Quadratic Invariants of Elasticity Tensor

by

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Abstract

This dissertation primarily deals with general theory of finding quadratic invariants of the elasticity tensors C_{ijkl} under the special rotation group for two and three-dimensional space $SO(2)$ and $SO(3)$ respectively. The method of diagonalization is used to determine the number of quadratic invariants. We find the transformation matrix for plane elasticity tensor and its invariants by using the same technique of diagonalization.

Acknowledgement

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May Allah bless them all (Ameen).

Gulfreen Riaz

Dedicated This Humble Task

To My

Loving Parents and Family.

Contents

Chapter 1

Introduction

1.1 Background

This dissertation entitled Quadratic Invariants of Elasticity tensor indicates that the dissertation revolves around three interconnected themes, each arising in a wide variety of mathematical sector.

The word tensor was first introduced in 1846 by William Rowan Hamilton to describe something different from what is now meant by a word tensor. After this Gregorio Ricci-Curbastro developed tensor calculus around 1890 under the title of absolute differential calculus, which was originally presented by Ricci in 1892. Later, it became accessible to scientist by the publication of Ricci and Tullio Levi-Civita's [1]. In the twentieth century, this came to be known as tensor analysis, and it achieved the broad acceptance with the introduction of Einstein's theory of general relativity, around 1915.

Generally tensors relate to arbitrary coordinate transformations, as used in general relativity. On the other hand, cartesian tensors relate to rotations between orthogonal axes, as in elasticity and other branches of classical physics. This makes tensor analysis an important tool in theoretical physics, continuum mechanics and many other fields of science and engineering.

1.2 Objective of the dissertation

In this dissertation, we consider the problem of finding the number of quadratic invariants of elasticity tensor under $SO(2)$ and $SO(3)$. The method to be used for obtaining the number of quadratic invariants of elasticity tensor is the method of diagonalization. We confirm the Norris [2] result about quadratic invariants and show that these invariants are essentially same as those of Ahmad [3].

1.3 Thesis layout

This dissertation is organized in the following way:

Chapter 2, deals with the basic notations along with examples, which provide us with necessary background for some later work. Also we introduce the terminology which will be used throughout this dissertation.

Chapter 3, comprises of some work already done in this field. The work of Ahmad [3] and Ahmad and Rashid [4] is reviewed for finding the number of independent linear invariants of an elasticity tensor under $SO(2)$ and $SO(3)$. Theorems presented in these papers are also reviewed.

Chapter 4, presented a method to find the quadratic invariants under $SO(2)$ and $SO(3)$. The results are provided to confirm the work of Norris [2], who proved his result about a complete set of quadratic invariants under $SO(2)$ and $SO(3)$. Further, the matrix of transformation for plane elasticity tensor and invariants of plane elasticity tensor are presented.

The last chapter of the dissertation summarize the earlier results and indicate the directions of further research.

Chapter 2

Mathematical preliminaries

Here we recall some fundamental concepts and relevant results used throughout this dissertation. We also provide few examples which make the definitions easily understandable. Necessary notations and the terms that are used are introduced.

2.1 Euclidean vector space

A vector space $\mathbf{V}$ associated with any pair of vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \ \epsilon \ \mathbf{V}$ by real number $\mathbf{u}.\mathbf{v}$ (defined as $\mathbf{u}.\mathbf{v} = u_1v_1 + u_2v_2... + u_nv_n$) is named as euclidean vector space if it satisfies the following axioms.

1) $\mathbf{u}.\mathbf{v} = \mathbf{v}.\mathbf{u}$,

2) $(\alpha\mathbf{u}).\mathbf{v} = \alpha(\mathbf{u}.\mathbf{v})$,

3) $\mathbf{u}.(\mathbf{v}+\mathbf{w}) = \mathbf{u}.\mathbf{v} + \mathbf{u}.\mathbf{w}$,

4) $\mathbf{u}.\mathbf{u} \geq 0$,

5) $\mathbf{u}.\mathbf{u} = \mathbf{0}$ if and only if $\mathbf{u} = \mathbf{0}$.

2.2 Orthonormal basis

A basis $\{\mathbf{e}_1, ..., \mathbf{e}_n\}$ in an n-dimensional euclidean vector space is called an orthonormal basis if its elements are unit vectors and also they are reciprocally orthogonal, i.e

$$\mathbf{e}_i.\mathbf{e}_j = \delta_{ij} = \begin{cases} 1 & \text{for} \quad i = j, \\ 0 & \text{for} \quad i \neq j. \end{cases}$$

where δ_{ij} is the kronecker delta.

Let us suppose that $\{\mathbf{e}_k\}$ and $\{\mathbf{e}_r^{'}\}$, $\{k, r\} = 1, 2, ..., n$ are two orthonormal basis of n-dimensional euclidean vector space. By expressing all vectors of one of the basis as linear combination of the vectors of the other basis, we have

$$\mathbf{e}_r^{'} = q_{kr}\mathbf{e}_k, \tag{2.1}$$

and

$$\mathbf{e}_k = q_{rk}^{'}\mathbf{e}_r^{'}, \tag{2.2}$$

where the matrix $Q^{'} = [q_{rk}^{'}]$ is the inverse of the matrix $Q = [q_{kr}]$. It follows that the transformation matrix Q and its inverse $Q^{'}$ have the following form.

$$Q = \begin{pmatrix} q_{11} & q_{12} & .... & q_{1r} \\ . & . & .... & . \\ . & . & .... & . \\ . & . & .... & . \\ q_{k1} & q_{k2} & .... & q_{kr} \end{pmatrix}, \tag{2.3}$$

$$Q^{'} = \begin{pmatrix} q_{11}^{'} & q_{12}^{'} & .... & q_{1r}^{'} \\ . & . & .... & . \\ . & . & .... & . \\ . & . & .... & . \\ q_{k1}^{'} & q_{k2}^{'} & .... & q_{kr}^{'} \end{pmatrix}. \tag{2.4}$$

where $QQ^{'} = I$ that is $Q^{'} = Q^{-1}$ and the transformation matrix Q has det$Q = \pm 1$. We restrict to those transformations in which det$Q = +1$. This corresponds to the right-handedness of the vector and Q is then called proper orthogonal matrix. A proper orthogonal transformation is the one which has det$Q = +1$. For a change of basis, in which this property is not preserved, det$Q = -1$, and Q is said to improper orthogonal matrix. For example, in $\mathbf{R}^3$

$$\begin{aligned} e_1^{'} &= \cos\theta e_1 + \sin\theta e_2, \\ e_2^{'} &= -\sin\theta e_1 + \cos\theta e_2, \\ e_3^{'} &= e_3, \end{aligned} \tag{2.5}$$

defines a rotation about the $x_3 - axis$ in an anticlockwise direction and has the rotation matrix as:

$$Q = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.6}$$

where $\det Q = 1$ and the transformation is proper orthogonal. On the other hand, a clockwise rotation will lead to

$$Q = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.7}$$

with $\det Q = -1$, an improper orthogonal matrix.

2.3 Tensor

A member of a vector space $\mathbf{V}$ is called a vector. When we consider tensors, a vector is termed as a tensor of order 1.

Tensor of order 2

Let $\mathbf{V}$ be a real vector space of dimension n and let vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$... be its elements. A linear transformation $\mathbf{T} : \mathbf{V} \to \mathbf{V}$ is called a tensor of order 2. If the value of $\mathbf{T}$ corresponding to a vector $\mathbf{u}$ is vector $\mathbf{v}$, then we can write $\mathbf{v} = \mathbf{T}(\mathbf{u}) = \mathbf{Tu}$. The linearity of the transformation $\mathbf{T}$ involves in the definition is expressed as

$$\mathbf{T}(\alpha\mathbf{u} + \beta\mathbf{v}) = \alpha\mathbf{T}(\mathbf{u}) + \beta\mathbf{T}(\mathbf{v}), \qquad \forall\ \ \mathbf{u}, \mathbf{v}\ \varepsilon\ \mathbf{V}, \qquad \forall\ \ \alpha, \beta\ \varepsilon\ \mathbf{R}.$$

Let the set of all second order tensors is denoted by L_2. In order to construct the basis in the vector space L_2, we define a tensor product of two vectors [5] $\mathbf{u}$,$\mathbf{v}$ ε $\mathbf{V}$ as $\mathbf{u} \otimes \mathbf{v} = \mathbf{uv}$ given by following rules:

$$(\mathbf{u} \otimes \mathbf{v})(\mathbf{w}) = \mathbf{uv}(\mathbf{w}) = \mathbf{u}(\mathbf{v}.\mathbf{w}). \qquad \forall\ \ \mathbf{w}\ \varepsilon\ V.$$

By using the above definition and axioms of vector space, we get

$$\begin{aligned} (\mathbf{u} \otimes \mathbf{v})(\alpha_1\mathbf{w}_1 + \alpha_2\mathbf{w}_2) &= (\mathbf{uv})(\alpha_1\mathbf{w}_1 + \alpha_2\mathbf{w}_2) \\ &= \mathbf{u}(\alpha_1\mathbf{v}.\mathbf{w}_1 + \alpha_2\mathbf{v}.\mathbf{w}_2) \\ &= \alpha_1\mathbf{u}(\mathbf{v}.\mathbf{w}_1) + \alpha_2\mathbf{u}(\mathbf{v}.\mathbf{w}_2) \\ &= \alpha_1(\mathbf{uv})\mathbf{w}_1 + \alpha_2(\mathbf{uv})\mathbf{w}_2. \end{aligned}$$

i.e the function $(\mathbf{u} \otimes \mathbf{v})$ satisfies the linearity property. Hence, the tensor product $\mathbf{uv}$ is a second order tensor.

Let us assume that $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\} = \{\mathbf{e}_i\}$ is an orthonormal basis in n dimensional vector space $\mathbf{V}$ and $\{\mathbf{e}_i\mathbf{e}_j\}$ is the set of all possible tensor product of $\{i, j\} = 1, \cdots, n$. Assuming that there exist n^2 real numbers λ_{ij} such that

$$\lambda_{ij}\mathbf{e}_i\mathbf{e}_j = 0. \tag{2.8}$$

Then we get,

$$\begin{aligned} (\lambda_{ij}\mathbf{e}_i\mathbf{e}_j).\mathbf{e}_k &= 0.\mathbf{e}_k, \\ (\lambda_{ij}\mathbf{e}_i)(\mathbf{e}_j.\mathbf{e}_k) &= \mathbf{0}, \\ \lambda_{ij}\mathbf{e}_i\delta_{jk} &= \mathbf{0}, \\ \lambda_{ik}\mathbf{e}_i &= \mathbf{0}. \qquad \text{for} \quad i = 1, 2, \cdots, n \end{aligned} \tag{2.9}$$

Hence $\lambda_{ik} = 0$, for $\{i, k\} = 1, \cdots, n$. Since $\{\mathbf{e}_i\}_{i=1}^n$ is a basis in n dimensional vector space $\mathbf{V}$, so $\{\mathbf{e}_i\mathbf{e}_j\}$ is a linearly independent set of the vector space L_2. Now we have to show that every member of L_2 is a linear combination of member of $\{\mathbf{e}_i\mathbf{e}_j\}$. For this, let us consider a tensor $\mathbf{T} \; \varepsilon \; L_2$. Since $\mathbf{T}\mathbf{e}_j$ is a vector of $\mathbf{V}$ and it can be expressed as linear combination of the vector $\mathbf{e}_1, \cdots, \mathbf{e}_n$.

$$\mathbf{T}\mathbf{e}_j = T_{ij}\mathbf{e}_j. \tag{2.10}$$

By using the orthonormality of the basis $\mathbf{e}_j$, properties of tensor product and the above relation for an arbitrary vector $\mathbf{v} = v_k\mathbf{e}_k$, we get

$$\begin{aligned} (\mathbf{T} - T_{ij}\mathbf{e}_i\mathbf{e}_j)\mathbf{v} &= (\mathbf{T} - T_{ij}\mathbf{e}_i\mathbf{e}_j)v_k\mathbf{e}_k, \\ &= v_k\mathbf{T}\mathbf{e}_k - v_kT_{ij}\mathbf{e}_i(\mathbf{e}_j.e_k), \\ &= v_k(T_{ik}\mathbf{e}_i - T_{ik}\mathbf{e}_i) \\ &= 0 \qquad \forall \;\; \mathbf{v} \; \varepsilon \; \mathbf{V}. \end{aligned} \tag{2.11}$$

Hence $\mathbf{T} - T_{ij}\mathbf{e}_i\mathbf{e}_j = 0$ (the second order null tensor). This tensor maps on arbitrary vectors into the zero vector or $\mathbf{T} = T_{ij}\mathbf{e}_i\mathbf{e}_j$.

Tensor of order 3

A linear transformation $\mathbf{A}$ define on $\mathbf{V}$ having values in L_2 is called a tensor of order 3 as $\mathbf{A} : \mathbf{V} \to L_2$. If the value of $\mathbf{A}$, corresponding to vector $\mathbf{v}$ is order 2 tensor $\mathbf{T}$, then we can write

$$\mathbf{T} = \mathbf{A}(\mathbf{v}) = \mathbf{A}\mathbf{v}.$$

The space L_3 denote the set of all third order tensors. Similarly the tensor of order n is a linear mapping from $\mathbf{V}$ into L_{n-1}.

