varpi + Gd\varpi^2, \tag{1.38}$$

where

$$E = S_v.S_v, \tag{1.39}$$

$$F = S_v.S_\varpi, \tag{1.40}$$

$$G = S_\varpi.S_\varpi. \tag{1.41}$$

This is called *First Fundamental Form* (FFF) of line element on the surface.

Now let $P(v_0, \varpi_0)$ be the point on the surface, and $\underline{\gamma}(v_0, \varpi_0)$ is the position vector, then $\underline{\gamma}_v$ and $\underline{\gamma}_\varpi$ at $P(v_0, \varpi_0)$ form a basis of tangent space. Thus the vector $\underline{\gamma}_v \wedge \underline{\gamma}_\varpi$ is orthognal to tangent plane. The unit normal to the surface can the be defined as

$$\underline{N}^* = \frac{\underline{\gamma}_v \wedge \underline{\gamma}_\varpi}{\left|\underline{\gamma}_v \wedge \underline{\gamma}_\varpi\right|}. \tag{1.42}$$

If

$$\underline{\gamma}(s) = \underline{\gamma}(v(s), \varpi(s)), \tag{1.43}$$

is a curve on this surface then

$$\frac{d\underline{\gamma}}{ds} = \frac{\partial\underline{\gamma}}{\partial v}\frac{dv}{ds} + \frac{\partial\underline{\gamma}}{\partial\varpi}\frac{d\varpi}{ds} = \underline{\gamma}_v \dot{v} + \underline{\gamma}_\varpi \dot{\varpi}, \tag{1.44}$$

and

$$\ddot{\underline{\gamma}}(s) = \underline{\gamma}_{vv}\dot{v}^2 + \underline{\gamma}_v \ddot{v} + \underline{\gamma}_{v\varpi}\dot{v}\dot{\varpi} + \underline{\gamma}_{\varpi v}\dot{v}\dot{\varpi} + \underline{\gamma}_{\varpi\varpi}\dot{\varpi}^2 + \underline{\gamma}_\varpi \ddot{\varpi}. \tag{1.45}$$

Taking dot product of eq.(1.45) with N^*, we get

$$\ddot{\underline{\gamma}}.\underline{N}^* = E^*\dot{v}^2 + 2F^*\dot{v}\dot{\varpi} + G^*\dot{\varpi}^2, \tag{1.46}$$

where

$$E^* = \underline{\gamma}_{vv}.\underline{N}^*, \tag{1.47}$$

$$F^* = \underline{\gamma}_{v\varpi}.\underline{N}^*, \tag{1.48}$$

$$G^* = \underline{\gamma}_{\varpi\varpi}.\underline{N}^*. \tag{1.49}$$

The *Second Fundamental Form* (SFF) can be written as

$$II = E^* dv^2 + 2F^* dv d\varpi + G^* d\varpi^2. \tag{1.50}$$

The normal curvature K_n is defined as

$$K_n = \frac{SFF}{FFF} = \frac{E^* dv^2 + 2F^* dv d\varpi + G^* d\varpi^2}{E dv^2 + 2F dv d\varpi + G d\varpi^2}. \tag{1.51}$$

The normal curvature has maximum and minimum values, denoted by k_1 and k_2 respectively, along two perpendicular directions, called principle curvatures. The product

$k_1 k_2$ is the *Gaussian* Curvature and the mean $(k_1 + k_2)/2$ is called *mean* curvature on the surface. Both *Gaussian* and *mean* curvatures are invariant under coordinate transformations.

The quantities that can be expressed in terms of FFF are called *intrinsic* to the surface. *Gaussian* curvature can purely be expressed in terms of of FFF, and therefore, is *intrinsic* property of the surface [16], while those expressed in terms of SFF are *extrinsic* to the surface, and therefore, *mean* curvature is often referred as the Mean Extrinsic Curvature (MEC).

The MEC in higher dimensions can be written as

$$K = \text{trace}(K) = K^b_b, \tag{1.52}$$

where the extrinsic curvature, K^c_d, is defined as

$$K^c_d = -n^c_{;d}. \tag{1.53}$$

Using eq.(1.53) in eq.(1.52), we get

$$K = -n^c_{;c}. \tag{1.54}$$

To obtain the generalization of the *Gaussian* curvature in higher dimensions, consider

$$V_{a;b;c} - V_{a;c;b} = \left(V_{a,b} - \Gamma^u_{ab} V_u\right)_{;c} - \left(V_{a,c} - \Gamma^u_{ac} V_u\right)_{;b}. \tag{1.55}$$

After simplification, we get

$$V_{a;b;c} - V_{a;c;b} = R^u_{abc} V_u, \tag{1.56}$$

where

$$R^u_{abc} = \Gamma^u_{ac,b} - \Gamma^u_{ab,c} + \Gamma^v_{ac}\Gamma^u_{vb} - \Gamma^v_{ab}\Gamma^u_{vc}, \tag{1.57}$$

is called *Riemann curvature tensor*. In lower indices it is given by

$$R_{abcd} = g_{ae} R^e_{bcd}. \tag{1.58}$$

The trace of R^u_{abc} is called *Ricci tensor*, R_{bc}, and, is given by

$$R_{bc} = \Gamma^u_{uc,b} - \Gamma^u_{ub,c} + \Gamma^v_{uc}\Gamma^u_{vb} - \Gamma^v_{ub}\Gamma^u_{vc}. \tag{1.59}$$

The trace of *Ricci tensor* is called *Ricci scalar*, R, and, is given by

$$R = g^{ab}R_{ab}. \tag{1.60}$$

The *Ricci scalar* is the higher dimensional generalization of *Gaussian* curvature.

1.6 Parallel Transport

We can understand parallel transport as transporting a tensor along a curve on a manifold while keeping it constant. On a flat space, a vector can be parallely transported along any curve from one point to another without depending on the geometry of the curve. This is easy to visualize if we use cartesian coordinates. Then we can parallel transport a vector by keeping its components constant.

In a curved space, parallel transport is not simple anymore. The easiest way to understand this, consider a vector on the sphere. If we move this vector along a closed curve while letting it point in the same direction, when it comes back to the starting point, it will not be parallel to the original vector. This means we have no well defined way to globally say that vectors are parallel or not.

To explain parallel transport mathematically, consider the curve $x^\alpha(\varpi)$ in the flat space, where ϖ is a parameter. On this flat space, the requirement that a tensor $T^{a_1a_2\ldots a_m}{}_{b_1b_2\ldots b_n}$ is constant along $x^\alpha(\varpi)$ is

$$0 = \frac{d}{d\varpi}T^{a_1a_2\ldots a_m}{}_{b_1b_2\ldots b_n}. \tag{1.61}$$

Using chain rule, we have

$$0 = \frac{dx^\alpha}{d\varpi}\frac{\partial}{\partial x^\alpha}\left(T^{a_1a_2\ldots a_m}{}_{b_1b_2\ldots b_n}\right) \tag{1.62}$$

$\forall\ \alpha$. To generalize this concept for curved space we need to replace partial derivative by covariant derivative. Therefore a tensor is called parallel transported along the curve if it satisfy

$$\frac{dx^\alpha}{d\varpi}T^{a_1a_2\ldots a_m}{}_{b_1b_2\ldots b_n;\alpha} = 0. \tag{1.63}$$

For simplicity, we consider a case of a vector (contravariant) $\underline{X}^\beta$, then eq.(1.63) becomes

$$\frac{dx^\alpha}{d\varpi}\underline{X}^\beta_{;\alpha} = \frac{dx^\alpha}{d\varpi}\left(\underline{X}^\beta,\alpha + \Gamma^\beta_{\alpha\nu}\underline{X}^\nu\right) = \frac{dX^\alpha}{d\varpi} + \Gamma^\beta_{\alpha\nu}\frac{dx^\alpha}{d\varpi}\underline{X}^\nu = 0. \tag{1.64}$$

1.7 Geodesics

A generalization of a straight line in a curved space is called a geodesic. It is the shortest distance between two points in a curved space. A geodesic is a curve that parallel transport its own tangent vector.

To obtain the geodesic equation, consider a curve $x^{\alpha}(\varpi)$, then the tangent vector along this curve is $\frac{dx^{\alpha}}{d\varpi}$. The requirement that this tangent vector is parallel transported along the curve is obtained by eq.(1.64). So if we take $x^{\alpha}(\varpi)$ be a geodesic then it must satisfy

$$\frac{d}{d\varpi}\left(\frac{dx^{\alpha}}{d\varpi}\right) + \Gamma^{\beta}_{\alpha\nu}\frac{dx^{\alpha}}{d\varpi}\frac{dx^{\nu}}{d\varpi} = 0, \tag{1.65}$$

$$\frac{d^2x^{\alpha}}{d\varpi^2} + \Gamma^{\beta}_{\alpha\nu}\frac{dx^{\alpha}}{d\varpi}\frac{dx^{\nu}}{d\varpi} = 0. \tag{1.66}$$

The eq.(1.66) is the required geodesic equation.

1.8 Metric Singularities

Singularity of a metric is a place where the curvature blows up [12]. If the coefficients of the metric become undefined at some point then the metric is called singular at that point. There are two kinds of singularities:

1) Coordinate (removable) Singularity

2) Essential Singularity

1.8.1 Coordinate (Removable) Singularity

Coordinate singularity appears due to inappropriate selection of coordinates. It can be removed by introducing new coordinates. For example, the metric of 2-sphere is

$$ds^2 = b^2 d\theta^2 + b^2 \sin^2\theta d\phi^2, \tag{1.67}$$

where b is the radius. If we introduce $bsin\theta = \mu$, then the metric becomes

$$ds^2 = \frac{d\mu^2}{1-\mu^2} + \mu^2 d\phi^2, \tag{1.68}$$

which is singular at $\mu = \pm 1$. This coordinate singularity appeared because of wrong selection of coordinates.

1.8.2 Essential Singularity

Essential (Crushing) or curvature singularity is the singularity where curvature tensor or equivalently the curvature invariants (mentioned in the next section) become infinite. Crushing singularity can never be removed by introducing new coordinates as we shall see this in the cases of the *Schwarzschild* and RN blackholes.

1.9 Curvature Invariants

Curvature invariants are used to determine whether the singularity is essential or coordinate. In general there are 14 curvature invariants [17, 18]. For a vacuum 10 of these automatically vanish as $R_{\mu\nu} = 0$.

$$I_1 = R = g^{ab} R_{ab}, \tag{1.69}$$

$$I_2 = R^{cd}_{ab} R^{ab}_{cd}, \tag{1.70}$$

$$I_3 = R^{abcd}_{efgh} R^{efgh}_{abcd}. \tag{1.71}$$

1.10 Energy-Momentum Tensor

GR comprise with the gravitational field which depends on matter distribution in space and its temporal evolution. Thus, we need a mathematical distribution of matter in spacetime. As SR provides a relation between mass and energy, the spacetime illustration must incorporate with the distribution of energy as well. The energy could either be carried by matter or accumulated in the field. Specifically, it could be contained in stresses arrangement in a medium approximate as continuum.

Before proceeding to the full relativistic description it is worthwhile to review the classical, 3-dimensional illustration of stresses."*Stress*" is a generalization of the concept of *pressure*, which is force per unit area. since *force* and *area* are vector quantities in 3-dimension. So, generally there is no meaning of dividing two vectors. The concept of *pressure* is applicable if the directions of the vector don't make a difference and only the magnitudes are relevant. Isotropic medium is the medium for which this requirement holds true. For example the pressure of a gas, as deduced from the kinetic energy of gases, is the same in all directions. Similarly, water is an isotropic medium. However, consider a helical spring stretched or squashed. The generalization of anisotropic media is called *stress*. The stress is given by stress tensor.

$$\sigma^{ab} = \frac{dF^a}{dS_b} \qquad (a, b = 1, 2, 3), \tag{1.72}$$

The skew part of this tensor $\sigma^{[a,b]}$ gives the rotation. From this quantity the vorticity vector can be defined by using the totally skew tensor in 3-dimensions,

$$\Omega_c = \frac{1}{2} e_{abc} \sigma^{ab}. \tag{1.73}$$

Assuming an irrotational medium, so that $\sigma_c = 0$ the stress tensor will be symmetric, $\sigma^{ab} = \sigma^{ba}$. It is symmetric part of the stress tensor that is the generalization of pressure. For the relativistic generalization of stress tensor we need to connect something to σ^{ab} to make it 4x4 matrix that will give the components of a 4-dimensional 2^{nd} rank tensor. Pressure (or stress) has units of energy density, ρc^2, in a manner consistent with the requirement of Lorentz covariance (i.e. invariance of the tensor under Lorentz transformations) is the time-time part. In the rest frame, there can only be the rest energy and the stresses with no kinetic component. Thus, in the rest frame in Minkowski spacetime the 4-dimensional tensor is

$$T^{\alpha\beta} = \rho c^2 \delta_0^\alpha \delta_0^\beta + \sigma^a b \delta_a^\alpha \delta_b^\beta. \tag{1.74}$$

In GR, the presence of matter curved the spacetime. (1.74) can be generalized for any arbitrary frame and arbitrary manifold by

$$T^{\alpha\beta} = \rho u^\alpha u^\beta + \sigma^a b \delta_a^\alpha \delta_b^\beta, \tag{1.75}$$

where u^β is the velocity 4-vector. In the case that there are no stresses in an arbitrary frame $T_{\alpha\beta}$ provides the momentum and total energy of any fluid portion. Therefore, it is called the energy-momentum tensor. As it generally give stresses it is also called stress energy tensor.

1.11 The Einstein Field Equations

To derive the Einstein Field Equations (EFEs) consider the action

$$P = P_G + P_M, \tag{1.76}$$

where P_M and P_G are components of the action because of matter and gravity. P is defined as

$$P = \int \mathcal{L}\sqrt{-g}d^4x, \tag{1.77}$$

where

$$\mathcal{L} = \mathcal{L}_G + \mathcal{L}_M, \tag{1.78}$$

$$\mathcal{L}_G = \frac{1}{2k}R, \tag{1.79}$$

and $k = \frac{8\pi G}{c^4}$ and R is *Ricci Scalar*. Let the action be varied due to a variation in $g^{\mu\nu}$ and $\delta g^{\mu\nu}$ and the total action must remain invariant *i.e*

$$0 = \delta P = \delta P_G + \delta P_M. \tag{1.80}$$

By using eq.(1.78) and eq.(1.79) in eq.(1.77) we obtain

$$P = \int_v \frac{1}{2k}R\sqrt{-g}d^4x + \int_v \mathcal{L}_M\sqrt{-g}d^4x. \tag{1.81}$$

Taking δP in eq.(1.81), we have

$$\delta P = \frac{1}{2k}\int_v \delta(R\sqrt{-g})d^4x + \int_v \delta(\mathcal{L}_M\sqrt{-g})d^4x. \tag{1.82}$$

Using eq.(1.60) in eq.(1.82), we have

$$\delta P = \tfrac{1}{2k}\int_v \delta(g^{\gamma\pi}R_{\gamma\pi}\sqrt{-g})d^4x + \int_v \delta(\mathcal{L}_M\sqrt{-g})d^4x,$$

$$\delta P = \frac{1}{2k}\int_v \left(R_{\gamma\pi}\delta(g^{\gamma\pi}\sqrt{-g}) + g^{\gamma\pi}\sqrt{-g}\delta R_{\gamma\pi}\right) d^4x$$
$$+ \int_v \left(\delta(\mathcal{L}_M)\sqrt{-g} + \mathcal{L}_M\delta\sqrt{-g}\right) d^4x,$$

$$\delta P = \frac{1}{2k}\int_v \left(R_{\gamma\pi}g^{\gamma\pi}\delta\sqrt{-g} + R_{\gamma\pi}\sqrt{-g}\delta(g^{\gamma\pi}) + \sqrt{-g}g^{\gamma\pi}\delta R_{\gamma\pi}\right) d^4x$$
$$+ \int_v \left(\delta(\mathcal{L}_M)\sqrt{-g} + \mathcal{L}_M\delta\sqrt{-g}\right) d^4x. \tag{1.83}$$

Using $\delta g = gg^{\gamma\pi}\delta g_{\gamma\pi} = -gg_{\gamma\pi}\delta g^{\gamma\pi}$

$$\delta\sqrt{-g} = -\frac{1}{2\sqrt{-g}}(-gg_{\gamma\pi}\delta g^{\gamma\pi}),$$
$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\gamma\pi}\delta g^{\gamma\pi}. \tag{1.84}$$

Using eq.(1.84) in eq.(1.83), we get

$$\int_v [\delta(\mathcal{L}_M)\sqrt{-g} + \mathcal{L}_M\delta\sqrt{-g}]d^4x = \delta S_M. \tag{1.85}$$

Using eq.(1.84) in eq.(1.85) we have,

$$\delta P_M = \int_v (\delta(\mathcal{L}_M)\sqrt{-g})d^4x + \int_v \mathcal{L}_M[\frac{-1}{2}\sqrt{-g}g_{\gamma\pi}\delta g^{\gamma\pi}]d^4x, \tag{1.86}$$

and

$$\delta\mathcal{L}_M = \frac{\partial\mathcal{L}_M}{\partial g^{\gamma\pi}}\delta g^{\gamma\pi}.$$

By using above equation in eq.(1.86), we have

$$\delta P_M = \int_v [\frac{\partial\mathcal{L}_M}{\partial g^{\mu\nu}} - \frac{1}{2}\mathcal{L}_M g_{\gamma\pi}]\delta g^{\gamma\pi}\sqrt{-g}d^4x, \tag{1.87}$$

$$\delta P_M = -\frac{1}{2}\int_v [-2\frac{\partial\mathcal{L}_M}{\partial g^{\gamma\pi}} + \mathcal{L}_M g_{\gamma\pi}]\delta g^{\gamma\pi}\sqrt{-g}d^4x. \tag{1.88}$$

We define the stress-energy momentum tensor $T_{\gamma\pi}$ as

$$T_{\gamma\pi} = -2\frac{\partial\mathcal{L}_M}{\partial g^{\gamma\pi}} + \mathcal{L}_M g_{\gamma\pi}.$$

Therefore eq.(1.88) become,

$$\delta P_M = \int_v \sqrt{-g}T_{\gamma\pi}\delta g^{\gamma\pi}d^4x. \tag{1.89}$$

Now, choosing *Reimann* normal coordinates, *i.e.* $\Gamma^a_{bc} = 0 \quad \Gamma^a_{bc,\alpha} \neq 0$, as

$$\begin{aligned} R_{\gamma\pi} &= \Gamma^{\lambda}_{\gamma\pi,\lambda} - \Gamma^{\lambda}_{\mu\lambda,\nu}, \\ \delta R_{\gamma\pi} &= \delta(\Gamma^{\lambda}_{\gamma\pi,\lambda}) - \delta(\Gamma^{\lambda}_{\mu\lambda,\nu}), \\ \delta R_{\gamma\pi} &= \delta(\Gamma^{\lambda}_{\gamma\pi})_{,\lambda} - \delta(\Gamma^{\lambda}_{\mu\lambda})_{,\nu}, \\ \delta R_{\gamma\pi} &= (\delta\Gamma^{\lambda}_{\gamma\pi})_{;\lambda} - (\delta\Gamma^{\lambda}_{\mu\lambda})_{;\nu}. \end{aligned}$$

Multiplying by $g^{\gamma\pi}$, we get,

$$g^{\gamma\pi}\delta R_{\gamma\pi} = g^{\gamma\pi}(\delta\Gamma^{\lambda}_{\gamma\pi})_{;\lambda} - g^{\gamma\pi}(\delta\Gamma^{\lambda}_{\mu\lambda})_{;\nu}. \tag{1.90}$$

By replacing λ by ν, eq.(1.90) becomes,

$$g^{\gamma\pi}\delta R_{\gamma\pi} = (g^{\gamma\pi}\delta\Gamma^{\lambda}_{\gamma\pi} - g^{\mu\lambda}\delta\Gamma^{\nu}_{\gamma\pi})_{;\lambda}.$$

We define a vector A^{λ} as,

$$A^{\lambda} = (g^{\gamma\pi}\delta\Gamma^{\lambda}_{\gamma\pi} - g^{\mu\lambda}\delta\Gamma^{\nu}_{\gamma\pi}),$$

$$g^{\gamma\pi}\delta R_{\gamma\pi} = A^{\lambda}_{;\lambda}. \tag{1.91}$$

Multiplying eq.(1.91) by $\sqrt{-g}$ and integrating over the 4-dimensional volume element we get

$$\int_v g^{\gamma\pi}\delta R_{\gamma\pi}\sqrt{-g}d^4x = \int_v A^{\lambda}_{;\lambda}\sqrt{-g}d^4x.$$

By using Gauss-divergence theorem, the R.H.S of above equation can be neglected and therefore the L.H.S is also neglected.

Now

$$\begin{aligned} \delta(g^{\gamma\pi}\sqrt{-g}) &= \sqrt{-g}\delta g^{\gamma\pi} + g^{\gamma\pi}\delta\sqrt{-g}, \\ \delta(g^{\gamma\pi}\sqrt{-g}) &= \sqrt{-g}\delta g^{\gamma\pi} + g^{\gamma\pi}(-\tfrac{1}{2}\sqrt{-g}g_{\alpha\beta}\delta g_{\alpha\beta}). \end{aligned} \tag{1.92}$$

By using eq.(1.89) and eq.(1.92) in eq.(1.83) we get,

$$\delta P = \frac{1}{2k}\int_v \sqrt{-g}R_{\gamma\pi}(\delta g^{\gamma\pi} - \frac{1}{2}g^{\gamma\pi}g_{\alpha\beta}\delta g^{\alpha\beta}) - \frac{1}{2}\int_v \sqrt{-g}T_{\gamma\pi}\delta g^{\gamma\pi}d^4x, \tag{1.93}$$

$$\delta P = \frac{1}{2k}\int_v \sqrt{-g}(R_{\gamma\pi}\delta g^{\gamma\pi} - \frac{1}{2}Rg_{\gamma\pi}\delta g^{\gamma\pi})d^4x - \frac{1}{2}\int_v \sqrt{-g}T_{\gamma\pi}\delta g^{\gamma\pi}d^4x,$$

$$\delta P = \frac{1}{2k}\int_v \sqrt{-g}[R_{\gamma\pi} - \frac{1}{2}g_{\gamma\pi}R - kT_{\gamma\pi}]\delta g^{\gamma\pi}d^4x. \tag{1.94}$$

Put eq.(1.94) in eq.(1.80) we get

$$R_{\gamma\pi} - \frac{1}{2}Rg_{\gamma\pi} - kT_{\gamma\pi} = 0,$$

$$R_{\gamma\pi} - \frac{1}{2}Rg_{\gamma\pi} = kT_{\gamma\pi}.$$

We can add a constant $\wedge$ (called the cosmological constant) to the action and thereby modify the gravitational Lagrangian density to

$$\mathcal{L}_G = \frac{1}{2k}(R + \wedge).$$

By following the above procedure we get the Einstein equations with a cosmological term

$$R_{\gamma\pi} - \frac{1}{2}Rg_{\gamma\pi} - \wedge g_{\gamma\pi} = kT_{\gamma\pi}. \tag{1.95}$$

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1.12 Black Holes

Stars have the ability to support themselves against gravity because of the pressure created by thermonuclear reactions. But these are not ever lasting continuing responses. At the point when nuclear fuel depleted, stars starts shrinking due to the pressure reaction and loosing its balance with gravity. In case of massive stars, all the forces acting outward get reduced by inward forces of gravity and disruption continues. As a result density of stars continues to increase and volume decreases, therefore the escape velocity surpasses the speed of light. At this stage, star is said to be a 'black hole'.

1.13 CP Diagrams

Eddington [19] was the first who discussed the two dimensional representation of space-time for the *Schwarzschild* Metric (SM). After this, Finkelstein [20] and Kruskal [21] describe the same work independently. In this regard Carter [22] and Graves and Brill [23] represents the same techniques for RN and Kerr solutions respectively. Carter [24] and Boyer and Lindquist [25] did the same. Martin Walker [26] took the duty of systematizing, formalizing and generalizing the techniques of above mentioned authors. Now these illustrations are known as Carter-Penrose (CP) or Penrose diagrams. Previously, these were known as block diagrams.

Chapter 2

Some Exact Solutions of the EFEs

Any spacetime metric can, in a sense, be considered as satisfying the EFEs [10] because, after obtaining the L.H.S. of the EFEs from the metric tensor, we can define energy-momentum tensor T_{ab} as the R.H.S. of the EFEs. T_{ab} in general have unreasonable physical properties, if the matter content is reasonable that the solution will be reasonable.

A spacetime for which the EFEs are satisfied with T_{ab} (for some reasonable matter content) is an exact solution of the EFEs. Due to the complexity of these equations, it is difficult to find the exact solutions except in space of rather high symmetry. In particular we can find exact solutions for vacuum ($T_{ab} = 0$). Exact solutions describe the qualitative features that can arise in GR. In this chapter our main focus will be on vacuum solutions of these equations.

These solutions can be divided into two categories:

1) Solutions with $\wedge = 0$

2) Solutions with $\wedge \neq 0$

2.1 Some Exact Solutions with $\wedge = 0$

In this section we first discuss the Minkowski metric as a solution of the EFEs, then the *Schwarzschild* metric, its singularities and some appropriate coordinates to study its geometry are discussed. After that we will do the same for RN metric. *Kerr* metric is discussed briefly.

2.1.1 The Minkowski Metric

Minkowski metric is the simplest metric in GR, and is infact the metric of Special Relativity. The topology of this metric is R^4. In terms of (x, y, z, w) coordinates on R^4, it is given as

$$ds^2 = dw^2 - dx^2 - dy^2 - dz^2. \tag{2.1}$$

If we use spherical polar coordinates (t, r, θ, ϕ) given by,

$$w = t,$$

$$x = r \sin\theta \sin\phi,$$

$$y = r \sin\theta \cos\phi,$$

$$z = r \cos\theta.$$

The metric in eq.(2.1), then becomes

$$ds^2 = dt^2 - dr^2 - r^2 d\Omega^2, \tag{2.2}$$

where

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2. \tag{2.3}$$

$r = 0$ and $\sin\theta = 0$ are the singularities of this metric, but these singularities appear due to the inappropriate choice of coordinates.
We can define new coordinates (Y, X, θ, ϕ), with 'θ' and 'ϕ' remain unchanged,

$$\begin{aligned} Y &= \tan^{-1}(r+t) + \tan^{-1}(t-r), \\ X &= \tan^{-1}(r+t) - \tan^{-1}(t-r). \end{aligned} \tag{2.4}$$

The metric given in eq.(2.2) becomes

$$ds^2 = \frac{1}{4}\sec^2\left(\frac{Y+X}{2}\right)\sec^2\left(\frac{Y-X}{2}\right)\left(dY^2 - dX^2\right) - r^2 d\Omega^2. \tag{2.5}$$

The CP diagram of Minkowski metric is shown in Figure. 2.1 and its maximal extension is shown in Figure. 2.2

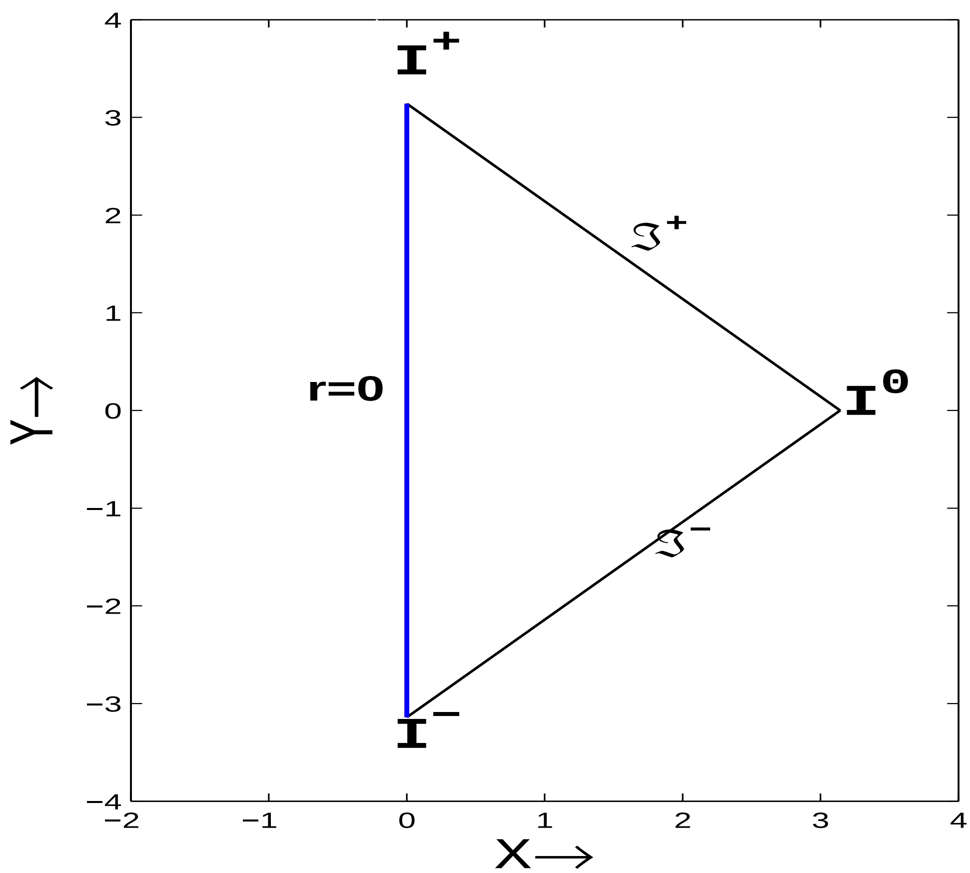

Figure 2.1: CP diagram of the Minkowski metric in (Y, X, θ, ϕ) coordinates. Notice that $r = 0$ is just a coordinate singularity.

