Strong metric dimension of generalized Petersen graph

by

Fazal Abbas

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Supervised by
Dr. Muhammad Ishaq

School of Natural Sciences,
National University of Sciences and Technology,
H-12, Islamabad, Pakistan

FORM TH-4

National University of Sciences & Technology

MS THESIS WORK

We hereby recommend that the dissertation prepared under our supervision by: FAZAL ABBAS, Regn No. 00000172923 Titled: Strong Metric dimension of Generalized Petersen Graph accepted in partial fulfillment of the requirements for the award of **MS** degree.

Examination Committee Members

1. Name: DR. RASHID FAROOQ Signature: ____________

2. Name: DR. MUJEEB UR REHMAN Signature: ____________

External Examiner: DR. NASIR REHMAN Signature: ____________

Supervisor's Name DR. MUHAMMAD ISHAQ Signature: ____________

Head of Department

05-11-19

Date

COUNTERSINGED

Date: 05-11-2019

Dean/Principal

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Signature: ________________

Name of Supervisor: Dr. Muhammad Ishaq

Date: 05-11-19

Signature (HoD): ________________

Date: 05-11-19

Signature (Dean/Principal): ________________

Date: 05-11-2019

Dedicated
to my
beloved Mother

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Abstract

Let G be a connected graph having n vertices and m edges. Let $x_1, y_1 \in V(G)$ and the distance between x_1, y_1 in a graph G is the number of edges on a shortest path from x_1 to y_1 in G. In a graph G, a set of vertices connected to a vertex v is called neighborhood of v. A vertex $y \in V(G)$ resolves two vertices $v_1, v_2 \in V(G)$ if $d(y, v_1) \neq d(y, v_2)$. An ordered subset S of vertices of G is called a resolving set for G if every pair of distinct vertices of G are resolved by some vertex of S. A resolving set of least cardinality is called metric dimension of a graph G. A vertex $y \in V(G)$ strongly resolves two distinct vertices $u_1, u_2 \in V(G)$ if u_1 lies on a shortest $u_2 - y$ path or u_2 lies on a shortest $u_1 - y$ path. An ordered set $S = \{s_1, \ldots, s_t\} \subseteq V(G)$ is a strong resolving set for G if every two distinct vertices $u, v \in V(G)$ are strongly resolved by some vertex of S. A strong metric basis of G is a strong resolving set of least cardinality. The cardinality of a strong metric basis is known as strong metric dimension of G. In this dissertation, we give some results related to metric dimension and strong metric dimension of graphs. We also compute strong metric dimension of generalized Petersen graph $GP(2m, m-1)$. We first compute the MMD vertices of $GP(2m, m-1)$ and using them, we construct strong resolving graph $GP_{SR}(2m, m-1)$ of $GP(2m, m-1)$. Then we find the vertex covering number of $GP_{SR}(2m, m-1)$, which is equal to $\text{sdim}(GP(2m, m-1))$.

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Chapter 1

Fundamentals of graph theory

In this section we discuss some essential definitions and primary concepts that will be used in this dissertation. In order to illuminate any uncertainty, supporting examples have also been discussed. The main idea is to discuss the properties of graphs, common classes of graphs and distances in graphs.

1.1 Introduction to graph

In our daily life we face many problems which can be modeled by means of a graph. Assume a popular publishing company has ten editors named as $\{1, 2 \ldots, 10\}$ in scientific, humanities and computing areas. These ten editors classify themselves into seven panels and decided to held a meeting on first Friday of each month to discuss topics of interest to company and to look for the solution of those problems which create troubles in their respective areas. This scenario leads us to our first example.

Example 1.1. Let $s_1 = \{1, 2, 3\}$, $s_2 = \{1, 3, 4, 5\}$, $s_3 = \{2, 5, 6, 7\}$, $s_4 = \{4, 7, 8, 9\}$, $s_5 = \{2, 6, 7\}$, $s_6 = \{8, 9, 10\}$ and $s_7 = \{1, 3, 9, 10\}$ are the seven panels of ten editors. They have fixed aside three time period for seven panels of editors to get together on those Friday when all ten editors are available. Notice that one or two editors are part of two or more panels. So these pair of panels can not meet at same time. This situation can be more understandable from the Figure 1.1.

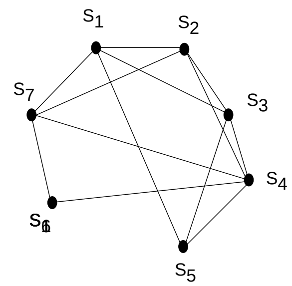

Figure 1.1: A graph G

Above figure contains the small dots and arcs. The small dots shows the seven panels and the arcs between two dots show that there is at least one common member in both panels. In fact the arcs between two dots say s_i, s_j show that these two panels of editors can not scheduled their meeting at same time. Above figure gives us the pictorial representation of seven panels and sharing of their members.

What we have constructed in Figure 1.1 is called a graph. A graph G is a mathematical structure consisting of non empty finite set $V(G)$ of small circles called vertices and set $E(G)$ (possibly empty) of some arcs connecting unordered pair of well defined small circles called edges. For a graph G of Figure 1.1 the vertex set of G is $V(G) = \{s_1, s_2, \ldots, s_7\}$ and the edge set $E(G) = \{s_1s_2, s_1s_3, s_1s_5, s_1s_7, s_2s_3, s_2s_4, s_2s_7, s_3s_4, s_3s_5, s_4s_5, s_4s_6, s_4s_7, s_6s_7\}$.

Let us consider an other situation. Suppose we have a list $2, 3, 5, 7, 11, 13, \ldots$ of integers. Every integer in the list is only divisible by 1 or itself. These numbers are building blocks of integers called prime numbers. Our next example relates to these numbers.

Example 1.2. Consider the set $P = \{2, 3, 5, 7, 11, 13\}$ of first six positive prime numbers. There are few pair of distinct integers contained in P whose sum or difference also contained in P, namely, $\{\{2, 3\}, \{2, 5\}, \{3, 5\}, \{2, 7\}, \{2, 13\}, \{2, 11\}, \{5, 7\}\}$. This situation can be transformed to a graph to identify these pairs, namely, by graph H as shown in Figure 1.2. In this instance $V(H) = \{2, 3, 5, 7, 11, 13\}$ and $E(H) = \{\{2, 3\}, \{2, 5\}, \{3, 5\}, \{2, 7\}, \{2, 13\}, \{2, 11\}, \{5, 7\}\}$.

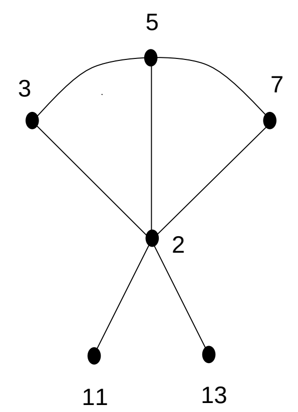

Figure 1.2: A graph G_1

Definition 1.3. The cardinality of vertex set $|V(G_1)|$ is known as order of a graph G_1, whereas the cardinality of edge set $|E(G_1)|$ is referred as size of G_1. The order and size of a graph are represented by n and m, respectively.

If two vertices v_1 and v_2 in a graph G are linked by an edge e, then v_1 and v_2 are called adjacent vertices otherwise non adjacent vertices. Let a vertex w is an end point of an edge e, then e is said to be incident on vertex w in the graph G. A graph is referred as trivial graph if the cardinality of its vertex set is one. The graph of Figure 1.2 has order $n = 6$ and size $m = 7$. If two or more edges are incident on common vertex v, then these edges are known as adjacent edges.

Definition 1.4. Let G_1 and H_1 be two graphs. If $V(H_1) \subseteq V(G_1)$ and $E(H_1) \subseteq E(G_1)$, then the graph H_1 is recognized as subgraph of G_1, symbolized as $H_1 \subseteq G_1$. If $H_1 \subseteq G_1$ and $|V(H_1)| = |V(G_1)|$, then H_1 is referred as a spanning subgraph of G_1. A graph F is said to be induced subgraph of a graph H whenever $x, y \in V(F)$ and x, y are adjacent in H then, x, y are also adjacent in F as well.

Examine the graph H in Figure 1.3. The graphs H_1, H_2 and H_3 shown in Figure 1.3 are subgraph, induced subgraph and spanning subgraph of H, respectively.

Definition 1.5. A pattern of vertices in G starting from u_1 and terminating at v_1 such

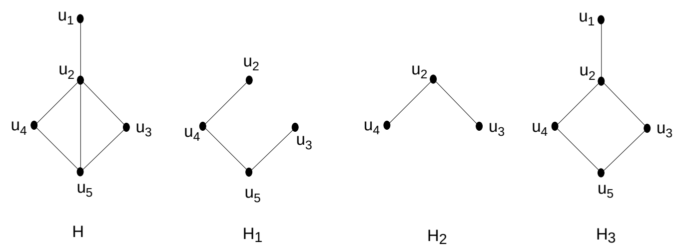

Figure 1.3: A graph H and some of its subgraphs

that consecutive vertices in the pattern are adjacent is called a $u_1 - v_1$ walk in G symbolized by W. If $u_1 = v_1$, then W is a closed walk. The total number of edges in a walk W is called length of W. If no edge is repeated in a walk W, then W is called a $u_1 - v_1$ trail. Similarly if there is no repetition of vertices in W, then W is called a $u_1 - v_1$ path and a path on n vertices is symbolized as P_n. Consequently, every path is a trail and every trail is a walk. A closed trail containing 3 or more edges is called a circuit in G. If there is no repetition of vertices except for the first and last in a circuit, then it is referred a cycle. A cycle of length l is called l-cycle, if l is odd, then we have odd cycle otherwise, even cycle. A cycle of order n is represented by C_n.

Definition 1.6. A graph G is said to be connected if there is a path between every pair of distinct vertices of G otherwise, disconnected graph. A connected subgraph of G that is not a proper subgraph of any other connected subgraph of G is called a component of G. In addition, a graph H is connected if and only if it has only one component. Let $w \in V(G)$. If deletion of a vertex w increases the cardinality of components of G, then w is referred as cut vertex. Likewise, if deletion of an edge e from G increases the cardinality of components of G, then e is said to be a cut edge. A graph F shown in Figure 1.4 is a connected graph containing cut vertex v and cut edge e.

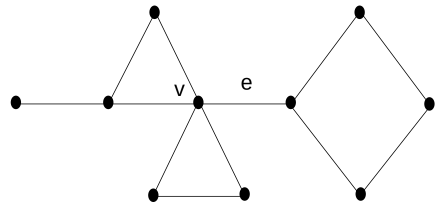

Figure 1.4: A connected graph F having cut vertex v and cut edge e.

1.2 Distance in graphs

In this portion, we study the basic concepts about the distance between vertices in a graph. Moreover, we state few results related to radius, diameter and eccentricity.

Definition 1.7. Let G_1 be a connected graph and $w_1, w_2 \in V(G_1)$. The distance between w_1 and w_2, symbolized by $d(w_1, w_2)$, is the number of edges of a shortest path from w_1 to w_2 in G_1.

Definition 1.8. Let F be a connected graph. The eccentricity $e(w)$ of a vertex w in F is expound by:

$$e(w) = \max\{d(w, x) \mid x \in V(F)\}.$$

Definition 1.9. The radius $rad(H)$ of a connected graph H is explicated as:

$$rad(H) = \min\{e(w) \mid w \in V(H)\}.$$

Definition 1.10. The diameter $diam(K)$ of a connected graph K is elucidated as:

$$diam(K) = \max\{e(w) \mid w \in V(K)\}.$$

Definition 1.11. Let $w \in V(K)$ of a connected graph K. If $e(w) = rad(K)$, then w is called a central vertex. A graph generated by central vertices of K is referred as center $cen(K)$ of K.

Example 1.12. Observe the graph G_1 shown in Figure 1.5. Here $e(v_1) = e(v_6) = 4$, $e(v_2) = 3 = e(v_4)$ and $e(v_3) = 2 = e(v_5)$. So $2rad(G_1) = diam(G_1)$.

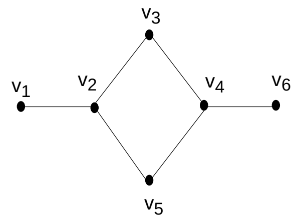

Figure 1.5: A graph G_1 of $rad(G_1) = 2$ and $diam(G_1) = 4$.

The next theorem tells us the correspondence between diameter and radius of a connected graph.

Theorem 1.13 (Chartrand and Zhang [9]). *For any non trivial connected graph H,*

$$rad(H) \leq diam(H) \leq 2rad(H).$$

1.3 Degrees

In this section, we define a parameter of a grpah G which is associated with each vertex of G. Moreover, we give some results related to these parameters.

Definition 1.14. Let H be a graph and $uv \in E(H)$, then u and v are said to be neighbors. A set of vertices connected to vertex w in a graph H is called neighborhood of w and is denoted by $N(w)$.

Definition 1.15. For a vertex w of a graph H, the cardinality of $|N(w)|$ in H is called degree of w signified by, $deg(w) = |N(w)|$.

If $deg(w) = 0$, then vertex w is referred as an isolated vertex, whereas if $deg(w) = 1$, then w is said to be leaf or a pendent vertex. A vertex w is an even vertex if $|N(w)|$ is even otherwise odd.

Definition 1.16. The minimum degree $\delta(H_1)$ of graph H_1 is defined as:

$$\delta(H_1) = \min\{deg(r) \mid r \in V(H_1)\}.$$

That is, $\delta(H_1)$ is the least degree among the vertices of H_1.

Definition 1.17. The maximum degree $\Delta(H_1)$ of graph H_1 is defined as:

$$\Delta(H_1) = \max\{deg(r) \mid r \in V(H_1)\}.$$

That is, $\Delta(H_1)$ is the greatest degree among the vertices of H_1.

The next inequality gives us the connection between maximum and minimum degrees of a graph F. If F is a simple graph having n vertices and $w \in V(F)$, then

$$0 \leq \delta(F) \leq deg(w) \leq \Delta(F) \leq n - 1.$$

Example 1.18. A graph F shown in Figure 1.5 has order $n = 9$ and size $m = 11$. Also, G has minimum degree $\delta(F) = 2$ and $\Delta(F) = 4 = deg(w)$. Following is the first theorem of graph theory.

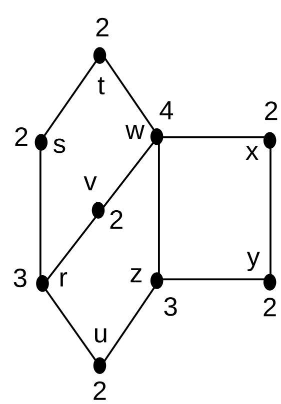

Figure 1.6: A graph F with $\delta(F) = 2$ and $\Delta(F) = 4$.

Theorem 1.19 (Chartrand and Zhang [9])**.** *If H is a graph of size m, then*

$$\sum_{v \in V(H)} deg(v) = 2m$$

Corollary 1.20 (Chartrand and Zhang [9])**.** *Every graph has an even number of odd vertices.*

Definition 1.21. Let G be a graph of order n and if $deg(u) = r$ for each $u \in V(G)$, then G is recognized as r-regular graph, otherwise, irregular graph.

By Corollary 1.3, if n and r both are odd, then it is impossible to create a regular graph on n vertices. The next theorem provides us the essential condition for the actuality of regular graph of on n vertices.

Theorem 1.22 (Chartrand and Zhang [9]). *Let r and n be integers, where $0 \leq r \leq n-1$. Then there exits an r-regular graph of order n if and only if at least one of r and n is even.*

Definition 1.23. If $V(G) = \{z_1, z_2, \ldots, z_n\}$, then the sequence $\{d(z_1), d(z_2), \ldots, d(z_n)\}$ is called the degree sequence of G. We often represent this sequence of non negative integers in ascending or descending order. If finite sequence s of non negative integers is a degree sequence of some graph, then it is called graphical.

1.4 Common classes of graphs

We continue to discuss primary concepts about graphs. In this portion we will see some common classes of graphs and the special notions reserved for these graphs.

Definition 1.24. If the cardinalities of a vertex set $V(H)$ and edge set $E(H)$ of a graph H are finite, then H is finite graph otherwise, infinite. If $E(H) = \phi$, then H is a null graph.

We have already discussed that paths and cycles are specific kind of walks in a graph. Paths and cycles are certain graphs that will be used throughout this dissertation.

Definition 1.25. A simple connected n vertex graph G is called complete if G is $n-1$ regular. That is, $\deg(t) = n-1$ for all $t \in V(G)$. An n vertex complete graph is symbolized by K_n having size $\frac{n(n-1)}{2}$.

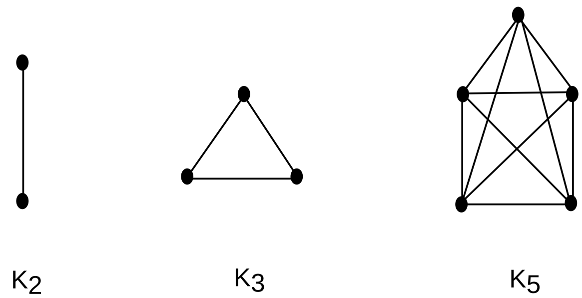

Figure 1.7: Complete graphs of order 2, 3 and 5.

The graphs shown in Figure 1.7 are the complete graphs of order 2, 3 and 5, respectively.

