On generalizations of fractional calculus

by
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A thesis
submitted for the degree of Master of Science
in
Mathematics

Supervised by
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Dedication

To my loving parents, brothers and prospective wife.

Acknowledgements

All praises are for Almighty Allah, the most gracious and the most merciful, Who created this entire universe. I am highly grateful to Him for showering His countless blessings upon me and giving me the ability and strength to complete this thesis successfully and blessing me more than I deserve.

I am greatly indebted to my supervisor Dr. Mujeeb ur Rehman, for his continuous support and guidance during my thesis. Without his expertise and help, this work would not have been possible. My understanding and appreciation of the subject are entirely due to his efforts and positive response to my queries. I would also like to thank the Head of Department and the entire faculty of the Mathematics Department for their kind help during my research work.

My studies at NUST have been made more memorable and enjoyable by my classmates and friends. I would like to give special thanks to Burhan Jafeer and Usman Rashid for valuable advices, suggestions, and friendship.

Finally, with the deepest gratitude, I acknowledge the support of my family. A huge thanks to my parents, my brothers and my wife-to-be Hafsa Bashir for supporting me all the way through my studies. Indeed words cannot express how grateful I am for all of the sacrifices they have made for me. Without their prayers and support, I would never be able to reach here. I am thankful to all those people who directly or indirectly helped me to complete my thesis.

Hafiz Muhammad Fahad

Abstract

The first part of this thesis focuses on substantial fractional operators which play an important role in the modeling of anomalous diffusion. We propose a new generalized substantial fractional integral. We also propose generalizations of fractional substantial derivatives both in the Riemann-Liouville and Caputo sense. Furthermore, we investigate elementary properties of these operators. In the end, we consider a family of generalized substantial fractional differential equations and discuss the existence, uniqueness and continuous dependence of solutions on initial data.

The second part of this thesis establishes a generalization of the Hadamard type fractional calculus which has been named as the Φ-Hadamard type fractional calculus. We give conditions for which the Φ-Hadamard type fractional integral is bounded in a generalized space. We develop sufficient conditions for the existence of the Φ-Hadamard type fractional derivative. Finally, some properties and integration by parts formulas of fractional calculus in the frame of these operators are established.

By inspiration of some new developments in Φ-fractional calculus, we develop, in the third part of the thesis, some new properties and uniqueness of the Φ-Laplace transform in the settings of Φ-fractional calculus. The ultimate goal of this part of the thesis is to reveal the efficiency of Φ-Laplace transform for solving Φ-fractional ordinary and partial differential equations.

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Chapter 1

Introduction

Fractional calculus has its roots in the end of the seventeenth century when Leibniz devised the notation $\frac{d^n}{dx^n}f(x)$ to denote the nth-order derivative of a function f. De l'Hospital then asked about the meaning of this derivative when $n = \frac{1}{2}$. Thus, this led to a new and emerging field called fractional calculus. Nowadays this field has become very popular amongst the researchers due to compulsive use of fractional differential equations (FDEs) in engineering, physics, economics and other branches of sciences [9, 15, 26, 33, 36, 38, 43].

Fractional integration and differentiation are generalizations of notions of integer-order integration and differentiation, of which nth derivatives and n-fold integrals are particular cases [34]. There are many different forms of fractional operators in the field of fractional calculus, including Riemann-Liouville(RL), Caputo, Erdelyi-Kober, Hadamard, Grunwald-Letnikov, Hilfer, Reisz, Katugampola and many others [1, 17, 20, 22, 23, 25, 26, 33, 36, 37]. Each definition has its own backgrounds and conditions, as a result of which these definitions are inequivalent to each other. Many of these definitions are specific cases of generalized fractional operators. As examples, we have the Katugampola, Φ-RL, Φ-Caputo and Φ-Hilfer operators which are generalized forms for the other classical operators. In consideration of the matter that there is a wide family of fractional operators available in literature which makes choosing the favorable operator a challenging effort while dealing with a physical problem. So it is logical to investigate and develop the generalizations of classical fractional operators to overcome the issue of choosing an appropriate operator.

The organization of this thesis is as follows. Chapter 2 is devoted to the preliminaries. In Chapter 3, we focus on substantial fractional operators which play an important role in modeling anomalous diffusion. We introduce the generalized substantial fractional integral, and derivatives both in RL and Caputo sense. Moreover, we obtain the relations between the generalized substantial fractional operators and some other fractional operators including Katugampola, standard RL, standard substantial and Hadamard fractional operators. Proofs of the composition rules in the settings of generalized substantial fractional operators also form the part of this chapter. Furthermore, while considering a class of generalized substantial FDEs, we discuss the existence, uniqueness and continuous dependence of solutions on initial data.

As of now, Hadamard-type fractional operators have not been investigated in much detail, as compared to the other classical fractional operators. A generalized form for the Hadamard-type fractional operators has not yet been developed by researchers. Thus, in Chapter 4, we present the Φ-Hadamard type fractional operators which generalize the classical Hadamard and Hadamard-type fractional operators. We give conditions under which the Φ-Hadamard type fractional integral is bounded in a generalized space and analyze sufficient conditions for the existence of the Φ-Hadamard type fractional derivative. Furthermore, we drive composition properties and several formulas of fractional integration by parts in the settings of these operators.

Chapter 5 of this thesis establishes some new properties and uniqueness of the Φ-Laplace transform in the settings of Φ-fractional calculus. This is significant because integral transforms like Laplace, Fourier, generalized Laplace and ρ-Laplace are effective tools for obtaining analytic solutions to some classes of FDEs [18, 19, 26, 33, 38, 40]. As of today, no research has been done to use integral transforms for obtaining analytic solutions to FDEs in the Φ-Hilfer fractional derivative settings. In this manner, we make use of Φ-Laplace transform for finding analytic solutions to some classes of FDEs in the settings of Φ-RL, Φ-Caputo and Φ-Hilfer fractional derivatives. Chapter 6 finally makes concluding statements about this thesis and its main results.

Chapter 2

Preliminaries

In this chapter, some prerequisite definitions of fractional operators are evoked that reader should familiar with. Moreover, we recall some definitions of special functions.

2.1 Substantial fractional operators

Before introducing the fractional operators, we first give some notations for the sake of convenience in developing the notations further. The notation ${}_{\sigma}\mathfrak{D}^{m} := \left(\frac{d}{dt}+\sigma\right)^{m}$ where $\left(\frac{d}{dt}+\sigma\right)^{m} = (\mathfrak{D}+\sigma)(\mathfrak{D}+\sigma)(\mathfrak{D}+\sigma)\cdots(\mathfrak{D}+\sigma)$ appears frequently in literature [8]. Together with the operator ${}_{\sigma}\mathfrak{D}$, in the sequel we shall use the operator ${}_{\sigma}\mathfrak{D}^{m,\rho} := (\frac{t^{1-\rho}}{\rho}\frac{d}{dt}+\sigma)^{m}$, where $\sigma \in \mathbb{R}$ and $\rho \neq 0$. Additionally, the generalized differential operator, defined as $\frac{t^{1-\rho}}{\rho}\frac{d^{m}}{dt^{m}}$, will be denoted by $\mathfrak{D}^{m,\rho}$. We define function spaces $\Omega^{m}_{\sigma,\rho}[a,b] := \left\{f : e^{\sigma t^{\rho}}t^{1-\rho}f(t) \in AC^{m}[a,b]\right\}$ and $\Lambda^{p}_{\sigma,\rho}[a,b] := \left\{f : e^{\sigma t^{\rho}}t^{1-\rho}f \in L_{p}[a,b]\right\}$ where $AC[a,b]$ is the space of absolutely continuous functions and $L_{p}[a,b]$ $(1 \leq p < \infty)$ denotes the space of measurable functions on $[a,b]$. For the sake of simplicity $\Omega_{\sigma,1}$ and $\Omega_{0,\rho}$ will be denoted by Ω_{σ} and Ω_{ρ}, respectively.

Definition 2.1.1. [8, 11] Let η and σ be real numbers such that $\eta > 0$ and $f \in \Lambda^{1}_{\sigma}[a,b]$ then substantial fractional integral operator is defined as $\tilde{\mathfrak{J}}^{\eta}_{a}f(t) = \frac{1}{\mathfrak{d}(\eta)}\int_{a}^{t}(t-s)^{\eta-1}e^{-\sigma(t-s)}f(s)ds$. Furthermore, for $f \in \Omega^{m}_{\sigma}[a,b]$, $m-1 < \eta \leq m$, the RL type substantial fractional derivative is defined as $\tilde{\mathfrak{D}}^{\eta}_{a}f(t) = \tilde{\mathfrak{D}}^{m}_{a}\tilde{\mathfrak{J}}^{m-\eta}_{a}f(t)$. On the other hand, the Caputo type substantial fractional derivative is defined as ${}^{c}\tilde{\mathfrak{D}}^{\eta}_{a}f(t) = \tilde{\mathfrak{J}}^{m-\eta}_{a}\tilde{\mathfrak{D}}^{m}_{a}f(t)$.

2.2 Katugampola fractional operators

For $\rho \neq 0$, let $\mathfrak{J}_a^{1,\rho} f(t) = \int_a^t f(s) d(s^\rho)$, where $d(s^\rho) = \rho s^{\rho-1} ds$. Then mth iterate of the integral operator $\mathfrak{J}_a^{1,\rho}$ is given by

$$\begin{aligned} \mathfrak{J}_a^{m,\rho} f(t) &= \int_a^t d(t_1^\rho) \int_a^{t_1} d(t_2^\rho) \int_a^{t_2} d(t_3^\rho) \cdots \int_a^{t_{n-1}} f(t_{n-1}) d(t_{n-1}^\rho) \\ &= \frac{\rho}{\mathfrak{d}(m)} \int_a^t (t^\rho - s^\rho)^{m-1} f(s) s^{\rho-1} ds. \end{aligned} \tag{2.2.1}$$

Replacing the m by real $\eta > 0$ in (2.2.1), the Katugampola fractional integral is defined as

Definition 2.2.1. [20] For $\rho \neq 0$, $\eta > 0$ and $f \in \Lambda_\rho^1[a,b]$, the Katugampola fractional integral is given by

$$\mathfrak{J}_a^{\eta,\rho} f(t) = \frac{\rho}{\mathfrak{d}(\eta)} \int_a^t (t^\rho - s^\rho)^{\eta-1} f(s) s^{\rho-1} ds.$$

Furthermore, for $m-1 < \eta \leq m$ and $f \in \Omega_\rho^m[a,b]$ the RL type Katugampola fractional derivative is defined as $\mathfrak{D}_a^{\eta,\rho} f(t) = \mathfrak{D}^{m,\rho} \mathfrak{J}_a^{m-\eta,\rho} f(t)$ and the Caputo-type Katugampola is defined as ${}^c\mathfrak{D}_a^\eta f(t) = \mathfrak{J}_a^{m-\eta,\rho} \mathfrak{D}^{m,\rho} f(t)$.

