SOLUTIONS OF THE EINSTEIN FIELD EQUATIONS WITH POLYTROPIC EQUATIONS OF STATE

By

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Dedicated to

my loving husband and daughter

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Abstract

In this thesis we study the solutions of the Einstein field equations with polytropic equation of state. By assuming a new gravitational potential, we extend the work presented in *Exact anisotropic sphere with polytropic equation of state* by S. Thirukkanesh and F. C. Ragel. Two different values of the polytropic index are assumed to obtain two exact solutions of the field equations utilizing the transformation. Both the models satisfy all the physical properties of the relativistic masses.

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Chapter 1

Introduction

Albert Einstein's theory of relativity gave new aspects of imagination and observations to the physicists and astronomers. He interlinked the force of gravitation to the geometrical properties of the spacetime continuum. The basic theme of his theory revolves around the idea of four-dimensional spacetime. The equivalence principle played a major role to formulate this theory. He gave entirely new description of the fundamental elements of the universe i.e. energy, matter, gravity, space and time.

He related geometry and matter in his field equations. These field equations help us to understand the geometry of manifolds and are important part of the theory of general relativity. The solutions of these field equations describe the geometry of celestial objects and black holes. In this thesis, we consider relativistic stellar models in static spherically symmetric field. We seek exact solutions to the Einstein field equations with anisotropic pressures.

To understand relativity theory more clearly, it is important to build a rigorous base of primary concepts of geometry and its elements like tensors. This chapter includes information about these basic ideas.

1.1 Differential geometry

Differential geometry is the branch of mathematics that deals with the application of the principles of differential and integral calculus along with the concepts of algebra to the study of curves and surfaces [1].

1.2 Curves and surfaces

A one dimensional object is called a *curve*. In geometry, curves are represented in a parameterized form and their geometric properties and various quantities such as the curvature and the arc length are expressed through derivatives and integrals using vector calculus. The Frenet-Seret frame i.e. ($\underline{\text{t}}$(s), $\underline{\text{n}}$(s), $\underline{\text{b}}$(s)), where $\underline{\text{t}}$(s) is the tangent vector, $\underline{\text{n}}$(s) is the normal vector and $\underline{\text{b}}$(s) is the binormal vector, is one of the most important tools used to analyze a curve.

A *surface* is the two-dimensional object. By "two-dimensional" we mean that at each point there is a coordinate patch on which a two-dimensional coordinate system is defined. The surface of the Earth is a two-dimensional sphere and latitude and longitude provide two-dimensional coordinates on it (except at the poles and along the 180th meridian) [2].

1.3 Manifolds

To understand manifolds, we need to recall a few concepts. A *topological space* is a set, say T, with a specified class of open subsets or neighbourhoods, such that:

1. T and the empty set ϕ are open;

2. the intersection of any two open sets is open;

3. the union of any number of open sets is open.

A topological space T is called *Hausdroff* if any two points of T possess non-intersecting neighbourhoods. A space is said to be *separable* if it has countable dense subsets. A *homeomorphism* is a bijective continuous map with continuous inverse.

A *Manifold* of dimension n is a separable, connected, Hausdroff space with a homeomorphism from each element of its open cover into $\mathbb{R}^n$ [3]. In other words we can say that a manifold is Hausdroff, it has a countable basis and it is homeomorphic to $\mathbb{R}^n$. A circle is a simple example of a manifold. $\mathbb{R}$ is also a manifold.

1.4 The first fundamental form

Consider a surface with two points $P(u,v)$ and $Q(u+du, v+dv)$. Let $\underline{x}_u$ and $\underline{x}_v$ be the basis vectors provided that $\underline{x}_u \times \underline{x}_v \neq 0$. Let $\underline{x}$ and $\underline{x}+d\underline{x}$ be the corresponding position vectors of P and Q respectively. Considering $\overrightarrow{PQ}$

$$\underline{x} + d\underline{x} - \underline{x} = d\underline{x} = \underline{x}_u du + \underline{x}_v dv, \tag{1.4.1}$$

$$\overrightarrow{PQ} = ds = |d\underline{x}|, \tag{1.4.2}$$

$$d\underline{x}.d\underline{x} = ds^2 = \underline{x}_u.\underline{x}_u du^2 + 2\underline{x}_u du \underline{x}_v dv + \underline{x}_v \underline{x}_v dv^2. \tag{1.4.3}$$

This is called *the first fundamental form* [3]. The first fundamental form is non-negative i.e.

$$ds^2 \geq 0.$$

The metric properties of a surface are defined by the first fundamental form. It provides the means to calculate the lengths of curves on the surface and the areas of regions on the surface. The line element given in equation (1.4.3) may be expressed in terms of the coefficients of the first fundamental form as

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2. \tag{1.4.4}$$

1.5 Tensors

A mathematical entity with components that change in a particular way in a transformation from one coordinate system to another is called Tensor [1]. The physical variables arising in a problem are represented by tensor fields. In other words, the physical phenomena can be shown mathematically by means of tensors whereas tensor fields indicate how tensor values vary in space and time. For solving problems, we need to express the tensor in a given coordinate system, hence we have the concept of tensor components, but while tensors are independent of the coordinate system, their components are not and change as the system change [4]. Thus a tensor is the quantity that remains the same in all coordinate systems.

The ***rank*** of a tensor is the total number of indices that it has. The rank is independent of the number of dimensions of the space. A rank "zero" tensor is called ***scalar***. These quantities have just magnitude and no direction, e.g. temperature, pressure etc. A rank "one" tensor is called ***vector***. These quantities have both magnitude and direction, e.g. force, velocity etc. The invariant quantities with rank ≥ 2 are generally called ***tensors***.

1.5.1 Covariant and contravariant vectors

Covariance and contravariance describe how the quantitative description of certain geometric or physical entities changes with a change of basis. If N quantities A^1, A^2, ..., A^N in a coordinate system $(x^1, x^2, ..., x^N)$ are related to N other quantities $\overline{A}^1$, $\overline{A}^2$, ... , $\overline{A}^N$ in another coordinate system $(\overline{x}^1, \overline{x}^2, ..., \overline{x}^N)$ by the transformation equations

$$\overline{A}^p = \sum_{q=1}^{N} \frac{\partial \overline{x}^p}{\partial x^q} A^q, \tag{1.5.1}$$

where $(p = 1, 2, , , , N)$, which by the conventions adopted can simply be written as

$$\overline{A}^p = \frac{\partial \overline{x}^p}{\partial x^q} A^q. \tag{1.5.2}$$

They are called components of a contravariant vector or contravariant tensor of the first rank [5].

If N quantities A_1, A_2, ... , A_N in a coordinate system $(x^1, x^2, ..., x^N)$ are related to N other quantities $\overline{A}_1$, $\overline{A}_2$, ... , $\overline{A}_N$ in another coordinate system $(\overline{x}^1, \overline{x}^2, ..., \overline{x}^N)$ by the transformation equations

$$\overline{A}_p = \sum_{q=1}^{N} \frac{\partial x^q}{\partial \overline{x}^p} A_q, \tag{1.5.3}$$

where $(p = 1, 2, ..., N)$, or

$$\overline{A}_p = \frac{\partial x^q}{\partial \overline{x}^p} A_q, \tag{1.5.4}$$

they are called components of a covariant vector or covariant tensor of the first rank [5].

