On a Complex Differential Riccati Equation

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Dedicated to

My Father, Mother,
and Sisters

Abstract

We consider a nonlinear partial differential equation for complex-valued functions which is related to the two-dimensional stationary Schrödinger equation and enjoys many properties similar to those of the Riccati equation such as the famous Euler theorems, the Picard theorem and others. Besides these generalizations of the classical "one-dimensional" results, we discuss new features of the considered equation including an analogue of the Cauchy integral theorem.

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May **Allah** bless them all (Ameen)
Farzana Kouser

Contents

Chapter 1

Introduction

In this dissertation, we review and present complete proofs appearing in some recent papers [1], [2] and [3] on the relationship of the Riccati equation and the stationary Schrödinger equation.

The ordinary Riccati equation

$$u'(x) = f_2(x)u^2(x) + f_1(x)u(x) + f_0(x), \tag{1.0.1}$$

can be reduced to the canonical form

$$y'(x) + y^2(x) = v(x), \tag{1.0.2}$$

by using an appropriate transformation.

This reduced form is related to the one-dimensional stationary Schrödinger equation

$$-\frac{d^2u(x)}{dx^2} + v(x)u(x) = 0, \tag{1.0.3}$$

in the sense that the solutions of these two equations are related. The study also allows for the factorization of the Schrödinger operator in terms of any solution of the Riccati equation.

Chapter 2 of this dissertation is devoted to this relationship. We first give an extensive treatment of the Riccati equation deriving its important properties and then discuss the relationship with the one-dimensional stationary Schrödinger equation exhibiting also the factorization of the Schrödinger operator.

The nonlinear complex Riccati equation is

$$\partial_{\bar{z}}Q + |Q|^2 = \frac{\nu}{4}, \tag{1.0.4}$$

where $\overline{z}$ is conjugate of a complex variable $z = x + \iota y$ and

$$\partial_{\overline{z}} = \frac{1}{2}(\partial_x + \iota\partial_y).$$

Here for convenience the factor $\frac{1}{4}$ was included. Q is a complex-valued analytic function of z and ν is a real-valued function. Note that this equation is different from the complex Riccati equation studied in various works (see, e.g. [4]) where it is supposed to have the form of Eq. (1.0.1) with complex analytic coefficients $f_0(x)$, $f_1(x)$ and $f_2(x)$ and a complex analytic solution u, as $|Q|^2$ is not an analytic function.

The stationary two-dimensional zero energy Schrödinger equation is

$$(-\Delta + \nu)\, u = 0, \tag{1.0.5}$$

where, $\Delta = \frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2}$. u and ν are real valued functions.

The appearance of $\partial_{\overline{z}}$ results in the development of pseudoanalytic function theory by Bers' which is one of the many generalizations of analytical function theory. A summary of Bers' theory of pseudoanalytic functions as a generalization of analytic functions appears in Chapter 3.

In the first section of Chapter 4, some preliminary results on the stationary two-dimensional Schrödinger equation and a class of pseudoanalytic functions are discussed. In fact the solution of the Vekua equation [3] and two-dimensional Schrödinger Eq. (1.0.5) are closely related to each other. Given a solution of the Schrödinger equation we can reconstruct a solution of the Vekua equation which is a pseudoanalytical function. Conversely given a pseudoanalytic function we can construct a solution of a related Schrödinger equation. In the second section we continue the discussion to obtain the relationship between the Riccati equation and the two-dimensional stationary Schrödinger equation. In fact given a solution of the Schrödinger equation in two-dimension, its logarithmic derivative is a solution of the complex Riccati equation. This result appears in Theorem 16. This leads also to the factorization of the Schrödinger operator in two-dimensions.

In the first section of Chapter 5, we discuss generalizations of both Euler's and Picard's theorems as related to the two-dimensional Riccati equation. These are also related to the two-dimensional Schrödinger equation and in the second section we discuss the generalizations of the Cauchy's integral formula for the complex Riccati equation.

Chapter 2

The Riccati equation and its relationship with the one-dimensional Schrödinger equation

2.1 The Riccati equation

The usual Riccati Eq. (1.0.1) has received a great deal of attention since a particular version was first studied by Count Riccati in 1724, because of both its specific properties and the wide range of applications in which it appears. For a survey of the history and classical results on this equation, see for example [5] and [6].

The Riccati Eq. (1.0.1), can always be reduced to its canonical form (1.0.2) by some suitable transformation. To see this, take

$$y(x) = -f_2(x)u(x) + \alpha(x), \tag{2.1.1}$$

where $\alpha(x)$ will be determined in terms of the functions $f_0(x)$, $f_1(x)$, $f_2(x)$ appearing in Eq. (1.0.1) to obtain the desired form. Differentiate Eq. (2.1.1) w.r.t. "x"

$$\begin{aligned}
y'(x) &= -f_2(x)u'(x) - f_2'(x)u(x) + \alpha'(x), \\
&= -f_2(x)\left\{f_2(x)u^2(x) + f_1(x)u(x) + f_0(x)\right\} - f_2'(x)u(x) + \alpha'(x), \\
&= -\left\{-f_2(x)u(x) + \alpha(x)\right\}^2 - 2\alpha(x)f_2(x)u(x) + \alpha^2(x) - f_2(x)f_1(x)u(x) \\
&\quad -f_2(x)f_0(x) - f_2'(x)u(x) + \alpha'(x), \\
&= -\left\{-f_2(x)u(x) + \alpha(x)\right\}^2 + u(x)\left\{-2\alpha(x)f_2(x) - f_2(x)f_1(x) - f_2'(x)\right\} \\
&\quad +\alpha^2(x) - f_2(x)f_0(x) + \alpha'(x).
\end{aligned} \tag{2.1.2}$$

To convert Eq. (2.1.2) into the canonical form (1.0.2) we determine $\alpha(x)$ such that

$$-2\alpha(x)f_2(x) - f_2(x)f_1(x) - f_2'(x) = 0,$$

i.e.

$$\alpha(x) = -\frac{1}{2}f_1(x) - \frac{1}{2}\frac{f_2'(x)}{f_2(x)}, \tag{2.1.3}$$

after substituting Eq. (2.1.3) in Eq. (2.1.2) we get

$$y'(x) = -y^2(x) + \left\{-\frac{1}{2}f_1(x) - \frac{1}{2}\frac{f_2'(x)}{f_2(x)}\right\}^2 - f_2(x)f_0(x) + \alpha'(x). \tag{2.1.4}$$

If we choose

$$\begin{aligned} v(x) &= \frac{1}{4}\left\{f_1(x) + \frac{f_2'(x)}{f_2(x)}\right\}^2 - f_2(x)f_0(x) + \alpha'(x), \\ &= \frac{1}{4}\left\{f_1(x) + \frac{f_2'(x)}{f_2(x)}\right\}^2 - f_2(x)f_0(x) - \frac{1}{2}f_1'(x) \\ &\quad -\frac{1}{2}\frac{d}{dx}\left\{\frac{f_2'(x)}{f_2(x)}\right\}, \end{aligned} \tag{2.1.5}$$

then we have the desired canonical form (1.0.2).

There is a procedure to find the general solution of the Riccati equation if we know a particular solution of it. The procedure converts the Riccati equation to a Bernoulli equation and then uses the standard procedure to solve it. Let $y = y_0 + z$ be a general solution of Riccati Eq. (1.0.2) where y_0 is a particular solution. Then,

$$y_0' + z' + (y_0 + z)^2 = v(x), \tag{2.1.6}$$

which, on expansion, gives

$$y_0' + z' + y_0^2 + z^2 + 2y_0z = v(x). \tag{2.1.7}$$

Now

$$y_0' + y_0^2 = v(x). \tag{2.1.8}$$

Substituting Eq. (2.1.8) in Eq. (2.1.7) we arrive at

$$z' + 2y_0z + z^2 = 0, \tag{2.1.9}$$

which is a homogeneous nonlinear ordinary differential equation.

Now using the standard Bernoulli procedure we put $z = \frac{1}{u}$. Differentiating gives $z' = -\frac{u'}{u^2}$ then Eq. (2.1.9) becomes

$$-\frac{u'}{u^2} + 2y_0\frac{1}{u} + \frac{1}{u^2} = 0,$$

or

$$u' - 2y_0 u - 1 = 0. \tag{2.1.10}$$

Now use the integrating factor to obtain the general solution

$$u(x) = \exp\left(\int 2y_0 dx\right)\left[\int \exp\left(-\int 2y_0 dx\right) dx + c\right]. \tag{2.1.11}$$

Thus,

$$\begin{aligned} y(x) &= y_0(x) + z \\ &= y_0(x) + \frac{1}{u}, \\ &= y_0(x) + \frac{\exp\left(-\int 2y_0 dx\right)}{\int \exp\left(-\int 2y_0 dx\right) dx + c}. \end{aligned} \tag{2.1.12}$$

Notice that this does not give a linear superposition of the particular solution y_0 with the complementary function but uses a nonlinear superposition. Thus, instead of obtaining $y(x) \to y_0(x)$ for $c \to 0$, we get it for $c \to \infty$.

The second of these theorems states that for the given two linearly independent particular solutions y_0 and y_1 of the Riccati Eq. (1.0.2), the general solution can be written as

$$y = \frac{cy_0 \exp\left(\int (y_0 - y_1) dx\right) - y_1}{c \exp\left(\int (y_0 - y_1) dx\right) - 1}, \tag{2.1.13}$$

where c is an arbitrary constant. Now we will show that Eq. (2.1.13) is a solution of Eq. (1.0.2). Eq. (2.1.13) implies

$$y = y_0 + \frac{y_0 - y_1}{c \exp(\int (y_0 - y_1) dx) - 1}.$$

Therefore,

$$y' = y_0' + \frac{y_0' - y_1'}{c \exp\left(\int (y_0 - y_1) dx\right) - 1} - \frac{(y_0 - y_1)^2 c \exp\left(\int (y_0 - y_1) dx\right)}{\left(c \exp\left(\int (y_0 - y_1) dx\right) - 1\right)^2}, \tag{2.1.14}$$

and

$$y^2 = y_0^2 + \frac{(y_0 - y_1)^2}{\left(c \exp\left(\int (y_0 - y_1) dx\right) - 1\right)^2} + \frac{2y_0(y_0 - y_1)}{c \exp\left(\int (y_0 - y_1) dx\right) - 1}. \tag{2.1.15}$$

Adding Eq. (2.1.14) and Eq. (2.1.15) we have

$$\begin{aligned} y' + y^2 &= y_0^2 + y_0' + \frac{y_0' - y_1'}{c \exp\left(\int (y_0 - y_1) dx\right) - 1} + \frac{2y_0(y_0 - y_1)}{c \exp\left(\int (y_0 - y_1) dx\right) - 1} \\ &\quad + \frac{(y_0 - y_1)^2 \left[1 - c \exp\left(\int (y_0 - y_1) dx\right)\right]}{\left(c \exp\left(\int (y_0 - y_1) dx\right) - 1\right)^2}. \end{aligned} \tag{2.1.16}$$

But

$$y_0' + y_0^2 = v(x) = y_1' + y_1^2,$$
$$y_0' - y_1' + y_0^2 - y_1^2 = 0,$$

or

$$y_0' - y_1' = -(y_0^2 - y_1^2). \tag{2.1.17}$$

Substitute Eq. (2.1.17) in Eq. (2.1.16)

$$\begin{aligned} y' + y^2 &= v(x) + \frac{-\left(y_0^2 - y_1^2\right)}{c \exp\left(\int (y_0 - y_1) dx\right) - 1} + \frac{2y_0(y_0 - y_1)}{c \exp\left(\int (y_0 - y_1) dx\right) - 1} \\ &\quad - \frac{(y_0 - y_1)^2}{c \exp\left(\int (y_0 - y_1) dx\right) - 1}, \\ &= v(x) + \frac{-(y_0^2 - y_1^2) + 2y_0 (y_0 - y_1) - (y_0 - y_1)^2}{c \exp\left(\int (y_0 - y_1) dx\right) - 1}, \\ &= v(x). \end{aligned}$$

Thus, Eq. (2.1.13) is a general solution of the Riccati Eq. (1.0.2). In Eq. (2.1.13) when $c \to \infty$ we see that $y(x) \to y_0(x)$ and when $c \to 0$ we see that $y(x) \to y_1(x)$.

Other interesting properties are those discovered by Weyl and Picard [5, 7]. The first is that given four particular solutions $y_1,\ y_2,\ y_3,\ y_4$ the cross ratio is

$$\frac{(y_1 - y_2)(y_3 - y_4)}{(y_1 - y_4)(y_3 - y_2)} = c, \tag{2.1.18}$$

where c is an arbitrary constant. Since the above ratio is equal to a constant, so the particular solutions can never be linearly independent. Therefore, using the Wronskian defined by

$$W(y_1,\ y_2,\ y_3,\ y_4) = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix},$$

if the four solutions are linearly dependent then $W(y_1,\ y_2,\ y_3,\ y_4) = 0$ and if $W(y_1,\ y_2,\ y_3,\ y_4) \neq 0$ then the four solutions are linearly independent. Given four solutions of Riccati Eq. (1.0.2) are $y_1,\ y_2,\ y_3,\ y_4$. Then, $y_1' + y_1^2 = v(x)$, $y_2' + y_2^2 = v(x)$, $y_3' + y_3^2 = v(x)$, and $y_4' + y_4^2 = v(x)$. To prove this consider

$$y_1' - y_2' = -(y_1^2 - y_2^2),$$

or

$$\frac{y_1' - y_2'}{y_1 - y_2} = -(y_1 + y_2). \tag{2.1.19}$$

Similarly,

$$\frac{y_3' - y_4'}{y_3 - y_4} = -(y_3 + y_4), \tag{2.1.20}$$

$$\frac{y_1' - y_4'}{y_1 - y_4} = -(y_1 + y_4), \tag{2.1.21}$$

$$\frac{y_2' - y_3'}{y_2 - y_3} = -(y_2 + y_3). \tag{2.1.22}$$

Adding Eq. (2.1.19) to Eq. (2.1.20) we have

$$\begin{aligned} \frac{y_1' - y_2'}{y_1 - y_2} + \frac{y_3' - y_4'}{y_3 - y_4} &= -(y_1 + y_2 + y_3 + y_4), \\ &= -(y_1 + y_4) - (y_2 + y_3), \end{aligned}$$

using Eq. (2.1.21) and Eq. (2.1.22) we have

$$\frac{y_1' - y_2'}{y_1 - y_2} + \frac{y_3' - y_4'}{y_3 - y_4} = \frac{y_1' - y_4'}{y_1 - y_4} + \frac{y_2' - y_3'}{y_2 - y_3}. \tag{2.1.23}$$