2.4 Elasticity tensor

In elasticity, the elasticity tensor (or the stiffness tensor) usually denoted C_{ijkl} plays a vital role. By definition, a medium is said to be elastic if it returns to its initial configuration when the external forces are removed. The generalized Hooke's law states:

$$T_{ij} = C_{ijkl} S_{kl}, \tag{2.12}$$

where T_{ij} and S_{kl} is a stress tensor and strain tensor respectively. C_{ijkl} is an elasticity tensor of order 4. The fourth order C_{ijkl} tensor has $(3^4 =)81$ components. Due to symmetries,

$$C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij} \tag{2.13}$$

C_{ijkl} leads to reduction and therefore the number of independent components of the elasticity tensor reduce from 81 to 21. Familiar Voigt two-index notation is used which is:

$$\begin{aligned} &11 \to 1, && 22 \to 2, && 33 \to 3, \\ &23 \to 4, && 13 \to 5, && 12 \to 6. \end{aligned} \tag{2.14}$$

Using this notation, we have

$$\begin{aligned} C_{3333} &= c_{33}, \\ C_{1212} &= c_{66}, \\ C_{1123} &= c_{14}, \\ C_{2233} &= c_{23}, \\ C_{2223} &= c_{24}, \\ C_{2313} &= c_{45}. \end{aligned} \tag{2.15}$$

and the tensor can be conveniently represented by a 6×6 symmetric matrix, whose first row is,

$$(c_{11}, c_{12}, c_{13}, c_{14}, c_{15}, c_{16}).$$

2.5 Invariants

An invariant is a function of components, which does not change under certain classes of transformations. Let $\mathbf{V}$ have an orthonormal basis $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$. Then the space L_2 has a basis $\{\mathbf{e}_i \otimes \mathbf{e}_j\}_{i,j=1}^{n}$ and a tensor $\mathbf{T}$ in L_2 can be expressed in term of its components with respect to this basis,

$$\mathbf{T} = \sum_{i,j=1}^{n} T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j. \tag{2.16}$$

Usually we omit the summation symbol because we use the convention that a repeated symbol is summed over. If we choose another orthonormal basis the components will change. A component or function of components of a tensor is called an invariant if this function is not affected by any change of basis. For example, let us consider a rank two tensor, its components transform like :

$$T^{'}_{ij} = q_{li}q_{mj}T_{lm}.$$

Therefore,

$$\begin{aligned} T^{'}_{ii} &= q_{li}q_{mi}T_{lm}, \\ T^{'}_{ii} &= \delta_{lm}T_{lm}, \\ T^{'}_{ii} &= T_{ll}. \end{aligned} \tag{2.17}$$

Thus Eq.(2.17) indicates:

$$T^{'}_{11} + T^{'}_{22} + T^{'}_{33} = T_{11} + T_{22} + T_{33} = \mathrm{tr}(T).$$

This is the only linear invariant of T in three dimension.

2.6 Isotropic tensor

A tensor is called an isotropic tensor if each of its components remain the same with respect to every basis. All the scalars are isotropic but, the rank-1 tensor are not isotropic. Moreover, the unique rank-2 isotropic tensor is kronecker delta δ_{ij}. As

$$\begin{aligned} \delta^{'}_{ij} &= q_{li}q_{mj}\delta_{lm}, \\ \delta^{'}_{ij} &= q_{li}q_{lj}, \\ \delta^{'}_{ij} &= \delta_{ij}. \end{aligned} \tag{2.18}$$

Also the unique rank-3 isotropic tensor is the permutation tensor ε_{ijk} [10].

2.7 Cartesian tensor

When the only transformation occurs from one orthogonal coordinate system to another orthogonal coordinate system, then the tensors involved are referred as cartesian tensor. In the three dimensional Euclidean space, cartesian coordinate system can be rectangular coordinate as (x, y, z) usual with the basis $\{(1,0,0),(0,1,0),(0,0,1)\}$.

Cartesian tensor $\mathbf{T}$ of rank n in form of components represented as $T_{ijk....m}$ with n indices $ijk\cdots m$. In three dimensional space, $\mathbf{T}$ has 3^n components and the transformation occur from coordinate system S to $S^{'}$ by a rotation are as follow

$$T^{'}_{ijk....m} \;=\; q_{ip}\, q_{jq}\, q_{kr} \,.......\, q_{ms}\, T_{pqr.....s} \tag{2.19}$$

where q_{ij} = cos(angle between the $x_i - axis$ and the transformed $x_j - axis$). As special cases, a scalar is a zero-th rank tensor $T^{'} = T$. A first rank tensor (vector) is transformed according to $T^{'}_i = q_{ij}T_j$. A second rank tensor is transformed according to $T^{'}_{ij} = q_{is}q_{jt}T_{st}$.

2.8 Orthogonal group

In mathematics, the orthogonal group in dimension n is the group which represents the distance preserving transformations of Euclidean space. It is denoted by $O(n)$. The orthogonal group $O(n)$ is the $n \times n$ orthogonal matrices and the group operation is matrix multiplication defined as

$$O(n) \;=\; \{\, A\, \epsilon\, GL(n) \mid A^T A \;=\; AA^T \;=\; I \,\}, \tag{2.20}$$

where A^T is the transpose of A and I is the identity. $GL(n)$ represents the general linear group. The general linear group in n dimensions is the set of $n \times n$ invertible matrices, with respect to the operation of ordinary matrix multiplication. The determinant of an orthogonal matrix $O(n)$ is either 1 or -1. Geometrically, elements of orthogonal matrix $O(n)$ are either rotations or combination of rotations and reflections.

2.9 Special orthogonal group

An important subgroup of orthogonal group $O(n)$ having determinant $+1$ is special orthogonal group. It is denoted by $SO(n)$ and defined as

$$SO(n) = \{\, Q \;\varepsilon\; O(n) \;\mid\; \det(Q) \;=\; 1 \,\}. \tag{2.21}$$

This group is also called the rotation group as in dimension 2 its elements are usual rotation about a fixed point and in dimension 3 its elements are usual rotation about a fixed line.

The special orthogonal group in two dimensions $SO(2)$, is defined by the set of real and orthogonal 2×2 matrices having determinant $+1$. In order to find the most general form of such matrices, let us assume that a, b, c and d represent four real numbers which satisfy the property of det = +1. With those numbers we construct a 2×2 matrix ρ, according to:

$$\rho = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \qquad \text{with } \det(\rho) = ad - bc = 1. \tag{2.22}$$

When we assume that ρ is orthogonal, then $\rho^{-1} = \rho^t$. This leads to the following relations for its components: $d = a$ and $c = b$. Using the condition of determinant, we find the following relation for a and b:

$$a^2 + b^2 = 1,$$

Now, recalling that a and b are real numbers, we obtain as a consequence of this relation that:

$$-1 \leqslant a \leqslant +1 \qquad \text{and} \qquad -1 \leqslant b \leqslant +1.$$

At this stage introduce a real parameter θ such that:

$$a = \cos\theta \qquad b = \sin\theta.$$

So, we obtain real and orthogonal 2×2 matrix. The general form is:

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \tag{2.23}$$

The group $SO(3)$ is the set of all rotations of three dimensional euclidean space. It is used to describe the possible rotational symmetries of an object and the possible orientations of an object in space. The orthogonal matrix under $SO(3)$ is

$$A = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.24}$$

2.10 Permutation symbol

The permutation symbol is a three index symbol used in mathematics particularly in linear algebra, differential geometry and tensor analysis. It represents a collection of numbers defined from the sign of a permutation of natural number $1, 2, ..., n$ where n is a positive integer. Permutation symbol is called the Levi-Civita symbol.

It is named after the Italian mathematician and physicist Tullio Levi-Civita. Its other names include the permutation symbol, anti-symmetric symbol or alternating symbol.

In two dimension, the permutation symbol is defined as

$$\epsilon_{ij} = \begin{cases} 1 & \text{for} \quad ij = 12, \\ -1 & \text{for} \quad ij = 21, \\ 0 & \text{for} \quad i = j. \end{cases} \tag{2.25}$$

In three dimension, an alternating symbol is defined as

$$\epsilon_{ijk} = \begin{cases} 1 & \text{for} \quad ijk = 123, \quad 231, \quad 312, \\ -1 & \text{for} \quad ijk = 132, \quad 321, \quad 213, \\ 0 & \text{for} \quad i = j \quad , j = k \quad , k = i \quad . \end{cases} \tag{2.26}$$

There are several common notations used for the permutation symbol also known as Levi-Civita symbol. The first one which is used as a permutation symbol is Greek epsilon character ϵ_{ijk} [10], the second one which uses the curly variant ε_{ijk} and the third of which uses a Latin lower case e_{ijk}. There are some useful identities involving pairs of Levi-Civita tensors. The most general one is

$$\begin{aligned} &\epsilon_{ijk}\epsilon_{pqr} = \delta_{ip}\delta_{jq}\delta_{kr} + \delta_{iq}\delta_{jr}\delta_{kp} + \delta_{ir}\delta_{jp}\delta_{kq} - \delta_{ir}\delta_{jq}\delta_{kp} - \delta_{iq}\ \delta_{jp}\delta_{kr} - \delta_{ip}\delta_{jr}\delta_{kq}, \\ &\epsilon_{ijk}\epsilon_{iqr} = \delta_{jq}\ \delta_{kr}\ -\ \delta_{jr}\ \delta_{kq}, \\ &\epsilon_{ijk}\epsilon_{ijr} = 2\ \delta_{kr}, \\ &\epsilon_{ijk}\epsilon_{ijk} = 6. \end{aligned} \tag{2.27}$$

2.11 Permutation tensor

A tensor $\mathbf{T}$ is said to be completely anti symmetric if it is anti symmetric in every pair of covariant indices i, j and in every pair of contravariant indices k, l:

$$t_{ij} = -t_{ji}, \qquad\qquad t^{kl} = -t^{kl}.$$

The completely antisymmetric tensor of rank n in the space $\mathbf{R}^n$ is called a permutation tensor. The coordinate of the covariant permutation tensor for a covariant basis $\mathbf{b}_1, ..., \mathbf{b}_n$ are uniquely determined. They are designed by $\epsilon_{i_1,...,i_n}$.

$$\epsilon(b_{i_1}, ..., b_{i_n}) = \epsilon_{i_1,...,i_n}.$$

The permutation tensor of rank four is important in general relativity, and its components are defined as

$$\epsilon_{ijkl} = \begin{cases} 1 & \text{if } ijkl \text{ is an even permutation of 0123,} \\ -1 & \text{if } ijkl \text{ is an odd permutation of 0123,} \\ 0 & \text{otherwise.} \end{cases} \tag{2.28}$$

Chapter 3

Linear invariants of elasticity tensor under SO(2) and SO(3)

This chapter provides a review of the number of linear invariants under $SO(3)$ as well as $SO(2)$ of elasticity tensor of an arbitrary rank. A linear form is defined in term of elements of a tensor. A group of special orthogonal transformation is considere here. The matrix A associated with an orthogonal coordinate transformation in three dimension is mentioned in Eq.(2.24). Under orthogonal group of transformation $SO(3)$, the elasticity tensor or stiffness tensor possesses only two linear invariants of rank 4, which are as follow.

$$\begin{aligned} A_1 &= C_{ijij} = c_{11} + c_{22} + c_{33} + 2(c_{44} + c_{55} + c_{66}), \\ A_2 &= C_{iijj} = c_{11} + c_{22} + c_{33} + 2(c_{12} + c_{23} + c_{13}). \end{aligned} \tag{3.1}$$

Here the familiar Voigt's two index notation is used. When the transformation is confined to the group of rotation about a fixed axis say the x_3-axis, the number of invariant increases to five. So under $SO(2)$ the five linear invariants are as follow:

$$\begin{aligned} L_1 &= c_{33}, \\ L_2 &= c_{13} + c_{23}, \\ L_3 &= c_{44} + c_{55}, \\ L_4 &= c_{11} + c_{22} + 2c_{12}, \\ L_5 &= c_{11} + c_{22} + 2c_{66}. \end{aligned} \tag{3.2}$$

It is well known [4] that set of invariants $\{A_1, A_2\}$ under $SO(3)$ is complete. Here the word complete means that any other linear invariants under $SO(3)$ is a linear combination of A_1 and A_2. Similarly the same result of completeness hold for linear invariants under $SO(2)$ and the set $\{L_1, L_2, L_3, L_4, L_5\}$ forms a basis.

3.1 Criterion for invariance

The transformation matrix A for a rotation through an angle θ about the $x_3 - axis$ is mentioned in Eq.(2.24) has eigenvalues $e^{i\theta}$, $e^{-i\theta}$ and 1 and their respective eigen vectors are:

$$e_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}, e_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

By the method of diagonalization we have, $Q = (B^t)^* AB$ where B is the matrix of eigen vectors of matrix A and $(B^t)^*$ is a complex conjugate of B.

$$Q = (B^t)^* AB = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \\ -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \\ -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}}\cos\theta + \frac{i}{\sqrt{2}}\sin\theta & \frac{i}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta & 0 \\ \frac{i}{\sqrt{2}}\cos\theta - \frac{1}{\sqrt{2}}\sin\theta & \frac{1}{\sqrt{2}}\cos\theta - \frac{i}{\sqrt{2}}\sin\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$Q = \begin{pmatrix} e^{i\theta} & 0 & 0 \\ 0 & e^{-i\theta} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.3}$$

Thus rotation matrix A takes the diagonal form.

Let us consider the transformation of a tensor of rank 4. If γ'_{pqrs} denotes the component after rotation through an angle θ. Then γ_{pqrs} transforms according to following rule.

$$\gamma'_{pqrs} = q_{pi}\, q_{qj}\, q_{rk}\, q_{sl}\, \gamma_{ijkl}, \qquad i,j,k,l,p,q,r,s = 1,2,3. \tag{3.4}$$

and $q's$ is the matrix entries of Eq.(3.3). Since Q is diagonal matrix, so it must be i = p, j = q, k = r, l = s. Otherwise it is equal to zero. Hence,

$$\gamma'_{pqrs} = e^{i(v_1-v_2)\theta}\,\gamma_{pqrs}. \tag{3.5}$$

In general, if $\gamma'_{i_1,....,i_n}$ of rank n denotes the component after rotation through an angle θ, then

$$\gamma'_{i_1,....,i_n} = e^{i(v_1-v_2)\theta}\,\gamma_{i_1,....,i_n}, \tag{3.6}$$

where
v_1 = numbers of $1's$ in the indices $i_1, ..., i_n$
v_2 = numbers of $2's$ in the indices $i_1, ..., i_n$

It is quite obvious that the component remains invariant for arbitrary θ if and only if number of $1's$ and $2's$ are equal i.e $v_1 = v_2$. The rest of components will be vanished.

3.2 Main results

In this section, some results are collected which help to find linear invariants of a tensor of an arbitrary rank in two and three dimensions respectively. Let $\mathbf{T} = T_{i_1,....,i_n}$ be a tensor of rank n in three dimensions. Let v_1 and v_2 denote respectively, as the number of $1's$ and $2's$ among the subscripts $i_1, ..., i_n$. For example in T_{212231}, $v_1 = 2$ and $v_2 = 3$, as the number of $1's$ in the subscipt is 2 and the number of $2's$ in the subscript is 3. The following theorems are presented by Ahmad [4].

Theorem 3.2.1. *The number of linear invariants of T under the group of rotations about a fixed axis, say the x_3-axis, is the same as the number of components $\gamma_{i_1,....,i_n}$ with $v_1 = v_2$.*

Theorem 3.2.2. *The number of invariants under $SO(2)$ is the same as the number of ways S_n, in which n elements containing $1s$, $2s$ and $3s$ can be arranged with the stipulation that the number of $1's$ and $2's$ are equal.*

It is easily see that

$$S_n = \sum_{r=0}^{k} \frac{(2k)!}{(2r)!(k-r)!(k-r)!}, \qquad \text{if } n = 2k,$$

and (3.7)

$$S_n = \sum_{r=0}^{k} \frac{(2k+1)!}{(2r+1)!(k-r)!(k-r)!}, \qquad \text{if } n = 2k+1.$$

Hence by using this formula we can find the invariants under $SO(2)$. For example, the number of linear invariants under $SO(2)$ of a tensor of rank 8 is 1051 by using the Eq.(3.7).