1.5.2 Contravariant, covariant and mixed tensors

If N^2 quantities A^{qs} in a coordinate system $(x^1, x^2, ..., x^N)$ are related to N^2 other quantities $\overline{A}^{pr}$ in another coordinate system $(\overline{x}^1, \overline{x}^2, ..., \overline{x}^N)$ by the transformation equations

$$\overline{A}^{pr} = \sum_{s=1}^{N} \sum_{q=1}^{N} \frac{\partial \overline{x}^p}{\partial x^q} \frac{\partial \overline{x}^r}{\partial x^s} A^{qs}, \tag{1.5.5}$$

where $(p, r = 1, 2, ..., N)$, or

$$\overline{A}^{pr} = \frac{\partial \overline{x}^p}{\partial x^q} \frac{\partial \overline{x}^r}{\partial x^s} A^{qs}, \tag{1.5.6}$$

by the adopted conventions, they are called contravariant components of a tensor of the second rank [5]. The N^2 quantities A_{qs} are called covariant components of a tensor of the second rank if

$$\overline{A}_{pr} = \frac{\partial x^q}{\partial \overline{x}^p} \frac{\partial x^s}{\partial \overline{x}^r} A_{qs}. \tag{1.5.7}$$

Similarly the N^2 quantities A^q_s are called components of a mixed tensor of the second rank if

$$\overline{A}^p_r = \frac{\partial \overline{x}^p}{\partial x^q} \frac{\partial x^s}{\partial \overline{x}^r} A^q_s. \tag{1.5.8}$$

If $A^{\bar{i}...\bar{j}}_{\bar{k}...\bar{l}}$ defines the components of a mixed tensor with $i...j$ as contravariant order and $k...l$ as covariant order, then they can be transformed as

$$\bar{A}^{\bar{i}...\bar{j}}_{\bar{k}...\bar{l}} = \frac{\partial \bar{x}^{\bar{i}}}{\partial x^i} ... \frac{\partial \bar{x}^{\bar{j}}}{\partial x^j} \frac{\partial x^k}{\partial \bar{x}^{\bar{k}}} ... \frac{\partial x^l}{\partial \bar{x}^{\bar{l}}} A^{i...j}_{k...l}. \tag{1.5.9}$$

1.5.3 Symmetric and skew symmetric tensors

A tensor can be broken down into its symmetric and anti-symmetric parts [3]. A tensor (covariant or contravariant) of rank ≥ 2 is said to be symmetric with respect to two covariant indices or contravariant indices if the components remain unchanged after interchange of the indices. A mixed tensor is symmetric in the indices if

$$A^{pqr}_{ij} = A^{qpr}_{ij},$$

or

$$A^{pqr}_{ij} = A^{pqr}_{ji}.$$

A tensor of rank ≥ 2 is said to be skew-symmetric with respect to two covariant indices or contravariant indices if the components get changed after interchange of the indices (lower or upper). The following mixed tensor is skew-symmetric in the indices

$$A^{pqr}_{ij} = -A^{qpr}_{ji}.$$

A tensor can be written as:

$$A_{ab} = A_{(ab)} + A_{[ab]}.$$

Here $A_{(ab)}$ is the symmetric part of the tensor A_{ab}, while $A_{[ab]}$ is the skew symmetric part of this tensor. The symmetric part can be written as

$$A_{(ab)} = A_{(ba)}.$$

While the skew symmetric part of tensor can be written as

$$A_{[ab]} = -A_{[ba]}.$$

Generalizing this concept to the tensor with rank r

$$A_{(a...c)} = \frac{1}{r!}(A_{a...c} + \text{all permutations of a . . . c}),$$

and

$$A_{[a...c]} = \frac{1}{r!}(A_{a...c} + (\text{all even permutations}) - (\text{all odd permutations})).$$

1.5.4 Operations with tensors

- **Addition**: Addition between two tensors of same rank result as the tensor of the same rank and type.

  $$A^{ac}_{bd} = B^{ac}_{bd} + C^{ac}_{bd},$$

  Commutative and associative properties hold in the case of addition.

- **Subtraction:** The difference of two tensors of the same rank and type is also a tensor of the same rank and type.

  $$A^{ac}_{bd} = B^{ac}_{bd} - C^{ac}_{bd}.$$

- **Outer multiplication:** The product of two tensors is a tensor whose rank is the sum of the ranks of the given tensors. This product which involves ordinary multiplication of the components of the tensor is called the *outer product* [5]. Outer multiplication of two tensors of different ranks result as the sum of the indices of these two tensors i.e.

$$A^{ab}_{e}B^{cd}_{fg} = C^{abcd}_{efg}.$$

Here it is important to remember that every tensor cannot be written as an outer product of two lower rank tensors. For this reason division of tensors is not always possible.

- **Contraction:** If one contravariant and one covariant index of a tensor are set equal, the result indicates that a summation over the equal indices is to be taken according to the summation convention. This resulting sum is a tensor of rank two less than that of the original tensor. The process is called *contraction* [5]. While doing contraction of tensor its rank is reduced. We can define contraction of T^{a}_{b} by

$$T = \delta^{b}_{a}T^{a}_{b},$$

where δ^{b}_{a} is the index substitution operator i.e. the upper and lower indices of tensor are set equal and then taking sum over will contract a tensor.

- **Inner product:** The inner product of tensors can be obtained by outer multiplication of tensors followed by contraction process, i.e.

$$A^{ab}_{e}B^{cd}_{fg} = C^{abcd}_{efg},$$

by putting $a = e$ and taking sum over its result would be C^{bcd}_{fg} i.e. a tensor of rank 5. The inner and the outer product both satisfy the properties of commutativity and associativity.

1.5.5 The metric tensor

Co-ordinates help us to locate some certain points in space but they do not provide sufficient information to analyze the geometry of the space. A metric tensor is a function that is defined at each point of space to provide the desired information. Assume a set of linearly independent generalized co-ordinates $(x^1, x^2, ..., x^n)$ which uniquely represent a point in space. A line segment between two points is formed if any infiniteisimal displacement is carried out simultaneously. Thus

$$d\underline{x} = \underline{e}_i dx^i,$$

where $\underline{e}_i$ are the basis vectors and x^i $(i = 1, 2..., n)$ are the co-ordinates of point. Here $d\underline{x}$ i.e. the line segment, is a vector quantity and if the dot product of $d\underline{x}$ is carried out with itself we get an interval

$$ds^2 = d\underline{x}.d\underline{x} = g_{ij}dx^i dx^j.$$

Here g_{ij} $(i, j = 1, 2, ..., n)$ is metric tensor and is defined as

$$g_{ij} = \underline{e}_i.\underline{e}_j = g_{ji}.$$

g_{ij} is symmetric tensor as it is the distance between the two points and is just dependent upon the co-ordinates and the position only. If the metric tensor is positive definite then the space is called **Riemannian Space** but if it is not positive definite then the space is called **Pseudo-Riemannian Space**. The **inverse** of g_{ij} is denoted by g^{ij} and is defined as

$$g^{ij} = \frac{\text{cofactor of } g_{ij}}{g},$$

where g is the determinant of g_{ij} i.e. $g = |g_{ij}|$ and $g \neq 0$. g^{ij} is symmetric contravariant tensor of rank two.

1.5.6 Christoffel symbols

Christoffel symbols are connection between the tangent spaces at different points of manifold and they enable us to link a vector in the tangent space at one point with the vector parallel to it at another point [6]. Symbolically it is written as Γ^a_{bc}

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad} (g_{bd,c} + g_{cd,b} - g_{bc,d}). \tag{1.5.10}$$

The Christoffel symbol is symmetric in the lower two indices i.e.

$$\Gamma^a_{bc} = \Gamma^a_{cb}.$$

1.5.7 Tensor differentiation

A scalar is a tensor of rank zero and is invariant quantity under co-ordinate transformation. A first order tensor i.e. a vector is obtained if we partially differentiate an invariant scalar i.e.

$$A_{,i} = \frac{\partial A}{\partial x^i} = B_i. \tag{1.5.11}$$

But if a tensor of rank ≥ 1 is partially differentiated its result is not a tensor. To avoid such situation we use covariant differentiation, which assures that the resulting differentiation is a tensor. The covariant differentiation for covariant and contravariant tensors with rank 1 are given as

$$A^a_{;b} = A^a_{,b} + \Gamma^a_{bc} A^c, \tag{1.5.12}$$

and

$$A_{a;b} = A_{a,b} - \Gamma^c_{ba} A_c, \tag{1.5.13}$$

where ‘,’ represents the usual partial differentiation and the Christoffel symbols ensure that this covariant differentiation transforms as a tensor. A tensor with rank 2 can have covariant differentiation as [2]

$$A^{ab}_{;c} = A^{ab}_{,c} + \Gamma^a_{dc} A^{db} + \Gamma^b_{dc} A^{ad}, \tag{1.5.14}$$

$$A_{ab;c} = A_{ab,c} - \Gamma^d_{ac} A_{db} - \Gamma^d_{bc} A_{ad}. \tag{1.5.15}$$

1.6 The Riemann curvature tensor

The covariant differentiation is generalization of partial differentiation, however, the order of differentiation is important in covariant differentiation i.e.

$$A^a_{;c;d} \neq A^a_{;d;c}. \tag{1.6.1}$$

Assuming the covariant derivation of rank 1 tensor i.e.