Integrating both sides of Eq. (2.1.23) we get

$$\ln|(y_1 - y_2)(y_3 - y_4)| = \ln|(y_1 - y_4)(y_2 - y_3)| + \ln|c|,$$

$$\ln\left|\frac{(y_1 - y_2)(y_3 - y_4)}{(y_1 - y_4)(y_2 - y_3)}\right| = \ln|c|,$$

i.e. Eq. (2.1.18). From here we obtain another important result that given three solutions then the fourth solution is obtained by a nonlinear superposition law. Thus given three solutions $y_1,\ y_2,\ y_3$, we may evaluate the general solution y_4 in terms of $y_1,\ y_2$, and y_3 with one constant involved as expected. From Eq. (2.1.18) we arrive at

$$y_4 = \frac{y_2(cy_1 + y_3) - (1 + c)y_1y_3}{y_2(1 - c) - y_1 - cy_3}. \tag{2.1.24}$$

Taking derivative on both sides of above cross ratio (2.1.18), then we have

$$\frac{(y_1-y_2)'(y_3-y_4)}{(y_1-y_4)(y_3-y_2)}+\frac{(y_1-y_2)(y_3-y_4)'}{(y_1-y_4)(y_3-y_2)}-\frac{(y_1-y_2)(y_3-y_4)(y_1-y_4)'}{(y_1-y_4)^2(y_3-y_2)}$$
$$-\frac{(y_1-y_2)(y_3-y_4)(y_3-y_2)'}{(y_1-y_4)(y_3-y_2)^2}=0. \tag{2.1.25}$$

Multiplying Eq. (2.1.25) by $(y_1-y_4)(y_3-y_2)$ we have

$$(y_1-y_2)'(y_3-y_4)+(y_1-y_2)(y_3-y_4)'-\frac{(y_1-y_2)(y_3-y_4)(y_1-y_4)'}{y_1-y_4}$$
$$-\frac{(y_1-y_2)(y_3-y_4)(y_3-y_2)'}{y_3-y_2}=0. \tag{2.1.26}$$

Divide Eq. (2.1.26) by $(y_3-y_4)(y_1-y_2)$, we see that the Picard's theorem is equivalent to the statement

$$\frac{(y_1-y_2)'}{y_1-y_2}+\frac{(y_3-y_4)'}{y_3-y_4}-\frac{(y_1-y_4)'}{y_1-y_4}-\frac{(y_3-y_2)'}{y_3-y_2}=0. \tag{2.1.27}$$

2.2 Relation between the Riccati equation and the one-dimensional Schrödinger equation

One of the reasons why the Riccati equation has so many applications is that it is related to the general second order homogeneous differential equation. In particular, the one-dimensional zero energy Schrödinger Eq. (1.0.5) can be converted to the Riccati Eq. (1.0.2) with the help of the transformation

$$y=\frac{u'}{u}. \tag{2.2.1}$$

Differentiating Eq. (2.2.1) and using Eq. (1.0.5) we have

$$\begin{aligned} y' &= \frac{u''}{u}-\frac{u'^2}{u^2}, \\ &= \frac{vu}{u}-y^2, \end{aligned}$$

which is same as Eq. (1.0.2). This substitution has a most spectacular application, it reduces Burger's equation to the standard one-dimensional heat equation. From [8] Burger's equation is

$$\partial_t u(x,t)+u(x,t)\partial_x u(x,t)-k\partial_{xx}u(x,t)=0. \tag{2.2.2}$$

Taking $u(x,t) = -2k\frac{\partial_x v(x,t)}{v(x,t)}$ and its derivatives w.r.t. "t" and "x"

$$\partial_t u(x,t) = -2k\frac{\partial_{xt}v}{v} + 2k\frac{\partial_x v\partial_t v}{v^2}, \tag{2.2.3}$$

$$\partial_x u(x,t) = -2k\frac{\partial_{xx}v}{v} + 2k\frac{(\partial_x v)^2}{v^2}, \tag{2.2.4}$$

$$\begin{aligned}\partial_{xx}u(x,t) &= -2k\frac{\partial_{xxx}v}{v} - 2k\left(-\frac{1}{v^2}\partial_x v\right)\partial_{xx}v + 4k\frac{\partial_x v\partial_{xx}v}{v^2} \\ &\quad +2k\left(-\frac{2}{v^3}\partial_x v\right)(\partial_x v)^2, \\ &= -2k\frac{\partial_{xxx}v}{v} + 6k\frac{\partial_x v\partial_{xx}v}{v^2} - 4k\frac{(\partial_x v)^3}{v^3}. \end{aligned} \tag{2.2.5}$$

Substituting Eq.'s (2.2.3), (2.2.4) and (2.2.5) in Eq. (2.2.2) we have

$$\begin{aligned}&\partial_t u(x,t) + u(x,t)\partial_x u(x,t) - k\partial_{xx}u(x,t) \\ =\ &\left(-2k\frac{\partial_{xt}v}{v} + 2k\frac{\partial_x v\partial_t v}{v^2}\right) + \left(-2k\frac{\partial_x v(x,t)}{v(x,t)}\right)\left[-2k\frac{\partial_{xx}v}{v} + 2k\frac{(\partial_x v)^2}{v^2}\right] \\ &-k\left[-2k\frac{\partial_{xxx}v}{v} + 6k\frac{\partial_x v\partial_{xx}v}{v^2} - 4k\frac{(\partial_x v)^3}{v^3}\right] \\ =\ &-2k\frac{\partial_{xt}v}{v} + 2k\frac{\partial_x v\partial_t v}{v^2} - 2k^2\frac{v_x v_{xx}}{v^2} + 2k^2\frac{v_{xxx}}{v} \\ =\ &-\frac{2k}{v^2}v_x\left(kv_{xx} - v_t\right) + \frac{2k}{v}\partial_x\left(kv_{xx} - v_t\right).\end{aligned}$$

This is zero if and only if

$$kv_{xx} - v_t = vg(t),$$

which has a special case as heat equation.

This is the basis of the well developed theory of logarithmic derivatives for the integration of nonlinear differential equations [8]. A generalization of this substitution will be used in this work.

A second relation between the one-dimensional Schrödinger equation and the Riccati equation is as follows. The one-dimensional Schrödinger operator can be factorized in the form

$$-\frac{d^2}{dx^2} + v(x) = -\left(\frac{d}{dx} + y(x)\right)\left(\frac{d}{dx} - y(x)\right). \tag{2.2.6}$$

This requires

$$\begin{aligned}\left(-\frac{d^2}{dx^2}+v(x)\right)u(x) &= -\left(\frac{d}{dx}+y(x)\right)\left(\frac{d}{dx}-y(x)\right)u(x),\\ &= -\frac{d^2u}{dx^2}-y(x)\frac{du}{dx}+\frac{d}{dx}\left(y(x)u(x)\right)+y^2u(x),\\ &= \left(-\frac{d^2}{dx^2}+\frac{dy}{dx}+y^2\right)u(x),\end{aligned}$$

or

$$v(x)=\frac{dy}{dx}+y^2=y'+y^2.$$

Thus the operator in the L.H.S. of Eq. (1.0.3) factorizes in the form given in Eq. (2.2.6) if and only if the Riccati Eq. (1.0.2) holds.

This observation is a key to a vast area of research related to the factorization method [9] and [10]. In the present work, we consider a result similar to Eq. (2.2.6) which is in a two-dimensional setting.

Chapter 3

Bers' theory

3.1 Some definitions and results from Bers' theory

Bers' theory of pseudoanalytic functions is a generalization of analytic function theory. It was essentially developed in [11] (see also [12]) and is based on the so-called generating pair: a pair of complex functions F and G satisfying the inequality

$$Im\left(\overline{F}G\right) > 0. \tag{3.1.1}$$

If $F = F_1 + \iota F_2$ and $G = G_1 + \iota G_2$, then

$$\overline{F}G = (F_1G_1 + F_2G_2) + \iota\left(F_1G_2 - F_2G_1\right),$$

where

$$Im\left(\overline{F}G\right) = F_1G_2 - F_2G_1 > 0,$$

in some domain of interest Ω which may coincide with the whole complex plane. F and G are assumed to possess partial derivatives with respect to the real variables x and y. In this case the operators $\partial_{\overline{z}} = \left(\frac{\partial}{\partial x} + \iota\frac{\partial}{\partial y}\right)$ and $\partial_z = \left(\frac{\partial}{\partial x} - \iota\frac{\partial}{\partial y}\right)$ can be applied (usually these operators are introduced with the factor $\frac{1}{2}$, nevertheless, here it is somewhat more convenient to consider them without it) and the following characteristic coefficients of the pair (F, G) can be defined,

$$a_{(F,G)} = -\frac{\overline{F}G_{\overline{z}} - F_{\overline{z}}\overline{G}}{F\overline{G} - \overline{F}G}, \qquad b_{(F,G)} = \frac{FG_{\overline{z}} - F_{\overline{z}}G}{F\overline{G} - \overline{F}G}, \tag{3.1.2}$$

$$A_{(F,G)} = -\frac{\overline{F}G_z - F_z\overline{G}}{F\overline{G} - \overline{F}G}, \qquad B_{(F,G)} = \frac{FG_z - F_zG}{F\overline{G} - \overline{F}G}, \tag{3.1.3}$$

where the subindex $\overline{z}$ or z means the application of $\partial_{\overline{z}}$ or ∂_z respectively.

Every complex function w defined in a subdomain of Ω admits the unique representation $w = \phi F + \psi G$ where the functions ϕ and ψ are real valued. Sometimes it is convenient to associate with the function w the function $\omega = \phi + \iota\psi$

where $F = 1$ and $G = \iota$.

Bers' introduces the notion of the (F, G) derivative of a function w which exists and has the form

$$\dot{w} = \phi_z F + \psi_z G = w_z - A_{(F,G)} w - B_{(F,G)} \overline{w}, \tag{3.1.4}$$

if and only if

$$\phi_{\overline{z}} F + \psi_{\overline{z}} G = 0. \tag{3.1.5}$$

This last equation can be rewritten in the following form

$$w_{\overline{z}} = a_{(F,G)} w + b_{(F,G)} \overline{w}, \tag{3.1.6}$$

which we call the Vekua equation. Solutions of this equation are called (F, G)-pseudoanalytic functions. If w is (F, G)-pseudoanalytic, the associated function ω is called (F, G)-pseudoanalytic of second kind.

Connection with the usual analytic function theory.

Taking $F = -\iota$ and $G = 1$,

$$Im(\overline{F} G) = Im(\iota) = 1 > 0.$$

Now we calculate $a_{(F,G)}$, $b_{(F,G)}$, $A_{(F,G)}$ and $B_{(F,G)}$

$$\begin{aligned} a_{(F,G)} &= -\frac{\overline{F} G_{\overline{z}} - F_{\overline{z}} \overline{G}}{F\overline{G} - \overline{F} G}, \\ &= -\frac{(\iota)(0) - (0)(1)}{(-\iota)(1) - (\iota)(1)}, \\ &= 0. \end{aligned}$$

$$\begin{aligned} b_{(F,G)} &= \frac{F G_{\overline{z}} - F_{\overline{z}} G}{F\overline{G} - \overline{F} G}, \\ &= \frac{(-\iota)(0) - (0)(1)}{(-\iota)(1) - (\iota)(1)}, \\ &= 0. \end{aligned}$$

$$\begin{aligned} A_{(F,G)} &= -\frac{\overline{F} G_z - F_z \overline{G}}{F\overline{G} - \overline{F} G}, \\ &= -\frac{(\iota)(0) - (0)(1)}{(-\iota)(1) - (\iota)(1)}, \\ &= 0. \end{aligned}$$

$$\begin{aligned} B_{(F,G)} &= \frac{FG_z - F_zG}{F\overline{G} - \overline{F}G}, \\ &= \frac{(-\iota)(0) - (0)(1)}{(-\iota)(1) - (\iota)(1)}, \\ &= 0. \end{aligned}$$

Remark 1: [1]
The functions F and G are (F, G)-pseudoanalytic, and $\dot{F} \equiv \dot{G} \equiv 0$.
Proof:
When

$$w = \phi F + \psi G = F, \qquad \text{iff} \;\; \phi = 1,\; \psi = 0,$$

$$\phi_{\overline{z}}F + \psi_{\overline{z}}G = 0,$$

it implies F is pseudoanalytic.
Similarly,

$$w = G, \quad \text{iff} \quad \phi = 0, \;\; \psi = 1,$$

$$\phi_{\overline{z}}F + \psi_{\overline{z}}G = 0,$$

implies G is pseudoanalytic.
Therefore, F and G are both pseudoanalytic.
Again with $w = F,\ \phi = 1,\ \psi = 0$

$$\dot{F} = \phi_z F + \psi_z G = 0.$$

Similarly with $w = G$, $\phi = 0$, $\psi = 1$

$$\dot{G} = \phi_z F + \psi_z G = 0.$$

Hence, F and G are pseudoanalytic with $\dot{F} = \dot{G} = 0$.

□

Definition 1: [1]
Let (F, G) and (F_1, G_1) be two generating pairs in Ω. (F_1, G_1) is called successor of (F, G) and (F, G) is called predecessor of (F_1, G_1) if

$$a_{(F_1,G_1)} = a_{(F,G)} \qquad \text{and} \qquad b_{(F_1,G_1)} = -B_{(F,G)}.$$

Theorem 1: [1]
Let w be an (F, G)-pseudoanalytic function and let (F_1, G_1) be a successor of (F, G). Then $\dot{w}$ is an (F_1, G_1)-pseudoanalytic function.