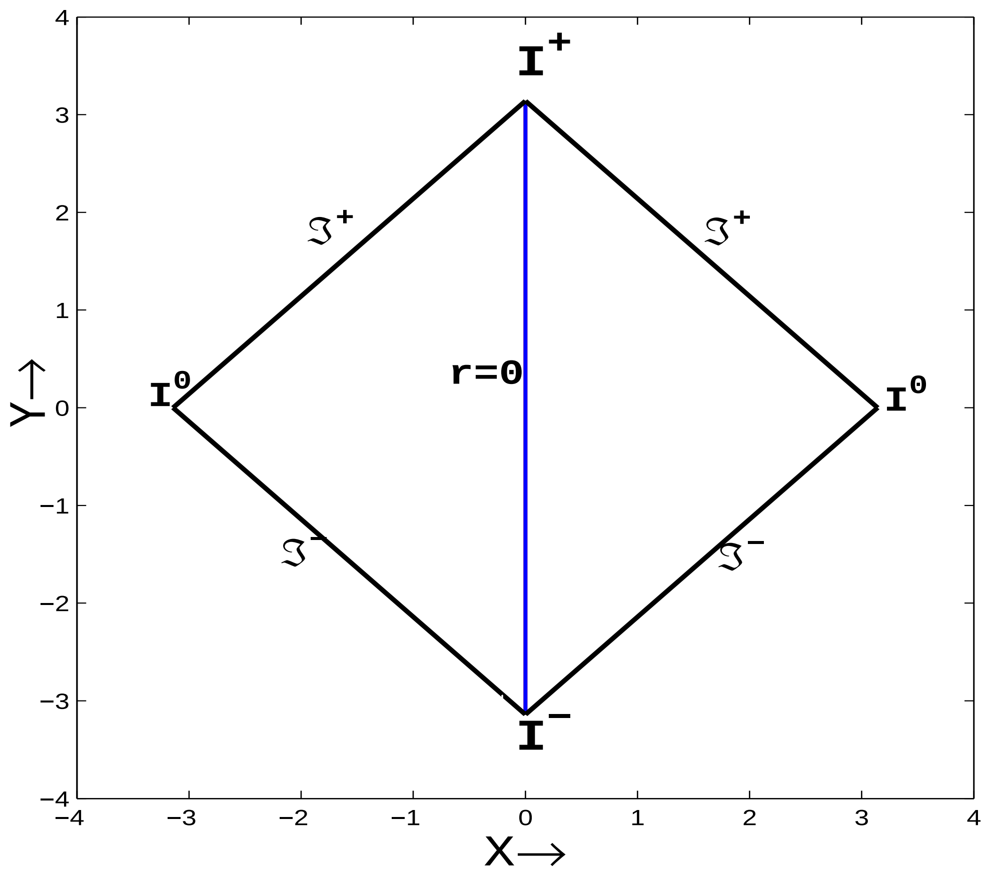

Figure 2.2: Maximal extension of the Minkowski metric.

2.1.2 The *Schwarzschild* Metric

When Einstein published his GR, it was thought that the exact solution of these equations can not be obtained due to their complexity. But soon after that, for Einstein's vacuum field equations, Karl Schwarzschild obtained the exact solution for exterior gravitational field for spherically symmetric static body. To derive this solution we consider the metric

$$ds^2 = e^{2\mu(r)}c^2dt^2 - e^{2\beta(r)}dr^2 - e^{2\delta(r)}r^2d\theta^2 - e^{2\psi(r)}r^2\sin^2\theta d\phi^2. \tag{2.6}$$

For spherical symmetry, we must have

$$\delta(r) = \psi(r). \tag{2.7}$$

Then eq.(2.6) becomes

$$ds^2 = e^{2\mu(r)}c^2dt^2 - e^{2\beta(r)}dr^2 - e^{2\delta(r)}r^2d\Omega^2. \tag{2.8}$$

Now let

$$\begin{aligned} r^* &= re^{\delta(r)}, \\ dr^* &= dre^{\delta(r)} + e^{\delta(r)}rd\delta, \\ &= e^{\delta}\left(1 + r\frac{d\delta}{dr}\right)dr. \end{aligned}$$

Or

$$dr = e^{-\delta}\frac{dr^*}{\left(1 + r\frac{d\delta}{dr}\right)}.$$

Put above equation in eq.(2.8)

$$ds^2 = e^{2\mu(r)}c^2dt^2 - e^{2\beta(r)-2\delta(r)}\left(1 + r\frac{d\delta}{dr}\right)^{-2}dr^{*^2} - r^{*^2}d\Omega^2. \tag{2.9}$$

Now define the relabeling

$$\begin{aligned} &e^{2\beta(r)-2\delta(r)}\left(1 + r\tfrac{d\delta}{dr}\right)^2 \to e^{\eta(r)}, \\ &e^{2\mu(r)} \to e^{\xi(r)}, \\ &r^* \to r, \end{aligned}$$

then eq.(2.9) becomes

$$ds^2 = e^{\xi(r)}c^2dt^2 - e^{\eta(r)}dr^2 - r^2d\Omega^2. \tag{2.10}$$

The metric tensor and its inverse are

$$g_{\alpha\beta} = \begin{pmatrix} e^{\xi} & & & \\ & -e^{\eta} & & \\ & & -r^2 & \\ & & & -r^2\sin^2\theta \end{pmatrix},$$

$$g^{\alpha\beta} = \begin{pmatrix} e^{-\xi} & & & \\ & -e^{-\eta} & & \\ & & -r^{-2} & \\ & & & \left(-r^2\sin^2\theta\right)^{-1} \end{pmatrix}.$$

The non vanishing components of R_{ab} are

$$R_{00} = \frac{1}{2}\left[\xi'' + \frac{1}{2}\xi'\left(\xi' - \eta'\right) + \frac{2}{r}\xi'\right]e^{2\xi-2\eta}, \tag{2.11}$$

$$R_{11} = \frac{1}{2}\left[-\xi'' - \frac{1}{2}\xi'\left(\xi' - \eta'\right) + \frac{2\eta'}{r}\right], \tag{2.12}$$

$$R_{22} = 1 - e^{-\xi} + \frac{1}{2}r(\eta' - \xi')\,e^{-\xi}, \tag{2.13}$$

while

$$R_{33} = R_{22}\sin^2\theta. \tag{2.14}$$

The EFEs in vacuum are

$$R_{\alpha\beta} = 0.$$

Using eqs.(2.11), (2.12) and (2.13), we get

$$R_{00} = \xi'' + \xi'\left(\xi' - \eta'\right) + \frac{2}{r}\xi' = 0, \tag{2.15}$$

$$R_{11} = \xi'' + \xi'\left(\xi' - \eta'\right) - \frac{2\eta'}{r} = 0, \tag{2.16}$$

$$R_{22} = 1 - e^{\xi} + \frac{1}{2} r(\eta' - \xi')\, e^{\xi} = 0. \tag{2.17}$$

Using eqs.(2.15) and (2.16), we obtain

$$\xi'(r) + \eta'(r) = 0,$$

or

$$\xi(r) + \eta(r) = constt.$$

We can define a function "$\xi(r) - \text{constt} = \bar{\xi}(r)$" and relabeling $\bar{\xi}(r) \to \xi(r)$, we get

$$\xi(r) = -\eta(r),$$

then eq.(2.17) becomes

$$R_{22} = \left(-re^{-\eta}\right)' + 1 = 0.$$

Integrating both sides and dividing by 'r'

$$e^{\xi(r)} = e^{-\eta(r)} = 1 + \frac{\alpha}{r}, \tag{2.18}$$

where α is a constant.
Using eq.(2.18) in eq.(2.10), we obtain

$$ds^2 = \left(1 + \frac{\alpha}{r}\right) c^2 dt^2 - \left(1 + \frac{\alpha}{r}\right)^{-1} dr^2 - r^2 d\Omega^2. \tag{2.19}$$

From [27], $\alpha = \dfrac{-2Gm}{c^2}$, where G is the Newton's gravitational constant, m is the point mass and c is the speed of light.
Eq.(2.19) becomes

$$ds^2 = \left(1 - \frac{2Gm}{rc^2}\right) c^2 dt^2 - \left(1 - \frac{2Gm}{rc^2}\right)^{-1} dr^2 - r^2 d\Omega^2,$$

or taking $G = c = 1$ (gravitational units), the above metric becomes

$$ds^2 = \left(1 - \frac{r_s}{r}\right) dt^2 - \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 - r^2 d\Omega^2, \tag{2.20}$$

where $r_s = 2m$ is the *Schwarzschild* radius.

Equation (2.20) represents the required *Schwarzschild* Metric (SM).

This metric has two singularities:

i) at $r = r_s$,

ii) at $r = 0$.

The curvature invariants for SM given in eq.(2.20) [13] are

$$I_1 = 0,$$
$$I_2 = \frac{48m^2}{r^4},$$
$$I_3 = \frac{64m^3}{r^6}.$$

The curvature invariants remain finite at $r = r_s$ while blows up at $r = 0$. Thus $r = r_s$ is a removable (coordinate) singularity while $r = 0$ is a crushing singularity. To remove the coordinate singularity at $r = r_s$, we first define the Eddington Finkelstein (EF) coordinates (w^*, z^*), while 'θ' and 'φ' remain unchanged.

$$w^* = t + r^*,$$
$$z^* = t - r^*,$$

where r^* is the "*Regge wheeler tortise* coordinate" [12] defined by

$$r^* = \int \frac{dr}{\left(1 - \frac{r_s}{r}\right)} = r + r_s \ln \left| \frac{r}{r_s} - 1 \right|.$$

The inverse transformations are

$$t = \frac{1}{2}\left(w^* + z^*\right), \tag{1a}$$
$$r^* = \frac{1}{2}\left(w^* - z^*\right). \tag{1b}$$

The metric in eq.(2.20) in these coordinates takes the form

$$ds^2 = \left(1 - \frac{r_s}{r}\right) dw^* dz^* - r^2 d\Omega^2, \tag{2.21}$$

which is still singular at $r = r_s$, since $g = 0$ there. We thus introduce Kruskal coordinates (W, Z) given by

$$W = \mu e^{\frac{w^*}{\lambda}}, \quad Z = -\mu e^{\frac{-z^*}{\lambda}}, \tag{2.22}$$

where μ and λ are the two constants which we shall fix later.
From eqs.(2.22) and (1b), we have relation

$$WZ = -\mu^2 e^{\frac{(w^*-z^*)}{\lambda}} = -\mu^2 e^{\frac{2r^*}{\lambda}}.$$

The metric in (W, Z) coordinates with $\mu = 1$ and $\lambda = 2r_s$ becomes

$$ds^2 = 4\frac{{r_s}^3}{r} e^{\frac{-r}{r_s}} dW dZ - r^2 d\Omega^2. \tag{2.23}$$

Now there is no singularity at $r = r_s$ as cleared from the metric given in eq.(2.23). Further, we define the Kruskal-Szekers (KS) coordinates (w, z), where 'w' is time like and 'z' is space like coordinates given by

$$w = \frac{1}{2}(W + Z) = \left(\frac{r}{r_s} - 1\right)^{\frac{1}{2}} e^{\frac{r}{2r_s}} \sinh\left(\frac{t}{2r_s}\right), \tag{2.24}$$

$$z = \frac{1}{2}(W - Z) = \left(\frac{r}{r_s} - 1\right)^{\frac{1}{2}} e^{\frac{r}{2r_s}} \cosh\left(\frac{t}{2r_s}\right). \tag{2.25}$$

The (w, z) plane has the four regions shown in Figure.2.3
For region I:-

$$r \geq r_s, \quad z \geq 0$$

$$z = \left(\frac{r}{r_s} - 1\right)^{\frac{1}{2}} e^{\frac{r}{2r_s}} \cosh\left(\frac{t}{2r_s}\right),$$

$$w = \left(\frac{r}{r_s} - 1\right)^{\frac{1}{2}} e^{\frac{r}{2r_s}} \sinh\left(\frac{t}{2r_s}\right).$$

For region II:-

$$r \leq r_s, \quad w \geq 0$$

$$z = \left(1 - \frac{r}{r_s}\right)^{\frac{1}{2}} e^{\frac{r}{2r_s}} \sinh\left(\frac{t}{2r_s}\right),$$

$$w = \left(1 - \frac{r}{r_s}\right)^{\frac{1}{2}} e^{\frac{r}{2r_s}} \cosh\left(\frac{t}{2r_s}\right).$$

For region III:-

$$r \geq r_s, \quad z \leq 0$$

$$z = -\left(\frac{r}{r_s} - 1\right)^{\frac{1}{2}} e^{\frac{r}{2r_s}} \cosh\left(\frac{t}{2r_s}\right),$$
$$w = -\left(\frac{r}{r_s} - 1\right)^{\frac{1}{2}} e^{\frac{r}{2r_s}} \sinh\left(\frac{t}{2r_s}\right).$$

For region IV:-

$$r \leq r_s, \quad w \geq 0$$

$$z = -\left(1 - \frac{r}{r_s}\right)^{\frac{1}{2}} e^{\frac{r}{2r_s}} \sinh\left(\frac{t}{2r_s}\right),$$
$$w = -\left(1 - \frac{r}{r_s}\right)^{\frac{1}{2}} e^{\frac{r}{2r_s}} \cosh\left(\frac{t}{2r_s}\right).$$

In these coordinates eq.(2.23) becomes

$$ds^2 = 4\frac{{r_s}^3}{r} e^{\frac{-r}{r_s}} \left(dw^2 - dz^2\right) - r^2 d\Omega^2. \tag{2.26}$$

The inverse transformations are

$$t = \begin{cases} 2r_s \tanh^{-1}\left(\frac{w}{z}\right), & \text{for region } I \text{ and } III, \\ 2r_s \tanh^{-1}\left(\frac{z}{w}\right), & \text{for region } II \text{ and } IV. \end{cases}$$

and $\left(\frac{r}{r_s} - 1\right) e^{\frac{r}{r_s}} = z^2 - w^2$ for all regions.
In KS coordinates, the crushing singularity becomes

$$w^2 - z^2 = 1,$$
$$\text{or} \quad w = \pm\sqrt{1 + z^2},$$

i.e. we have two singularities, $w = \sqrt{1 + z^2}$ is a future space like singularity, while $w = -\sqrt{1 + z^2}$ is a past space like singularity.
Also the region $r \geq r_s$ is actually $z^2 \geq w^2$, which indicates that there are two regions

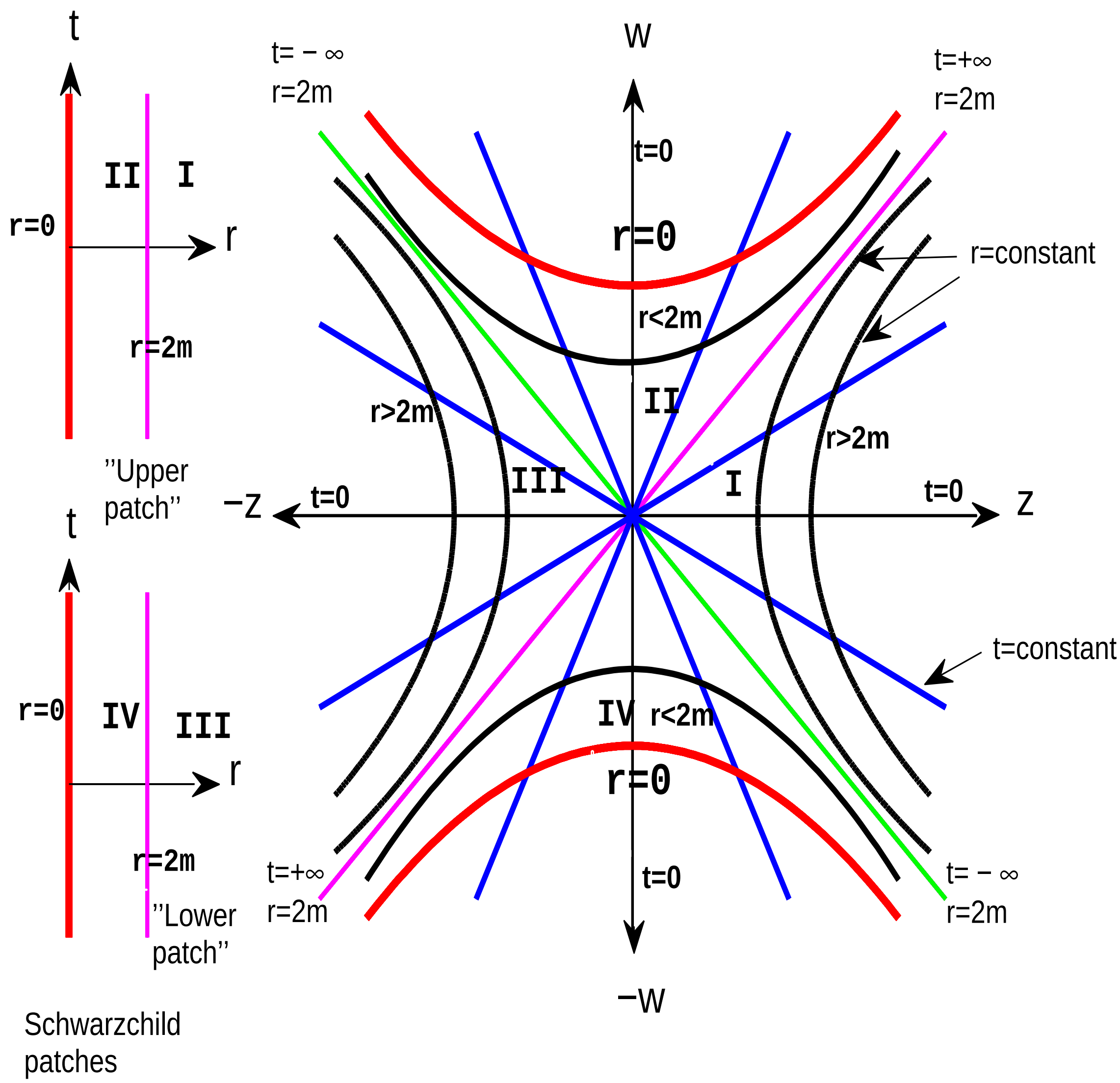

Figure 2.3: The transformation of (t, r) and (w, z) coordinates for SM.

outside the black hole *i.e* $z \geq |w|$ and $z \leq -|w|$, both corresponds to $r \geq r_s$. Same as for $r \leq r_s$, the region is $w^2 \geq z^2$, which also show there are two interior regions of black hole.

The KS coordinates are non-compact ranges from $(-\infty, +\infty)$. These are not suitable for calculations because of their asymptotic range. Penrose constructed a method to analyze the asymptotic properties of space time. He introduced conformal transformation for infinity to make it finite and then the asymptotic calculations are converted into finite calculations. By using this method KS coordinates are converted into compactified KS coordinates (CKS) $(Y,\ X,\ \theta,\ \varphi)$ [1] defined as

$$\begin{aligned} Y &= \tan^{-1}(w+z) + \tan^{-1}(w-z)\,, \\ X &= \tan^{-1}(w+z) - \tan^{-1}(w-z)\,, \end{aligned} \tag{2.27}$$

and the implicit relation for r and $Y,\ X$ is

$$\left(\frac{r}{r_s} - 1\right) e^{\frac{r}{r_s}} = z^2 - w^2 = -\tan\left(\frac{Y-X}{2}\right)\tan\left(\frac{Y+X}{2}\right). \tag{2.28}$$

Inserting eq.(2.27) in eq.(2.26), we get

$$ds^2 = \frac{{r_s}^3}{r} e^{\frac{-r}{r_s}} \sec^2\left(\frac{Y-X}{2}\right)\sec^2\left(\frac{Y+X}{2}\right)\left(dY^2 - dX^2\right) - r^2 d\Omega^2. \tag{2.29}$$

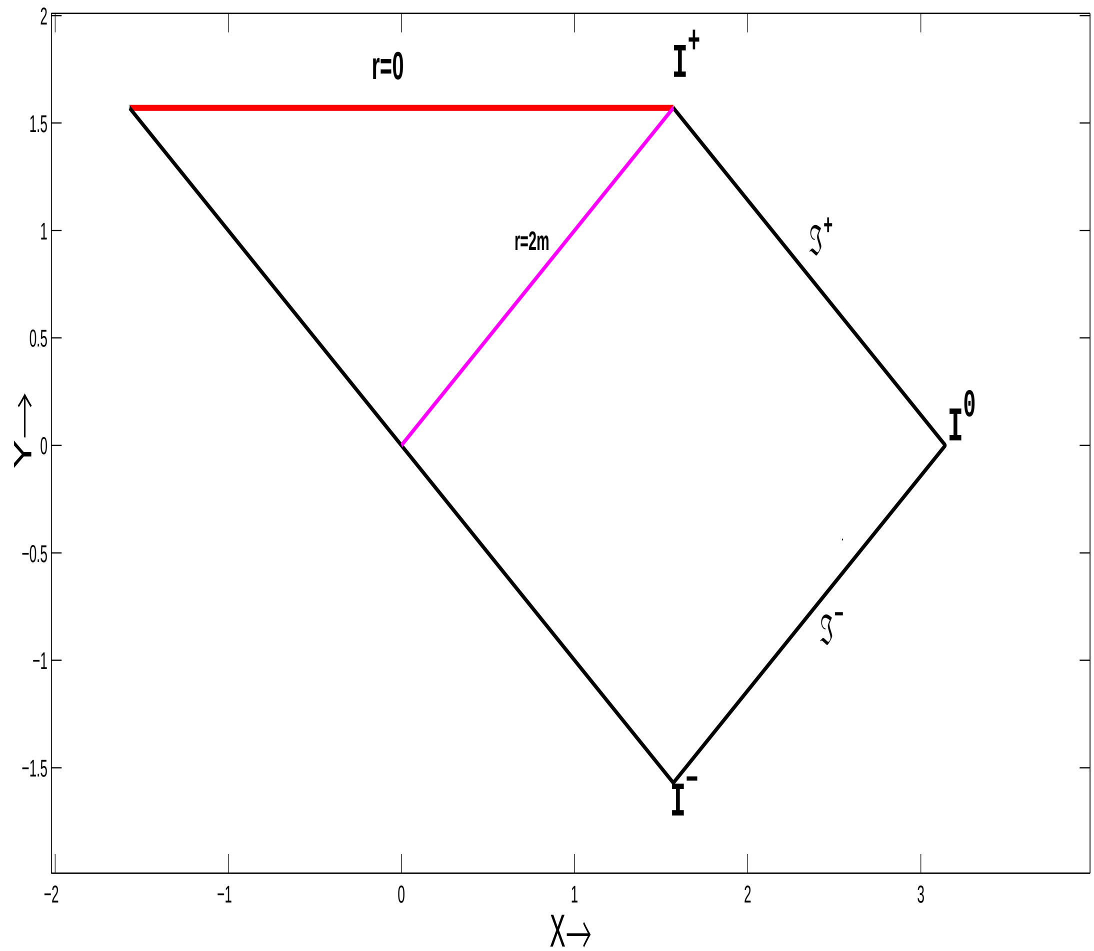

Figure 2.4: CP diagram of the SM.

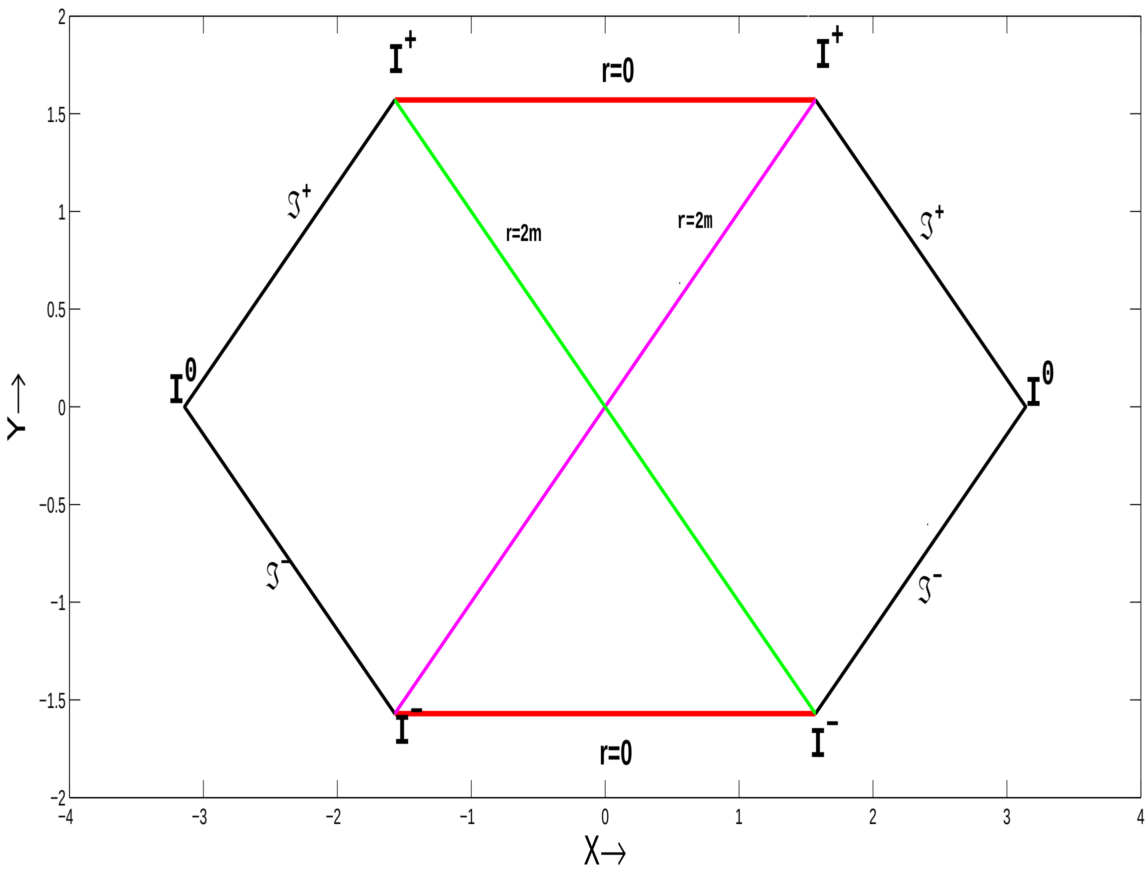

Figure 2.5: The maximal extension of the SM.