$$A^a_{;c} = A^a_{,c} + \Gamma^a_{bc} A^b, \tag{1.6.2}$$

differentiating again

$$(A^a_{;c})_{;d} = (A^a_{;c})_{,d} + \Gamma^a_{ed}(A^e_{;c}) - \Gamma^e_{cd}(A^a_{;e}). \tag{1.6.3}$$

Inserting the values of $A^a_{;c}$, $A^e_{;c}$ and $A^a_{;e}$ in equation (1.6.3) we get

$$A^a_{;c;d} = (A^a_{,c} + \Gamma^a_{bc} A^b)_{,d} + \Gamma^a_{ed}(A^a_{,c} + \Gamma^e_{bc} A^b) - \Gamma^e_{cd}(A^a_{,e} + \Gamma^a_{be} A^b). \tag{1.6.4}$$

By repeating the process but changing the order of differentiation we get

$$A^a_{;d;c} = (A^a_{,d} + \Gamma^a_{bd} A^b)_{,c} + \Gamma^a_{ec}(A^e_{,d} + \Gamma^e_{bd} A^b) - \Gamma^e_{dc}(A^a_{,e} + \Gamma^a_{be} A^b), \tag{1.6.5}$$

subtracting equation (1.6.4) from equation (1.6.5) and equating the dummy indices, we have

$$A^a_{;d;c} - A^a_{;c;d} = R^a_{bcd} A^b + (\Gamma^e_{cd} - \Gamma^e_{dc}) \nabla_e A^a, \tag{1.6.6}$$

where

$$R^a_{bcd} = (\Gamma^a_{bd})_e - (\Gamma^a_{bc})_d + \Gamma^e_{bd}\Gamma^a_{ec} - \Gamma^e_{bc}\Gamma^a_{ed}. \tag{1.6.7}$$

As

$$\Gamma^e_{cd} = \Gamma^e_{dc}, \tag{1.6.8}$$

so equation (1.6.6) becomes

$$A^a_{;d;c} - A^a_{;c;d} = R^a_{bcd}A^b. \tag{1.6.9}$$

R^a_{bcd} is called Riemann curvature tensor and possess rank 4. The Riemann curvature tensor measures the flatness of the space. If $R^a_{bcd} = 0$, it shows that the specific space is flat and if $R^a_{bcd} \neq 0$, it means that the space is curved. By using the transformation

$$R_{abcd} = g_{ae}R^e_{bcd},$$

the Riemann curvature tensor becomes a rank 4 covariant tensor.

1.6.1 Symmetry properties of R_{abcd}

(1) R_{abcd} is symmetric under permutation of the first and the second pair of indices i.e.

$$R_{abcd} = R_{cdab}.$$

(2) R_{abcd} is skew symmetric in the first two indices i.e.

$$R_{abcd} = -R_{bacd}.$$

(3) R_{abcd} is skew symmetric in the last two indices i.e.

$$R_{abcd} = -R_{abdc}.$$

1.7 The Bianchi identities

Due to the symmetry of the connection symbols, the Riemann curvature tensors satisfy the property

$$R^a_{bcd} + R^a_{cdb} + R^a_{dbc} = 0. \quad (1.7.1)$$

It is called *"The First Bianchi Identity"*. R^a_{bcd} also satisfies *"The Second Bianchi Identity"* by covariant differentiation i.e.

$$R^a_{p[bc;d]} = 0, \quad (1.7.2)$$

$$R^a_{pbc;d} + R^a_{pcd;b} + R^a_{pdb;c} = 0. \quad (1.7.3)$$

1.8 The Ricci tensor

The Ricci tensor is defined as the trace of the Riemann tensor

$$R^a_{bad} = R_{bd}, \quad (1.8.1)$$

i.e. by applying the contraction on the first and the third indices, a rank 2 symmetric tensor is obtained which is called *the Ricci tensor*.

1.8.1 The Ricci scalar

The Ricci scalar is a scalar quantity and is obtained by applying contraction on the Ricci tensor along with g^{ab} i.e.

$$R = g^{ab} R_{ab}. \quad (1.8.2)$$

The scalar entities are invariant under coordinate transformations so the Ricci scalar helps us to check the singularity. The singularity is coordinate if the curvature invariant are finite otherwise its essential.

Chapter 2

Elements of relativity

Aristotle, the Greek philosopher in the 4th century BC believed that there exists a "cause" without which there is no possibility of effects and motions of bodies. The cause of the downward motion of heavy bodies, such as the element earth, was related to their nature, which caused them to move downward toward the center of the universe, which was their natural place. Conversely, light bodies such as the element fire, move by their nature upward toward the inner surface of the sphere of the Moon. Thus in Aristotle's system heavy bodies are not attracted to the earth by an external force of gravity, but tend toward the center of the universe because of an inner gravitas or heaviness [6]. Brahmagupta, an Indian mathematician in 7th century stated "Bodies fall towards the earth as it is in the nature of the earth to attract bodies, just as it is in the nature of water to flow."

Ibn al-Haytham in 1020s and 1030s, wrote numerous books on astronomy and provided a more realistic view of the way the universe works. He also wrote about the laws governing the movement of bodies (later known as Newton's 3 laws of motion) and the attraction between two bodies i.e. gravity. In 11th century, an Indian Mathematician Bhaskaracharya in his book called *Siddhantha Siromani* described gravity. According to him the earth attracts the solid objects in the sky by its own force towards itself. Bhaskaracharya further discusses the forces between the celestial bodies using a question: Where can the celestial bodies fall since they attract each other? [7], "all objects accelerated equally when falling" was Galileo's concept presented in 17th century. In the late 17th century, Issac Newton deduced that there is a gravitational force which depends on the inverse square of the distance by using Robert Hooke's suggestions. The Newtonian original formula of gravity is

$$\text{force of gravity} \propto \frac{\text{mass of object 1} * \text{mass of object 2}}{\text{distance of centers}^2}. \tag{2.0.1}$$

In early 19th century, Albert Einstein developed the equivalence principle which suggests that gravitation was exactly equivalent to acceleration. This principle later helped him to develop the theory of general relativity.

2.1 Introducing relativity

Issac Newton's formulated laws of gravitation were enough to pen down the physical events that are occurring on Earth. Newton's laws are applied to objects which are idealized as single point masses, in the sense that the size and shape of the object's body are neglected in order to focus on its motion more easily. This can be done when the object is small compared to the distances involved in its analysis, or the deformation and rotation of the body are of no importance. In this way, even a planet can be idealized as a particle for analysis of its orbital motion around a star [2]. Newton's laws of motion are not adequate to characterize the motion of rigid bodies and deformable bodies. Newton's laws hold only with respect to a certain set of frames of reference.

Albert Einstein, in 1905 determined that the laws of physics are same for all non-accelerating observers, and that the speed of light in a vacuum is independent of the motion of all observers. This is the theory of special relativity. He suggested that matter and energy are interlinked and mathematically gave its formula

$$E = mc^2,$$

where E is the energy, m is the mass of the object and c is the speed of light. It introduced a new framework for all of physics and proposed new concepts of space and time.

Einstein then spent ten years trying to include acceleration in his theory and published the theory of general relativity (GR) in 1915. He determined that massive objects cause a distortion in spacetime, which is felt as gravity [9]. The theory of relativity leads us to the concept of four-dimensional spacetime continuum. It explains that the time and position of an event is relative to the frame of reference of the observers.

2.2 Postulates of special relativity (SR)

The fundamental components of universe i.e. *space, time, matter, gravity, mass, energy and light* act in very surprising manner in certain specific *relativistic* situations, in special relativity. Understanding this relativistic effect on each of these components provide understanding to SR. The theory of SR is limited to the description of events as they appear to observers in a state of uniform motion relative to one another. The two important postulates of SR are [10]

- The laws of physics hold true for all inertial frames of references.
- In all frames of references, the speed of light is measured as constant.

By the *frame of reference* we mean, a framework that is used for the observation and mathematical description of physical phenomena and the formulation of physical laws, usually consisting of an observer, a coordinate system, and a clock or clocks assigning times at positions with respect to the coordinate system [1]. It is important to evolve this concept here that the universe has no absolute frame of reference i.e. there is not a single stationary place in the universe. The theory of SR is limited to the observers moving with the uniform velocity in relation to each other.