□

Definition 2: [1]
Let (F, G) be a generating pair. Its adjoint generating pair $(F, G)^* = (F^*, G^*)$ is

defined by the formulae

$$F^* = -\frac{2\overline{F}}{F\overline{G} - \overline{F}G}, \qquad G^* = \frac{2\overline{G}}{F\overline{G} - \overline{F}G}.$$

Theorem 2: [1]

$$\begin{aligned} &(F,G)^{**} = (F,G), \\ &a_{(F^*,G^*)} = -a_{(F,G)}, \qquad A_{(F^*,G^*)} = -A_{(F,G)}, \\ &b_{(F^*,G^*)} = -\overline{B}_{(F,G)}, \qquad B_{(F^*,G^*)} = -\overline{b}_{(F,G)}. \end{aligned}$$

Proof:
Since,

$$(F,G)^* = (F^*,G^*),$$

where,

$$F^* = -\frac{2\overline{F}}{F\overline{G} - \overline{F}G}, \qquad G^* = \frac{2\overline{G}}{F\overline{G} - \overline{F}G},$$

we have to show that

$$(F,G)^{**} = (F^{**},G^{**}) = (F,G).$$

Indeed,

$$\begin{aligned} F^{**} &= -\frac{2\overline{F^*}}{F^*\overline{G^*} - \overline{F^*}G^*}, \\ &= \frac{\frac{4F}{\overline{F}G - F\overline{G}}}{-\frac{4\overline{F}G}{(F\overline{G}-\overline{F}G)(\overline{F}G-F\overline{G})} + \frac{4F\overline{G}}{(F\overline{G}-\overline{F}G)(\overline{F}G-F\overline{G})}}, \\ &= \frac{4F}{\overline{F}G - F\overline{G}} \times \frac{(F\overline{G}-\overline{F}G)(\overline{F}G-F\overline{G})}{4(-\overline{F}G + F\overline{G})}, \\ &= F. \end{aligned}$$

$$\begin{aligned} G^{**} &= \frac{2\overline{G^*}}{F^*\overline{G^*} - \overline{F^*}G^*}, \\ &= \frac{\frac{4G}{\overline{F}G - F\overline{G}}}{-\frac{4\overline{F}G}{(F\overline{G}-\overline{F}G)(\overline{F}G-F\overline{G})} + \frac{4F\overline{G}}{(F\overline{G}-\overline{F}G)(\overline{F}G-F\overline{G})}}, \\ &= \frac{4G}{\overline{F}G - F\overline{G}} \times \frac{(F\overline{G}-\overline{F}G)(\overline{F}G-F\overline{G})}{4(F\overline{G} - \overline{F}G)}, \\ &= G. \end{aligned}$$

$$
\begin{aligned}
A_{(F^*,G^*)} &= -\frac{\overline{F^*}G^*_z - F^*_z\overline{G^*}}{F^*\overline{G^*} - \overline{F^*}G^*}, \\
&= \Big[(\frac{2F}{\overline{F}G - F\overline{G}}) \frac{(F\overline{G} - \overline{F}G)(2\overline{G}_z) - 2\overline{G}(F_z\overline{G} + F\overline{G}_z - \overline{F}_zG - \overline{F}G_z)}{(F\overline{G} - \overline{F}G)^2} \\
&\quad - \frac{(F\overline{G} - \overline{F}G)(2\overline{F}_z) - 2\overline{F}(F_z\overline{G} + F\overline{G}_z - \overline{F}_zG - \overline{F}G_z)}{(F\overline{G} - \overline{F}G)^2} (\frac{2G}{\overline{F}G - F\overline{G}}) \Big] \\
&\quad \Big[-\frac{4\overline{F}G}{(F\overline{G} - \overline{F}G)(\overline{F}G - F\overline{G})} + \frac{4F\overline{G}}{(F\overline{G} - \overline{F}G)(\overline{F}G - F\overline{G})} \Big]^{-1}, \\
&= \frac{4}{(\overline{F}G - F\overline{G})(F\overline{G} - \overline{F}G)^2} \left[-F\overline{F}G\overline{G}_z - FF_z\overline{G}\overline{G} + F\overline{F}_zG\overline{G} + F\overline{F}\overline{G}G_z \right. \\
&\quad \left. -F\overline{F}_zG\overline{G} + \overline{F}F_zG\overline{G} + F\overline{F}G\overline{G}_z - \overline{F}\overline{F}GG_z \right] \left[\frac{(F\overline{G} - \overline{F}G)(\overline{F}G - F\overline{G})}{4(F\overline{G} - \overline{F}G)} \right], \\
&= \frac{1}{(F\overline{G} - \overline{F}G)^2} \left[-F_z\overline{G}(F\overline{G} - \overline{F}G) + \overline{F}G_z(F\overline{G} - \overline{F}G) \right], \\
&= \frac{\overline{F}G_z - F_z\overline{G}}{F\overline{G} - \overline{F}G}, \\
&= -A_{(F,G)}.
\end{aligned}
$$

$$
\begin{aligned}
a_{(F^*,G^*)} &= -\frac{\overline{F^*}G^*_{\overline{z}} - F^*_{\overline{z}}\overline{G^*}}{F^*\overline{G^*} - \overline{F^*}G^*}, \\
&= \Big[(\frac{2F}{\overline{F}G - F\overline{G}}) \frac{(F\overline{G} - \overline{F}G)(2\overline{G}_{\overline{z}}) - 2\overline{G}(F_{\overline{z}}\overline{G} + F\overline{G}_{\overline{z}} - \overline{F}_{\overline{z}}G - \overline{F}G_{\overline{z}})}{(F\overline{G} - \overline{F}G)^2} \\
&\quad - \frac{(F\overline{G} - \overline{F}G)(2\overline{F}_{\overline{z}}) - 2\overline{F}(F_{\overline{z}}\overline{G} + F\overline{G}_{\overline{z}} - \overline{F}_{\overline{z}}G - \overline{F}G_{\overline{z}})}{(F\overline{G} - \overline{F}G)^2} (\frac{2G}{\overline{F}G - F\overline{G}}) \Big] \\
&\quad \Big[-\frac{4\overline{F}G}{(F\overline{G} - \overline{F}G)(\overline{F}G - F\overline{G})} + \frac{4F\overline{G}}{(F\overline{G} - \overline{F}G)(\overline{F}G - F\overline{G})} \Big]^{-1}, \\
&= \frac{4}{(\overline{F}G - F\overline{G})(F\overline{G} - \overline{F}G)^2} \left[F\overline{F}G\overline{G}_{\overline{z}} - FF_{\overline{z}}\overline{G}\overline{G} + F\overline{F}_{\overline{z}}G\overline{G} + F\overline{F}\overline{G}G_{\overline{z}} \right. \\
&\quad \left. -F\overline{F}_{\overline{z}}G\overline{G} + \overline{F}F_{\overline{z}}G\overline{G} + F\overline{F}G\overline{G}_{\overline{z}} - \overline{F}\overline{F}GG_{\overline{z}} \right] \left[\frac{(F\overline{G} - \overline{F}G)(\overline{F}G - F\overline{G})}{4(F\overline{G} - \overline{F}G)} \right], \\
&= \frac{1}{(F\overline{G} - \overline{F}G)^2} \left[-F_{\overline{z}}\overline{G}(F\overline{G} - \overline{F}G) + \overline{F}G_{\overline{z}}(F\overline{G} - \overline{F}G) \right], \\
&= \frac{\overline{F}G_{\overline{z}} - F_{\overline{z}}\overline{G}}{F\overline{G} - \overline{F}G}, \\
&= -a_{(F,G)}.
\end{aligned}
$$

$$
\begin{aligned}
b_{(F^*,G^*)} &= \frac{F^*G^*_{\overline{z}} - F^*_{\overline{z}}G^*}{F^*\overline{G^*} - \overline{F^*}G^*}, \\
&= \Bigg[\Big(\frac{-2\overline{F}}{F\overline{G} - \overline{F}G}\Big)\frac{(F\overline{G} - \overline{F}G)(2\overline{G}_{\overline{z}}) - 2\overline{G}(F_{\overline{z}}\overline{G} + F\overline{G}_{\overline{z}} - \overline{F}_{\overline{z}}G - \overline{F}G_{\overline{z}})}{(F\overline{G} - \overline{F}G)^2} \\
&\quad + \frac{(F\overline{G} - \overline{F}G)(2\overline{F}_{\overline{z}}) - 2\overline{F}(F_{\overline{z}}\overline{G} + F\overline{G}_{\overline{z}} - \overline{F}_{\overline{z}}G - \overline{F}G_{\overline{z}})}{(F\overline{G} - \overline{F}G)^2}\Big(\frac{2\overline{G}}{F\overline{G} - \overline{F}G}\Big)\Bigg] \\
&\quad \left[-\frac{4\overline{F}G}{(F\overline{G} - \overline{F}G)(\overline{F}G - F\overline{G})} + \frac{4F\overline{G}}{(F\overline{G} - \overline{F}G)(\overline{F}G - F\overline{G})}\right]^{-1}, \\
&= \frac{4}{(F\overline{G} - \overline{F}G)^3}\Big[\overline{FF}G\overline{G}_{\overline{z}} + \overline{F}F_{\overline{z}}\overline{GG} - \overline{FF}_{\overline{z}}G\overline{G} - \overline{FFG}G_{\overline{z}} \\
&\quad + F\overline{F}_{\overline{z}}\overline{GG} - \overline{F}F_{\overline{z}}\overline{GG} - F\overline{FGG}_{\overline{z}} + \overline{FFG}G_{\overline{z}}\Big]\left[\frac{(F\overline{G} - \overline{F}G)(\overline{F}G - F\overline{G})}{4(F\overline{G} - \overline{F}G)}\right], \\
&= \frac{-1}{(F\overline{G} - \overline{F}G)^2}\left[-F_{\overline{z}}\overline{G}(F\overline{G} - \overline{F}G) + \overline{F}G_{\overline{z}}(F\overline{G} - \overline{F}G)\right], \\
&= \frac{-1}{(F\overline{G} - \overline{F}G)}\left(-\overline{FG}_{\overline{z}} + \overline{F}_{\overline{z}}\overline{G}\right), \\
&= \frac{\overline{FG}_{\overline{z}} - \overline{F}_{\overline{z}}\overline{G}}{(F\overline{G} - \overline{F}G)}, \\
&= -\overline{\left(\frac{FG_z - F_zG}{F\overline{G} - \overline{F}G}\right)}, \\
&= -\overline{B}_{(F,G)}.
\end{aligned}
$$

$$
\begin{aligned}
B_{(F^*,G^*)} &= \frac{F^*G^*_{z} - F^*_{z}G^*}{F^*\overline{G^*} - \overline{F^*}G^*}, \\
&= \Bigg[\Big(\frac{-2\overline{F}}{F\overline{G} - \overline{F}G}\Big)\frac{(F\overline{G} - \overline{F}G)(2\overline{G}_{z}) - 2\overline{G}(F_{z}\overline{G} + F\overline{G}_{z} - \overline{F}_{z}G - \overline{F}G_{z})}{(F\overline{G} - \overline{F}G)^2} \\
&\quad + \frac{(F\overline{G} - \overline{F}G)(2\overline{F}_{z}) - 2\overline{F}(F_{z}\overline{G} + F\overline{G}_{z} - \overline{F}_{z}G - \overline{F}G_{z})}{(F\overline{G} - \overline{F}G)^2}\Big(\frac{2\overline{G}}{F\overline{G} - \overline{F}G}\Big)\Bigg] \\
&\quad \left[-\frac{4\overline{F}G}{(F\overline{G} - \overline{F}G)(\overline{F}G - F\overline{G})} + \frac{4F\overline{G}}{(F\overline{G} - \overline{F}G)(\overline{F}G - F\overline{G})}\right]^{-1}, \\
&= \frac{4}{(F\overline{G} - \overline{F}G)^3}\Big[\overline{FF}G\overline{G}_{z} + \overline{F}F_{z}\overline{GG} - \overline{FF}_{z}G\overline{G} - \overline{FFG}G_{z} \\
&\quad + F\overline{F}_{z}\overline{GG} - \overline{F}F_{z}\overline{GG} - F\overline{FG}_{z}\overline{G} + \overline{FFG}G_{z}\Big]\left[\frac{(F\overline{G} - \overline{F}G)(\overline{F}G - F\overline{G})}{4(F\overline{G} - \overline{F}G)}\right], \\
&= \frac{-1}{(F\overline{G} - \overline{F}G)^2}\left[\overline{F}_{z}\overline{G}(F\overline{G} - \overline{F}G) - \overline{FG}_{z}(F\overline{G} - \overline{F}G)\right], \\
&= -\frac{\overline{F}_{z}\overline{G} - \overline{FG}_{z}}{F\overline{G} - \overline{F}G},
\end{aligned}
$$

$$\begin{aligned} B_{(F^*,G^*)} &= \frac{\overline{F}_z\overline{G} - \overline{F}\overline{G}_z}{\overline{F}G - F\overline{G}}, \\ &= -\left(\overline{\frac{FG_{\overline{z}} - F_{\overline{z}}G}{F\overline{G} - \overline{F}G}}\right), \\ &= -\overline{b}_{(F,G)}. \end{aligned}$$

□

Lemma 3: [1]
If (F_1, G_1) is a successor of (F, G) then $(F, G)^*$ is a successor of $(F_1, G_1)^*$.

□

The (F, G)-integral of W on a rectifiable curve Γ is, by definition,

$$\int_\Gamma W d_{(F,G)} z = Re \int_\Gamma F^* W dz - \iota Re \int_\Gamma G^* W dz.$$

Another important integral is also needed

$$\int_\Gamma W d_{(F,G)} z = Re \int_\Gamma G^* W dz + \iota Re \int_\Gamma F^* W dz,$$

(we follow the notation of Bers').
A continuous function w defined in a domain Ω is called (F, G)-integrable if for every closed curve Γ situated in a simply connected subdomain of Ω,

$$\int_\Gamma W d_{(F,G)} z = 0.$$

THEOREM 4: [1]
An (F, G) derivative $\dot{w}$ of an (F, G)-pseudoanalytic function w is (F, G)-integrable and

$$\int_{z_0}^{z_1} \dot{w} d_{(F,G)} z = \omega(z_1) - \omega(z_0).$$

The integral $\int_{z_0}^{z_1} \dot{w} d_{(F,G)} z$ is called (F, G)-antiderivative of $\dot{w}$.

□

THEOREM 5: [1]
Let (F, G) be a predecessor of (F_1, G_1). A continuous function is (F_1, G_1)-pseudoanalytic if and only if it is (F, G)-integrable.