Definition 1.26. The complement $\overline{H}$ of a graph H is a graph with vertex set $V(H)$ and $z_1z_2 \in E(H)$ if and only if $z_1z_2 \notin E(\overline{H})$. If H is n vertex graph and having m edges, then $|V(\overline{H})| = n$ and $|E(\overline{H})| = \frac{n(n-1)}{2} - m$. The complement of a complete graph is a null graph.

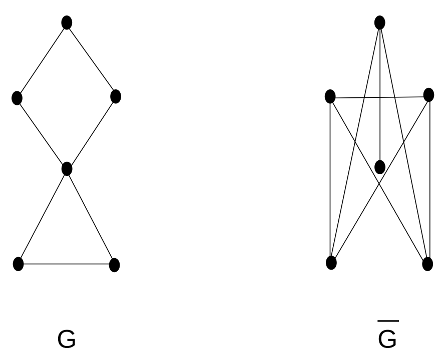

Figure 1.8: A graph G and its complement.

Noticed that if a graph H is connected, then $\overline{H}$ need not to be connected. But if H itself is disconnected, then its complement must be connected. The next theorem illustrates it more precisely.

Theorem 1.27 (Chartrand and Zhang [9]). *If H is a disconnected graph, then $\overline{H}$ is a connected graph.*

In next definition, we discuss a special class of graphs whose vertex set can be partitioned in a particular ways.

1.4.1 Bipartite graphs

Definition 1.28. A graph H is known as a bipartite graph if its vertex set can be divided into two nonempty disjoint sets P_1 and P_2 called partite sets in such a way if $z_1z_2 \in E(H)$, then its end points z_1, z_2 does not belong to same partite sets.

Often it is not easy to claim whether the given graph is bipartite or not. The next theorem will be very helpful to identify the bipartite nature of a graph.

Theorem 1.29 (Chartrand and Zhang [9])**.** *A graph H is bipartite graph if and only if H possesses no odd cycles.*

Definition 1.30. Let H be a bipartite graph having partite sets P_1 and P_2. If each vertex of P_1 is connected to each vertex of P_2, then H is a complete bipartite graph denoted as $K_{s,t}$, where s, t are the number of vertices in partite sets P_1 and P_2, respectively. If, in addition, any of s or t is 1, then $K_{s,t}$ is called a star.

Definition 1.31. Let H be a graph whose vertex set can be divided into k non empty disjoint subsets P_1, P_2, P_3, $\ldots$, P_k such that every $e \in E(H)$ have both ends in different partite sets then H is called k-partite graph. Furthermore, if every pair of vertices belonging to different partite sets are adjacent, then H is called a complete k-partite graph. If $|P_i| = s_i$, then we denote complete k-partite graph as $K_{s_1,s_2,\ldots,s_k}$.

1.4.2 Multigraphs

Definition 1.32. If the end points of two or more edges are same, then these edges are called multiple edges. Let H_1 be a graph and an edge l connects a vertex u to itself, then l is called a loop at vertex u. An n vertex graph G having size m, is a multigraph if it contains multiple edges.

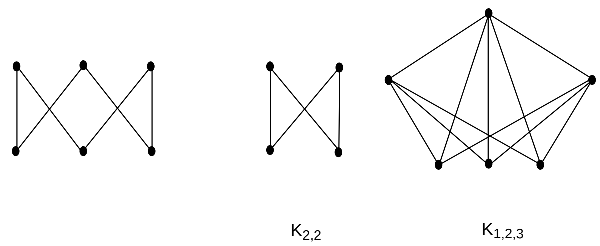

Figure 1.9: Bipartite graphs.

1.4.3 Eulerian Graphs

Definition 1.33. A circuit C in a finite graph G_1 is an Eulerian if it holds every edge of G_1. Furthermore, if a finite connected graph G_1 possesses an Eulerian circuit, then G_1 is called an Eulerian graph.

Example 1.34. The graph shown in Figure 1.10 is Eulerian graph containig Eulerian circuit $C = u_1u_2u_3u_4u_5u_6u_7u_2u_4u_7u_8u_6u_9u_8u_{11}u_9u_{10}u_{11}u_1$.

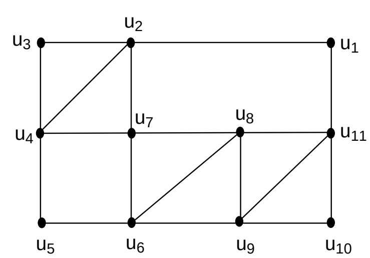

Figure 1.10: An Eulerian graph

By observing a graph, it is not possible to identify whether the graph is Eulerian or not. Next theorem gives the necessary condition for a connected graph H to be an Eulerian.

Theorem 1.35 (Chartrand and Zhang [9])**.** *A non-trivial connected graph H is Eulerian if and only if every vertex of H is even.*

1.4.4 Hamiltonian graphs

Definition 1.36. A cycle C in a graph H is Hamiltonian if contains each vertex of H. In addition, if a finite connected graph H possesses a Hamiltonian cycle, then H is said to be a Hamiltonian graph. Obviously, an n-cycle and a complete graph on n vertices are Hamiltonian graphs.

The sufficient condition for a graph H to be Hamiltonian is presented in next theorem.

Theorem 1.37 (Chartrand and Zhang [9]). *Let H be a graph of order $n \geq 3$. If*

$$deg(z_1) + deg(z_2) \geq n$$

for every pair z_1, z_2 of nonadjacent vertices of H, then H is Hamiltonian.

1.4.5 Planar graphs

Definition 1.38. A graph G is a planar graph if G can be drawn in the plane without edge crossing. A graph that is not planar is called non-planar graph.

A planar graph partitions the plane into connected pieces called regions, denoted by r. The graph shown in Figure 1.11 has 6 vertices, 12 edges and 8 regions. Therefore in this case, $n - m + r = 2$. This always holds, which leads us to present Euler identity.

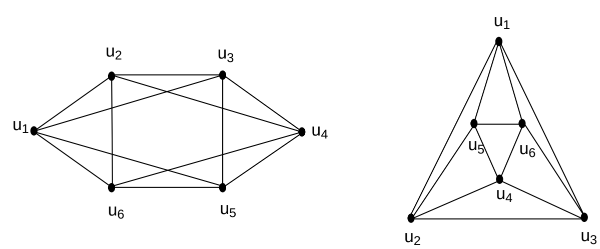

Figure 1.11: A graph G and its plane drawing.

Theorem 1.39 (Chartrand and Zhang [9]). *If G is an n vertex connected planar graph having m edges and r regions, then $n - m + r = 2$.*

1.4.6 Trees

Definition 1.40. A graph having no cycle is said to be acyclic graph. A tree is an acyclic connected graph. A tree T of order greater than 3 is called a caterpillar if the removal of pendent vertices of T creates a path, called spine of T.

The next theorem provide us a necessary condition for a graph G to ba a tree.

Theorem 1.41 (Chartrand and Zhang [9])**.** *A graph H is a tree if and only if each pair of vertices of H are connected by a unique path.*

1.5 Isomorphism

Definition 1.42. Two simple graphs A and B are said to be isomorphic if there exist a one to one correspondence $\varphi : V(A) \to V(B)$ such that $xy \in E(A)$ if and only if $\varphi(x)\varphi(y) \in E(B)$. We write $A \cong B$ if A and B are isomorphic.

Definition 1.43. Let G be a connected graph with vertex set $V(G) = \{u_1, u_2, \ldots, u_n\}$. The adjacency matrix $A(G) = [a_{ij}]_{n\times n}$ of G is an $n \times n$ matrix defined as $a_{ij} = 1$, if and only if u_i and u_j are adjacent; otherwise $a_{ij} = 0$.

Definition 1.44. Let G be a connected graph with vertex set $V(G) = \{u_1, u_2, \ldots, u_n\}$ and edges set $E(G) = \{e_1, e_2, \ldots, e_m\}$. The incidence matrix $B(G) = [b_{ij}]_{m\times n}$ of G is an $m \times n$ matrix described as $b_{ij} = 1$, if and only if v_i is incidence on e_j; otherwise $b_{ij} = 0$. The adjacency matrix $A(G)$ and incidence matrix $B(G)$ of graph G shown in Figure 1.12 is given by

$$A(G) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix},$$

$$B(G) = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

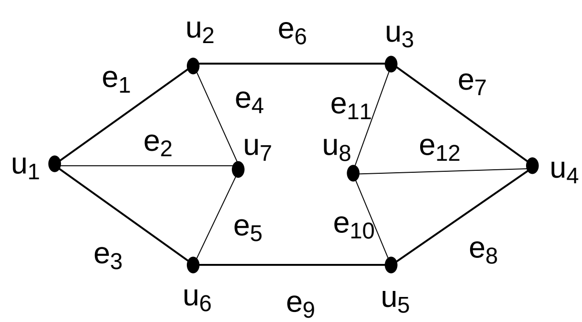

Figure 1.12: A graph of order 8 and size 12.

1.6 Overview

The plan of dissertation is given as:

In Chapter 2, we review some elementary results associated with metric dimension and strong metric dimension of a graph.

In Chapter 3, we find strong metric dimension of the Petersen graph $GP(2m, m-1)$ for $m \geq 6$. First, we compute the mutually maximally distant vertices of the Petersen graph and then using them, we construct strong resolving graph of $GP(2m, m-1)$. Next we find the independence number of strong resolving graph $GP(2m, m-1)$. Then, by using Theorem 3.1, we determine its vertex covering number which is equal to the strong metric dimension of $GP(2m, m-1)$.

Chapter 2

Resolvability in graphs

In this chapter, we study the notion of resolving sets, metric basis and strong metric basis of graphs. We also review some well known results related to these resolvability parameters and association between them.

2.1 Introduction to resolvability

Let w_1 and w_2 be any two vertices of a simple connected graph G. The distance $d(w_1, w_2)$ between w_1 and w_2 is the length of a shortest path from w_1 to w_2 in G. A vertex v of G resolves two vertices u_1 and u_2 if the distance between u_1 and v is not same as the distance between u_2 and v, that is, $d(v, u_1) \neq d(v, u_2)$. For an ordered subset $S = \{s_1, s_2, \ldots, s_k\}$ of vertices of G and a vertex $w \in V(G)$, we represent the ordered k-tuple $r_S(w)$ as the representation w with respect to S, where

$$r_S(w) = (d(w, s_1), d(w, s_2), \ldots, d(w, s_k)).$$

If all vertices of G have distinct representation with respect to S, then set S is referred as a resolving set for G. We have another equivalent definition of resolving set. A set $S \subseteq V(G)$ is said to be a resolving set (or metric generator) for G, if any two distinct vertices of G are resolved by some vertex of S. A resolving set containing least number of vertices is called metric basis for G. The cardinality of metric basis of G is called metric dimension of a graph G and is denoted by $dim(G)$. In addition, the metric dimension of G is also called location number of G denoted by $loc(G)$. In 1970, the idea of metric basis of graphs was proposed by Harary et al. [6]. We present an example to demonstrate this concept.

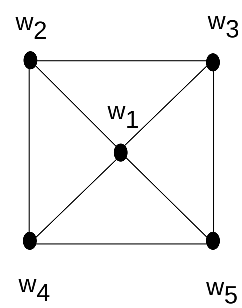

Figure 2.1: A graph $H = (V(H), E(H))$.

Example 2.1. For a graph H of Figure 2.1, take ordered set $Z_1 = \{w_2, w_3, w_4\}$. The representations of vertices of H with respect to Z_1 are given by:

$$r(w_1|Z_1) = \{1, 1, 1\},\ r(w_2|Z_1) = \{0, 1, 1\},$$
$$r(w_3|Z_1) = \{1, 0, 2\}\ ,\ r(w_4|Z_1) = \{1, 2, 0\},$$
$$r(w_5|Z_1) = \{2, 1, 1\}.$$

The vertices of H have distinct representations with respect to Z_1. Thus, Z_1 is a resolving set. Now take $Z_2 = \{w_2, w_3\}$. Then the representations of vertices of H with respect to Z_2 are given by:

$$r(w_1|Z_2) = \{1, 1\},\ r(w_2|Z_2) = \{0, 1\},$$
$$r(w_3|Z_2) = \{1, 0\},\ r(w_4|Z_2) = \{1, 2\},$$
$$r(w_5|Z_2) = \{2, 1\},$$

where all these 5 representations are distinct. Therefore, Z_2 is the resolving set. It can be easily seen that there is no metric basis of cardinality 1. Hence $dim(G) = 2$.

In graph theory, the metric dimension is a configuration that has been proved very useful in different fields of applied sciences. For instance, a model application of these distance associated parameters to robot navigation in networks is studied in [20]. The creativity of these concepts like metric basis comes from the chemistry. In such a situation, the chemical compounds are transformed through mathematical tools and these chemical compounds are analyzed by mathematical objects. The chemical structures are mostly represented by graphs. In a graph of chemical compound, the vertices indicate the atoms of a molecule while the edges of a graph indicate valence bond between pairs of atoms. For example, a propane molecule has the chemical formula C_3H_8, where C_3 represents the three atoms of carbon and H_8 represents the eight atoms of hydrogen. A propane molecule can be constituted by a graph manifested in Figure 2.2.

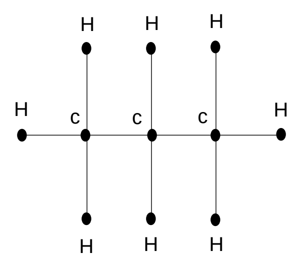

Figure 2.2: A Propane molecule.

In chemistry, an isomer is a molecule having distinct structural arrangements but identical number of atoms. Let us consider an example. The atomic formula of both butane and isobutane is identical, that is, C_4H_{10} but both have distinct chemical properties. These chemical compound are exhibited in Figure 2.3.

In order to examine whether the given ordered subset $W \subseteq V(H)$ is a resolving set for a graph H, we only have to authenticate the vertices of H which are not contained in W, as the vertices of H contained in W have distinct codes with respect to W. A helpful characteristic for calculating the metric basis of a graph H is presented in next lemma.

Lemma 2.2 (Ahmad et al. [33]). *Let H be a connected graph and W be a resolving set*

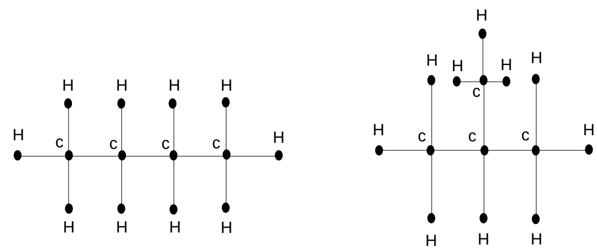

Figure 2.3: Butane and Isobutane molucule.

for H. Suppose $x, y \in V(H)$ if $d(x, w_1) = d(y, w_1)$ for all $w_1 \in V(H) \backslash \{x, y\}$, then $x \in W$ or $y \in W$.

2.2 Some known results on metric basis of graphs

The following two theorems characterize the metric dimension of path P_n and complete graph K_n of order n.

Theorem 2.3 (Chartrand et al. [7]). *A connected n vertex graph H has metric dimension 1 if and only if $H \cong P_n$.*

Theorem 2.4 (Chartrand et al. [7]). *A connected graph H containing n vertices has metric dimension $n-1$ if and only if $H \cong K_n$.*

Let H be a non trivial connected graph and also let a vertex $u \in V(H)$. Then it is common observation that $V(H)$ and $V(H) - \{u\}$ are resolving sets for G. This shows that if H is non-trivial connected graph containing n vertices, then $1 \leq dim(H) \leq n-1$. On the other hand, if H is an n vertex graph having diameter d, then we attain an improved extremal bounds for metric basis of H. For positive integer d, we define $g(n, d)$ to the smallest natural number l for which $l + d^l \geq n$.

Theorem 2.5 (Chartrand et al. [7]). *If H is a non-trivial connected graph containing n and having diameter d, then*

$$f(n,d) \le dim(H) \le n - d.$$

Theorem 2.6 (Sudhakara and Kumar [12]). *Let H be a simple connected graph having metric dimension 2 and let $W = \{u_1, u_2\} \subseteq V(H)$ is a metric basis for H, then the degree of each u_1 and u_2 is at most 3 and there exists a unique path connecting u_1 and u_2.*

Theorem 2.7 (Sudhakara and Kumar [12]). *A simple connected graph H having metric dimension 2 can not possess the following:*

(a). *K_5 as a subgraph.*

(b). *$K_5 - f$ as a subgraph, where $f \in E(H)$.*

(c). *$K_{3,3}$ as a subgraph.*

(d). *The Petersen graph as a subgraph.*

Furthermore, the least bound in Theorem 2.2, is achievable only for the graphs having diameter 2 or 3. Next theorem gives the sharp least bound for metric basis of a connected graph H in terms of $\Delta(H)$, and this bound can be improved.

Theorem 2.8 (Chartrand et al. [10]). *Let H be a connected graph containing n vertices, where $n \ge 2$, then*

$$\lceil log_3(\Delta(H)+1) \rceil \le dim(H) \le n - diam(H).$$

2.3 Strong metric dimension of graphs

In this portion, we discuss some fundamental concepts of strong metric basis of graphs and give few results about strong metric basis of few familiar families of graphs.