2.1.3 The *Reissner Nordstorm* Metric

Reissner [28] and Nordstorm [29] derive the solution of the EFEs, for outside a spherically symmetric charged point with non zero energy momentum tensor which arises from electromegnetic field, called RN metric, given as

$$ds^2 = \varrho(r)dt^2 - \varrho^{-1}(r)dr^2 - r^2 d\Omega^2, \tag{2.30}$$

where

$$\varrho(r) = 1 - \frac{2m}{r} + \frac{Q^2}{r^2}. \tag{2.31}$$

For $m > Q$, RN metric has two coordinate singularities,

$$r_1 = m + \sqrt{m^2 - Q^2}, \tag{2.32}$$

and

$$r_2 = m - \sqrt{m^2 - Q^2}, \tag{2.33}$$

called outer and inner horizons respectively, and one curvature singularity $r = 0$, as curvature invariants become infinite there [13]. For the removal of singularities at $r = r_1$ and r_2, we first define EF like coordinates (w^*, z^*)

$$\begin{aligned} w^* &= t + r^*, \\ z^* &= t - r^*, \end{aligned} \tag{2.34}$$

where

$$r^* = \int \frac{r^2}{(r - r_2)(r - r_1)} dr = r + \frac{r_1^2}{r_1 - r_2} \ln \left| \frac{r - r_1}{d_1} \right| - \frac{r_2^2}{r_1 - r_2} \ln \left| \frac{r - r_2}{d_2} \right|, \tag{2.35}$$

where d_1 and d_2 are constants. To make the arguments of log functions dimensionless, we choose $d_1 = r_1$ and $d_2 = r_2$. Using these values in eq.(2.35), we get

$$r^* = r + \frac{r_1^2}{r_1 - r_2} \ln \left| \frac{r}{r_1} - 1 \right| - \frac{r_2^2}{r_1 - r_2} \ln \left| \frac{r}{r_2} - 1 \right|. \tag{2.36}$$

The metric given in eq.(2.30) in (w^*, z^*) coordinates becomes

$$ds^2 = \left(1 - \frac{r_1}{r}\right)\left(1 - \frac{r_2}{r}\right) dw^* dz^* - r^2 d\Omega^2. \tag{2.37}$$

The metric in above equation remains singular because $g = 0$ at $r = r_1$ and r_2. We thus introduce Kruskal coordinates (W, Z) given by

$$\begin{aligned} W &= \mu e^{\frac{w^*}{\lambda}}, \\ Z &= -\mu e^{\frac{-z^*}{\lambda}}. \end{aligned} \tag{2.38}$$

Using eq.(2.34) along with eq.(2.36) in eq.(2.38), we get

$$W = \mu e^{\frac{t}{\lambda}} e^{\frac{r}{\lambda}} \left|\frac{r}{r_1} - 1\right|^{\frac{r_1^2}{\lambda(r_1-r_2)}} \left|\frac{r}{r_2} - 1\right|^{\frac{-r_2^2}{\lambda(r_1-r_2)}}, \tag{2.39}$$

$$Z = -\mu e^{\frac{-t}{\lambda}} e^{\frac{r}{\lambda}} \left|\frac{r}{r_1} - 1\right|^{\frac{r_1^2}{\lambda(r_1-r_2)}} \left|\frac{r}{r_2} - 1\right|^{\frac{-r_2^2}{\lambda(r_1-r_2)}}. \tag{2.40}$$

And

$$WZ = -\mu^2 e^{\frac{2r^*}{\lambda}} = -\mu^2 e^{\frac{2r}{\lambda}} \left|\frac{r}{r_1} - 1\right|^{\frac{2r_1^2}{\lambda(r_1-r_2)}} \left|\frac{r}{r_2} - 1\right|^{\frac{-2r_2^2}{\lambda(r_1-r_2)}}.$$

For the removal of the singularities at $r = r_1$ and r_2, we need to cancel the factors involving $\left(1 - \frac{r_1}{r}\right)$ and $\left(1 - \frac{r_2}{r}\right)$. But these cannot be canceled simultaneously. Therefore two coordinate patches are required to cover the entire RN-geometry. Thus the (W_1, Z_1) coordinates (called Kruskal like coordinates) are defined as

$$W_1 = \mu_1 e^{\frac{t}{\lambda_1}} e^{\frac{r}{\lambda_1}} \left|\frac{r}{r_1} - 1\right|^{\frac{r_1^2}{\lambda_1(r_1-r_2)}} \left|\frac{r}{r_2} - 1\right|^{\frac{-r_2^2}{\lambda_1(r_1-r_2)}}, \tag{2.41}$$

$$Z_1 = -\mu_1 e^{\frac{-t}{\lambda_1}} e^{\frac{r}{\lambda_1}} \left|\frac{r}{r_1} - 1\right|^{\frac{r_1^2}{\lambda_1(r_1-r_2)}} \left|\frac{r}{r_2} - 1\right|^{\frac{-r_2^2}{\lambda_1(r_1-r_2)}}, \tag{2.42}$$

where

$$\lambda_1 = \frac{2r_1^2}{r_1 - r_2}. \tag{2.43}$$

The metric for this region is given by

$$ds_1^2 = \frac{\lambda_1^2 r_1 r_2}{r^2 \mu_+^2} e^{-2r/\lambda_1} \left(\frac{r}{r_2} - 1\right)^{\frac{r_1^2+r_2^2}{r_1^2}} dW_1 dZ_1 - r^2 d\Omega^2. \tag{2.44}$$

The (W_2, Z_2) are defined as

$$W_2 = \mu_2 e^{\frac{t}{\lambda_2}} e^{\frac{r}{\lambda_2}} \left|\frac{r}{r_1} - 1\right|^{\frac{r_1^2}{\lambda_2(r_1-r_2)}} \left|\frac{r}{r_2} - 1\right|^{\frac{-r_2^2}{\lambda_2(r_1-r_2)}}, \tag{2.45}$$

$$Z_2 = -\mu_2 e^{\frac{-t}{\lambda_2}} e^{\frac{r}{\lambda_1}} \left|\frac{r}{r_1} - 1\right|^{\frac{r_1^2}{\lambda_2(r_1-r_2)}} \left|\frac{r}{r_2} - 1\right|^{\frac{-r_2^2}{\lambda_2(r_1-r_2)}}, \tag{2.46}$$

where

$$\lambda_2 = \frac{-2r_2^2}{r_1 - r_2}. \tag{2.47}$$

The metric for this region is

$$ds_2^2 = \frac{\lambda_2^2 r_2 r_1}{r^2 \mu_2^2} e^{-2r/\lambda_2} \left(\frac{r}{r_1} - 1\right)^{\frac{r_1^2 + r_2^2}{r_2^2}} dW_2 dZ_2 - r^2 d\Omega^2. \tag{2.48}$$

We now define KS coordinates (w_1, z_1) which covers the region $r_1 < r < \infty$

$$w_1 = \frac{1}{2}(W_1 + Z_1) = \mu_1 \sinh\left(\frac{t}{\lambda_1}\right) e^{\frac{r}{\lambda_1}} \left|\frac{r}{r_1} - 1\right|^{\frac{r_1^2}{\lambda_1(r_1-r_2)}} \left|\frac{r}{r_2} - 1\right|^{\frac{-r_2^2}{\lambda_1(r_1-r_2)}}, \tag{2.49}$$

$$z_1 = \frac{1}{2}(W_1 - Z_1) = \mu_1 \cosh\left(\frac{t}{\lambda_1}\right) e^{\frac{r}{\lambda_1}} \left|\frac{r}{r_1} - 1\right|^{\frac{r_1^2}{\lambda_1(r_1-r_2)}} \left|\frac{r}{r_2} - 1\right|^{\frac{-r_2^2}{\lambda_1(r_1-r_2)}}. \tag{2.50}$$

These are non-singular at $r = r_1$ and for the region $r_2 < r < r_1$, w_1, z_1 are interchanged. The metric for this region in these coordinates is given by

$$ds_1^2 = \frac{\lambda_1^2 r_1 r_2}{r^2 \mu_1^2} e^{-2r/\lambda_1} \left(\frac{r}{r_2} - 1\right)^{\frac{r_1^2 + r_2^2}{r_1^2}} \left(dw_1^2 - dz_1^2\right) - r^2 d\Omega^2. \tag{2.51}$$

The region $0 < r < r_2$ is covered by (w_2, z_2), where

$$w_2 = \frac{1}{2}(W_2 + Z_2) = \mu_2 \sinh\left(\frac{t}{\lambda_2}\right) e^{\frac{r}{\lambda_2}} \left|\frac{r}{r_1} - 1\right|^{\frac{r_1^2}{\lambda_2(r_1-r_2)}} \left|\frac{r}{r_2} - 1\right|^{\frac{-r_2^2}{\lambda_2(r_1-r_2)}}, \tag{2.52}$$

$$z_2 = \frac{1}{2}(W_2 - Z_2) = \mu_2 \cosh\left(\frac{t}{\lambda_2}\right) e^{\frac{r}{\lambda_2}} \left|\frac{r}{r_1} - 1\right|^{\frac{r_1^2}{\lambda_2(r_1-r_2)}} \left|\frac{r}{r_2} - 1\right|^{\frac{-r_2^2}{\lambda_2(r_1-r_2)}}. \tag{2.53}$$

These are non-singular at $r = r_2$, and for the region $r_2 < r < r_1$, w_2,z_2 are interchanged. Here the metric becomes

$$ds_2^2 = \frac{\lambda_2^2 r_2 r_1}{r^2 \mu_2^2} e^{-2r/\lambda_2} \left(\frac{r}{r_1} - 1\right)^{\frac{r_1^2 + r_2^2}{r_2^2}} \left(dw_2^2 - dz_2^2\right) - r^2 d\Omega^2. \tag{2.54}$$

The region $r_2 < r < \infty$ is covered by the compactified KS coordinates (Y_1, X_1), where

$$\begin{aligned} Y_1 &= \tan^{-1}(w_1 + z_1) + \tan^{-1}(w_1 - z_1), \\ X_1 &= \tan^{-1}(w_1 + z_1) - \tan^{-1}(w_1 - z_1). \end{aligned} \tag{2.55}$$

Here the metric is

$$ds_1^2 = \frac{\lambda_1^2 r_1 r_2}{4r^2\mu_1^2} e^{-2r/\lambda_1} \left(\frac{r}{r_2} - 1\right)^{\frac{r_1^2+r_2^2}{r_1^2}} \sec^2\left(\frac{Y_1 - X_1}{2}\right) \sec^2\left(\frac{Y_1 + X_1}{2}\right) \left(dY_1^2 - dX_1^2\right) - r^2 d\Omega^2. \tag{2.56}$$

The region $0 < r < r_1$ is covered by (Y_2, X_2), where

$$\begin{aligned} Y_2 &= \tan^{-1}(w_2 + z_2) + \tan^{-1}(w_2 - z_2), \\ X_2 &= \tan^{-1}(w_2 + z_2) - \tan^{-1}(w_2 - z_2). \end{aligned} \tag{2.57}$$

Here the metric is

$$ds_2^2 = \frac{\lambda_2^2 r_2 r_1}{4r^2\mu_2^2} e^{-2r/\lambda_2} \left(\frac{r}{r_1} - 1\right)^{\frac{r_1^2+r_2^2}{r_2^2}} \sec^2\left(\frac{Y_2 - X_2}{2}\right) \sec^2\left(\frac{Y_2 + X_2}{2}\right) \left(dY_2^2 - dX_2^2\right) - r^2 d\Omega^2. \tag{2.58}$$

The implicit relations for r are

$$\begin{aligned} z_1^2 - w_1^2 &= \mu_1^2 e^{\frac{2r}{\lambda_1}} \left|\frac{r}{r_1} - 1\right|^{\frac{2r_1^2}{\lambda_1(r_1-r_2)}} \left|\frac{r}{r_2} - 1\right|^{\frac{-2r_2^2}{\lambda_1(r_1-r_2)}} \\ &= -\tan\left(\frac{Y_1 - X_1}{2}\right) \tan\left(\frac{Y_1 + X_1}{2}\right), \end{aligned} \tag{2.59}$$

and

$$\begin{aligned} z_2^2 - w_2^2 &= \mu_2^2 e^{\frac{2r}{\lambda_2}} \left|\frac{r}{r_1} - 1\right|^{\frac{2r_1^2}{\lambda_2(r_1-r_2)}} \left|\frac{r}{r_2} - 1\right|^{\frac{-2r_2^2}{\lambda_2(r_1-r_2)}} \\ &= -\tan\left(\frac{Y_2 - X_2}{2}\right) \tan\left(\frac{Y_2 + X_2}{2}\right). \end{aligned} \tag{2.60}$$

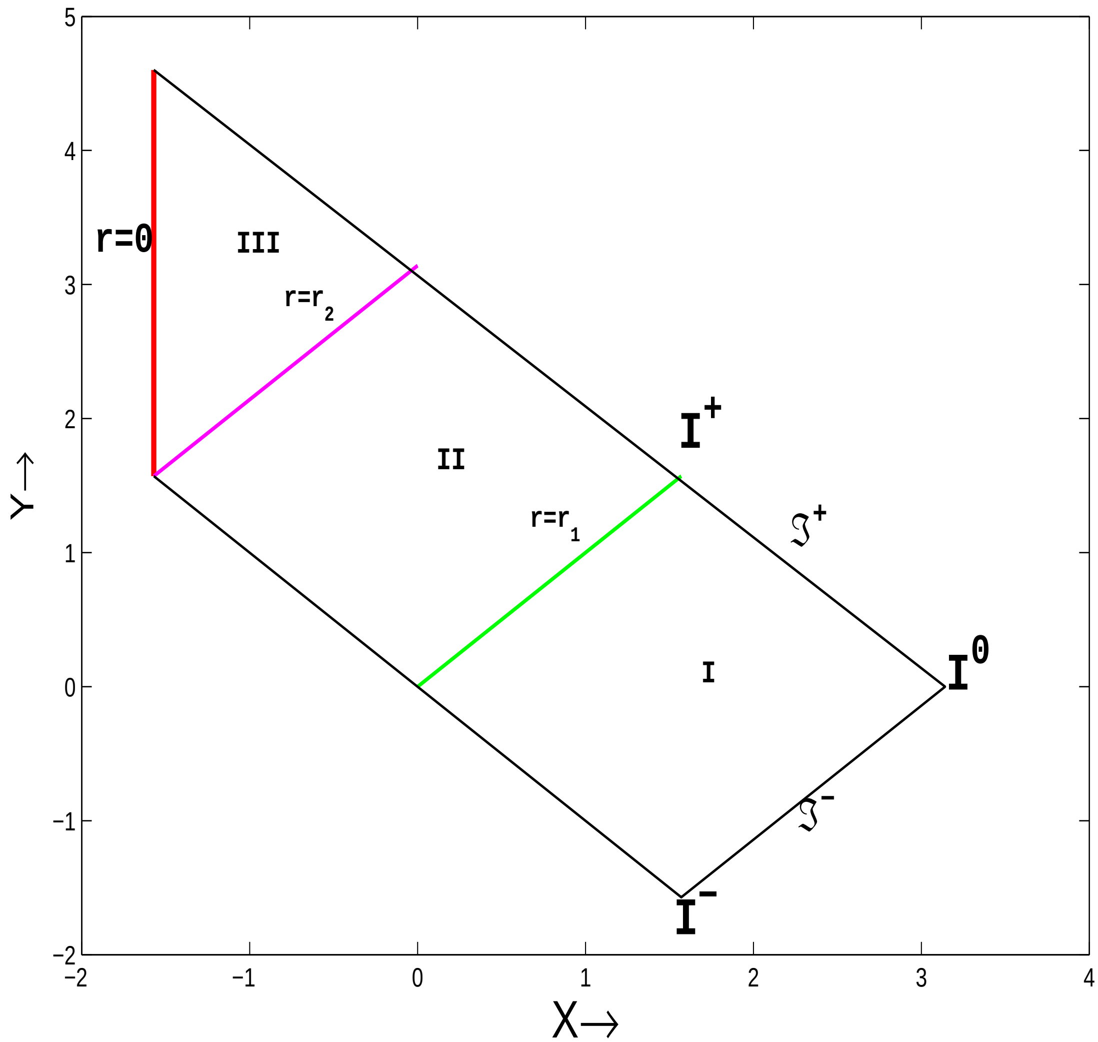

Figure 2.6: CP diagram of the RN metric for $m > Q$.

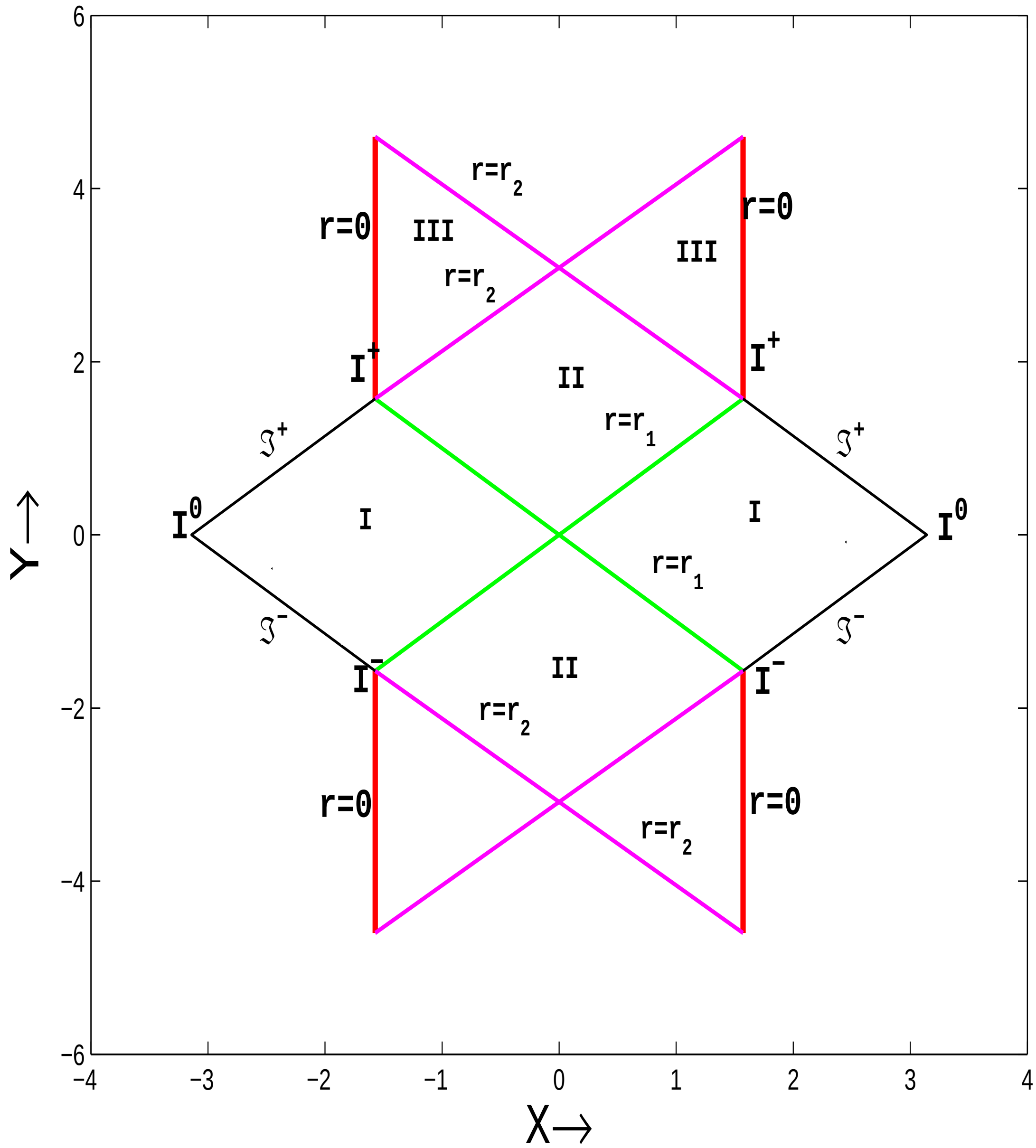

Figure 2.7: The Maximal extension of the RN metric for $m > Q$ in (Y, X, θ, ϕ) coordinates.

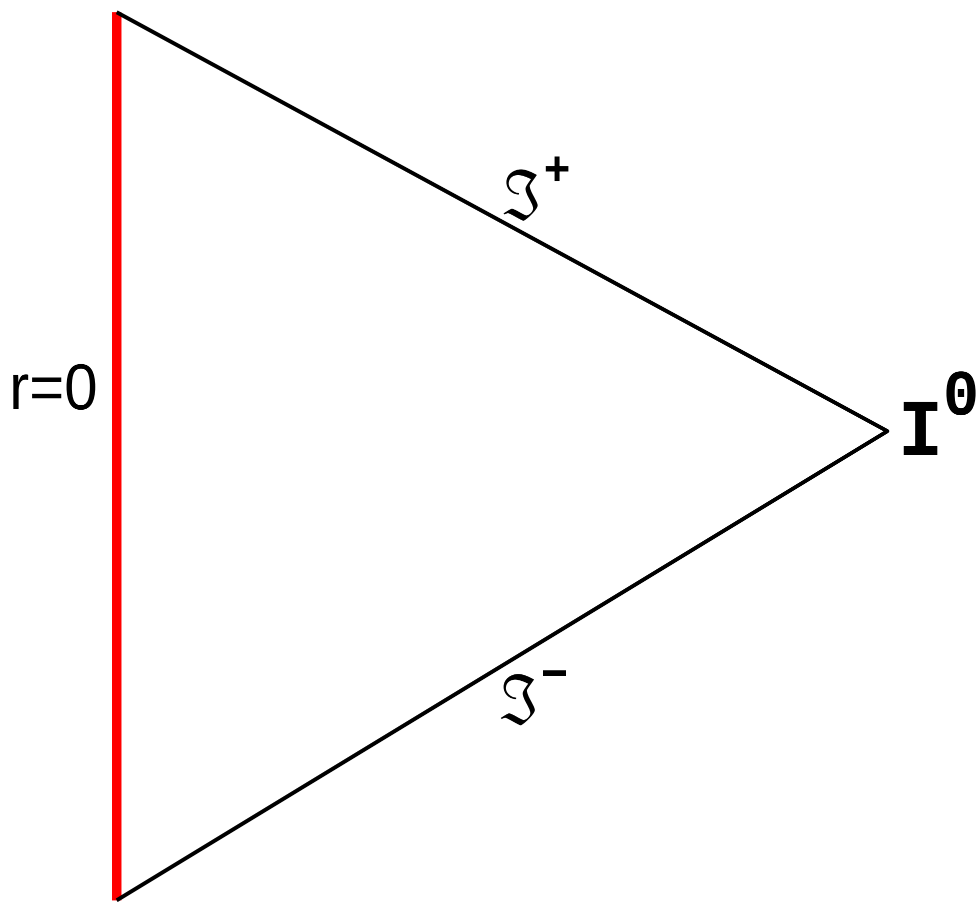

Figure 2.8: CP diagram of the RN metric for $m < Q$.

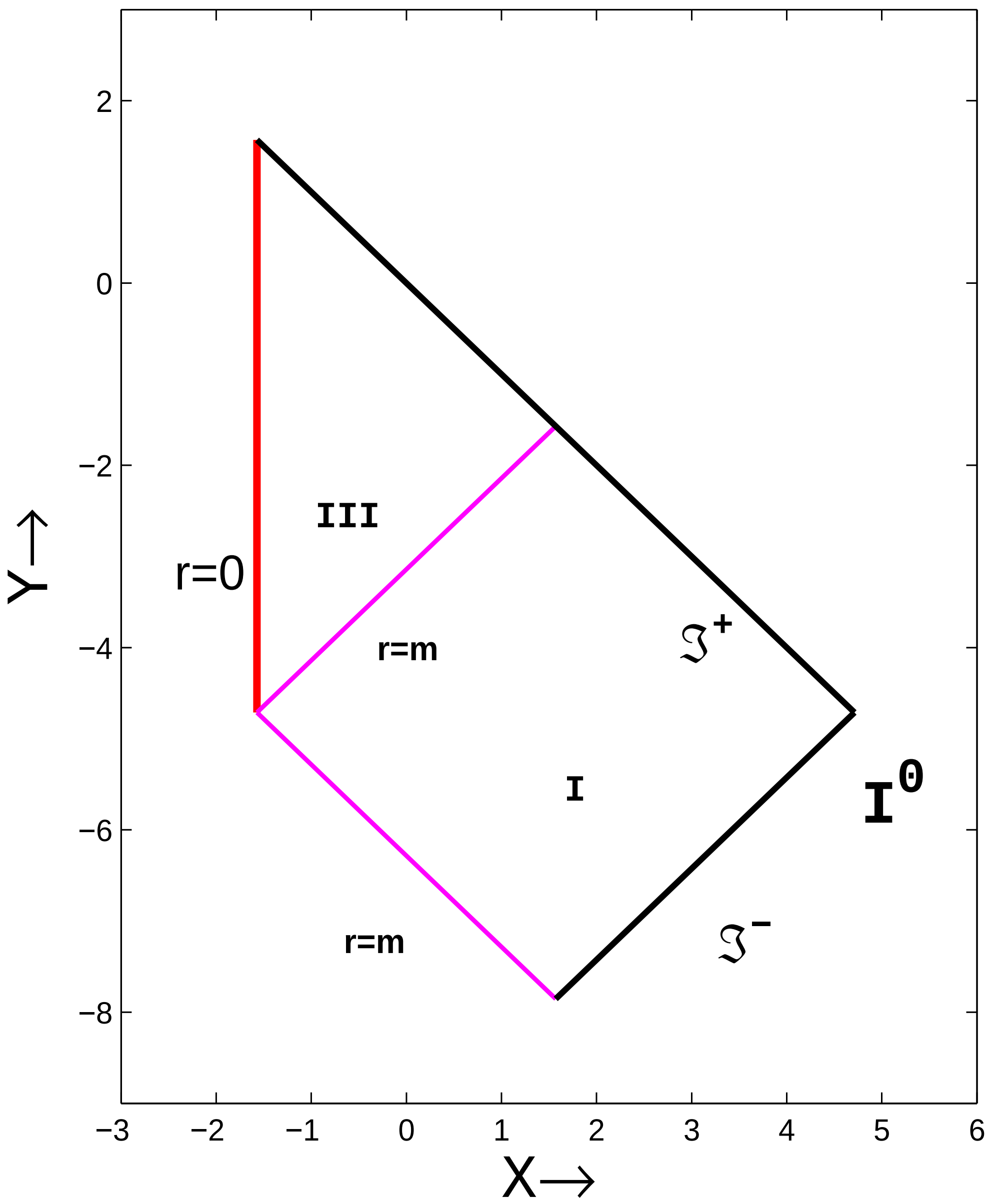

Figure 2.9: CP diagram of the eRN metric.

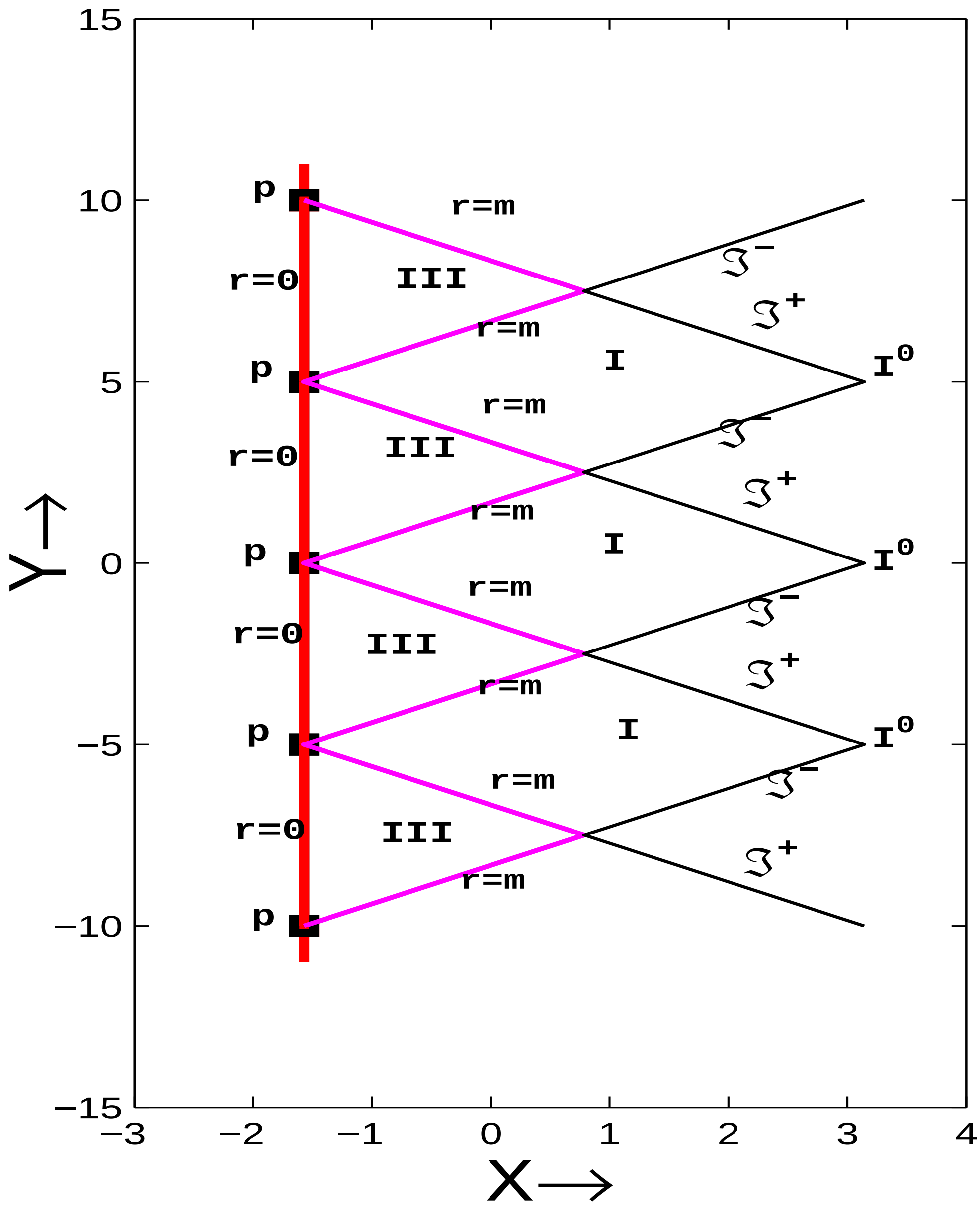

Figure 2.10: The Maximal extension of the eRN metric. The points 'p' are not the part of essential singularity but are infact exceptional points at infinity.