□

3.2 Applications of Bers' theory to the stationary Schrödinger equation

Let us start from

$$w_{\overline{z}} = b\overline{w}, \tag{3.2.1}$$

$$v_{\overline{z}} = \overline{b}\overline{v}, \tag{3.2.2}$$

where $z = x + \iota y$, $b = -\frac{\partial_{\overline{z}} f_0}{f_0}$. Then,

$$F = \frac{1}{f_0} \quad \text{and} \quad G = \iota f_0, \tag{3.2.3}$$

are two solutions of Eq. (3.2.1). Now by taking

$$\begin{aligned}
w &= \frac{1}{f_0} & &(w = \iota f_0),\\
w_{\overline{z}} &= -\frac{1}{f_0^2}\partial_{\overline{z}} f_0 & &(w_{\overline{z}} = \iota \partial_{\overline{z}} f_0),\\
w_{\overline{z}} &= -\frac{\partial_{\overline{z}} f_0}{f_0}\frac{1}{f_0} & &(w_{\overline{z}} = -\iota f_0(-\frac{\partial_{\overline{z}} f_0}{f_0})),\\
w_{\overline{z}} &= b\overline{w} & &(w_{\overline{z}} = b\overline{w}),
\end{aligned}$$

which obviously satisfy Eq. (3.1.1). Since,

$$Im\left(\overline{F}G\right) = Im\left(\frac{1}{f_0}\ \iota f_0\right) = Im\left(\iota\right) = 1 > 0.$$

Thus, (F, G) is a generating pair corresponding to Eq. (3.2.1). With the generating functions, we have

$$\begin{aligned}
a_{(F,G)} &= -\frac{\overline{F}G_{\overline{z}} - F_{\overline{z}}\overline{G}}{F\overline{G} - \overline{F}G},\\
&= -\frac{(\frac{1}{f_0})\partial_{\overline{z}}(\iota f_0) - (-\frac{1}{f_0^2}\partial_{\overline{z}} f_0)(-\iota f_0)}{(\frac{1}{f_0})(-\iota f_0) - \frac{1}{f_0}(\iota f_0)},\\
&= -\frac{\frac{\iota\partial_{\overline{z}} f_0}{f_0} - \frac{\iota\partial_{\overline{z}} f_0}{f_0}}{-\iota - \iota},\\
&= 0.
\end{aligned} \tag{3.2.4}$$

$$\begin{aligned}
b_{(F,G)} &= \frac{FG_{\overline{z}} - F_{\overline{z}}G}{F\overline{G} - \overline{F}G}, \\
&= \frac{(\frac{1}{f_0})\partial_{\overline{z}}(\iota f_0) - (-\frac{1}{f_0^2}\partial_{\overline{z}} f_0)(\iota f_0)}{(\frac{1}{f_0})(-\iota f_0) - \frac{1}{f_0}(\iota f_0)}, \\
&= \frac{\frac{\iota\partial_{\overline{z}} f_0}{f_0} + \frac{\iota\partial_{\overline{z}} f_0}{f_0}}{-\iota - \iota}, \\
&= \left(\frac{2\iota\partial_{\overline{z}} f_0}{f_0}\right)\left(\frac{-1}{2\iota}\right), \\
&= -\frac{\partial_{\overline{z}} f_0}{f_0}, \\
&= b. \qquad (3.2.5)
\end{aligned}$$

$$\begin{aligned}
A_{(F,G)} &= -\frac{\overline{F}G_z - F_z\overline{G}}{F\overline{G} - \overline{F}G}, \\
&= -\frac{(\frac{1}{f_0})\partial_z(\iota f_0) - (-\frac{1}{f_0^2}\partial_z f_0)(-\iota f_0)}{(\frac{1}{f_0})(-\iota f_0) - \frac{1}{f_0}(\iota f_0)}, \\
&= -\frac{\frac{\iota\partial_z f_0}{f_0} - \frac{\iota\partial_z f_0}{f_0}}{-\iota - \iota}, \\
&= 0. \qquad (3.2.6)
\end{aligned}$$

$$\begin{aligned}
B_{(F,G)} &= \frac{FG_z - F_zG}{F\overline{G} - \overline{F}G}, \\
&= \frac{(\frac{1}{f_0})\partial_z(\iota f_0) - (-\frac{1}{f_0^2}\partial_z f_0)(\iota f_0)}{(\frac{1}{f_0})(-\iota f_0) - \frac{1}{f_0}(\iota f_0)}, \\
&= \frac{\frac{\iota\partial_z f_0}{f_0} + \frac{\iota\partial_z f_0}{f_0}}{-\iota - \iota}, \\
&= -\frac{\partial_z f_0}{f_0}, \\
&= \overline{b}. \qquad (3.2.7)
\end{aligned}$$

According to Definition 1, the characteristic coefficients for a successor of (F, G) have the form

$$a_{(F_1,G_1)} = a_{(F,G)} = 0 \quad \text{and} \quad b_{(F_1,G_1)} = -B_{(F,G)} = \frac{\partial_z f_0}{f_0} = -\overline{b}.$$

If w is a solution of Eq. (3.2.1) then its (F, G) derivative $\dot{w}$ is a solution of the equation

$$W_{\overline{z}} = -\overline{b}\overline{W}, \qquad (3.2.8)$$

we know that $w = \frac{1}{f_0}$ is a solution of Eq. (3.2.1). Now we show that $\dot{w}$ is a solution of Eq. (3.2.8).

$$\begin{aligned}\dot{w} &= w_z - A_{(F,G)}w - B_{(F,G)}\overline{w},\\ &= w_z - (0)w - \overline{b}\overline{w},\\ &= w_z - \overline{b}\overline{w}.\end{aligned}$$

We have to show that $W = \dot{w} = w_z - \overline{b}\overline{w}$ is a solution of Eq. (3.2.8).

$$\begin{aligned}W_{\overline{z}} &= w_{z\overline{z}} - \overline{b}\overline{w}_{\overline{z}} - \overline{b}_{\overline{z}}\overline{w},\\ &= w_{z\overline{z}} + \frac{\partial_z f_0}{f_0}\overline{w}_{\overline{z}} + \partial_{\overline{z}}\left(\frac{\partial_z f_0}{f_0}\right)\overline{w},\\ &= w_{z\overline{z}} + \frac{\partial_z f_0}{f_0}\overline{w}_{\overline{z}} - \frac{1}{f_0^2}\left(\partial_{\overline{z}} f_0\right)\left(\partial_z f_0\right)\overline{w} + \frac{1}{f_0}\left(\partial_{z\overline{z}} f_0\right)\overline{w},\\ &= \partial_{\overline{z}}\left(-\frac{1}{f_0^2}\partial_z f_0\right) + \frac{\partial_z f_0}{f_0}\left(-\frac{1}{f_0^2}\partial_{\overline{z}} f_0\right) - \frac{1}{f_0^2}\left(\partial_{\overline{z}} f_0\right)\left(\partial_z f_0\right)\frac{1}{f_0} + \frac{1}{f_0}\left(\partial_{z\overline{z}} f_0\right)\frac{1}{f_0},\\ &= -\frac{1}{f_0^2}\partial_{z\overline{z}} f_0 + \frac{2}{f_0^3}\partial_{\overline{z}} f_0 \cdot \partial_z f_0 - \frac{1}{f_0^3}\partial_{\overline{z}} f_0 \cdot \partial_z f_0 - \frac{1}{f_0^3}\partial_{\overline{z}} f_0 \cdot \partial_z f_0 + \frac{1}{f_0^2}\partial_{z\overline{z}} f_0,\\ &= 0.\end{aligned}$$

$$\begin{aligned}-\overline{bW} &= -\overline{\left(-\frac{\partial_{\overline{z}} f_0}{f_0}\right)\left(w_z - \overline{b}\overline{w}\right)},\\ &= \overline{\left(\frac{\partial_{\overline{z}} f_0}{f_0}\right)\left(w_z + \frac{\partial_z f_0}{f_0}\overline{w}\right)},\\ &= \left(\frac{\partial_z f_0}{f_0}\right)\left(w_{\overline{z}} + \frac{\partial_{\overline{z}} f_0}{f_0}w\right),\\ &= \frac{\partial_z f_0}{f_0}\left(-\frac{1}{f_0^2}\partial_{\overline{z}} f_0 + \frac{\partial_{\overline{z}} f_0}{f_0}\frac{1}{f_0}\right),\\ &= 0.\end{aligned}$$

Hence $\dot{w}$ is a solution of Eq. (3.2.8). But solutions of Eq. (3.2.8) multiplied by ι become solutions of Eq. (3.2.2). Thus we obtain the following statement.

THEOREM 6: [1]
Let w be a solution of Eq. (3.2.1). Then the function

$$v = \iota\dot{w} = \iota\left(w_z + \frac{\partial_z f_0}{f_0}\overline{w}\right), \tag{3.2.9}$$

is a solution of Eq. (3.2.2).

Proof:
To prove that v is a solution of Eq. (3.2.2) we start from its R.H.S.

$$\begin{aligned}
\overline{b}\overline{v} &= \overline{\left(-\frac{\partial_{\overline{z}} f_0}{f_0}\right)\left[\iota\left(w_z + \frac{\partial_z f_0}{f_0}\overline{w}\right)\right]}, \\
&= \left(-\frac{\partial_z f_0}{f_0}\right)\left[-\iota(w_{\overline{z}} + \frac{\partial_{\overline{z}} f_0}{f_0} w)\right], \\
&= \iota\frac{\partial_z f_0}{f_0}\left(w_{\overline{z}} + \frac{\partial_{\overline{z}} f_0}{f_0} w\right), \\
&= \iota\frac{\partial_z f_0}{f_0}\left(-\frac{1}{f_0^2}\partial_{\overline{z}} f_0 + \frac{\partial_{\overline{z}} f_0}{f_0}\frac{1}{f_0}\right), \\
&= 0.
\end{aligned}$$

Now we take L.H.S. of (3.2.2)

$$\begin{aligned}
v_{\overline{z}} &= \iota\left[w_{z\overline{z}} - \frac{1}{f_0^2}\partial_{\overline{z}} f_0 \partial_z f_0 \overline{w} + \frac{\partial_{z\overline{z}} f_0}{f_0}\overline{w} + \frac{\partial_z f_0}{f_0}\overline{w_{\overline{z}}}\right], \\
&= \iota\left[\partial_{\overline{z}}\left(-\frac{1}{f_0^2}\partial_z f_0\right) - \frac{1}{f_0^2}\partial_{\overline{z}} f_0 \cdot \partial_z f_0 \frac{1}{f_0} + \frac{\partial_{z\overline{z}} f_0}{f_0}\frac{1}{f_0} + \frac{\partial_z f_0}{f_0}\left(-\frac{1}{f_0^2}\partial_{\overline{z}} f_0\right)\right], \\
&= \iota\left[-\frac{1}{f_0^2}\partial_{z\overline{z}} f_0 + \frac{2}{f_0^3}\partial_{\overline{z}} f_0 \cdot \partial_z f_0 - \frac{1}{f_0^3}\partial_{\overline{z}} f_0 \cdot \partial_z f_0 + \frac{1}{f_0^2}\partial_{z\overline{z}} f_0 - \frac{1}{f_0^3}\partial_z f_0 \cdot \partial_{\overline{z}} f_0\right], \\
&= 0.
\end{aligned}$$

Hence v is a solution of (3.2.2).

□

By using Eq. (3.2.3) in F^* and G^* we have

$$\begin{aligned}
F^* &= -\frac{2\overline{F}}{F\overline{G} - \overline{F}G}, \\
&= -\frac{\frac{1}{f_0}}{\frac{1}{f_0}(-\iota f_0) - \frac{1}{f_0}(\iota f_0)}, \\
&= -\frac{\iota}{f_0}.
\end{aligned}$$

$$\begin{aligned}
G^* &= \frac{2\overline{G}}{F\overline{G} - \overline{F}G}, \\
&= \frac{2(-\iota f_0)}{\frac{1}{f_0}(-\iota f_0) - \frac{1}{f_0}(\iota f_0)}, \\
&= \frac{-2\iota f_0}{-2\iota}, \\
&= f_0.
\end{aligned}$$

From Theorem 2, we have $b_{(F^*,G^*)} = -\overline{B}_{(F,G)}$ and from Eq. (3.2.7) we have $B_{(F,G)} = \overline{b}$. Therefore,

$$b_{(F^*,G^*)} = -\overline{B}_{(F,G)} = -b.$$

Thus the (F,G)-integral of a function W is defined as follows

$$\begin{aligned}\int_\Gamma W d_{(F,G)} z &= Re \int_\Gamma F^* W dz - \iota Re \int_\Gamma G^* W dz, \\ &= -Re \int_\Gamma \frac{\iota}{f_0} W dz - \iota Re \int_\Gamma f_0 W dz, \\ &= Im \int_\Gamma \frac{W}{f_0} dz - \iota Re \int_\Gamma f_0 W dz. \qquad (3.2.10)\end{aligned}$$

Where we have used the relation

$$Re\,(\iota W) = Re\,(\iota w_1 - w_2) = -w_2 = -ImW.$$

From Theorems 4 and 5, we obtain the following results.

Theorem 7: [1]
Let v be a solution of Eq. (3.2.2) in a domain Ω. Then for every closed curve Γ situated in a simply connected subdomain of Ω,

$$Re \int_\Gamma \frac{v}{f_0} dz + \iota Im \int_\Gamma f_0 v dz = 0. \qquad (3.2.11)$$

Proof:
For any solution v of Eq. (3.2.2), the function $W = \iota v$ is a solution of Eq. (3.2.8). Since $v = \iota \dot{w}$ is a solution of Eq. (3.2.2) $\iota v = -\dot{w}$ is a solution of Eq. (3.2.8). As Eq. (3.2.8) corresponds to a successor of (F,G), by Theorem 4, W is (F,G)-integrable, *i.e.*,

$$Im \int_\Gamma \frac{W}{f_0} dz - \iota Re \int_\Gamma f_0 W dz = 0.$$

Replacing W by ιv, we obtain

$$Im \int_\Gamma \frac{\iota v}{f_0} dz - \iota Re \int_\Gamma f_0 (\iota v) dz = 0,$$

or

$$Re \int_\Gamma \frac{v}{f_0} dz + \iota Im \int_\Gamma f_0 v dz = 0. \qquad (3.2.12)$$

In Eq. (3.2.12) we have used

$$Re(\iota w) = -w_2 = -ImW, \quad \text{and} \quad Im(\iota w) = w_1 = ReW.$$

□

In order to understand this result for solutions of the Schrödinger equation, let us rewrite some statements in our "two-dimensional" notation. Consider the stationary two dimensional Schrödinger Eq. (1.0.5) where u and ν depend on x and y. For simplicity we consider u and ν to be real-valued functions.