3.2.1 Linear invariants under SO(2)

We apply the Theorem 3.2.1 to find some invariants under $SO(2)$. The linear invariants of a tensor of rank n with $2 \leq n \leq 4$ have been mentioned below.

1) **For** $n = 2$

In this case, with $v_1 = v_2$ we have three cases γ_{33}, γ_{12}, γ_{21}. In the case of symmetric tensor, i.e $\gamma_{ij} = \gamma_{ji}$, the number of invariant decreases to two which are γ_{33}, γ_{12}.

2) **For** $n = 3$

In this case with $v_1 = v_2$ we have, γ_{333}, γ_{312}, γ_{321}, γ_{132}, γ_{231}, γ_{123} and γ_{213}. Thus there are seven linear invariants of rank 3 under $SO(2)$.

3) **For** $n = 4$

In this case, with $v_1 = v_2$ the number of linear invariants of a tensor of rank 4, with no symmetry are:

γ_{3333}, γ_{3312}, γ_{3321}, γ_{3132}, γ_{3123}, γ_{3213}, γ_{3231}, γ_{1122}, γ_{1212}, γ_{2121}, γ_{1221}, γ_{2211}, γ_{2112}, γ_{2313}, γ_{2331}, γ_{2133}, γ_{1332}, γ_{1323}, γ_{1233}.

But the elasticity tensor has symmetry $\gamma_{ijkl} = \gamma_{jikl} = \gamma_{ijlk} = \gamma_{klij}$. Due to this symmetry the number of invariant reduces to five. Thus only five invariants are left in this case, which are

γ_{3333}, γ_{3312}, γ_{1332}, γ_{1122}, γ_{1212}.

To check that the above five linear invariants are same as the invariants found by Ahmad [3], we consider the transformation rule.

$$\gamma'_{pqrs} = b_{pi}\, b_{qj}\, b_{rk}\, b_{sl}\, \gamma_{ijkl}, \tag{3.8}$$

where

$$B = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i & 0 \\ i & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.9}$$

So, by the transformation rule firstly we transform γ_{3333}.

1) $$\gamma'_{3333} = b_{3i}\ b_{3j}\ b_{3k}\ b_{3l}\ c_{ijkl},$$

$$= 1.1.1.1.c_{3333},$$

$$= c_{33}, \qquad \text{where } i,j,k,l = 3$$

To verify whether c_{33} is invariant or not, we use transformation matrix A and transform c_{33} according to same transformation rule which we used above.

$$c''_{33} = a_{3i}\ a_{3j}\ a_{3k}\ a_{3l}\ c_{ijkl}, \qquad \text{where } i,j,k,l = 3$$

$$c''_{33} = c_{33}.$$

2) $$\gamma'_{1233} = b_{1i}\ b_{2j}\ b_{3k}\ b_{3l}\ c_{ijkl},$$

Possibility for i and j is $i = j = 1, 2$ and for $k, l = 3$.

$$\gamma'_{1233} = b_{1i} b_{2j} c_{ij33},$$

$$= b_{11} b_{12} c_{1133} + b_{11} b_{22} c_{1233} + b_{21} b_{12} c_{213} + b_{21} b_{22} c_{2233},$$

$$= i c_{13} + c_{63} + i^2 c_{63} + i c_{23},$$

$$= i(c_{13} + c_{23}).$$

Again we need to verify here whether $c_{13} + c_{23}$ is invariant or not.

$$c''_{13} = a_{1i}\ a_{1j}\ a_{3k}\ a_{3l}\ c_{ijkl},$$

$$= c_{13} \cos^2\theta - 2c_{36} \cos\theta \sin\theta + c_{23} \sin^2\theta,$$

and

$$c''_{23} = a_{2i}\ a_{2j}\ a_{3k}\ a_{3l}\ c_{ijkl},$$

$$= c_{23}\cos^2\theta + 2c_{36}\cos\theta\sin\theta + c_{13}\sin^2\theta.$$

$$c''_{13} + c''_{23} = (c_{13}\cos^2\theta - 2c_{36}\cos\theta\sin\theta + c_{23}\sin^2\theta)$$

$$+ (c_{23}\cos^2\theta + 2c_{36}\cos\theta\sin\theta + c_{13}\sin^2\theta),$$

$$= c_{13}(\cos^2\theta + \sin^2\theta) + c_{23}(\cos^2\theta + \sin^2\theta),$$

$$= c_{13} + c_{23}.$$

3) $$\gamma'_{1323} = b_{1i}\ b_{3j}\ b_{2k}\ b_{3l}\ c_{ijkl}.$$

Possibility for i and j is $i = k = 1, 2$ and for j and l only possibility is 3.

$$\gamma'_{1323} = b_{1i}b_{2k}c_{i3k3},$$

$$= b_{11}b_{12}c_{1313} + b_{11}b_{22}c_{1323} + b_{21}b_{12}c_{2313} + b_{21}b_{22}c_{2323},$$

$$= ic_{55} + c_{54} + i^2c_{45} + ic_{44},$$

$$= i(c_{44} + c_{55}).$$

To verify $c_{44} + c_{55}$ is invariant or not, we have

$$c''_{44} = a_{2i}\ a_{3j}\ a_{2k}\ a_{3l}\ c_{ijkl},$$

$$= c_{44}\cos^2\theta + 2c_{45}\cos\theta\sin\theta + c_{55}\sin^2\theta.$$

and

$$c''_{55} = a_{1i}\ a_{3j}\ a_{1k}\ a_{3l}\ c_{ijkl},$$

$$= c_{55}\cos^2\theta - 2c_{45}\cos\theta\sin\theta + c_{44}\sin^2\theta.$$

$$c''_{44} + c''_{55} = (c_{44}\cos^2\theta + 2c_{45}\cos\theta\sin\theta + c_{55}\sin^2\theta) + (c_{55}\cos^2\theta - 2c_{45}\cos\theta\sin\theta + c_{44}\sin^2\theta),$$

$$= c_{44}(\cos^2\theta + \sin^2\theta) + c_{55}(\cos^2\theta + \sin^2\theta),$$

$$c''_{44} + c''_{55} = c_{44} + c_{55}.$$

4) $\gamma'_{1212} = b_{1i}\ b_{2j}\ b_{1k}\ b_{2l}\ c_{ijkl},$

Only possibilty for i, j, k, l is 1 and 2. So,

$$\begin{aligned}
\gamma'_{1212} &= b_{11}b_{12}b_{11}b_{12}c_{1111} + b_{11}b_{12}b_{11}b_{22}c_{1112} + b_{11}b_{12}b_{21}b_{12}c_{1121} + b_{11}b_{12}b_{21}b_{22}c_{1122} \\
&\quad b_{11}b_{22}b_{11}b_{12}c_{1211} + b_{11}b_{22}b_{11}b_{22}c_{1212} + b_{11}b_{22}b_{21}b_{12}c_{1221} + b_{11}b_{21}b_{22}b_{22}c_{1222} \\
&\quad b_{21}b_{12}b_{11}b_{12}c_{2111} + b_{21}b_{12}b_{11}b_{22}c_{2112} + b_{21}b_{12}b_{21}b_{12}c_{2121} + b_{21}b_{12}b_{21}b_{22}c_{2122} \\
&\quad b_{21}b_{22}b_{11}b_{12}c_{2211} + b_{21}b_{22}b_{11}b_{22}c_{2212} + b_{21}b_{22}b_{21}b_{12}c_{2221} + b_{21}b_{21}b_{21}b_{22}c_{2222}, \\
&= -\tfrac{1}{4}c_{11} - \tfrac{1}{4}c_{12} - \tfrac{1}{4}c_{21} - \tfrac{1}{4}c_{22}, \\
&= -\tfrac{1}{4}(c_{11} + 2c_{12} + c_{22}), \\
&= c_{11} + 2c_{12} + c_{22}.
\end{aligned}$$

Here we verify $(c_{11} + 2c_{12} + c_{22})$ is invariant or not. Thus

$$\begin{aligned}
c''_{11} &= a_{1i}\ a_{1j}\ a_{1k}\ a_{1l}\ c_{ijkl}, \\
&= c_{11}\cos^4\theta - 4c_{66}\cos^3\theta\sin\theta + 2c_{12}\cos^2\theta\sin^2\theta + 4c_{66}\cos^2\theta\sin^2\theta - 4c_{26}\cos\theta\sin^3\theta \\
&\quad + c_{22}\sin^4\theta.
\end{aligned}$$

and

$$\begin{aligned}
c''_{22} &= a_{2i}\ a_{2j}\ a_{2k}\ a_{2l}\ c_{ijkl}, \\
&= c_{22}\cos^4\theta + 4c_{26}\cos^3\theta\sin\theta + 2c_{12}\cos^2\theta\sin^2\theta + 4c_{66}\cos^2\theta\sin^2\theta + 4c_{66}\cos\theta\sin^3\theta \\
&\quad + c_{11}\sin^4\theta.
\end{aligned}$$

and

$$\begin{aligned}
c''_{12} &= a_{1i}\ a_{1j}\ a_{1k}\ a_{2l}\ c_{ijkl}, \\
&= c_{12}\cos^4\theta + 2c_{66}\cos^3\theta\sin\theta - 2c_{26}\cos^3\theta\sin\theta + c_{11}\cos^2\theta\sin^2\theta - 4c_{66}\cos^2\theta\sin^2\theta
\end{aligned}$$

$$+ c_{22}\cos^2\theta\sin^2\theta - 2c_{66}\cos\theta\sin^3\theta + 2c_{26}\cos\theta\sin^3\theta + c_{12}\sin^4\theta.$$

Now

$c_{11}^{''} + c_{22}^{''} + 2c_{12}^{''} = c_{11}\cos^4\theta - 4c_{66}\cos^3\theta\sin\theta + 2c_{12}\cos^2\theta\sin^2\theta + 4c_{66}\cos^2\theta\sin^2\theta -$

$4c_{26}\cos\theta\sin^3\theta + c_{22}\sin^4\theta + c_{22}\cos^4\theta + 4c_{26}\cos^3\theta\sin\theta + 2c_{12}\cos^2\theta\sin^2\theta + 4c_{66}\cos^2\theta\sin^2\theta +$

$4c_{66}\cos\theta\sin^3\theta + c_{11}\sin^4\theta + 2(c_{12}\cos^4\theta + 2c_{66}\cos^3\theta\sin\theta - 2c_{26}\cos^3\theta\sin\theta + c_{11}\cos^2\theta\sin^2\theta -$

$4c_{66}\cos^2\theta\sin^2\theta + c_{22}\cos^2\theta\sin^2\theta - 2c_{66}\cos\theta\sin^3\theta + 2c_{26}\cos\theta\sin^3\theta + c_{12}\sin^4\theta),$

By combining the coefficients of c_{11}, c_{22}, etc. We have

$c_{11}^{''} + c_{22}^{''} + 2c_{12}^{''} = c_{11}(\cos^2\theta + \sin^2\theta)^2 + 2c_{12}(\cos^2\theta + \sin^2\theta)^2 + c_{22}(\cos^2\theta + \sin^2\theta)^2,$

$c_{11}^{''} + c_{22}^{''} + 2c_{12}^{''} = c_{11} + 2c_{12} + c_{22}.$

Lastly, on the same pattern we have

5) $\quad \gamma_{1122}^{'} = c_{11} + 2c_{66} + c_{22}.$

and

$c_{11}^{''} + c_{22}^{''} + 2c_{66}^{''} = c_{11} + 2c_{66} + c_{22}.$

This completes the result of Theorem 3.2.1.

The following theorem helps us to find linear invariants of tensor of an arbitrary rank in three dimension. Let T $= T_{i_1,\ldots,i_n}$ be a tensor of rank n in three dimension [4].

Theorem 3.2.3. *The number of linear invariants of* $\boldsymbol{T}$ *under* $SO(3)$ *is the same as the dimension of the space of isotropic tensors of rank n possessing the same symmetries, if any, as* $\boldsymbol{T}$.

The following formula determines the precise number of linear invariants of a tensor under $SO(3)$.

Theorem 3.2.4. *The number* $I_3(n)$ *of independent linear invariants for a tensor of arbitrary rank n in three dimensions is given by*

$$I_3(n) = \sum_{i=0}^{\lfloor \frac{n+1}{2} \rfloor} \frac{n!(n+1-3i)}{i!i!(n+1-2i)!}, \tag{3.10}$$

where $\lfloor \frac{n+1}{2} \rfloor$ represents the largest integer less than or equal to $\frac{n+1}{2}$.

3.2.2 Linear Invariants under $SO(3)$

Here, we verify that the number of invariants calculated by finding the dimension of the space of isotropic tensors agrees with the same number produced in Eq.(3.10). The direct approach of Eq.(3.10) has the benefits that not only it give the number of invariants but also, at the same time, it produces the corresponding invariants as well. However, for large n, it is quite difficult to find the invariants by using this approach. Above formula becomes handy in such a situation. For example, for $n = 10$, it produces 603 linear invariants.

Now we apply Theorem 3.2.3 to find the linear invariants of tensor of rank n, with $2 \leq n \leq 6$.

1) For $n = 2$:

In this case $T_{ij} = \mu\delta_{ij}$, so there will be only one linear invariants of T_{ij}, called T_{ii}. As $T_{ii} = T_{11} + T_{22} + T_{33}$.

2) For $n = 3$:

Here, $T_{ijk} = \mu\epsilon_{ijk}$. Again there will be only one linear invariant $\varepsilon_{ijk}T_{ijk}$ is possible.

3) For $n = 4$:

For this case, $T_{ijkl} = \mu_1\delta_{ij}\delta_{kl} + \mu_2\delta_{ik}\delta_{jl} + \mu_3\delta_{il}\delta_{jk}$. Thus the 4 rank tensor possesses three linear invariant without symmetries. Symmetry in the first two indices or the last two indices of the tensor make the last two tensors of the above expression equals. This reducing the number of invariants from three to two. For elasticity tensor C_{ijkl} , these two invariants C_{iijj} and C_{ijij} are same as mentioned in Eq.(3.1) and Eq.(3.2).

4) For $n = 5$:

Here,

$$T_{ijklm} \;=\; \mu_1 \delta_{ij} \epsilon_{klm} \;+\; .... + \; \mu_{10} \delta_{lm} \epsilon_{ijk},$$

Although, there are 10 terms on the right side of the above expression, but only six of them are linearly independent. Therefore tensor of rank 5 has 6 linear invariants.