For $m < Q$, the metric is regular everywhere except at the curvature singularity $r = 0$. The CP diagram for this case is shown in Figure.2.8
For $m = Q$, RN metric is called extreme RN (eRN), the metric in eq.(2.30) then becomes

$$ds^2 = k(r)dt^2 - k^{-1}(r)dr^2 - r^2 d\Omega^2, \tag{2.61}$$

where

$$k(r) = \left(1 - \frac{m}{r}\right)^2. \tag{2.62}$$

This metric has coordinate singularity at $r = m$, and essential singularity at $r = 0$. For eRN, Kruskal like coordinates does not exist [30], then we use Carter coordinates [22] (Y, X), given by

$$Y = \tan^{-1} w^* + \cot^{-1} z^*, \tag{2.63}$$

$$X = \tan^{-1} w^* - \cot^{-1} z^*, \tag{2.64}$$

where

$$w^* = r^* + t,$$
$$z^* = r^* - t,$$

where r^* is

$$r^* = \int \frac{1}{\left(1 - \frac{m}{r}\right)^2} dr = r - \frac{rm}{r - m} + 2m \ln \left| \frac{r - m}{m} \right| \tag{2.65}$$

The metric in these coordinates becomes

$$ds^2 = \left(\frac{r - m}{2r}\right)^2 \csc^2\left(\frac{Y - X}{2}\right) \sec^2\left(\frac{Y + X}{2}\right) \left(dY^2 - dX^2\right) - r^2 d\Omega^2. \tag{2.66}$$

The CP diagram of eRN metric and its maximal extension are shown in Figures.2.9 and 2.10 respectively.

2.1.4 The *Kerr* Metric

Kerr [31] presented the first axisymmetric stationary asymptotically flat metric which describes a spinning black hole. The *Kerr* metric in Boyer-Lindquist coordinates is given by

$$ds^2 = \frac{\triangle}{\rho^2}\left(dt - a\sin^2\theta d\phi\right)^2 + \frac{\sin^2\theta}{\rho^2}\left((r^2 + a^2)d\phi - adt\right)^2 + \frac{\rho^2}{\triangle}dr^2 + \rho^2 d\theta^2, \tag{2.67}$$

where

$$\triangle = r^2 - 2mr + a^2, \tag{2.68}$$

$$\rho^2 = r^2 + a^2 \cos^2\theta. \tag{2.69}$$

In this metric 'a' is the angular momentum and m is the mass as measured from infinity. When $a = 0$ the metric reduces to the SM given in eq.(2.20).
When $a^2 > m^2, \triangle > 0$, the metric in eq.(2.67) becomes singular at $r = 0$. This singularity is not in fact a point but a ring.
When $a^2 < m^2$, $\triangle(r) = 0$ at

$$r_1 = m + \sqrt{m^2 - a^2}, \tag{2.70}$$

and

$$r_2 = m - \sqrt{m^2 - a^2}. \tag{2.71}$$

These are the outer and inner horizons respectively analogous to the outer and inner horizons of RN metric. When $m^2 = a^2$, r_1 and r_2 coincide.

2.2 Some Exact Solutions with $\wedge \neq 0$

In this section we first discuss *de Sitter* (dS) and *Anti-de Sitter* (AdS) metrics, and then the derivation of the *Schwarzschild de Sitter* (SdS) and *Schwarzschild Anti-de Sitter* (SAdS) spacetimes.
For non-zero cosmological constant, the the EFEs for vacuum are

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - \wedge g_{\mu\nu} = 0, \tag{2.72}$$

contracting indices and again using eq.(2.72), we obtain

$$R_{\mu\nu} = - \wedge g_{\mu\nu}. \tag{2.73}$$

2.2.1 *De Sitter* Metric

De Sitter [32] first discover this solution for $\wedge > 0$. It has the topology $R^1 \times S^3$ [10]. After the discovery of this solution, *de Sitter* [33] and Lanczos [34] realized that it can be visualized as the hyperboloid

$$y_0^2 - y_1^2 - y_2^2 - y_3^2 - y_4^2 = a^2, \tag{2.74}$$

where

$$a = \sqrt{\frac{3}{\wedge}}, \tag{2.75}$$

embedded in a five-dimensional Minkowski space

$$ds^2 = dy_0^2 - dy_1^2 - dy_2^2 - dy_3^2 - dy_4^2. \tag{2.76}$$

Here, we first introduce the $(\tau, \chi, \theta, \phi)$ coordinates to cover the entire hyperboloid,

$$\begin{aligned} y_0 &= a \sinh\left(\frac{\tau}{a}\right), \\ y_1 &= a \cosh\left(\frac{\tau}{a}\right) \cos\chi, \\ y_2 &= a \cosh\left(\frac{\tau}{a}\right) \sin\chi \cos\theta, \\ y_3 &= a \cosh\left(\frac{\tau}{a}\right) \sin\chi \cos\phi \sin\theta, \\ y_4 &= a \cosh\left(\frac{\tau}{a}\right) \sin\chi \sin\phi \sin\theta. \end{aligned}$$

In these coordinates the metric in eq.(2.76) becomes

$$ds^2 = d\tau^2 + a^2 \cosh^2 \frac{\tau}{a} \left(d\chi^2 + \sin^2 \chi d\Omega^2\right). \tag{2.77}$$

The metric in eq.(2.77) has coordinate singularities at $\chi = 0, \pi$ and $\theta = 0, \pi$, which are due to the choice of polar coordinates. Further we can introduce spherically symmetric coordinates (t, r, θ, ϕ) by

$$y_0 = \sqrt{a^2 - r^2} \sinh\left(\frac{t}{a}\right), \tag{2.78}$$

$$y_1 = \sqrt{a^2 - r^2} \cosh\left(\frac{t}{a}\right), \tag{2.79}$$

$$y_2 = r\cos\theta, \tag{2.80}$$

$$y_3 = r\cos\phi\sin\theta, \tag{2.81}$$

$$y_4 = r\sin\phi\sin\theta. \tag{2.82}$$

Using eqs.(2.78)-(2.82) in eq.(2.76) we get

$$ds^2 = \vartheta(r)dt^2 - \vartheta^{-1}(r)dr^2 - r^2d\Omega^2, \tag{2.83}$$

where

$$\vartheta(r) = 1 - \frac{\wedge}{3}r^2. \tag{2.84}$$

The metric becomes singular at $r = \sqrt{\frac{3}{\wedge}}$, which is the cosmological horizon.

2.2.2 *Anti-de Sitter* Metric

For $\wedge < 0$, we obtain *Anti-de Sitter* metric. It has the topology $S^1 \times R^3$, and can be represented as the hyperboloid

$$y_0^2 - y_1^2 - y_2^2 - y_3^2 + y_4^2 = -a^2, \tag{2.85}$$

where

$$a = \sqrt{\frac{-3}{\wedge}}, \tag{2.86}$$

embedded in a five-dimensional Minkowski space

$$ds^2 = dy_0^2 - dy_1^2 - dy_2^2 - dy_3^2 + dy_4^2. \tag{2.87}$$

Notice that AdS has two timelike dimensions y_0 and y_4.
This hyperboloid can be completely covered by the coordinates $(\tau, \chi, \theta, \phi)$

$$y_0 = a\sin\left(\frac{\tau}{a}\right)\cosh\chi, \tag{2.88}$$

$$y_1 = a\cos\theta\sinh\chi, \tag{2.89}$$

$$y_2 = a\sin\theta\cos\phi\sinh\chi, \tag{2.90}$$

$$y_3 = a\sin\theta\sin\phi\sinh\chi, \tag{2.91}$$

$$y_4 = a\cos\left(\frac{\tau}{a}\right)\cosh\chi. \tag{2.92}$$

Using eqs.(2.88)-(2.92) in (2.87), we get

$$ds^2 = \cosh^2\chi d\tau^2 + a^2(d\chi^2 + \sinh^2\chi d\Omega^2) \tag{2.93}$$

Here, coordinate singularities are at $\chi = 0$, and $\theta = 0, \pi$.

To introduce spherically symmetric coordinates (t, r, θ, ϕ), define a transformation

$$r = a\sinh\chi \tag{2.94}$$

Using eqs.(2.88)-(2.92), we obatin

$$y_0 = \sqrt{a^2 + r^2}\sinh\left(\frac{t}{a}\right), \tag{2.95}$$

$$y_1 = r\cos\theta, \tag{2.96}$$

$$y_2 = r\cos\phi\sin\theta, \tag{2.97}$$

$$y_3 = r\sin\phi\sin\theta, \tag{2.98}$$

$$y_4 = \sqrt{a^2 - r^2}\cosh\left(\frac{t}{a}\right). \tag{2.99}$$

Using eqs.(2.95)-(2.99) in eq.(2.87), we get

$$ds^2 = \vartheta^*(r)dt^2 - \vartheta^{*-1}(r)dr^2 - r^2 d\Omega^2, \tag{2.100}$$

where

$$\vartheta^*(r) = 1 + \frac{r^2}{a^2}. \tag{2.101}$$

Notice that there are no horizons for AdS metric because $(1 + \frac{r^2}{a^2}) > 0$ everywhere.

2.2.3 SdS and SAdS Metrics

SdS and SAdS are the generalizations of dS and AdS metrics respectively in *Schwarzschild* geometry. This metric was first discovered by Kottler [35], Weyl [36] and Trefftz [37]. Using eqs.(2.11), (2.12), (2.13) in eqs.(2.73), we get

$$R_{00} = \frac{1}{2}\left[\xi'' + \frac{1}{2}\xi'(\xi' - \eta') + \frac{2}{r}\xi'\right]e^{\xi-\eta} = -\wedge e^{\xi}, \tag{2.102}$$

which gives

$$R_{00} = \frac{1}{2}\left[\xi'' + \frac{1}{2}\xi'\left(\xi' - \eta'\right) + \frac{2}{r}\xi'\right] = -\wedge e^{\eta}, \tag{2.103}$$

$$R_{11} = \frac{1}{2}\left[-\xi'' - \frac{1}{2}\xi'\left(\xi' - \eta'\right) + \frac{2\eta'}{r}\right] = \wedge e^{\eta}, \tag{2.104}$$

$$R_{22} = 1 - e^{\xi} + \frac{1}{2}r(\eta' - \xi')\, e^{\xi} = \wedge r^2. \tag{2.105}$$

Adding eq.(2.103) and eq.(2.104), we get

$$\xi'(r) + \eta'(r) = 0,$$

or

$$\xi(r) + \eta(r) = constt.$$

Again we can define a function "$\xi(r)$ − constt = $\bar{\xi}(r)$" and relabeling $\bar{\xi}(r) \to \xi(r)$, we get

$$\xi(r) = -\eta(r),$$

then R_{22} becomes

$$\left(re^{-\eta}\right)' = 1 - \wedge r^2. \tag{2.106}$$

Now integrating both sides of eq.(2.106) and dividing by 'r', we get

$$e^{\xi(r)} = e^{-\eta(r)} = 1 - \wedge\frac{r^2}{3} + \frac{\beta_0}{r}. \tag{2.107}$$

For small r, the Newtonian approximation gives $\beta_0 = -2m$, so eq.(2.107) becomes

$$e^{2\xi(r)} = e^{-2\eta(r)} = 1 - \wedge\frac{r^2}{3} - \frac{2m}{r}. \tag{2.108}$$

Finally, using eq.(2.108) in eq.(2.10), we get

$$ds^2 = d(r)dt^2 - d^{-1}(r)dr^2 - r^2 d\Omega^2, \tag{2.109}$$

where

$$d(r) = 1 - \wedge \frac{r^2}{3} - \frac{2m}{r}. \tag{2.110}$$

The metric in eq.(2.109) can be classified into two types based on the nature of $\wedge$.
1) If $\wedge < 0$, then eqs.(2.109) and (2.110) takes the form

$$ds^2 = d(r)dt^2 - d^{-1}(r)dr^2 - r^2 d\Omega^2, \tag{2.111}$$

where

$$d(r) = 1 + \frac{r^2}{a^2} - \frac{2m}{r}. \tag{2.112}$$

Here we use the relation given in eq.(2.86). The metric in (2.111) is called *Schwarzschild Anti de Sitter* (SAdS) metric. If $\wedge > 0$, eqs.(2.109) and (2.110) represents *Schwarzschild de Sitter* (SdS) metric. This metric has three horizons, from which two of them are positive (blackhole and cosmological horizon(s)) while the third one is negative and is neglected being nonphysical. Here the curvature singularity is at $r = 0$.

Chapter 3

Foliation

The word 'foliation' is derived from Latin word 'folia' which means leaf. The idea of foliation starts with the origin of the theory of differential equations. Where the solution curves taken as the foliation leaves. Poincare developed methods to study the global and qualitative properties of solutions of dynamical systems in situations where the implicit solution methods failed. He further discovered that the study of geometry of the space of solution curves of a dynamical system unveil complex phenomena. He accentuated the qualitative nature of phenomena which gives the powerful drive to the topological methods, which led to the subject of foliation. The pioneers of foliation theory were Reeb [38] and Ehresman [39].

The best way to understand the concept of foliation is to consider the simplest example of 2-dimensional xy-plane, $\Re^2$. It can be foliated by the straight lines, $y = mx + c$, with any fixed m and varying c as shown in Figure.3.1. The xy-plane can also be foliated by circles, $x^2 + y^2 = b^2$ as shown in Figure.3.2, but in this case the origin is left out unless we include the case of degenerate circle $b = 0$.

A foliation is called complete if it covers the entire space by a sequence of disjoint subspaces. For example. a disc of radius d can be completely foliated by circles but cannot be completely foliated by squares as there would be some part of the disc left uncovered.

Foliation of the black hole metrics by hypersurfaces may be obtained by null, space or timelike hypersurfaces. Foliation is not unique, a spacetime metric can be foliated by more than one ways, also the foliation procedure does not ensure the achievement of the complete foliation of the metric. A local or global approach can be adopted to foliate the metric. The former approach describes that, locally the hypersurfaces look flat to an observer, *i.e* they have zero intrinsic (*Reimann*) curvature while the later approach describes that hypersurfaces have constant or zero MEC.

3.1 K-Slicing

For the variational principle of K-surfaces, BCI [3] generalized an extremum property that the spheres in *Eucleadean* space have constant MEC and have least surface area for a fixed enclosed volume. Then by following the same procedure for the spacetime they extremized the 3-dimensional area $\Pi(\Sigma)$ of a hypersurface Σ, by keeping constant the 4-volume, $\Upsilon(\Sigma, \Sigma_1)$, enclosed by Σ together with any fixed surface Σ_1. To include this constraint in the variational principle, they used a Langrange multiplier and obtain

$$\delta L = 0, \tag{3.1}$$

with

$$L = \Pi(\Sigma) + \eta\Upsilon(\Sigma, \Sigma_1) = \int_{\Sigma} n^{\beta} d^3\Sigma_{\beta} + \eta \int_{\Upsilon} d^4\Upsilon. \tag{3.2}$$

Here n^{β} represents the field of unit vectors normal to Σ and Σ_1.

To show that eqs.(3.1) and (3.2) leads to the K-surface, notice that the boundary of Υ is $\Sigma - \Sigma_1$, so rewrite L as

$$\begin{aligned} L &= \Pi(\Sigma) + \eta\Upsilon(\Sigma, \Sigma_1) + \Pi(\Sigma) - \Pi(\Sigma_1) = \Pi(\Sigma) - \Pi(\Sigma_1) + \eta\Upsilon(\Sigma, \Sigma_1) + \Pi(\Sigma_1) \\ &= \int_{\Sigma}^{\Sigma_1} n^{\beta} d^3\Sigma_{\beta} + \eta \int_{\Upsilon} d^4\Upsilon + \Pi(\Sigma_1). \end{aligned} \tag{3.3}$$

Using divergence theorem in eq.(3.3), we get

$$L = \int_{\Upsilon} n^{\beta}_{;\beta} d^4\Upsilon + \eta \int_{\Upsilon} d^4\Upsilon + \Pi(\Sigma_1) = \int_{\Upsilon} \left(n^{\beta}_{;\beta} + \eta\right) d^4\Upsilon + \Pi(\Sigma_1). \tag{3.4}$$

The variation of this expression can vanish for arbitrary variation of Σ only if the integrand vanishes every where on Σ. Thus

$$n^{\beta}_{;\beta} + \eta = 0. \tag{3.5}$$

Using eq.(1.54) in eq.(3.5), we get

$$\eta = K. \tag{3.6}$$

This shows that K is constant and its value is just the *Langrange* multiplier.

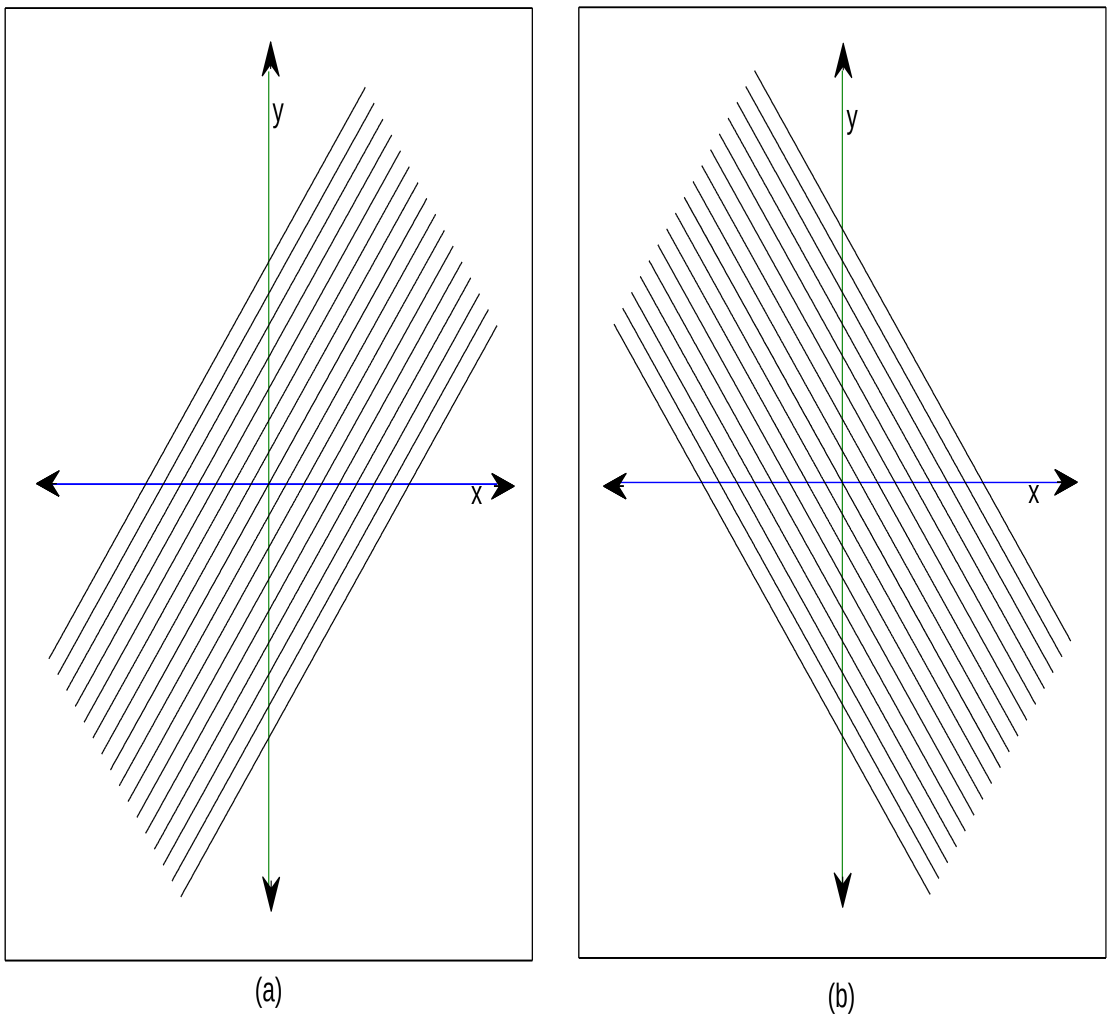

Figure 3.1: Foliation of xy-plane by straight lines $y = mx + c$. Figure (a) represents foliation of xy-plane for $m = 1$ and figure (b) for $m = -1$.

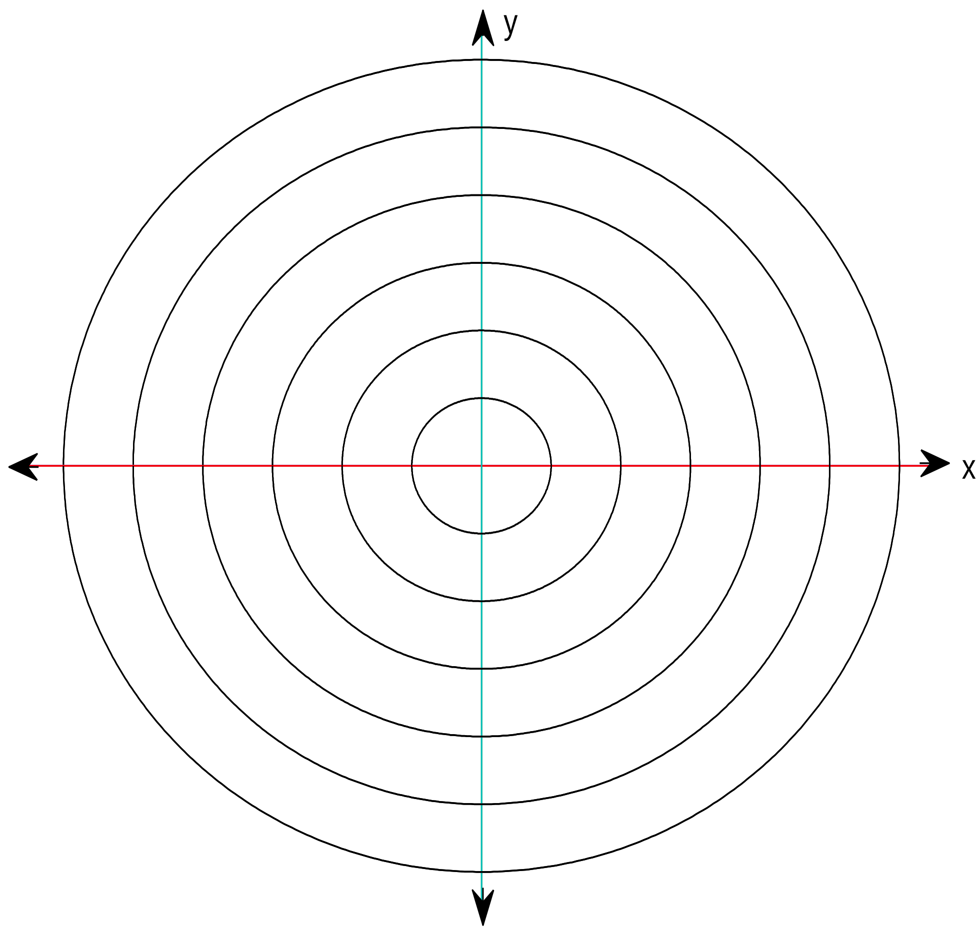

Figure 3.2: Foliation of xy-plane by circles.

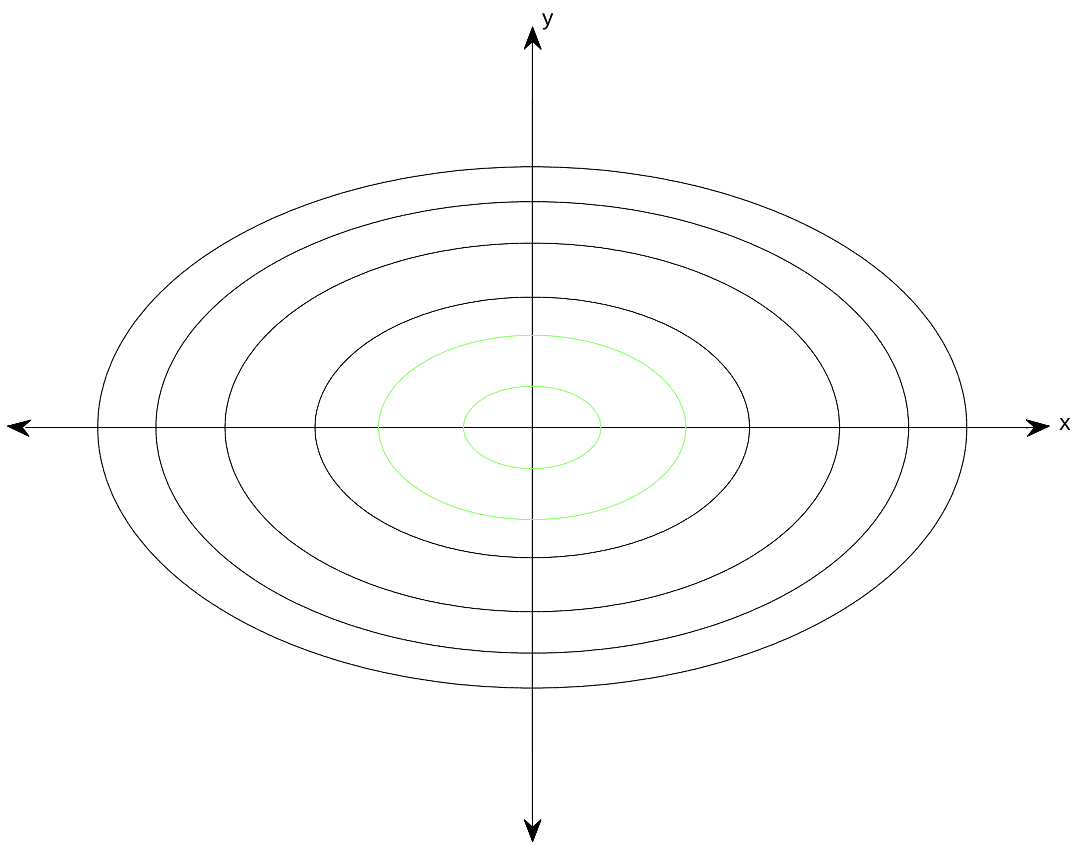

Figure 3.3: Foliation of xy-plane by ellipses.