Proposition 8: [1]
Let f_1 be another nonvanishing solution of the Schrödinger Eq. (1.0.5). Then the ratio $\Psi = \frac{f_1}{f_0}$ is a solution of the equation

$$div\left(f_0^2 grad\Psi\right) = 0.$$

□

Proposition 9: [1]
Let f_1 be another solution of Eq. (1.0.5). Then the function

$$v = f_0 \partial_z \left(\frac{f_1}{f_0}\right), \tag{3.2.13}$$

is a solution of Eq. (3.2.2), where $b = -\frac{\partial_{\bar{z}} f_0}{f_0}$.
Proof:
It is given that v is a solution of Eq. (3.2.2). To prove this take L.H.S. of Eq. (3.2.2)

$$v_{\bar{z}} = \partial_{\bar{z}} f_0 \cdot \partial_z \left(\frac{f_1}{f_0}\right) + f_0 \partial_{z\bar{z}} \left(\frac{f_1}{f_0}\right).$$

Now,

$$\begin{aligned} \text{R.H.S. of } (3.2.2) = \bar{b}\bar{v} &= \overline{\left(-\frac{\partial_{\bar{z}} f_0}{f_0}\right)\left(f_0 \partial_z \left(\frac{f_1}{f_0}\right)\right)}, \\ &= -\frac{\partial_z f_0}{f_0} \cdot f_0 \partial_{\bar{z}}(\frac{f_1}{f_0}), \\ &= -\partial_z f_0 \cdot \partial_{\bar{z}}(\frac{f_1}{f_0}). \end{aligned}$$

To see that v satisfies Eq. (3.2.2) we require

$$\partial_{\bar{z}} f_0 \cdot \partial_z \left(\frac{f_1}{f_0}\right) + f_0 \partial_{z\bar{z}} \left(\frac{f_1}{f_0}\right) = -\partial_z f_0 \cdot \partial_{\bar{z}} \left(\frac{f_1}{f_0}\right),$$

or

$$\partial_{\bar{z}} f_0 \cdot \partial_z \left(\frac{f_1}{f_0}\right) + f_0 \nabla^2 \left(\frac{f_1}{f_0}\right) + \partial_z f_0 \cdot \partial_{\bar{z}} \left(\frac{f_1}{f_0}\right) = 0. \tag{3.2.14}$$

By using Proposition 8,

$$div\left(f_0^2 grad\Psi\right) = 0 \quad \text{where } \Psi = \frac{f_1}{f_0},$$

we have

$$\begin{aligned} div\left(\phi\nabla\psi\right) &= \phi\left(\nabla\cdot\nabla\right)\psi + \nabla\phi\cdot\nabla\psi, \\ f_0^2 div\left(\nabla\Psi\right) + \nabla\left(f_0^2\right)\cdot\nabla\Psi &= 0, \\ f_0^2\nabla^2\Psi + 2f_0\nabla f_0\cdot\nabla\Psi &= 0, \end{aligned}$$

or

$$f_0\left[f_0\nabla^2\Psi + 2\nabla f_0\cdot\nabla\Psi\right] = 0.$$

Since $f_0 \neq 0$ we have

$$f_0\nabla^2\Psi + 2\nabla f_0\cdot\nabla\Psi = 0. \tag{3.2.15}$$

Now we show that Eq. (3.2.14) follows from Eq. (3.2.15). Indeed,

$$\begin{aligned} & \partial_{\overline{z}} f_0\cdot\partial_z\left(\frac{f_1}{f_0}\right) + f_0\nabla^2\left(\frac{f_1}{f_0}\right) + \partial_z f_0\cdot\partial_{\overline{z}}\left(\frac{f_1}{f_0}\right) \\ = & \left(\frac{\partial}{\partial x} + \iota\frac{\partial}{\partial y}\right) f_0\cdot\left(\frac{\partial}{\partial x} - \iota\frac{\partial}{\partial y}\right)\left(\frac{f_1}{f_0}\right) + \left(\frac{\partial}{\partial x} - \iota\frac{\partial}{\partial y}\right) f_0\cdot\left(\frac{\partial}{\partial x} + \iota\frac{\partial}{\partial y}\right)\left(\frac{f_1}{f_0}\right) \\ & + f_0\nabla^2\left(\frac{f_1}{f_0}\right), \\ = & \left(\frac{\partial f_0}{\partial x} + \iota\frac{\partial f_0}{\partial y}\right)\left(\frac{\partial}{\partial x}(\frac{f_1}{f_0}) - \iota\frac{\partial}{\partial y}(\frac{f_1}{f_0})\right) + \left(\frac{\partial f_0}{\partial x} - \iota\frac{\partial f_0}{\partial y}\right)\left(\frac{\partial}{\partial x}(\frac{f_1}{f_0}) + \iota\frac{\partial}{\partial y}(\frac{f_1}{f_0})\right) \\ & + f_0\nabla^2\left(\frac{f_1}{f_0}\right), \\ = & 2\left[\frac{\partial f_0}{\partial x}\cdot\frac{\partial}{\partial x}\left(\frac{f_1}{f_0}\right) + \frac{\partial f_0}{\partial y}\cdot\frac{\partial}{\partial y}\left(\frac{f_1}{f_0}\right)\right] + f_0\nabla^2\left(\frac{f_1}{f_0}\right), \\ = & 2\nabla f_0\cdot\nabla\Psi + f_0\nabla^2\left(\frac{f_1}{f_0}\right), \\ = & 0. \quad \text{by using Eq. (3.2.15)} \end{aligned}$$

□

From this Proposition we have, if we have two solutions of stationary Schrödinger Eq. (1.0.5) then v in terms of f_0 and f_1 becomes the solution of Eq. (3.2.2). This Proposition will be used in proof of Cauchy's integral theorem for the Schrödinger equatöion.

Chapter 4

Stationary Schrödinger equation, class of pseudoanalytic functions, two-dimensional stationary Schrödinger equation and the complex Riccati equation

4.1 Some preliminary results on the stationary Schrödinger equation and a class of pseudo-analytic functions

The theory of pseudoanalytic functions is a generalization of the theory of analytic functions. Consider the equation $\partial_{\overline{z}}\varphi = \Phi$ in the whole complex plane, where φ is a real valued function and $\Phi = \Phi_1 + \iota\Phi_2$ is a given complex-valued function.

4.1.1 Cauchy-Riemann equations

$$\partial_{\overline{z}}\varphi = \Phi = \Phi_1 + \iota\Phi_2,$$

i.e.

$$\frac{1}{2}\left(\frac{\partial\varphi}{\partial x} + \iota\frac{\partial\varphi}{\partial y}\right) = \Phi_1 + \iota\Phi_2.$$

Comparing real and imaginary parts, we have

$$\frac{1}{2}\frac{\partial\varphi}{\partial x} = \Phi_1, \qquad \frac{1}{2}\frac{\partial\varphi}{\partial y} = \Phi_2. \tag{4.1.1}$$

From there

$$\partial_y \Phi_1 = \frac{1}{2}\frac{\partial^2 \varphi}{\partial x \partial y} = \partial_x \Phi_2,$$

or

$$\partial_y \Phi_1 - \partial_x \Phi_2 = 0. \tag{4.1.2}$$

This is the corresponding Cauchy-Riemann equation. This equation can be used to construct $\varphi(x, y)$ as,

$$\varphi(x, y) = 2\left[\int_{x_0}^{x} \Phi_1(\eta, y)d\eta + \int_{y_0}^{y} \Phi_2(x_0, \xi)d\xi\right] + c, \tag{4.1.3}$$

which satisfies Eq. (4.1.1) as can be easily seen. Where (x_0, y_0) is an arbitrary fixed point in the domain of interest. By $\overline{A}$ we denote the integral operator in Eq. (4.1.3)

$$\overline{A}[\Phi](x, y) = 2\left[\int_{x_0}^{x} \Phi_1(\eta, y)d\eta + \int_{y_0}^{y} \Phi_2(x_0, \xi)d\xi\right] + c.$$

Note that formula (4.1.3) can be easily extended to any simply connected domain by considering the integral along an arbitrary rectifiable curve Γ leading from (x_0, y_0) to (x, y)

$$\varphi(x, y) = 2\left(\int_{\Gamma} \Phi_1 dx + \Phi_2 dy\right) + c.$$

Thus if Φ satisfies Eq. (4.1.2), there exists a family of real-valued functions φ such that $\partial_{\overline{z}}\varphi = \Phi$ given by the formula $\varphi = \overline{A}[\Phi]$.

In a similar way, we introduce the operator

$$A[\Phi](x, y) = 2\left(\int_{\Gamma} \Phi_1 dx - \Phi_2 dy\right) + c,$$

which is applicable to complex functions satisfying the condition

$$\partial_y \Phi_1 + \partial_x \Phi_2 = 0, \tag{4.1.4}$$

and corresponds to the operator ∂_z.

4.1.2 Vekua equation and its relationship with Schrödinger equation

Let f denote a positive twice continuously differentiable function defined in a domain $\Omega \subset \mathbb{C}$. Consider the following Vekua equation:

$$W_{\overline{z}} = \frac{f_{\overline{z}}}{f}\overline{W} \qquad \text{in } \Omega, \tag{4.1.5}$$

where f is a real valued function. W is a complex-valued function and $\overline{W} = C[W]$ is its complex conjugate function. As was shown in [2], Eq. (4.1.5) is closely related to the second-order equation of the form (1.0.5), where $\nu = \frac{\Delta f}{f}$ and u are real-valued functions. In particular, the following statements are valid which we show below.

THEOREM 10: [3]
Let W be a solution of Eq. (4.1.5). Then $u = ReW$ is a solution of Eq. (1.0.5) and $v = ImW$ is a solution of the equation

$$(-\Delta + \eta)\, v = 0, \tag{4.1.6}$$

where

$$\eta = 2\left(\frac{|\nabla f|^2}{f}\right)^2 - \nu.$$

Proof:
We have

$$W_{\overline{z}} = \frac{f_{\overline{z}}}{f}\overline{W},$$

where f is a real function. This becomes on multiplying by f

$$fW_{\overline{z}} = f_{\overline{z}}\overline{W},$$

or

$$f(x,y)\frac{1}{2}\left(\frac{\partial}{\partial x} + \iota\frac{\partial}{\partial y}\right)(u + \iota v) = \frac{1}{2}\left(\frac{\partial f}{\partial x} + \iota\frac{\partial f}{\partial y}\right)(u - \iota v),$$

or

$$f\left(u_x + \iota v_x + \iota u_y - v_y\right) = f_x u - \iota f_x v + \iota f_y u + f_y v.$$

Comparing real and imaginary parts we have,

$$f\left(u_x - v_y\right) = f_x u + f_y v, \tag{4.1.7}$$

$$f\left(v_x + u_y\right) = f_y u - f_x v. \tag{4.1.8}$$

Differentiating Eq. (4.1.7) w.r.t. x

$$-f_x v_y + f\left(u_{xx} - v_{xy}\right) = f_{xx}u + f_{xy}v + f_y v_x. \tag{4.1.9}$$

Again differentiating Eq. (4.1.8) w.r.t. y

$$f_y v_x + f\left(v_{xy} + u_{yy}\right) = f_{yy}u - f_{xy}v - f_x v_y. \tag{4.1.10}$$

From Eq. (4.1.9), we have

$$-f_x v_y = f_{xx}u + f_{xy}v + f_y v_x - f\left(u_{xx} - v_{xy}\right).$$

This we substitute in the above Eq. (4.1.10) to arrive at

$$f_y v_x + f(v_{xy} + u_{yy}) = f_{yy} u - f_{xy} v + f_{xx} u + f_{xy} v + f_y v_x - f(u_{xx} - v_{xy}),$$

or

$$f(u_{xx} + u_{yy}) = (f_{xx} + f_{yy}) u,$$

or

$$f(\Delta u) = (\Delta f) u,$$

or

$$\Delta u = \frac{\Delta f}{f} u,$$

or

$$\left(\Delta - \frac{\Delta f}{f}\right) u = 0,$$

which is Eq. (1.0.5).
Next we differentiate Eq. (4.1.7) and Eq. (4.1.8) w.r.t. y and x respectively to find,

$$f_y(u_x - v_y) + f(u_{xy} - v_{yy}) = f_{xy} u + f_x u_y + f_{yy} v + f_y v_y, \tag{4.1.11}$$

$$f_x(v_x + u_y) + f(v_{xx} + u_{xy}) = f_{xy} u + f_y u_x - f_{xx} v - f_x v_x. \tag{4.1.12}$$

From the above two equations, on subtraction we find

$$f_x(v_x + u_y) + f(v_{xx} + u_{xy}) - f_y(u_x - v_y) - f(u_{xy} - v_{yy}) = f_{xy} u + f_y u_x$$
$$- f_{xx} v - f_x v_x - f_{xy} u - f_x u_y - f_{yy} v - f_y v_y,$$

or

$$f(v_{xx} + v_{yy}) + (f_{xx} + f_{yy}) v = 2\left[f_y(u_x - v_y) - f_x(v_x + u_y)\right],$$

or

$$f(\Delta v) + (\Delta f) v = 2\left[f_y(u_x - v_y) - f_x(v_x + u_y)\right]. \tag{4.1.13}$$

From Eq. (4.1.7)

$$u_x - v_y = \frac{f_x}{f} u + \frac{f_y}{f} v$$

and from Eq. (4.1.8)

$$v_x + u_y = \frac{f_y}{f} u - \frac{f_x}{f} v$$

Thus from Eq. (4.1.13) we have

$$\begin{aligned} f(\Delta v) + (\Delta f)\, v &= 2\left[f_y\left(\frac{f_x}{f}u + \frac{f_y}{f}v\right) - f_x\left(\frac{f_y}{f}u - \frac{f_x}{f}v\right)\right], \\ &= 2\left[\frac{1}{f}\left(f_x f_y u + f_y^2 v - f_x f_y u + f_x^2 v\right)\right], \\ &= 2\left[\frac{f_x^2 + f_y^2}{f}\right] v, \end{aligned}$$

or

$$f(\Delta v) + (\Delta f)\, v = 2\frac{(\nabla f)^2}{f} v,$$

or

$$\left(-\Delta + 2\frac{(\nabla f)^2}{f^2} - \frac{\Delta f}{f}\right) v = 0,$$

which gives Eq. (4.1.6) for

$$\eta = 2\frac{(\nabla f)^2}{f^2} - \frac{\Delta f}{f}.$$

□

Proposition-11: [2]
Let b be a complex function such that b_z is real valued, and let $W = u + \iota v$ be a solution of the equation

$$W_{\overline{z}} = b\overline{W}. \tag{4.1.14}$$

Then u is a solution of the equation

$$\partial_{\overline{z}}\partial_z u - \left(b\overline{b} + b_z\right) u = 0, \tag{4.1.15}$$

and v is a solution of the equation

$$\partial_{\overline{z}}\partial_z v - \left(b\overline{b} - b_z\right) v = 0. \tag{4.1.16}$$

Proof:
Eq. (4.1.14) yields

$$\partial_z\left(W_{\overline{z}}\right) = \partial_z\left(b\overline{W}\right),$$

or

$$W_{\overline{z}z} = b_z\overline{W} + b\overline{W}_z. \tag{4.1.17}$$

Eq. (4.1.14) also gives

$$\left(\frac{\partial}{\partial x} + \iota\frac{\partial}{\partial y}\right)(u + \iota v) = b\,(u - \iota v).$$