Remark 2.2.2. We have introduced a slight modification in the definition of Katugampola fractional integral operator. The factor $\rho^{1-\eta}$ in the original definition is now replaced with ρ. This avoids the repeated reappearance of some factors of ρ in calculations [26, p. 103].

It is to be noted that $\mathfrak{D}^{1,\rho} \mathfrak{J}_a^{1,\rho} f(t) = f(t)$. A repeated application of this identity leads us to the identity $\mathfrak{D}^{m,\rho} \mathfrak{J}_a^{m,\rho} f(t) = f(t)$. Furthermore $\mathfrak{J}_a^{1,\rho} \mathfrak{D}^{1,\rho} f(t) = \int_a^t \mathfrak{D}^\rho f(s) d(s^\rho)$ $= \int_a^t d(f(s)) = f(t) - f(a)$. Similarly $\mathfrak{J}_a^{2,\rho} \mathfrak{D}^{2,\rho} f(t) = f(t) - f(a) - (t^\rho - a^\rho) \mathfrak{D}^{1,\rho} f(a)$. Generally speaking, a repeated application of the preceding steps leads to a Taylor type expansion of f as

$$f(t) = \sum_{k=0}^{m-1} \frac{\mathfrak{D}^{k,\rho} f(a)}{k!} (t^\rho - a^\rho)^k + \mathfrak{J}_a^{m,\rho} \mathfrak{D}^{m,\rho} f(t). \tag{2.2.2}$$

The Katugampola fractional differential and integral operators satisfy following properties [5, 31]:

(P1) For $f \in \Lambda_\rho^1[a,b]$, $\mathfrak{J}_a^{\eta,\rho}\mathfrak{J}_a^{\zeta,\rho}f(t) = \mathfrak{J}_a^{\zeta,\rho}\mathfrak{J}_a^{\eta,\rho}f(t) = \mathfrak{J}_a^{\eta+\zeta,\rho}f(t)$.

(P2) For $\zeta \geq \eta$, and $f \in \Lambda_\rho^1[a,b]$, $\mathfrak{D}_a^{\eta,\rho}\mathfrak{J}_a^{\zeta,\rho}f(t) = \mathfrak{J}_a^{\zeta-\eta,\rho}f(t)$ and ${}^c\mathfrak{D}_a^{\eta,\rho}\mathfrak{J}_a^{\zeta,\rho}f(t) = \mathfrak{J}_a^{\zeta-\eta,\rho}f(t)$.

(P3) For $\eta > 1$ and $m-1 < \eta \leq m$ and $f \in \Omega_\rho^m[a,b]$, we have

$$\mathfrak{J}_a^{\eta,\rho}\mathfrak{D}_a^{\eta,\rho}f(t) = f(t) - \sum_{k=1}^{m-1} \frac{\lim_{s\to a^+}\mathfrak{D}_a^{\eta-k,\rho}f(s)}{\mathfrak{d}(\eta-k+1)}(t^\rho - a^\rho)^{\eta-1}.$$

Specifically, for $0 < \eta < 1$, $\mathfrak{J}_a^{\eta,\rho}\mathfrak{D}_a^{\eta,\rho}f(t) = f(t) - \frac{\mathfrak{J}_a^{1-\eta,\rho}f(a)}{\mathfrak{d}(\eta)}(t^\rho - a^\rho)^{\eta-1}$.

(P4) For $f \in \Omega_\rho^m[a,b]$ and $\zeta \geq \eta$,

$$\mathfrak{J}_a^{\eta,\rho c}\mathfrak{D}_a^{\eta,\rho}f(t) = f(t) - \sum_{k=0}^{m-1} \frac{\mathfrak{D}^{k,\rho}f(a)}{k!}(t^\rho - a^\rho)^k.$$

Lemma 2.2.1. *Assume that $f \in \Omega_\rho^m[a,b]$. Then ${}_\sigma\mathfrak{D}^{m,\rho}(e^{-\sigma t^\rho}f(t)) = e^{-\sigma t^\rho}\mathfrak{D}^{m,\rho}f(t)$ and $e^{\sigma t^\rho}{}_\sigma\mathfrak{D}^{m,\rho}(f(t)) = \mathfrak{D}^{m,\rho}(e^{\sigma t^\rho}f(t))$.*

Proof. We prove this Lemma by induction. For $m = 1$, we have

$${}_\sigma\mathfrak{D}^{1,\rho}(e^{-\sigma t^\rho}f(t)) = \frac{t^{1-\rho}}{\rho}\frac{d}{dt}\left(e^{-\sigma t^\rho}f(t)\right) + \sigma e^{-\sigma t^\rho}f(t) = e^{-\sigma t^\rho}\mathfrak{D}^{1,\rho}f(t).$$

Assume the conclusion follows for $m-1$. Then,

$$\begin{aligned}{}_\sigma\mathfrak{D}^{m,\rho}(e^{-\sigma t^\rho}f(t)) = {}_\sigma\mathfrak{D}^{1,\rho}{}_\sigma\mathfrak{D}^{m-1,\rho}(e^{-\sigma t^\rho}f(t)) &= {}_\sigma\mathfrak{D}^{1,\rho}\left(e^{-\sigma t^\rho}\mathfrak{D}^{m-1,\rho}f(t)\right)\\ &= e^{-\sigma t^\rho}\mathfrak{D}^{m,\rho}f(t).\end{aligned}$$

Second identity can be obtained in a similar way. □

2.3 A review of Hadamard type fractional calculus

In this section, we give a brief overview of Hadamard type fractional calculus. We recall the definitions of Hadamard type fractional operators and state some basic results in the settings of Hadamard type fractional calculus. The proofs of these results can be seen in [6, 24, 27].

Definition 2.3.1. [6, 24] The Hadamard type fractional integral with parameter $\mu \in \mathbb{R}$ of the given function $f(t)$ with order $\eta > 0$ is defined as

$$ {}_{H}\mathfrak{J}^{\eta}_{a^+,\mu} f(t) = \frac{1}{\mathfrak{d}(\eta)} \int_a^t \left(\frac{s}{t}\right)^{\mu} \left(\log \frac{t}{s}\right)^{\eta-1} f(s) \frac{ds}{s} $$

where $t \in (a, b)$ and $0 \leq a < b \leq \infty$.

Definition 2.3.2. [6, 24] The Hadamard type fractional derivative with parameter $\mu \in \mathbb{R}$ of the given function $f(t)$ is defined as

$$ {}_{H}\mathfrak{D}^{\eta}_{a^+,\mu} f(t) = t^{-\mu} \delta^m t^{\mu} ({}_{H}\mathfrak{J}^{m-\eta}_{a^+,\mu} f(t)) $$

where $\delta = t\frac{d}{dt}$, $m - 1 < \eta \leq m \in \mathbb{N}$, $t \in (a, b)$ and $0 \leq a < b \leq \infty$.

In Definitions 2.3.1 and 2.3.2, for the particular case that $\mu = 0$, the Hadamard type fractional integral and derivative reduce back to the classical Hadamard fractional integral and derivative, respectively.

Lemma 2.3.1. *For Hadamard type fractional integral* ${}_{H}\mathfrak{J}^{\eta}_{a^+,\mu}$ *and fractional derivative* ${}_{H}\mathfrak{D}^{\eta}_{a^+,\mu}$, *the following properties hold*

(a) $\lim_{\eta \to 1} {}_{H}\mathfrak{J}^{\eta}_{a^+,\mu} f(t) = \int_a^t \left(\frac{s}{t}\right)^{\mu} f(s) \frac{ds}{s}$,

(b) $\lim_{\eta \to 0^+} {}_{H}\mathfrak{J}^{\eta}_{a^+,\mu} f(t) = f(t)$,

(c) $\lim_{\eta \to 0^+} {}_{H}\mathfrak{D}^{\eta}_{a^+,\mu} f(t) = f(t)$,

(d) $\lim_{\eta \to (m-1)^+} {}_{H}\mathfrak{D}^{\eta}_{a^+,\mu} f(t) = t^{-\mu} \delta^{m-1} t^{\mu} f(t)$,

(e) $\lim_{\eta \to m^-} {}_{H}\mathfrak{D}^{\eta}_{a^+,\mu} f(t) = t^{-\mu} \delta^{m} t^{\mu} f(t)$.

Lemma 2.3.2. *If* $0 < \eta < 1$ *and* $\zeta > 0$. *Then, for the Hadamard type fractional integral and derivative, the following relations hold*

$$ {}_{H}\mathfrak{J}^{\eta}_{a^+,\mu} \left\{ t^{-\mu} \left(\log \frac{t}{a}\right)^{\zeta-1} \right\} = \frac{\mathfrak{d}(\zeta)}{\mathfrak{d}(\zeta+\eta)} t^{-\mu} \left(\log \frac{t}{a}\right)^{\zeta+\eta-1}. $$

$$ {}_{H}\mathfrak{D}^{\eta}_{a^+,\mu} \left\{ t^{-\mu} \left(\log \frac{t}{a}\right)^{\zeta-1} \right\} = \frac{\mathfrak{d}(\zeta)}{\mathfrak{d}(\zeta-\eta)} t^{-\mu} \left(\log \frac{t}{a}\right)^{\zeta-\eta-1}. $$

Theorem 2.3.3. *Let $m-1<\eta<\zeta\leq m\in\mathbb{N}$, $1\leq p\leq\infty$, $0\leq a<b\leq\infty$ and let $\mu,c\in\mathbb{R}$, such that $\mu\geq c$. Then for $f\in X_c^p(a,b)$ and ${}_H\mathfrak{J}^{\eta}_{a^+,\mu}f\in AC^m_{\delta;\mu}[a,b]$, the following relation holds*

$$ {}_H\mathfrak{D}^{\zeta}_{a^+,\mu}\,{}_H\mathfrak{J}^{\eta}_{a^+,\mu}f(t)={}_H\mathfrak{D}^{\zeta-\eta}_{a^+,\mu}f(t). $$

Theorem 2.3.4. *Let $\zeta\geq\eta>0$, $m_1-1<\eta\leq m_1\in\mathbb{N}$, $m_2-1<\zeta\leq m_2\in\mathbb{N}$, $0\leq a<b\leq\infty$, $1\leq p\leq\infty$ and let $\mu,c\in\mathbb{R}$ with $\mu\geq c$. Then for $f\in AC^{m_1}_{\delta;\mu}[a,b]$ and ${}_H\mathfrak{D}^{\eta}_{a^+,\mu}f\in X_c^p(a,b)$, there holds*

$$ {}_H\mathfrak{J}^{\zeta}_{a^+,\mu}\,{}_H\mathfrak{D}^{\eta}_{a^+,\mu}f(t)={}_H\mathfrak{J}^{\zeta-\eta}_{a^+,\mu}f(t). $$

Prior to introducing the Φ-Laplace transform, we first recall some definitions from the classical and fractional calculus.