3.1.1 K-Slicing Equation for Spherically Symmetric Static Spacetimes

The variational principle given in eqs.(3.1) and (3.2) gives a convenient method for the derivation of K-surface equation in spherically symmetric static (SSS) metric whose metric in (t, r, θ, ϕ) takes the form

$$ds^2 = B(r)dt^2 - C(r)dr^2 - r^2 d\Omega^2. \tag{3.7}$$

Let the spacelike surface Σ be described as

$$t = t(r, \theta, \phi), \tag{3.8}$$

and for Σ_1, we choose

$$t = 0. \tag{3.9}$$

To calculate the variational principle for the metric given in eq.(3.7) we need to find the values of $\Pi(\Sigma)$ and $\Upsilon(\Sigma, \Sigma_1)$. Since $\Pi(\Sigma)$ is the 3-dimensional area of Σ, and Σ is given in eq.(3.8) so

$$\Pi(\Sigma) = \int \sqrt{|g^*|} d^3\Pi, \tag{3.10}$$

where g^* is the determinant of induced metric defined on eq.(3.7).
And $\Upsilon(\Sigma, \Sigma_1)$ is the 4-dimensional volume given by

$$\Upsilon(\Sigma, \Sigma_1) = \int \sqrt{|g|} d^4\Upsilon, \tag{3.11}$$

where g is the metric determinant given in eq.(3.7). Now for induced metric, from eq.(3.8), we have

$$dt = t_r dr + t_\theta d\theta + t_\phi d\phi. \tag{3.12}$$

Inserting eq.(3.12) in eq.(3.7) and after simplification we get

$$\begin{aligned} ds^2 = &- \left(-Bt_r^2 + C\right) dr^2 - \left(-Bt_\theta^2 + r^2\right) d\theta^2 - \left(-Bt_\phi^2 + r^2 \sin^2\theta\right) d\phi^2 \\ &+ 2Bt_r t_\theta dr d\theta + 2Bt_\theta t_\phi d\theta d\phi + 2Bt_\phi t_r d\phi dr. \end{aligned} \tag{3.13}$$

Then

$$g^* = \begin{vmatrix} Bt_r^2 - C & Bt_r t_\theta & Bt_\phi t_r \\ Bt_r t_\theta & Bt_\theta^2 - r^2 & Bt_\theta t_\phi \\ Bt_\phi t_r & Bt_\theta t_\phi & Bt_\phi^2 - r^2 \sin^2\theta \end{vmatrix}, \tag{3.14}$$

$$
\begin{aligned}
&= \left(Bt_r^2 - C\right) \begin{vmatrix} Bt_\theta^2 - r^2 & Bt_\theta t_\phi \\ Bt_\theta t_\phi & Bt_\phi^2 - r^2\sin^2\theta \end{vmatrix} - Bt_r t_\theta \begin{vmatrix} Bt_r t_\theta & Bt_\theta t_\phi \\ Bt_\phi t_r & Bt_\phi^2 - r^2\sin^2\theta \end{vmatrix} \\
&\quad + Bt_\phi t_r \begin{vmatrix} Bt_r t_\theta & Bt_\theta^2 - r^2 \\ Bt_\phi t_r & Bt_\theta t_\phi \end{vmatrix}.
\end{aligned} \tag{3.15}
$$

After simplification, we get

$$
g^* = \left(C - Bt_r^2 - \frac{BC}{r^2}\left(t_\theta^2 + \frac{t_\phi^2}{\sin^2\theta}\right)\right) r^4 \sin^2\theta. \tag{3.16}
$$

Then

$$
\sqrt{|g^*|} = r^2 \sin\theta \sqrt{C - Bt_r^2 - \frac{BC}{r^2}\left(t_\theta^2 + \frac{t_\phi^2}{\sin^2\theta}\right)}. \tag{3.17}
$$

Using eq.(3.17) in eq.(3.10), we get

$$
\Pi(\Sigma) = \int r^2 \sin\theta \sqrt{C - Bt_r^2 - \frac{BC}{r^2}\left(t_\theta^2 + \frac{t_\phi^2}{\sin^2\theta}\right)} dr d\theta d\phi. \tag{3.18}
$$

Using eq.(3.7) in eq.(3.11), we get

$$
\Upsilon(\Sigma, \Sigma_1) = \int \sqrt{BC} r^2 \sin\theta dt dr d\theta d\phi. \tag{3.19}
$$

Using eqs.(3.18) and (3.19) in eq.(3.2), we get

$$
\begin{aligned}
L &= \int r^2 \sin\theta \sqrt{C - Bt_r^2 - \frac{BC}{r^2}\left(t_\theta^2 + \frac{t_\phi^2}{\sin^2\theta}\right)} dr d\theta d\phi + \eta \int \sqrt{BC} r^2 \sin\theta dt dr d\theta d\phi \\
&= \int \left(\mho + \eta\sqrt{BC} t\right) r^2 \sin\theta dr d\theta d\phi,
\end{aligned} \tag{3.20}
$$

where

$$
\mho = \sqrt{C - Bt_r^2 - \frac{BC}{r^2}\left(t_\theta^2 + \frac{t_\phi^2}{\sin^2\theta}\right)}. \tag{3.21}
$$

Using eq.(3.20) in eq.(3.1), we get

$$
\delta\left(\int \left(\mho + \eta\sqrt{BC} t\right) r^2 \sin\theta dr d\theta d\phi\right) = 0. \tag{3.22}
$$

By varying ‘t’

$$
\int \left(\delta\mho + \eta\sqrt{BC}\delta t\right) r^2 \sin\theta dr d\theta d\phi = 0, \tag{3.23}
$$

where

$$\delta\mho = \frac{1}{2\mho}\left(-2Bt_r\delta t_r - \frac{BC}{r^2}\left(2t_\theta\delta t_\theta + \frac{2t_\phi\delta t_\phi}{\sin^2\theta}\right)\right), \\ = \frac{1}{\mho}\left(-Bt_r\delta t_r - \frac{BC}{r^2}\left(t_\theta\delta t_\theta + \frac{t_\phi\delta t_\phi}{\sin^2\theta}\right)\right). \tag{3.24}$$

For continuity,

$$\delta t_w = \delta\frac{\partial t}{\partial w} = \frac{\partial}{\partial w}\delta t. \tag{3.25}$$

Using eq.(3.25) in eq.(3.24), we have

$$\delta\mho = \mho^{-1}\left(-Bt_r\frac{\partial}{\partial r} - \frac{BC}{r^2}\left(t_\theta\frac{\partial}{\partial\theta} + \frac{t_\phi}{\sin^2\theta}\frac{\partial}{\partial\phi}\right)\right)\delta t. \tag{3.26}$$

Inserting eq.(3.26) in eq.(3.23), we get

$$\int\left(\mho^{-1}\left(-Bt_r\frac{\partial}{\partial r} - \frac{BC}{r^2}\left(t_\theta\frac{\partial}{\partial\theta} + \frac{t_\phi}{\sin^2\theta}\frac{\partial}{\partial\phi}\right)\right) + \eta\sqrt{BC}\right)\delta t r^2\sin\theta dr d\theta d\phi = 0. \tag{3.27}$$

Now consider

$$\int\frac{Bt_r r^2\sin\theta}{\mho}\frac{\partial}{\partial r}dr = \frac{Bt_r r^2\sin\theta}{\mho}\int\frac{\partial}{\partial r}dr - \int\left(\left(\int\frac{\partial}{\partial r}dr\frac{d}{dr}\left(\frac{Bt_r r^2\sin\theta}{\mho}\right)\right)\right)dr \\ = \left|\frac{Bt_r r^2\sin\theta}{\mho}\right|_{initial}^{final} - \int\left(\frac{Bt_r r^2\sin\theta}{\mho}\right)_{,r}dr. \tag{3.28}$$

Since there is no variation w.r.t 'r', so the term $\left|\frac{Bt_r r^2\sin\theta}{\mho}\right|_{initial}^{final}$ in the above equation is zero and we get,

$$\int\frac{Bt_r r^2\sin\theta}{\mho}\frac{\partial}{\partial r}dr = -\int\left(\frac{Bt_r r^2\sin\theta}{\mho}\right)_{,r}dr. \tag{3.29}$$

Similarly,

$$\int\frac{BCt_\theta\sin\theta}{\mho}\frac{\partial}{\partial\theta}d\theta = -\int\left(\frac{BCt_\theta\sin\theta}{\mho}\right)_{,\theta}d\theta, \tag{3.30}$$

and

$$\int\frac{BCt_\phi}{\mho\sin\theta}\frac{\partial}{\partial\phi}d\phi = -\int\left(\frac{BCt_\phi}{\mho\sin\theta}\right)_{,\phi}d\phi. \tag{3.31}$$

Using eqs.(3.29)-(3.31) in eq.(3.27), we get

$$\int\left[\left(\frac{Bt_r r^2\sin\theta}{\mho}\right)_{,r}+\left(\frac{BCt_\theta\sin\theta}{\mho}\right)_{,\theta}+\left(\frac{BCt_\phi}{\mho\sin\theta}\right)_{,\phi}+\eta r^2\sqrt{BC}\sin\theta\right]\delta t dr d\theta d\phi=0. \tag{3.32}$$

Now, by variational principle, setting the integrand to be zero, we get

$$\left(\frac{Bt_r r^2\sin\theta}{\mho}\right)_{,r}+\left(\frac{BCt_\theta\sin\theta}{\mho}\right)_{,\theta}+\left(\frac{BCt_\phi}{\mho\sin\theta}\right)_{,\phi}+\eta r^2\sqrt{BC}\sin\theta=0. \tag{3.33}$$

Using eq.(3.6) in eq.(3.33), we get

$$\left(\frac{Bt_r r^2\sin\theta}{\mho}\right)_{,r}+\left(\frac{BCt_\theta\sin\theta}{\mho}\right)_{,\theta}+\left(\frac{BCt_\phi}{\mho\sin\theta}\right)_{,\phi}=-Kr^2\sqrt{BC}\sin\theta. \tag{3.34}$$

For spherically symmetric hypersurface

$$t\equiv t(r). \tag{3.35}$$

Using eq.(3.35) in eq.(3.34), we get

$$\frac{d}{dr}\left(\frac{\frac{dt}{dr}Br^2\sin\theta}{\mho^*}\right)=-Kr^2\sqrt{BC}\sin\theta, \tag{3.36}$$

where $\mho^*$ is

$$\mho^*=\sqrt{C-B\left(\frac{dt}{dr}\right)^2}. \tag{3.37}$$

Integrating both sides of eq.(3.36), we get

$$\frac{B\frac{dt}{dr}r^2}{\sqrt{C-B\left(\frac{dt}{dr}\right)^2}}=-KJ+H, \tag{3.38}$$

where H is a constant and

$$J=\int^r\sqrt{B(u)C(u)}u^2du. \tag{3.39}$$

Now multiplying both sides by $\sqrt{C-B\left(\frac{dt}{dr}\right)^2}$ to eq.(3.38) and solving for $\frac{dt}{dr}$, we get

$$\frac{dt}{dr}=\left(\frac{C}{B}\right)^{\frac{1}{2}}\frac{H-KJ}{\left(Br^4+(H-KJ)^2\right)^{\frac{1}{2}}}. \tag{3.40}$$

Which is the required K-slicing equation for the metric given in eq.(3.7).

3.2 K-Slicing of the *Schwarzschild* Metric

For *Schwarzschild* metric, put

$$B = \frac{1}{C} = 1 - \frac{r_s}{r}, \tag{3.41}$$

in eqs.(3.39) and (3.40), we get

$$\frac{dt}{dr} = \frac{F}{G}\frac{r}{(r - r_s)}, \tag{3.42}$$

or

$$\frac{dr}{dt} = \frac{F}{G}\frac{(r - r_s)}{r}, \tag{3.43}$$

where

$$G = H - \frac{Kr^3}{3}, \tag{3.44}$$

and

$$F = \sqrt{E^2 + r^3(r - r_s)}. \tag{3.45}$$

Now we convert the eq.(3.42) in KS coordinates, given by eq.(2.24). We need to calculate $\frac{dw}{dz}$, for this purpose we consider

$$\frac{dw}{dz} = \frac{w_r dr + w_t dt}{z_r dr + z_t dt}. \tag{3.46}$$

Using eq.(2.24), we get

$$w_t = \frac{1}{2r_s}z, \quad z_t = \frac{1}{2r_s}w, \tag{3.47}$$

and

$$w_r = \frac{1}{2r_s}\left(\left(\frac{r}{r_s} - 1\right)^{-1} + 1\right)w. \tag{3.48}$$

Similarly

$$z_r = \frac{1}{2r_s}\left(\left(\frac{r}{r_s} - 1\right)^{-1} + 1\right)z. \tag{3.49}$$

Using eqs.(3.47)-(3.49) in eq.(3.46), we get

$$\frac{dw}{dz} = \frac{\left(\frac{1}{2r_s}\left(\left(\frac{r}{r_s} - 1\right)^{-1} + 1\right)w\right)\frac{dr}{dt} + \frac{1}{2r_s}z}{\left(\frac{1}{2r_s}\left(\left(\frac{r}{r_s} - 1\right)^{-1} + 1\right)z\right)\frac{dr}{dt} + \frac{1}{2r_s}w}. \tag{3.50}$$

Inserting eq.(3.43) in eq.(3.50), we get

$$\frac{dw}{dz}=\frac{\left(\frac{1}{2r_s}\left(\left(\frac{r}{r_s}-1\right)^{-1}+1\right)w\right)\left(\frac{F}{G}\frac{(r-r_s)}{r}\right)+\frac{1}{2r_s}z}{\left(\frac{1}{2r_s}\left(\left(\frac{r}{r_s}-1\right)^{-1}+1\right)z\right)\left(\frac{F}{G}\frac{(r-r_s)}{r}\right)+\frac{1}{2r_s}w}, \tag{3.51}$$

$$=\frac{\left(\left(\frac{r_s}{r-r_s}\right)+1\right)wF(r-r_s)+Grz}{\left(\left(\frac{r_s}{r-r_s}\right)+1\right)zF(r-r_s)+Grw}, \tag{3.52}$$

$$=\frac{Fr_sw+F(r-r_s)w+Grz}{Fr_sz+F(r-r_s)z+Grw}. \tag{3.53}$$

Finally, we get

$$\frac{dw}{dz}=\frac{Fw+Gz}{Fz+Gw} \tag{3.54}$$

This is the K-slicing equation for *Schwarzschild* metric in KS coordinates, these coordinates are non-compact, to compactify eq.(3.54), we convert this equation into compactified KS coordinates (Y,X) given by eq.(2.27). For this purpose, using eq.(2.27), we have

$$dY=\frac{1}{1+(w+z)^2}(dw+dz)+\frac{1}{1+(w-z)^2}(dw-dz), \tag{3.55}$$

and

$$dX=\frac{1}{1+(w+z)^2}(dw+dz)-\frac{1}{1+(w-z)^2}(dw-dz). \tag{3.56}$$

Dividing eq.(3.55) by eq.(3.56), we get

$$\frac{dY}{dX}=\frac{\frac{1}{1+(w+z)^2}(dw+dz)+\frac{1}{1+(w-z)^2}(dw-dz)}{\frac{1}{1+(w+z)^2}(dw+dz)-\frac{1}{1+(w-z)^2}(dw-dz)}, \tag{3.57}$$

$$=\frac{\left(1+(w-z)^2\right)(dw+dz)+\left(1+(w+z)^2\right)(dw-dz)}{\left(1+(w-z)^2\right)(dw+dz)-\left(1+(w+z)^2\right)(dw-dz)}, \tag{3.58}$$

$$=\frac{\left(1+(w-z)^2\right)\left(\frac{dw}{dz}+1\right)+\left(1+(w+z)^2\right)\left(\frac{dw}{dz}-1\right)}{\left(1+(w-z)^2\right)\left(\frac{dw}{dz}+1\right)-\left(1+(w+z)^2\right)\left(\frac{dw}{dz}-1\right)}, \tag{3.59}$$

$$=\frac{\left(1+(w-z)^2+1+(w+z)^2\right)\frac{dw}{dz}+1+(w-z)^2-1-(w+z)^2}{\left(1+(w-z)^2-1-(w+z)^2\right)\frac{dw}{dz}+1+(w-z)^2+1+(w+z)^2}, \tag{3.60}$$

$$\implies \frac{dY}{dX} = \frac{\left(1+w^2+z^2\right)\frac{dw}{dz} - 2zw}{-2zw\frac{dw}{dz} + \left(1+w^2+z^2\right)}. \tag{3.61}$$

Now using eq.(3.54) in eq.(3.61), we get

$$\frac{dY}{dX} = \frac{Fw\left(1+w^2-z^2\right) + Gz\left(1+z^2-w^2\right)}{Fz\left(1+z^2-w^2\right) + Gw\left(1+w^2-z^2\right)}. \tag{3.62}$$

Now we neet to find the values of $1+w^2-z^2$ and $1+z^2-w^2$ in terms of (Y, X). From eq.(2.27) we have

$$w + z = \tan\left(\frac{Y+X}{2}\right), \tag{3.63}$$

and

$$w - z = \tan\left(\frac{Y-X}{2}\right). \tag{3.64}$$

From eqs.(3.63) and (3.64), we obtain

$$\begin{aligned} w &= \tfrac{1}{2}\tan\left(\tfrac{Y+X}{2}\right) + \tfrac{1}{2}\tan\left(\tfrac{Y-X}{2}\right), \\ z &= \tfrac{1}{2}\tan\left(\tfrac{Y+X}{2}\right) - \tfrac{1}{2}\tan\left(\tfrac{Y-X}{2}\right). \end{aligned} \tag{3.65}$$

Multiplying eqs.(3.63) and (3.64) and adding 1 to both sides, we get

$$1+w^2-z^2 = 1 + \tan\left(\frac{Y+X}{2}\right)\tan\left(\frac{Y-X}{2}\right). \tag{3.66}$$

After simplification, we get

$$1+w^2-z^2 = \frac{\cos X}{\cos\left(\frac{Y+X}{2}\right)\cos\left(\frac{Y-X}{2}\right)}. \tag{3.67}$$

Using eq.(3.65) along with eq.(3.67), we get

$$w\left(1+w^2-z^2\right) = \frac{\sin Y \cos X}{2\cos^2\left(\frac{Y+X}{2}\right)\cos^2\left(\frac{Y-X}{2}\right)}. \tag{3.68}$$

Similarly,

$$z\left(1+z^2-w^2\right) = \frac{\sin X \cos Y}{2\cos^2\left(\frac{Y+X}{2}\right)\cos^2\left(\frac{Y-X}{2}\right)}. \tag{3.69}$$

Utilizing eqs.(3.68) and (3.69) in eq.(3.62), we get

$$\frac{dY}{dX} = \frac{F\sin Y\cos X + G\sin X\cos Y}{F\sin X\cos Y + G\sin Y\cos X}, \tag{3.70}$$

where G and F are given by eqs.(3.44) and (3.45) respectively. Eq.(3.70) is the required K-slicing equation for *Schwarzschild* metric in compactified KS coordinates.

BCI [3] solved the eq.(3.54) for $m = 0.5$ with the condition $\frac{dY}{dX}(X = 0) = 0$, *i.e*, hypersurfaces must be flat at $r = 2m$. They use only non-negative values of K and did not provide the complete foliation for it. Later Qadir *et.al* [5] convert the eq.(3.54) into eq.(3.70) and foliate the CP diagram by using same conditions and obtained the complete foliation for both the *Schwarzschild* and RN metrics [6]. Further they obatined the foliation of eRN metric [7]. We have solved eq.(3.70) for $m = 0.5$ numerically using Runge-Kutta 4^{th} order (RK4) method with the implicit relation given in eq.(2.28) and require that

$$\frac{dY}{dX}(X = 0) = 0. \tag{3.71}$$

This gives

$$F = 0, \tag{3.72}$$

this implies

$$H = \frac{Kr_i^3}{3} \pm \sqrt{r_i^3(r_s - r_i)}. \tag{3.73}$$

The K-surfaces are shown in Figure3.4 and their corresponding values are shown in Table3.1.

No	K	r_i	H	Y_i
0	0.0	1.0	0	0.0
$\pm$ 1	$\pm$ 0.01	0.9827	$\mp$ 0.1250	$\pm$ 0.4237
$\pm$ 2	$\pm$ 0.02	0.9602	$\mp$ 0.1817	$\pm$ 0.6235
$\pm$ 3	$\pm$ 0.03	0.9243	$\mp$ 0.2366	$\pm$ 0.8237
$\pm$ 4	$\pm$ 0.05	0.8668	$\mp$ 0.2837	$\pm$ 1.0255
$\pm$ 5	$\pm$ 0.1	0.7897	$\mp$ 0.3054	$\pm$ 1.1952
$\pm$ 6	$\pm$ 1.0	0.6004	$\mp$ 0.2219	$\pm$ 1.4130

Table 3.1: For $m = 0.5$, thirteen K-surfaces for various values of MEC , K, are described by their corresponding initial values of r and Y.

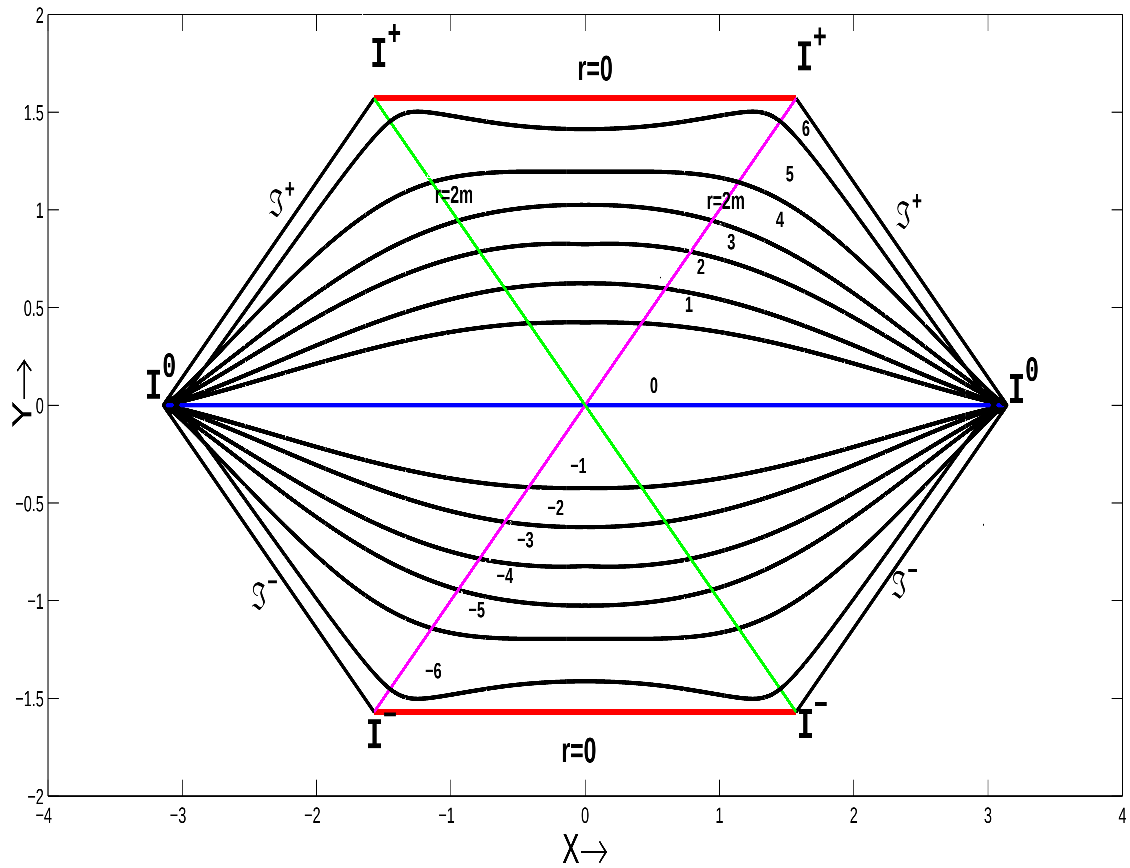

Figure 3.4: K-Slicing of the SM for m=0.5. Few spacelike hypersurfaces are shown.

3.3 K-Slicing of the *Reissner Nordstorm* Metric

For RN Metric, we have

$$B = \frac{1}{C} = \left(1 - \frac{r_1}{r}\right)\left(1 - \frac{r_2}{r}\right), \tag{3.74}$$

where r_1 and r_2 are given in eqs.(2.32) and (2.33) respectively.

Using values of B and C in eq.(3.40) and following the procedure mentioned in the previous section, we obtain the K-slicing equation for RN metric in compactified Kruskal-Szekers like coordinates (Y, X) given in eqs.(2.55)

$$\frac{dY}{dX} = \frac{F \sin Y \cos X + G \sin X \cos Y}{F \sin X \cos Y + G \sin Y \cos X}, \tag{3.75}$$

where

$$G = H - \frac{Kr^3}{3}, \tag{3.76}$$

$$F = \sqrt{G^2 + r^2(r - r_2)(r - r_1)}, \tag{3.77}$$

and

$$H = \frac{Kr_i^3}{3} \pm r_i\sqrt{(r_1 - r_i)(r_2 - r_i)}. \tag{3.78}$$

We have solved eq.(3.75) with the implicit relation given in eq.(2.59) using RK4 method for $Q/m = 0.6,\ 0.8$ and 0.94. K-surfaces for $Q/m = 0.6,\ 0.8$ and 0.94 are shown in Figures. 3.5, 3.6 and 3.7 respectively.

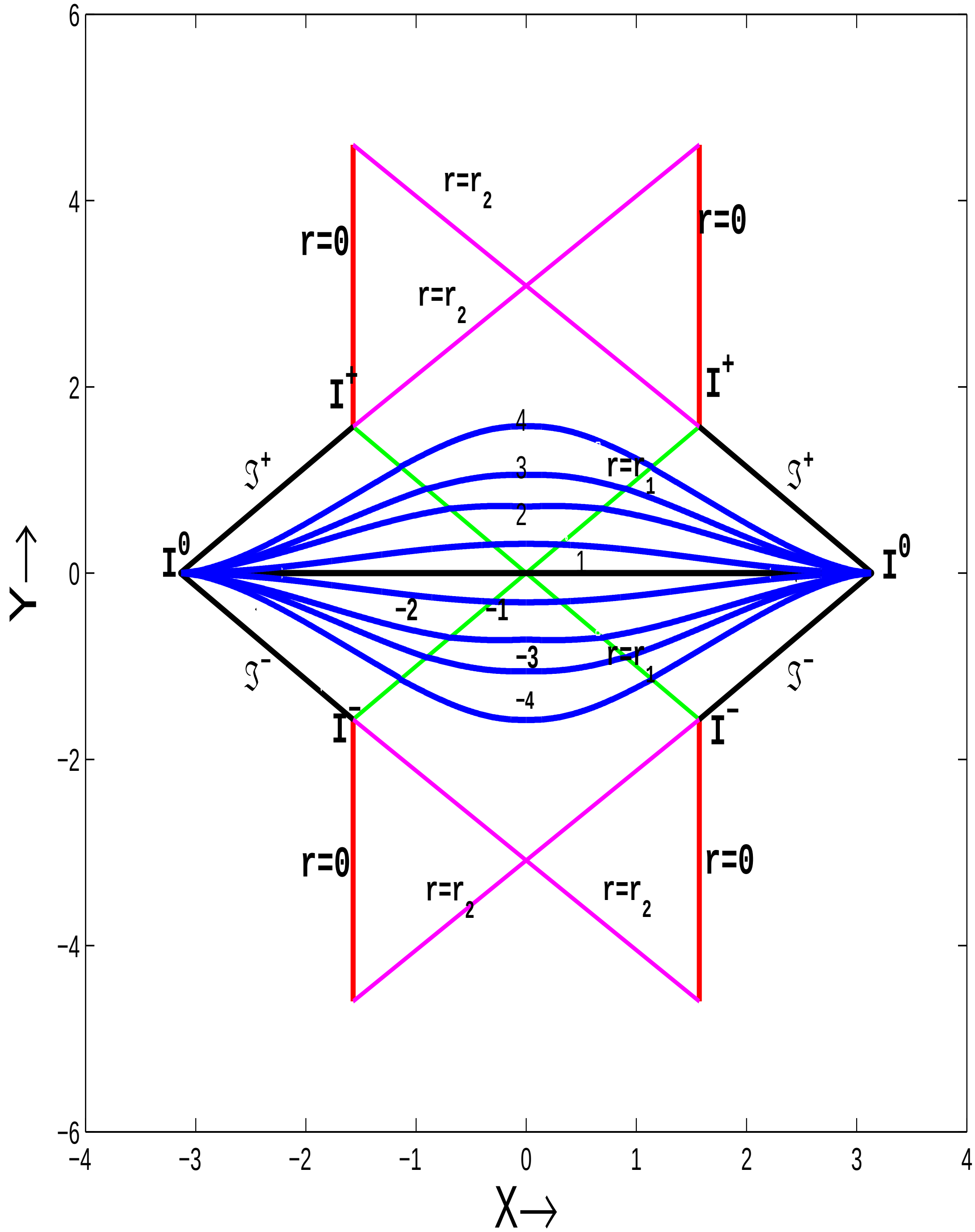

Figure 3.5: K-Slicing of the RN metric for $Q/m = 0.6$. Few spacelike hypersurfaces are shown.

No	K	r_i	H	Y_i
$\pm$ 1	$\pm$ 0.01	0.8944	$\mp$ 0.0160	$\pm$ 0.3138
$\pm$ 2	$\pm$ 0.02	0.8970	$\mp$ 0.3848	$\pm$ 0.7138
$\pm$ 3	$\pm$ 0.06	0.8928	$\mp$ 0.0528	$\pm$ 1.0529
$\pm$ 4	$\pm$ 0.1	0.8783	$\mp$ 0.0913	$\pm$ 1.5760

Table 3.2: For $Q/m = 0.6$, eight K-surfaces for various values of MEC , K, are described by their corresponding initial values of r and Y.