5) For $n = 6$:

Since $6!/2!2!2!3! = 15$, so there will be 15 terms of the form $\mu \delta_{ij} \delta_{kl} \delta_{mn}$ in the expansion of T_{ijklmn}. This set of tensors is linearly independent. Hence a tensor of rank 6 has 15 linear invariants.

Chapter 4

Quadratic invariants of elasticity tensor under SO(2) and SO(3)

A quadratic invariant is defined as a quadratic form in the elements of a tensor that remains invariant under a group of coordinate transformations. Ting [7] and Ahmad [3] considered quadratic invariants of elasticity tensor. In 1987, Ting [7] reported 2 second-order invariants of elasticity tensor C_{ijkl} under $SO(3)$ and 15 second-order invariants under $SO(2)$. After that in 2002, Ahmad [3] increased these numbers to 4 and 17, respectively. Furthermore he demonstrated that the seven quadratic invariants are independent but still it was an open question whether the list of these invariants is complete or not. In 2007, Norris[2] showed that the seven invariants identified by Ahmad [3] form a complete basis. He also proved that the corresponding set under SO(2) consists of 35 quadratic invariants.

4.1 Quadratic invariants under SO(3)

In this section, a quadratic invariants under $SO(3)$ are studied. For a fourth order elasticity tensor C_{ijkl}, there are four quadratic invariants under $SO(3)$ [3].

$$I_1 = C_{ijkl}C_{ijkl}, \quad I_2 = C_{iikl}C_{jjkl}, \quad I_3 = C_{iikl}C_{jkjl}, \quad I_4 = C_{kiil}C_{kjjl} \quad i,j,k,l = 1,2,3. \tag{4.1}$$

Expanding the above expressions one by one, we have

$$I_1 = c_{11}^2 + c_{22}^2 + c_{33}^2 + 2(c_{12}^2 + c_{23}^2 + c_{13}^2) + 4(c_{16}^2 + c_{15}^2 + c_{14}^2 + c_{66}^2 + c_{26}^2 + c_{36}^2 + c_{55}^2 + c_{35}^2$$

$$+ c_{34}^2 + c_{25}^2 + c_{24}^2 + c_{44}^2) + 8(c_{45}^2 + c_{46}^2 + c_{56}^2),$$

$$I_2 = \sum_{i=1}^{3}(c_{i1} + c_{i2} + c_{i3})^2 + 2\sum_{i=4}^{6}(c_{i1} + c_{i2} + c_{i3})^2,$$

$$I_3 = (c_{11}+c_{12}+c_{13})(c_{11}+c_{55}+c_{66}) + (c_{12}+c_{22}+c_{23})(c_{22}+c_{44}+c_{66})$$

$$+ (c_{13}+c_{23}+c_{33})(c_{33}+c_{44}+c_{55}) + 2[(c_{14}+c_{24}+c_{34})(c_{24}+c_{34}+c_{56})$$

$$+ (c_{15}+c_{25}+c_{35})(c_{15}+c_{35}+c_{46}) + (c_{16}+c_{26}+c_{36})(c_{16}+c_{26}+c_{45})],$$

$$I_4 = (c_{11}+c_{55}+c_{66})^2 + (c_{22}+c_{44}+c_{66})^2 + (c_{33}+c_{44}+c_{55})^2$$

$$+ 2[(c_{16}+c_{26}+c_{45})^2 + (c_{15}+c_{35}+c_{46})^2 + (c_{24}+c_{34}+c_{56})^2],$$

where we have used the Voigt two index notation mentioned in Eq.(2.14). Now we have a tensor defined by [3],

$$T_{kn} = \frac{1}{2}\varepsilon_{ijk}\varepsilon_{lmn}c_{iljm}, \qquad \{i,j,k,l,m,n\} = 1,2,3. \tag{4.2}$$

ε_{ijk} is permutation symbol defined in Eq.(2.26). The components of tensor T_{kn} are found by giving variation to Eq.(4.2), e.g

$$T_{11} = \tfrac{1}{2}\varepsilon_{ij1}\varepsilon_{lm1}C_{iljm},$$

$$= \tfrac{1}{2}\varepsilon_{231}\varepsilon_{lm1}C_{2l3m} + \tfrac{1}{2}\varepsilon_{321}\varepsilon_{lm1}C_{3l2m},$$

$$= \tfrac{1}{2}\varepsilon_{231}\varepsilon_{231}C_{2233} + \tfrac{1}{2}\varepsilon_{231}\varepsilon_{321}C_{2332} + \tfrac{1}{2}\varepsilon_{321}\varepsilon_{231}C_{3223} + \tfrac{1}{2}\varepsilon_{321}\varepsilon_{321}C_{3322},$$

$$= \tfrac{1}{2}c_{23} - \tfrac{1}{2}c_{44} - \tfrac{1}{2}c_{44} + \tfrac{1}{2}c_{23},$$

$$T_{11} = c_{23} - c_{44}.$$

Similarly,

$$T_{12} = \tfrac{1}{2}\varepsilon_{ij1}\varepsilon_{lm2}c_{iljm},$$

$$= \tfrac{1}{2}\varepsilon_{231}\varepsilon_{lm2}c_{2l3m} + \tfrac{1}{2}\varepsilon_{321}\varepsilon_{lm2}c_{3l2m},$$

$$= \tfrac{1}{2}\varepsilon_{231}\varepsilon_{132}c_{2l33} + \tfrac{1}{2}\varepsilon_{231}\varepsilon_{312}c_{2331} + \tfrac{1}{2}\varepsilon_{321}\varepsilon_{132}c_{3l23} + \tfrac{1}{2}\varepsilon_{321}\varepsilon_{312}c_{3321},$$

$$= -\tfrac{1}{2}c_{36} + \tfrac{1}{2}c_{45} + \tfrac{1}{2}c_{54} - \tfrac{1}{2}c_{36},$$

$$T_{12} = c_{45} - c_{36}.$$

$$T_{13} = \tfrac{1}{2}\varepsilon_{ij1}\varepsilon_{lm3}c_{iljm},$$

$$= \tfrac{1}{2}\varepsilon_{231}\varepsilon_{lm3}c_{2l3m} + \tfrac{1}{2}\varepsilon_{321}\varepsilon_{lm3}c_{3l2m},$$

$$= \tfrac{1}{2}\varepsilon_{231}\varepsilon_{123}c_{2l32} + \tfrac{1}{2}\varepsilon_{231}\varepsilon_{213}c_{2231} + \tfrac{1}{2}\varepsilon_{321}\varepsilon_{123}c_{3l22} + \tfrac{1}{2}\varepsilon_{321}\varepsilon_{213}c_{3221},$$

$$= \tfrac{1}{2}c_{46} - \tfrac{1}{2}c_{25} - \tfrac{1}{2}c_{52} + \tfrac{1}{2}c_{46},$$

$$T_{13} = c_{46} - c_{25}.$$

$$T_{22} = \tfrac{1}{2}\varepsilon_{ij2}\varepsilon_{lm2}c_{iljm},$$

$$= \tfrac{1}{2}\varepsilon_{132}\varepsilon_{lm2}c_{1l3m} + \tfrac{1}{2}\varepsilon_{312}\varepsilon_{lm2}c_{3l1m},$$

$$= \tfrac{1}{2}\varepsilon_{132}\varepsilon_{132}c_{1l33} + \tfrac{1}{2}\varepsilon_{132}\varepsilon_{312}c_{1331} + \tfrac{1}{2}\varepsilon_{312}\varepsilon_{132}c_{3113} + \tfrac{1}{2}\varepsilon_{312}\varepsilon_{312}c_{3311},$$

$$= \tfrac{1}{2}c_{13} - \tfrac{1}{2}c_{55} - \tfrac{1}{2}c_{55} + \tfrac{1}{2}c_{13},$$

$$T_{22} = c_{13} - c_{55}.$$

$$T_{23} = \tfrac{1}{2}\varepsilon_{ij2}\varepsilon_{lm3}c_{iljm},$$

$$= \tfrac{1}{2}\varepsilon_{132}\varepsilon_{lm3}c_{1l3m} + \tfrac{1}{2}\varepsilon_{312}\varepsilon_{lm3}c_{3l1m},$$

$$= \tfrac{1}{2}\varepsilon_{132}\varepsilon_{123}c_{1l32} + \tfrac{1}{2}\varepsilon_{132}\varepsilon_{213}c_{1231} + \tfrac{1}{2}\varepsilon_{312}\varepsilon_{123}c_{3112} + \tfrac{1}{2}\varepsilon_{312}\varepsilon_{213}c_{3211},$$

$$= -\tfrac{1}{2}c_{14} + \tfrac{1}{2}c_{65} + \tfrac{1}{2}c_{56} - \tfrac{1}{2}c_{41},$$

$$T_{23} = c_{56} - c_{14}.$$

and,

$$T_{31} = \tfrac{1}{2}\varepsilon_{ij3}\varepsilon_{lm1}c_{iljm},$$

$$= \tfrac{1}{2}\varepsilon_{123}\varepsilon_{lm1}c_{1l2m} + \tfrac{1}{2}\varepsilon_{213}\varepsilon_{lm1}c_{2l1m},$$

$$= \tfrac{1}{2}\varepsilon_{123}\varepsilon_{231}c_{1223} + \tfrac{1}{2}\varepsilon_{123}\varepsilon_{321}c_{1322} + \tfrac{1}{2}\varepsilon_{213}\varepsilon_{231}c_{2213} + \tfrac{1}{2}\varepsilon_{213}\varepsilon_{321}c_{2312},$$

$$= \tfrac{1}{2}c_{64} - \tfrac{1}{2}c_{52} - \tfrac{1}{2}c_{25} + \tfrac{1}{2}c_{64},$$

$T_{31} = c_{46} - c_{25}.$

$T_{33} = \frac{1}{2}\varepsilon_{ij3}\varepsilon_{lm3}c_{iljm},$

$= \frac{1}{2}\varepsilon_{123}\varepsilon_{lm3}c_{1l2m} + \frac{1}{2}\varepsilon_{213}\varepsilon_{lm3}c_{2l1m},$

$= \frac{1}{2}\varepsilon_{123}\varepsilon_{123}c_{1122} + \frac{1}{2}\varepsilon_{123}\varepsilon_{213}c_{1221} + \frac{1}{2}\varepsilon_{213}\varepsilon_{123}c_{2112} + \frac{1}{2}\varepsilon_{213}\varepsilon_{213}c_{2211},$

$= \frac{1}{2}c_{12} - \frac{1}{2}c_{66} - \frac{1}{2}c_{66} + \frac{1}{2}c_{12},$

$T_{33} = c_{12} - c_{66}.$

By writing out all the components in matrix form we have matrix T as :

$$T = \begin{bmatrix} c_{23} - c_{44} & c_{45} - c_{36} & c_{46} - c_{25} \\ c_{45} - c_{36} & c_{13} - c_{55} & c_{56} - c_{14} \\ c_{46} - c_{25} & c_{56} - c_{14} & c_{12} - c_{66} \end{bmatrix}$$

And the square of this matrix T^2 is :

$$T^2 = \begin{bmatrix} c_{23} - c_{44} & c_{45} - c_{36} & c_{46} - c_{25} \\ c_{45} - c_{36} & c_{13} - c_{55} & c_{56} - c_{14} \\ c_{46} - c_{25} & c_{56} - c_{14} & c_{12} - c_{66} \end{bmatrix}^2 \tag{4.3}$$

$\text{tr}(T^2) = (c_{23}-c_{44})^2+(c_{13}-c_{55})^2+(c_{12}-c_{66})^2+2[(c_{45}-c_{36})^2+(c_{46}-c_{25})^2+(c_{56}-c_{14})^2].$

The above expression is a second-order invariant of T^2 denoted as I_5 :

$$I_5 = (c_{23}-c_{44})^2+(c_{13}-c_{55})^2+(c_{12}-c_{66})^2+2[(c_{45}-c_{36})^2+(c_{46}-c_{25})^2+(c_{56}-c_{14})^2]. \tag{4.4}$$

Because of the symmetries of elasticity tensor $C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij}$, the elasticity tensor possesses 21 independent components. Among the above 5 quadratic invariants, I_1 and I_3 depends on all 21 component, while I_2 and I_4 depends on 15 component. In this respect the invariant I_5 may be considered as superior to rest of invariants as it only depends on 12 components of the tensor. However I_5 is not an independent invariant. To show that I_5 is not an independent invariant, consider a linear combination of I_5:

$$I_5 = \alpha I_1 + \beta I_2 + \gamma I_3 + \delta I_4 + \lambda A_1^2 + \mu A_2^2 + \eta A_1 A_2. \tag{4.5}$$

By the method of comparing the coefficients, we have eight equation corresponding to $c_{23}^2, c_{44}^2, c_{22}c_{23}, c_{23}c_{44}, c_{45}^2, c_{36}^2, c_{44}c_{55}, c_{45}c_{36}$. These equations are

$$\begin{aligned} 2\alpha + 2\beta + 4\mu &= 1, \\ 4\alpha + 2\delta + 4\lambda &= 1, \\ \beta + 4\mu + 2\eta &= 0, \\ \gamma + 2\eta &= -1, \\ 4\alpha + \delta &= 1, \\ 2\alpha + \beta &= 1, \\ 2\delta + 8\lambda &= 0, \\ 2\gamma &= -4. \end{aligned} \tag{4.6}$$

We get $\gamma = -2$ from the last equation. By putting value of γ in the fourth equation, we get value of η which is $\frac{1}{2}$. Substitute the value of δ from the fifth equation into the second equation, we have:

$$\begin{aligned} 4\alpha + 2(1 - 4\alpha) + 4\lambda &= 1, \\ 4\alpha + 2 - 8\alpha + 4\lambda &= 1, \\ -4\alpha + 4\lambda &= -1, \\ \alpha - \lambda &= \tfrac{1}{4}. \end{aligned}$$

Similarly, using the same value of δ in the seventh equation, we have :

$$\begin{aligned} 1 - 4\alpha + 4\lambda &= 0, \\ -4\alpha + 4\lambda &= -1, \\ \alpha - \lambda &= \tfrac{1}{4}. \end{aligned}$$

Let us assume $\alpha = 0$, then we have $\lambda = -\frac{1}{4}$. After a few manipulations, we get remaining values as

$$\delta = 1, \qquad \beta = 1, \qquad \mu = -\tfrac{1}{4}.$$

Thus Eq.(4.5) become:

$I_5 = I_2 - 2I_3 + I_4 - \frac{1}{4}A_1^2 - \frac{1}{4}A_2^2 + \frac{1}{2}A_1A_2,$

$= I_2 - 2I_3 + I_4 - \frac{1}{4}(A_1^2 + A_2^2 - 2A_1A_2),$

$I_5 = I_2 - 2I_3 + I_4 - \frac{1}{4}(A_1 - A_2)^2.$

This shows that I_5 is not an independent invariant as it can be written as linear combination of A_1^2, A_2^2 and so on. Hence the seven independent quadratic invariants are $\{A_1^2, A_2^2, A_1A_2, I_1, I_2, I_3, I_4\}$. Norris [4] showed that this is the complete set of second order invariants under $SO(3)$.