2.3 Theory of general relativity (GR) in a glance

The theory of GR is extension of SR and was presented by Einstein after 10 years of presentation of SR. In SR, the velocity of light was taken as constant for every observer but in GR this law is not taken as it is. The theory of GR leads us to an important result that velocity of light must always depend on the co-ordinates when a gravitational field is present. GR also applies to observers with non-uniform or accelerated relative motion. It provides the result that matter and empty space influence each other in a complex fashion. Einstein's theory of GR has a set of three assumptions [11]

1. We live in a four-dimensional spacetime with a local, flat Minkowski space.

2. The structure of the space is given by the Einstein field equations

   $$G^{\mu\nu} = \frac{8\pi G}{c^4} T^{\mu\nu}.$$

   Here $G^{\mu\nu}$ is the Einstein tensor derived from the Riemann curvature tensor and $T^{\mu\nu}$ is the energy-momentum tensor for the medium under consideration. $G^{\mu\nu}$ and $T^{\mu\nu}$ are mentioned with detail in the coming section.

3. Particles follow geodesics in spacetime.

The following five principles also helped Einstein to formulate the theory of GR [12]:

- **Mach's principle:** "The matter distribution defines geometry. If there is no matter then there is no geometry and body in an otherwise empty universe should possess no inertial properties."
- **The principle of equivalence:** "Inertial and gravitational mass are equivalent."
- **The principle of covariance:** "All frames of reference are equivalent and the equations of physics have tensorial form."
- **The principle of minimal gravitational coupling:** "No term explicitly containing the curvature should be added to the tensor equation in making the transition from the special to general relativity."
- **The correspondence principle:** "General relativity must agree with special relativity in the absence of gravitation and with Newtonian gravitational theory in the limit of weak gravitational field and low velocities."

2.4 The Einstein field equations (EFEs)

Theory of relativity states that it is matter and energy that create curves in spacetime as a result of which the fundamental interaction of gravitation occurs. To describe this fundamental interaction of gravity Einstein derived the partial differential equations in 1915 and they are called *the Einstein field equations*. The formulation of these equations is based on Riemannian geometry i.e. the curved space geometry without straight lines. These equations describe how matter creates gravity and conversely how gravity affects matter.

Einstein equated the local spacetime curvature which is expressed by the Einstein tensor with the stress-energy tensor that consists of the local energy and momentum within that spacetime. The presence of mass-energy and linear momentum gives the spacetime geometry and the EFEs can be used to determine this spacetime geometry. The EFEs are essentially the equations of motion. The solutions of these equations explain spacetime geometry in a particular medium. Recall equation (1.7.2), the second Bianchi identity

$$R^a_{bcd;e} + R^a_{bec;d} + R^a_{bde;c} = 0, \tag{2.4.1}$$

contracting over a and c, we get

$$R_{bd;e} - R_{be;d} + R^a_{bde;a} = 0, \tag{2.4.2}$$

multiplying with g^{bd}

$$(\frac{1}{2}\delta^a_e R - R^a_e)_{;a} = 0. \tag{2.4.3}$$

Multiplying with $-g^{eb}$ on both sides

$$(R^{ab} - \frac{1}{2}g^{ab}R)_{;a} = 0. \tag{2.4.4}$$

Here

$$R^{ab} - \frac{1}{2}g^{ab}R = G^{ab},$$

is called the **Einstein tensor**. We define the Einstein tensor as the divergence free part of the Ricci tensor. The geometric behavior of the spacetime is specified by the Einstein tensor and it is the combination of the curvature tensor that contains the geometry and the Riemann tensor. The equation (2.4.4) can be written as

$$G^{ab}{}_{;a} = 0. \tag{2.4.5}$$

This equation (2.4.5) shows that rate of change is conserved in curved surface. The **energy-momentum tensor** is given by T^{ab} and it describes the distribution of energy and momentum in a spacetime. It is a symmetric, rank 2 tensor i.e.

$$T^{ab} = T^{ba}. \tag{2.4.6}$$

Its components can be represented in the matrix form i.e.

$$T^{ab} = \begin{pmatrix} T^{00} & T^{01} & T^{02} & T^{03} \\ T^{10} & T^{11} & T^{12} & T^{13} \\ T^{20} & T^{21} & T^{22} & T^{23} \\ T^{30} & T^{31} & T^{32} & T^{33} \end{pmatrix}.$$

Here time-time component i.e. T^{00} is the energy density of relativistic mass [13]. The components T^{ii} (i = 1, 2, 3) represents pressure. The components T^{i0} (i = 1, 2, 3) represents the momentum density and the remaining components represent momentum flux. The energy-momentum tensor must be conserved to fulfil the physical requirements i.e.

$$T^{ab}{}_{;a} = 0, \tag{2.4.7}$$

using equation (2.4.5) and equation (2.4.7), we get

$$G^{ab} \propto T^{ab}, \tag{2.4.8}$$

$$G^{ab} = \kappa T^{ab}. \tag{2.4.9}$$

The above equation provides us a relation between geometry and energy-momentum tensor. Thus if the geometry of any surface is known then its energy and momentum can also be calculated. Equation (2.4.9) can be written as

$$R^{ab} - \frac{1}{2} R g^{ab} = \kappa T^{ab}. \tag{2.4.10}$$

It is the system of 10 non-linear partial differential equations with 10 unknowns, which are functions of 4 independent variables (x, y, z, t). The above equation can also be written in the following forms

$$R_{ab} - \frac{1}{2} R g_{ab} = \kappa T_{ab}, \tag{2.4.11}$$

and

$$R^a_b - \frac{1}{2} \delta^a_b R = \kappa T^a_b. \tag{2.4.12}$$

2.5 Significance of the field equations

The theory of GR can be applied to the universe as a whole. The observations made by physicists and scientists have proven that the construction of universe appears to be following simple lines. In other words, the universe is isotropic and homogeneous. By *Isotropic* we mean an object exhibiting uniform properties throughout in all directions. They suggest that the cosmological structure should also be relatively same. The simple solutions of the EFEs can be applied to describe the structure of simple universes. The current cosmological models of the universe are obtained by combining these simple solutions to GR with theories describing the properties of the universe's matter content, namely thermodynamics, nuclear and particle physics [14].

All branches of physics dealing with electromagnetic radiation and high speed particles have been influenced with the relativity theory. The concepts of astronomy and astrophysics are also effected by the theory of relativity. The behavior of astrophysical objects can be explained using the solutions of the field equations.

2.6 The Schwarzschild solution

The German scientist Karl Schwarzschild gave the first exact solution of the EFEs. The Schwarzschild solution is the simplest of all exact solutions of the EFEs. This is static, spherically symmetric vacuum solution. Since we are considering the vacuum solution so

$$T^{ab} = 0. \tag{2.6.1}$$

Thus equation (2.4.10) reduces to

$$R^{ab} = 0. \tag{2.6.2}$$

Thus equation (2.6.2) is the vacuum Einstein field equation. The most general spherically symmetric metric is

$$ds^2 = -c^2 e^{\nu} dt^2 + e^{\lambda} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \tag{2.6.3}$$

For vacuum Einstein field equations the Schwarzschild solution is

$$ds^2 = -c^2 \left(1 - \frac{2Gm}{c^2 r^2}\right) dt^2 + \left(\frac{1}{1 - \frac{2Gm}{c^2 r^2}}\right) dr^2 + r^2 d\Omega^2. \tag{2.6.4}$$

2.7 Other solutions of the field equations

Solutions of the EFEs play a vital role to study the structure and geometry of the universe. Their solutions provide us not only information regarding the geometry of our earth but also of stellar objects in the universe. These equations are non linear in nature so cannot be completely solved without making assumptions. In a spacetime, for two massive bodies there is no known complete solution. However, in these cases usually approximations are made. Apart from this, the field equations have been completely solved in many cases and such solutions are called *Exact solutions* [14]. Under simplifying assumptions e.g. symmetry, the exact solutions for the field equations can be found.

Solutions of the field equations that are just approximated but are not exact are called *non-exact solutions.* Such solutions mainly arise due to the difficulty of solving the field equation in closed form and often take the form of approximations to ideal systems. Many non-exact solutions may be devoid of physical content, but serve as useful counter examples to theoretical conjectures [15].

The Reissner-Nordstrom solution describes the gravitational field outside a static spherically charged body. In the absence of charge the Reissner-Nordstrom solution reduces to the Schwarzschild exterior solution. *The Kerr solution* describes the exterior of a rotating body. This rotating solution has a complex form with interesting physical features. The Kerr solution reduces to the Schwarzschild solution in the appropriate limit. Some other special solutions are *the Weyl solutions*, *Israel-Khan solutions*.