Sebö and Tannier [1] inaugurated the problem of strong metric dimension of graphs. The strong metric dimension problem can be interpreted as: a vertex w of graph H

strongly resolves two vertices x and y if x lies on a shortest $y - w$ path or y lies on a shortest $x - w$ path in H. A set $S = \{s_1, s_2, \ldots, s_k\} \subset V(H)$, is a strong resolving set for H if any two distinct vertices of H are strongly resolved by some vertex of S. If S contains minimum number of vertices of H, then S is referred as strong metric basis for H and cardinality of strong metric basis for H is said to be strong metric dimension of H represented by $sdim(H)$. Observe that if a vertex $s_1 \in S$ strongly resolves two vertices u_1 and v_1, then s_1 also resovles these vertices. For instance, if u_1 lies on a $v_1 - s_1$ path of smallest length, then $d(u_1, s_1) < d(v_1, s_1)$ and thus $d(u_1, s_1) \neq d(v_1, s_1)$. This shows that each strong resolving set is also resolving set, therefore, $dim(H) \leq sdim(H)$. In the sequel, we propose several definitions which will be helpful in computing the strong metric dimension of graphs. In any graph H, the set of vertices connected with a vertex w by an edge e is called neighborhood of w and is denoted by $N(w)$.

Definition 2.9. In a connected graph H, a pair of vertices $u_1, u_2 \in V(H)$ are said to be maximally distant if $d(u_1, v) \leq d(u_1, u_2)$ for each $v \in N(u_1)$ written as, $u_1\text{MD}u_2$.

A pair of vertices u_1 and u_2 of a graph H are mutually maximally distant if and only if u_1 is maximally distant from u_2 and vice versa, written as $u_1\text{MMD}u_2$.

Lemma 2.10 (Kratica et al. [19]). *If $W \subset V(H)$ is a strong resolving set for a graph H, then for each pair of MMD vertices $u_1, v_1 \in V(H)$, then essentially either $u_1 \in W$ or $v_1 \in W$.*

Lemma 2.11 (Kratica et al. [19]). *If $W \subset V(H)$ is a strong resolving set for a graph H, then for each pair of vertices $u_1, v_1 \in V(H)$ such that $d(u_1, v_1) = diam(H)$, then it is essential that either $u_1 \in W$ or $v_1 \in W$.*

2.4 Strong resolving graph

Graphs are fundamental combinatorial shapes and modification of these shapes plays an important role in the enlargement of mathematics. Remarkably, in graph theory, some primary amendments originate a new graph from initial, such as insertion or removal of a vertex or an edge, amalgamating and splitting of vertices, edge contraction, etc. Further advanced amendments produces a new graph from the actual one by composite changes, such as subdivision of a graph, cartesian product of graphs, graceful graph, complement of a graph, strong resolving graph, weighted graphs, etc. Infrequently, these modification of graphs appeared as a natural mechanism to resolve experimental complication. On contrary, the problem of computing a particular property of a graph has been transformed into the problem of computing another property of another graph acquired from the actual one. This is the case of the strong resolving graph G_{SR} of a connected graph G which was initiated by Oellermann and Peters-Fransen [28], as a mechanism to interpret the strong metric dimension of G. Predominantly, it was manifested that the problem of computing the strong metric dimension of G can be transmuted to the problem of computing the vertex covering number of G_{SR}.

Definition 2.12. A subset $W = \{u_1, u_2, \ldots, u_k\} \subseteq V(H)$ is called a vertex cover of graph H if each edge of H is incident to at least one vertex of W. A vertex cover of H with minimum cardinality over all vertex covers of H is called the vertex covering number of H and is represented by $\alpha(G)$.

A subset $Q \subseteq V(H)$ is called an independent set of a graph H such that whenever $u, v \in Q$ then $uv \notin E(H)$. The maximum cardinality of an independent set over all the independent sets of a graph H is called the independence number of H and is represented $\beta(H)$.

Definition 2.13. Let H be a connected graph. Then the strong resolving graph H_{SR} of H is a graph with vertex set $V(H_{SR}) = V(H)$ and $u_1u_2 \in E(H_{SR})$ if and only if u_1MMDu_2.

The following theorem tells us that the problem of finding the strong metric dimension of a graph G is equivalent to determining the $\alpha(G_SR)$.

Theorem 2.14 (Oellermann et al. [28])**.** *Let H be a connected graph, then*

$$sdim(H) = \alpha(H_{SR}).$$

The *sdim* of few well known families of graphs have been determined. We state them in next section.

2.5 Strong metric dimension of few well known families of graphs

Theorem 2.15 (Sebö et al. [1]). *Let H be a non-trivial connected graph of order n, then $sdim(H) = 1$ if and only if $H \cong P_n$.*

Theorem 2.16 (Sebö et al. [1]). *Let H be an n vertex connected graph, then $sdim(H) = n - 1$ if and only if $H \cong K_n$.*

Theorem 2.17 (Sebö et al. [1]). *Let C be a cycle containing n vertices, then $sdim(C) = \lceil \frac{n}{2} \rceil$.*

Theorem 2.18 (Sebö et al. [1]). *Let T be a tree having n vertices and l leaves, then $sdim(T) = l - 1$.*

Chapter 3

Strong metric dimension of generalized Petersen graph $GP(2m, m-1)$

In this chapter we compute the strong metric dimension of special class of graph so called generalized Petersen graph.

3.1 Introduction

In chapter 2 we discussed the vertex covering number and vertex independence number of graph G. In the next theorem we present the famous identity named as Gallai identity that gives us the relation between vertex covering number, vertex independence number and order of a graph H.

Theorem 3.1 (Chartrand and Zhang [9]). *For every graph H of order n containing no isolated vertices,*

$$\alpha(H) + \beta(H) = n.$$

The generalized Petersen graph represented as $GP(r,s)$, where $r \geq 3$ and $1 \leq s \leq \lfloor \frac{r-1}{2} \rfloor$, is a 3-regular graph having vertex set $\{u_1, u_2, \ldots, u_r, v_1, v_2, \ldots, v_r\}$ and edge set $\{u_i u_{i+1}, u_i v_i, v_i v_{i+s}, 1 \leq i \leq r\}$, where the indices larger than r will be taken as modulo r. The generalized Petersen graph was first defined by Watkins [22].

We find the strong metric dimension of generalized Petersen graph $GP(2m, m-1)$ having vertex set $V(GP(2m, m-1)) = \{u_1, v_1, u_2, v_2, \ldots, u_{2m}, v_{2m}\}$ and edge set $\{u_i u_{i+1}, u_i v_i, v_i v_{i+m-1}, 1 \leq i \leq 2m\}$, where the indices greater than $2m$ will be taken as modulo $2m$. For our convenience, we call $\{u_1, u_2, \ldots, u_{2m}\}$ as outer vertices and $\{v_1, v_2, \ldots, v_{2m}\}$ as inner vertices of $GP(2m, m-1)$.

3.2 Strong metric dimension of $GP(2m, m-1)$ when m is odd.

In this constituent, we calculate the strong metric dimension of $GP(2m, m-1)$ for all $m \geq 7$ and $m \equiv 1 \pmod 2$. Let $i, j \in \{1, 2, \ldots, 2m\}$. We define $\mathcal{F}_1$ and $\mathcal{F}_2$ as follows:

$$\mathcal{F}_1 = \left\{ k(m-1), k(m+1) \mid 0 \leq k \leq \frac{m-1}{2} \right\}, \tag{3.1}$$

$$\mathcal{F}_2 = \left\{ k(m-1)-1, k(m+1)+1 \mid 0 \leq k \leq \frac{m-1}{2} - 1 \right\}, \tag{3.2}$$

in such a way that if $l \in \mathcal{F}_1 \cup \mathcal{F}_2$, then $1 \leq l \leq 2m$, or l is modulo $2m$. By varying k from 0 to $\frac{m-1}{2}$, the set $\mathcal{F}_1$ contains all even integers from $1, \ldots, 2m$. Similarly, by varying k from 0 to $\frac{m-1}{2} - 1$, the set $\mathcal{F}_2$ contains all odd integers from $1, \ldots, 2m$ except m. This shows that $|j - i| \in \mathcal{F}_1 \cup \mathcal{F}_2 \cup \{m\}$.

Firstly, we compute the mutually maximally distant (hereafter, MMD) vertices in the generalized Petersen graph $GP(2m, m-1)$. By using MMD vertices, we construct the strong resolving graph $GP_{SR}(2m, m-1)$ of $GP(2m, m-1)$ and then we find its vertex covering number. By Theorem 2.4, the problem of calculating strong metric dimension of a graph is converted into determining the vertex covering number of a strong resolving graph.

Using structure of $GP(2m, m-1)$, the distances between vertices of $GP(2m, m-1)$ are given by:

(a). When $|j-i| \in \mathcal{F}_1$:

$$d(v_i, v_j) = k, \tag{3.3}$$

$$d(v_i, u_j) = k+1. \tag{3.4}$$

If $\frac{m-1}{2} \equiv 0 \pmod 2$ or $[\frac{m-1}{2} \equiv 1 \pmod 2$ and $k \neq \frac{m-1}{2}]$, then

$$d(u_i, u_j) = \begin{cases} k & k \equiv 0 (\text{mod } 2), \\ k+2 & k \equiv 1 (\text{mod } 2). \end{cases} \tag{3.5}$$

If $\frac{m-1}{2} \equiv 1 (\text{mod } 2)$ and $k = \frac{m-1}{2}$, then

$$d(u_i, u_j) = k+1. \tag{3.6}$$

(b). When $|j-i| \in \mathcal{F}_2$:

$$d(v_i, v_j) = k+3, \tag{3.7}$$

$$d(v_i, u_j) = k+2. \tag{3.8}$$

If $\frac{m-1}{2} \equiv 1 \pmod 2$ or $[\frac{m-1}{2} \equiv 0 \pmod 2$ and $k \neq \frac{m-1}{2} - 1]$, then

$$d(u_i, u_j) = \begin{cases} k+1 & k \equiv 0 (\text{mod } 2), \\ k+3 & k \equiv 1 (\text{mod } 2). \end{cases} \tag{3.9}$$

If $\frac{m-1}{2} \equiv 0 \pmod 2$ and $k = \frac{m-1}{2} - 1$, then

$$d(u_i, u_j) = k+2. \tag{3.10}$$

(c). When $|j-i|=m$:

$$d(v_i, v_{i+m}) = d(u_i, u_{i+m}) = 4,\ d(v_i, u_{i+m}) = 3. \tag{3.11}$$

We have the following equivalence classes to modulo $2m$ which will be used in computing the $sdim(GP(2m, m-1))$:

$$1 \equiv (0)(m+1)+1 \in \mathcal{F}_2, \tag{3.12}$$

$$-1 \equiv (0)(m-1)-1 \in \mathcal{F}_2, \tag{3.13}$$

If $\frac{m-1}{2} \equiv 0 \pmod 2$, then

$$\frac{m+1}{2}+1 \equiv \left(\frac{m-1}{2}-1\right)(m-1) \in \mathcal{F}_1, \tag{3.14}$$

$$m+\frac{m+1}{2}-2 \equiv \left(\frac{m-1}{2}-1\right)(m+1) \in \mathcal{F}_1, \tag{3.15}$$

$$m+1 \equiv \left(\frac{m-1}{2}-\frac{m-3}{2}\right)(m+1) \in \mathcal{F}_1, \tag{3.16}$$

$$m-1 \equiv \left(\frac{m-1}{2}-\frac{m-3}{2}\right)(m-1) \in \mathcal{F}_1, \tag{3.17}$$

$$\frac{m+1}{2}-1 \equiv \left(\frac{m-1}{2}\right)(m+1) \in \mathcal{F}_1, \tag{3.18}$$

$$m+\frac{m+1}{2} \equiv \left(\frac{m-1}{2}\right)(m-1) \in \mathcal{F}_1, \tag{3.19}$$

$$\frac{-m+1}{2}-2 \equiv \left(\frac{m-1}{2}-1\right)(m+1) \in \mathcal{F}_1, \tag{3.20}$$

$$\frac{-m+1}{2} \equiv \left(\frac{m-1}{2}\right)(m-1) \in \mathcal{F}_1, \tag{3.21}$$

$$\frac{m+1}{2} \equiv \left(\frac{m-1}{2} - 1\right)(m-1) - 1 \in \mathcal{F}_2, \tag{3.22}$$

$$m + \frac{m+1}{2} - 1 \equiv \left(\frac{m-1}{2} - 1\right)(m+1) + 1 \in \mathcal{F}_2, \tag{3.23}$$

$$\frac{m+1}{2} + 2 \equiv \left(\frac{m-1}{2} - 3\right)(m-1) - 1 \in \mathcal{F}_2, \tag{3.24}$$

$$m + \frac{m+1}{2} - 3 \equiv \left(\frac{m-1}{2} - 3\right)(m+1) + 1 \in \mathcal{F}_2, \tag{3.25}$$

$$\frac{m+1}{2} - 2 \equiv \left(\frac{m-1}{2} - 2\right)(m+1) + 1 \in \mathcal{F}_2, \tag{3.26}$$

$$m + \frac{m+1}{2} + 1 \equiv \left(\frac{m-1}{2} - 2\right)(m-1) - 1 \in \mathcal{F}_2, \tag{3.27}$$

$$\frac{-m+1}{2} + 1 \equiv \left(\frac{m-1}{2} - 2\right)(m-1) - 1 \in \mathcal{F}_2, \tag{3.28}$$

$$\frac{-m+1}{2} - 1 \equiv \left(\frac{m-1}{2} - 1\right)(m+1) + 1 \in \mathcal{F}_2. \tag{3.29}$$

If $\frac{m-1}{2} \equiv 1 \pmod 2$, then

$$\frac{m+1}{2} \equiv \left(\frac{m-1}{2}\right)(m-1) \in \mathcal{F}_1, \tag{3.30}$$

$$m + \frac{m+1}{2} - 1 \equiv \left(\frac{m+1}{2} - 1\right)(m+1) \in \mathcal{F}_1, \tag{3.31}$$

$$m + 1 \equiv \left(\frac{m+1}{2} - \frac{m-1}{2}\right)(m+1) \in \mathcal{F}_1, \tag{3.32}$$

$$m - 1 \equiv \left(\frac{m+1}{2} - \frac{m-1}{2}\right)(m-1) \in \mathcal{F}_1, \tag{3.33}$$

$$\frac{m+1}{2} + 2 \equiv \left(\frac{m-1}{2} - 2\right)(m-1) \in \mathcal{F}_1, \tag{3.34}$$

$$m + \frac{m+1}{2} - 3 \equiv \left(\frac{m-1}{2} - 2\right)(m+1) \in \mathcal{F}_1, \tag{3.35}$$

$$\frac{-m+1}{2} - 1 \equiv \left(\frac{m-1}{2}\right)(m+1) \in \mathcal{F}_1, \tag{3.36}$$

$$\frac{-m+1}{2} + 1 \equiv \left(\frac{m-1}{2} - 1\right)(m-1) \in \mathcal{F}_1, \tag{3.37}$$

$$\frac{m+1}{2} - 2 \equiv \left(\frac{m-1}{2} - 1\right)(m+1) \in \mathcal{F}_1, \tag{3.38}$$

$$\frac{m+1}{2} + 1 \equiv \left(\frac{m-1}{2} - 2\right)(m-1) - 1 \in \mathcal{F}_2, \tag{3.39}$$

$$m + \frac{m+1}{2} - 2 \equiv \left(\frac{m-1}{2} - 2\right)(m+1) + 1 \in \mathcal{F}_2, \tag{3.40}$$

$$m + \frac{m+1}{2} \equiv \left(\frac{m-1}{2} - 1\right)(m-1) - 1 \in \mathcal{F}_2, \tag{3.41}$$

$$m + \frac{m-1}{2} + 1 \equiv \left(\frac{m-1}{2} - 1\right)(m-1) - 1 \in \mathcal{F}_2, \tag{3.42}$$

$$\frac{m-1}{2} \equiv \left(\frac{m-1}{2} - 1\right)(m+1) + 1 \in \mathcal{F}_2, \tag{3.43}$$

$$\frac{-m-1}{2} - 1 \equiv \left(\frac{m-1}{2} - 2\right)(m+1) + 1 \in \mathcal{F}_2, \tag{3.44}$$

$$\frac{-m-1}{2} + 1 \equiv \left(\frac{m-1}{2} - 1\right)(m-1) - 1 \in \mathcal{F}_2, \tag{3.45}$$

$$m + \frac{m-1}{2} - 1 \equiv \left(\frac{m-1}{2} - 2\right)(m+1) + 1 \in \mathcal{F}_2, \tag{3.46}$$

$$\frac{m-1}{2} + 2 \equiv \left(\frac{m-1}{2} - 2\right)(m-1) - 1 \in \mathcal{F}_2, \tag{3.47}$$

$$\frac{m+1}{2} - 1 \equiv \left(\frac{m+1}{2} - 2\right)(m+1) + 1 \in \mathcal{F}_2. \tag{3.48}$$

Theorem 3.2. *Let $GP(2m, m-1)$ be the generalized Petersen graph, where $m \geq 7$ and $m \equiv 1 \pmod 2$. Then for each $i, j \in \{1 \ldots, 2m\}$, the following hold:*

(a). *u_iMMDu_j if and only if $|j-i| \in \{m, \frac{m+1}{2}, \frac{m+1}{2}+1, m+\frac{m+1}{2}-1, m+\frac{m+1}{2}-2\}$.*

(b). *Let $\frac{m-1}{2}$ is odd. Then v_iMMDu_j if and only if $|j-i| \in \{\frac{m+1}{2}, m+\frac{m+1}{2}-1\}$.*

(c). *Let $\frac{m-1}{2}$ is even. Then v_iMMDu_j if and only if $|j-i| \in \{\frac{m+1}{2}-1, m+\frac{m+1}{2}\}$.*

(d). *Let $\frac{m-1}{2}$ is odd. Then v_iMMDv_j if and only if $|j-i| \in \{m, \frac{m-1}{2}, m+\frac{m-1}{2}+1\}$.*

(e). *Let $\frac{m-1}{2}$ is even. Then v_iMMDv_j if and only if $|j-i| \in \{m, \frac{m+1}{2}, m+\frac{m+1}{2}-1\}$.*

Proof. Let $i, j \in \{1, 2, \ldots, 2m\}$. Without loss of generality, assume that $i \leq j$.