4.2 Rotation about the fixed $x_3 - axis$

In this section, we discus about the quadratic invariants under $SO(2)$. We presented the results of Ahmad [3] about quadratic invariants under $SO(2)$ and confirms the result of Norris [2].

Let θ be the angle of rotation about a fixed axis say x_3-axis. The matrix A associated with this transformation having entries as (a_{ij}) is referred to Eq.(2.24). The elasticity tensor C_{ijkl} has $3^4(= 81)$ components, out of which we consider the subset of $(2^4 = 16)$ elements $C^{(2)} = \{C_{ijkl},\ i, j, k, l \ = \ 1, 2\}$. Due to symmetry 16 elements of $C^{(2)}$ reduces to 6 distinct element. For example, the component C_{1122} transforms as C'_{1122} according to tranformation matrix A. So, we have

$$C'_{1122} \ = \ a_{1m}a_{1n}a_{2p}a_{2q}\, C_{mnpq}, \quad m, n, p, q \ = \ 1, 2, 3.$$

Since the entries a_{13} and a_{23} are zero, the above equation reduces to

$$C'_{1122} \ = \ a_{1m}a_{1n}a_{2p}a_{2q}\, C_{mnpq}, \quad m, n, p, q \ = \ 1, 2.$$

It is quite obvious that any component of $C^{(2)}$ transforms according to following rule

$$C'_{ijkl} \ = \ a_{im}a_{jn}a_{kp}a_{lq}\, C_{mnpq}, \quad i, j, k, l, m, n, p, q \ = \ 1, 2. \tag{4.7}$$

Thus $C^{(2)}$ may be regarded as tensor of rank 4 in two dimensions. Similarly C_{ijk3} and C_{ij33}, where $\{i, j, k = 1, 2\}$ both are tensor in two dimensions having rank 3 and 2 respectively. Here it is assumed that all indices to take values 1 and 2 only.

Now Ahmad [3] consider three vectors C_{i333}, C_{iij3} and C_{iji3} ; denoting as V_1, V_2, and V_3 respectively. These vectors are

$$\begin{aligned}
V_1 &= C_{i333}, \\
&= (C_{1333}, C_{2333}) = (c_{35}, c_{34}). \\
V_2 &= C_{iij3}, \\
&= (C_{ii13}, C_{ii23}) = (C_{1113} + C_{2213}, C_{1123} + C_{2223}) = (c_{15} + c_{25}, c_{14} + c_{24}). \\
V_3 &= C_{iji3}, \\
&= (C_{i1i3}, C_{i2i3}) = (C_{1113} + C_{2123}, C_{1213} + C_{2223}) = (c_{15} + c_{46}, c_{56} + c_{24}).
\end{aligned}$$

Also Ahmad [3] presented four tensors by $C_i's$, we have :

$$\begin{aligned}
C_1 &= C_{ij33}, \\
&= (C_{1133}, C_{1233}, C_{2133}, C_{2233}),
\end{aligned}$$

$$C_1 = \begin{bmatrix} c_{13} & c_{c36} \\ c_{36} & c_{23} \end{bmatrix}$$

$$\begin{aligned}
C_2 &= C_{i3j3}, \\
&= (C_{1313}, C_{1323}, C_{2313}, C_{2323}),
\end{aligned}$$

$$C_2 = \begin{bmatrix} c_{55} & c_{c45} \\ c_{45} & c_{44} \end{bmatrix}$$

$$\begin{aligned}
C_3 &= C_{ijkk}, \\
&= (C_{11kk}, C_{12kk}, C_{21kk}, C_{22kk}), \\
&= (C_{1111} + C_{1122}, C_{1211} + C_{1222}, C_{2111} + C_{2122}, C_{2211} + C_{2222}),
\end{aligned}$$

$$C_3 = \begin{bmatrix} c_{11} + c_{12} & c_{16} + c_{26} \\ c_{16} + c_{26} & c_{12} + c_{22} \end{bmatrix}$$

$$
\begin{aligned}
C_4 &= C_{ikkj}, \\
&= (C_{1kk1}, C_{1kk2}, C_{2kk1}, C_{2kk2}), \\
&= (C_{1111} + C_{1221}, C_{1112} + C_{1222}, C_{2111} + C_{2221}, C_{2112} + C_{2222}),
\end{aligned}
$$

$$
C_4 = \begin{bmatrix} c_{11} + c_{66} & c_{16} + c_{26} \\ c_{16} + c_{26} & c_{22} + c_{66} \end{bmatrix}.
$$

The five linear invariants under $SO(2)$ are mentioned in Eq.(3.2). Ahmad [3] denoted these linear invariants corresponding to above tensors by D_i, we have

$$
\begin{aligned}
&D_1 = c_{33}, \\
&D_2 = tr(C_1) = c_{13} + c_{23}, \\
&D_3 = tr(C_2) = c_{44} + c_{55}, \\
&D_4 = tr(C_3) = c_{11} + c_{22} + 2c_{12}, \\
&D_5 = \frac{1}{2} tr(C_3 - C_4) = c_{12} + c_{66}.
\end{aligned}
$$

Ahmad [3] recorded all second order invariants under $SO(2)$ and denoted these by E_i, we have

$$
\begin{aligned}
E_1 &= V_1.V_1 = c_{34}^2 + c_{35}^2, \\
E_2 &= V_2.V_2 = (c_{15} + c_{25})^2 + (c_{14} + c_{24})^2, \\
E_3 &= V_3.V_3 = (c_{15} + c_{46})^2 + (c_{24} + c_{56})^2, \\
E_4 &= V_1.V_2 = c_{35}(c_{15} + c_{25}) + c_{34}(c_{14} + c_{24}), \\
E_5 &= V_1.V_3 = c_{35}(c_{15} + c_{46}) + c_{34}(c_{24} + c_{56}), \\
E_6 &= V_2.V_3 = (c_{15} + c_{25})(c_{15} + c_{46}) + (c_{14} + c_{24})(c_{24} + c_{56}), \\
E_7 &= \mathrm{tr}(C_1^2) = c_{13}^2 + c_{23}^2 + 2c_{36}^2, \\
E_8 &= \mathrm{tr}(C_2^2) = c_{44}^2 + c_{55}^2 + 2c_{45}^2, \\
E_9 &= \mathrm{tr}(C_3^2) = (c_{11} + c_{12})^2 + (c_{12} + c_{22})^2 + 2(c_{16} + c_{26})^2, \\
E_{10} &= \mathrm{tr}(C_1 C_2) = c_{13}c_{55} + c_{23}c_{44} + 2c_{36}c_{45},
\end{aligned}
$$

$$E_{11} = \mathrm{tr}(C_1C_3) = c_{13}(c_{11}+c_{12}) + c_{23}(c_{12}+c_{22}) + 2c_{36}(c_{16}+c_{26}),$$

$$E_{12} = \mathrm{tr}(C_2C_3) = c_{55}(c_{11}+c_{12}) + c_{44}(c_{12}+c_{22}) + 2c_{45}(c_{16}+c_{26}),$$

We notice that since $C_4 = C_3 - D_5 I$, it shows that $\mathrm{tr}(C_4)$, $\mathrm{tr}(C_1C_4)$, $\mathrm{tr}(C_2C_4)$ etc, are not independent invariants. Two more invariants namely E_{13} and E_{14} are given by contracting the product of $C_{ijkl}C_{mnpq}$. Thus

$$E_{13} = C_{ijkl}C_{ijkl} = c_{11}^2 + c_{22}^2 + 2c_{12}^2 + 4(c_{16}^2 + c_{26}^2 + c_{66}^2),$$

$$E_{14} = C_{ijk3}C_{ijk3} = c_{14}^2 + c_{24}^2 + c_{15}^2 + c_{25}^2 + 2(c_{56}^2 + c_{46}^2),$$

Definition: We call an entity U_i, where $i = 1,2$, a quasivector if a rotation of the coordinate axes through an angle θ transforms the component of U_i as follows:

$$U_1^{'} = U_1\cos 2\theta + U_2 \sin 2\theta,$$
$$U_2^{'} = -U_1\sin 2\theta + U_2 \cos 2\theta.$$

Thus for the quasivector transformation matrix is the square of the matrix for an ordinary vector. Since the transformation is orthogonal, so U_iU_i, U_iV_i and $U_1V_2 - U_2V_1$ are invariant, where U_i and V_i are any quasivectors. Here we state the following lemma which shows that a quasivector can be associated with any symmetric tensor in two dimensions.

Lemma 4.2.1. *Let T_{ij} be a symmetric tensor in two dimensions, then $(2T_{12}, T_{22} - T_{11})$ is a quasivector.*

Proof: The component T_{ij} in two dimensions associated with the transformation matrix A mentioned in Eq.(2.24) can be transforms as:

$$T_{ij}^{'} = a_{il}a_{jm}T_{lm}, \tag{4.8}$$

Thus the transformed components of T_{ij} are as :

$$T_{11}^{'} = a_{1l}a_{1m}T_{lm},$$

$$= a_{11}a_{11}T_{11} + a_{11}a_{12}T_{12} + a_{12}a_{11}T_{21} + a_{12}a_{12}T_{22},$$

$$= \cos^2\theta T_{11} + \cos\theta\sin\theta T_{12} + \cos\theta\sin\theta T_{21} + \sin^2\theta T_{22},$$

$$T_{11}^{'} = \cos^2\theta T_{11} + 2\cos\theta\sin\theta T_{12} + \sin^2\theta T_{22},$$

Similarly the other components T_{22} and T_{12} are transformed as:

$$
\begin{aligned}
T'_{22} &= a_{2l}a_{2m}T_{lm}, \\
&= a_{21}a_{21}T_{11} + a_{21}a_{22}T_{12} + a_{22}a_{21}T_{21} + a_{22}a_{22}T_{22}, \\
&= \sin^2\theta T_{11} - \cos\theta\sin\theta T_{12} - \cos\theta\sin\theta T_{21} + \cos^2\theta T_{22}, \\
T'_{22} &= \sin^2\theta T_{11} - 2\cos\theta\sin\theta T_{12} + \cos^2\theta T_{22},
\end{aligned}
$$

and

$$
\begin{aligned}
T'_{12} &= a_{1l}a_{2m}T_{lm}, \\
&= a_{11}a_{21}T_{11} + a_{11}a_{22}T_{12} + a_{12}a_{21}T_{21} + a_{12}a_{22}T_{22}, \\
&= -\cos\theta\sin\theta T_{11} + \cos^2\theta T_{12} - \sin^2\theta T_{21} + \cos\theta\sin\theta T_{22}, \\
T'_{12} &= (\cos^2\theta - \sin^2\theta)T_{12} + \cos\theta\sin\theta(T_{22} - T_{11}),
\end{aligned}
$$

Thus all three components are transformed. Let us define U_1 and U_2 as

$$
\begin{aligned}
U_1 &= 2T_{12}, \\
U_2 &= T_{22} - T_{11}.
\end{aligned}
$$

With the help of the transformed equations T'_{ij}, U'_i becomes

$$
\begin{aligned}
U'_1 &= 2T'_{12}, \\
&= 2(\cos^2\theta - \sin^2\theta)T_{12} + 2\sin\theta\cos\theta(T_{22} - T_{11}), \\
&= 2\cos 2\theta T_{12} + \sin 2\theta(T_{22} - T_{11}), \\
U'_1 &= \cos 2\theta U_1 + \sin 2\theta U_2.
\end{aligned}
$$

Similarly U'_2 becomes :

$$
\begin{aligned}
U'_2 &= T'_{22} - T'_{11}, \\
&= (\sin^2\theta T_{11} - 2\cos\theta\sin\theta T_{12} + \cos^2\theta T_{22}) - (\cos^2\theta T_{11} + 2\cos\theta\sin\theta T_{12} + \sin^2\theta T_{22}), \\
&= (\cos^2\theta - \sin^2\theta)T_{22} - (\cos^2\theta - \sin^2\theta)T_{11} - \sin 2\theta T_{12} - \sin 2\theta T_{12},
\end{aligned}
$$

$$
\begin{aligned}
&= (\cos^2\theta - \sin^2\theta)(T_{22} - T_{11}) - 2\sin 2\theta T_{12}, \\
U_2' &= \cos 2\theta U_2 - \sin 2\theta U_1.
\end{aligned}
$$

This proves the lemma. By keeping in mind the definition of U_1 and U_2 and by referring to the tensors C_1, C_2 and C_3, it is seen that

$$
\mathbf{U} = (2c_{36}, c_{23} - c_{13}), \quad \mathbf{V} = (2c_{45}, c_{44} - c_{55}), \quad \mathbf{W} = (2(c_{16} + c_{26}), c_{22} - c_{11})
$$

are quasivectors. This gives three additional invariants and the number of invariants increases to 17. Three additional invariants are:

$$
\begin{aligned}
E_{15} &= \tfrac{1}{2}(U_1V_2 - U_2V_1) = c_{36}(c_{44} - c_{55}) + c_{45}(c_{13} - c_{23}), \\
E_{16} &= \tfrac{1}{2}(U_1W_2 - U_2W_1) = (c_{13} - c_{23})(c_{16} + c_{26}) - c_{36}(c_{11} - c_{22}), \\
E_{17} &= \tfrac{1}{2}(W_1V_2 - W_2V_1) = (c_{44} - c_{55})(c_{16} + c_{26}) + c_{45}(c_{11} - c_{22}).
\end{aligned}
$$

This completes the set of 17 invariants of second order with respect to rotation about fixed axis as x_3-axis.