Chapter 3

Solutions of the Einstein field equations

Solutions of the EFEs depend on the stress-energy tensor, which depends on the dynamics of matter and energy (such as trajectories of moving particles), which in turn depends on the gravitational field. Models of stars and compact bodies are idealized by using the perfect fluid solution in a static, symmetric space-time equipped with suitable tensor fields. The *perfect fluids* are often used to model the distribution of matter in the interior of a star and *the fluid solutions* are the exact solutions of the field equations in which the gravitational field is produced entirely by mass, momentum and stress density of a fluid [16]. Pressure and density do not vary from point to point and velocity is co-moving. Many times imperfect fluid is considered, as a result pressure and density vary from point to point. In imperfect fluids, there is no isotropy around a point moving with the fluid.

To get the solutions of the field equations and relativistic models sometimes the condition of the equation of state is used. The equation of state is discussed in the coming section. The relativistic models have to satisfy certain physical conditions, to check if they are physically reasonable models and confirms that the exact solutions found are physically acceptable. These physical properties are mentioned in the later section.

3.1 Brief introduction to thermodynamics

Thermodynamics is the branch of natural sciences concerned with heat and its relation to energy and work. It defines the macroscopic variables (such as temperature, internal energy, entropy and pressure) that characterize materials and radiations and explains how they are related and by what laws they change with time. *Entropy* is the measure of the number of specific ways in which a thermodynamic system may be arranged, often taken to be a measure of disorder or a measure of progressing towards thermodynamic equilibrium. In a state of *thermodynamic equilibrium* there is no net flow of matter or of energy, no phase changes and no unbalanced potentials within the system [2].

3.1.1 Equation of state

The stress-energy tensor involves both the pressure p and the energy density ρ but these are related by an equation of state. For a simple fluid in local thermodynamics equilibrium, there always exist a relation of the form

$$p = p(\rho, s), \tag{3.1.1}$$

which gives the pressure of the energy density and specific entropy s. We often deal with situations in which the entropy can be considered to be a constant (in particular, negligibly small), so that we have a relation

$$p = p(\rho). \tag{3.1.2}$$

These relations will of course have different functional forms for different fluids [17].

3.1.2 Hydrostatic equilibrium and equation

To understand the structure of the stellar bodies we need to understand the concept of hydrostatic equilibrium. Hydrostatic equilibrium in stars include the following ideas:

- The energy is carried outward towards the cooler surface but it is generated in the hot core of star.
- The inward force of gravity balances the outward force of pressure inside a star.
- By pressure-temperature thermostat, the star is stabilized.

Using these concepts the equation of hydrostatic equilibrium is derived i.e.

$$\frac{dp}{dr} = -\frac{GM(r)}{r^2}\rho, \tag{3.1.3}$$

where M(r) is the mass of the object and both $\frac{GM(r)}{r^2}$ and ρ are positive. This shows that $\frac{dp}{dr}$ is less than or equal to zero thus the pressure must decrease outward everywhere for a gravitating system to be in hydrostatic equilibrium. This will in turn imply that density and temperature must increase toward the center of a star [18]. Thus the hydrostatic conditions are enough to assure that the star must be much more dense and more hot near their centers than surface.

3.1.3 Polytropic equation of state

A polytrope is a spherical self-gravitating body in which the pressure relates to the density in a particularly simple way

$$p(r) \propto \rho(r)^{1+\frac{1}{n}},$$

$$p(r) = K\rho(r)^{1+\frac{1}{n}},$$

here n is called the polytropic index and K is the constant. This relationship has its origins in thermodynamics and results from the notion of polytropic change. Equation of state of this form, when coupled with the equation of hydrostatic equilibrium, will provide a single relation for the run of pressure or density with position. The solution of this equation basically solves the fundamental problem of stellar structure in so far as the equation of state correctly represents the behavior of the stellar gas. Such solutions are called polytropes of a particular index n [19].

3.2 Physical properties of a relativistic star

The physical properties that should be satisfied by a realistic model are [20]:

1. regularity of the gravitational potentials at the origin;
2. positive definiteness of the energy density and the radial pressure at the origin i.e. ρ and p_r are positive at origin;
3. vanishing of the pressure at some finite radius;
4. monotonic decrease of the energy density and the radial pressure with increasing radius i.e. $\frac{d\rho}{dr}$ and $\frac{dp_r}{dr}$ are less than zero;
5. the interior metric of static, spherically symmetric model, in the absence of charge matches smoothly with the Schwarzschild exterior metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (3.2.1)$$

across the boundary $r = R$ and M is the mass of the sphere. For the charged spherically symmetric body, the interior metric must match with the Reissner-Nordstrom metric

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (3.2.2)$$

where Q is the charge of the body. For the rotating models without charge, the interior metric should match the Kerr metric while in the presence of charge the interior metric should match smoothly with the Kerr-Newman metric.

3.3 Review of an isotropic solution of the Einstein field equations

A. J. John and S. D. Maharaj [21] derived a new exact solution of the field equations which is useful to describe the interior of the star. The field equations are written in new co-ordinates and the gravitational potential assumed is in cubic form. The condition of pressure isotropy is simplified using this gravitational potential. The line element in static, spherically symmetric spacetime is given as

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \tag{3.3.1}$$

The energy-momentum tensor is given as

$$T^i_j = diag(-\rho, p, p, p),$$

here the energy density is represented by ρ and the pressure p is measured relative to comoving fluid 4-velocity $u^a = e^{-\nu}\delta^a_0$. The field equations (2.4.12) for neutral perfect fluid are written as

$$\frac{1}{r^2}\left[r(1-e^{-2\lambda})\right]' = \rho, \tag{3.3.2}$$

$$-\frac{1}{r^2}\left[1-e^{-2\lambda}\right] + 2\frac{\nu'}{r}e^{-2\lambda} = p, \tag{3.3.3}$$

$$e^{-2\lambda}\left[\nu'' + \nu'^2 + \frac{\nu'}{r} - \nu'\lambda' - \frac{\lambda'}{r}\right] = p. \tag{3.3.4}$$

The differentiation with respect to r is denoted by '$\prime$'. The coupling constant and speed of light is taken equal to one. By using the transformation suggested by Durgapal and Bannerji [22] i.e.

$$x = Cr^2, Z(x) = e^{-2\lambda(r)}, A^2y^2(x) = e^{2\nu(r)},$$

where A and C are arbitrary constants, the above field equations (3.6.2)-(3.6.4) become

$$\frac{1-Z}{x} - 2\dot{Z} = \frac{\rho}{C}, \tag{3.3.5}$$

$$4Z\frac{\dot{y}}{y} + \frac{Z-1}{x} = \frac{p}{C}, \tag{3.3.6}$$

$$4Zx^2\ddot{y} + 2\dot{Z}x^2\dot{y} + (\dot{Z}x - Z + 1) = 0. \tag{3.3.7}$$

The overdot represents the differentiation with respect to x. Since there is four unknowns i.e. ρ, p, y and Z and the number of equations is three, so assuming the ansatz Z as

$$Z = ax^3 + 1, \tag{3.3.8}$$

where a is any constant. Using equation (3.3.8) in equation (3.3.7)

$$2(ax^3 + 1)\ddot{y} + 3ax^2\dot{y} + axy = 0. \tag{3.3.9}$$

The Frobenius method is then used to find out the recurrence relation and get the solution of the above second degree linear equation. The solution is given by the equation

$$y(x) = c_0 y_1(x) + c_1 y_2(x), \tag{3.3.10}$$

where

$$y_1(x) = \left(1 + \sum_{n=0}^{\infty} (-1)^{n+1} (\frac{a}{4})^{n+1} \prod_{k=0}^{n} \frac{2(3k)^2 + 3k + 1}{(3k+3)(3k+2)} x^{3n+3}\right),$$

and

$$y_2(x) = \left(x + \sum_{n=0}^{\infty} (-1)^{n+1} (\frac{a}{2})^{n+1} \prod_{k=0}^{n} \frac{2(3k+1)^2 + (3k+1) + 1}{(3k+4)(3k+3)} x^{3n+4}\right).$$

Using equation (3.3.10) to satisfy the physical properties given in section 3.2 and obtain the following results

$$e^{2\lambda} = \frac{1}{ax^3 + 1}, \tag{3.3.11}$$

$$e^{2\nu} = A^2 y^2, \tag{3.3.12}$$

$$\frac{\rho}{C} = -7ax^2, \tag{3.3.13}$$

$$\frac{p}{C} = 4(ax^3 + 1)\frac{\dot{y}}{y} + ax^2. \tag{3.3.14}$$

The value of y is equation (3.3.10). The above calculated values satisfy all the physical properties of stellar star. The functions ν and λ have constant values at the center i.e. at $x = 0$. At the origin the function ρ vanishes and the pressure p has constant value. Thus the gravitational potentials and the matter variables are finite at the origin. As $y(x) = c_o y_1(x) + c_1 y_2(x)$ is well defined on the interval $[0, b]$ and the quantities ν, λ, ρ and p are nonsingular and continuous. The energy density and the pressure are positive on this interval. At the boundary $x = b$ the interior spacetime matches smoothly to the Schwarzschild exterior metric.