(a). We will show that u_iMMDu_j if and only if $j-i \in \{m, \frac{m+1}{2}, \frac{m+1}{2}+1, m+\frac{m+1}{2}-1, m+\frac{m+1}{2}-2\}$. On contrary, assume $j-i \notin \{m, \frac{m+1}{2}, \frac{m+1}{2}+1, m+\frac{m+1}{2}-1, m+\frac{m+1}{2}-2\}$.

Case 1. When $\frac{m-1}{2}$ is odd. From equations (3.30), (3.31), (3.39) and (3.40), it holds that

$$j-i \in \left\{\mathcal{F}_1 \setminus \left\{\frac{m+1}{2}, m+\frac{m+1}{2}-1\right\}\right\} \cup \left\{\mathcal{F}_2 \setminus \left\{\frac{m+1}{2}+1, m+\frac{m+1}{2}-2\right\}\right\}. \tag{3.49}$$

Subcase I. First assume that $j-i \in \mathcal{F}_1 \setminus \left\{\frac{m+1}{2}, m+\frac{m+1}{2}-1\right\}$. If k is even, then from equation (3.5) we get

$$d(u_i, u_j) = k.$$

Also, $v_i \in N(u_i)$ and from equation (3.4), we have

$$d(v_i, u_j) = k + 1 > d(u_i, u_j).$$

If k is odd, then by equation (3.5)

$$d(u_i, u_j) = k + 2.$$

Let $j-i = k(m-1)$. Note that $u_{i+1} \in N(u_i)$ and we have $j-(i+1) = k(m-1)-1 \in \mathcal{F}_2$. From equation (3.9), we obtain $d(u_{i+1}, u_j) = k + 3 > d(u_i, u_j)$. This shows that u_i and u_j are not MMD. If $j - i = k(m + 1)$, then we can write $j - (i - 1) = k(m + 1) + 1$, that is, $j - (i - 1) \in \mathcal{F}_2$. Note that $u_{i-1} \in N(u_i)$. From equation (3.9), we have $d(u_{i-1}, u_j) = k + 3 > d(u_i, u_j)$. This proves u_i and u_j are not MMD.

Subcase II. Next, assume that $j - i \in \mathcal{F}_2 \setminus \left\{\frac{m+1}{2} + 1, m + \frac{m+1}{2} - 2\right\}$. If k is even, then from equation (3.9), we obtain

$$d(u_i, u_j) = k + 1.$$

We know that $v_i \in N(u_i)$. Therefore equation (3.8) yields

$$d(v_i, u_j) = k + 2 > d(u_i, u_j).$$

If k is odd, then equation (3.9) tells us that

$$d(u_i, u_j) = k + 3.$$

First note that u_{i+1}, $u_{i-1} \in N(u_i)$. Let $j - i = k(m - 1) - 1$. Then $j - (i + 1) = k(m - 1) - 2 = k(m - 1) + 2(m - 1) = (k + 2)(m - 1) \in \mathcal{F}_1$. From equation (3.5), we have $d(u_{i+1}, u_j) = k + 4 > d(u_i, u_j)$. If $j - i = k(m + 1) + 1$, then $j - (i - 1) = k(m + 1) + 2 = k(m + 1) + 2(m + 1) = (k + 2)(m + 1) \in \mathcal{F}_1$. From equation (3.5), we get $d(u_{i-1}, u_j) = k + 4 > d(u_i, u_j)$. Thus u_i and u_j are not MMD.

Case 2. When $\frac{m-1}{2}$ is even. From equations (3.14), (3.15), (3.22) and (3.23), it holds that

$$j - i \in \left\{\mathcal{F}_1 \setminus \left\{\frac{m+1}{2} + 1, m + \frac{m+1}{2} - 2\right\}\right\} \cup \left\{\mathcal{F}_2 \setminus \left\{\frac{m+1}{2}, m + \frac{m+1}{2} - 1\right\}\right\}. \tag{3.50}$$

Subcase I. First assume that $j - i \in \mathcal{F}_1 \setminus \left\{\frac{m+1}{2} + 1, m + \frac{m+1}{2} - 2\right\}$. If k is even, then from equation (3.5), it is clear that

$$d(u_i, u_j) = k.$$

We know that $v_i \in N(u_i)$. Hence from equation (3.4), we obtain

$$d(v_i, u_j) = k + 1 > d(u_i, u_j).$$

If k is odd, then equation (3.5) yields

$$d(u_i, u_j) = k + 2.$$

Let $j - i = k(m-1)$. Then $j-(i+1) = k(m-1)-1$, that is, $j-(i+1) \in \mathcal{F}_2$. We know that $u_{i+1} \in N(u_i)$. By using equation (3.9), we get $d(u_{i+1}, u_j) = k + 3 > d(u_i, u_j)$. Note that $u_{i-1} \in N(u_i)$. Therefore if $j - i = k(m+1)$, then we have $j - (i-1) = k(m+1)+1 \in \mathcal{F}_2$. Again using equation (3.9), we note that $d(u_{i-1}, u_j) = k + 3 > d(u_i, u_j)$. This shows u_i and u_j are not MMD.

Subcase II. Assume that $j - i \in \mathcal{F}_2 \setminus \left\{\frac{m+1}{2}, m + \frac{m+1}{2} - 1\right\}$. If k is even, then from equation (3.9), we have

$$d(u_i, u_j) = k + 1.$$

Also $v_i \in N(u_i)$ and from equation (3.8), we get

$$d(v_i, u_j) = k + 2 > d(u_i, u_j).$$

If k is odd, then equation (3.9) tells us that

$$d(u_i, u_j) = k + 3.$$

Note that u_{i+1}, $u_{i-1} \in N(u_i)$. Let $j - i = k(m-1)-1$. Then we can write $j-(i+1) = k(m-1) - 2 = (k+2)(m-1) \in \mathcal{F}_1$. From equation (3.5), we have $d(u_{i+1}, u_j) = k + 4 > d(u_i, u_j)$. If $j - i = k(m+1) + 1$, then $j - (i-1) = k(m+1) + 2 = (k+2)(m+1) \in \mathcal{F}_1$. Again from equation (3.5), we have $d(u_{i-1}, u_j) = k + 4 > d(u_i, u_j)$. From above we conclude that u_i and u_j are not MMD.

Now we prove the converse. As $GP(2m, m-1)$ is 3-regular graph, so for proving u_iMMDu_j we show that for each $w_i \in N(u_i)$ the distance $d(w_i, u_j) \leq d(u_i, u_j)$ and vice versa.

Case 1. When $\frac{m-1}{2}$ is odd. Let $j - i = m$. We know that

$$N(u_i) = \{v_i, u_{i+1}, u_{i-1}\},$$
$$N(u_{i+m}) = \{v_{i+m}, u_{i+m+1}, u_{i+m-1}\}.$$

From equation (3.11), we have

$$d(u_i, u_j) = 4.$$

From equations (3.32), (3.33), (3.5) and (3.11), we get

$d(u_{i-1}, u_j) = d(u_{i+1}, u_j) = d(v_i, u_j) = d(u_i, u_{j+1}) = d(u_i, u_{j-1}) = d(u_i, v_j) = 3.$

Thus u_iMMDu_{i+m}.

Now let $j - i = \frac{m+1}{2}$. We know that

$$N(u_i) = \{v_i, u_{i+1}, u_{i-1}\},$$
$$N(u_{i+\frac{m+1}{2}}) = \{v_{i+\frac{m+1}{2}}, u_{i+\frac{m+1}{2}+1}, u_{i+\frac{m+1}{2}-1}\}.$$

Equations (3.30) and (3.6) yields

$$d(u_i, u_j) = \tfrac{m+1}{2}.$$

We first show u_iMDu_j.

(1). Note that $j-(i-1) = \frac{m+1}{2}+1$. From equations (3.39) and (3.9), $d(u_{i-1}, u_j) = \frac{m+1}{2}$.

(2). We have $j-(i+1) = \frac{m+1}{2}-1$. From equations (3.48) and (3.9), $d(u_{i+1}, u_j) = \frac{m-1}{2}$.

(3). From equations (3.30) and (3.4), $d(v_i, u_j) = \frac{m+1}{2}$.

Next we show that u_jMDu_i.

(1). We have $(j+1)-i = \frac{m+1}{2}+1$. From equations (3.39) and (3.9), $d(u_i, u_{j+1}) = \frac{m+1}{2}$.

(2). Also $(j-1)-i = \frac{m+1}{2}-1$. From equations (3.48) and (3.9), $d(u_i, u_{j-1}) = \frac{m-1}{2}$.

(3). From equations (3.30) and (3.4), $d(u_i, v_j) = \frac{m+1}{2}$.

From above, u_i and u_j are MMD.

Now let $j - i = \frac{m+1}{2} + 1$. We know that

$$N(u_i) = \{v_i, u_{i+1}, u_{i-1}\},$$
$$N(u_{i+\frac{m+1}{2}+1}) = \{v_{i+\frac{m+1}{2}+1}, u_{i+\frac{m+1}{2}+2}, u_{i+\frac{m+1}{2}}\}.$$

From equations (3.39) and (3.9), we obtain

$$d(u_i, u_j) = \tfrac{m+1}{2}.$$

First we prove u_iMDu_j.

(1). Note that $j-(i-1) = \frac{m+1}{2}+2$. Using equations (3.34) and (3.5), we get $d(u_{i-1}, j) = \frac{m-1}{2}$.

(2). We have $j-(i+1) = \frac{m+1}{2}$. By equations (3.30) and (3.6), we obtain $d(u_{i+1}, u_j) = \frac{m+1}{2}$.

(3). From equations (3.39) and (3.8), $d(v_i, u_j) = \frac{m-1}{2}$.

We next prove that u_jMDu_i.

(1). Since $(j+1)-i = \frac{m+1}{2}+2$. From equations (3.34) and (3.5), $d(u_i, u_{j+1}) = \frac{m-1}{2}$.

(2). We have $(j-1)-i = \frac{m+1}{2}$. From equations (3.30) and (3.6), $d(u_i, u_{j-1}) = \frac{m+1}{2}$.

(3). By using equations (3.39) and (3.8), $d(u_i, v_j) = \frac{m-1}{2}$.

Thus u_i and u_j are MMD.

Now assume that $j-i = m+\frac{m+1}{2}-1$. We know that

$$N(u_i) = \{v_i, u_{i+1}, u_{i-1}\},$$
$$N(u_{i+m+\frac{m+1}{2}-1}) = \{v_{i+m+\frac{m+1}{2}-1}, u_{i+m+\frac{m+1}{2}-2}, u_{i+m+\frac{m+1}{2}}\}.$$

Equations (3.31) and (3.6) tells us that

$$d(u_i, u_j) = \tfrac{m+1}{2}.$$

We first prove that u_iMDu_j.

(1). We have $j-(i-1) = m+\frac{m+1}{2}$. From equations (3.41) and (3.9), $d(u_{i-1}, u_j) = \frac{m-1}{2}$.

(2). Also $j-(i+1) = m+\frac{m+1}{2}-2$. From equations (3.40) and (3.9), $d(u_{i+1}, u_j) = \frac{m+1}{2}$.

(3). By using equations (3.31) and (3.4), we obtain $d(v_i, u_j) = \frac{m+1}{2}$.

Now we prove that u_jMDu_i.

(1). Note that $(j+1)-i = m+\frac{m+1}{2}$. From equations (3.41) and (3.9), $d(u_i, u_{j+1}) = \frac{m-1}{2}$.

(2). We have $(j-1)-i = m+\frac{m+1}{2}-2$. From equations (3.40) and (3.9), $d(u_i, u_{j-1}) = \frac{m+1}{2}$.

(3). From equations (3.31) and (3.4), $d(u_i, v_j) = \frac{m+1}{2}$.

This shows u_i and u_j are MMD.

Suppose $j-i = m+\frac{m+1}{2}-2$. We know that

$$N(u_i) = \{v_i, u_{i+1}, u_{i-1}\},$$
$$N(u_{i+m+\frac{m+1}{2}-2}) = \{v_{i+m+\frac{m+1}{2}-2}, u_{i+m+\frac{m+1}{2}-3}, u_{i+m+\frac{m+1}{2}-1}\}.$$

From equations (3.40) and (3.9), we get

$$d(u_i, u_j) = \frac{m+1}{2}.$$

We first prove that $u_i\text{MD}u_j$.

(1). We have $j-(i-1) = m+\frac{m+1}{2}-1$. From equation (3.31) and (3.6), $d(u_{i-1}, u_j) = \frac{m+1}{2}$.

(2). Note that $j-(i+1) = m+\frac{m+1}{2}-3$. By using equations (3.35) and (3.5), we get $d(u_{i+1}, u_j) = \frac{m-1}{2}$.

(3). From equations (3.40) and (3.8), $d(v_i, u_j) = \frac{m-1}{2}$.

Now we show that $u_j\text{MD}u_i$.

(1). We have $(j+1)-i = m+\frac{m+1}{2}-1$. From equations (3.31) and (3.6), $d(u_i, u_{j+1}) = \frac{m+1}{2}$.

(2). Note that $(j-1)-i = m+\frac{m+1}{2}-3$. By using equations (3.35) and (3.5), we have $d(u_i, u_{j-1}) = \frac{m-1}{2}$.

(3). From equations (3.40) and (3.8), $d(u_i, v_j) = \frac{m-1}{2}$.

Thus u_i and u_j are MMD.

Case 2. When $\frac{m-1}{2}$ is even.

Assume $j-i = m$. We know that

$$N(u_i) = \{v_i, u_{i+1}, u_{i-1}\},$$
$$N(u_{i+m}) = \{v_{i+m}, u_{i+m+1}, u_{i+m-1}\}.$$

From equation (3.11), we have

$$d(u_i, u_j) = 4.$$

We first show that u_iMDu_j.

(1). Note that $j - (i-1) = m+1$. From equations (3.16) and (3.5), $d(u_{i-1}, u_j) = 3$.

(2). Also $j - (i+1) = m-1$. From equations (3.17) and (3.5), $d(u_{i+1}, u_j) = 3$.

(3). From equation (3.11), $d(v_i, u_j) = 3$.

Now we show that u_jMDu_i.

(1). We have $(j+1) - i = m+1$. From equations (3.16) and (3.5), $d(u_i, u_{j+1}) = 3$.

(2). Note that $(j-1) - i = m-1$. From equations (3.17) and (3.5), $d(u_i, u_{j-1}) = 3$.

(3). From equation (3.11), $d(u_i, v_j) = 3$.

Thus u_i and u_{i+m} are MMD.

Now assume $j - i = \frac{m+1}{2}$. We know that

$$N(u_i) = \{v_i, u_{i+1}, u_{i-1}\},$$
$$N(u_{i+\frac{m+1}{2}}) = \{v_{i+\frac{m+1}{2}}, u_{i+\frac{m+1}{2}+1}, u_{i+\frac{m+1}{2}-1}\}.$$

From equations (3.22) and (3.10), we obtain

$$d(u_i, u_j) = \tfrac{m+1}{2}.$$

We first show u_iMDu_j.

(1). As $j-(i-1) = \frac{m+1}{2}+1$, therefore equations (3.14) and (3.5), gives us $d(u_{i-1}, u_j) = \frac{m+1}{2}$.

(2). Using equations (3.18) and (3.5), we have $d(u_{i+1}, u_j) = \frac{m-1}{2}$ for $j-(i+1) = \frac{m+1}{2}-1$.

(3). Using equations (3.22) and (3.8), we obtain $d(v_i, u_j) = \frac{m+1}{2}$.

Now we show that u_jMDu_i. We consider three cases,

(1). We have $(j+1)-i = \frac{m+1}{2}+1$. From equations (3.14) and (3.5), $d(u_i, u_{j+1}) = \frac{m+1}{2}$.

(2). Note that $(j-1)-i = \frac{m+1}{2}-1$. From equations (3.18) and (3.5), $d(u_i, u_{j-1}) = \frac{m-1}{2}$.

(3). Using equations (3.22) and (3.8), we obtain $d(u_i, v_j) = \frac{m+1}{2}$.

This shows u_iMMDu_j.

Suppose $j - i = \frac{m+1}{2} + 1$. We know that

$$N(u_i) = \{v_i, u_{i+1}, u_{i-1}\},$$
$$N(u_{i+\frac{m+1}{2}+1}) = \{v_{i+\frac{m+1}{2}+1}, u_{i+\frac{m+1}{2}+2}, u_{i+\frac{m+1}{2}}\}.$$

From equations (3.39) and (3.5), we obtain

$$d(u_i, u_j) = \tfrac{m+1}{2}.$$

We first prove u_i is maximally distant from u_j.

(1). As $j - (i-1) = \frac{m+1}{2} + 2$, therefore from equations (3.34) and (3.9), we have $d(u_{i-1}, u_j) = \frac{m-1}{2}$.

(2). Note that $j - (i+1) = \frac{m+1}{2}$. From equations (3.30) and (3.10), $d(u_{i+1}, u_j) = \frac{m+1}{2}$.

(3). By equations (3.39) and (3.4), we get $d(v_i, u_j) = \frac{m-1}{2}$.