4.2.1 Norris's result

Ahmad [3] demonstrated that the 32 quadratic invariants formed from the 15 quadratic combinations of $\{L_1, \cdots, L_5\}$ mentioned in Eq.(3.2) plus the 17 invariants $\{E_1, \cdots, E_{17}\}$ were independent of one another. Norris [2] improved the above result by showing that a complete set of quadratic invariants requires three additional invariants to make the total number 35. A complete set for the 35-dimensional space of quadratic invariants under $SO(2)$ is formed by the 32 of Ahmad [3] augmented by:

$$
\begin{aligned}
E_{18} &= (c_{15} + c_{46})c_{14}(c_{24} + c_{56})c_{25}c_{15}c_{56} + c_{24}c_{46}\ , \\
E_{19} &= (c_{15} + c_{46})c_{34}(c_{24} + c_{56})c_{35}\ , \\
E_{20} &= (c_{15} + c_{25})c_{34}(c_{14} + c_{24})c_{35}.
\end{aligned}
$$

4.2.2 Quadratic invariants under $SO(2)$: An alternate approach

There are five linear invariants under rotation about a fixed axis [3]. Taking the axis as e_3, these are:

γ_{3333} , γ_{3312} , γ_{1332} , γ_{1122} , γ_{1212}.

Previously, we were able to find quadratic invariants by employing several techniques. However, it was by no means clear that we had exhausted the list. Now we repeat the process by employing the criteria already described. We diagonalize the rotation matrix, and components of the elasticity tensor transform in such a manner that the invariants are easily identified. To find the quadratic invariants under $SO(2)$, we have to identify the components with $v_1 = v_2$ of $\gamma_{ijkl}\gamma_{pqrs}$ with the same symmetries which described earlier in Theorem 3.2.1.

a) The component with $v_1 = v_2 = 0$ is γ_{3333}^2.

b) The components with $v_1 = v_2 = 1$ are $\gamma_{3333}\gamma_{1233}$, $\gamma_{3333}\gamma_{1323}$ and $\gamma_{3331}\gamma_{3332}$.

c) The components with $v_1 = v_2 = 2$ are $\gamma_{3333}\gamma_{1212}$, $\gamma_{3333}\gamma_{1122}$, γ_{1233}^2, $\gamma_{1323}\gamma_{1233}$, γ_{1323}^2, $\gamma_{3311}\gamma_{3322}$, $\gamma_{3131}\gamma_{3232}$, $\gamma_{3311}\gamma_{3232}$, $\gamma_{3131}\gamma_{3322}$.

d) The components with $v_1 = v_2 = 3$ are $\gamma_{1233}\gamma_{1212}$, $\gamma_{1233}\gamma_{1122}$, $\gamma_{1323}\gamma_{1212}$, $\gamma_{1323}\gamma_{1122}$, $\gamma_{3111}\gamma_{3222}$, $\gamma_{1321}\gamma_{2312}$, $\gamma_{1322}\gamma_{2311}$, $\gamma_{2312}\gamma_{2311}$, $\gamma_{1321}\gamma_{1322}$, $\gamma_{3223}\gamma_{1112}$, $\gamma_{3113}\gamma_{2212}$, $\gamma_{3311}\gamma_{1222}$.

e) The components with $v_1 = v_2 = 4$ are $\gamma_{1212}\gamma_{1122}$, γ_{1212}^2, $\gamma_{2222}\gamma_{1111}$, $\gamma_{2111}\gamma_{1222}$.

Above components are 29 in numbers, whereas Norris stated 35 quadratic invariants under $SO(2)$. Some of the above invariants give us a complex number, so it split into two parts real and imaginary part. Their imaginary parts also give an invariant. So, remaining 6 invariants are added to the number 29. For example, after transformation $\gamma_{3311}\gamma_{3232}$ give us:

$$\gamma_{3311}^{'}\gamma_{3232}^{'} = [(c_{44} - c_{55}) + 2ic_{45}][(c_{13} - c_{23}) + 2ic_{36}].$$

This is the complex number and their real and imaginary parts both give us invariant. Hence one invariant is :

$$(c_{44} - c_{55})(c_{13} - c_{23}) - 4c_{45}c_{36},$$

And the second one is :

$$2[c_{36}(c_{44} - c_{55}) + c_{45}(c_{13} - c_{23})].$$

Similarly $\gamma_{3223}\gamma_{1112}$, $\gamma_{3113}\gamma_{2212}$, $\gamma_{3311}\gamma_{1222}$, $\gamma_{3131}\gamma_{3322}$, $\gamma_{2312}\gamma_{2311}$ and $\gamma_{1321}\gamma_{1322}$ give us one additional invariant. Thus the set of quadratic invariants of elasticity tensor C_{ijkl} under $SO(2)$ becomes 35. This confirms the result of Norris.

Let $\gamma'_{ijkl}\gamma'_{pqrs}$ represent the transformed components under transformation matrix A mentioned in Eq.(2.24). After transformation these components give us 35 invariants. The list of these invariants denoted by $\tau_i's$ in which first 15 are simply formed by multiples of linear invariant $L_i's$ stated in Eq.(3.2). Thus

1) $\tau_1 = c_{33}^2$,

2) $\tau_2 = c_{33}(c_{13} + c_{23})$,

3) $\tau_3 = c_{33}(c_{55} + c_{44})$,

4) $\tau_4 = -c_{33}(c_{11} + c_{22} + 2c_{12})$,

5) $\tau_5 = -c_{33}(c_{11} + c_{22} - 2c_{12} + 4c_{66})$,

6) $\tau_6 = -(c_{13} + c_{23})^2$,

7) $\tau_7 = -(c_{44} + c_{55})(c_{13} + c_{23})$,

8) $\tau_8 = -(c_{13} + c_{23})(c_{11} + c_{22} + 2c_{12})$,

9) $\tau_9 = -(c_{13} + c_{23})(c_{11} + c_{22} - 2c_{12} + 4c_{66})$,

10) $\tau_{10} = -(c_{44} + c_{55})^2$,

11) $\tau_{11} = -(c_{44} + c_{55})(c_{11} + c_{22} + 2c_{12})$,

12) $\tau_{12} = -(c_{44} + c_{55})(c_{11} + c_{22} - 2c_{12} + 4c_{66})$,

13) $\tau_{13} = (c_{11} + c_{22} + 2c_{12})^2$,

14) $\tau_{14} = (c_{11} + c_{22} + 2c_{12})(c_{11} + c_{22} - 2c_{12} + 4c_{66})$,

15) $\tau_{15} = (c_{11} + c_{22} - 2c_{12} + 4c_{66})^2$.

Here we do the same task which we have done in section 3.1.1. By using transformation matrix A Eq.(2.24) and their corresponding eigen vector matrix B mentioned in Eq.(3.8), we transform $\gamma_{3331}\gamma_{3332}$.

$$\gamma'_{3331}\gamma'_{3332} = i(c_{35}^2 + c_{34}^2).$$

where γ'_{3331} and γ'_{3332} are

$$\gamma'_{3331} = b_{3i}\ b_{3j}\ b_{3k}\ b_{1l}\ c_{ijkl}, \qquad \text{where } i,j,k,l = 1,2,3$$

$$= c_{35} + ic_{34}.$$

and

$$\gamma'_{3332} = b_{3i}\ b_{3j}\ b_{3k}\ b_{2l}\ c_{ijkl}, \qquad \text{where } i,j,k,l = 1,2,3$$

$$= ic_{35} + c_{34}.$$

Now to verify whether $c_{35}^2 + c_{34}^2$ is invariant or not, we consider transformation matrix A :

$$c''_{35} = a_{3i}\ a_{3j}\ a_{1k}\ a_{3l}\ c_{ijkl}, \qquad \text{where } i,j,k,l = 1,2,3$$

$$= c_{35}\cos\theta - c_{34}\sin\theta.$$

$$c''_{34} = a_{3i}\ a_{3j}\ a_{2k}\ a_{3l}\ c_{ijkl}, \qquad \text{where } i,j,k,l = 1,2,3$$

$$= c_{34}\cos\theta + c_{35}\sin\theta.$$

Now

$$c_{35}^{2''} + c_{34}^{2''} = (c_{35}\cos\theta - c_{34}\sin\theta)^2 + (c_{34}\cos\theta + c_{35}\sin\theta)^2,$$

$$= c_{35}^2\cos^2\theta + c_{34}^2\sin^2\theta - 2c_{35}c_{34}\sin\theta\cos\theta + c_{34}^2\cos^2\theta + c_{35}^2\sin^2\theta$$

$$+ 2c_{35}c_{34}\sin\theta\cos\theta,$$

$$= c_{35}^2(\cos^2\theta + \sin^2\theta) + c_{34}^2(\cos^2\theta + \sin^2\theta),$$

$c_{35}^{2''} + c_{34}^{2''} = c_{35}^2 + c_{34}^2$.

Similarly, on the same pattern, we found the remaining invarinats. These invariants are :

16) $\tau_{16} = (c_{35}^2 + c_{34}^2)$,

17) $\tau_{17} = -[(c_{13} - c_{23})^2 + 4c_{36}^2]$,

18) $\tau_{18} = -[(c_{44} - c_{55})^2 + 4c_{45}^2]$,

19) $\tau_{19} = 2[(c_{16} + c_{26})(c_{55} - c_{44}) - c_{45}(c_{11} - c_{22})]$,

20) $\tau_{20} = 2[c_{36}(c_{44} - c_{55}) + c_{45}(c_{13} - c_{23})]$,

21) $\tau_{21} = -2[(c_{16} + c_{26})(c_{55} - c_{44}) - c_{45}(c_{11} - c_{22})]$,

22) $\tau_{22} = [-4c_{45}(c_{16} + c_{26}) + (c_{22} - c_{11})(c_{55} - c_{44})]$,

23) $\tau_{23} = -[(c_{23} - c_{13})(c_{44} - c_{55}) + 4c_{36}c_{45}]$,

24) $\tau_{24} = 2[c_{36}(c_{44} - c_{55}) - c_{45}(c_{23} - c_{13})]$,

25) $\tau_{25} = -[(c_{15} - c_{25} - 2c_{46})^2 + (c_{14} - c_{24} + 2c_{56})^2]$,

26) $\tau_{26} = -[(c_{15} + c_{25})^2 + (c_{14} + c_{24})^2]$,

27) $\tau_{27} = -[(c_{15} - c_{25} + 2c_{46})^2 + (c_{14} - c_{24} - 2c_{56})^2]$,

28) $\tau_{28} = [(c_{15} + c_{25})(c_{24} - c_{14} + 2c_{56}) - (c_{14} + c_{24})(c_{15} - c_{25} + 2c_{46})]$,

29) $\tau_{29} = [(c_{14} + c_{24})(c_{14} - c_{24} - 2c_{56}) + (c_{15} + c_{25})(c_{15} - c_{25} + 2c_{46})]$,

30) $\tau_{30} = [(c_{15} + c_{25})(c_{14} - c_{24} - 2c_{56}) + (c_{14} + c_{24})(c_{15} - c_{25} + 2c_{46})]$,

31) $\tau_{31} = [(c_{14} + c_{24})(c_{14} - c_{24} - 2c_{56}) - (c_{15} + c_{25})(c_{15} - c_{25} + 2c_{46})]$,

32) $\tau_{32} = -2(c_{16} + c_{26})(c_{13} - c_{23}) + 2c_{36}(c_{11} - c_{22})$,

33) $\tau_{33} = -(c_{11} - c_{22})(c_{13} - c_{23}) - 4c_{36}(c_{16} + c_{26})$,

34) $\tau_{34} = c_{11}^2 + 16c_{16}^2 + 16c_{66}^2 + 4c_{12}^2 + 16c_{26}^2 + c_{22}^2 + 8c_{66}(2c_{12} - c_{22}) - 32c_{16}c_{26} - 4c_{12}c_{22} + c_{11}(-8c_{66} - 4c_{12} + 2c_{22})$,

35) $\tau_{35} = (c_{11} - c_{22})^2 + 4(c_{16} + c_{26})^2$.

Thus the set of 35 quadratic invariants of elasticity tensor under $SO(2)$ is complete.

4.2.3 Comparison with earlier work

The invariants found by the two approaches are equivalent, because each invariants in the second list can be expressed as a linear combination of members of the first list and viceversa. The first fifteen invariants $\{\tau_1, ..., \tau_{15}\}$ are found by making the quadratic combinations of $\{L_1, ..., L_5\}$. For the remaining 20 invariants we can make the different identifications. For example, let us firstly consider τ_{33}. We can write τ_{33} as a combination of D_2D_4 and E_{11}. Let us assume following relation.

$$\tau_{33} = \alpha D_2D_4 + \beta E_{11}, \tag{4.9}$$

By comparing the coefficients of R.H.S with τ_{33}, we get equations which give us the coefficients. As by comparison of coefficient $c_{22}c_{13}$, we found $\alpha = 1$. Similarly

$$c_{36}c_{16}: \qquad -4 = 2\beta,$$

This give $\beta = -2$. Substitute the value of α and β in Eq.(4.9)

$$\tau_{33} = D_2D_4 - 2E_{11},$$

Again, let us consider τ_{34}. It can be written as a combination of $D_4^2, D_5^2, D_4D_5, E_9$ and E_{13}. Assume the following relation

$$\tau_{34} = \alpha D_4^2 + \beta D_5^2 + \gamma D_4D_5 + \delta E_9 + \tau E_{13}, \tag{4.10}$$

By comparison of coefficients, following equations are formed.

$$\begin{array}{ll} c_{16}^2 & 16 = 2\delta + \tau, \\ c_{66}c_{12} & 16 = -2\beta - 2\gamma, \\ c_{11}^2 & 1 = \alpha + \delta + \tau, \\ c_{12}c_{22} & -4 = 4\alpha + \gamma + 2\tau, \\ c_{12}^2 & 4 = 4\alpha + \beta + 2\gamma + 2\delta + 2\tau. \end{array} \tag{4.11}$$

Substitute the value of δ from the first equation into the third equation. This give us

$$\alpha + \delta + \tau = 1,$$

$$\alpha + 8 - 2\tau + \tau = 1,$$

$$\alpha - \tau = -7,$$

$$\alpha = \tau - 7.$$

Both coefficients δ and α values are in the form of τ. Substitute the value of δ and α in fourth equation.

$$4\alpha + \gamma + 2\tau = -4,$$

$$4(\tau - 7) + \gamma + 2(8 - 2\tau) = -4,$$

$$4\tau - 28 + \gamma + 16 - 4\tau = -4,$$

$$\gamma = 8.$$

By putting value of γ in second equation, we have

$$-2\beta - 2\gamma = 16,$$

$$-2\beta - 2(8) = 16,$$

$$\beta = -16.$$

After getting the value of γ and β, substitute the value of α, δ, β and γ in the last equation and we get,

$$\begin{aligned} 4\alpha + \beta + 2\gamma + 2\delta + 2\tau &= 4, \\ 4(\tau - 7) - 16 + 2(8) + 2(8 - 2\tau) + 2\tau &= 4, \\ 4\tau - 28 - 16 + 16 + 16 - 4\tau + 2\tau &= 4, \\ \tau &= 8. \end{aligned}$$