Taking complex conjugate of both sides

$$\left(\frac{\partial}{\partial x} - \iota \frac{\partial}{\partial y}\right)(u - \iota v) = \overline{b}\,(u + \iota v)\,,$$

i.e.

$$\overline{W}_z = \overline{b}W. \tag{4.1.18}$$

After substituting Eq. (4.1.18) in Eq. (4.1.17) we get

$$W_{z\overline{z}} = b_z\overline{W} + b\overline{b}W,$$

or

$$\partial_z\partial_{\overline{z}}\,(u + \iota v) = b_z\,(u - \iota v) + b\overline{b}\,(u + \iota v)\,,$$

or

$$\partial_z\partial_{\overline{z}}u + \iota\partial_z\partial_{\overline{z}}v = \left(b_z + b\overline{b}\right)u + \iota\left(b\overline{b} - b_z\right)v.$$

Comparing real and imaginary parts we get,

$$\partial_z\partial_{\overline{z}}u = \left(b_z + b\overline{b}\right)u,$$

$$\partial_z\partial_{\overline{z}}v = \left(b\overline{b} - b_z\right)v.$$

□

Proposition 12: [2]
Let W be a solution of

$$\left(\partial_{\overline{z}} - \frac{\partial_{\overline{z}}f_0}{f_0}C\right)W = 0. \tag{4.1.19}$$

Then $u = ReW$ is a solution of Eq. (1.0.5) and $v = ImW$ is a solution of the equation

$$\left(\Delta + \nu - 2\left(\frac{|\nabla f_0|}{f_0}\right)^2\right)v = 0. \tag{4.1.20}$$

Proof:
Observe that the coefficient $b = \frac{\partial_{\overline{z}}f_0}{f_0}$ in Eq. (4.1.19) satisfies the condition of Proposition 11

$$\begin{aligned} b_z &= \partial_z\left(\frac{\partial_{\overline{z}}f_0}{f_0}\right), \\ &= \frac{\partial_z\partial_{\overline{z}}f_0}{f_0} - \frac{1}{f_0^2}\left(\partial_z f_0\right)\left(\partial_{\overline{z}}f_0\right), \\ &= \frac{\Delta f_0}{f_0} - \frac{|\partial_{\overline{z}}f_0|^2}{f_0^2}, \\ &= \nu - \frac{\left(\partial_{\overline{z}}f_0\right)\left(\overline{\partial_{\overline{z}}f_0}\right)}{f_0^2}, \end{aligned}$$

$$
\begin{aligned}
b_z &= \nu - \left(\frac{\partial_{\bar{z}} f_0}{f_0}\right)\left(\frac{\partial_z f_0}{f_0}\right), \\
&= \nu - b\bar{b}.
\end{aligned}
$$

Substitute $b\bar{b} + b_z = \nu$ in Eq. (1.0.5) which is the same as Eq. (4.1.15) and by Proposition 11 u is solution of Eq. (4.1.15).
Similarly,

$$
\begin{aligned}
b_z &= \partial_z \left(\frac{\partial_{\bar{z}} f_0}{f_0}\right), \\
&= \frac{\partial_z \partial_{\bar{z}} f_0}{f_0} - \frac{1}{f_0^2}\left(\partial_z f_0\right)\left(\partial_{\bar{z}} f_0\right), \\
&= \frac{\Delta f_0}{f_0} - \frac{|\partial_{\bar{z}} f_0|^2}{f_0^2}, \\
&= \nu - 2\left(\frac{|\partial_{\bar{z}} f_0|}{f_0}\right)^2 + \left(\frac{|\partial_{\bar{z}} f_0|}{f_0}\right)^2, \\
&= \nu - 2\left(\frac{|\nabla f_0|}{f_0}\right)^2 + \left(\frac{\partial_{\bar{z}} f_0}{f_0}\right)\left(\frac{\partial_z f_0}{f_0}\right), \\
&= \nu - 2\left(\frac{|\nabla f_0|}{f_0}\right)^2 + b\bar{b}.
\end{aligned}
$$

Substitute $b\bar{b} - b_z = 2\left(\frac{|\nabla f_0|}{f_0}\right)^2 - \nu$ in Eq. (4.1.20) which is the same as Eq. (4.1.16) and from Proposition 11 v is solution of Eq. (4.1.16).

□

THEOREM 13: [2]
Let u be a solution of Eq. (1.0.5) in a simply connected domain Ω. Then the function

$$
v \in ker\left(\Delta + \nu - 2\left(\frac{|\nabla f|}{f}\right)^2\right), \tag{4.1.21}
$$

such that $W = u + \iota v$ be a solution of Eq. (4.1.5) is constructed according to the formula

$$
v = f^{-1}\overline{A}\left(\iota f^2 \partial_{\bar{z}}\left(f^{-1}u\right)\right). \tag{4.1.22}
$$

It is unique up to an additive term cf^{-1}, where c is an arbitrary real constant.
Given $v \in ker\left(\Delta + \nu - 2\left(\frac{|\nabla f|}{f}\right)^2\right)$, the corresponding $u \in ker\left(\Delta - \nu\right)$ can be constructed as follows:

$$
u = -f\overline{A}\left(\iota f^{-2}\partial_{\bar{z}}\left(fv\right)\right), \tag{4.1.23}
$$

up to an additive term cf.

Proof:
Consider equation,

$$\left(\partial_{\overline{z}} - \frac{\partial_{\overline{z}} f_0}{f_0} C\right) W = 0.$$

Let $W = \phi f_0 + \iota \frac{\psi}{f_0}$ be its solution then it becomes,

$$\left(\partial_{\overline{z}} - \frac{\partial_{\overline{z}} f_0}{f_0} C\right)\left(\phi f_0 + \iota \frac{\psi}{f_0}\right) = 0,$$

$$\left(\partial_{\overline{z}}\phi\right) f_0 + \phi\left(\partial_{\overline{z}} f_0\right) + \iota \frac{\partial_{\overline{z}}\psi}{f_0} - \iota \frac{\psi}{f_0^2}\left(\partial_{\overline{z}} f_0\right) - \frac{\partial_{\overline{z}} f_0}{f_0}\left(\phi f_0\right) - \frac{\partial_{\overline{z}} f_0}{f_0}\left(-\iota \frac{\psi}{f_0}\right) = 0,$$

$$\left(\partial_{\overline{z}}\phi\right) f_0 + \phi\left(\partial_{\overline{z}} f_0\right) + \iota \frac{\partial_{\overline{z}}\psi}{f_0} - \iota \frac{\psi \partial_{\overline{z}} f_0}{f_0^2} - \phi\left(\partial_{\overline{z}} f_0\right) + \iota \frac{\psi \partial_{\overline{z}} f_0}{f_0^2} = 0,$$

or

$$\left(\partial_{\overline{z}}\phi\right) f_0 + \iota \frac{\partial_{\overline{z}}\psi}{f_0} = 0,$$

or

$$\psi_{\overline{z}} - \iota f_0^2 \phi_{\overline{z}} = 0.$$

Since $u = ReW = \phi f_0$, so $\phi = \frac{u}{f_0}$

$$\psi_{\overline{z}} = \iota f_0^2 \phi_{\overline{z}},$$

or

$$\psi = \overline{A}\left(\iota f_0^2 \phi_{\overline{z}}\right).$$

By using this result

$$\left[\partial_y \Phi_1 - \partial_x \Phi_2 = 0, \quad \text{since} \quad \partial_{\overline{z}}\varphi = \Phi, \quad \text{implies} \quad \varphi(x,y) = \overline{A}[\Phi]\right],$$

it can be verified that the expression $\overline{A}\left(\iota f_0^2 \phi_{\overline{z}}\right)$ makes sense i.e.,

$$\partial_y\left(-f_0^2 \phi_y\right) - \partial_x\left(f_0^2 \phi_x\right) = 0,$$

$$\partial_y\left(f_0^2 \phi_y\right) + \partial_x\left(f_0^2 \phi_x\right) = 0.$$

By Proposition 12, $v = f_0^{-1}\psi$ is a solution of

$$\left(\Delta + \nu - 2\left(\frac{|\nabla f|}{f}\right)^2\right) v = 0,$$

$$\begin{aligned} v &= f_0^{-1}\overline{A}\left(\iota f_0^2 \phi_{\overline{z}}\right), \\ &= f_0^{-1}\overline{A}\left(\iota f_0^2 \partial_{\overline{z}}\left(f_0^{-1} u\right)\right) \qquad \text{as } \phi = \frac{u}{f_0}. \end{aligned}$$

Let us notice that as the operator $\overline{A}$ reconstructs the scalar function up to an arbitrary real constant, the function v in the formula

$$v = f^{-1}\overline{A}\left(\iota f^2 \partial_{\overline{z}}\left(f^{-1}u\right)\right),$$

is uniquely determined up to an additive term cf_0^{-1} where c is an arbitrary real constant.
Similarly, $v = ImW$, $v = f_0^{-1}\psi$ or $\psi = vf_0$

$$\iota f_0^2 \phi_{\overline{z}} = \psi_{\overline{z}},$$

$$\phi_{\overline{z}} = \frac{\psi_{\overline{z}}}{\iota f_0^2} = -\iota f_0^{-2}\psi_{\overline{z}},$$

$$\phi = \overline{A}\left[-\iota f_0^{-2}\psi_{\overline{z}}\right],$$

$$\partial_y(f_0^{-2}\psi_y) - \partial_x(-f_0^{-2}\psi_x) = 0,$$

or

$$\partial_y(f_0^{-2}\psi_y) + \partial_x(f_0^{-2}\psi_x) = 0.$$

But $u = f_0\phi$ is a solution of $(-\Delta + \nu)u = 0$ where

$$\begin{aligned} u &= f_0\overline{A}\left[-\iota f_0^{-2}\psi_{\overline{z}}\right], \\ &= -f_0\overline{A}\left[\iota f_0^{-2}\partial_{\overline{z}}(vf_0)\right]. \end{aligned}$$

□

Thus the relation between Eq. (4.1.5) and Eq. (1.0.5) is similar to that of Cauchy-Riemann system and the Laplace equation. For a Vekua equation of the form

$$W_{\overline{z}} = aW + b\overline{W},$$

where a and b are arbitrary complex-valued functions from an appropriate function space [13]. A well-developed theory of Taylor and Laurent series in formal powers has been created (see [11]). We recall that a formal power $Z^{(n)}(a, z_0; z)$ corresponding to a coefficient a and a centre z_0 is a solution of the Vekua equation satisfying the asymptotic formula

$$Z^{(n)}(a, z_0; z) \sim a(z - z_0)^n \qquad z \rightarrow z_0. \tag{4.1.24}$$

In Chapter 5, we show how this theory is applied for generalizing the second Euler theorem. For this we need the expansion theorem. For the Vekua equation of the form (4.1.5), this expansion theorem reads as follows (for more details, we refer the reader to [14]).

THEOREM 14: [3]
Let W be a regular solution of Eq. (4.1.5) defined for $|z - z_0| < R$. Then it admits

a unique expansion of the form

$$W(z) = \sum_{n=0}^{\infty} Z^n(a_n, z_0, z), \tag{4.1.25}$$

which converges normally for $|z - z_0| < R$.

□

THEOREM 15: [3] (Cauchy's integral theorem for the Schrödinger equation).
Let f be a nonvanishing solution of the Schrödinger Eq. (1.0.5) in a domain Ω and u be another arbitrary solution of the same equation in Ω. Then for every closed curve Γ situated in a simply connected subdomain of Ω,

$$Re \int_\Gamma \partial_z \left(\frac{u}{f}\right) dz + \iota Im \int_\Gamma f^2 \partial_z \left(\frac{u}{f}\right) dz = 0, \tag{4.1.26}$$

or equivalently

$$Re \int_\Gamma \partial_z \left(\frac{u}{f}\right) dz = 0 \qquad \text{and} \qquad Im \int_\Gamma f^2 \partial_z \left(\frac{u}{f}\right) dz = 0.$$

Proof:
Eq. (3.2.13) is

$$v = f \partial_z \left(\frac{u}{f}\right),$$

and Eq. (3.2.11) is

$$Re \int_\Gamma \frac{v}{f} dz + \iota Im \int_\Gamma f v dz = 0.$$

From these two results

$$Re \int_\Gamma \frac{f \partial_z \left(\frac{u}{f}\right)}{f} dz + \iota Im \int_\Gamma f \cdot f \partial_z \left(\frac{u}{f}\right) dz = 0,$$

or

$$Re \int_\Gamma \partial_z \left(\frac{u}{f}\right) dz + \iota Im \int_\Gamma f^2 \partial_z \left(\frac{u}{f}\right) dz = 0.$$

□

Another result which will be used in the present work (Section 5.2) is a Cauchy-type integral theorem for the stationary Schrödinger equation. It was obtained in [1] with the aid of the pseudoanalytic function theory.

4.2 The two-dimensional stationary Schrödinger equation and the complex Riccati equation

Consider the complex differential Riccati Eq. (1.0.4) and stationary Schrödinger Eq. (1.0.5), where u and ν are real valued. Both equations are studied in a domain $\Omega \subset \mathbb{C}$.