3.4 Review of exact anisotropic sphere with polytropic equation of state

S. Thirukkanesh and F. C. Ragel [23] found an exact solution of the field equations with imperfect fluid and by taking polytropic equation of state. The interior of a static, spherically symmetric star is described by the line element given in equation (3.3.1). The energy-momentum tensor taken is given as

$$T^i_j = diag(-\rho, p_r, p_t, p_t), \tag{3.4.1}$$

where ρ is the energy density. The radial pressure is p_r and p_t denotes the tangential pressure. The Einstein field equations (2.4.12) are then expressed as

$$\frac{1}{r^2}\left[r(1 - e^{-2\lambda})\right]' = \rho, \tag{3.4.2}$$

$$-\frac{1}{r^2}\left[1 - e^{-2\lambda}\right] + 2\frac{\nu'}{r}e^{-2\lambda} = p_r, \tag{3.4.3}$$

$$e^{-2\lambda}\left[\nu'' + \nu'^2 + \frac{\nu'}{r} - \nu'\lambda' - \frac{\lambda'}{r}\right] = p_t, \tag{3.4.4}$$

the coupling constant and speed of light is taken equal to one. Here '$\prime$' denotes the differentiation with respect to r.

$$m(r) = \frac{1}{2}\int_0^r w^2\rho(w)dw, \tag{3.4.5}$$

defines the mass of the sphere with radius r. Using the transformation by Durgapal and Bannerji [22], the above equations convert as

$$\frac{1-Z}{x} - 2\dot{Z} = \frac{\rho}{C}, \tag{3.4.6}$$

$$4Z\frac{\dot{y}}{y} + \frac{Z-1}{x} = \frac{p_r}{C}, \tag{3.4.7}$$

$$4xZ\frac{\ddot{y}}{y} + (4Z + 2x\dot{Z})\frac{\dot{y}}{y} + \dot{Z} = \frac{p_t}{C}. \tag{3.4.8}$$

The differentiation with respect to x is denoted by dot. The mass function under this transformation becomes

$$m(x) = \frac{1}{4C^{\frac{3}{2}}}\int_0^x \sqrt{w}\rho(w)dw. \tag{3.4.9}$$

There is five unknown variables i.e. ρ, p_r, p_t, y and Z whereas the number of equations is three. Assuming that the matter distribution must satisfy the barotropic equation of state, the polytropic equation of state taken is

$$p_r = k\rho^{1+(\frac{1}{n})}, \tag{3.4.10}$$

where k is real constant and n is the polytropic index. The difference between the tangential pressure and the radial pressure gives the measure of anisotropy and is denoted by $\triangle$. The system of equations is

$$\frac{1-Z}{x} - 2\dot{Z} = \frac{\rho}{C}, \tag{3.4.11}$$

$$p_r = k\rho^{1+\frac{1}{n}}, \tag{3.4.12}$$

$$p_t = p_r + \triangle, \tag{3.4.13}$$

$$\frac{\triangle}{C} = 4xZ\frac{\ddot{y}}{y} + \dot{Z}\left(1 + 2x\frac{\dot{y}}{y}\right) + \frac{1-Z}{x}, \tag{3.4.14}$$

$$\frac{\dot{y}}{y} = \frac{kC^{\frac{1}{n}}}{4Z}\left[\frac{1-Z}{x} - 2\dot{Z}\right]^{1+\frac{1}{n}} + \frac{1-Z}{4xZ}. \tag{3.4.15}$$

Equation (3.4.15) is integrable if the values of n are specified. Here the assumed values are $n = 1$ and $n = 2$.

3.5 Exact models

Equation (3.4.15) can be integrated if we specify some value of the gravitational potential Z such that it is regular at the origin. The gravitational potential assumed is

$$Z = (1 - ax)^2, \tag{3.5.1}$$

where a is the real constant and Z is well behaved at the origin.

3.5.1 Model 1

Putting $n = 1$ in equation (3.4.15) and using equation (3.5.1), to get

$$\frac{\dot{y}}{y} = \frac{a(2-ax)}{4(1-ax)^2} + \frac{a^2Ck(6-5ax)^2}{4(1-ax)^2}. \tag{3.5.2}$$

Integration gives

$$y = d_1(1-ax)^{-\frac{1+10aCk}{4}} exp\left[\frac{1 + aCk(1 - 25(1-ax)^2)}{4(1-ax)}\right], \tag{3.5.3}$$

where d_1 is the constant of integration. This solution provides the values of unknowns as

$$e^{2\lambda} = \frac{1}{(1-ax)^2}, \tag{3.5.4}$$

$$e^{2\nu} = A^2 d_1^2 (1-ax)^{-\frac{1+10aCk}{2}} exp\left[\frac{1+aCk(1-25(1-ax)^2)}{2(1-ax)}\right], \tag{3.5.5}$$

$$\rho = aC(6-5ax), \tag{3.5.6}$$

$$p_r = k\rho^2, \tag{3.5.7}$$

$$p_t = p_r + \triangle, \tag{3.5.8}$$

$$\triangle = \frac{a^2Cx(6-5ax)}{4(1-ax)^2} 2 - a[x - Ck(8 + a[Ck(6-5ax)^3 + 2x(2-5ax)])]. \tag{3.5.9}$$

The above equations also satisfy all the physical properties for a relativistic star. The gravitational potentials i.e. $e^{2\nu}$ and $e^{2\lambda}$ are regular at the origin. The energy density and radial pressure provide positive values at the origin i.e. $\rho(0) = 6aC$ and $p_r(0) = 36a^2C^2k$. The radial pressure vanishes at finite radius i.e. at $R = \sqrt{\frac{6}{5aC}}$, $p_r = 0$. $\frac{d\rho}{dr}$ and $\frac{dp_r}{dr}$ provide negative values and the interior Schwarzschild metric matches smoothly with the exterior Schwarzschild metric.

3.5.2 Model 2

Putting $n = 2$ in equation (3.4.15) to get

$$\frac{\dot{y}}{y} = \frac{a(2-ax)}{4(1-ax)^2} + \frac{\sqrt{C}ka^{\frac{3}{2}}(6-5ax)^{\frac{3}{2}}}{4(1-ax)^2}. \tag{3.5.10}$$

Integration yields

$$y = d_2 \frac{1}{(1-ax)^{\frac{1}{4}}} \left[\frac{\sqrt{6-5ax}+1}{\sqrt{6-5ax}-1}\right]^{\frac{15k\sqrt{aC}}{8}} exp\left[\frac{1-k\sqrt{aC(6-5ax)}(9-10ax)}{4(1-ax)}\right], \tag{3.5.11}$$

d_2 is the constant of integration. The values of unknowns are given as

$$e^{2\lambda} = \frac{1}{(1-ax)^2}, \tag{3.5.12}$$

$$e^{2\nu} = A^2 d_2^2 \frac{1}{(1-ax)^{\frac{1}{4}}} \left[\frac{\sqrt{6-5ax}+1}{\sqrt{6-5ax}-1} \right]^{\frac{15k\sqrt{aC}}{8}} exp \left[\frac{1-k\sqrt{aC(6-5ax)}(9-10ax)}{4(1-ax)} \right], \tag{3.5.13}$$

$$\rho = aC(6-5ax), \tag{3.5.14}$$

$$p_r = k\rho^{\frac{3}{2}}, \tag{3.5.15}$$

$$p_t = p_r + \triangle, \tag{3.5.16}$$

$$\triangle = \frac{a^2Cx}{4(1-ax)^2}[12 + 5a^2x^2 + 8a(27Ck^2 - 2x) + 2\sqrt{aCk}\sqrt{6-5ax}(9-8ax) - 5a^2Ck^2x(108 - 90ax + 25a^2x^2)]. \tag{3.5.17}$$

This model also satisfies all the physical properties i.e. the gravitational potentials $e^{2\nu}$ and $e^{2\lambda}$ are regular at the origin. The energy density and radial pressure provide positive values at the origin i.e. $\rho(0) = 6aC$ and $p_r(0) = 6\sqrt{6}a^{\frac{3}{2}}C^{\frac{3}{2}}k$. The radial pressure vanishes at finite radius i.e. at $R = \sqrt{\frac{6}{5aC}}$, $p_r = 0$. $\frac{d\rho}{dr}$ and $\frac{dp_r}{dr}$ provide negative values and the interior Schwarzschild metric matches smoothly with the exterior Schwarzschild metric. The solutions generated fulfill all major requirements of a star to be realistic and also give mass and densities comparable with the experimental observations, it may be helpful to study the behavior of a realistic polytropic star.