Next we show that u_j is maximally distant from u_i.

(1). For $(j+1) - i = \frac{m+1}{2} + 2$. We get $d(u_i, u_{j+1}) = \frac{m-1}{2}$ by using equations (3.34) and (3.9).

(2). Since $(j-1) - i = \frac{m+1}{2}$, so from equations (3.30) and (3.10), $d(u_i, u_{j-1}) = \frac{m+1}{2}$.

(3). Equations (3.39) and (3.4) yields that $d(u_i, v_j) = (\frac{m-1}{2})$.

This shows u_i and u_j are MMD.

Let $j - i = m + \frac{m+1}{2} - 1$. We have

$$N(u_i) = \{v_i, u_{i+1}, u_{i-1}\},$$
$$N(u_{i+m+\frac{m+1}{2}-1}) = \{v_{i+m+\frac{m+1}{2}-1}, u_{i+m+\frac{m+1}{2}-2}, u_{i+m+\frac{m+1}{2}}\}.$$

From equations (3.23) and (3.10), we get

$$d(u_i, u_j) = \tfrac{m+1}{2}.$$

We first prove that $u_i \mathrm{MD} u_j$.

(1). As $j - (i-1) = m + \frac{m+1}{2}$, therefore from equations (3.19) and (3.5), we obtain $d(u_{i-1}, u_j) = \frac{m-1}{2}$.

(2). Note that $j - (i+1) = m + \frac{m+1}{2} - 2$. From equations (3.15) and (3.5), we get $d(u_{i+1}, u_j) = \frac{m+1}{2}$.

(3). Using equations (3.23) and (3.8), we obtain $d(v_i, u_j) = \frac{m+1}{2}$.

Next we show that u_j MD u_i.

(1). We have $(j+1) - i = m + \frac{m+1}{2}$. Using equations (3.19) and (3.5), we get $d(u_i, u_{j+1}) = \frac{m-1}{2}$.

(2). For $(j-1) - i = m + \frac{m+1}{2} - 2$, equations (3.15) and (3.5) yields that $d(u_i, u_{j-1}) = \frac{m+1}{2}$.

(3). From equations (3.23) and (3.8), $d(u_i, v_j) = \frac{m+1}{2}$.

Thus u_i and u_j are MMD.

Assume $j - i = m + \frac{m+1}{2} - 2$. We know that

$$N(u_i) = \{v_i, u_{i+1}, u_{i-1}\},$$
$$N(u_{i+m+\frac{m+1}{2}-2}) = \{v_{i+m+\frac{m+1}{2}-2}, u_{i+m+\frac{m+1}{2}-3}, u_{i+m+\frac{m+1}{2}-1}\}.$$

By using equations (3.15) and (3.5), we obtain

$$d(u_i, u_j) = \frac{m+1}{2}.$$

First we show that u_iMDu_j.

(1). We have $j-(i-1) = m + \frac{m+1}{2} - 1$. From equations (3.23) and (3.10), $d(u_{i-1}, u_j) = \frac{m+1}{2}$.

(2). Note that $j-(i+1) = m + \frac{m+1}{2} - 3$. From equations (3.25) and (3.9), $d(u_{i+1}, u_j) = \frac{m-1}{2}$.

(3). Using equations (3.15) and (3.4), we have $d(v_i, u_j) = \frac{m-1}{2}$.

Next we show that u_jMDu_i.

(1). We have $(j+1)-i = m + \frac{m+1}{2} - 1$. From equations (3.23) and (3.10), $d(u_i, u_{j+1}) = \frac{m+1}{2}$.

(2). Note that $(j-1)-i = m + \frac{m+1}{2} - 3$. From equations (3.25) and (3.9), $d(u_i, u_{j-1}) = \frac{m-1}{2}$.

(3). By using equations (3.15) and (3.4), we obtain $d(u_i, v_j) = \frac{m-1}{2}$.

Thus u_i MMD u_j.

(b). Let $\frac{m-1}{2}$ is odd. Then we will prove that $v_i \text{MMD} u_j$ if and only if $j-i \in \{\frac{m+1}{2}, m+\frac{m+1}{2}-1\}$.

On contrary, assume that $j-i \notin \{\frac{m+1}{2}, m+\frac{m+1}{2}-1\}$. Since $\frac{m-1}{2}$ is odd, therefore equations (3.30) and (3.31) tells us that

$$j-i \in \left\{\mathcal{F}_1 \setminus \left\{\frac{m+1}{2}, m+\frac{m+1}{2}-1\right\}\right\} \cup \{\mathcal{F}_2\} \cup \{m\}. \tag{3.51}$$

Subcase I. First assume that $j-i \in \mathcal{F}_1 \setminus \left\{\frac{m+1}{2}, m+\frac{m+1}{2}-1\right\}$. Then from equation (3.4), we get

$$d(v_i, u_j) = k+1.$$

When k is even. We know that u_{j-1} and $u_{j+1} \in N(u_j)$. Suppose $j-i = k(m-1)$. We can write $(j-1)-i = k(m-1)-1 \in \mathcal{F}_2$. From equations (3.8), $d(v_i, u_{j-1}) = k+2 > d(v_i, u_j)$. Let $j-i = k(m+1)$. Then $(j+1)-i = k(m+1)+1 \in \mathcal{F}_2$. From equation (3.8) $d(v_i, u_{j+1}) = k+2 > d(v_i, u_j)$.

Next, suppose k is odd. Also $u_i \in N(v_i)$. Then from equation (3.5), we have

$$d(u_i, u_j) = k+2 > d(v_i, u_j).$$

Thus v_i and u_j are not MMD.

Subcase II. Now suppose that $j-i \in \mathcal{F}_2$. Then equation (3.8) yields

$$d(v_i, u_j) = k+2.$$

When k is even. Let $j-i = k(m-1)-1$. Then $(j-1)-i = k(m-1)-2 = (k+2)(m-1) \in \mathcal{F}_1$. Also $u_{j-1} \in N(u_j)$. From equation (3.4), we get $d(v_i, u_{j-1}) = k+3 > d(v_i, u_j)$. If $j-i = k(m+1)+1$, then we know that $u_{j+1} \in N(u_j)$. Also $(j+1)-i = k(m+1)+2 = (k+2)(m+1) \in \mathcal{F}_1$. From equation (3.4), $d(v_i, u_{j+1}) = k+3 > d(v_i, u_j)$.

Next, suppose k is odd and we know that $u_i \in N(v_i)$. From equation (3.9), we get

$$d(u_i, u_j) = k+3 > d(v_i, u_j).$$

Summing up, we conclude that v_i and u_j are not MMD.

Subcase III.

Let $j - i \in m$. Then from equation (3.11), we have

$$d(v_i, u_{i+m}) = 3.$$

We know that $u_i \in N(v_i)$. Then again from equation (3.11), we obtain

$$d(u_i, u_{i+m}) = 4.$$

This shows v_i and u_j are not MMD.

Now we prove the converse.

Let $j - i = \frac{m+1}{2}$. We know that

$$N(v_i) = \{u_i, v_{i+m+1}, v_{i+m-1}\},$$
$$N(u_{i+\frac{m+1}{2}}) = \{v_{i+\frac{m+1}{2}}, u_{i+\frac{m+1}{2}+1}, u_{i+\frac{m+1}{2}-1}\}.$$

From equations (3.30) and (3.4), we obtain

$$d(v_i, u_j) = \frac{m+1}{2}.$$

We first show that v_iMDu_j.

(1). Since $j - (i + m + 1) = \frac{-m+1}{2} - 1$, therefore using equations (3.36) and (3.4), we get $d(v_{i+m+1}, u_j) = \frac{m+1}{2}$.

(2). Note that $j - (i + m - 1) = \frac{-m+1}{2} + 1$. From equations (3.37) and (3.4), $d(v_{i+m-1}, u_j) = \frac{m-1}{2}$.

(3). By equations (3.30) and (3.6), $d(u_i, u_j) = \frac{m+1}{2}$.

Next we show that u_jMDv_i.

(1). We have $(j+1)-i = \frac{m+1}{2}+1$. From equations (3.39) and (3.8), $d(v_i, u_{j+1}) = \frac{m-1}{2}$.

(2). Note that $(j-1)-i = \frac{m+1}{2}-1$. From equations (3.48) and (3.8), $d(v_i, u_{j-1}) = \frac{m+1}{2}$.

(3). Equations (3.30) and (3.3) yields $d(v_i, v_j) = \frac{m-1}{2}$.

This proves v_i and u_j are MMD.

Suppose $j - i = m + \frac{m+1}{2} - 1$. We know that

$$N(v_i) = \{u_i, v_{i+m+1}, v_{i+m-1}\},$$
$$N(u_{i+m+\frac{m+1}{2}-1}) = \{v_{i+m+\frac{m+1}{2}-1}, u_{i+m+\frac{m+1}{2}-2}, u_{i+m+\frac{m+1}{2}}\}.$$

From equation (3.31) and (3.4), we get

$$d(v_i, u_j) = \tfrac{m+1}{2}.$$

We will show that $v_i\text{MD}u_j$.

(1). From equations (3.38) and (3.4), $d(v_{i+m+1}, u_j) = \frac{m-1}{2}$ for $j-(i+m+1) = \frac{m+1}{2}-2$.

(2). For $j-(i+m-1) = \frac{m+1}{2}$, we get $d(v_{i+m-1}, u_j) = \frac{m+1}{2}$ by using equations (3.30) and (3.4).

(3). By using equations (3.31) and (3.6), $d(u_i, u_j) = \frac{m+1}{2}$.

Next, we prove $u_j\text{MD}v_i$.

(1). We have $(j+1)-i = m+\frac{m+1}{2}$. From equations (3.41) and (3.8), $d(v_i, u_{j+1}) = \frac{m+1}{2}$.

(2). Note that $(j-1)-i = m+\frac{m+1}{2}-2$. From equations (3.40) and (3.8), $d(v_i, u_{j-1}) = \frac{m-1}{2}$.

(3). By using equations (3.31) and (3.3), $d(v_i, v_j) = \frac{m-1}{2}$.

This proves $v_i\text{MMD}u_j$.

(c). Let $\frac{m-1}{2}$ is even. Then we will prove that $v_i\text{MMD}u_j$ if and only if $j-i \in \{\frac{m+1}{2}-1, m+\frac{m+1}{2}\}$. On contrary, assume that $j-i \notin \{\frac{m+1}{2}-1, m+\frac{m+1}{2}\}$.

Since $\frac{m-1}{2}$ is even, therefore from equations (3.18) and (3.19), it holds that

$$j-i \in \left\{\mathcal{F}_1 \setminus \left\{\frac{m+1}{2}-1, m+\frac{m+1}{2}\right\}\right\} \cup \{\mathcal{F}_2\} \cup \{m\}. \tag{3.52}$$

Subcase I. Suppose that $j-i \in \mathcal{F}_1 \setminus \{\frac{m+1}{2}-1, m+\frac{m+1}{2}\}$. Then from equation (3.4), we have

$$d(v_i, u_j) = k+1.$$

Let k is even. Note that u_{j-1} and $u_{j+1} \in N(u_j)$. Let $j-i = k(m-1)$. We can write $(j-1)-i = k(m-1)-1 \in \mathcal{F}_2$. From equation (3.8), we obtain $d(v_i, u_{j-1}) = k+2 >$

$d(v_i, u_j)$. If $j-i = k(m+1)$, then we have $(j+1)-i = k(m+1)+1 \in \mathcal{F}_2$. From equation (3.8), we get $d(v_i, u_{j+1}) = k+2 > d(v_i, u_j)$.

Now suppose k is odd and $u_i \in N(v_i)$. From equation (3.5), we get

$$d(u_i, u_j) = k + 2 > d(v_i, u_j).$$

This shows v_i and u_j are not MMD.

Subcase II.

Suppose that $j - i \in \mathcal{F}_2$. Equation (3.8) tells us

$$d(v_i, u_j) = k + 2.$$

Suppose k is even. Let $j - i = k(m-1) - 1$. Note that $u_{j-1} \in N(u_j)$. We have $(j-1) - i = k(m-1) - 2 = (k+2)(m-1) \in \mathcal{F}_1$. From equation (3.4), we obtain $d(v_i, u_{j-1}) = k+3 > d(v_i, u_j)$. Assume $j - i = k(m+1)+1$. Then we have $(j+1)-i = k(m+1)+2 = (k+2)(m+1) \in \mathcal{F}_1$. As we know that $u_{j+1} \in N(u_j)$, therefore from equation (3.4), $d(v_i, u_{j+1}) = k+3 > d(v_i, u_j)$.

Next, suppose k is odd. We know that $u_i \in N(v_i)$. From equation (3.9), we obtain

$$d(u_i, u_j) = k + 3 > d(v_i, u_j).$$

This shows v_i and u_j are not MMD.

Subcase III.

Assume that $j - i \in m$. From equation (3.11), we have

$$d(v_i, u_{i+m}) = 3.$$

We know that $u_i \in N(v_i)$. By using equation (3.11), we get

$$d(u_i, u_{i+m}) = 4 > d(v_i, u_{i+m}).$$

Thus v_i and u_j are not MMD.

Now we prove the converse.

Suppose $j - i = \frac{m+1}{2} - 1$. We know that

$$N(v_i) = \{u_i, v_{i+m+1}, v_{i+m-1}\},$$

$$N(u_{i+\frac{m+1}{2}-1}) = \{v_{i+\frac{m+1}{2}-1}, u_{i+\frac{m+1}{2}}, u_{i+\frac{m+1}{2}-2}\}.$$

From equations (3.18) and (3.4), it holds that

$$d(v_i, u_j) = \tfrac{m+1}{2}.$$

We show v_iMDu_j.

(1). We have $j-(i+m+1) = \frac{-m+1}{2}-2$. From equations (3.20) and (3.4), $d(v_{i+m+1}, u_j) = \frac{m-1}{2}$.

(2). Note that $j-(i+m-1) = \frac{-m+1}{2}$. From equations (3.21) and (3.4), $d(v_{i+m-1}, u_j) = \frac{m+1}{2}$.

(3). By using equations (3.18) and (3.5), we obtain $d(u_i, u_j) = \frac{m-1}{2}$.

Next, we prove that u_jMDv_i.

(1). As $(j+1)-i = \frac{m+1}{2}$, therefore using equations (3.22) and (3.8), we have $d(v_i, u_{j+1}) = \frac{m+1}{2}$.

(2). Note that $(j-1)-i = \frac{m+1}{2}-2$. From equations (3.26) and (3.8), $d(v_i, u_{j-1}) = \frac{m-1}{2}$.

(3). Equations (3.18) and (3.3), tells us that $d(v_i, v_j) = \frac{m-1}{2}$.

Thus v_iMMDu_j.

Now suppose that $j - i = m + \frac{m+1}{2}$. We know that

$$N(v_i) = \{u_i, v_{i+m+1}, v_{i+m-1}\},$$

$$N(u_{i+m+\frac{m+1}{2}}) = \{v_{i+m+\frac{m+1}{2}}, u_{i+m+\frac{m+1}{2}-1}, u_{i+m+\frac{m+1}{2}+1}\}.$$

Equations (3.19) and (3.4), yields

$$d(v_i, u_j) = \tfrac{m+1}{2}.$$

We show that v_iMDu_j.

(1). We have $j-(i+m+1) = \frac{m+1}{2}-1$. From equations (3.18) and (3.4), $d(v_{i+m+1}, u_j) = \frac{m+1}{2}$.

(2). Note that $j-(i+m-1) = \frac{m+1}{2}+1$. From equations (3.14) and (3.4), $d(v_{i+m-1}, u_j) = \frac{m-1}{2}$.

(3). By using equations (3.19) and (3.5), we get $d(u_i, u_j) = \frac{m-1}{2}$.

Now we prove that u_j MD v_i.

(1). We have $(j+1)-i = m+\frac{m+1}{2}+1$. From equations (3.27) and (3.8), $d(v_i, u_{j+1}) = \frac{m-1}{2}$.

(2). Note that $(j-1)-i = m+\frac{m+1}{2}-1$. From equations (3.23) and (3.8), $d(v_i, u_{j-1}) = \frac{m+1}{2}$.

(3). By using equations (3.19) and (3.3), $d(v_i, v_j) = \frac{m-1}{2}$.

This proves v_i and u_j are MMD.

(d). Let $\frac{m-1}{2}$ is odd. Then we will prove that v_iMMDv_j if and only if $j-i \in \{m, \frac{m-1}{2}, m+\frac{m-1}{2}+1\}$.

On contrary, suppose that $j-i \notin \{m, \frac{m-1}{2}, m+\frac{m-1}{2}+1\}$. Since $\frac{m-1}{2}$ is odd, therefore from equations (3.43) and (3.42), it holds that

$$j-i \in \mathcal{F}_1 \cup \left\{\mathcal{F}_2 \setminus \left\{\frac{m-1}{2}, m+\frac{m-1}{2}+1\right\}\right\}. \tag{3.53}$$

Subcase I. First assume that $j-i \in \mathcal{F}_1$. Then from equation (3.3), we have

$$d(v_i, v_j) = k.$$

We know that $u_j \in N(v_j)$. From equation (3.4), we get

$$d(v_i, u_j) = k+1 > d(v_i, v_j).$$

Hence v_i and v_j are not MMD.

Subcase II.