2.4 Fractional integrals and derivatives of a function with respect to another function

In view of the fact that there is a large class of fractional operators available in literature which makes choosing the appropriate approach a difficult task while dealing with a given problem. So it becomes important to introduce the generalizations of classical fractional operators to overcome the issue of choosing an adequate operator. In this section we invoke some generalized definitions of fractional integrals and derivatives.

Definition 2.4.1. [26, 32, 38] Let η be a real number such that $\eta>0$, $-\infty\leq a<b\leq\infty$, $m=[\eta]+1$, f be an integrable function defined on $[a,b]$ and $\Phi\in C^1([a,b])$ be an increasing function such that $\Phi'(t)\neq 0$ for all $t\in[a,b]$. Then, the Φ-RL fractional integral and Φ-RL fractional derivative of a function f of order η are defined as

$$ \mathfrak{J}^{\eta,\Phi}_a f(t)=\frac{1}{\mathfrak{d}(\eta)}\int_a^t\Big(\Phi(t)-\Phi(s)\Big)^{\eta-1}\Phi'(s)f(s)ds \tag{2.4.1} $$

and

$$ \mathfrak{D}^{\eta,\Phi}_a f(t)=\left(\frac{1}{\Phi'(t)}\frac{d}{dt}\right)^m\mathfrak{J}^{m-\eta,\Phi}_a f(t) \tag{2.4.2} $$

respectively.

It is to be noted that for $\Phi(t) \to t$, $\mathfrak{I}_a^{\eta,\Phi} f(t) \to \mathfrak{I}_a^{\eta} f(t)$ which is the standard RL integral. Moreover for $\Phi(t) \to \ln(t)$ the integral defined in (2.4.1) approaches to the Hadamard fractional integral.

By inspiration from Caputo's concept [7] of the fractional derivative, Almeida [2] presents the following Caputo version of (2.4.2) and studies some important properties.

Definition 2.4.2. Let η be a real number such that $\eta > 0$, $-\infty \leq a < b \leq \infty$, $m = [\eta] + 1$, f, $\Phi \in C^m([a, b])$ be the functions such that Φ is increasing and $\Phi'(t) \neq 0$ for all $t \in [a, b]$. Then, the Φ-Caputo fractional derivative of a function f of order η is defined as

$$ {}^{C}\mathfrak{D}_a^{\eta,\Phi} f(t) = \mathfrak{I}_a^{m-\eta,\Phi} \left(\frac{1}{\Phi'(t)} \frac{d}{dt} \right)^m f(t). \tag{2.4.3} $$

Taking $\Phi(t) \to \ln(t)$ and $\Phi(t) \to t$, we get the Caputo-type Hadamard fractional derivative [12] and Caputo fractional derivative [38] respectively.

Motivated by the definitions of Φ-RL and Hilfer fractional derivatives, Sousa and Oliveira [41] introduce the Φ-Hilfer fractional derivative, which we recall in the following definition.

Definition 2.4.3. [41] Let η be a real number such that $\eta > 0$, $-\infty \leq a < b \leq \infty$, $m = [\eta]+1$, f, $\Phi \in C^m([a, b], \mathbb{R})$ be the functions such that Φ is increasing and $\Phi'(t) \neq 0$ for all $t \in [a, b]$. Then, the Φ-Hilfer fractional derivative of a function f of order η and type $0 \leq \zeta \leq 1$ is given by

$$ \mathfrak{D}_a^{\eta,\zeta,\Phi} f(t) = \mathfrak{I}_a^{\zeta(m-\eta),\Phi} \left(\frac{1}{\Phi'(t)} \frac{d}{dt} \right)^m \mathfrak{I}_a^{(1-\zeta)(m-\eta),\Phi} f(t). \tag{2.4.4} $$

2.5 Laplace and Fourier transforms

Definition 2.5.1. Assuming the function f is defined for $t \geq 0$, the Laplace transform of a function f, denoted by $\mathfrak{L}\{f\}$, is defined by the improper integral

$$ \mathfrak{L}\left\{f(t)\right\} = \int_0^\infty e^{-\nu t} f(t) dt \tag{2.5.1} $$

provided that the integral in (2.5.1) exists for all ν larger than or equal to some ν_0.

Definition 2.5.2. Assume that f is a piecewise smooth, continuous and absolutely integrable function. Then the Fourier transform of a function f, denoted by $\mathfrak{F}\{f\}$ or $\tilde{f}(k)$, is defined by

$$\mathfrak{F}\{f(t)\} = \int_{-\infty}^{\infty} e^{-ikt} f(t) dt \tag{2.5.2}$$

where k is the Fourier transform variable. The inverse Fourier transform of $\mathfrak{F}\{f(t)\}$ is defined by

$$\mathfrak{F}^{-1}\Big\{\mathfrak{F}\{f(t)\}\Big\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikt} \tilde{f}(k) dk. \tag{2.5.3}$$

2.6 Some special functions

There are several special functions that are considered to be helpful for finding the solutions of FDEs. In the following definitions, we state a few of them.

Definition 2.6.1. The entire function

$$W(z,\eta,\zeta) = \sum_{j=0}^{\infty} \frac{z^j}{j!\mathfrak{d}(\eta j+\zeta)}, \quad \text{where } \eta > -1, \quad \zeta \in \mathbb{C} \tag{2.6.1}$$

which is valid in the whole complex plane, is known as the Wright function. It appeared for the first time in [46, 47] in connection with E. M. Wright's investigations in the asymptotic theory of partitions.

In [28], Gosta Mittag-Leffler introduced the well-known Mittag-Leffler function $\mathfrak{E}_\eta(z)$, given by

$$\mathfrak{E}_\eta(z) = \sum_{j=0}^{\infty} \frac{z^j}{\mathfrak{d}(\eta j+1)}, \quad \eta \in \mathbb{C}, \quad \mathrm{Re}(\eta) > 0. \tag{2.6.2}$$

Later on, a natural generalization of $\mathfrak{E}_\eta(z)$ was discussed by Wiman in [45]. He introduced the function $\mathfrak{E}_{\eta,\zeta}(z)$ as

$$\mathfrak{E}_{\eta,\zeta}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\mathfrak{d}(\eta j+\zeta)}, \quad \eta,\zeta \in \mathbb{C}, \quad \mathrm{Re}(\eta) > 0. \tag{2.6.3}$$

If we consider $\zeta = 1$ in (2.6.3), we obtain the Mittag-Leffler function (2.6.2). In [35], Prabhakar presented the more generalized version of (2.6.2)-(2.6.3), which we recall in the following definition.

Definition 2.6.2. The Prabhakar function is defined by the series representation

$$\mathfrak{E}^{\gamma}_{\eta,\zeta}(z) = \frac{1}{\mathfrak{d}(\gamma)} \sum_{j=0}^{\infty} \frac{\mathfrak{d}(\gamma + j) z^j}{j! \mathfrak{d}(\eta j + \zeta)}, \quad \eta, \zeta, \gamma \in \mathbb{C}, \quad \mathrm{Re}(\eta) > 0. \tag{2.6.4}$$

It is an entire function of order $1/\mathrm{Re}(\eta)$, which is also known as the three parameter Mittag-Leffler function. This function plays a necessary role in the explanation of the anomalous dielectric properties in heterogeneous systems. Some important properties of this function can be seen in [13, 25, 30, 39].

Definition 2.6.3. The Gamma function in the half-plane is defined by the integral

$$\mathfrak{d}(z) = \int_0^{\infty} t^{z-1} e^{-t} dt, \quad \mathrm{Re}(z) > 0. \tag{2.6.5}$$

It was introduced by the famous mathematician L. Euler as a natural extension of the factorial operation $n!$ from positive integers n to real and even complex values of the argument n.

Chapter 3

Generalized substantial fractional operators and well-posedness of Cauchy problem

In this chapter, we introduce the generalized substantial fractional operators and analyze the fundamental properties of fractional calculus in the frame of these new generalized operators. Furthermore, well-posedness results for a class of generalized substantial FDEs constitute part of this chapter.