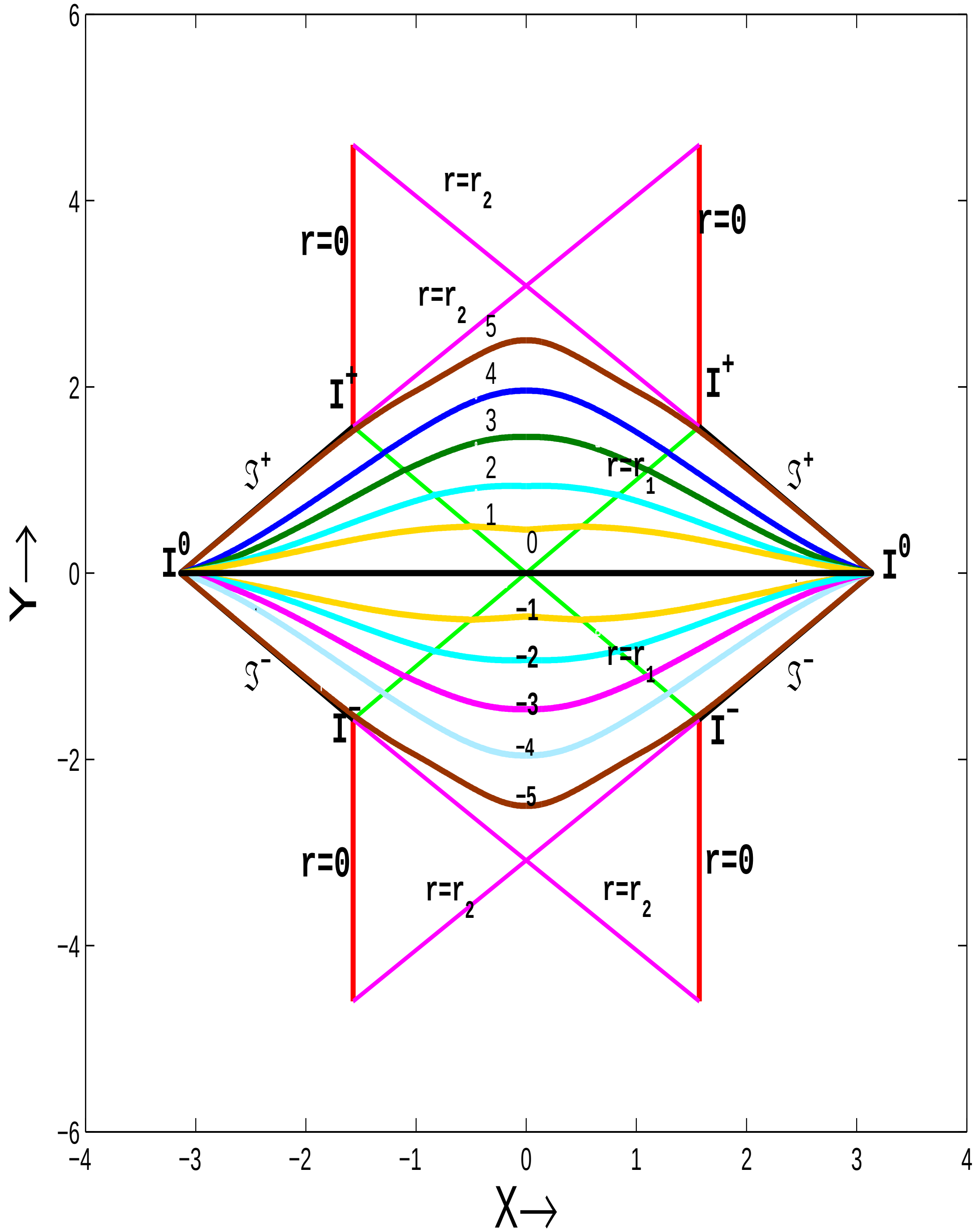

Figure 3.6: K-Slicing of the RN metric for $Q/m = 0.8$. Few spacelike hypersurfaces are shown.

No	K	r_i	H	Y_i
0	0.0	0.8	0	0.0
$\pm$ 1	$\pm$ 0.01	0.7987	$\mp$ 0.0202	$\pm$ 0.4670
$\pm$ 2	$\pm$ 0.02	0.7942	$\mp$ 0.0429	$\pm$ 0.9340
$\pm$ 3	$\pm$ 0.04	0.7817	$\mp$ 0.0741	$\pm$ 1.4623
$\pm$ 4	$\pm$ 0.07	0.7483	$\mp$ 0.1162	$\pm$ 1.9585
$\pm$ 5	$\pm$ 0.1	0.5531	$\mp$ 0.1576	$\pm$ 2.5000

Table 3.3: For $Q/m = 0.8$, eleven K-surfaces for various values of MEC , K, are described by their corresponding initial values of r and Y.

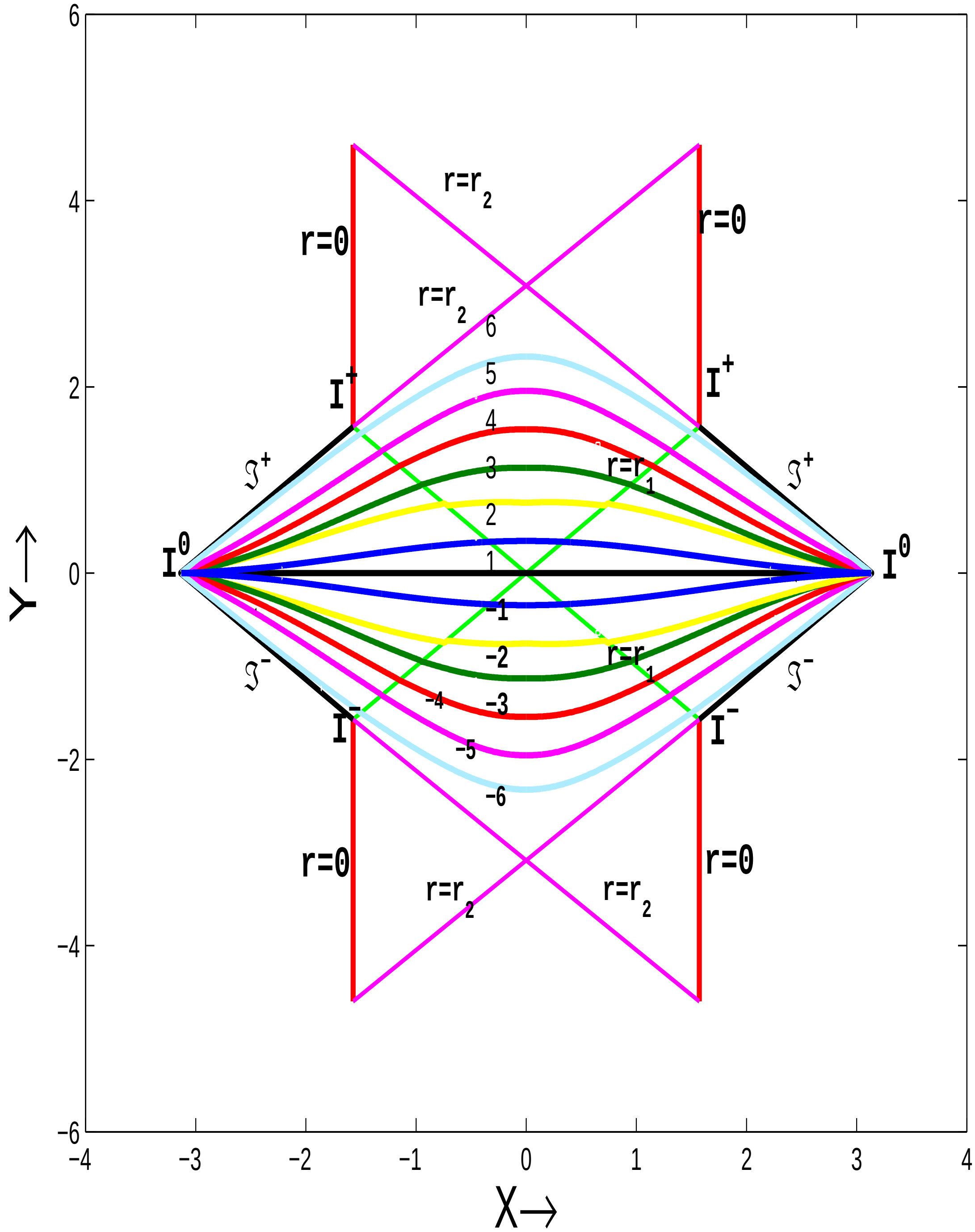

Figure 3.7: K-Slicing of the RN metric for $Q/m = 0.94$. Few spacelike hypersurfaces are shown.

No	K	r_i	H	Y_i
$\pm$ 1	$\pm$ 0.02	0.6699	$\mp$ 0.0082	$\pm$ 0.3450
$\pm$ 2	$\pm$ 0.05	0.6670	$\mp$ 0.0181	$\pm$ 0.7563
$\pm$ 3	$\pm$ 0.09	0.6615	$\mp$ 0.0275	$\pm$ 1.1294
$\pm$ 4	$\pm$ 0.095	0.6493	$\mp$ 0.0448	$\pm$ 1.5427
$\pm$ 5	$\pm$ 0.1	0.6211	$\mp$ 0.0666	$\pm$ 1.9560
$\pm$ 6	$\pm$ 0.15	0.5521	$\mp$ 0.0812	$\pm$ 2.3250

Table 3.4: For $Q/m = 0.94$, twelve K-surfaces for various values of MEC , K, are described by their corresponding initial values of r and Y.

3.4 Generalized K-Surface Equation

In order to obtain generalized K-surface equation, we consider the general spacetime metric

$$
\begin{aligned}
ds^2 = Bdt^2 - Cdr^2 - Dd\theta^2 - Ed\phi^2 + 2Fdtdr + 2Gdtd\theta \\
+ 2Hdtd\phi + 2Idrd\theta + 2Jdrd\phi + 2Ld\theta d\phi,
\end{aligned} \tag{3.79}
$$

where $B,\ C,\ D,\ E,\ G,\ H,\ I,\ J,\ K,\ L$ may depend on $r,\ \theta,\ \phi$. The 4-volume of the metric is

$$\Upsilon = \int \sqrt{|g|}dtdrd\theta d\phi. \tag{3.80}$$

Consider the foliating hypersurface be u as

$$u\left(t, r, \theta, \phi\right) = 0. \tag{3.81}$$

Inserting eq.(3.81) in eq.(3.80), we get $\Upsilon\left(\Sigma, \Sigma_1\right)$ as $\Sigma_1(t = 0)$ is

$$\Upsilon\left(\Sigma, \Sigma_1\right) = \int \sqrt{|g|}u\left(t, r, \theta, \phi\right) drd\theta d\phi. \tag{3.82}$$

From eq.(3.81), we have

$$du = 0, \tag{3.83}$$

which gives eq.(3.12). Now inserting eq.(3.12) in eq.(3.79), we obtain g^* as

$$ds^{*2} = -Pdr^2 - Qd\theta^2 - Td\phi^2 + 2Udrd\theta + 2Wdrd\phi + 2Yd\theta d\phi, \tag{3.84}$$

where

$$P = C - Bt_r^2 - 2Ft_r, \tag{3.85}$$

$$Q = D - Bt_\theta^2 - 2Gt_\theta, \tag{3.86}$$

$$T = E - Bt_\phi^2 - 2Ht_\phi, \tag{3.87}$$

$$U = I - Bt_r t_\theta - Ft_\theta - Gt_r, \tag{3.88}$$

$$W = J - Bt_r t_\phi - Ft_\phi - Ht_r, \tag{3.89}$$

$$Y = L - Bt_\phi t_\theta - Gt_\phi - Ht_\theta. \tag{3.90}$$

The determinant of the induced metric is

$$g^* = \begin{vmatrix} -C^* & I^* & J^* \\ I^* & -D^* & L^* \\ j^* & L^* & -E^* \end{vmatrix}, \tag{3.91}$$

Hence the 3-area, $\Pi(\Sigma)$ of the hypersurface Σ, is

$$\Pi(\Sigma) = \int \sqrt{|g^*|} dr d\theta d\phi. \tag{3.92}$$

Using eqs.(3.92) and (3.82) in eqs.(3.1)-(3.2) along with eq.(3.6), we get

$$\delta \left(\int \left(\sqrt{|g^*|} + K\sqrt{|g|}t \right) dr d\theta d\phi \right) = 0, \tag{3.93}$$

$$\int \left(\delta\sqrt{|g^*|} + K\sqrt{|g|}\delta t \right) dr d\theta d\phi = 0. \tag{3.94}$$

For simplicity, let

$$|g^*| = \mho^2, \tag{3.95}$$

then

$$\delta\sqrt{|g^*|} = \delta\mho. \tag{3.96}$$

Notice that

$$\mho = \mho\left(r, \theta, \phi, t_r, t_\theta, t_\phi\right), \tag{3.97}$$

therefore the variations w.r.t 't' gives

$$\delta\mho = \mho_{,t_r}\delta t_r + \mho_{,t_\theta}\delta t_\theta + \mho_{,t_\phi}\delta t_\phi. \tag{3.98}$$

For continuity

$$\delta t_r = \delta\frac{\partial t}{\partial r} = \frac{\partial}{\partial r}\delta t. \tag{3.99}$$

Inserting eq.(3.99) in eq.(3.98), we get

$$\begin{aligned} \delta\mho &= \mho_{,t_r}\frac{\partial}{\partial r}\delta t + \mho_{,t_\theta}\frac{\partial}{\partial\theta}\delta t + \mho_{,t_\phi}\frac{\partial}{\partial\phi}\delta t, \\ &= \left(\mho_{,t_r}\frac{\partial}{\partial r} + \mho_{,t_\theta}\frac{\partial}{\partial\theta} + \mho_{,t_\phi}\frac{\partial}{\partial\phi} \right)\delta t. \end{aligned} \tag{3.100}$$

Using eqs.(3.100) and (3.96) in eq.(3.94), we get

$$\int \left(\mho_{,t_r} \frac{\partial}{\partial r} + \mho_{,t_\theta} \frac{\partial}{\partial \theta} + \mho_{,t_\phi} \frac{\partial}{\partial \phi} + \sqrt{|g|} K \right) \delta t dr d\theta d\phi = 0. \tag{3.101}$$

Integrating above equation implies

$$\int \left(-\left(\mho_{,t_r}\right)_{,r} - \left(\mho_{,t_\theta}\right)_{,\theta} - \left(\mho_{,t_\phi}\right)_{,\phi} + \sqrt{|g|} K \right) \delta t dr d\theta d\phi = 0. \tag{3.102}$$

The above integral vanishes, if the integrand is zero, therefore, we have

$$\left(\mho_{,t_r}\right)_{,r} + \left(\mho_{,t_\theta}\right)_{,\theta} + \left(\mho_{,t_\phi}\right)_{,\phi} = \sqrt{|g|} K. \tag{3.103}$$

Eq.(3.103) is the required Differential equation satisfied by K-surfaces.

3.5 Flat Hypersurfaces for the *Schwarzschild* Metric

Flat foliation is used to study Hawking radiation [40, 41, 42]. Observers falling freely from infinity along the spacelike hypersurfaces, starting at rest, observe Minkowski space about them and these hypersurfaces are orthogonal to the *world-lines* of such observers. Hence instead of solving ${}^3R_{abcd} = 0$, we need to calculate only free fall geodesics [8].

In order to obtain the maximal slices or flat spacelike hypersurfaces of the blackhole metric, we require that the *world-lines* of the observers falling freely from infinity, starting at rest, must be orthogonal to the flat foliating hypersurfaces. This requirement gives the tangent vectors of the hypersurfaces and then using these vectors, we obtain the differential equation for maximal slices. To view these hypersurfaces in CP diagram, we, then convert this equation into CKS coordinates.

Let the unit tangent vector to the *world-line* of the freely falling observer be v^a and the unit tangent vector to the flat hyprsurfaces be V^a, our requirements gives us

$$v^a v_a = 1, \tag{3.104}$$

$$V^a V_a = -1, \tag{3.105}$$

$$V^a v_a = 0. \tag{3.106}$$

The SM can be written as

$$ds^2 = e^{\lambda} dt^2 - e^{-\lambda} dr^2 - r^2 d\Omega^2, \tag{3.107}$$

where

$$e^{\lambda} = 1 - \frac{r_s}{r}. \tag{3.108}$$

The SM is spherically symmetric, so without loss of generality we fixed ϕ and θ, we can write down the tangent vectors to the geodesics which is in fact the path of freely falling observers, as

$$v^a = (v^0, v^1, 0, 0) = \left(\left(\frac{dt}{d\varpi}\right)_o, \left(\frac{dr}{d\varpi}\right)_o, 0, 0 \right), \tag{3.109}$$

and for the flat hypersurfaces as

$$V^a = (V^0, V^1, 0, 0) = \left(\left(\frac{dt}{d\varpi}\right)_h, \left(\frac{dr}{d\varpi}\right)_h, 0, 0 \right), \tag{3.110}$$

where the subscripts h and o represents the hypersurface and the observer respectively. To obtain the tangent vector we solve the geodesic equation given in eq.(1.66) with eqs.(3.104)-(3.106).

The geodesic equation for first coordinate $i.e$ for t is

$$\left(\frac{d^2 t}{d\varpi^2}\right)_0 + \Gamma^0_{\alpha\nu} \left(\frac{dt^{\alpha}}{d\varpi}\right)_o \left(\frac{dt^{\nu}}{d\varpi}\right)_o = 0. \tag{3.111}$$

The only nonzero $Christoffel$ symbols for the above equation are

$$\Gamma^0_{10} = \Gamma^0_{01} = \frac{d\lambda}{2dr}. \tag{3.112}$$

Using eq.(3.112) in eq.(3.111), we get

$$\left(\frac{d^2 t}{d\varpi^2}\right)_o + \left(\frac{d\lambda}{dr}\right)_o \left(\frac{dt}{d\varpi}\right)_o \left(\frac{dr}{d\varpi}\right)_o = 0. \tag{3.113}$$

$$\left(\frac{d^2 t}{d\varpi^2}\right)_o + \left(\frac{d\lambda}{d\varpi}\right)_o \left(\frac{dt}{d\varpi}\right)_o = 0. \tag{3.114}$$

Multiplying both sides by e^{λ}, we get

$$\frac{d}{d\varpi}\left(e^{\lambda}\left(\frac{dt}{d\varpi}\right)_o\right) = 0. \tag{3.115}$$

Integrating both sides, we get

$$v^0 = \left(\frac{dt}{d\varpi}\right)_o = je^{-\lambda}, \tag{3.116}$$

where 'j' is constant.
In order to find $\left(\frac{dr}{d\varpi}\right)_o$, consider

$$d\varpi^2 = e^{\lambda}dt^2 - e^{-\lambda}dr^2. \tag{3.117}$$

Dividing both sides by $d\varpi^2$, we get

$$1 = e^{\lambda}\left(\frac{dt}{d\varpi}\right)_o^2 - e^{-\lambda}\left(\frac{dr}{d\varpi}\right)_o^2. \tag{3.118}$$

Using eq.(3.116) into eq.(3.118), we get

$$\left(v^1\right)^2 = \left(\frac{dr}{d\varpi}\right)_o^2 = j^2 - e^{\lambda}. \tag{3.119}$$

Thus

$$v^a = \left(je^{-\lambda}, \pm\sqrt{j^2 - e^{\lambda}}, 0, 0\right). \tag{3.120}$$

From eq.(3.106), we have

$$V^a v_a = g_{ai}V^a v^i = g_{00}V^0 v^0 + g_{11}V^1 v^1 = 0. \tag{3.121}$$

Inserting eq.(3.120) with eq.(3.117) in eq.(3.121), we obtain

$$V^0 = \frac{e^{-2\lambda}V^1 v^1}{v^0}. \tag{3.122}$$

From eq.(3.105), we have

$$e^{\lambda}\left(V^0\right)^2 - e^{-\lambda}\left(V^1\right)^2 = -1. \tag{3.123}$$

Using eqs.(3.116), (3.119) and (3.122) in eq.(3.123), we get

$$V^1 = \pm j. \tag{3.124}$$

Using eq.(3.124) in eq.(3.122), we get

$$V^0 = \pm e^{-\lambda}\sqrt{j^2 - e^\lambda}. \tag{3.125}$$

Finally, we have

$$V^a = \left(\pm e^{-\lambda}\sqrt{j^2 - e^\lambda}, \pm j, 0, 0\right). \tag{3.126}$$

From eq.(3.110), we have

$$\frac{V^0}{V^1} = \frac{\left(\frac{dt}{d\varpi}\right)_h}{\left(\frac{dr}{d\varpi}\right)_h} = \left(\frac{dt}{dr}\right)_h. \tag{3.127}$$

In order to obtain the flat hypersurfaces differential equation, we use eqs.(3.124) and (3.125) in eq.(3.127) and get

$$\left(\frac{dt}{dr}\right)_h = \frac{\pm e^{-\lambda}\sqrt{j^2 - e^\lambda}}{j}. \tag{3.128}$$

Similarly, the Differential equations for the *world-lines* of freely falling observers, *i.e*, the geodesics, are

$$\frac{v^0}{v^1} = \frac{\left(\frac{dt}{d\varpi}\right)_o}{\left(\frac{dr}{d\varpi}\right)_o} = \left(\frac{dt}{dr}\right)_o. \tag{3.129}$$

Using eqs.(3.116) and (3.119) in eq.(3.129), we get

$$\left(\frac{dt}{dr}\right)_o = \frac{\pm j e^{-\lambda}}{\sqrt{j^2 - e^\lambda}}. \tag{3.130}$$

The negative and positive signs correspond to paths of incoming and outgoing observers respectively. The constant j corresponds to the energy of the observer at infinity.
The equation for flat hypersurfaces for decreasing values of r is given by

$$\left(\frac{dt}{dr}\right)_h = \frac{-\sqrt{j^2 - e^\lambda}}{j e^\lambda} = \frac{F^*}{G^*}, \tag{3.131}$$

where

$$F^* = -\sqrt{j^2 - e^\lambda}, \tag{3.132}$$

and

$$G^* = j e^\lambda. \tag{3.133}$$

In order to convert the eq.(3.131) into CKS (Y, X) coordinates, we first convert this equation into KS, (w, z) coordinates, for this purpose, we put eq.(3.131) in eq.(3.50), we get

$$\frac{dw}{dz} = \frac{\left(\frac{1}{2r_s}\left(\left(\frac{r}{r_s}-1\right)^{-1}+1\right)w\right)\frac{G^*}{F^*}+\frac{1}{2r_s}z}{\left(\frac{1}{2r_s}\left(\left(\frac{r}{r_s}-1\right)^{-1}+1\right)z\right)\frac{G^*}{F^*}+\frac{1}{2r_s}w}, \tag{3.134}$$

after simplification, we get

$$\frac{dw}{dz} = \frac{G^* r w + F^*\left(r-r_s\right)z}{G^* r z + F^*\left(r-r_s\right)w}. \tag{3.135}$$

$$\frac{dw}{dz} = \frac{G^{**}w + F^{**}z}{G^{**}z + F^{**}w}, \tag{3.136}$$

where

$$G^{**} = G^* r = jre^{\lambda} = jr\left(1-\frac{r_s}{r}\right), \tag{3.137}$$

and

$$F^{**} = F^*\left(r-r_s\right) = -\left(r-r_s\right)\sqrt{j^2-e^{\lambda}} = -\left(r-r_s\right)\sqrt{j^2-1+\frac{r_s}{r}}. \tag{3.138}$$

Eq.(3.136) in CKS coordinates takes the form

$$\frac{dY}{dX} = \frac{G^{**}\sin Y\cos X + F^{**}\sin X\cos Y}{G^{**}\sin X\cos Y + F^{**}\sin Y\cos X}, \tag{3.139}$$

where F^{**} and G^{**} are given in eqs.(3.138) and (3.137) respectively.
To check whether the foliating hypersurfaces are flat or not, we need to determine the curvature invariants for these flat hypersurfaces. For this purpose, we use eq.(3.128) with eq.(3.108) in eq.(3.107) and obtain the induced metric

$$ds_h^2 = \frac{-1}{j^2}dr^2 - r^2 d\Omega^2. \tag{3.140}$$

The curvature invariants for this metric are

$$I_1 = \frac{2(1-j^2)}{r^2}, \tag{3.141}$$

$$I_2 = \frac{2(j^2-1)^2}{r^4}. \tag{3.142}$$

From eqs.(3.141) and (3.142), it is clear that the hypersurfaces are flat only for $j^2 = 1.0$, while for $j \lesseqgtr 1$ they are not flat.

In order to obtain the flat hypersurfaces, we solve the eq.(3.139) by using RK4 method for $j = 0.6,\ 0.8,\ 1.0,\ 1.25$ and 1.5 with $m = 0.5$, with the implicit relation given in eq.(2.28). We move in X direction from $-\pi/2$ to $\pi/2$ and for every X, say X_i, we get a new hypersurface. For $j \geq 1$, the foliating hypersurfaces completely foliate the spacetime. We found a barrier for the hypersurfaces for $j < 1$ to reach I^0 because the expression inside the square root in eq.(3.138) becomes complex. For this case, the hypersurfaces are defined only for $r \leq \frac{r_s}{1-j^2}$. These hypersurfaces are shown in Figures.3.8 and 3.9. The hypersurfaces in the maximal extension are obtained by reversing the diagrams Figure.3.8-3.12 and adjoining it to the original copy [30]. The flat hypersurfaces ($j = 1$) in the maximal extension are shown in Figure.3.15.

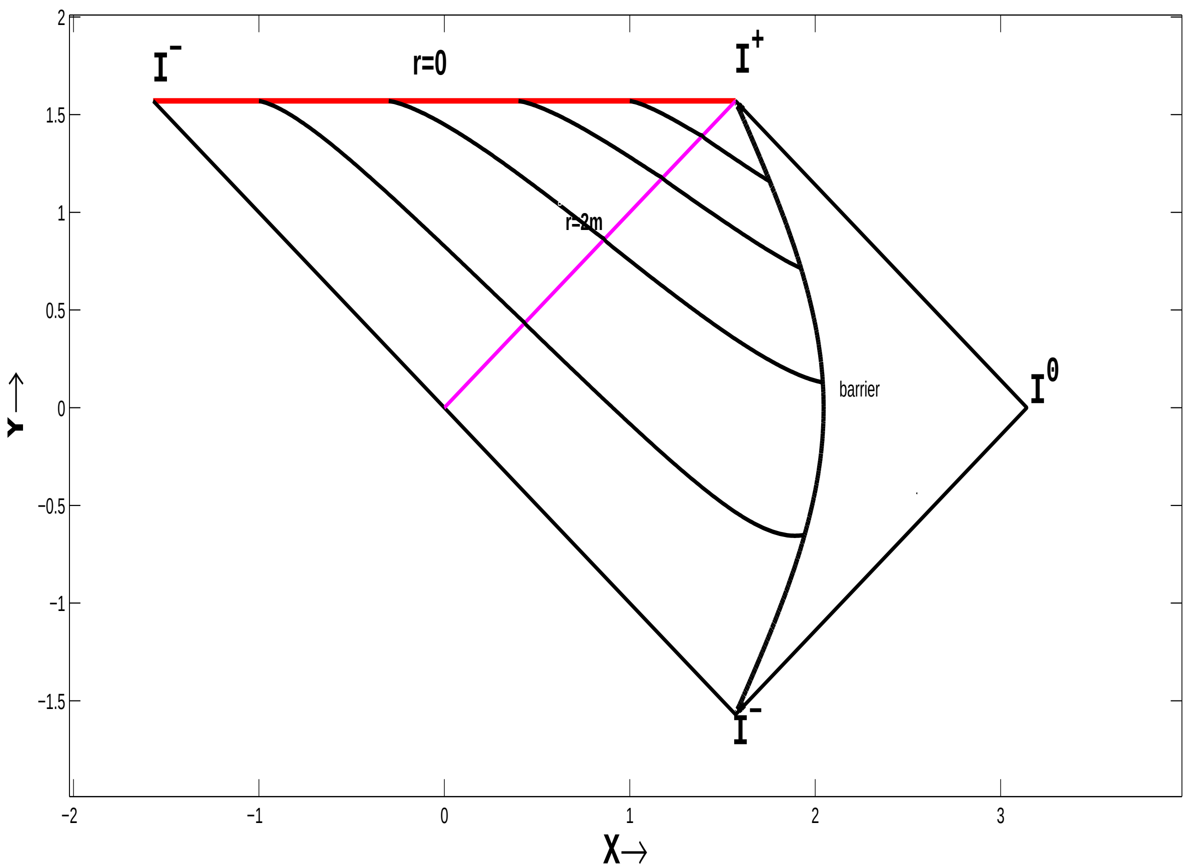

Figure 3.8: Few Foliating hypersurfaces for $j = 0.6$ are shown for SM. In this case the hypersurfaces do not reach I^0 due to a barrier. However, they can be analytically continued to reach I^0.