Next, assume that $j-i \in \mathcal{F}_2 \setminus \left\{\frac{m-1}{2}, m+\frac{m-1}{2}+1\right\}$. Then from equation (3.7), we obtain

$$d(v_i, v_j) = k+3.$$

Note that $v_{j+m+1}, v_{j+m-1} \in N(v_j)$. Let $j-i = k(m+1)+1$. We can write $(j+m+1)-i = k(m+1)+1+m+1 = (k+1)(m+1)+1 \in \mathcal{F}_2$. From equation (3.7), we have $d(v_{j+m+1}, v_i) = k+4 > d(v_i, v_j)$. If $j-i = k(m-1)-1$, then we have

$(j+m-1)-i = k(m-1)-1+m-1 = (k+1)(m-1)-1 \in \mathcal{F}_2$. By equation (3.7), $d(v_{j+m-1}, v_i) = k+4 > d(v_i, v_j)$.

This shows that v_i and v_j are not MMD.

Now we prove the converse. Let $j-i=m$. We know that

$$N(v_i) = \{u_i, v_{i+m+1}, v_{i+m-1)}\},$$
$$N(v_{i+m}) = \{v_{i+1}, v_{i-1}, u_{i+m}\}.$$

From equation (3.11), we get

$$d(v_i, v_j) = 4.$$

We first show that $v_i \mathrm{MD} v_j$.

(1). We have $j-(i+m-1)=1$. From equations (3.12) and (3.7), $d(v_{i+m-1}, v_j) = 3$.

(2). Note that $j-(i+m+1) = -1$. From equations (3.13) and (3.7), $d(v_{i+m+1}, v_j) = 3$.

(3). From equation (3.11), $d(u_i, v_j) = 3$.

Next we prove that $v_j \mathrm{MD} v_i$.

(1). We have $(j-m+1)-i = 1$. From equations (3.12) and (3.7), $d(v_i, v_{i+1}) = 3$.

(2). Note $(j-m-1)-i = -1$. From equations (3.13) and (3.7), $d(v_i, v_{i-1}) = 3$.

(3). From equation (3.11), $d(v_i, u_j) = 3$.

This shows that v_i and v_j are MMD.

Suppose $j-i = \frac{m-1}{2}$. We know that

$$N(v_i) = \{u_i, v_{i+m+1}, v_{i+m-1}\},$$
$$N(v_{i+\frac{m-1}{2}}) = \{v_{i+\frac{m-1}{2}+m+1}, v_{i+\frac{m-1}{2}+m-1}, u_{i+\frac{m-1}{2}}\}.$$

From equations (3.43) and (3.7), we obtain

$$d(v_i, v_j) = \tfrac{m+3}{2}.$$

We first show that v_i MD v_j.

(1). As $j-(i+m+1) = \frac{-m-1}{2} - 1$, therefore equations (3.44) and (3.7) tells us that $d(v_{i+m+1}, v_j) = \frac{m+1}{2}$.

(2). From equations (3.45) and (3.7), we have $d(v_{i+m-1}, v_j) = \frac{m+3}{2}$ for $j-(i+m-1) = \frac{-m-1}{2} + 1$.

(3). Using equations (3.43) and (3.8), we get $d(u_i, v_j) = \frac{m+1}{2}$.

Now we show that v_jMDv_i.

(1). We have $(j + m + 1) - i = m + \frac{m-1}{2} + 1$. From equations (3.42) and (3.7), $d(v_i, v_{j+m+1}) = \frac{m+3}{2}$.

(2). Note that $(j + m - 1) - i = m + \frac{m-1}{2} - 1$. From equations (3.46) and (3.7), $d(v_i, v_{j+m-1}) = \frac{m+1}{2}$.

(3). By using equations (3.43) and (3.8), we obtain $d(v_i, u_j) = \frac{m+1}{2}$.

Hence v_i and v_j are MMD.

Suppose $j - i = m + \frac{m-1}{2} + 1$. We have

$$N(v_i) = \{u_i, v_{i+m+1}, v_{i+m-1}\},$$
$$N(v_{i+m+\frac{m-1}{2}+1}) = \{v_{i+\frac{m-1}{2}+2}, v_{i+\frac{m-1}{2}}, u_{i+m+\frac{m-1}{2}}\}.$$

Equations (3.42) and (3.7) yields

$$d(v_i, v_j) = \frac{m+3}{2}.$$

We first show that v_iMDv_j.

(1). We have $j-(i+m-1) = \frac{m-1}{2}+2$. From equations (3.47) and (3.7), $d(v_{i+m-1}, v_j) = \frac{m+1}{2}$.

(2). Note that $j-(i+m+1) = \frac{m-1}{2}$. From equations (3.43) and (3.7), $d(v_{i+m+1}, v_j) = \frac{m+3}{2}$.

(3). Using equations (3.42) and (3.8), we have $d(u_i, v_j) = \frac{m+1}{2}$.

Next we show that v_jMDv_i.

(1). We have $(j-m+1)-i = \frac{m-1}{2}+2$. From equations (3.47) and (3.7), $d(v_i, v_{j-m+1}) = \frac{m+1}{2}$.

(2). Note that $(j-m-1)-i = \frac{m-1}{2}$. From equations (3.43) and (3.7), $d(v_i, v_{j-m-1}) = \frac{m+3}{2}$.

(3). We have $d(v_i, u_j) = \frac{m+1}{2}$ by using equations (3.42) and (3.8).

This proves that v_i and v_j are MMD.

(e). Let $\frac{m-1}{2}$ is even. Then we will prove that v_iMMDv_j if and only if $j-i \in \{m, \frac{m+1}{2}, m+\frac{m+1}{2}-1\}$.

On contrary, assume that $j-i \notin \{m, \frac{m+1}{2}, m+\frac{m+1}{2}-1\}$. Since $\frac{m-1}{2}$ is even, therefore from equations (3.22) and (3.23), it holds that

$$j-i \in \mathcal{F}_1 \cup \left\{\mathcal{F}_2 \setminus \left\{\frac{m+1}{2}, m+\frac{m+1}{2}-1\right\}\right\}. \tag{3.54}$$

Subcase I. First assume that $j-i \in \mathcal{F}_1$. From equation (3.3), we have

$$d(v_i, v_j) = k.$$

We know that $u_i \in N(v_i)$, therefore from equation (3.4), we get

$$d(u_i, v_j) = k+1 > d(v_i, v_j).$$

This shows v_i and v_j are not MMD.

Subcase II.

Suppose that $j-i \in \mathcal{F}_2 \setminus \left\{\frac{m+1}{2}, m+\frac{m+1}{2}-1\right\}$.

Then from equation (3.7), we obtain

$$d(v_i, v_j) = k+3.$$

Note $v_{j+m+1} \in N(v_j)$. Let $j-i = k(m+1)+1$. We can write $(j+m+1)-i = k(m+1)+1+m+1 = (k+1)(m+1)+1 \in \mathcal{F}_2$. From equation (3.7), we have $d(v_{j+m+1}, v_i) = k+4 > d(v_i, v_j)$. If $j-i = k(m-1)-1$, then $(j+m-1)-i = k(m-1)-1+m-1$=$(k+1)(m-1)-1 \in \mathcal{F}_2$. Using equation (3.7), we obtain $d(v_{j+m-1}, v_i) = k+4 > d(v_i, v_j)$. This shows that v_i and v_j are not MMD.

Now we prove the converse.

Let $j-i = m$. We know that

$$N(v_i) = \{u_i, v_{i+m+1}, v_{i+m-1}\},$$
$$N(v_{i+m}) = \{v_{i+1}, v_{i-1}, u_{i+m}\}.$$

From equation (3.11), we have

$$d(v_i, v_j) = 4.$$

We first show that v_iMDv_j.

(1). We have $j - (i + m - 1) = 1$. From equations (3.12) and (3.7), $d(v_{i+m-1}, v_j) = 3$.

(2). Note that $j-(i+m+1) = -1$. From equations (3.13) and (3.7), $d(v_{i+m+1}, v_j) = 3$.

(3). By using equation (3.11), we obtain $d(u_i, v_j) = 3$.

Next we show that v_jMDv_i.

(1). Since $(j - m + 1) - i = 1$, therefore from equations (3.12) and (3.7), we have $d(v_i, v_{i+1}) = 3$.

(2). From equations (3.13) and (3.7), we get $d(v_i, v_{i-1}) = 3$ for $(j - m - 1) - i = -1$.

(3). Equation (3.11) yields that $d(v_i, u_j) = 3$.

This shows that v_i and v_j are MMD.

Suppose $j - i = \frac{m+1}{2}$. We know that

$$N(v_i) = \{u_i, v_{i+m+1}, v_{i+m-1}\},$$
$$N(v_{i+\frac{m+1}{2}}) = \{v_{i+\frac{m+1}{2}+m+1}, v_{i+\frac{m+1}{2}+m-1}, u_{i+\frac{m+1}{2}}\}.$$

From equations (3.22) and (3.7), we get

$$d(v_i, v_j) = \frac{m+3}{2}.$$

We first show that v_iMDv_j.

(1). We have $j-(i+m+1) = \frac{-m+1}{2}-1$. From equations (3.29) and (3.7), $d(v_{i+m+1}, v_j) = \frac{m+3}{2}$.

(2). Note that $j - (i + m - 1) = \frac{-m+1}{2} + 1$. From equations (3.28) and (3.7), $d(v_{i+m-1}, v_j) = \frac{m+1}{2}$.

(3). From equations (3.22) and (3.8), $d(u_i, v_j) = \frac{m+1}{2}$.

Next We show that v_jMDv_i.

(1). Using equations (3.27) and (3.7), we get $d(v_i, v_{j+m+1}) = \frac{m+1}{2}$ for $(j+m+1)-i = m + \frac{m+1}{2} + 1$.

(2). As $(j+m-1)-i = m + \frac{m+1}{2} - 1$, therefore from equations (3.23) and (3.7), tells us that $d(v_i, v_{j+m-1}) = \frac{m+3}{2}$.

(3). Using equations (3.22) and (3.8), we obtain $d(v_i, u_j) = \frac{m+1}{2}$.

Hence v_i and v_j are MMD.

Suppose that $j - i = m + \frac{m+1}{2} - 1$. We know that

$$N(v_i) = \{u_i, v_{i+m+1}, v_{i+m-1}\},$$
$$N(v_{i+m+\frac{m+1}{2}-1}) = \{v_{i+\frac{m+1}{2}-2}, v_{i+\frac{m+1}{2}}, u_{i+m+\frac{m+1}{2}-1}\}.$$

From equations (3.23) and (3.7), we get

$$d(v_i, v_j) = \frac{m+3}{2}.$$

We first show that v_iMDv_j.

(1). We have $j-(i+m-1) = \frac{m+1}{2}$. From equations (3.22) and (3.7), $d(v_{i+m-1}, v_j) = \frac{m+3}{2}$.

(2). Note that $j-(i+m+1) = \frac{m+1}{2}-2$. From equations (3.26) and (3.7), $d(v_{i+m+1}, v_j) = \frac{m+1}{2}$.

(3). By using equations (3.23) and (3.8), we get $d(u_i, v_j) = \frac{m+1}{2}$.

Next, we show that v_j is maximally distant from v_i.

(1). We have $(j-m+1)-i = \frac{m+1}{2}$. From equations (3.22) and (3.7), $d(v_i, v_{j-m+1}) = \frac{m+3}{2}$.

(2). Note that $(j-m-1)-i = \frac{m+1}{2}-2$. From equations (3.26) and (3.7), $d(v_i, v_{j-m-1}) = \frac{m+1}{2}$.

(3). By using equations (3.23) and (3.8), we have $d(v_i, u_j) = \frac{m+1}{2}$.

Hence v_i and v_j are MMD.

This completes the proof.

$\square$

In Theorem 3.2, we have seen that only these pair of vertices are mutually maximally distant vertices in $GP(2m, m-1)$. We know that only these vertices will be adjacent in strong resolving graph $GP_{SR}(2m, m-1)$ of $GP(2m, m-1)$.

Theorem 3.3. *Let $m \geq 7$ and $m \equiv 1 \bmod 2$. Then $\beta(GP_{SR}(2m, m-1)) = m$.*

Proof. Let $i, j \in \{1, 2, \ldots, 2m\}$ and we construct the largest independent set for $GP_{SR}(2m, m-1)$ from $S = V(GP_{SR}(2m, m-1)) = \{u_1, u_2, \ldots, u_{2m}, v_1, v_2, \ldots, v_{2m}\}$.

Suppose $\frac{m-1}{2}$ is odd. By Theorem 3.2, the vertices u_i and u_j are adjacent in $GP_{SR}(2m, m-1)$ if and only if $|j-i| \in \{m, \frac{m+1}{2}, \frac{m+1}{2}+1, m+\frac{m+1}{2}-1, m+\frac{m+1}{2}-2\}$. Similarly vertices v_i and v_j are adjacent in $GP_{SR}(2m, m-1)$ if and only if $|j-i| \in \{m, \frac{m-1}{2}, m+\frac{m-1}{2}+1\}$. Also the vertices v_i and u_j are adjacent in $GP_{SR}(2m, m-1)$ if and only if $|j-i| \in \{\frac{m+1}{2}, m+\frac{m+1}{2}-1\}$. Suppose u_i is contained in an independent set S, then $u_j \notin S$ where $j = i + \frac{m+1}{2}$ and $i \in \{1, 2, \ldots, 2m\}$. Otherwise, $u_i \sim u_j$ contradicts the definition of independent set. Likewise, let v_i is contained in an independent set S, then $v_j \notin S$ where $j = i + \frac{m-1}{2}$ and $i \in \{1, 2, \ldots, 2m\}$. By removing these vertices from S we construct a new set $S_1 = \{u_1, u_2, \ldots, u_{\frac{m+1}{2}}, v_1, v_2, \ldots, v_{\frac{m-1}{2}}\}$. It is straight forward to observe that S_1 is largest independent set.

Now suppose that $\frac{m-1}{2}$ is even.

By Theorem 3.2, the vertices u_i and u_j are adjacent in $GP_{SR}(2m, m-1)$ if and only if $|j-i| \in \{m, \frac{m+1}{2}, \frac{m+1}{2}+1, m+\frac{m+1}{2}-1, m+\frac{m+1}{2}-2\}$. Similarly, vertices v_i and v_j are adjacent in $GP_{SR}(2m, m-1)$ if and only if $|j-i| \in \{m, \frac{m+1}{2}, m+\frac{m+1}{2}-1\}$. Suppose u_i, v_i are contained in an independent set S, then $u_j, v_j \notin S$ where $j = i + \frac{m+1}{2}$ and $i \in \{1, 2, \ldots, 2m\}$. Otherwise, $u_i \sim u_j$ and $v_i \sim v_j$ which contradicts the definition of an independent set. By removing u_j and v_j from S, we construct a new set, $S_1 = \{u_1, u_2, \ldots, u_{\frac{m+1}{2}}, v_1, v_2, \ldots, v_{\frac{m+1}{2}}\}$. Also note that v_i and $u_{i+\frac{m+1}{2}-1}$ are adjacent in $GP_{SR}(2m, m-1)$. Since we assumed $v_i \in S_1$, then $u_{i+\frac{m+1}{2}-1} \notin S_1$. Thus we have a new set $S_2 = \{u_1, u_2, \ldots, u_{\frac{m-1}{2}}, v_1, v_2, \ldots, v_{\frac{m+1}{2}}\}$. It is straight forward to observe that S_2 is largest independent set.

Hence $\beta(GP_{SR}(2m, m-1)) = m$.

$\square$

Theorem 3.4. *Let $GP(2m, m-1)$ be the generalized Petersen graph, where $m \geq 7$ and $m \equiv 1 \pmod 2$. Then $(sdimGP(2m, m-1))$ is $3m$.*

Proof. The order of $GP(2m, m-1)$ is $4m$. By Theorem 3.1 and Theorem 3.3, the vertex covering number of $GP_{SR}(2m, m-1)$ is $3m$. Hence by Theorem 2.4, $sdim(GP(2m, m-1)) = \alpha(G_{SR}(2m, m-1)) = 3m$. $\square$

In next section, we compute strong metric dimension of $GP(2m, m-1)$ for $m \geq 6$ and $m \equiv 0 \pmod 2$.

3.3 Strong metric dimension of $GP(2m, m-1)$ when m is even.

Let $m \equiv 0 \pmod 2$ and $m \geq 6$ and also let $i, j \in \{1, 2, \ldots, 2m\}$. We define $\mathcal{F}_3$ and $\mathcal{F}_4$ by:

$$\mathcal{F}_3 = \left\{k(m-1), k(m+1) \mid 0 \leq k \leq \frac{m}{2}\right\}, \tag{3.55}$$

$$\mathcal{F}_4 = \left\{k(m-1)-1, k(m+1)+1 \mid 0 \leq k \leq \frac{m}{2}-2\right\}, \tag{3.56}$$

in such a way that if $l \in \mathcal{F}_3$ or $\mathcal{F}_4$, then $1 \leq l \leq 2m$, otherwise l is modulo $2m$. When $0 \leq k \leq \frac{m}{2}$, and let $k \equiv 0 \pmod 2$ or $k \equiv 1 \pmod 2$, then the set $\mathcal{F}_3$ contains even and odd integers respectively of the form $k(m-1)$ or $k(m+1)$ from $1, \ldots, 2m$. Similarly, when $0 \leq k \leq \frac{m}{2} - 2$, and let $k \equiv 0 \pmod 2$ or $k \equiv 1 \pmod 2$, then the set $\mathcal{F}_4$ contains odd and even integers respectively of the form $k(m-1)-1$ or $k(m+1)+1$ from $1, \ldots, 2m$. This shows that $|j-i| \in \mathcal{F}_3 \cup \mathcal{F}_4 \cup \{m\}$.