3.1 Generalized substantial fractional integral and derivatives

By motivation of the definitions of substantial fractional operators, here we introduce new definitions for the substantial fractional operators by establishing generalizations of the Katugampola fractional operators. We also establish a relation between the generalized substantial fractional operators and the Katugampola fractional operators. For $\rho \neq 0$ and $\sigma \in \mathbb{R}$, define ${}_{\sigma}\mathfrak{J}_a^{1,\rho} f(t) = \int_a^t f(s)e^{-\sigma(t^\rho - s^\rho)}d(s^\rho)$. Then generalized substantial integral of order m is given by mth iterate of the integral ${}_{\sigma}\mathfrak{J}_a^{1,\rho}$ as

$$
\begin{aligned}
{}_{\sigma}\mathfrak{J}_a^{m,\rho} f(t) &= \int_a^t e^{-\sigma(t^\rho - t_1^\rho)}d(t_1^\rho)\int_a^{t_1} e^{-\sigma(t^\rho - t_2^\rho)}d(t_2^\rho)\cdots\int_a^{t_{n-1}} e^{-\sigma(t^\rho - t_{n-1}^\rho)}f(t_{n-1})d(t_{n-1}^\rho) \\
&= \frac{\rho}{\mathfrak{d}(m)}\int_a^t (t^\rho - s^\rho)^{m-1}e^{-\sigma(t^\rho - s^\rho)}f(s)s^{\rho-1}ds.
\end{aligned}
\tag{3.1.1}
$$

We observe that ${}_{\sigma}\mathfrak{D}^{1,\rho}{}_{\sigma}\mathfrak{J}_a^{1,\rho}f(t) = f(t)$. By repeated application of this identity, we are lead to the identity ${}_{\sigma}\mathfrak{D}^{m,\rho}{}_{\sigma}\mathfrak{J}_a^{m,\rho}f(t) = f(t)$. Thus for $m > n$, we have ${}_{\sigma}\mathfrak{D}^{m-n,\rho}{}_{\sigma}\mathfrak{J}_a^{m-n,\rho}f(t) = f(t)$. Application of the operator ${}_{\sigma}\mathfrak{D}^{n,\rho}$ to both sides of this identity leads to the identity ${}_{\sigma}\mathfrak{D}^{n,\rho}f(t) = {}_{\sigma}\mathfrak{D}^{m,\rho}{}_{\sigma}\mathfrak{J}_a^{m-n,\rho}f(t)$. This relation will lead us to the definition of generalized fractional derivative. Moreover, ${}_{\sigma}\mathfrak{J}_a^{1,\rho}{}_{\sigma}\mathfrak{D}^{1,\rho}f(t) = \int_a^t(\frac{s^{1-\rho}}{\rho}\frac{d}{ds}+\sigma)f(s)d(s^\rho) = f(t) - f(a)e^{-\sigma(t^\rho - a^\rho)}$. In general, the repeated application of this process leads us to the generalized Taylor expansion involving generalized operators

$$ {}_{\sigma}\mathfrak{J}_a^{m,\rho}{}_{\sigma}\mathfrak{D}^{m,\rho}f(t) = f(t) - e^{-\sigma(t^\rho - a^\rho)}\sum_{k=1}^{m}\frac{(t^\rho - a^\rho)^{m-k}}{\mathfrak{d}(m-k+1)}\lim_{s\to a^+}{}_{\sigma}\mathfrak{D}^{m-k,\rho}f(s) $$

provided $f \in \Omega_\rho^m[a,b]$.

Definition 3.1.1. For real numbers σ, $\rho \neq 0$, $\eta > 0$ and $f \in \Lambda^1_{\sigma,\rho}[a,b]$, we define generalized substantial integral as

$$ {}_{\sigma}\mathfrak{J}_a^{\eta,\rho}f(t) = \frac{\rho}{\mathfrak{d}(\eta)}\int_a^t \frac{s^{\rho-1}e^{-\sigma(t^\rho - s^\rho)}}{(t^\rho - s^\rho)^{1-\eta}}f(s)ds. $$

Furthermore, the RL type generalized substantial fractional derivative is defined as ${}_{\sigma}\mathfrak{D}_a^{\eta,\rho}f(t) = {}_{\sigma}\mathfrak{D}^{m,\rho}{}_{\sigma}\mathfrak{J}_a^{m-\eta,\rho}f(t)$ where $m-1 < \eta \leq m$.

It is to be noted that for $\sigma \to 0$, ${}_{\sigma}\mathfrak{J}_a^{\eta,\rho}f(t) \to \mathfrak{J}_a^{\eta,\rho}f(t)$, which is the Katugampola fractional integral. Furthermore, for $\sigma = 0$ and $\rho \to 1$, the generalized substantial integral approaches the standard RL integral and the lower limit $a \to -\infty$ leads to the Weyl fractional integral. For $\sigma \neq 0$ and $\rho = 1$ the generalized substantial integral becomes the standard substantial integral. Finally for $\sigma = 0$ and $\rho \to 0$, we get the Hadamard fractional integral.

Definition 3.1.2. For $m-1 < \eta \leq m$, $a < b < \infty$ and $f \in \Omega^m_{\sigma,\rho}[a,b]$. Then the generalized Caputo type substantial derivative is defined as

$$ {}_{\sigma}^{c}\mathfrak{D}_a^{\eta,\rho}f(t) = {}_{\sigma}\mathfrak{D}_a^{\eta,\rho}\Big(f(t) - \sum_{k=0}^{m-1}\frac{{}_{\sigma}\mathfrak{D}^{k,\rho}f(a)}{k!}(t^\rho - a^\rho)^k e^{-\sigma(t^\rho - a^\rho)}\Big). $$

Theorem 3.1.3. *Assume $\eta > 0$, $\sigma > 0$, $\rho > 0$ and $\{f_k\}_{k=1}^\infty$ is a uniformly convergent sequence of continuous functions on $[0,b]$. Then*

$$ ({}_{\sigma}\mathfrak{J}_0^{\eta,\rho}\lim_{k\to\infty}f_k)(t) = (\lim_{k\to\infty}{}_{\sigma}\mathfrak{J}_0^{\eta,\rho}f_k)(t). $$

Proof. We denote the limit of sequence $\{f_k\}_{k=1}^{\infty}$ by f. It is well-known that f is continuous. We then have following estimates

$$\begin{aligned}|{}_\sigma\mathfrak{J}_0^{\eta,\rho}f_k(t) - {}_\sigma\mathfrak{J}_0^{\eta,\rho}f(t)| &\leq \frac{\rho}{\mathfrak{d}(\eta)}\int_0^t s^{\rho-1}(t^\rho - s^\rho)^{\eta-1}|(e^{-\sigma(t^\rho - s^\rho)})(f_k(s) - f(s))|ds \\ &\leq \frac{\rho}{\mathfrak{d}(\eta)}\|f_k - f\|_\infty \int_0^t s^{\rho-1}(t^\rho - s^\rho)^{\eta-1}ds \\ &= \frac{b^\rho}{\mathfrak{d}(\eta+1)}\|f_k - f\|_\infty.\end{aligned}$$

The conclusion follows, since $\|f_k - f\|_\infty \to 0$ as $k \to \infty$ uniformly on $[0, b]$. □

In the forthcoming results, we shall demonstrate the relationship between RL type Katugampola fractional operators and the generalized substantial fractional operators.

Lemma 3.1.1. *Assuming* $f \in \Lambda^1_{\sigma,\rho}[a,b]$. *Then* ${}_\sigma\mathfrak{J}_a^{\eta,\rho}f(t) = e^{-\sigma t^\rho}\mathfrak{J}_a^{\eta,\rho}(e^{\sigma t^\rho}f(t))$.

Theorem 3.1.4. *Assuming* $f \in \Omega^m_{\sigma,\rho}[a,b]$. *Then* ${}_\sigma\mathfrak{D}_a^{\eta,\rho}f(t) = e^{-\sigma t^\rho}\mathfrak{D}_a^{\eta,\rho}(e^{\sigma t^\rho}f(t))$.

Proof. By the definition (3.1.1) of the substantial fractional differential operator, Lemma 2.2.1, Lemma 3.1.1 and definition 2.2.1 we have

$$\begin{aligned}{}_\sigma\mathfrak{D}_a^{\eta,\rho}f(t) &= {}_\sigma\mathfrak{D}^{m,\rho}{}_\sigma\mathfrak{J}_a^{m-\eta,\rho}f(t) = {}_\sigma\mathfrak{D}^{m,\rho}\left(e^{-\sigma t^\rho}\mathfrak{J}_a^{m-\eta,\rho}(e^{\sigma t^\rho}f(t))\right) \\ &= e^{-\sigma t^\rho}\mathfrak{D}^{m,\rho}\mathfrak{J}_a^{m-\eta,\rho}(e^{\sigma t^\rho}f(t)) = e^{-\sigma t^\rho}\mathfrak{D}_a^{\eta,\rho}(e^{\sigma t^\rho}f(t)).\end{aligned}$$

□

Now we will introduce the composition properties of the generalized substantial operators. First we show that the generalized integral satisfies the semi-group property.

Theorem 3.1.5. *Let* $\eta, \zeta > 0$ *and* $f \in \Lambda^1_{\sigma,\rho}[a,b]$. *Then* ${}_\sigma\mathfrak{J}_a^{\eta,\rho}{}_\sigma\mathfrak{J}_a^{\zeta,\rho}f(t) = {}_\sigma\mathfrak{J}_a^{\eta+\zeta,\rho}f(t)$.

Proof. Using $(P1)$ and Lemma 3.1.1 repeatedly we have

$${}_\sigma\mathfrak{J}_a^{\eta,\rho}({}_\sigma\mathfrak{J}_a^{\zeta,\rho}f(t)) = {}_\sigma\mathfrak{J}_a^{\eta,\rho}(e^{-\sigma t^\rho}\mathfrak{J}_a^{\zeta,\rho}(e^{\sigma t^\rho}f(t))) = e^{-\sigma t^\rho}\mathfrak{J}_a^{\eta+\zeta,\rho}(e^{\sigma t^\rho}f(t)) = {}_\sigma\mathfrak{J}_a^{\eta+\zeta,\rho}f(t).$$

□

Theorem 3.1.6. *Let $m-1<\eta\leq m$, $\zeta\geq\eta$ and $f\in\Lambda^1_{\sigma,\rho}[a,b]$. Then*

$$ {}_{\sigma}\mathfrak{D}_a^{\eta,\rho}\,{}_{\sigma}\mathfrak{J}_a^{\zeta,\rho}f(t)={}_{\sigma}\mathfrak{J}_a^{\zeta-\eta,\rho}f(t). $$

The proof of Theorem 3.1.6 is the same as the proof of the Theorem 3.1.5. Therefore we omit it.