Chapter 4

Solutions of the Einstein field equations with polytropic equations of state

4.1 Introduction

Along with polytropic equation of state, the system of field equations is written in simpler form and two classes of exact models have been obtained. The physical properties of a realistic star are satisfied in both cases. We extend the work of S. Thirukkanesh and F. C. Ragel [23] by introducing the new values of gravitational potential metric function $Z(x)$.

4.2 The field equations

In Schwarzschild co-ordinates $(x^a) = (t, r, \theta, \phi)$ the interior of a spherically symmetric, static star is described by the line segment given in equation (3.3.1) and equation (3.4.1) represents the energy momentum tensor for an anisotropic perfect fluid sphere. Here ρ is the energy density. The radial pressure is represented by p_r and p_t denotes the tangential pressure. The Einstein field equations, for the line segment (3.3.1) and the matter distribution (3.4.1) can be written as equations (3.4.2)-(3.4.4). The behavior of the gravitational field for an anisotropic imperfect fluid is determined by the above equations.

The expression (3.4.5) defines the mass of the sphere with radius r. The transformation by Durgapal and Bannerji [22] transforms the above system of equations (3.4.2) – (3.4.4) into the equations given as (3.4.6)-(3.4.8). Here A and C are arbitrary constants and the differentiation with respect to x is denoted by '.'. Under this transformation, the mass function (3.4.5) becomes equation (3.4.9).

In equations (3.4.6) – (3.4.8) the number of unknowns is five i.e. $\rho,\ p_r,\ p_t,\ y$ and Z but the number of equations is three. It is not possible to determine all the unknowns using these equations. For a physically realistic relativistic star the matter distribution should satisfy a barotropic equation of state [24] i.e.

$$p_r = p_r(\rho).$$

As mentioned in introduction, we assume the polytropic equation of state as

$$p_r = K\rho^{1+\frac{1}{n}} - l, \tag{4.2.1}$$

where K is real constant and n is the polytropic index. The constant l is greater than or equal to zero. Here it is important to mention that n is the natural number as if n is taken to be zero, an undefined number is gained. Using equation (4.2.1) in equation (3.4.7)

$$\frac{\dot{y}}{y} = \frac{KC^{\frac{1}{n}}}{4Z}\left[\frac{1-Z}{x} - 2\dot{Z}\right]^{1+\frac{1}{n}} + \frac{1-Z}{4xZ} - \frac{l}{4CZ}. \tag{4.2.2}$$

If we take energy density ρ equal to zero in equation (4.2.1) then the negativity of the pressure corresponds to a repulsive gravitational force. We specify the gravitational potential Z, which is well behaved and regular at the origin and use it to integrate equation (4.2.2). Considering

$$Z = \frac{1}{1+ax}, \tag{4.2.3}$$

where a is the real constant. Put equation (4.2.3) in equation (4.2.2) to get

$$\frac{\dot{y}}{y} = \frac{KC^{\frac{1}{n}}a^{1+\frac{1}{n}}(3+ax)^{1+\frac{1}{n}}}{4(1+ax)^{1+\frac{2}{n}}} + \frac{a}{4} - \frac{l(1+ax)}{4C}. \tag{4.2.4}$$

Equation (4.2.4) is integrable if the values of n are specified. Here we are taking two values of n.

4.3 Exact models

Considering the two values of n for equation (4.2.4) i.e. 1 and 2.

4.3.1 Model 1

Taking $n = 1$ in equation (4.2.4), so it becomes

$$\frac{\dot{y}}{y} = \frac{a}{4} + \frac{KCa^2(3+ax)^2}{4(1+ax)^3} - \frac{l(1+ax)}{4C}. \tag{4.3.1}$$

Integration yields

$$\ln y = \frac{ax}{4} - \frac{l}{4C}\left[x + \frac{ax^2}{2}\right] + \frac{KCa^2}{4}\int \frac{(3+ax)^2}{(1+ax)^3}dx + \ln c_0, \tag{4.3.2}$$

where c_0 is the constant of integration. Further integration gives

$$\ln y = \frac{ax}{4} - \frac{l}{4C}\left(x + \frac{ax^2}{2}\right) + \frac{KCa^2}{4}\left[\frac{ln(1+ax)}{a} - \frac{2}{a(1+ax)^2} - \frac{4}{a(1+ax)}\right] + \ln c_0, \tag{4.3.3}$$

and simplifying it to get

$$\ln y = \ln c_0 + \ln{(1+ax)}^{\frac{KCa}{4}} + \frac{ax}{4} - \frac{KCa(3+2ax)}{2(1+ax)^2} - \frac{l}{4C}\left(x + \frac{ax^2}{2}\right), \tag{4.3.4}$$

$$y = c_0(1+ax)^{\frac{KCa}{4}} exp\left[\frac{ax}{4} - \frac{Kca(3+2ax)}{2(1+ax)^2} - \frac{l}{4C}(x + \frac{ax^2}{2})\right]. \tag{4.3.5}$$

Using equation (3.7.6) to obtain the value of ρ i.e.

$$\rho = \frac{aC(3+ax)}{(1+ax)^2}. \tag{4.3.6}$$

To obtain the value of p_r using $n = 1$ in equation (4.2.1)

$$p_r = K\rho^2 - l, \tag{4.3.7}$$

For p_r, using the value of ρ given in equation (4.3.6) i.e.

$$p_r = K\left(\frac{aC(3+ax)}{(1+ax)^2}\right)^2 - l. \tag{4.3.8}$$

The value of p_t is obtained using equation (3.7.6)

$$4x\left(\frac{1}{1+ax}\right)\frac{\ddot{y}}{y} + \left(\frac{4}{1+ax} - \frac{2ax}{(1+ax)^2}\right)\frac{\dot{y}}{y} - \frac{a}{(1+ax)^2} = \frac{p_t}{C}, \tag{4.3.9}$$

and the value of y is equation (4.3.5).

4.3.2 Model 2

Taking $n = 2$ in equation (4.2.4) to get

$$\frac{\dot{y}}{y} = \frac{a}{4} - \frac{l}{4C}(1+ax) + \frac{KC^{\frac{1}{2}}a^{\frac{3}{2}}(3+ax)^{\frac{3}{2}}}{4(1+ax)^2}. \tag{4.3.10}$$

Integration gives

$$\ln y = \frac{ax}{4} - \frac{l}{4C}\left(x + \frac{ax^2}{2}\right) + \frac{KC^{\frac{1}{2}}a^{\frac{3}{2}}}{4}\left[\left(\frac{3}{a\sqrt{2}}\ln\frac{\sqrt{3+ax}-\sqrt{2}}{\sqrt{3+ax}+\sqrt{2}}\right) + \frac{2x\sqrt{3+ax}}{(1+ax)}\right] + \ln c_1, \tag{4.3.11}$$

c_1 is the constant of integration. Further integration gives

$$y = c_1\left(\frac{\sqrt{3+ax}-\sqrt{2}}{\sqrt{3+ax}+\sqrt{2}}\right)^{3K\sqrt{\frac{aC}{2}}} exp\left[\frac{ax}{4} + \frac{2xKC^{\frac{1}{2}}a^{\frac{3}{2}}\sqrt{3+ax}}{(1+ax)} - \frac{l}{4C}\left(x + \frac{ax^2}{2}\right)\right]. \tag{4.3.12}$$

The expression for the density ρ is same as in equation (4.3.6) and the radial pressure p_r in this case is given by

$$p_r = K\rho^{\frac{3}{2}} - l. \tag{4.3.13}$$

In this Model the tangential pressure p_t is same as in Model 1 i.e. (4.3.9) and the value of y is given in equation (4.3.12). The mass function in both the cases is given by equation (3.7.9) and the value of ρ is equation (4.3.8).