Let $GP(2m, m-1)$ be the generalized Petersen graph, where $m \geq 6$ and $m \equiv 0 \pmod 2$. Let $i, j \in \{1, 2, \ldots, 2m\}$. Then using structure of $GP(2m, m-1)$, the distances between vertices of $GP(2m, m-1)$ are given by:

(a). When $|j - i| \in \mathcal{F}_3$:

$$d(v_i, v_j) = k, \tag{3.57}$$

$$d(v_i, u_j) = k + 1, \tag{3.58}$$

$$d(u_i, u_j) = \begin{cases} k & k \equiv 0 (\text{mod } 2), \\ k + 2 & k \equiv 1 (\text{mod } 2). \end{cases} \tag{3.59}$$

If $\frac{m}{2} \equiv 1 \pmod 2$ and $k = \frac{m}{2}$, then

$$d(u_i, u_j) = k. \tag{3.60}$$

(b). When $|j - i| \in \mathcal{F}_4$:

$$d(v_i, v_j) = k + 3, \tag{3.61}$$

$$d(v_i, u_j) = k + 2, \tag{3.62}$$

$$d(u_i, u_j) = \begin{cases} k + 1, & k \equiv 0 (\text{mod } 2), \\ k + 3, & k \equiv 1 (\text{mod } 2). \end{cases} \tag{3.63}$$

(c). When $|j - i| = m$:

$$d(v_i, v_j) = 4,\ d(u_j, v_i) = d(u_i, v_j) = 3,\ d(u_i, u_j) = 4. \tag{3.64}$$

We have the following equivlance classes to modulo $2m$ which are helpful for computing $sdim(GP(2m, m-1))$ where m is an even integer.

$$m + 1 \equiv \left(\frac{m}{2} - \frac{m-2}{2}\right)(m + 1) \in \mathcal{F}_3, \tag{3.65}$$

$$m - 1 \equiv \left(\frac{m}{2} - \frac{m-2}{2}\right)(m - 1) \in \mathcal{F}_3, \tag{3.66}$$

$$1 \equiv (0)(m+1)+1 \in \mathcal{F}_4, \tag{3.67}$$

$$-1 \equiv (0)(m-1)-1 \in \mathcal{F}_4. \tag{3.68}$$

If $\frac{m}{2} \equiv 0 \pmod 2$, then

$$\frac{m}{2}+1 \equiv \left(\frac{m}{2}-1\right)(m-1) \in \mathcal{F}_3, \tag{3.69}$$

$$m+\frac{m}{2}-1 \equiv \left(\frac{m}{2}-1\right)(m+1) \in \mathcal{F}_3, \tag{3.70}$$

$$\frac{m}{2}+2 \equiv \left(\frac{m}{2}-3\right)(m-1)-1 \in \mathcal{F}_4, \tag{3.71}$$

$$\frac{m}{2} \equiv \left(\frac{m}{2}\right)(m+1) \in \mathcal{F}_3, \tag{3.72}$$

$$m+\frac{m}{2} \equiv \left(\frac{m}{2}\right)(m-1) \in \mathcal{F}_3, \tag{3.73}$$

$$m+\frac{m}{2}-2 \equiv \left(\frac{m}{2}-3\right)(m+1)+1 \in \mathcal{F}_4, \tag{3.74}$$

$$\frac{-m}{2}+1 \equiv \left(\frac{m}{2}-2\right)(m-1)-1 \in \mathcal{F}_4, \tag{3.75}$$

$$\frac{-m}{2}-1 \equiv \left(\frac{m}{2}-1\right)(m+1) \in \mathcal{F}_3, \tag{3.76}$$

$$\frac{m}{2}-1 \equiv \left(\frac{m}{2}-2\right)(m+1)+1 \in \mathcal{F}_4, \tag{3.77}$$

$$m+\frac{m}{2}+1 \equiv \left(\frac{m}{2}-2\right)(m-1)-1 \in \mathcal{F}_4. \tag{3.78}$$

If $\frac{m}{2} \equiv 1 \pmod 2$, then

$$m + \frac{m}{2} - 2 \equiv \left(\frac{m}{2} - 2\right)(m+1) \in \mathcal{F}_3, \tag{3.79}$$

$$m + \frac{m}{2} \equiv \left(\frac{m}{2}\right)(m+1) \in \mathcal{F}_3, \tag{3.80}$$

$$m + \frac{m}{2} - 1 \equiv \left(\frac{m}{2} - 2\right)(m+1) + 1 \in \mathcal{F}_4, \tag{3.81}$$

$$\frac{m}{2} + 1 \equiv \left(\frac{m}{2} - 2\right)(m-1) - 1 \in \mathcal{F}_4, \tag{3.82}$$

$$\frac{m}{2} \equiv \left(\frac{m}{2}\right)(m-1) \in \mathcal{F}_3, \tag{3.83}$$

$$\frac{m}{2} + 2 \equiv \left(\frac{m}{2} - 2\right)(m-1) \in \mathcal{F}_3, \tag{3.84}$$

$$\frac{-m+1}{2} \equiv \left(\frac{m}{2} - 1\right)(m+1) \in \mathcal{F}_3, \tag{3.85}$$

$$\frac{-m}{2} - 1 \equiv \left(\frac{m}{2} - 2\right)(m+1) + 1 \in \mathcal{F}_4, \tag{3.86}$$

$$\frac{m}{2} - 1 \equiv \left(\frac{m}{2} - 1\right)(m+1) \in \mathcal{F}_3, \tag{3.87}$$

$$m + \frac{m}{2} + 1 \equiv \left(\frac{m}{2} - 1\right)(m-1) \in \mathcal{F}_3, \tag{3.88}$$

$$\frac{-m}{2} + 1 \equiv \left(\frac{m}{2} - 1\right)(m-1) \in \mathcal{F}_3. \tag{3.89}$$

Theorem 3.5. *Let $GP(2m, m-1)$ be the generalized Petersen graph, where $m \geq 6$ and $m \equiv 0 \pmod 2$. Then for each $i, j \in \{1, \ldots, 2m\}$, the following holds:*

(a). u_iMMDu_j *if and only if* $|j-i| \in \{m, \frac{m}{2}+1, m+\frac{m}{2}-1\}$.

(b). v_iMMDu_j *if and only if* $|j-i| \in \{\frac{m}{2}, m+\frac{m}{2}\}$.

(c). v_iMMDv_j *if and only if* $|j-i| = m$.

Proof. Let $i, j \in \{1, 2, \dots, 2m\}$. Without loss of generality, assume that $i \leq j$.

(a). We prove u_iMMDu_j if and only if $j-i \in \{m, \frac{m}{2}+1, m+\frac{m}{2}-1\}$.

Suppose that $j-i \notin \{m, \frac{m}{2}+1, \frac{m}{2}+m-1\}$.

Case 1. When $\frac{m}{2}$ is odd. From equations (3.81) and (3.82), it holds that

$$j-i \in \left\{\mathcal{F}_3 \cup \mathcal{F}_4 \setminus \left\{\frac{m}{2}+1, \frac{m}{2}+m-1\right\}\right\}. \tag{3.90}$$

Subcase 1. Let $j-i \in \mathcal{F}_3$. If k is even, then from equation (3.59), we have

$$d(u_i, u_j) = k.$$

Also $v_i \in N(u_i)$. Equation (3.58) yields

$$d(v_i, u_j) = k+1.$$

Thus $d(v_i, u_j) > d(u_i, u_j)$, that is, u_i and u_j are not MMD.

If k is odd, then equation (3.59) tells us that

$$d(u_i, u_j) = k+2.$$

Let $j-i = k(m-1)$. Note that $u_{i+1} \in N(u_i)$ and $j-(i+1) = k(m-1)-1$, that is, $j-(i+1) \in \mathcal{F}_4$. By equation (3.63), $d(u_{i+1}, u_j) = k+3$. Thus $d(u_{i+1}, u_j) > d(u_i, u_j)$. Let $j-i = k(m+1)$. Then $(j+1)-i = k(m+1)+1$, that is, $(j+1)-i \in \mathcal{F}_4$. Also $u_{j+1} \in N(u_j)$. By equation (3.63), $d(u_i, u_{j+1}) = k+3 > d(u_i, u_j)$. Thus u_i and u_j are not MMD.

Subcase 2. Next, assume that $j-i \in \mathcal{F}_4 \setminus \left\{\frac{m}{2}+1, \frac{m}{2}+m-1\right\}$. If k is even, then from equation (3.63), we have

$$d(u_i, u_j) = k+1.$$

Note that $v_i \in N(u_i)$. Using equation (3.62), we have

$$d(v_i, u_j) = k + 2.$$

Thus $d(v_i, u_j) > d(u_i, u_j)$, that is, u_i and u_j are not MMD.

If k is odd then, from equation (3.63), we have

$$d(u_i, u_j) = k + 3.$$

Let $j - i = k(m-1) - 1$. Then

$$(j-1) - i = k(m-1) - 2 = k(m-1) + 2(m-1) = (k+2)(m-1).$$

That is, $(j-1) - i \in \mathcal{F}_3$. But $d(u_i, u_{j-1}) = (k+2) + 2 > d(u_i, u_j)$.

Let $j - i = k(m+1) + 1$. Then

$$(j+1) - i = k(m+1) + 2 = k(m+1) + 2(m+1) = (k+2)(m+1).$$

Thus $(j+1) - i \in \mathcal{F}_3$. Then $k+2$ is odd and by equation (3.59), we obtain

$$d(u_i, u_{j+1}) = (k+2) + 2 > d(u_i, u_j).$$

This shows that u_i and u_j are not MMD.

Case 2. When $\frac{m}{2}$ is even. From equations (3.69) and (3.70), it holds that

$$j - i \in \left\{\mathcal{F}_3 \setminus \left\{\frac{m}{2} + 1, \frac{m}{2} + m - 1\right\}\right\} \cup \mathcal{F}_4. \tag{3.91}$$

Subcase 1. Let $j - i \in \mathcal{F}_3 \setminus \left\{\frac{m}{2} + 1, \frac{m}{2} + m - 1\right\}$. If k is even, then from equation (3.59), we obtain

$$d(u_i, u_j) = k.$$

Also $v_i \in N(u_i)$. Using equation (3.58), we have

$$d(v_i, u_j) = k + 1.$$

Thus $d(v_i, u_j) > d(u_i, u_j)$, that is, u_i and u_j are not MMD.

If k is odd, then equation (3.59) tells us that

$$d(u_i, u_j) = k + 2.$$

Let $j - i = k(m-1)$. Note that $u_{i+1} \in N(u_i)$ and $j - (i+1) = k(m-1) - 1$, that is, $j-(i+1) \in \mathcal{F}_4$. By equation (3.63), we get $d(u_{i+1}, u_j) = k+3$. Thus $d(u_{i+1}, u_j) > d(u_i, u_j)$. Let $j - i = k(m+1)$. Then $(j+1) - i = k(m+1) + 1$, that is, $(j+1) - i \in \mathcal{F}_4$. Also $u_{j+1} \in N(u_j)$ and by using equation (3.63), we have $d(u_i, u_{j+1}) = k + 3 > d(u_i, u_j)$. Thus u_i and u_j are not MMD.

Subcase 2. Suppose that $j - i \in \mathcal{F}_4$. If k is even, then from equation (3.63), we obtain

$$d(u_i, u_j) = k + 1.$$

Note that $v_i \in N(u_i)$ and from equation (3.62), we have

$$d(v_i, u_j) = k + 2.$$

Thus $d(v_i, u_j) > d(u_i, u_j)$, that is, u_i and u_j are not MMD.

If k is odd, then equation (3.63) yields that

$$d(u_i, u_j) = k + 3.$$

Let $j - i = k(m-1) - 1$. Then

$$(j-1) - i = k(m-1) - 2 = k(m-1) + 2(m-1) = (k+2)(m-1).$$

That is, $(j-1) - i \in \mathcal{F}_3$. But from (3.59), $d(u_i, u_{j-1}) = (k+2) + 2 > d(u_i, u_j)$.

Let $j - i = k(m+1) + 1$. Then

$$(j+1) - i = k(m+1) + 2 = k(m+1) + 2(m+1) = (k+2)(m+1).$$

Thus $(j+1) - i \in \mathcal{F}_3$. Then $k + 2$ is odd and by using equation (3.59), we obtain

$$d(u_i, u_{j+1}) = (k+2) + 2 > d(u_i, u_j).$$

This shows that u_i and u_j are not MMD.

Now we prove the converse. We know that $GP(2m, m-1)$ is 3-regular graph, so for proving u_iMMDu_j we show that for each $w_i \in N(u_i)$ the distance $d(w_i, u_j) \leq d(u_i, u_j)$ and vice versa.

Let $j - i = m$. We know that

$$N(u_i) = \{v_i, u_{i+1}, u_{i-1}\}, \tag{3.92}$$

$$N(u_{i+m}) = \{v_{i+m}, u_{i+m+1}, v_{i+m-1}\}. \tag{3.93}$$

From equation (3.64), we have

$$d(u_i, u_{i+m}) = 4.$$

Also from equations (3.57)∼(3.63) and equations (3.65)∼(3.66), we have

$$d(u_i, v_{i+m}) = d(u_i, u_{i+m+1}) = d(u_i, u_{i+m-1}) = d(u_{i+m}, v_i) \tag{3.94}$$

$$= d(u_{i+m}, u_{i+1}) = d(u_{i+m}, u_{i-1}) = 3. \tag{3.95}$$

Thus u_iMMDu_{i+m}.

Case 1. When $\frac{m}{2}$ is odd.

Now let $j - i = \frac{m}{2} + 1$. We know that

$$N(u_i) = \{v_i, u_{i+1}, u_{i-1}\}, \tag{3.96}$$

$$N(u_{i+\frac{m}{2}+1}) = \{v_{i+\frac{m}{2}+1}, u_{i+\frac{m}{2}+2}, u_{i+\frac{m}{2}}\}. \tag{3.97}$$

From equations (3.63) and (3.82), we have

$$d(u_i, u_{i+\frac{m}{2}+1}) = \frac{m}{2} + 1.$$

We first show that u_iMDu_j.

(1). As $j - (i-1) = \frac{m}{2} + 2$, therefore from equations (3.84) and (3.59), we get $d(u_{i-1}, u_j) = \frac{m}{2}$.

(2). For $j - (i+1) = \frac{m}{2}$. By using equations (3.83) and (3.60), we get $d(u_{i+1}, u_j) = \frac{m}{2}$.

(3). From equations (3.82) and (3.62), we have $d(v_i, u_j) = \frac{m}{2}$.

Now we prove that u_jMDu_i.

(1). We have $(j+1)-i = \frac{m}{2}+2$. From equations (3.84) and (3.59), we obtain $d(u_i, u_{j+1}) = \frac{m}{2}$.

(2). Note that $(j-1)-i = \frac{m}{2}$. Equations (3.83) and (3.60) tells us that $d(u_i, u_{j-1}) = \frac{m}{2}$.

(3). From equations (3.82) and (3.62), we get $d(u_i, v_j) = \frac{m}{2}$.

Thus u_i and u_j are MMD.

Let $j-i = m+\frac{m}{2}-1$. We know that

$$N(u_i) = \{v_i, u_{i+1}, u_{i-1}\}, \tag{3.98}$$

$$N(u_{i+m+\frac{m}{2}-1}) = \{v_{i+m+\frac{m}{2}-1}, u_{i+m+\frac{m}{2}-2}, u_{i+m+\frac{m}{2}}\}. \tag{3.99}$$

From equations (3.63) and (3.81), we get

$$d(u_i, u_{i+m+\frac{m}{2}-1}) = \frac{m}{2}+1.$$

We first show that $u_i\text{MD}u_j$.

(1). We have $j-(i-1) = m+\frac{m}{2}$. From equations (3.80) and (3.60), $d(u_{i-1}, u_j) = \frac{m}{2}$.

(2). For $j-(i+1) = m+\frac{m}{2}-2$, from equations (3.79) and (3.59), we obtain $d(u_{i+1}, u_j) = \frac{m}{2}$.

(3). Using equations (3.81) and (3.62), we have $d(v_i, u_j) = \frac{m}{2}$.

Now we prove that $u_j\text{MD}u_i$.

(1). We have $(j+1)-i = m+\frac{m}{2}$. From equations (3.80) and (3.60), $d(u_i, u_{j+1}) = \frac{m}{2}$.

(2). Note that $(j-1)-i = m+\frac{m}{2}-2$. From equations (3.79) and (3.59), $d(u_i, u_{j-1}) = \frac{m}{2}$.

(3). From equations (3.81) and (3.62), we have $d(u_i, v_j) = \frac{m}{2}$.

Thus u_i and u_j are MMD.

Case 2. When $\frac{m}{2}$ is even.

Let $j-i = \frac{m}{2}+1$. We know that

$$N(u_i) = \{v_i, u_{i+1}, u_{i-1}\}, \tag{3.100}$$

$$N(u_{i+\frac{m}{2}+1}) = \{v_{i+\frac{m}{2}+1}, u_{i+\frac{m}{2}+2}, u_{i+\frac{m}{2}}\}. \tag{3.101}$$

From equations (3.59) and (3.69), we have

$$d(u_i, u_{i+\frac{m}{2}+1}) = \frac{m}{2} + 1.$$

We first show that u_iMDu_j.

(1). As $j-(i-1) = \frac{m}{2}+2$, therefore from equations (3.71) and (3.63), $d(u_{i-1}, u_j) = \frac{m}{2}$.