THEOREM 16: [3]
Let u be a solution of Eq. (1.0.5). Then its logarithmic derivative

$$Q = \frac{d}{dz}(\log u) = \frac{u_z}{u}, \tag{4.2.1}$$

is a solution of Eq. (1.0.4).
Proof:
Let u is a solution of Eq. (1.0.5), then we have to show that $Q = \frac{u_z}{u}$ is a solution of Eq. (1.0.4).
If it is solution of Eq. (1.0.4) then it must satisfy Eq. (1.0.4).
Now

$$\begin{aligned}
\partial_{\bar{z}} Q &= \frac{1}{2}\left(\frac{\partial}{\partial x} + \iota\frac{\partial}{\partial y}\right)\left(\frac{u_z}{u}\right), \\
&= \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{u_z}{u}\right) + \iota\frac{1}{2}\frac{\partial}{\partial y}\left(\frac{u_z}{u}\right), \\
&= \frac{1}{2}\frac{\partial}{\partial x}\left[\frac{1}{2u}\left(\frac{\partial u}{\partial x} - \iota\frac{\partial u}{\partial y}\right)\right] + \iota\frac{1}{2}\frac{\partial}{\partial y}\left[\frac{1}{2u}\left(\frac{\partial u}{\partial x} - \iota\frac{\partial u}{\partial y}\right)\right], \\
&= \frac{1}{4}\left[-\frac{1}{u^2}\frac{\partial u}{\partial x}\left(\frac{\partial u}{\partial x} - \iota\frac{\partial u}{\partial y}\right) + \frac{1}{u}\left(\frac{\partial^2 u}{\partial x^2} - \iota\frac{\partial^2 u}{\partial x \partial y}\right)\right] \\
&\quad + \iota\frac{1}{4}\left[-\frac{1}{u^2}\frac{\partial u}{\partial y}\left(\frac{\partial u}{\partial x} - \iota\frac{\partial u}{\partial y}\right) + \frac{1}{u}\left(\frac{\partial^2 u}{\partial x \partial y} - \iota\frac{\partial^2 u}{\partial y^2}\right)\right], \\
&= \frac{1}{4}\left[-\frac{1}{u^2}\left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right) + \frac{1}{u}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\right].
\end{aligned}$$

The L.H.S. of Eq. (1.0.4) becomes

$$\begin{aligned}
\partial_{\bar{z}} Q + |Q|^2 &= \frac{1}{4}\left[-\frac{1}{u^2}\left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right) + \frac{1}{u}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\right] + \left|\frac{u_z}{u}\right|^2, \\
&= \frac{1}{4}\left[-\frac{1}{u^2}\left(\frac{\partial u}{\partial x}\right)^2 - \frac{1}{u^2}\left(\frac{\partial u}{\partial y}\right)^2 + \frac{1}{u}\frac{\partial^2 u}{\partial x^2} + \frac{1}{u}\frac{\partial^2 u}{\partial y^2}\right] \\
&\quad + \left|\frac{1}{2u}\left(\frac{\partial u}{\partial x} - \iota\frac{\partial u}{\partial y}\right)\right|^2,
\end{aligned}$$

$$\begin{aligned}
\partial_{\bar{z}}Q + |Q|^2 &= \frac{1}{4}\left[-\frac{1}{u^2}\left(\frac{\partial u}{\partial x}\right)^2 - \frac{1}{u^2}\left(\frac{\partial u}{\partial y}\right)^2 + \frac{1}{u}\frac{\partial^2 u}{\partial x^2} + \frac{1}{u}\frac{\partial^2 u}{\partial y^2} + \frac{1}{u^2}\left(\frac{\partial u}{\partial x}\right)^2 \right. \\
&\quad \left. + \frac{1}{u^2}\left(\frac{\partial u}{\partial y}\right)^2\right], \\
&= \frac{1}{4u}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \\
&= \frac{1}{4u}(\Delta u), \\
&= \frac{\nu}{4}. \qquad \text{using Eq. (1.0.5)}
\end{aligned}$$

which proves the result that Eq. (4.2.1) is a solution of Eq. (1.0.4).

□

Remark-2: [3]
Any solution of Eq. (1.0.4) satisfies Eq. (4.1.4) where $Q = Q_1 + \iota Q_2$. Indeed, the Eq. (1.0.4) yields

$$\frac{1}{2}\left(\frac{\partial}{\partial x} + \iota\frac{\partial}{\partial y}\right)(Q_1 + \iota Q_2) + |Q_1 + \iota Q_2|^2 = \frac{\nu}{4},$$

$$\left(\frac{\partial Q_1}{\partial x} - \frac{\partial Q_2}{\partial y}\right) + \iota\left(\frac{\partial Q_2}{\partial x} + \frac{\partial Q_1}{\partial y}\right) + 2Q_1^2 + 2Q_2^2 = \frac{\nu}{2}.$$

Comparing the imaginary part, we find

$$\frac{\partial Q_2}{\partial x} + \frac{\partial Q_1}{\partial y} = 0,$$

$$\partial_x Q_2 + \partial_y Q_1 = 0.$$

□

THEOREM 17: [3]
Let Q be a solution of Eq. (1.0.4). Then the function

$$u = \exp\left(A[Q]\right), \tag{4.2.2}$$

is a solution of Eq. (1.0.5).
Proof:
Let Q be a solution of Eq. (1.0.4). Then Eq. (1.0.5) yields

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u - \nu u = 0,$$

or

$$\left(\frac{\partial}{\partial x} + \iota\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - \iota\frac{\partial}{\partial y}\right)u - \nu u = 0,$$

or

$$4\partial_z \partial_{\bar{z}} u - \nu u = 0. \tag{4.2.3}$$

Substitute Eq. (4.2.2) in Eq. (4.2.3)

$$\begin{aligned}
\partial_{\bar{z}} \partial_z u &= \partial_{\bar{z}} \partial_z \left(\exp\left(A[Q]\right)\right), \\
&= \partial_{\bar{z}} \left[\partial_z \left(A[Q]\right) \exp\left(A[Q]\right)\right],
\end{aligned}$$

$$\partial_{\bar{z}} \left(A[Q]\right) = \overline{\partial_z \left(A[Q]\right)} = \overline{Q},$$

$$\Big[\text{since } \partial_{\bar{z}} \varphi = \Phi \quad \text{and} \quad \varphi = \overline{A}[\Phi] \quad \text{then} \quad \partial_{\bar{z}} \varphi = \partial_{\bar{z}} \left(\overline{A}[\Phi]\right) \quad \text{implies} \quad \Phi = \partial_z \left(A[\Phi]\right)\Big]. \tag{4.2.4}$$

$$\begin{aligned}
\partial_{\bar{z}} \partial_z u &= \partial_{\bar{z}} \left[Q \exp\left(A[Q]\right)\right], \\
&= \left(\partial_{\bar{z}} Q\right) \exp\left(A[Q]\right) + Q \left[\partial_{\bar{z}} \left(A[Q]\right) \exp\left(A[Q]\right)\right], \\
&= \left(\partial_{\bar{z}} Q\right) \exp\left(A[Q]\right) + Q\overline{Q} \exp\left(A[Q]\right), \\
&= \exp\left(A[Q]\right) \left(\partial_{\bar{z}} Q + |Q|^2\right), \\
&= u\frac{\nu}{4}. \qquad \text{using Eq. (1.0.4) and Eq. (4.2.2)}
\end{aligned}$$

Therefore,

$$\left(4\partial_{\bar{z}} \partial_z - \nu\right) u = 0.$$

□

Theorem 17 shows that if Q is a solution of Eq. (1.0.4) then there exists a solution u of Eq. (1.0.5) such that Eq. (4.2.1) is valid.

THEOREM 18: [3]
Given a complex function Q, then for any real-valued twice continuously differentiable function φ, the following equality is valid:

$$\begin{aligned}
\frac{1}{4}\left(\Delta - \nu\right) \varphi &= \left(\partial_{\bar{z}} + QC\right) \left(\partial_z - QC\right) \varphi, && (4.2.5) \\
&= \left(\partial_z + \overline{Q}C\right) \left(\partial_{\bar{z}} - \overline{Q}C\right) \varphi, && (4.2.6)
\end{aligned}$$

if and only if Q is a solution of the Riccati Eq. (1.0.4).
Proof:
First we calculate the term $\partial_{\bar{z}} \partial_z \varphi$, i.e.,

$$\begin{aligned}
\partial_{\bar{z}} \partial_z \varphi &= \frac{1}{2}\left(\frac{\partial}{\partial x} + \iota \frac{\partial}{\partial y}\right) \frac{1}{2}\left(\frac{\partial}{\partial x} - \iota \frac{\partial}{\partial y}\right) \varphi, \\
&= \frac{1}{4}\Delta \varphi. && (4.2.7)
\end{aligned}$$

Taking R.H.S. of Eq. (4.2.5)

$$\begin{aligned}
\left(\partial_{\overline{z}}+QC\right)\left(\partial_z-QC\right)\varphi &= \left(\partial_{\overline{z}}+QC\right)\left(\partial_z\varphi-Q\varphi\right), \\
&= \partial_{\overline{z}}\partial_z\varphi-\partial_{\overline{z}}(Q\varphi)+Q\overline{\partial_z\varphi}-Q\overline{Q\varphi}, \\
&= \partial_{\overline{z}}\partial_z\varphi-\left(\partial_{\overline{z}}Q\right)\varphi-Q\left(\partial_{\overline{z}}\varphi\right)+Q\left(\partial_{\overline{z}}\varphi\right)-Q\overline{Q}\varphi, \\
&= \frac{\Delta\varphi}{4}-\left(\partial_{\overline{z}}Q-|Q|^2\right)\varphi, \qquad \text{using Eq. (4.2.7)}
\end{aligned}$$

using the Riccati Eq. (1.0.4) we have

$$\left(\partial_{\overline{z}}+QC\right)\left(\partial_z-QC\right)\varphi=\frac{1}{4}\left(\Delta-\nu\right)\varphi.$$

Taking R.H.S. of Eq. (4.2.6)

$$\begin{aligned}
\left(\partial_z+\overline{Q}C\right)\left(\partial_{\overline{z}}-\overline{Q}C\right)\varphi &= \left(\partial_z+\overline{Q}C\right)\left(\partial_{\overline{z}}\varphi-\overline{Q}\varphi\right), \\
&= \partial_z\partial_{\overline{z}}\varphi-\partial_z\left(\overline{Q}\varphi\right)+\overline{Q}\left(\overline{\partial_{\overline{z}}\varphi}\right)-\overline{Q}\left(\overline{\overline{Q}\varphi}\right), \\
&= \partial_z\partial_{\overline{z}}\varphi-\left(\partial_z\overline{Q}\right)\varphi-\overline{Q}\left(\partial_z\varphi\right)+\overline{Q}\left(\partial_z\varphi\right)-\overline{Q}Q\varphi, \\
&= \partial_z\partial_{\overline{z}}\varphi-\left(\partial_z\overline{Q}\right)\varphi-|Q|^2\varphi, \\
&= \frac{1}{4}\Delta\varphi-\left(\partial_z\overline{Q}-|Q|^2\right)\varphi, \qquad \text{using Eq. (4.2.7)}
\end{aligned}$$

using the Riccati Eq. (1.0.4) we have

$$\left(\partial_z+\overline{Q}C\right)\left(\partial_{\overline{z}}-\overline{Q}C\right)\varphi=\frac{1}{4}\left(\Delta-\nu\right)\varphi.$$

□

In Theorem 18 we have discussed the factorization of two-dimensional Schrödinger equation.

Chapter 5

Generalizations of classical theorems and Cauchy's integral theorem

5.1 Generalizations of classical theorems

In this section, we give generalizations of both the Euler's theorem and the Picard's theorem.

5.1.1 Generalization of the first and second Euler theorems

In this section we will give generalization of first Euler's theorem and second Euler's theorem using the Riccati Eq. (1.0.4).

THEOREM 19: [3] (First Euler's theorem)
Let Q_0 be a bounded particular solution of the Riccati Eq. (1.0.4). Then this equation reduces to the following first order (real-linear) equation:

$$W_{\overline{z}} = \overline{Q_0 W}, \tag{5.1.1}$$

in the following sense. Any solution of Eq. (1.0.4) has the form

$$Q = \frac{\partial_z ReW}{ReW}, \tag{5.1.2}$$

and conversely, any solution of Eq. (5.1.1) can be expressed as a corresponding solution Q of Eq. (1.0.4) in the form

$$W = e^{A[Q]} + \iota e^{-A[Q_0]}\overline{A}\left[\iota e^{zA[Q_0]}\partial_{\overline{z}} e^{A[Q-Q_0]}\right]. \tag{5.1.3}$$

Proof:
Let Q_0 be a bounded solution of Eq. (1.0.4). Then by Theorem 16, there exists a

non-vanishing real-valued solution f of Eq. (1.0.5) such that, $Q_0 = \frac{f_z}{f}$. In other words, Eq. (5.1.1) can be written as

$$\begin{aligned} W_{\overline{z}} &= \overline{\frac{f_z}{f}W}, \\ &= \frac{f_{\overline{z}}}{f}\overline{W}. \end{aligned}$$

Now, let Q be any solution of Eq. (1.0.4). Then $Q = \frac{u_z}{u}$ where u is a solution of Eq. (1.0.5). According to Theorem 13, $u = ReW$ and W is solution of Eq. (4.1.5) which also can be written in the form of Eq. (5.1.1). Thus we have proved, the first part of the theorem.

From Theorem 13, we have $W = u + \iota v$, where v is solution of Eq. (4.1.21) which can be written in the form of Eq. (4.1.22) and from Theorem 17, we have u is solution of Eq. (1.0.5) which can be written in the form of Eq. (4.2.2).

$$\begin{aligned} W &= e^{A[Q]} + \iota f^{-1}\overline{A}\left[\iota f^2 \partial_{\overline{z}}\left(f^{-1}u\right)\right], \\ &= e^{A[Q]} + \iota e^{-A[Q_0]}\overline{A}\left[\iota e^{2A[Q_0]}\partial_{\overline{z}}\left(e^{-A[Q_0]}e^{A[Q]}\right)\right], \\ &= e^{A[Q]} + \iota e^{-A[Q_0]}\overline{A}\left[\iota e^{2A[Q_0]}\partial_{\overline{z}}\left(e^{A[Q-Q_0]}\right)\right]. \end{aligned}$$

□

Thus, the Riccati Eq. (1.0.4) is equivalent to the Vekua equation of the form (4.1.5). This follows that in the domain of interest Ω, there exists a bounded solution of Eq. (1.0.4).

THEOREM 20: [3] (Second Euler's theorem)
Any solution Q of Eq. (1.0.4) defined for $|z - z_0| < R$ can be represented in the form

$$Q = \frac{\sum_{n=0}^{\infty} Q_n e^{A[Q_n]}}{\sum_{n=0}^{\infty} e^{A[Q_n]}}, \tag{5.1.4}$$

where $\{Q_n\}_{n=0}^{\infty}$ is the set of particular solutions of the Riccati Eq. (1.0.4) obtained as follows:

$$Q_n(z) = \frac{\partial_z ReZ^{(n)}(a_n, z_0, z)}{ReZ^{(n)}(a_n, z_0, z)},$$

and $Z^{(n)}(a_n, z_0, z)$ are formal powers corresponding to Eq. (5.1.1) and both series in Eq. (5.1.4) converge normally for $|z - z_0| < R$.
Proof:
From Theorem 19, we have Eq. (5.1.2) where W is a solution of Eq. (5.1.1). From Theorem 14, we have Eq. (4.1.25). After substituting Eq. (4.1.25) in Eq. (5.1.2) we have

$$Q(z) = \frac{\partial_z \sum_{n=0}^{\infty} ReZ^{(n)}(a_n, z_0, z)}{\sum_{n=0}^{\infty} ReZ^{(n)}(a_n, z_0, z)}.$$

Every formal power $Z^{(n)}(a_n, z_0, z)$ corresponds to a solution of Eq. (1.0.4), i.e.

$$Q_n(z) = \frac{\partial_z Re Z^{(n)}(a_n, z_0, z)}{Re Z^{(n)}(a_n, z_0, z)}.$$

From Theorem 17, we have $u = ReW$ can be written in the form of Eq. (4.2.2) implies

$$Re\ Z^{(n)}(a_n, z_0, z) = e^{A[Q_n](z)},$$

$$Q(z) = \frac{\partial_z \sum_{n=0}^{\infty} e^{A[Q_n](z)}}{\sum_{n=0}^{\infty} e^{A[Q_n](z)}},$$

$$Q = \frac{\partial_z \sum_{n=0}^{\infty} e^{A[Q_n]}}{\sum_{n=0}^{\infty} e^{A[Q_n]}},$$

$$Q = \frac{\sum_{n=0}^{\infty} e^{A[Q_n]} \partial_z \left(A[Q_n]\right)}{\sum_{n=0}^{\infty} e^{A[Q_n]}}.$$

By using Eq. (4.2.4) we have

$$Q = \frac{\sum_{n=0}^{\infty} Q_n e^{A[Q_n]}}{\sum_{n=0}^{\infty} e^{A[Q_n]}}.$$

□

5.1.2 Generalization of Picard's theorem

In this section we give a generalization of Picard's theorem written in the form of Eq. (2.1.27) using Riccati Eq. (1.0.4).