Theorem 3.1.7. *Assume $\eta>0$, $m-1<\eta\leq m$ and ${}_{\sigma}\mathfrak{J}_a^{m-\eta,\rho}f\in\Omega^m_{\sigma,\rho}[a,b]$. Then*

$$ {}_{\sigma}\mathfrak{J}_a^{\eta,\rho}\,{}_{\sigma}\mathfrak{D}_a^{\eta,\rho}f(t)=f(t)-e^{-\sigma(t^\rho-a^\rho)}\sum_{k=1}^{m}\frac{(t^\rho-a^\rho)^{\eta-k}}{\mathfrak{d}(\eta-k+1)}\lim_{s\to a^+}{}_{\sigma}\mathfrak{D}_a^{\eta-k,\rho}f(s). $$

Specifically, for $0<\eta<1$ we have

$$ {}_{\sigma}\mathfrak{J}_a^{\eta,\rho}\,{}_{\sigma}\mathfrak{D}_a^{\eta,\rho}f(t)=f(t)-e^{-\sigma(t^\rho-a^\rho)}\frac{(t^\rho-a^\rho)^{\eta-1}}{\mathfrak{d}(\eta)}\lim_{s\to a^+}{}_{\sigma}\mathfrak{J}_a^{1-\eta,\rho}f(s). $$

Proof. Using Leibniz rule, the following relation can be established.

$$ \Big(\frac{t^{1-\rho}}{\rho}\frac{d}{dt}+\sigma\Big)\int_a^t\frac{s^{\rho-1}e^{-\sigma(t^\rho-s^\rho)}}{(t^\rho-s^\rho)^{-\eta}}\,{}_{\sigma}\mathfrak{D}_a^{\eta,\rho}f(s)ds=\eta\int_a^t\frac{s^{\rho-1}e^{-\sigma(t^\rho-s^\rho)}}{(t^\rho-s^\rho)^{1-\eta}}\,{}_{\sigma}\mathfrak{D}_a^{\eta,\rho}f(s)ds. \tag{3.1.2} $$

By definition of ${}_{\sigma}\mathfrak{J}_a^{\eta,\rho}$, we have

$$ {}_{\sigma}\mathfrak{J}_a^{\eta,\rho}\,{}_{\sigma}\mathfrak{D}_a^{\eta,\rho}f(t)=\frac{\rho}{\mathfrak{d}(\eta)}\int_a^t\frac{s^{\rho-1}e^{-\sigma(t^\rho-s^\rho)}}{(t^\rho-s^\rho)^{1-\eta}}\,{}_{\sigma}\mathfrak{D}_a^{\eta,\rho}f(s)ds. \tag{3.1.3} $$

From Eq. (3.1.2) and (3.1.3), we get

$$ {}_{\sigma}\mathfrak{J}_a^{\eta,\rho}\,{}_{\sigma}\mathfrak{D}_a^{\eta,\rho}f(t)=\frac{\rho}{\mathfrak{d}(\eta+1)}\Big(\frac{t^{1-\rho}}{\rho}\frac{d}{dt}+\sigma\Big)\int_a^t\frac{s^{\rho-1}e^{-\sigma(t^\rho-s^\rho)}}{(t^\rho-s^\rho)^{-\eta}}\,{}_{\sigma}\mathfrak{D}_a^{\eta,\rho}f(s)ds. \tag{3.1.4} $$

From Definition 3.1.1 and Eq. (3.1.4), we find

$$ \begin{aligned} {}_{\sigma}\mathfrak{J}_a^{\eta,\rho}\,{}_{\sigma}\mathfrak{D}_a^{\eta,\rho}f(t)=\frac{\rho}{\mathfrak{d}(\eta+1)}\Big(\frac{t^{1-\rho}}{\rho}\frac{d}{dt}+\sigma\Big)\int_a^t\frac{s^{\rho-1}e^{-\sigma(t^\rho-s^\rho)}}{(t^\rho-s^\rho)^{-\eta}}\Big(\frac{s^{1-\rho}}{\rho}\frac{d}{ds}+\sigma\Big)\\ \times\,{}_{\sigma}\mathfrak{D}^{m-1,\rho}\,{}_{\sigma}\mathfrak{J}_a^{m-\eta,\rho}f(s)ds. \end{aligned} $$

Applying integration by parts and the product rule for classical derivatives, we have

$$ {}_{\sigma}\mathfrak{J}_a^{\eta,\rho}\,{}_{\sigma}\mathfrak{D}_a^{\eta,\rho}f(t)=-\frac{e^{-\sigma(t^\rho-a^\rho)}}{\mathfrak{d}(\eta)(t^\rho-s^\rho)^{1-\eta}}\lim_{s\to a^+}{}_{\sigma}\mathfrak{D}^{m-1,\rho}\,{}_{\sigma}\mathfrak{J}_a^{m-\eta,\rho}f(s)+\frac{\rho}{\mathfrak{d}(\eta)}\Big(\frac{t^{1-\rho}}{\rho}\frac{d}{dt}+\sigma\Big) $$

$$\times \int_a^t \frac{s^{\rho-1}e^{-\sigma(t^\rho - s^\rho)}}{(t^\rho - s^\rho)^{1-\eta}} \Big(\frac{s^{1-\rho}}{\rho}\frac{d}{ds} + \sigma\Big) {}_\sigma\mathfrak{D}^{m-2,\rho} {}_\sigma\mathfrak{J}_a^{m-\eta,\rho} f(s)ds.$$

Continuing in this manner, we get

$$\begin{aligned} {}_\sigma\mathfrak{J}_a^{\eta,\rho} {}_\sigma\mathfrak{D}_a^{\eta,\rho} f(t) &= -e^{-\sigma(t^\rho - a^\rho)} \sum_{k=1}^{m} \frac{(t^\rho - a^\rho)^{\eta-k}}{\mathfrak{d}(\eta - k + 1)} \lim_{s\to a^+} {}_\sigma\mathfrak{D}_a^{\eta-k,\rho} f(s) \\ &+ \frac{\rho}{\mathfrak{d}(\eta - m + 1)} \Big(\frac{t^{1-\rho}}{\rho}\frac{d}{dt} + \sigma\Big) \int_a^t \frac{s^{\rho-1}e^{-\sigma(t^\rho - s^\rho)}}{(t^\rho - s^\rho)^{m-\eta}} {}_\sigma\mathfrak{J}_a^{m-\eta,\rho} f(s)ds, \end{aligned}$$

where

$$\frac{\rho}{\mathfrak{d}(\eta - m + 1)} \Big(\frac{t^{1-\rho}}{\rho}\frac{d}{dt} + \sigma\Big) \int_a^t \frac{s^{\rho-1}e^{-\sigma(t^\rho - s^\rho)}}{(t^\rho - s^\rho)^{m-\eta}} {}_\sigma\mathfrak{J}_a^{m-\eta,\rho} f(s)ds = f(t).$$

Finally, we get the desired result

$${}_\sigma\mathfrak{J}_a^{\eta,\rho} {}_\sigma\mathfrak{D}_a^{\eta,\rho} f(t) = f(t) - e^{-\sigma(t^\rho - a^\rho)} \sum_{k=1}^{m} \frac{(t^\rho - a^\rho)^{\eta-k}}{\mathfrak{d}(\eta - k + 1)} \lim_{s\to a^+} {}_\sigma\mathfrak{D}_a^{\eta-k,\rho} f(s).$$

□

Theorem 3.1.8. *Assume $f \in \Omega^m_{\sigma,\rho}[a,b]$. Then the generalized Caputo type substantial derivative can be written as*

$${}^c_\sigma\mathfrak{D}_a^{\eta,\rho} f(t) = {}_\sigma\mathfrak{J}_a^{m-\eta,\rho} {}_\sigma\mathfrak{D}^{m,\rho} f(t) = \frac{\rho}{\mathfrak{d}(m-\eta)} \int_a^t \frac{e^{-\sigma(t^\rho - s^\rho)}}{(t^\rho - s^\rho)^{1+\eta-m}} {}_\sigma\mathfrak{D}^{m,\rho}(f(s))ds.$$

Proof. By using the Definition 3.1.2 and Eq. (3.1.2), we have

$${}^c_\sigma\mathfrak{D}_a^{\eta,\rho} f(t) = {}_\sigma\mathfrak{D}_a^{\eta,\rho} {}_\sigma\mathfrak{J}^{m,\rho} {}_\sigma\mathfrak{D}^{m,\rho} f(t).$$

By applying the Definition 3.1.2, and the properties $(P1)$ and $(P2)$, we get

$$\begin{aligned} {}^c_\sigma\mathfrak{D}_a^{\eta,\rho} f(t) &= {}_\sigma\mathfrak{D}^{m,\rho} {}_\sigma\mathfrak{J}_a^{m-\eta,\rho} {}_\sigma\mathfrak{J}_a^{m,\rho} {}_\sigma\mathfrak{D}^{m,\rho} f(t) = {}_\sigma\mathfrak{D}^{m,\rho} {}_\sigma\mathfrak{J}_a^{m,\rho} {}_\sigma\mathfrak{J}_a^{m-\eta,\rho} {}_\sigma\mathfrak{D}^{m,\rho} f(t) \\ &= {}_\sigma\mathfrak{J}_a^{m-\eta,\rho} {}_\sigma\mathfrak{D}^{m,\rho} f(t). \end{aligned}$$

□

Lemma 3.1.2. *For $f \in \Omega^m_\rho[a,b]$, the operator ${}^c_\sigma\mathfrak{D}_a^{\eta,\rho}$ satisfies the relation*

$${}^c_\sigma\mathfrak{D}_a^{\eta,\rho} f(t) = e^{-\sigma t^\rho} {}^c\mathfrak{D}_a^{\eta,\rho}(e^{\sigma t^\rho} f(t)).$$

Proof. By using the Lemma 2.2.1, Lemma 3.1.1 and Theorem 3.1.8 we have

$$
\begin{aligned}
{}_{\sigma}^{c}\mathfrak{D}_a^{\eta,\rho} f(t) &= {}_{\sigma}\mathfrak{I}_a^{m-\eta,\rho}\, {}_{\sigma}\mathfrak{D}^{m,\rho} f(t) = e^{-\sigma t^\rho}\mathfrak{I}_a^{m-\eta,\rho}(e^{\sigma t^\rho}\, {}_{\sigma}\mathfrak{D}^{m,\rho} f(t)) \\
&= e^{-\sigma t^\rho}\mathfrak{I}_a^{m-\eta,\rho}\mathfrak{D}^{m,\rho}(e^{\sigma t^\rho} f(t)) = e^{-\sigma t^\rho}\, {}^{c}\mathfrak{D}_a^{\eta,\rho}(e^{\sigma t^\rho} f(t)).
\end{aligned}
$$

□

Theorem 3.1.9. *Let $m-1<\eta\le m$, $\zeta\ge\eta$ and $f\in\Omega^m_{\sigma,\rho}[a,b]$. Then*

$$
{}_{\sigma}^{c}\mathfrak{D}_a^{\eta,\rho}\, {}_{\sigma}\mathfrak{I}_a^{\zeta,\rho} f(t) = {}_{\sigma}\mathfrak{I}_a^{\zeta-\eta,\rho} f(t).
$$

By use of Lemma 3.1.1 and Lemma 3.1.2, the result can be easily proved.