(2). Using equations (3.72) and (3.59), we get $d(u_{i+1}, u_j) = \frac{m}{2}$ for $j - (i + 1) = \frac{m}{2}$.

(3). Equations (3.69) and (3.58) yields $d(v_i, u_j) = \frac{m}{2}$.

Next, we prove that u_jMDu_i.

(1). Equations (3.71) and (3.63) tells us that $d(u_i, u_{j+1}) = \frac{m}{2}$ for$(j + 1) - i = \frac{m}{2} + 2$.

(2). Note that $(j-1)-i = \frac{m}{2}$. From equations (3.72) and (3.59), we have $d(u_i, u_{j-1}) = \frac{m}{2}$.

(3). Using equations (3.69) and (3.58), we have $d(u_i, v_j) = \frac{m}{2}$.

Thus u_i and u_j are MMD.

Let $j - i = m + \frac{m}{2} - 1$. We know that

$$N(u_i) = \{v_i, u_{i+1}, u_{i-1}\}, \tag{3.102}$$

$$N(u_{i+m+\frac{m}{2}-1}) = \{v_{i+m+\frac{m}{2}-1}, u_{i+m+\frac{m}{2}-2}, u_{i+m+\frac{m}{2}}\}. \tag{3.103}$$

From equations (3.59) and (3.70), we obtain

$$d(u_i, u_{i+m+\frac{m}{2}-1}) = \frac{m}{2} + 1.$$

We first show that u_iMDu_j.

(1). We have $j - (i - 1) = m + \frac{m}{2}$. From equations (3.73) and (3.59), $d(u_{i-1}, u_j) = \frac{m}{2}$.

(2). We note that $j - (i + 1) = m + \frac{m}{2} - 2$. From equations (3.74) and (3.63), $d(u_{i+1}, u_j) = \frac{m}{2}$.

(3). From equations (3.70) and (3.58), we get $d(v_i, u_j) = \frac{m}{2}$.

Now we prove that u_jMDu_i.

(1). We have $(j + 1) - i = m + \frac{m}{2}$. From equations (3.73) and (3.59), $d(u_i, u_{j+1}) = \frac{m}{2}$.

(2). Note that $(j-1)-i = m+\frac{m}{2}-2$. From equations (3.74) and (3.63), $d(u_i, u_{j-1}) = \frac{m}{2}$.

(3). Using equations (3.70) and (3.58), we obtain $d(u_i, v_j) = \frac{m}{2}$.

Thus u_i and u_j are MMD.

(b). We prove v_iMMDu_j if and only if $j-i \in \{\frac{m}{2}, m+\frac{m}{2}\}$.

Suppose that $j-i \notin \{\frac{m}{2}, m+\frac{m}{2}\}$.

Case 1.

Let $j-i=m$. From equation (3.64), we have

$$d(v_i, u_j) = 3.$$

Also $u_i \in N(v_i)$. By equation (3.64), we have $d(u_i, u_j) = 4 > d(v_i, u_j)$. It is easily seen that v_i and u_j are not MMD.

Case 2.

If $\frac{m}{2}$ is odd, then from equations (3.83) and (3.80) [or if $\frac{m}{2}$ is even, then from equations (3.72) and (3.73)], it holds that

$$j-i \in \left\{\mathcal{F}_3 \setminus \left\{\frac{m}{2}, m+\frac{m}{2}\right\} \cup \mathcal{F}_4\right\}. \tag{3.104}$$

Subcase 1. Let $j-i \in \mathcal{F}_3 \setminus \left\{\frac{m}{2}, m+\frac{m}{2}\right\}$. From equation (3.58), we get

$$d(v_i, u_j) = k+1.$$

Suppose k is odd. We know that $u_i \in N(v_i)$. Therefore from equation (3.59), we have

$$d(u_i, u_j) = k+2.$$

Thus $d(u_i, u_j) > d(v_i, u_j)$, that is, v_i and u_j are not MMD.

Let k is even and $j-i = k(m-1)$. Note that $u_{j-1} \in N(u_j)$ and $(j-1)-i = k(m-1)-1$, that is, $(j-1)-i \in \mathcal{F}_4$. By equation (3.62), $d(v_i, u_{j-1}) = k+2$. Thus $d(u_i, u_{j-1}) > d(v_i, u_j)$. Let $j-i = k(m+1)$. Then $(j+1)-i = k(m+1)+1$, that is, $(j+1)-i \in \mathcal{F}_4$. Also $u_{j+1} \in N(u_j)$ and by equation (3.62), we have $d(v_i, u_{j+1}) = k+2 > d(v_i, u_j)$. Thus v_i and u_j are not MMD.

Subcase 2. Next, assume that $j-i \in \mathcal{F}_4$. From equation (3.62), we have

$$d(v_i, u_j) = k+2.$$

Suppose k is odd. Note that $u_i \in N(v_i)$ and from equation (3.63), we have

$$d(u_i, u_j) = k + 3.$$

Thus $d(u_i, u_j) > d(v_i, u_j)$, that is, v_i and u_j are not MMD.

Let k is even and $j - i = k(m-1) - 1$. Then

$$(j-1) - i = k(m-1) - 2 = k(m-1) + 2(m-1) = (k+2)(m-1).$$

That is, $(j-1)-i \in \mathcal{F}_3$. From equation (3.58), we have $d(v_i, u_{j-1}) = (k+2)+1 > d(v_i, u_j)$. Let $j - i = k(m+1) + 1$. Then

$$(j+1) - i = k(m+1) + 2 = k(m+1) + 2(m+1) = (k+2)(m+1).$$

Thus $(j+1) - i \in \mathcal{F}_3$. By equation (3.58), we obtain

$$d(v_i, u_{j+1}) = (k+2) + 1 > d(v_i, u_j).$$

This shows that v_i and u_j are not MMD.

Now we prove the converse.

Case 1. Let $\frac{m}{2}$ is odd. Also let $j - i = \frac{m}{2}$. We know that

$$N(v_i) = \{u_i, v_{i+m+1}, v_{i+m-1}\}, \tag{3.105}$$

$$N(u_{i+\frac{m}{2}}) = \{v_{i+\frac{m}{2}}, u_{i+\frac{m}{2}+1}, u_{i+\frac{m}{2}-1}\}. \tag{3.106}$$

From equations (3.58) and (3.83), we have

$$d(v_i, u_{i+\frac{m}{2}}) = \frac{m}{2} + 1.$$

We first show that v_iMDu_j.

(1). We have $j-(i+m-1) = \frac{-m}{2}+1$. From equations (3.89) and (3.58), $d(v_{i+m-1}, u_j) = \frac{m}{2}$.

(2). Note that $j-(i+m+1) = \frac{-m}{2}-1$. From equations (3.86) and (3.62), $d(v_{i+m+1}, u_j) = \frac{m}{2}$.

(3). Using equations (3.83) and (3.60), we get $d(u_i, u_j) = \frac{m}{2}$.

Now we prove that u_jMDv_i.

(1). Since $(j+1)-i=\frac{m}{2}+1$, therefore from equations (3.82) and (3.62), we obtain $d(v_i,u_{j+1})=\frac{m}{2}$.

(2). Using equations (3.87) and (3.58), we get $d(v_i,u_{j-1})=\frac{m}{2}$ for $(j-1)-i=\frac{m}{2}-1$.

(3). From equations (3.83) and (3.57), we obtain $d(v_i,v_j)=\frac{m}{2}$.

Thus v_i and u_j are MMD.

Next assume $j-i=m+\frac{m}{2}$. We know that

$$N(v_i)=\{u_i,v_{i+m+1},v_{i+m-1}\}, \tag{3.107}$$

$$N(u_{i+m+\frac{m}{2}})=\{v_{i+m+\frac{m}{2}},u_{i+m+\frac{m}{2}-1},u_{i+m+\frac{m}{2}+1}\}. \tag{3.108}$$

From equations (3.58) and (3.80), we have

$$d(v_i,u_{i+m+\frac{m}{2}})=\frac{m}{2}+1.$$

We first show that $v_i\text{MD}u_j$.

(1). We have $j-(i+m-1)=\frac{m}{2}+1$. From equations (3.82) and (3.62), $d(v_{i+m-1},u_j)=\frac{m}{2}$.

(2). We can obtain $j-(i+m+1)=\frac{m}{2}-1$. From equations (3.87) and (3.58), $d(v_{i+m+1},u_j)=\frac{m}{2}$.

(3). Equations (3.80) and (3.60) yields that $d(u_i,u_j)=\frac{m}{2}$.

Now we prove that $u_j\text{MD}v_i$.

(1). We have $(j+1)-i=m+\frac{m}{2}+1$. From equations (3.88) and (3.58), $d(v_i,u_{j+1})=\frac{m}{2}$.

(2). Note that $(j-1)-i=m+\frac{m}{2}-1$. From equations (3.81) and (3.62), $d(v_i,u_{j-1})=\frac{m}{2}$.

(3). Using equations (3.80) and (3.57), we obtain $d(v_i,v_j)=\frac{m}{2}$.

Thus v_i and u_j are MMD.

Case 2. Let $\frac{m}{2}$ is even. Also let $j-i=\frac{m}{2}$. We know that

$$N(v_i)=\{u_i,v_{i+m+1},v_{i+m-1}\}, \tag{3.109}$$

$$N(u_{i+\frac{m}{2}})=\{v_{i+\frac{m}{2}},u_{i+\frac{m}{2}+1},u_{i+\frac{m}{2}-1}\}. \tag{3.110}$$

From equations (3.58) and (3.72), we have

$$d(v_i, u_{i+\frac{m}{2}}) = \tfrac{m}{2} + 1.$$

We first show that v_i is maximally distant from u_j.

(1). Using equations (3.75) and (3.62), we get $d(v_{i+m-1}, u_j) = \frac{m}{2}$ for $j-(i+m-1) = \frac{-m}{2}+1$.

(2). Since $j-(i+m+1) = \frac{-m}{2}-1$, therefore equations (3.76) and (3.58) tells us that $d(v_{i+m+1}, u_j) = \frac{m}{2}$.

(3). From equations (3.72) and (3.59), we get $d(u_i, u_j) = \frac{m}{2}$.

Now we prove that u_j is maximally distant from v_i.

(1). We have $(j+1)-i = \frac{m}{2}+1$. From equations (3.69) and (3.58), $d(v_i, u_{j+1}) = \frac{m}{2}$.

(2). Note that $(j-1)-i = \frac{m}{2}-1$. From equations (3.77) and (3.62), $d(v_i, u_{j-1}) = \frac{m}{2}$.

(3). From equations (3.72) and (3.57), we get $d(v_i, v_j) = \frac{m}{2}$.

Thus v_i and u_j are MMD.

Suppose that $j-i = m+\frac{m}{2}$. We know that

$$N(v_i) = \{u_i, v_{i+m+1}, v_{i+m-1)}\}, \tag{3.111}$$

$$N(u_{i+m+\frac{m}{2}}) = \{v_{i+m+\frac{m}{2}}, u_{i+m+\frac{m}{2}-1}, u_{i+m+\frac{m}{2}+1}\}. \tag{3.112}$$

From equations (3.58) and (3.73), we have

$$d(v_i, u_{i+m+\frac{m}{2}}) = \tfrac{m}{2} + 1.$$

We first show that v_iMDu_j.

(1). We have $j-(i+m-1) = \frac{m}{2}+1$. From equations (3.69) and (3.58), $d(v_{i+m-1}, u_j) = \frac{m}{2}$.

(2). Since $j-(i+m+1) = \frac{m}{2}-1$, therefore from equations (3.77) and (3.62), we obtain $d(v_{i+m+1}, u_j) = \frac{m}{2}$.

(3). Equations (3.73) and (3.59) yields that $d(u_i, u_j) = \frac{m}{2}$.

Now we prove that u_jMDv_i.

(1). We have $(j+1)-i = m+\frac{m}{2}+1$. From equations (3.78) and (3.62), $d(v_i, u_{j+1}) = \frac{m}{2}$.

(2). Note that $(j-1)-i = m+\frac{m}{2}-1$. From equations (3.70) and (3.58), $d(v_i, u_{j-1}) = \frac{m}{2}$.

(3). Using equations (3.73) and (3.57), we get $d(v_i, v_j) = \frac{m}{2}$.

Thus v_i and u_j are MMD.

(c). We will prove that $v_i \text{MMD} v_j$ if and only if $j - i = m$.

Suppose that $j - i \neq m$. Then $j - i \in \mathcal{F}_3 \cup \mathcal{F}_4$.

Case 1. Let $j - i \in \mathcal{F}_3$. Then from equation (3.57), we have

$$d(v_i, v_j) = k.$$

Note that $u_j \in N(v_j)$ and from equation (3.58), we get

$$d(v_i, u_j) = k + 1.$$

Thus $d(v_i, u_j) > d(v_i, v_j)$. This shows that v_i and v_j are not MMD.

Case 2. Suppose that $j - i \in \mathcal{F}_4$. Then from equation (3.61), we have

$$d(v_i, v_j) = k + 3.$$

First note that $v_{j+(m-1)}, v_{j+(m+1)} \in N(v_j)$. If $j-i = k(m-1)-1$, then let $j' = j+(m-1)$. Also $j'-i = j-i+(m-1) = k(m-1)-1+m-1 = (k+1)(m-1)-1$. Thus $j'-i \in \mathcal{F}_4$. By equation (3.61), we have $d(v_i, v_{j'}) = (k+1)+3 > d(v_i, v_j)$. If $j-i = k(m+1)+1$, then let $j' = j+(m+1)$. Also $j'-i = j-i+(m+1) = k(m+1)+1+(m+1) = (k+1)(m+1)+1$. Thus $j' - i \in \mathcal{F}_4$. By equation (3.61), we have $d(v_i, v_{j'}) = (k + 1) + 3 > d(v_i, v_j)$. This proves that v_i and v_j are not MMD.

Now we prove tha converse. Let $j - i = m$. Then

$$\begin{aligned} N(v_i) &= \{u_i, v_{i+(m+1)}, v_{i+(m-1)}\}, \\ N(v_j) &= \{u_j, v_{j+1}, v_{j-1}\}. \end{aligned} \tag{3.113}$$

From equations (3.67), (3.68) and equations (3.57), (3.64), we have

$$\begin{aligned} &d(v_i, v_j) = 4, \\ &d(u_i, v_j) = d(v_{i+(m+1)}, v_j) = d(v_{i+(m-1)}, v_j) = d(u_j, v_i) = d(v_{i+1}, v_i) = d(v_{i-1}, v_i) = 3. \end{aligned}$$

Thus $v_i \text{MMD} v_{i+m}$. This completes the proof.

□

Theorem 3.6. *Let $m \geq 6$ and $m \equiv 0 \bmod 2$. Then $\beta(GP_{SR}(2m, m-1)) = m$.*

Proof. Let $i, j \in \{1, 2, \ldots, 2m\}$ and we construct the largest independent set for $GP_{SR}(2m, m-1)$ from $S = V(GP_{SR}(2m, m-1)) = \{u_1, u_2, \ldots, u_{2m}, v_1, v_2, \ldots, v_{2m}\}$. By Theorem 3.5, the vertices u_i and u_j are adjacent in $GP_{SR}(2m, m-1)$ if and only if $|j-i| \in \{m, \frac{m}{2}+1, m+\frac{m}{2}-1\}$. Similarly, the vertices v_i and v_j are adjacent in $GP_{SR}(2m, m-1)$ if and only if $|j-i| = m$. Suppose u_i is contained in an independent set S, then $u_j \notin S$ where $|j-i| = \frac{m}{2}+1$ and $i \in \{1, 2, \ldots, 2m\}$. Likewise, let v_i is contained in an independent set S, then $v_j \notin S$ where $|j-i| = m$ and $i \in \{1, 2, \ldots, 2m\}$. By removing these vertices from S we construct a new set $S_1 = \{u_1, u_2, \ldots, u_{\frac{m}{2}+1}, v_1, v_2, \ldots, v_m\}$. We know that u_i and $v_{i+\frac{m}{2}}$ are adjacent in $GP_{SR}(2m, m-1)$. Since we assumed $u_i \in S_1$, then $v_j \notin S_1$ for $|j-i| = \frac{m}{2}$. By deleting v_j from S_1, we obtain a new set $S_2 = \{u_1, u_2, \ldots, u_{\frac{m}{2}+1}, v_1, v_2, \ldots, v_{\frac{m}{2}}\}$. Also note that u_i and $v_{i+m+\frac{m}{2}}$ are adjacent in $GP_{SR}(2m, m-1)$. For $i = \frac{m}{2}+1$, we have $u_{\frac{m}{2}+1} \sim v_1$. This shows $u_{\frac{m}{2}+1} \notin S_2$. Thus we have $S_3 = \{u_1, u_2, \ldots, u_{\frac{m}{2}}, v_1, v_2, \ldots, v_{\frac{m}{2}}\}$. It is straight forward to observe that S_3 is largest independent set.

Hence $\beta(GP_{SR}(2m, m-1)) = m$.

□

Theorem 3.7. *Let $GP(2m, m-1)$ be the generalized Petersen graph, where $m \geq 6$ and $m \equiv 0 \pmod 2$. Then $sdimGP(2m, m-1)$ is $3m$.*

Proof. The order of $GP(2m, m-1)$ is $4m$. By Theorem 3.3 and Theorem 3.1, the vertex covering number of $GP_{SR}(2m, m-1)$ is $3m$. Hence by Theorem 2.4, $sdim(G) = \alpha(G_{SR}(2m, m-1)) = 3m$. □

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