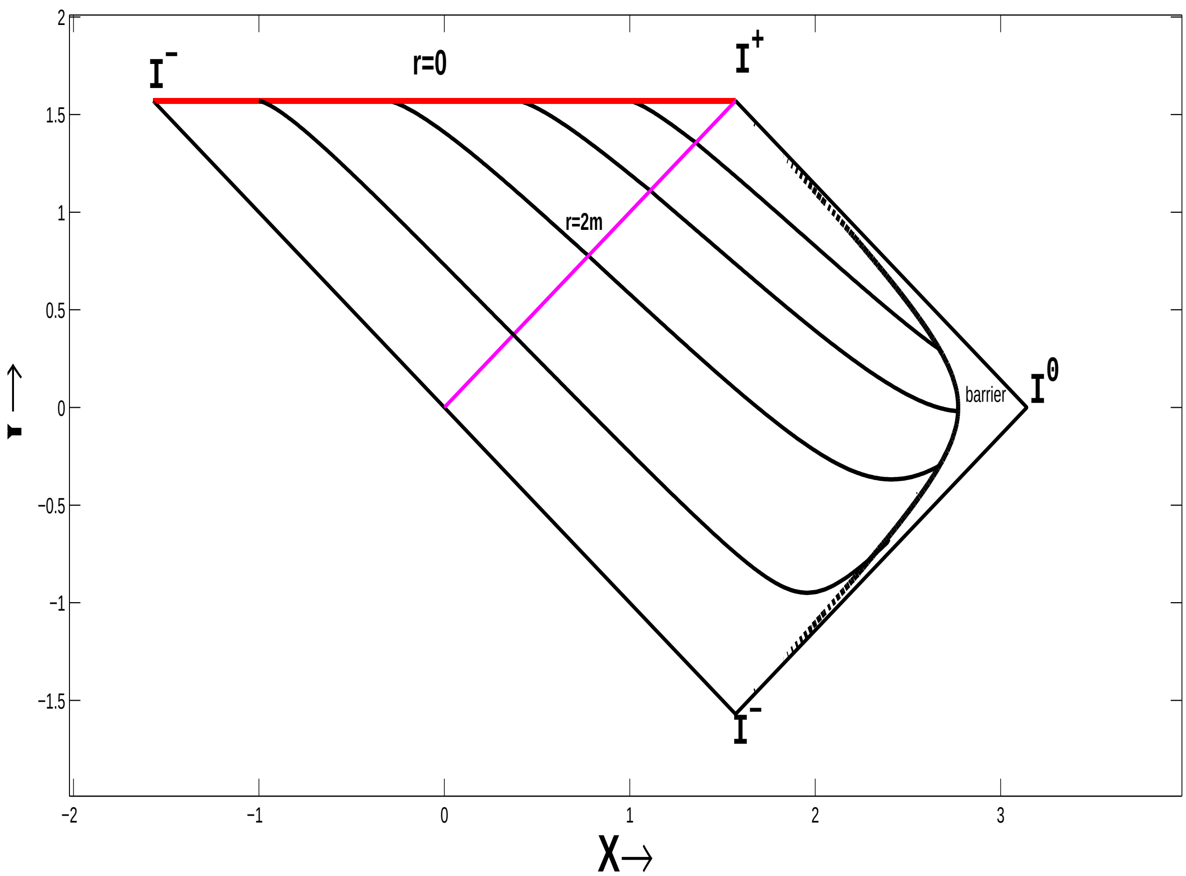

Figure 3.9: Few Foliating hypersurfaces for $j = 0.8$ are shown for SM. In this case the hypersurfaces do not reach I^0 due to a barrier. However, they can be analytically continued to reach I^0.

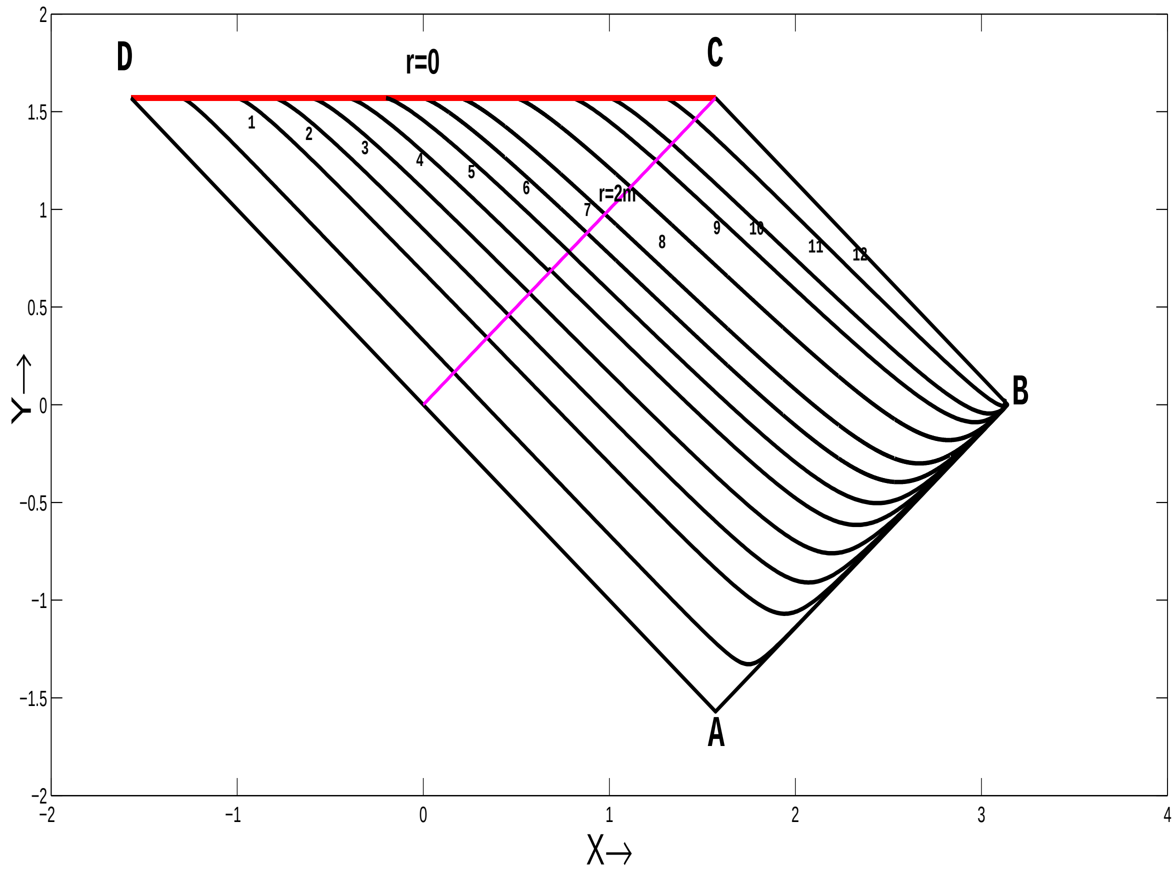

Figure 3.10: Few Flat hypersurfaces for $j = 1$ are shown for SM.

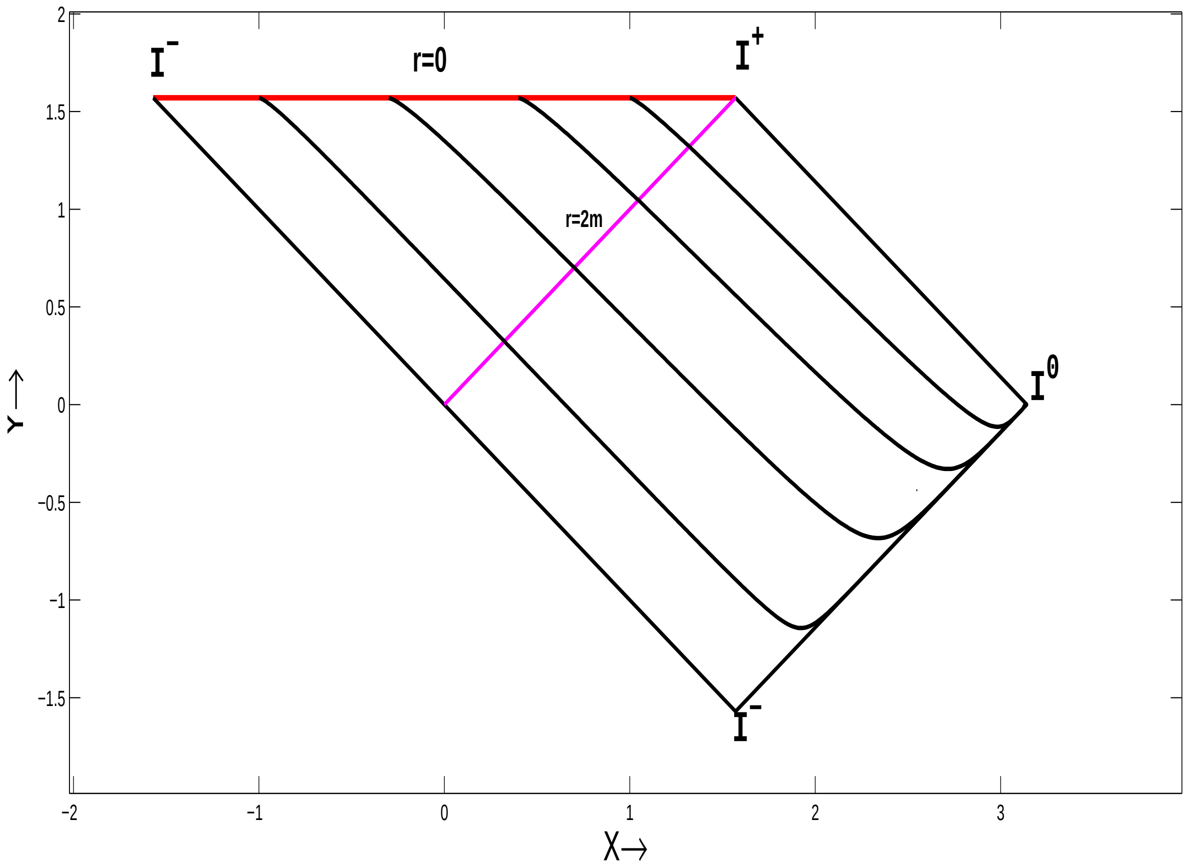

Figure 3.11: Few spacelike foliating hypersurfaces for $j = 1.25$ are shown for SM.

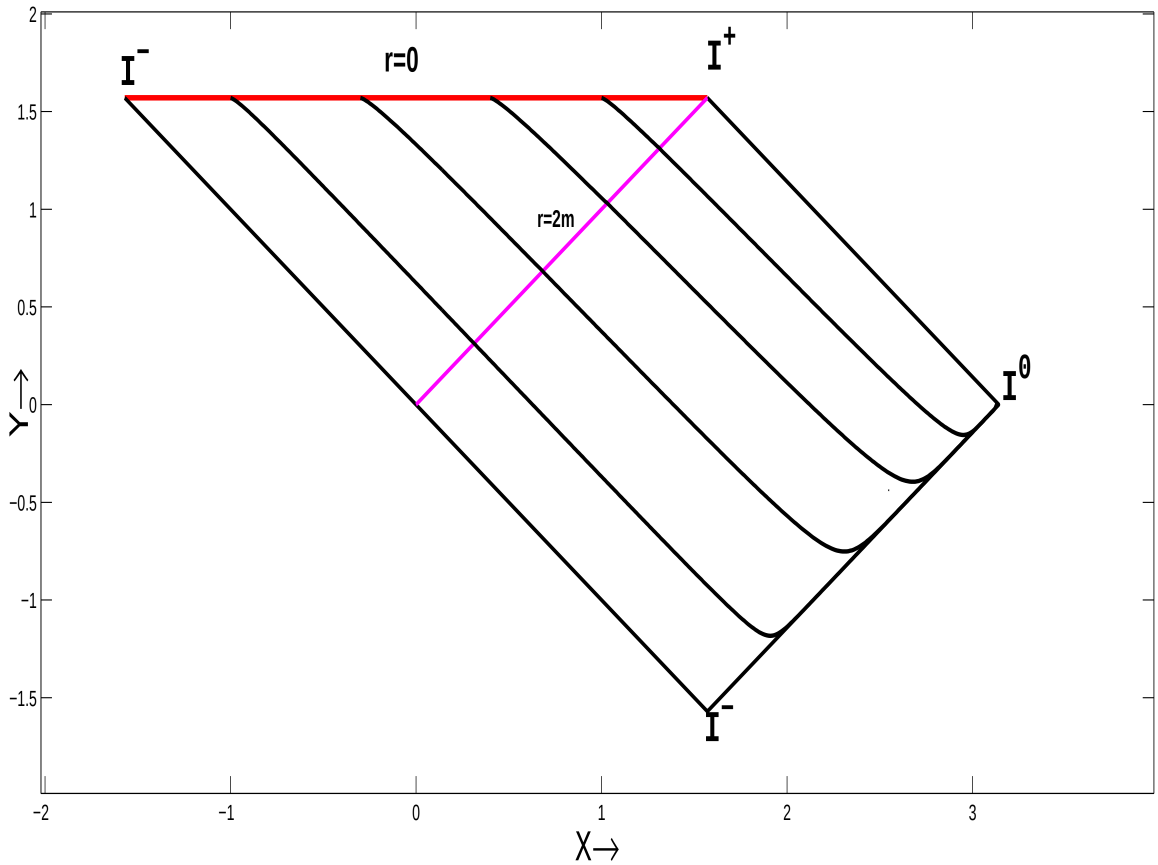

Figure 3.12: Few spacelike foliating hypersurfaces for $j = 1.5$ are shown for SM.

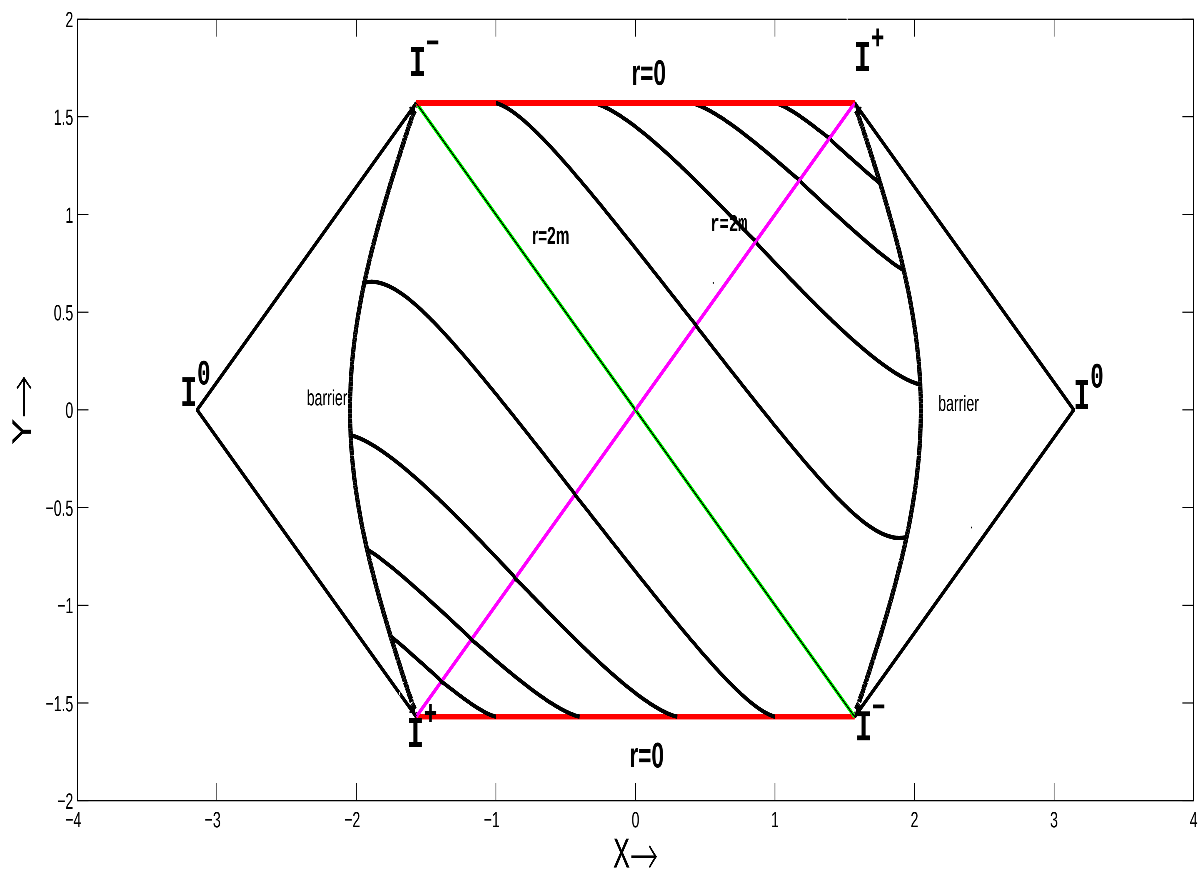

Figure 3.13: Foliating hypersurfaces for $j = 0.6$ in the maximal extension of SM are shown.

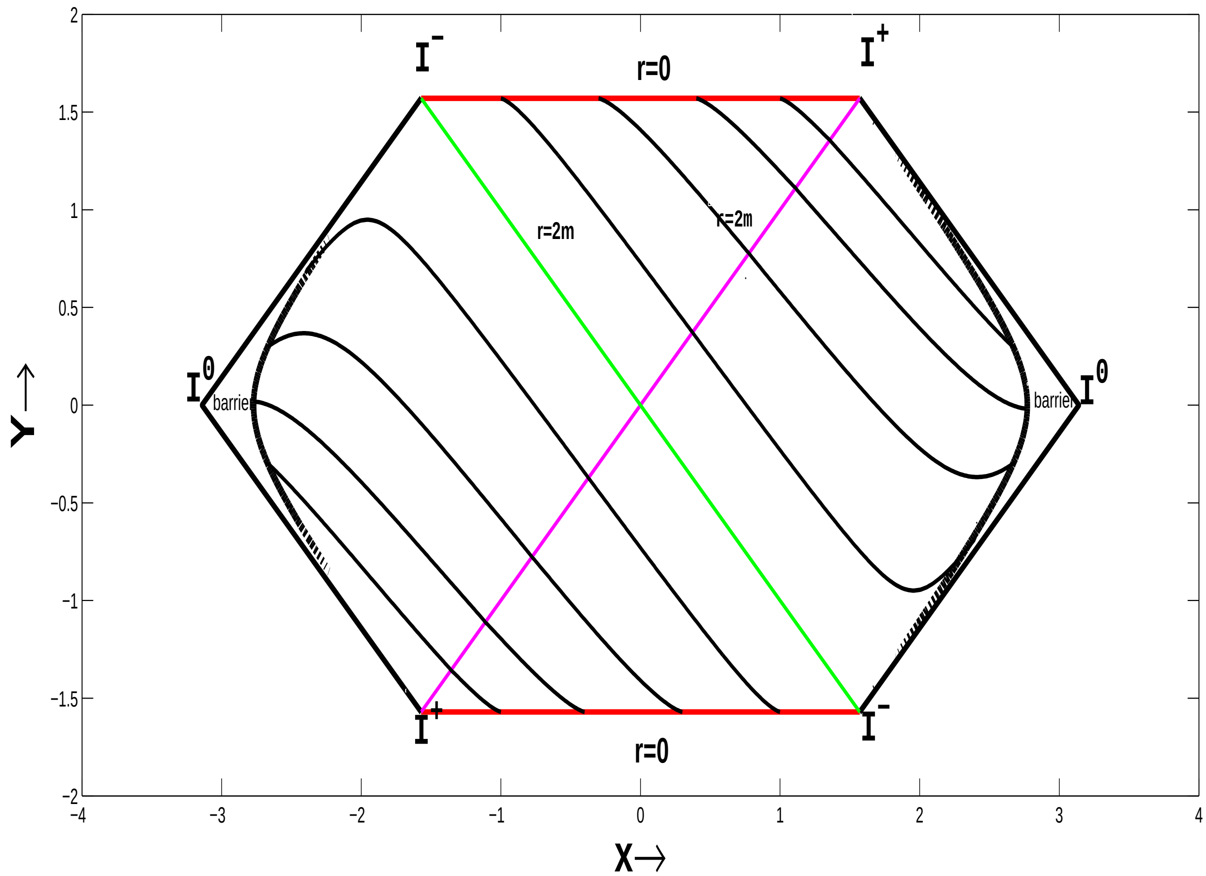

Figure 3.14: Foliating hypersurfaces for $j = 0.8$ in the maximal extension of SM are shown.

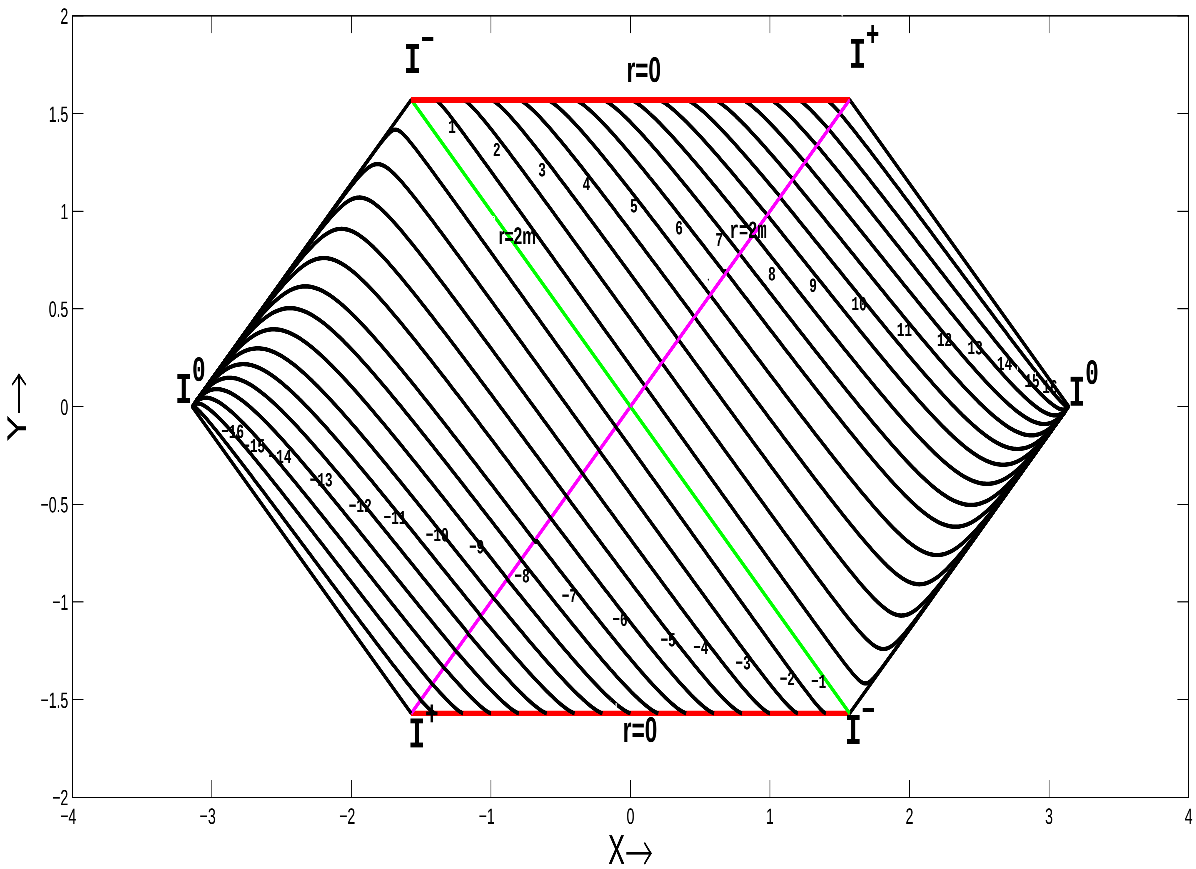

Figure 3.15: Flat hypersurfaces for $j = 1$ in the maximal extension of SM are shown.

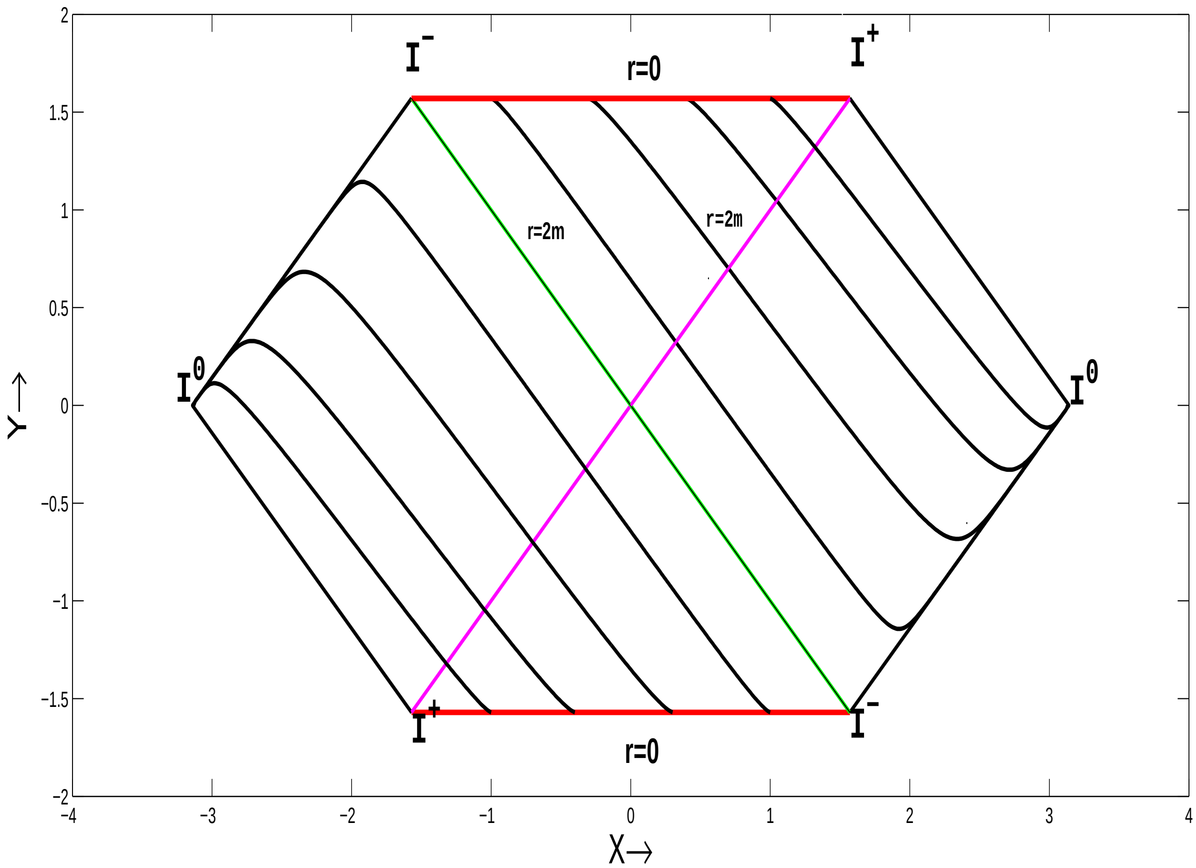

Figure 3.16: Foliating hypersurfaces for $j = 1.25$ in the maximal extension of SM are shown.