At the end, α and δ becomes 1 and -8 respectively. Hence, all the coefficients are found so substitute all the values in Eq.(4.10). Thus we have

$$\tau_{34} = D_4^2 - 16D_5^2 + 8D_4D_5 - 8E_9 + 8E_{13}.$$

Similarly, Consider τ_{35}. It can be written as a combination of D_4^2, E_9 and E_{13}. Thus we have

$$\tau_{35} = \alpha D_4^2 + \beta E_9 + \gamma E_{13}, \tag{4.12}$$

By comparing the coefficients, following equations are found.

$$\begin{aligned} c_{11}^2 \qquad\qquad & 1 = \alpha + \beta + \gamma, \\ c_{11}c_{22} \qquad\qquad & -2 = 2\alpha, \\ c_{16}^2 \qquad\qquad & 4 = 2\beta + 4\gamma. \end{aligned}$$

From the second equation, we have $\alpha = -1$. By substituting value of α in the first equation,

$$\beta + \gamma = 2$$

Solving this equation simultaneously with third equation, then it gives $\gamma = 0$ and $\beta = 2$. Thus Eq.(4.12) becomes

$$\tau_{35} = -D_4^2 + 2E_9.$$

On the same pattern, we can make the remaining identifications. These are

$$\begin{aligned} \tau_{16} &= E_1, \\ \tau_{17} &= D_2^2 - 2E_7, \\ \tau_{18} &= D_3^2 - 2E_8, \\ \tau_{19} &= -2E_{17}, \\ \tau_{20} &= 2E_{15}, \\ \tau_{21} &= 2E_{17}, \\ \tau_{22} &= D_3D_4 - 2E_{12}, \end{aligned}$$

$$
\begin{aligned}
\tau_{23} &= D_2 D_3 - 2E_{10}, \\
\tau_{24} &= -2E_{15}, \\
\tau_{25} &= 3E_2 + 4(E_3 - E_6 - E_{14}), \\
\tau_{26} &= -E_2, \\
\tau_{27} &= -E_2 - 4(E_3 - E_6), \\
\tau_{28} &= -2E_{18}, \\
\tau_{29} &= -E_2 - 2(E_3 - E_6 - E_{14}), \\
\tau_{30} &= 2E_{18}, \\
\tau_{31} &= E_2 - 2E_6, \\
\tau_{32} &= -2E_{16}, \\
\tau_{33} &= D_2 D_4 - 2E_{11}, \\
\tau_{34} &= D_4^2 - 16D_5^2 + 8D_4 D_5 - 8E_9 + 8E_{13}, \\
\tau_{35} &= -D_4^2 + 2E_9.
\end{aligned}
$$

For the elasticity tensor, our analysis confirms the result of Norris. Hence, it is proved that there are seven quadratic invariants of the 21-element elastic modulus tensor under $SO(3)$ and 35 under $SO(2)$.

4.3 Plane Elasticity tensor

We shall now briefly describe the analysis of the plane (i.e. $2D$) elasticity tensor. The elasticity tensor C_{ijkl} in two dimension is called plane elasticity tensor. Let us consider the Q be the transformation matrix and it not necessary to be symmetric.

$$
Q = \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{bmatrix}, \tag{4.13}
$$

Also C be the matrix of components of plane elasticity tensor and it should be symmetric.

$$C = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{12} & c_{22} & c_{23} \\ c_{13} & c_{23} & c_{33} \end{bmatrix}. \tag{4.14}$$

Firstly, our task here is to find the transformation matrix Q. For this, we use transformation rule

$$C^{'} = Q^t C Q, \tag{4.15}$$

$$C^{'} = \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{bmatrix}^t \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{12} & c_{22} & c_{23} \\ c_{13} & c_{23} & c_{33} \end{bmatrix} \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{bmatrix},$$

After simplification, we have matrix $C^{'}$ and its entries are represented by R_{ij}, where i and j shows the i^{th} row and j^{th} column of $C^{'}$.

$R_{11} = c_{11}q_{11}^2 + 2c_{12}q_{11}q_{21} + c_{22}q_{21}^2 + 2c_{13}q_{11}q_{31} + 2c_{23}q_{21}q_{31} + c_{33}q_{31}^2,$

$R_{12} = c_{11}q_{11}q_{12} + c_{12}q_{12}q_{21} + c_{12}q_{21}q_{22} + c_{22}q_{21}q_{22} + c_{13}q_{12}q_{31} + c_{23}q_{22}q_{31}$

$\qquad + c_{13}q_{11}q_{32} + c_{23}q_{21}q_{32} + c_{33}q_{31}q_{32},$

$R_{13} = c_{11}q_{11}q_{13} + c_{12}q_{13}q_{21} + c_{12}q_{11}q_{23} + c_{22}q_{21}q_{23} + c_{13}q_{13}q_{31} + c_{23}q_{23}q_{31}$

$\qquad + c_{13}q_{11}q_{33} + c_{23}q_{21}q_{33} + c_{33}q_{31}q_{33},$

$R_{21} = c_{11}q_{11}q_{12} + c_{12}q_{12}q_{21} + c_{12}q_{21}q_{22} + c_{22}q_{21}q_{22} + c_{13}q_{12}q_{31} + c_{23}q_{22}q_{31}$

$\qquad + c_{13}q_{11}q_{32} + c_{23}q_{21}q_{32} + c_{33}q_{31}q_{32},$

$R_{22} = c_{11}q_{12}^2 + 2c_{12}q_{12}q_{22} + c_{22}q_{22}^2 + 2c_{13}q_{12}q_{32} + 2c_{23}q_{21}q_{32} + c_{33}q_{32}^2,$

$R_{23} = c_{11}q_{12}q_{13} + c_{12}q_{13}q_{22} + c_{12}q_{12}q_{23} + c_{22}q_{22}q_{23} + c_{13}q_{13}q_{32} + c_{23}q_{23}q_{32}$

$\qquad + c_{13}q_{12}q_{33} + c_{23}q_{22}q_{33} + c_{33}q_{32}q_{33},$

$R_{31} = c_{11}q_{11}q_{13} + c_{12}q_{13}q_{21} + c_{12}q_{11}q_{23} + c_{22}q_{21}q_{23} + c_{13}q_{13}q_{31} + c_{23}q_{23}q_{31}$

$$+ c_{13}q_{11}q_{33} + c_{23}q_{21}q_{33} + c_{33}q_{31}q_{33},$$

$$R_{32} = c_{11}q_{12}q_{13} + c_{12}q_{13}q_{22} + c_{12}q_{12}q_{23} + c_{22}q_{22}q_{23} + c_{13}q_{13}q_{32} + c_{23}q_{23}q_{32}$$

$$+ c_{13}q_{12}q_{33} + c_{23}q_{22}q_{33} + c_{33}q_{32}q_{33},$$

$$R_{33} = c_{11}q_{13}^2 + 2c_{12}q_{13}q_{23} + c_{22}q_{23}^2 + 2c_{13}q_{13}q_{33} + 2c_{23}q_{23}q_{33} + c_{33}q_{33}^2.$$

Let us consider

$$R_{ij}^{'} = c_{ij}^{'}, \qquad\qquad i, j = 1, 2. \tag{4.16}$$

where R_{ij} are the matrix entries of i^{th} row and j^{th} column of $C^{'}$, and $c_{ij}^{'}$ are the transformed components for a rotation in $2D$ through an angle θ. The rotation matrix in $2D$ is:

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Now, we find the components q_{ij} of transformation matrix Q by giving variation to Eq.(4.16). After that, compare the coefficients of L.H.S with R.H.S. It give us components of Q.

Introduce the two index notation

$$\begin{aligned} c_{1111} &= c_{11}, \\ c_{2222} &= c_{22}, \\ c_{1122} &= c_{12}, \\ \sqrt{2}c_{1112} &= c_{13}, \\ \sqrt{2}c_{1222} &= c_{23}, \\ 2c_{1212} &= c_{33}. \end{aligned} \tag{4.17}$$

Let $i = j = 1$, then Eq.(4.16) becomes

$$R_{11}^{'} = c_{11}^{'},$$

where

$$c_{11}^{'} = c_{11}\cos^4\theta - 4c_{16}\cos^3\theta\sin\theta + 2c_{12}\cos^2\theta\sin^2\theta + 4c_{66}\cos^2\theta\sin^2\theta - 4c_{26}\cos\theta\sin^3\theta$$

$$+ c_{22} \sin^4 \theta,$$

After comparing the coefficients, it gives two equations.

$$\begin{aligned} c_{11} : \qquad & q_{11}^2 = \cos^4 \theta, \\ 2c_{12} : \qquad & q_{11} q_{21} = \cos^2 \theta \sin^2 \theta. \end{aligned}$$

from both of the above equations, we have

$$\begin{aligned} q_{11} &= \cos^2 \theta, \\ q_{21} &= \sin^2 \theta, \end{aligned}$$

Again, for $i = 1$ and $j = 2$ in Eq.(4.16).

$$R_{12}^{'} = c_{12}^{'},$$

where

$$c_{12}^{'} = c_{12} \cos^4 \theta - 2c_{26} \cos^3 \theta \sin \theta + 2c_{16} \cos^3 \theta \sin \theta + c_{11} \cos^2 \theta \sin^2 \theta - 4c_{66} \cos^2 \theta \sin^2 \theta$$

$$+ c_{22} \cos^2 \theta \sin^2 \theta + 2c_{26} \cos \theta \sin^3 \theta - 2c_{16} \cos \theta \sin^3 \theta + c_{12} \sin^4 \theta,$$

After comparing the coefficients, it gives two equations.

$$c_{11} : \qquad q_{11} q_{12} = \cos^2 \theta \sin^2 \theta,$$

$$c_{22} : \qquad q_{21} q_{22} = \cos^2 \theta \sin^2 \theta.$$

Substitute the value of q_{11} and q_{21} in the above equations, we have

$$\begin{aligned} q_{12} &= \sin^2 \theta, \\ q_{22} &= \cos^2 \theta. \end{aligned}$$

Now, for $i = 1$ and $j = 3$, Eq.(4.16) becomes

$$R_{13}^{'} = c_{13}^{'},$$

Here we have a notation $c_{13} = bc_{1112}$, where $b = \sqrt{2}$. By using this notation we have

$$R_{13}^{'} = bc_{16}^{'},$$

where $b = \sqrt{2}$ and

$$c_{16}^{'} = c_{16}\cos^4\theta + c_{11}\cos^3\theta\sin\theta - c_{12}\cos^3\theta\sin\theta - 2c_{66}\cos^3\theta\sin\theta + 3c_{26}\cos^2\theta\sin^2\theta$$

$$-3c_{16}\cos^4\theta\sin^2\theta + c12\cos\theta\sin^3\theta + 2c_{66}\cos\theta\sin^3\theta - c_{22}\cos\theta\sin^3\theta - c_{26}\sin^4\theta,$$

By comparing the coefficients, we have

$$c_{11}: \qquad q_{11}q_{13} = b\cos^3\theta\sin\theta,$$

$$q_{13} = \tfrac{b\cos^3\theta\sin\theta}{\cos^2\theta},$$

$$q_{13} = \sqrt{2}\cos\theta\sin\theta.$$

Similarly,

$$c_{22}: \qquad q_{21}q_{23} = -b\cos\theta\sin^3\theta,$$

$$q_{23} = \tfrac{b\cos\theta\sin^3\theta}{\sin^2\theta},$$

$$q_{23} = -\sqrt{2}\cos\theta\sin\theta.$$

Since $c_{13} = \sqrt{2}c_{1112}$, so by comparing the coefficient of $\sqrt{2}c_{16}$, we have

$$\sqrt{2}c_{16}: \qquad q_{13}q_{31} + q_{11}q_{33} = \cos^4\theta - 3\cos^2\theta\sin^2\theta,$$

Substitute the value of q_{13} and q_{11}, we have

$$\begin{aligned} \sqrt{2}\cos\theta\sin\theta q_{31} + \cos^2\theta q_{33} &= \cos^4\theta - 3\cos^2\theta\sin^2\theta, \\ \sqrt{2}\sin\theta q_{31} + \cos\theta q_{33} &= \cos\theta(\cos^2\theta - 3\sin^2\theta), \end{aligned} \tag{4.18}$$

Again, since $c_{23} = \sqrt{2}c_{2212}$, so by comparing the coefficient of $\sqrt{2}c_{26}$, we have

$$\sqrt{2}c_{26}: \qquad q_{23}q_{31} + q_{21}q_{33} = 3\cos^2\theta\sin^2\theta - \sin^4\theta,$$

Substitute the value of q_{23} and q_{21}, we have

$$\begin{aligned} -\sqrt{2}\cos\theta\sin\theta q_{31} + \sin^2\theta q_{33} &= 3\cos^2\theta\sin^2\theta - \sin^4\theta, \\ -\sqrt{2}\cos\theta q_{31} + \sin\theta q_{33} &= \sin\theta(3\cos^2\theta - \sin^2\theta), \end{aligned} \tag{4.19}$$

Multiply Eq.(4.18) with $\sin\theta$ and Eq.(4.19) with $\cos\theta$, then subtract the both equations. After some simplifications, it gives

$$\sqrt{2}\sin^2\theta q_{31} + \sqrt{2}\sin^2\theta q_{31} = \sin\theta\cos\theta(\cos^2\theta - 3\sin^2\theta) - \cos\theta\sin\theta(3\cos^2\theta - \sin^2\theta),$$

$$\sqrt{2}q_{31} = -2\sin^3\theta\cos\theta - 2\sin\theta\cos^3\theta,$$

$$q_{31} = -\sqrt{2}\sin\theta\cos\theta.$$

Substitute the value of q_{31} in (4.18), we have

$$\begin{aligned} \sqrt{2}\sin\theta(-\sqrt{2}\sin\theta\cos\theta) + \cos\theta q_{33} &= \cos\theta(\cos^2\theta - 3\sin^2\theta), \\ -2\sin^2\theta\cos\theta + \cos\theta q_{33} &= \cos\theta(\cos^2\theta - 3\sin^2\theta), \\ q_{33} &= \cos^2\theta - \sin^2\theta. \end{aligned}$$