Theorem 3.1.10. *Assume $m-1<\eta\le m$ and $f\in\Omega^m_{\sigma,\rho}[a,b]$. Then*

$$
{}_{\sigma}\mathfrak{I}_a^{\eta,\rho}\, {}_{\sigma}^{c}\mathfrak{D}_a^{\eta,\rho} f(t) = f(t) - \sum_{k=0}^{m-1} \frac{{}_{\sigma}\mathfrak{D}^{k,\rho} f(a)}{k!} e^{-\sigma(t^\rho - a^\rho)}(t^\rho - a^\rho)^k.
$$

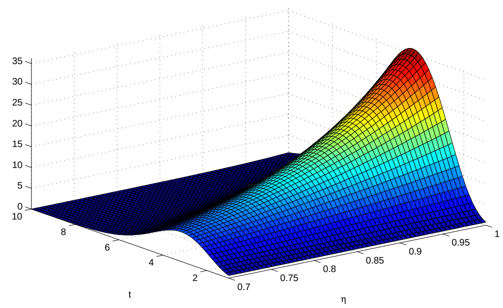
Proof. From Lemma 3.1.2 and Lemma 3.1.1, we have

$$
{}_{\sigma}\mathfrak{I}_a^{\eta,\rho}\, {}_{\sigma}^{c}\mathfrak{D}_a^{\eta,\rho} f(t) = {}_{\sigma}\mathfrak{I}_a^{\eta,\rho}\left(e^{-\sigma t^\rho}\, {}^{c}\mathfrak{D}_a^{\eta,\rho}(e^{\sigma t^\rho} f(t))\right) = e^{-\sigma t^\rho}\mathfrak{I}_a^{\eta,\rho}({}^{c}\mathfrak{D}_a^{\eta,\rho}(e^{\sigma t^\rho} f(t))).
$$

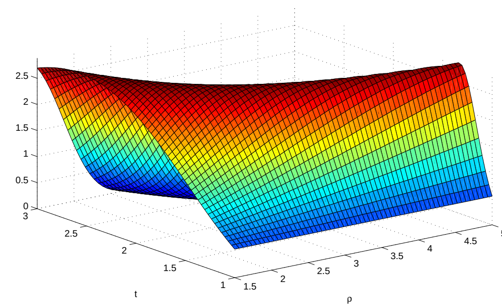
Now by property $(P4)$, we have

$$
\begin{aligned}
{}_{\sigma}\mathfrak{I}_a^{\eta,\rho}\, {}_{\sigma}^{c}\mathfrak{D}_a^{\eta,\rho} f(t) &= e^{-\sigma t^\rho}\left(e^{\sigma t^\rho} f(t) - \sum_{k=0}^{m-1} \frac{\mathfrak{D}^{k,\rho}(e^{\sigma t^\rho} f(t))|_{t=a}}{k!}(t^\rho - a^\rho)^k\right) \\
&= f(t) - \sum_{k=0}^{m-1} \frac{{}_{\sigma}\mathfrak{D}^{k,\rho} f(a)}{k!} e^{-\sigma(t^\rho - a^\rho)}(t^\rho - a^\rho)^k.
\end{aligned}
$$

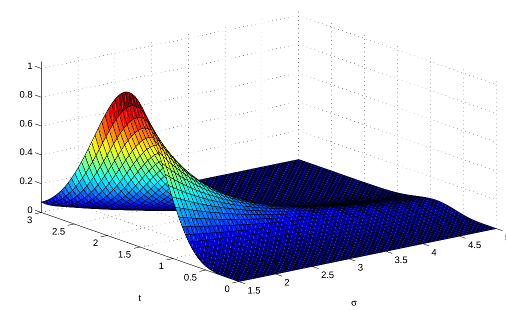
□



(a) $\zeta = 2, \sigma = 1, \rho = 1.5,\ 0.7 \leq \eta \leq 1.$



(b) $\eta = 0.5, \zeta = 2, \sigma = 1, 1 < \rho \leq 5.$



(c) $\eta = 0.5, \zeta = 2, \rho = 2, 1.5 \leq \sigma \leq 5.$

Figure 3.1: Fractional integrals ${}_{\sigma}\mathfrak{J}_a^{\eta,\rho}$ of $f(t) = (t^\rho - a^\rho)^\zeta e^{-\sigma t^\rho}$.

Example 3.1.11. Consider $f(t) = (t^\rho - a^\rho)^\zeta e^{-\sigma t^\rho}$. Then from Lemma 3.1.1 we have ${}_{\sigma}\mathfrak{J}_a^{\eta,\rho} f(t) = e^{-\sigma t^\rho} \mathfrak{J}_a^{\eta,\rho} (t^\rho - a^\rho)^\zeta$. Now by Lemma 3 in [31], we have

$$ {}_{\sigma}\mathfrak{J}_a^{\eta,\rho} f(t) = \frac{\mathfrak{d}(\zeta+1) e^{-\sigma t^\rho}}{\mathfrak{d}(\eta+\zeta+1)} (t^\rho - a^\rho)^{\eta+\zeta}. \tag{3.1.5} $$

Fractional integrals of $f(t)$, for different values of η, ζ, σ and ρ are graphically illustrated in Fig. 3.1. Now we compute the RL substantial derivative of $f(t) = (t^\rho - a^\rho)^\zeta e^{-\sigma t^\rho}$. Note that

$$ \mathfrak{D}^{1,\rho} (t^\rho - a^\rho)^\zeta = \frac{t^{1-\rho}}{\rho} \frac{d}{dt} (t^\rho - a^\rho)^\zeta = \zeta (t^\rho - a^\rho)^{\zeta-1}. \tag{3.1.6} $$

Therefore, from the definition of the RL substantial derivative, Lemma 2.2.1 and Eq.

(3.1.6), we have

$$
\begin{aligned}
{}_{\sigma}\mathfrak{D}_a^{\eta,\rho} f(t) &= {}_{\sigma}\mathfrak{D}^{1,\rho}{}_{\sigma}\mathfrak{J}_a^{1-\eta,\rho}\left[e^{-\sigma t^{\rho}}(t^{\rho}-a^{\rho})^{\zeta}\right] = \frac{\mathfrak{d}(\zeta+1)}{\mathfrak{d}(\zeta-\eta+2)}{}_{\sigma}\mathfrak{D}^{1,\rho}\left[e^{-\sigma t^{\rho}}(t^{\rho}-a^{\rho})^{\zeta-\eta+1}\right] \\
&= \frac{\mathfrak{d}(\zeta+1)e^{-\sigma t^{\rho}}}{\mathfrak{d}(\zeta-\eta+2)}\mathfrak{D}^{1,\rho}(t^{\rho}-a^{\rho})^{\zeta-\eta+1} = \frac{\mathfrak{d}(\zeta+1)e^{-\sigma t^{\rho}}}{\mathfrak{d}(\zeta-\eta+1)}(t^{\rho}-a^{\rho})^{\zeta-\eta}.
\end{aligned}
$$

Similarly, Caputo type substantial derivative of $f(t) = (t^{\rho}-a^{\rho})^{\zeta}e^{-\sigma t^{\rho}}$ can be computed as

$$
\begin{aligned}
{}_{\sigma}^{c}\mathfrak{D}_a^{\eta,\rho} f(t) &= {}_{\sigma}\mathfrak{J}_a^{1-\eta,\rho}{}_{\sigma}\mathfrak{D}^{1,\rho}\left[e^{-\sigma t^{\rho}}(t^{\rho}-a^{\rho})^{\zeta}\right] = {}_{\sigma}\mathfrak{J}_a^{1-\eta,\rho}\left[e^{-\sigma t^{\rho}}\mathfrak{D}^{1,\rho}((t^{\rho}-a^{\rho})^{\zeta})\right] \\
&= {}_{\sigma}\mathfrak{J}_a^{1-\eta,\rho}e^{-\sigma t^{\rho}}\mathfrak{D}^{1,\rho}(t^{\rho}-a^{\rho})^{\zeta} = e^{-\sigma t^{\rho}}\mathfrak{J}_a^{1-\eta,\rho}\mathfrak{D}^{1,\rho}(t^{\rho}-a^{\rho})^{\zeta} \\
&= \zeta e^{-\sigma t^{\rho}}\mathfrak{J}_a^{1-\eta,\rho}(t^{\rho}-a^{\rho})^{\zeta-1} = \frac{\mathfrak{d}(\zeta+1)e^{-\sigma t^{\rho}}}{\mathfrak{d}(\zeta-\eta+1)}(t^{\rho}-a^{\rho})^{\zeta-\eta}.
\end{aligned}
$$

3.2 Existence and uniqueness of solutions

When solving a FDE, the existence and uniqueness results have their own significance. It becomes necessary to notice, in advance, whether or not there is a solution to a given FDE. Keeping this in view, here we prove the equivalence between the initial value problem (IVP) and the Volterra equation. Then, using this equivalence along with Weissinger's fixed point theorem, we prove the existence and uniqueness of solution for the following IVP

$$
{}_{\sigma}^{c}\mathfrak{D}_0^{\eta,\rho} f(t) = g(t, f(t)), \quad t > 0, \tag{3.2.1}
$$

$$
{}_{\sigma}\mathfrak{D}^{k,\rho} f(0) = b_k, \quad k \in \{0, 1, 2, ..., m-1\}, \tag{3.2.2}
$$

where $\sigma > 0$, $\rho > 0$, $\eta > 0$, $m = \lceil \eta \rceil$, ${}_{\sigma}^{c}\mathfrak{D}_0^{\eta,\rho}$ is the generalized Caputo-type substantial fractional derivative and $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$.