4.4 Physical analysis

4.4.1 Physical analysis for Model 1

1. In this case

$$e^{2\nu(0)} = A^2 c_0^2 exp(-3KCa), \tag{4.4.1}$$

and

$$e^{2\lambda(0)} = 1. \tag{4.4.2}$$

The differentiation with respect to r is denoted by '$\prime$'. Equations (4.4.1)−(4.4.2) show that the gravitational potentials at the origin are regular.

2. At the origin the energy density is

$$\rho(0) = 3aC, \tag{4.4.3}$$

and the radial pressure is given by the expression

$$p_r(0) = 9a^2C^2K - l, \tag{4.4.4}$$

where C, K are arbitrary constants, $a, l > 0$ and $9a^2C^2K > l$. Equations (4.4.3) and (4.4.4) show that the energy density and the radial pressure at the origin are positive.

3. If $r = R$ is the boundary and using the condition

$$p_r = 0, \tag{4.4.5}$$

then equation (4.3.6) becomes

$$\frac{a^2C^2(3 + aCR^2)}{(1 + aCR^2)^4} = \frac{l}{K}. \tag{4.4.6}$$

For the possible positive values of R in expression (4.4.6), Model 1 is valid. Thus satisfying the third physical property of relativistic model.

4. Differentiating equation (4.3.6) and equation (4.3.8) with respect to r to check if $\frac{d\rho}{dr}$ and $\frac{dp}{dr}$ are negative at $x = Cr^2$.

$$\frac{d\rho}{dr} = \frac{-2a^2C^2r(5 + aCr^2)}{(1 + aCr^2)^4}, \tag{4.4.7}$$

and

$$\frac{dp_r}{dr} = \frac{-4Ka^3C^3r(3 + aCr^2)(5 + aCr^2)}{(1 + aCr^2)^5}, \tag{4.4.8}$$

i.e.

$$\frac{d\rho}{dr} < 0, \frac{dp_r}{dr} < 0.$$

The equation (4.4.7) and equation (4.4.8) indicate the monotonic decrease in the energy density and the radial pressure $\forall\ 0 < r < R$, where $r = R$ is the boundary.

5. The expressions

$$\left(1 - \frac{2M}{R}\right) = A^2y^2(x)|_{x=CR^2}, \tag{4.4.9}$$

and

$$\frac{1}{(1 + aCR^2)} = \left(1 - \frac{2M}{R}\right)^{-1}, \tag{4.4.10}$$

show that the interior metric matches with the Schwarzschild exterior metric hence satisfying the fifth property.

In the view of the theory of general relativity, the physical properties of a realistic model are satisfied in this Model.

4.4.2 Physical analysis for Model 2

1. In this case, the gravitational potentials at origin are

$$e^{2\nu(0)} = A^2c_1^2(\sqrt{3} - \sqrt{2})^{6K\sqrt{2aC}}, \tag{4.4.11}$$

and

$$e^{2\lambda(0)} = 1. \tag{4.4.12}$$

The regularity of gravitational potentials are given by equation (4.4.11) and equation (4.4.12).

2. The energy density in this model is same as given in expression (4.4.3) i.e.

$$\rho(0) = 3aC,$$

while the radial pressure at the origin is given by

$$p_r(0) = 3\sqrt{3}Ka^{\frac{3}{2}}C^{\frac{3}{2}} - l, \tag{4.4.13}$$

with $a, l > 0$, and $3\sqrt{3}Ka^{\frac{3}{2}}C^{\frac{3}{2}} > l$, the above expression is positive thus satisfying the second condition.

3. At the boundary $r = R$, the radial pressure vanishes and gives the expression

$$\frac{a^{\frac{3}{2}}C^{\frac{3}{2}}(3 + aCR^2)^{\frac{3}{2}}}{(1 + aCR^2)^3} = \frac{l}{K}, \tag{4.4.14}$$

for the possible positive value of R in expression (4.4.14), this model is valid.

4. The energy density and the radial pressure at $x = Cr^2$ are given by

$$\rho = \frac{aC(3 + aCr^2)}{(1 + aCr^2)^2}, \tag{4.4.15}$$

and

$$p_r = \frac{Ka^{\frac{3}{2}}C^{\frac{3}{2}}(3 + aCr^2)^{\frac{3}{2}}}{(1 + aCr^2)^3} - l. \tag{4.4.16}$$

Differentiating equation (4.4.15) and equation (4.4.16) with respect to r to check if they are negative.

$$\frac{d\rho}{dr} = \frac{-2a^2C^2r(5 + aCr^2)}{(1 + aCr^2)^3}, \tag{4.4.17}$$

and

$$\frac{dp_r}{dr} = \frac{-3rKa^{\frac{5}{2}}C^{\frac{5}{2}}(3+aCr^2)^{\frac{1}{2}}(5+aCr^2)}{(1+aCr^2)^4}, \tag{4.4.18}$$

i.e. if $a, C > 0$,

$$\frac{d\rho}{dr}, \frac{dp_r}{dr} < 0,$$

thus $\forall\ 0 < r < R$, the monotonic decrease in the energy density and the radial pressure from the center to the boundary of the star $r = R$ is given by the expressions (4.4.17) and (4.4.18).

5. The expressions

$$\left(1 - \frac{2M}{R}\right) = A^2 y^2(x)|_{x=CR^2}, \tag{4.4.19}$$

where the value of y is given in equation (4.3.12).

$$\left(1 - \frac{2M}{R}\right)^{-1} = \frac{1}{(1+aCR^2)}, \tag{4.4.20}$$

show that the interior metric matches with the Schwarzschild exterior metric. This fulfils the fifth condition of relativistic model.

This Model too satisfies all the physical properties of relativistic model.

Chapter 5

Conclusion

In this thesis we have found the solutions of the Einstein field equations with polytropic equation of state to study the general situation of a relativistic object for the matter distribution.

The basics of the differential geometry are discussed briefly in the first chapter. These basic concepts are required to understand the theory of relativity presented by Einstein. Tensors are discussed as they play a key role in the formation of the field equations. The first fundamental form is also described in this chapter for a better understanding to the solutions of the Einstein field equations.

The second chapter is fully based on the theory of relativity. A brief background is given in the beginning of this chapter. This theory has changed the face of physics and related studies. A quick review of SR is also included in it to give broader view of this theory. GR is discussed with its important principles that helped Einstein to develop this theory. This chapter also contains the derivation of field equations. The field equations are hub of GR.

The vacuum solution of field equations in Schwarzschild co-ordinates is also the part of the second chapter. The Schwarzschild metric helps us to analyze the physical properties that should be satisfied by the star. A brief discussion of the importance of solutions of these field equations is included along with the role of these solutions in astrophysical objects are also highlighted.

Third chapter includes the information about the stellar objects. How general relativity influences the astrophysics and stellar bodies is also highlighted. Few type of exact solutions are also discussed in its section. The main theme of this chapter is to provide a foundation to the work, for which this dissertation is all about.

The final chapter consists of the solutions of the EFEs. Using the co-ordinate transformation, the EFEs are written as a new system of differential equations. Two distinct values of polytropic index are taken in the polytropic equation of state to get exact models. These models describe, in the static spherically symmetric spacetime, the behavior of anisotropic compact sphere. For each model the values of unknown parameters and quantities are found. These quantities help us to fulfil the physical properties of the realistic masses. Both models satisfy these physical features i.e. the regularity of the gravitational potentials at the origin, positive definiteness of the energy density and the radial pressure at the origin, monotonic decrease of the energy density and the radial pressure with increasing radius, vanishing of radial pressure at some finite radius. The exterior Schwarzschild metric is also matched smoothly in the interior one to satisfy the final condition. In *Model* 1 of section 4.3 of this dissertation, the solution obtained matches to the *Model* 1 solution discussed by T. Feroze and Azad A. Siddiqui [25], if we take $\alpha = K$, $\beta = k = 0$ and $\gamma = l$ in the first solution of [25] .

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