THEOREM 21: [3] (Picard's theorem)
Let Q_k, $k = 1, 2, 3, 4$ be four solutions of Eq. (1.0.4). Then

$$\frac{\partial_{\overline{z}}\left(Q_1 - Q_2\right) + 2\iota Im\left(\overline{Q_1}Q_2\right)}{Q_1 - Q_2} + \frac{\partial_{\overline{z}}\left(Q_3 - Q_4\right) + 2\iota Im\left(\overline{Q_3}Q_4\right)}{Q_3 - Q_4}$$
$$-\frac{\partial_{\overline{z}}\left(Q_1 - Q_4\right) + 2\iota Im\left(\overline{Q_1}Q_4\right)}{Q_1 - Q_4} - \frac{\partial_{\overline{z}}\left(Q_3 - Q_2\right) + 2\iota Im\left(\overline{Q_3}Q_2\right)}{Q_3 - Q_2} = 0.$$

Proof:

$$\begin{aligned}
\left(\overline{Q_1} + \overline{Q_4}\right)\left(Q_1 - Q_4\right) &= \overline{Q_1}Q_1 - \overline{Q_1}Q_4 + \overline{Q_4}Q_1 - \overline{Q_4}Q_4, \\
&= |Q_1|^2 - \overline{Q_1}Q_4 + \overline{Q_4}Q_1 - |Q_4|^2, \\
&= \left(\frac{\nu}{4} - \partial_{\overline{z}}Q_1\right) - \overline{Q_1}Q_4 + \overline{Q_4}Q_1 - \left(\frac{\nu}{4} - \partial_{\overline{z}}Q_4\right),
\end{aligned}$$

since $\partial_{\overline{z}}Q_k + |Q_k|^2 = \frac{\nu}{4}$, for $k = 1, 2, 3, 4$. Thus,

$$\begin{aligned}\left(\overline{Q_1} + \overline{Q_4}\right)(Q_1 - Q_4) &= -\partial_{\overline{z}}Q_1 + \partial_{\overline{z}}Q_4 - \left(\overline{Q_1}Q_4 - \overline{Q_4}Q_1\right), \\ &= -\partial_{\overline{z}}(Q_1 - Q_4) - 2\iota Im\left(\overline{Q_1}Q_4\right).\end{aligned}$$

Similarly,

$$\begin{aligned}\left(\overline{Q_2} + \overline{Q_3}\right)(Q_3 - Q_2) &= -\partial_{\overline{z}}(Q_3 - Q_2) - 2\iota Im\left(\overline{Q_3}Q_2\right), \\ \left(\overline{Q_1} + \overline{Q_2}\right)(Q_1 - Q_2) &= -\partial_{\overline{z}}(Q_1 - Q_2) - 2\iota Im\left(\overline{Q_1}Q_2\right), \\ \left(\overline{Q_3} + \overline{Q_4}\right)(Q_3 - Q_4) &= -\partial_{\overline{z}}(Q_3 - Q_4) - 2\iota Im\left(\overline{Q_3}Q_4\right).\end{aligned}$$

From these equations we get,

$$\begin{aligned}\left(\overline{Q_1} + \overline{Q_4}\right) &= \frac{-\partial_{\overline{z}}(Q_1 - Q_4) - 2\iota Im\left(\overline{Q_1}Q_4\right)}{(Q_1 - Q_4)}, \\ \left(\overline{Q_2} + \overline{Q_3}\right) &= \frac{-\partial_{\overline{z}}(Q_3 - Q_2) - 2\iota Im\left(\overline{Q_3}Q_2\right)}{(Q_3 - Q_2)}, \\ \left(\overline{Q_1} + \overline{Q_2}\right) &= \frac{-\partial_{\overline{z}}(Q_1 - Q_2) - 2\iota Im\left(\overline{Q_1}Q_2\right)}{(Q_1 - Q_2)}, \\ \left(\overline{Q_3} + \overline{Q_4}\right) &= \frac{-\partial_{\overline{z}}(Q_3 - Q_4) - 2\iota Im\left(\overline{Q_3}Q_4\right)}{(Q_3 - Q_4)},\end{aligned}$$

or

$$\begin{aligned}&\frac{-\partial_{\overline{z}}(Q_1 - Q_4) - 2\iota Im\left(\overline{Q_1}Q_4\right)}{Q_1 - Q_4} + \frac{-\partial_{\overline{z}}(Q_3 - Q_2) - 2\iota Im\left(\overline{Q_3}Q_2\right)}{Q_3 - Q_2} \\ &- \frac{-\partial_{\overline{z}}(Q_1 - Q_2) - 2\iota Im\left(\overline{Q_1}Q_2\right)}{Q_1 - Q_2} - \frac{-\partial_{\overline{z}}(Q_3 - Q_4) - 2\iota Im\left(\overline{Q_3}Q_4\right)}{Q_3 - Q_4} \\ &= \left(\overline{Q_1} + \overline{Q_4}\right) + \left(\overline{Q_2} + \overline{Q_3}\right) - \left(\overline{Q_1} + \overline{Q_2}\right) - \left(\overline{Q_3} + \overline{Q_4}\right) = 0,\end{aligned}$$

or

$$\begin{aligned}&\frac{\partial_{\overline{z}}(Q_1 - Q_2) + 2\iota Im\left(\overline{Q_1}Q_2\right)}{Q_1 - Q_2} + \frac{\partial_{\overline{z}}(Q_3 - Q_4) + 2\iota Im\left(\overline{Q_3}Q_4\right)}{Q_3 - Q_4} \\ &- \frac{\partial_{\overline{z}}(Q_1 - Q_4) + 2\iota Im\left(\overline{Q_1}Q_4\right)}{Q_1 - Q_4} - \frac{\partial_{\overline{z}}(Q_3 - Q_2) + 2\iota Im\left(\overline{Q_3}Q_2\right)}{Q_3 - Q_2} = 0.\end{aligned}$$

□

5.2 Cauchy's integral theorem

In this section we give generalization of Cauchy's integral theorem.

THEOREM 22: [3] (Cauchy's integral theorem for the complex Riccati equation)
Let Q_0 and Q_1 be bounded solutions of Eq. (1.0.4) in Ω. Then for every closed curve Γ lying in a simply connected subdomain of Ω,

$$Re\int_{\Gamma}(Q_1-Q_0)\,e^{A[Q_1-Q_0]}dz+\iota Im\int_{\Gamma}(Q_1-Q_0)\,e^{A[Q_1+Q_0]}dz=0, \qquad (5.2.1)$$

or equivalently

$$Re\int_{\Gamma}(Q_1-Q_0)\,e^{A[Q_1-Q_0]}dz=0, \quad \text{and} \quad Im\int_{\Gamma}(Q_1-Q_0)\,e^{A[Q_1+Q_0]}dz=0,$$

Proof:
From Theorem 17, we have that $f=e^{A[Q_0]}$ and $u=e^{A[Q_1]}$ are solutions of Eq. (1.0.5). Now, applying Theorem 15, we have

$$Re\int_{\Gamma}\partial_z\left(\frac{u}{f}\right)dz+\iota Im\int_{\Gamma}f^2\partial_z\left(\frac{u}{f}\right)dz=0,$$

for every closed curve Γ situated in a simply connected subdomain of Ω, which gives us the equality

$$Re\int_{\Gamma}\partial_z\left(\frac{e^{A[Q_1]}}{e^{A[Q_0]}}\right)dz+\iota Im\int_{\Gamma}e^{2A[Q_0]}\partial_z\left(\frac{e^{A[Q_1]}}{e^{A[Q_0]}}\right)dz=0,$$

or

$$Re\int_{\Gamma}\partial_z\left(e^{A[Q_1-Q_0]}\right)dz+\iota Im\int_{\Gamma}e^{2A[Q_0]}\partial_z\left(e^{A[Q_1-Q_0]}\right)dz=0,$$

or

$$Re\int_{\Gamma}(Q_1-Q_0)\,e^{A[Q_1-Q_0]}dz+\iota Im\int_{\Gamma}e^{2A[Q_0]}\,(Q_1-Q_0)\,e^{A[Q_1-Q_0]}dz=0,$$

where we have used Eq. (4.2.4). Thus,

$$Re\int_{\Gamma}(Q_1-Q_0)\,e^{A[Q_1-Q_0]}dz+\iota Im\int_{\Gamma}(Q_1-Q_0)\,e^{A[Q_1+Q_0]}dz=0.$$

□

As a particular case, let us analyze the Riccati Eq. (1.0.4) with $v\equiv 0$ which is related to the Laplace equation. If in Eq. (5.2.1) we assume that $Q_0\equiv 0$, then Eq. (5.2.1) takes the form

$$Re\int_{\Gamma}Q_1e^{A[Q_1]}dz+\iota Im\int_{\Gamma}Q_1e^{A[Q_1]}dz=0,$$

which can be represented as,

$$\int_{\Gamma} Q_1 e^{A[Q_1]} dz = 0.$$

This is obviously valid because if Q_1 is a bounded solution of the Riccati equation with $\nu \equiv 0$, then according to Theorem 17, we have that $u = e^{A[Q_1]}$ is a function and this equation becomes,

$$\int_{\Gamma} u_z dz = 0, \tag{5.2.2}$$

which represents the analyticity of u_z.
Now, if in Eq. (5.2.1) we assume that $Q_1 \equiv 0$, then Eq. (5.2.1) takes the form

$$Re \int_{\Gamma} Q_0 e^{-A[Q_0]} dz + \iota Im \int_{\Gamma} Q_0 e^{A[Q_0]} dz = 0.$$

Rewriting this equality in terms of function $f = e^{A[Q_0]}$

$$f_z = \partial_z \left(A[Q_0] \right) e^{A[Q_0]} = Q_0 e^{A[Q_0]},$$

we obtain

$$Re \int_{\Gamma} \frac{f_z}{f^2} dz + \iota Im \int_{\Gamma} f_z dz = 0,$$

which, on using Eq. (5.2.2) gives

$$Re \int_{\Gamma} \frac{f_z}{f^2} dz = 0,$$

or

$$Re \int_{\Gamma} \partial_z \left(\frac{1}{f} \right) dz = 0. \tag{5.2.3}$$

This equality is a simple corollary of a complex version of the Green-Gauss theorem, [15], according to which we have

$$\frac{1}{2\iota} \int_{\Gamma} \partial_z \left(\frac{1}{f} \right) dz = \int_{\Omega} \partial_{\bar{z}} \partial_z \left(\frac{1}{f} \right) dxdy.$$

For f to be real the R.H.S. of above equation is real-valued and we obtain Eq. (5.2.3).

Chapter 6

Conclusions

We have shown that the stationary Schrödinger equation in a two-dimensional case is related to a complex differential Riccati equation which possesses many interesting properties similar to its one-dimensional prototype. Besides the generalizations of the famous Euler theorems, we have obtained the generalization of Picard's theorem and the Cauchy integral theorem for solutions of the complex Riccati equation. The theory of pseudoanalytic functions has been intensively used.

Chapter 2 of this dissertation is devoted to the relationship between Riccati Eq. (1.0.2) and the one-dimensional Schrödinger Eq. (1.0.3). We first give an extensive treatment of the Riccati equation deriving its important properties and then discuss the relationship with the one-dimensional stationary Schrödinger equation exhibiting also the factorization of the Schrödinger operator.

The appearance of $\partial_{\overline{z}}$ results in the development of pseudoanalytic function theory by Bers' which is one of the many generalizations of analytical function theory. A summary of Bers' theory of pseudoanalytic functions as a generalization of analytic functions appears in Chapter 3. In the first section, we discuss the pseudoanalytic functions, generating pair (F, G), (F, G)-derivative, characteristic coefficients $a_{(F,G)}$, $b_{(F,G)}$, $A_{(F,G)}$, $B_{(F,G)}$ and adjoint generating pair $(F, G)^* = (F^*, G^*)$ and in the second section we discuss the applications of Bers' theory to the stationary Schrödinger equation.

In the first section of Chapter 4, some preliminary results on the stationary two-dimensional Schrödinger equation and a class of pseudoanalytic functions are discussed. In fact the solutions of the Vekua Eq. (4.1.5) and the two-dimensional Schrödinger Eq. (1.0.5) are closely related to each other. Given a solution of the Schrödinger equation we can reconstruct a solution of the Vekua equation which is a pseudoanalytical function. Conversely given a pseudoanalytic function we can construct a solution of a related Schrödinger equation. In the second section we continue the discussion to obtain the relationship between the Riccati equation and the two-dimensional stationary Schrödinger equation. In fact given a solution of the Schrödinger equation in two dimension, its logarithmic derivative is a solution of the complex Riccati equation. This leads also to the factorization of the Schrödinger operator in two-dimensions.

Chapter 5 is devoted to some famous theorems from analytic function theory to the corresponding pseudoanalytic function theory. In first section we discuss generalizations of both Euler's and Picard's theorems as related to the two-dimensional Riccati equation. These are also related to the two-dimensional Schrödinger equation. In the second section we discussed the generalization of the Cauchy integral formula for the complex Riccati equation.
Chapter 6 is devoted to the conclusion.

We have discussed Schrödinger equation upto two dimensions. Further, if we extend the work to higher dimensions, then it leads to the introduction of quaternions. Which form a non-commutative field.

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