For $K > 0$, $h^* > 0$ and $b_1, ..., b_m \in \mathbb{R}$, define the set

$$
H := \left\{ (t, f(t)) : 0 \le t \le h^*, \left| f(t) - e^{-\sigma t^{\rho}} \sum_{k=0}^{m-1} \frac{b_k}{\mathfrak{d}(k+1)} t^{\rho k} \right| \le K \right\}.
$$

We assume the following while establishing the subsequent results of this section:

(H1) $g : H \to \mathbb{R}$ is both continuous and bounded in H;

(H2) g satisfies the Lipschitz condition with respect to the second variable, i.e. for some constant $L > 0$ and for all $(t, f(t)), (t, \tilde{f}(t)) \in H$, we have

$$|g(t, f(t)) - g(t, \tilde{f}(t))| \leq L|f(t) - \tilde{f}(t)|.$$

We introduce some notations, for the sake of convenience.
Let $h := \min \left\{ h^*, \tilde{h}, \left(\frac{\mathfrak{d}(\eta+1)K}{M} \right)^{\frac{1}{\rho\eta}} \right\}$ where $M := \sup_{(x,y)\in H} |g(x, y)|$ and $\tilde{h}$ being a positive real number, fulfills the inequality $\tilde{h} < \left(\frac{\mathfrak{d}(\eta+1)}{L} \right)^{\frac{1}{\rho\eta}}$. These notations occur frequently in this section. The generalizations of the existence and uniqueness results presented in [21, 29, 10], are the main results of this section.

Theorem 3.2.1. *Assume that $h > 0$ and $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is continuous. Then $f \in C[0, h]$ is the solution of IVP (3.2.1)–(3.2.2) if and only if $f \in C[0, h]$ satisfies the Volterra equation*

$$f(t) = e^{-\sigma t^\rho} \sum_{k=0}^{m-1} \frac{b_k}{\mathfrak{d}(k+1)} t^{\rho k} + \frac{\rho}{\mathfrak{d}(\eta)} \int_0^t \frac{e^{-\sigma(t^\rho - s^\rho)}}{(t^\rho - s^\rho)^{1-\eta}} s^{\rho-1} g(s, f(s)) ds.$$

Proof. Let $f \in C[0, h]$ be a solution of the Volterra equation

$$f(t) = e^{-\sigma t^\rho} \sum_{k=0}^{m-1} \frac{b_k}{\mathfrak{d}(k+1)} t^{\rho k} + {}_\sigma\mathfrak{J}_0^{\eta,\rho} g(t, f(t)).$$

Apply ${}^c_\sigma\mathfrak{D}_0^{\eta,\rho}$ to both sides of the above equation. Using Theorem 3.1.5 and Example 3.1.11, we get

$$\begin{aligned} {}^c_\sigma\mathfrak{D}_0^{\eta,\rho} f(t) &= \sum_{k=0}^{m-1} \frac{b_k}{\mathfrak{d}(k+1)} {}^c_\sigma\mathfrak{D}_0^{\eta,\rho} e^{-\sigma t^\rho} t^{\rho k} + {}^c_\sigma\mathfrak{D}_0^{\eta,\rho} {}_\sigma\mathfrak{J}_0^{\eta,\rho} g(t, f(t)) \\ &= g(t, f(t)). \end{aligned}$$

Now we apply ${}_\sigma\mathfrak{D}^{k,\rho}$ to both sides of the Volterra equation, where $0 \leq k \leq m - 1$.

Using Theorem 3.1.5, Theorem 3.1.9 and Example 3.1.11, we have

$$\begin{aligned}{}_{\sigma}\mathfrak{D}_0^{k,\rho} f(t) &= \sum_{j=0}^{m-1} \frac{b_j}{\mathfrak{d}(j+1)} {}_{\sigma}\mathfrak{D}^{k,\rho} e^{-\sigma t^\rho} t^{\rho j} + {}_{\sigma}\mathfrak{D}^{k,\rho} {}_{\sigma}\mathfrak{J}_0^{\eta,\rho} g(t, f(t)) \\ &= \sum_{j=0}^{m-1} \frac{b_j}{\mathfrak{d}(j+1)} \Big(\frac{\mathfrak{d}(j+1)}{\mathfrak{d}(j-k+1)} e^{-\sigma t^\rho} t^{\rho(j-k)} \Big) + {}_{\sigma}\mathfrak{D}^{k,\rho} {}_{\sigma}\mathfrak{J}^{k,\rho} {}_{\sigma}\mathfrak{J}_0^{\eta-k,\rho} g(t, f(t)) \\ &= e^{-\sigma t^\rho} \sum_{j=0}^{m-1} \frac{b_j}{\mathfrak{d}(j-k+1)} t^{\rho(j-k)} + \frac{\rho}{\mathfrak{d}(\eta-k)} \int_0^t \frac{e^{-\sigma(t^\rho - s^\rho)}}{(t^\rho - s^\rho)^{1-\eta+k}} s^{\rho-1} g(s, f(s)) ds.\end{aligned}$$

It is clear that for $j < k$, the summands become identically zero because the reciprocal of the Gamma function for non-positive integers, vanishes. Furthermore, for $k < j$, the summands vanish if $t = 0$. Since $\eta - k$ is a positive real number, so the integral also vanishes when $t = 0$. Thus, we are left with the case $j = k$.

$${}_{\sigma}\mathfrak{D}^{k,\rho} f(0) = \frac{b_k}{\mathfrak{d}(k-k+1)} e^{-\sigma t^\rho} t^{\rho(k-k)} \Big|_{t=0} = b_k.$$

Conversely, we assume that $f \in C[0, h]$ is the solution of the given IVP. Using the initial conditions (3.2.2) and result of Theorem 3.1.10 and applying ${}_{\sigma}\mathfrak{J}_0^{\eta,\rho}$ to both sides of the FDE (3.2.1), we get the Volterra equation. □

Theorem 3.2.2. *Assume that f satisfies $(H1)$ and $(H2)$. Then, the Volterra equation*

$$f(t) = e^{-\sigma t^\rho} \sum_{k=0}^{m-1} \frac{b_k}{\mathfrak{d}(k+1)} t^{\rho k} + \frac{\rho}{\mathfrak{d}(\eta)} \int_0^t \frac{e^{-\sigma(t^\rho - s^\rho)}}{(t^\rho - s^\rho)^{1-\eta}} s^{\rho-1} g(s, f(s)) ds$$

possesses a uniquely determined solution $f \in C[0, h]$.

Proof. Define a set

$$B := \left\{ f \in C[0, h] : \sup_{0 \le t \le h} \left| f(t) - e^{-\sigma t^\rho} \sum_{k=0}^{m-1} \frac{b_k}{\mathfrak{d}(k+1)} t^{\rho k} \right| \le K \right\}$$

equipped with the norm $||.||_B$

$$||f||_B := \sup_{0<t\le h} \Big| f(t) \Big|.$$

It can be seen that $(B, ||.||_B)$ is a Banach space. Define the operator E by

$$Ef(t) := e^{-\sigma t^\rho} \sum_{k=0}^{m-1} \frac{b_k}{\mathfrak{d}(k+1)} t^{\rho k} + \frac{\rho}{\mathfrak{d}(\eta)} \int_0^t \frac{e^{-\sigma(t^\rho - s^\rho)}}{(t^\rho - s^\rho)^{1-\eta}} s^{\rho-1} g(s, f(s)) ds.$$

It is easy to check that Ef is continuous on the interval $[0, h]$ for $f \in B$. Furthermore,

$$\begin{aligned}
\Big|Ef(t) - e^{-\sigma t^\rho} \sum_{k=0}^{m-1} \frac{b_k}{\mathfrak{d}(k+1)} t^{\rho k}\Big| &= \Big|\frac{\rho}{\mathfrak{d}(\eta)} \int_0^t \frac{e^{-\sigma(t^\rho - s^\rho)}}{(t^\rho - s^\rho)^{1-\eta}} s^{\rho-1} g(s, f(s)) ds\Big| \\
&\leq \frac{\rho}{\mathfrak{d}(\eta)} M \int_0^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\eta}} ds \\
&= \frac{\rho}{\mathfrak{d}(\eta)} M \frac{t^{\rho\eta}}{\rho\eta} = \frac{M}{\mathfrak{d}(\eta+1)} t^{\rho\eta} \leq K
\end{aligned}$$

for $t \in [0, h]$, the last step follows from the definition of h. The result of this is that $Ef \in B$ for $f \in B$, i.e. E is the self-map.

From the definition of the operator E and the Volterra equation, it follows that the fixed points of E are the solutions of the Volterra equation.

To prove that the operator E has a unique fixed point, we use Weissinger's fixed point theorem. For $f_1, f_2 \in B$, first we will show the following inequality

$$||E^j f_1 - E^j f_2||_B \leq \Big(\frac{L h^{\rho\eta}}{\mathfrak{d}(\eta+1)}\Big)^j ||f_1 - f_2||_B.$$

Clearly, the above inequality is true for the case $j = 0$. Assuming that it is true for

$j = k-1$. For $j = k$, we have

$$\begin{aligned}
||E^k f_1 - E^k f_2||_B &= \sup_{0\le t\le h} \left| E^k f_1(t) - E^k f_2(t) \right| \\
&= \sup_{0\le t\le h} \left| EE^{k-1} f_1(t) - EE^{k-1} f_2(t) \right| \\
&= \sup_{0\le t\le h} \frac{1}{\mathfrak{d}(\eta)} \Big| \int_0^t \frac{\rho e^{-\sigma(t^\rho - s^\rho)} s^{\rho-1}}{(t^\rho - s^\rho)^{1-\eta}} \Big(g(s, E^{k-1} f_1(s)) - g(s, E^{k-1} f_2(s)) \Big) ds \Big| \\
&\le \sup_{0\le t\le h} \frac{L}{\mathfrak{d}(\eta)} \left\{ \int_0^t \frac{\rho s^{\rho-1}}{(t^\rho - s^\rho)^{1-\eta}} ds \right\} ||E^{k-1} f_1 - E^{k-1} f_2||_B \\
&= \frac{L}{\mathfrak{d}(\eta)} \left\{ \frac{h^{\rho\eta}}{\eta} \right\} ||E^{k-1} f_1 - E^{k-1} f_2||_B \\
&= \left(\frac{L h^{\rho\eta}}{\mathfrak{d}(\eta+1)} \right)^k ||f_1 - f_2||_B.
\end{aligned}$$

Since $h \le \tilde{h}$, we have $\left(\frac{Lh^{\rho\eta}}{\mathfrak{d}(\eta+1)} \right) < 1$. Thus, the series $\sum_{j=0}^{\infty} \left(\frac{Lh^{\rho\eta}}{\mathfrak{d}(\eta+1)} \right)^j$ is convergent. This completes the proof. □

The following presents an example for which a general method to determine the analytical solution is not available, but Theorem 3.2.1 and Theorem 3.2.2 allow us to comment on the existence of its unique solution.