Lastly, since $c_{33} = 2c_{1212}$, so by comparing coefficients of c_{33} we use $2c_{66}$ instead of c_{33}. Thus

$$2c_{66}: \qquad q_{31}q_{32} = -2\cos^2\theta\sin^2\theta,$$

Substitute the value of q_{31} in above equation. We have

$$q_{32} = \sqrt{2}\cos\theta\sin\theta.$$

Thus all the components of transformation matrix Q are formed. Let us assume

$$a = \cos\theta,$$

$$b = \sin\theta,$$

$$k = \sqrt{2},$$

Hence transformation matrix Q Eq.(4.13) will be:

$$Q = \begin{bmatrix} a^2 & b^2 & kab \\ b^2 & a^2 & -kab \\ -kab & kab & a^2 - b^2 \end{bmatrix}$$

and

$$Q = \begin{bmatrix} \cos^2\theta & \sin^2\theta & \sqrt{2}\cos\theta\sin\theta \\ \sin^2\theta & \cos^2\theta & -\sqrt{2}\cos\theta\sin\theta \\ -\sqrt{2}\cos\theta\sin\theta & \sqrt{2}\cos\theta\sin\theta & \cos^2\theta-\sin^2\theta \end{bmatrix}. \qquad (4.20)$$

$$\det Q = \begin{vmatrix} \cos^2\theta & \sin^2\theta & \sqrt{2}\cos\theta\sin\theta \\ \sin^2\theta & \cos^2\theta & -\sqrt{2}\cos\theta\sin\theta \\ -\sqrt{2}\cos\theta\sin\theta & \sqrt{2}\cos\theta\sin\theta & \cos^2\theta-\sin^2\theta \end{vmatrix}.$$

$$\begin{aligned}
\det Q = {} & \cos^2\theta[\cos^2\theta(\cos^2\theta-\sin^2\theta)+2\sin^2\theta\cos^2\theta]-\sin^2\theta[\sin^2\theta(\cos^2\theta-\sin^2\theta) \\
& -2\sin^2\theta\cos^2\theta]+\sqrt{2}\sin\theta\cos\theta[\sqrt{2}\sin^3\theta\cos\theta+\sqrt{2}\sin\theta\cos^3\theta], \\
& =(\cos^4\theta-\sin^4\theta)(\cos^2\theta-\sin^2\theta)+4\sin^2\theta\cos^4\theta+4\sin^4\theta\cos^2\theta, \\
& =\cos^4\theta+\sin^4\theta+2\sin^2\theta\cos^2\theta, \\
& =(\sin^2\theta+\cos^2\theta), \\
\det Q = {} & 1.
\end{aligned}$$

4.3.1 Diagonalization of the transformation matrix

Transformation matrix Q for a rotation through an angle θ is mentioned in Eq.(4.20). To proceed further, we need to diagonalize Q. Here we follows the same technique of diagonalization which was employed by Ahmad and Rashid [4] to investigate invariants of elasticity tensor under $SO(2)$.

The transformation matrix Q (Eq.(4.20)) has eigenvalues 1, $e^{-2i\theta}$ and $e^{2i\theta}$ with respective eigenvectors, denoted as $W_i's$.

$$W_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$W_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{i}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \\ 1 \end{pmatrix}$$

$$W_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} -\frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \\ 1 \end{pmatrix}$$

By the method of diagonalization, we have $D' = (K^*)^t QK$, where K is the matrix of eigen vectors of matrix Q and $(K^*)^t$ represents the transpose of complex conjugate of K. Thus

$$D' = (K^*)^t QK = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{i}{2} & \frac{i}{2} & \frac{1}{\sqrt{2}} \\ \frac{i}{2} & -\frac{i}{2} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \cos^2\theta & \sin^2\theta & \sqrt{2}\cos\theta\sin\theta \\ \sin^2\theta & \cos^2\theta & -\sqrt{2}\cos\theta\sin\theta \\ -\sqrt{2}\cos\theta\sin\theta & \sqrt{2}\cos\theta\sin\theta & \cos^2\theta - \sin^2\theta \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{2} & -\frac{i}{2} \\ \frac{1}{\sqrt{2}} & -\frac{i}{2} & \frac{i}{2} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix},$$

$$
= \begin{pmatrix} \frac{1}{\sqrt{2}}(\cos^2\theta + \sin^2\theta) & \frac{1}{\sqrt{2}}(\cos^2\theta + \sin^2\theta) & 0 \\ -\frac{i}{2}\cos^2\theta + \frac{i}{2}\sin^2\theta - \cos\theta\sin\theta & -\frac{i}{2}\sin^2\theta + \frac{i}{2}\cos^2\theta + \cos\theta\sin\theta & \frac{1}{\sqrt{2}}e^{-2i\theta} \\ \frac{i}{2}\cos^2\theta - \frac{i}{2}\sin^2\theta - \cos\theta\sin\theta & \frac{i}{2}\sin^2\theta - \frac{i}{2}\cos^2\theta + \cos\theta\sin\theta & \frac{1}{\sqrt{2}}e^{2i\theta} \end{pmatrix}
$$

$$
\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{2} & -\frac{i}{2} \\ \frac{1}{\sqrt{2}} & -\frac{i}{2} & \frac{i}{2} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix},
$$

By following simply multiplication rule, diagonal entries of matrix D' are:

$D'_{11} = \frac{1}{2} + \frac{1}{2} + 0,$

$D'_{11} = 1.$

$D'_{22} = \frac{i}{2}(-\frac{i}{2}\cos 2\theta - \sin\theta\cos\theta) - \frac{i}{2}(\frac{i}{2}\cos 2\theta + \sin\theta\cos\theta) + \frac{1}{2}e^{-2i\theta},$

$\quad = \frac{1}{4}e^{-2i\theta} + \frac{1}{4}e^{-2i\theta} + \frac{1}{2}e^{-2i\theta},$

$D'_{22} = e^{-2i\theta}.$

also

$D'_{33} = -\frac{i}{2}(-\frac{i}{2}\cos 2\theta - \sin\theta\cos\theta) + \frac{i}{2}(-\frac{i}{2}\cos 2\theta + \sin\theta\cos\theta) + \frac{1}{2}e^{2i\theta},$

$\quad = \frac{1}{4}e^{2i\theta} + \frac{1}{4}e^{2i\theta} + \frac{1}{2}e^{2i\theta},$

$D'_{33} = e^{2i\theta}.$

Similarly

$D'_{12} = \frac{i}{2}\frac{1}{\sqrt{2}} - \frac{i}{2}\frac{1}{\sqrt{2}},$

$D'_{12} = 0.$

All the remaining entries will be vanished. Hence,

$$D' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-2i\theta} & 0 \\ 0 & 0 & e^{2i\theta} \end{pmatrix}. \tag{4.21}$$

In $2D$, consider the transformation of a tensor of rank 2. If Γ'_{pq} denotes a component after rotation through an angle θ. Then Γ_{pq} transforms according to following rule.

$$\Gamma'_{pq} \quad = \quad k_{pi}\ k_{qj}\ \Gamma_{ij}, \qquad\qquad i,j,p,q \ = \ 1,2,3. \tag{4.22}$$

Since D' is diagonal matrix, So it must be i = p, j = q. Otherwise it is equal to zero. Hence,

$$\Gamma'_{pq} \quad = \quad e^{2i(v_1-v_2)\theta}\ \Gamma_{pq} \tag{4.23}$$

4.3.2 Criterion for invariance

Elastic moduli Γ_{pq} before and after rotation are related by Eq.(4.21),

where

v_1 = numbers of $1's$ in the indices
v_2 = numbers of $2's$ in the indices

Hence Γ will be invariant if and only if, among the subscripts, the number of ones and twos are equal. So, we immediately get the following set of invariants.

$\Gamma_{33},\ \ \Gamma_{12},\ \ \Gamma_{13}\ \Gamma_{23},\ \ \Gamma_{11}\ \Gamma_{22}$ and $\Gamma_{13}^2\Gamma_{22} + \Gamma_{23}^2\Gamma_{11}$.

The elastic muoduli Γ_{pq} are given by the following transformation equation.

$$\Gamma'_{pq} \ = k_{pi}k_{qj}\Gamma_{ij}, \qquad\qquad \text{where } i,j = 1,2,3$$

where $k_i's$ are elements of the matrix K (matrix of eigen vectors of K). Now, let us firstly consider Γ_{12},

$$\Gamma'_{12} \ = k_{1i}k_{2j}\Gamma_{ij},$$

$$= k_{11}k_{21}c_{11} + k_{11}k_{22}c_{12} + k_{11}k_{23}c_{13} + k_{12}k_{21}c_{21} + k_{12}k_{22}c_{22} + k_{12}k_{23}c_{23} + k_{13}k_{21}c_{31}$$

$$+ k_{13}k_{22}c_{32} + k_{13}k_{23}c_{33},$$

$$= \tfrac{1}{2}c_{11} - \tfrac{1}{2}c_{12} + \tfrac{i}{\sqrt{2}}c_{13} - \tfrac{1}{2}c_{21} + \tfrac{1}{2}c_{22} - \tfrac{i}{\sqrt{2}}c_{23} - \tfrac{i}{\sqrt{2}}c_{31} + \tfrac{i}{\sqrt{2}}c_{32} + c_{33},$$

$$\Gamma'_{12} = \tfrac{1}{2}(c_{11} + c_{22} - 2c_{12} + 2c_{33}).$$

Similarly, Γ_{13} Γ_{23}

$$\Gamma'_{13} = k_{1i}k_{3j}\Gamma_{ij},$$

$$= k_{11}k_{31}c_{11} + k_{11}k_{32}c_{12} + k_{11}k_{33}c_{13} + k_{12}k_{31}c_{21} + k_{12}k_{32}c_{22} + k_{12}k_{33}c_{23} + k_{13}k_{31}c_{31}$$

$$+ k_{13}k_{32}c_{32} + k_{13}k_{33}c_{33},$$

$$= \tfrac{i}{\sqrt{2}}c_{11} + \tfrac{i}{\sqrt{2}}c_{12} - \tfrac{i}{\sqrt{2}}c_{21} - \tfrac{i}{\sqrt{2}}c_{22} + \tfrac{1}{\sqrt{2}}c_{31} + \tfrac{1}{\sqrt{2}}c_{32},$$

$$\Gamma'_{13} = \tfrac{i}{\sqrt{2}}c_{11} - \tfrac{i}{\sqrt{2}}c_{22} + \tfrac{1}{\sqrt{2}}c_{13} + \tfrac{1}{\sqrt{2}}c_{32}.$$

and

$$\Gamma'_{23} = k_{2i}k_{3j}\Gamma_{ij},$$

$$= k_{21}k_{31}c_{11} + k_{21}k_{32}c_{12} + k_{21}k_{33}c_{13} + k_{22}k_{31}c_{21} + k_{22}k_{32}c_{22} + k_{22}k_{33}c_{23} + k_{23}k_{31}c_{31}$$

$$+ k_{23}k_{32}c_{32} + k_{23}k_{33}c_{33},$$

$$= -\tfrac{i}{\sqrt{2}}c_{11} - \tfrac{i}{\sqrt{2}}c_{12} + \tfrac{i}{\sqrt{2}}c_{21} + \tfrac{i}{\sqrt{2}}c_{22} + \tfrac{1}{\sqrt{2}}c_{31} + \tfrac{1}{\sqrt{2}}c_{32},$$

$$\Gamma'_{23} = -\tfrac{i}{\sqrt{2}}c_{11} + \tfrac{i}{\sqrt{2}}c_{22} + \tfrac{1}{\sqrt{2}}c_{13} + \tfrac{1}{\sqrt{2}}c_{32}.$$

$$\Gamma'_{13}\ \Gamma'_{23} = (\tfrac{i}{\sqrt{2}}c_{11} - \tfrac{i}{\sqrt{2}}c_{22} + \tfrac{1}{\sqrt{2}}c_{13} + \tfrac{1}{\sqrt{2}}c_{32})(-\tfrac{i}{\sqrt{2}}c_{11} + \tfrac{i}{\sqrt{2}}c_{22} + \tfrac{1}{\sqrt{2}}c_{13} + \tfrac{1}{\sqrt{2}}c_{32}),$$

$$= \tfrac{1}{2}[(c_{11} - c_{22})^2 + (c_{13} + c_{23})^2].$$

Similarly, on the same pattern the rest of invariants will be :

$$c_{11} + c_{22} + 6c_{12} - c_{33},$$

$$\tfrac{1}{8}[(c_{11} + c_{22}) - 2c_{12}c_{33}]^2 + (c_{13} - c_{23})^2,$$

$$(c_{11} + c_{22} - 2c_{12} - 2c_{33})[(c_{11} - c_{22})^2 - 2(c_{13} + c_{23})^2] + 8(c_{13}^2 - c_{23}^2)(c_{11} - c_{22}).$$

Hence, these above five invariants are the only independent ones.

Chapter 5

Conclusion and future work

In the end, we conclude this dissertation.

Ahmad and Rashid [4] have already studied the independent linear invariants under $SO(2)$ as well as $SO(3)$ of an elasticity tensor. They defined a linear form in terms of elements of a tensor. Formula's were shown for finding the number of independent linear invariants for an arbitrary tensor in two and three dimensions. They have also obtained explicit expressions for the simple cases.

In this dissertation, we found linear and quadratic invariants of an arbitrary tensor of rank $r = 4$. Here, we have not only found the number of invariants, but also have calculated the linearly independent invariants. We studied the Ahmad [3] and Norris [2] results about quadratic invariants under $SO(2)$ and $SO(3)$. We have seen that under $SO(3)$, there are two linear invariants A_1, A_2 and seven quadratic invariants, their complete set is $\{A_1^2, A_2^2, A_1A_2, I_1, I_2, I_3, I_4\}$ but, out of seven three are derivable from linear ones. So there are four independent quadratic invariants under $SO(3)$.

However, the number of independent quadratic invariants under $SO(2)$ with rank $r = 4$ is found to be 35 which proves the Norris result. Also, by comparing these invariants with earlier work, it was shown that the invariants found by the two approaches are equivalent.

We derived a 3×3 transformation matrix for plane elasticity tensor. This reduces the calculation to simple matrix multiplication. We used the same technique which was employed by Ahmad and Rashid [4] to find invariants of elasticity tensor about a fixed axis. Five invariants are found for plane elasticity tensor: Γ_{33}, Γ_{12}, $\Gamma_{13}\,\Gamma_{23}$, $\Gamma_{11}\,\Gamma_{22}$ and $\Gamma_{13}^2\Gamma_{22} + \Gamma_{23}^2\Gamma_{11}$.

A finite set of invariants is called a functional basis if each invariant is a function of the elements of that set. If these elements are polynomials and all polynomial invariants are also expressible as polynomial functions of them, then this is called an integrity basis. Hilbert's basis theorem [14] states that the set of polynomial invariant is generated by a finite number of them. The set of five invariants of plane elasticity tensor in which two linear, two quadratic and one cubic invariant form an integrity basis [12]. However, finding an integrity basis for elasticity tensors in three dimensions is as yet an open problem.

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