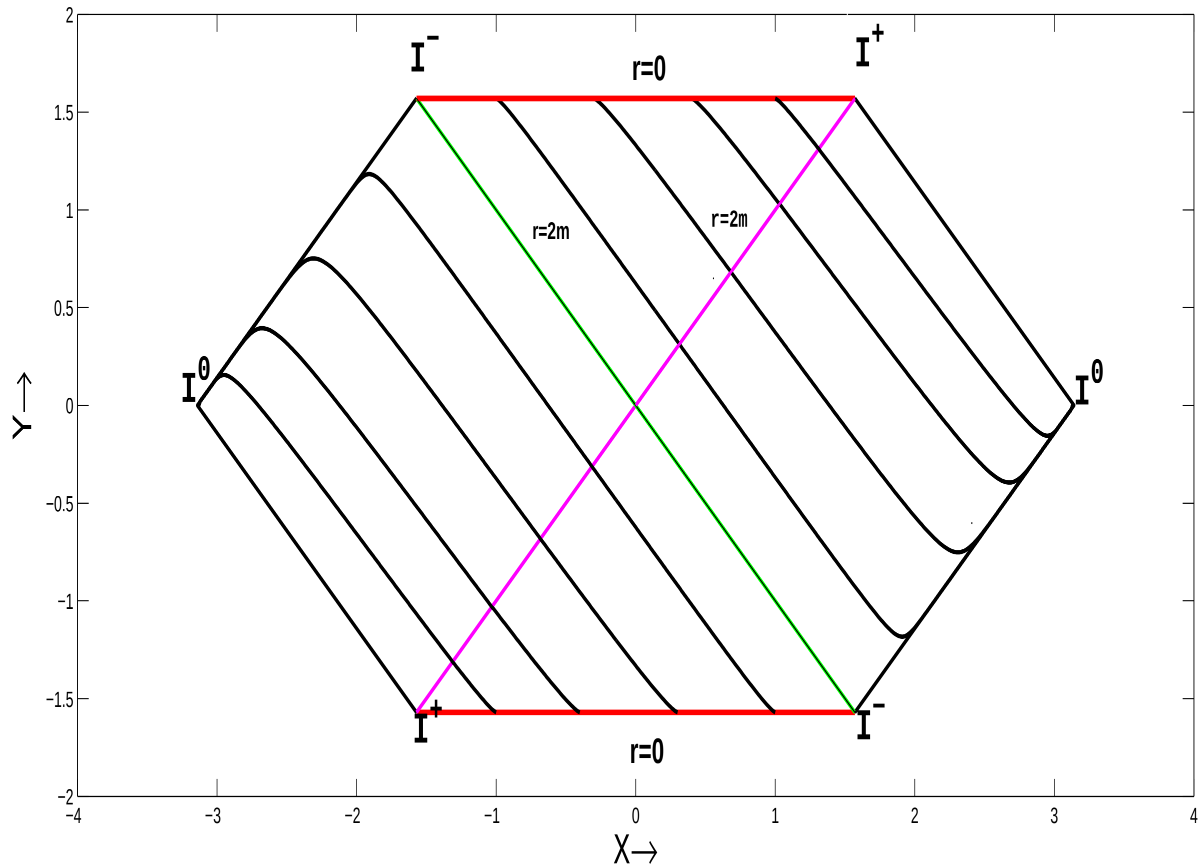

Figure 3.17: Foliating hypersurfaces for $j = 1.5$ in the maximal extension of SM are shown.

Chapter 4

Non-Singular Coordinates of the Taub-Nut Metric

In this chapter we will discuss appropriate coordinates to study the geometry of Taub-Nut (TN) metric. For this purpose, we first discuss its coordinate singularities and then discuss appropriate coordinates to study its geometry.

4.1 The Taub-NUT Metric

In 1951, Taub first discovered this solution [43]. Later in 1963 this solution was subsequently rediscovered by Newman, Tamburino and Unti [44] as a simple generalization of the *Schwarzschild* metric. The metric is

$$ds^2 = \sigma(r)\left(dt + 4l\sin^2\frac{\theta}{2}d\phi\right)^2 - \sigma^{-1}(r)dr^2 - \left(r^2 + l^2\right)\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{4.1}$$

where m and l are constants of which l is the NUT parameter and

$$\sigma(r) = \frac{r^2 - 2mr - l^2}{r^2 + l^2}. \tag{4.2}$$

For this metric all the components of *Riemann* tensor are analytic functions of r [45], no curvature singularity exists for any value of r. It is therefore natural for r to cover the full range $r\varepsilon(-\infty, \infty)$. However this metric has coordinate singularities at

$$\sigma(r) = 0, \tag{4.3}$$

$$r_1 = m + \sqrt{m^2 + l^2}. \tag{4.4}$$

and

$$r_2 = m - \sqrt{m^2 + l^2}. \tag{4.5}$$

Now $\sigma(r)$ can be rewritten as

$$\sigma(r) = \frac{(r - r_1)(r - r_2)}{r^2 + l^2}. \tag{4.6}$$

Purely radial lines with θ and ϕ constant have

$$ds^2 = \sigma(r)dt^2 - \sigma^{-1}(r)dr^2. \tag{4.7}$$

In order to describe the global structure of this metric, it is necessary to investigate the continuation of the metric across horizons.

4.2 Eddington-Finklestein Like Coordinates

For the removal of the coordinate singularities at $r = r_1$ and r_2, we first define EF like coordinates (w^*, z^*)

$$\begin{aligned} w^* &= t + r^*, \\ z^* &= t - r^*, \end{aligned} \tag{4.8}$$

where

$$r^* = \int \frac{dr}{\sigma(r)} = r + r_2 \ln\left|\frac{r}{r_2} - 1\right| + r_1 \ln\left|\frac{r}{r_1} - 1\right|. \tag{4.9}$$

The two dimensional metric given in eq.(4.7) in these coordinates takes the form

$$ds^2 = \frac{(r - r_1)(r - r_2)}{r^2 + l^2} dw^* dz^*. \tag{4.10}$$

4.3 Kruskal Like Coordinates

Kruskal like (W, Z) coordinates are defined as

$$\begin{aligned} W &= e^{\frac{w^*}{\lambda}}, \\ Z &= -e^{\frac{-z^*}{\lambda}}. \end{aligned} \tag{4.11}$$

Inserting eq.(4.8) in eq.(4.11), we get

$$W = e^{\frac{t}{\lambda}} e^{\frac{r}{\lambda}} \left|\frac{r}{r_1} - 1\right|^{\frac{r_1}{\lambda}} \left|\frac{r}{r_2} - 1\right|^{\frac{r_2}{\lambda}}, \tag{4.12}$$

$$Z = -e^{\frac{-t}{\lambda}} e^{\frac{r}{\lambda}} \left|\frac{r}{r_1} - 1\right|^{\frac{r_1}{\lambda}} \left|\frac{r}{r_2} - 1\right|^{\frac{r_2}{\lambda}}. \tag{4.13}$$

And

$$WZ = -e^{\frac{2r^*}{\lambda}} = -\mu^2 e^{\frac{2r}{\lambda}} \left|\frac{r}{r_1} - 1\right|^{\frac{2r_1}{\lambda}} \left|\frac{r}{r_2} - 1\right|^{\frac{2r_2}{\lambda}}.$$

For the removal of the singularities at $r = r_\pm$, we need to cancel the factors involving $\left(1 - \frac{r_1}{r}\right)$ and $\left(1 - \frac{r_2}{r}\right)$. But these cannot be canceled simultaneously. Therefore we required two coordinate patches to cover the entire geometry. Thus we define (W_1, Z_1) given by

$$W_1 = e^{\frac{t}{\lambda_1}} e^{\frac{r}{\lambda_1}} \left|\frac{r}{r_1} - 1\right|^{\frac{r_1}{\lambda_1}} \left|\frac{r}{r_2} - 1\right|^{\frac{r_2}{\lambda_1}}, \tag{4.14}$$

$$Z_1 = -e^{\frac{-t}{\lambda_1}} e^{\frac{r}{\lambda_1}} \left|\frac{r}{r_1} - 1\right|^{\frac{r_1}{\lambda_1}} \left|\frac{r}{r_2} - 1\right|^{\frac{r_2}{\lambda_1}}, \tag{4.15}$$

where

$$\lambda_1 = 2r_1. \tag{4.16}$$

The metric for this region is given by

$$ds_1^2 = \frac{4r_1^2 r_2}{r^2 + l^2} e^{-r/r_1} \left(\frac{r - r_2}{r_2}\right)^{\frac{r_1 - r_2}{r_1}} dW_1 dZ_1. \tag{4.17}$$

The (W_2, Z_2) are defined by

$$W_2 = e^{\frac{t}{\lambda_2}} e^{\frac{r}{\lambda_2}} \left|\frac{r}{r_1} - 1\right|^{\frac{r_1}{\lambda_2}} \left|\frac{r}{r_2} - 1\right|^{\frac{r_2}{\lambda_2}}, \tag{4.18}$$

$$Z_2 = -e^{\frac{-t}{\lambda_2}} e^{\frac{r}{\lambda_2}} \left|\frac{r}{r_1} - 1\right|^{\frac{r_1}{\lambda_2}} \left|\frac{r}{r_2} - 1\right|^{\frac{r_2}{\lambda_2}}, \tag{4.19}$$

where

$$\lambda_2 = 2r_2. \tag{4.20}$$

The metric for this region is given by

$$ds_2^2 = \frac{4r_2^2 r_1}{r^2 + l^2} e^{-r/r_2} \left(\frac{r - r_1}{r_1}\right)^{\frac{r_2 - r_1}{r_2}} dW_2 dZ_2. \tag{4.21}$$

4.4 Kruskal-Szekers Like Coordinates

We now define KS like coordinates (w_1, z_1) for the region $r_1 < r < \infty$ (known as NUT_+ region)

$$w_1 = \frac{1}{2}(W_1 + Z_1) = \sinh\left(\frac{t}{\lambda_1}\right) e^{\frac{r}{\lambda_1}} \left|\frac{r}{r_1} - 1\right|^{\frac{r_1}{\lambda_1}} \left|\frac{r}{r_2} - 1\right|^{\frac{r_2}{\lambda_1}}, \tag{4.22}$$

$$z_1 = \frac{1}{2}(W_1 - Z_1) = \cosh\left(\frac{t}{\lambda_1}\right) e^{\frac{r}{\lambda_1}} \left|\frac{r}{r_1} - 1\right|^{\frac{r_1}{\lambda_1}} \left|\frac{r}{r_2} - 1\right|^{\frac{r_2}{\lambda_1}}. \tag{4.23}$$

These are non-singular at $r = r_1$ and for the region $r_2 < r < r_1$ (known as Taub region), w_1, z_1 are interchanged. The metric for this region in these coordinates is given by

$$ds_1^2 = \frac{4r_1^2 r_2}{r^2 + l^2} e^{-r/r_1} \left(\frac{r - r_2}{r_2}\right)^{\frac{r_1 - r_2}{r_1}} \left(dw_1^2 - dz_1^2\right). \tag{4.24}$$

The region $-\infty < r < r_2$ (known as NUT_- region) is covered by (w_2, z_2), where

$$w_2 = \frac{1}{2}(W_2 + Z_2) = \sinh\left(\frac{t}{\lambda_2}\right) e^{\frac{r}{\lambda_2}} \left|\frac{r}{r_1} - 1\right|^{\frac{r_1}{\lambda_2}} \left|\frac{r}{r_2} - 1\right|^{\frac{r_2}{\lambda_2}}, \tag{4.25}$$

$$z_2 = \frac{1}{2}(W_2 - Z_2) = \cosh\left(\frac{t}{\lambda_2}\right) e^{\frac{r}{\lambda_2}} \left|\frac{r}{r_1} - 1\right|^{\frac{r_1}{\lambda_2}} \left|\frac{r}{r_2} - 1\right|^{\frac{r_2}{\lambda_2}}. \tag{4.26}$$

These are non-singular at $r = r_2$ and for the region $r_2 < r < r_1$, w_2, z_2 are interchanged The metric for this region is given by

$$ds_2^2 = \frac{4r_2^2 r_1}{r^2 + l^2} e^{-r/r_2} \left(\frac{r - r_1}{r_1}\right)^{\frac{r_2 - r_1}{r_2}} \left(dw_2^2 - dz_2^2\right). \tag{4.27}$$

4.5 Compactified Kruskal-Szekers Like Coordinates

The region $r_2 < r < \infty$ is covered by the compactified KS coordinates (Y_1, X_1), where

$$\begin{aligned} Y_1 &= \tan^{-1}(w_1 + z_1) + \tan^{-1}(w_1 - z_1), \\ X_1 &= \tan^{-1}(w_1 + z_1) - \tan^{-1}(w_1 - z_1). \end{aligned} \tag{4.28}$$

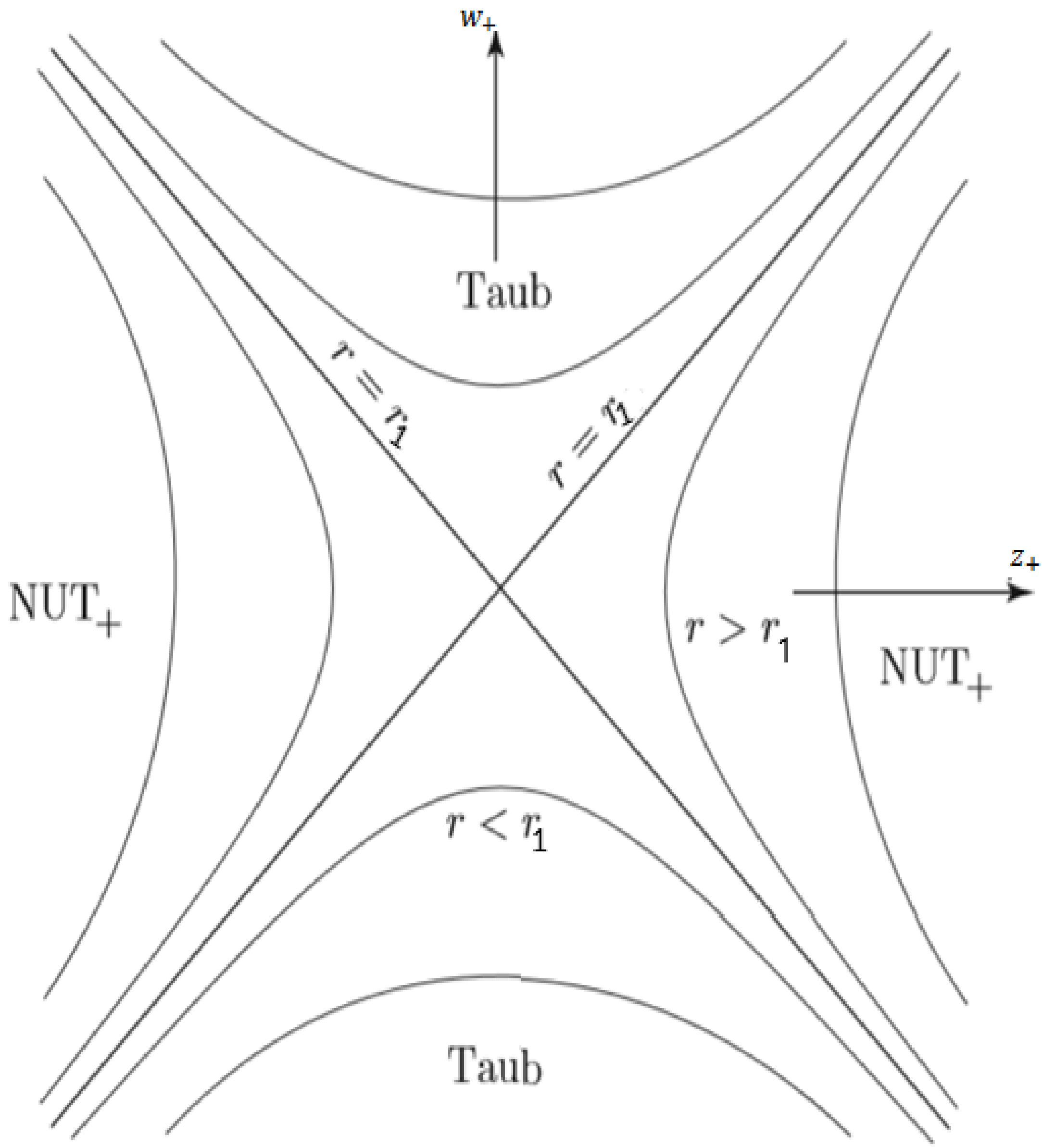

Figure 4.1: The (w_1, z_1) coordinates covers two separate Taub and two Nut$_+$ regions. $r = constant$ hypersurfaces are shown by hyperbolae.

The metric for these coordinates is given by

$$ds_1^2 = \frac{r_1^2 r_2}{r^2 + l^2} e^{-r/r_1} \left(\frac{r - r_2}{r_2}\right)^{\frac{r_1 - r_2}{r_1}} \sec^2\left(\frac{Y_1 - X_1}{2}\right) \sec^2\left(\frac{Y_1 + X_1}{2}\right) \left(dY_1^2 - dX_1^2\right). \tag{4.29}$$

The region $-\infty < r < r_1$ is covered by (Y_2, X_2), where

$$\begin{aligned} Y_2 &= \tan^{-1}(w_2 + z_2) + \tan^{-1}(w_2 - z_2), \\ X_2 &= \tan^{-1}(w_2 + z_2) - \tan^{-1}(w_2 - z_2). \end{aligned} \tag{4.30}$$

Here the metric is

$$ds_2^2 = \frac{r_2^2 r_1}{r^2 + l^2} e^{-r/r_2} \left(\frac{r - r_1}{r_1}\right)^{\frac{r_2 - r_1}{r_2}} \sec^2\left(\frac{Y_2 - X_2}{2}\right) \sec^2\left(\frac{Y_2 + X_2}{2}\right) \left(dY_2^2 - dX_2^2\right). \tag{4.31}$$

The implicit relations for r are

$$\begin{aligned} z_1^2 - w_1^2 &= e^{\frac{2r}{\lambda_1}} \left|\frac{r}{r_1} - 1\right|^{\frac{2r_1}{\lambda_1}} \left|\frac{r}{r_2} - 1\right|^{\frac{2r_2}{\lambda_1}} \\ &= -\tan\left(\frac{Y_1 + X_1}{2}\right) \tan\left(\frac{Y_1 - X_1}{2}\right), \end{aligned} \tag{4.32}$$

and

$$\begin{aligned} z_2^2 - w_2^2 &= e^{\frac{2r}{\lambda_2}} \left|\frac{r}{r_1} - 1\right|^{\frac{2r_1}{\lambda_2}} \left|\frac{r}{r_2} - 1\right|^{\frac{2r_2}{\lambda_2}} \\ &= -\tan\left(\frac{Y_2 + X_2}{2}\right) \tan\left(\frac{Y_2 - X_2}{2}\right). \end{aligned} \tag{4.33}$$

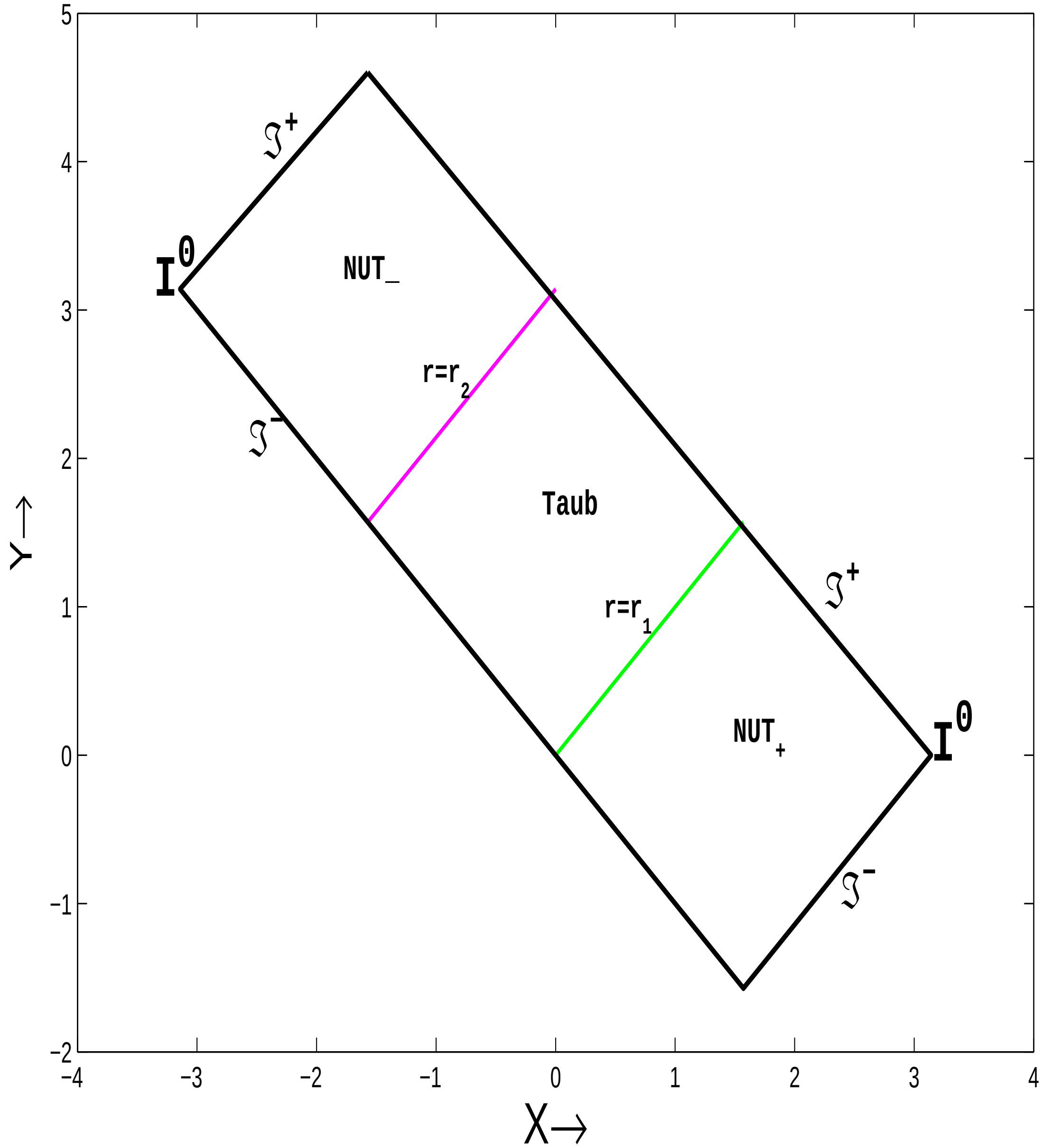

Figure 4.2: CP diagram of the TN metric in (Y, X) coordinates. Notice that there is no essential singularity.

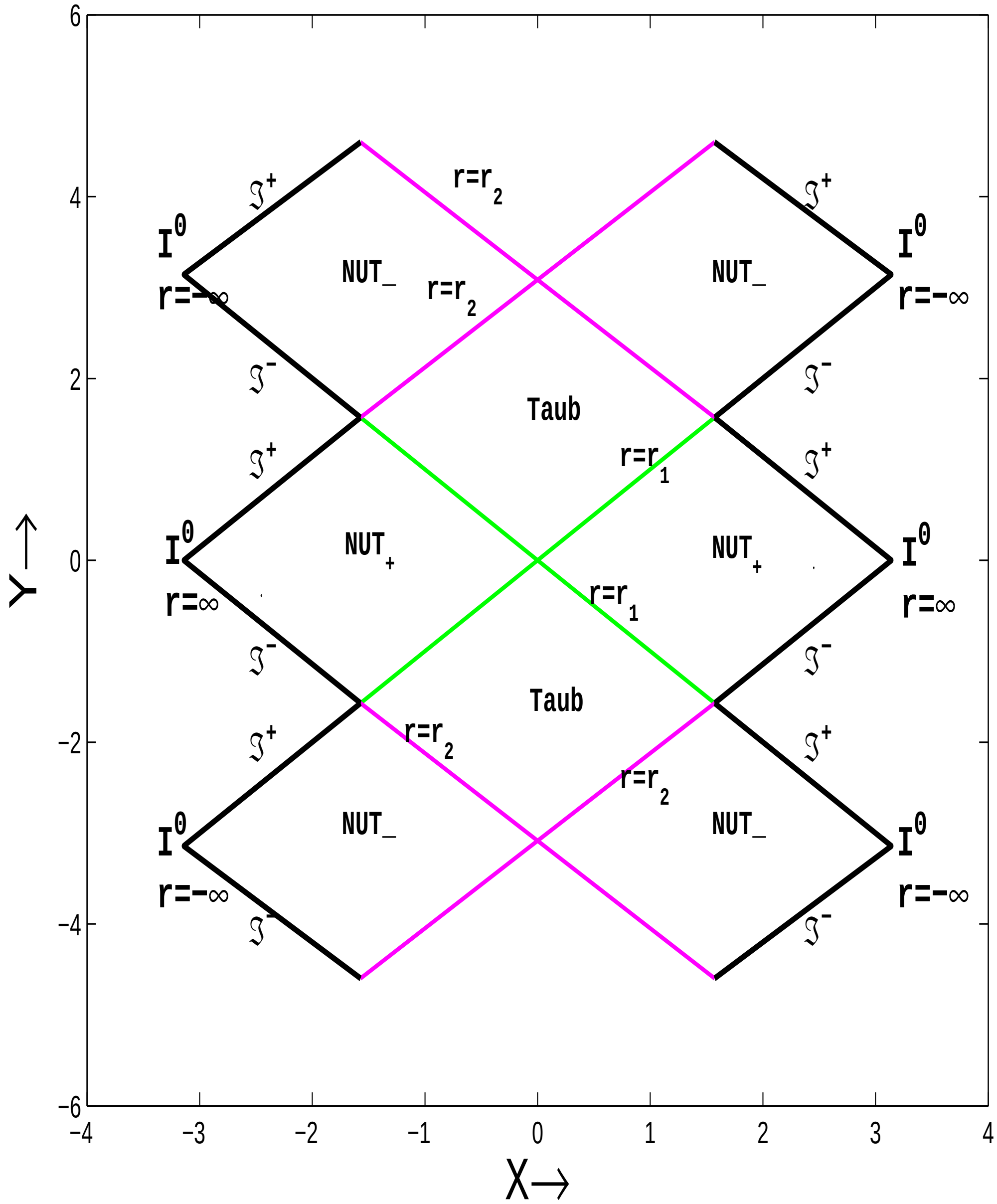

Figure 4.3: Maximal extension of the TN metric.

Chapter 5

Summary and Conclusion

In this thesis, we have discussed the compactification of the *Schwarzschild*, RN and TN metrics and discussed the K-slicing of the *Schwarzschild* and RN metrics. We have also discussed the flat foliation of the *Schwarzschild* metric. In Chapter 1, we have first discussed basics of differential geometry which were useful for the GR and then derived the EFEs and gave a brief introduction of CP diagrams. In Chapter 2, we have discussed some exact solutions of the EFEs, which we have classified into two categories based on the zero or nonzero cosmological constant $\wedge$. In the former category we have discussed the Minkowski metric as a solution of the EFEs, then the *Schwarzschild* metric, its singularities and some appropriate coordinates to study its geometry are discussed. After that we have done the same for RN metric. *Kerr* metric is discussed briefly. While in the later category we have discussed *de Sitter* and *Anti-de Sitter* metrics, and then the derivation of the *Schwarzschild de Sitter* and *Schwarzschild Anti-de Sitter* spacetimes. In Chapter 3, after defining foliation in general and explaining it with the help of some examples, we have reviewed BCI procedure for SSS metrics and then using this procedure we have discussed the K-slicing of the *Schwarzschild* and the RN metrics. We have used RK4 method with step size of 0.001 for solving K-slicing equations given in eqs.(3.70) and (3.75). The main difficulty for solving these equations was that they contain F and G, which are the functions of r that is implicitly related with X and Y. So we cannot write explicitly F and G in terms of X and Y. We had to solve implicit relations given in eqs.(2.28) and

(2.59) for r for both the *Schwarzschild* and RN cases respectively. For flat foliation, we have used the simple fact that the $world - lines$ of the observers falling freely from infinity, starting at rest must be orthogonal to the flat foliating hypersurfaces. We have noticed that the foliating hypersurfaces are flat only for $j = 1.0$ and for $j \neq 1$ they are not flat. Further we have noticed that for $j < 1$, these hypersurfaces do not foliate the complete CP diagram. We found a barrier for the hypersurfaces for $j < 1$ to reach I^0 because the expression inside the square root in eq.(3.138) becomes complex. For this case, the hypersurfaces are defined only for $r \leq \frac{r_s}{1-j^2}$ but they can be analytically continued to reach I^0. For smaller values of j we have larger barrier. An interesting feature of flat hypersurfaces is that they never enter into the maximal extension so for maximal extension we reverse the diagrams Figure.3.8-3.12 and adjoined them to their original copy. In Chapter 4, we have discussed the non-singular coordinates for the TN metric and its CP diagram. TN metric is singularity free spacetime but this solution is also being reconsidered in the context of higher-dimensional theories of semi-classical quantum gravity. It would be interesting to foliate this metric and observe the behavior of K-surfaces.

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