Example 3.2.3. Consider the IVP

$$ {}_1^c\mathfrak{D}_0^{0.5,2} f(t) = te^{-t^2} \frac{(f(t))^2}{1+(f(t))^2}, \tag{3.2.3}$$

$$ f(0) = b_0. \tag{3.2.4}$$

It can easily verifiable that $g(t, f(t)) = te^{-t^2} \frac{(f(t))^2}{1+(f(t))^2}$ is both, continuous and bounded in H. Furthermore, we show that f satisfies the Lipschitz condition

$$\begin{aligned}
|g(t, f(t)) - g(t, \tilde{f}(t))| &= \left| te^{-t^2} \frac{(f(t))^2}{1+(f(t))^2} - te^{-t^2} \frac{(\tilde{f}(t))^2}{1+(\tilde{f}(t))^2} \right| \\
&= \left| te^{-t^2} \frac{(\tilde{f}(t))^2 - (f(t))^2}{(1+(f(t))^2)(1+(\tilde{f}(t))^2)} \right|.
\end{aligned}$$

Since $1+(f(t))^2 \geq 1$ and $1+(\tilde{f}(t))^2 \geq 1$, so

$$\begin{aligned}|g(t,f(t))-g(t,\tilde{f}(t))| &\leq \Big\{\sup_{0\leq t\leq h}\Big|te^{-t^2}(\tilde{f}(t)+f(t))\Big|\Big\}|\tilde{f}(t)-f(t)|\\ &\leq h\Big\{\Big|\sup_{0\leq t\leq h}|\tilde{f}(t)|+\sup_{0\leq t\leq h}|f(t)|\Big\}|\tilde{f}(t)-f(t)|\\ &= h(K_1+K_2)|f(t)-\tilde{f}(t)|,\end{aligned}$$

where $L := h(K_1+K_2)$ is the Lipschitz constant. Thus, the hypotheses $(H1)$ and $(H2)$ hold. From Theorem 3.2.1 and Theorem 3.2.2, we can deduce that there exists a unique solution of IVP (3.2.3)-(3.2.4).

3.3 Continuous dependence of solutions on the given data

In this section, first we prove a Gronwall-type inequality, which is the generalized version of Gronwall-type inequalities, presented in [48, 14, 3]. Without any doubt, this inequality plays a significant role in the qualitative theory of integral and differential equations. Furthermore, we present an analysis of the continuous dependence of the solutions of FDEs on the given data.

Theorem 3.3.1. *Assume that p and q are non-negative integrable functions and g is a non-negative and non-decreasing continuous function on $[a,b]$.*

If

$$p(t) \leq q(t) + \rho^{1-\eta} g(t)\int_a^t \frac{s^{\rho-1}e^{-\sigma(t^\rho-s^\rho)}}{(t^\rho-s^\rho)^{1-\eta}}p(s)ds, \quad \forall t\in[a,b],$$

then

$$p(t) \leq q(t) + \int_a^t \sum_{k=1}^{\infty}\frac{\rho^{1-k\eta}[g(t)\mathfrak{d}(\eta)]^k}{\mathfrak{d}(k\eta)}e^{-\sigma(t^\rho-s^\rho)}(t^\rho-s^\rho)^{k\eta-1}s^{\rho-1}q(s)ds, \quad \forall\ t\in[a,b].$$

Moreover, if q is non-decreasing, then

$$p(t) \leq q(t)\mathfrak{E}_\eta\Big[g(t)\mathfrak{d}(\eta)\Big(\frac{(t^\rho-a^\rho)}{\rho}\Big)^\eta\Big], \quad \forall\ t\in[a,b].$$

Proof. Define operator A as

$$Af(t) := \rho^{1-\eta} g(t) \int_a^t \frac{s^{\rho-1} e^{-\sigma(t^\rho - s^\rho)}}{(t^\rho - s^\rho)^{1-\eta}} f(s) ds.$$

Then,

$$p(t) \le q(t) + Ap(t).$$

Iterating successively, for $n \in \mathbb{N}$, we obtain

$$p(t) \le \sum_{k=0}^{n-1} A^k q(t) + A^n p(t).$$

By mathematical induction, we show that if f is non-negative, then,

$$A^k f(t) \le \rho^{1-k\eta} \int_a^t \frac{[g(t)\mathfrak{d}(\eta)]^k}{\mathfrak{d}(k\eta)} e^{-\sigma(t^\rho - s^\rho)} (t^\rho - s^\rho)^{k\eta - 1} s^{\rho-1} f(s) ds.$$

For $k = 1$, the equality holds. Assuming that it is true for $k \in \mathbb{N}$

$$\begin{aligned} A^{k+1} f(t) =& A(A^k f(t)) \le A\Big(\rho^{1-k\eta} \int_a^\tau \frac{[g(\tau)\mathfrak{d}(\eta)]^k}{\mathfrak{d}(k\eta)} e^{-\sigma(\tau^\rho - s^\rho)} (\tau^\rho - s^\rho)^{k\eta-1} s^{\rho-1} f(s) ds\Big) \\ =& \rho^{1-\eta} g(t) \int_a^t \frac{\tau^{\rho-1} e^{-\sigma(t^\rho - \tau^\rho)}}{(t^\rho - \tau^\rho)^{1-\eta}} \Big(\rho^{1-k\eta} \int_a^\tau \frac{[g(\tau)\mathfrak{d}(\eta)]^k}{\mathfrak{d}(k\eta)} \\ & \times e^{-\sigma(\tau^\rho - s^\rho)} (\tau^\rho - s^\rho)^{k\eta-1} s^{\rho-1} f(s) ds\Big) d\tau. \end{aligned}$$

By assumption, g is non-decreasing, so $g(\tau) \le g(t)$, $\forall\ \tau \le t$. Thus, we have

$$\begin{aligned} A^{k+1} f(t) \le \frac{(\mathfrak{d}(\eta))^k}{\mathfrak{d}(k\eta)} \rho^{2-(k+1)\eta} (g(t))^{k+1} \int_a^t \int_a^\tau & e^{-\sigma(t^\rho - s^\rho)} (t^\rho - \tau^\rho)^{\eta-1} \tau^{\rho-1} \\ & \times (\tau^\rho - s^\rho)^{k\eta-1} s^{\rho-1} f(s) ds d\tau. \end{aligned}$$

Using the Fubini's Theorem and the Dirichlet's technique, we get

$$\begin{aligned} A^{k+1} f(t) \le & \frac{(\mathfrak{d}(\eta))^k}{\mathfrak{d}(k\eta)} \rho^{2-(k+1)\eta} (g(t))^{k+1} \int_a^t e^{-\sigma(t^\rho - s^\rho)} s^{\rho-1} f(s) \\ & \times \int_s^t (t^\rho - \tau^\rho)^{\eta-1} \tau^{\rho-1} (\tau^\rho - s^\rho)^{k\eta-1} d\tau ds \\ = & \frac{(\mathfrak{d}(\eta))^k}{\mathfrak{d}(k\eta)} \rho^{2-(k+1)\eta} (g(t))^{k+1} \int_a^t e^{-\sigma(t^\rho - s^\rho)} s^{\rho-1} f(s) \\ & \times \left(\frac{\mathfrak{d}(\eta)\mathfrak{d}(k\eta)}{\rho \mathfrak{d}(k\eta + \eta)} (t^\rho - s^\rho)^{(k+1)\eta - 1} \right) ds \end{aligned}$$

$$=\rho^{1-(k+1)\eta}\int_a^t \frac{[g(t)\mathfrak{d}(\eta)]^{(k+1)}}{\mathfrak{d}((k+1)\eta)}e^{-\sigma(t^\rho-s^\rho)}(t^\rho-s^\rho)^{(k+1)\eta-1}s^{\rho-1}f(s)ds.$$

Now we prove that $A^n p(t) \to 0$ as $n \to \infty$. Since g is continuous on $[a,b]$, so by Weierstrass theorem, $\exists$ a constant $M>0$ such that $g(t) \leq M, \quad \forall\, t \in [a,b]$.

$$\implies A^n p(t) \leq \rho^{1-n\eta}\int_a^t \frac{[M\mathfrak{d}(\eta)]^n}{\mathfrak{d}(n\eta)}e^{-\sigma(t^\rho-s^\rho)}(t^\rho-s^\rho)^{n\eta-1}s^{\rho-1}p(s)ds.$$

Consider the series

$$\sum_{n=1}^{\infty}\frac{[M\mathfrak{d}(\eta)]^n}{\mathfrak{d}(n\eta)}.$$

Using the relation

$$\lim_{n\to\infty}\frac{\mathfrak{d}(n\eta)(n\eta)^\eta}{\mathfrak{d}(n\eta+\eta)}=1,$$

and the ratio test, we deduce that the series converges and therefore $A^n p(t) \to 0$ as $n \to \infty$. Thus,

$$\begin{aligned} p(t) &\leq \sum_{k=0}^{\infty}A^k q(t) \\ &\leq q(t)+\int_a^t \sum_{k=1}^{\infty}\frac{\rho^{1-k\eta}[g(t)\mathfrak{d}(\eta)]^k}{\mathfrak{d}(k\eta)}e^{-\sigma(t^\rho-s^\rho)}(t^\rho-s^\rho)^{k\eta-1}s^{\rho-1}q(s)ds. \end{aligned}$$

Additionally, if q is non-decreasing, then, $q(s) \leq q(t)$, $\forall\, s \in [a,t]$. So,

$$\begin{aligned} p(t) &\leq q(t)\Big[1+\sum_{k=1}^{\infty}\frac{\rho^{1-k\eta}[g(t)\mathfrak{d}(\eta)]^k}{\mathfrak{d}(k\eta)}\int_a^t e^{-\sigma(t^\rho-s^\rho)}(t^\rho-s^\rho)^{k\eta-1}s^{\rho-1}ds\Big] \\ &\leq q(t)\Big[1+\sum_{k=1}^{\infty}\frac{\rho^{1-k\eta}[g(t)\mathfrak{d}(\eta)]^k}{\mathfrak{d}(k\eta)}\int_a^t (t^\rho-s^\rho)^{k\eta-1}s^{\rho-1}ds\Big] \\ &= q(t)\Big[1+\sum_{k=1}^{\infty}\frac{\rho^{-k\eta}[g(t)\mathfrak{d}(\eta)(t^\rho-a^\rho)^\eta]^k}{\mathfrak{d}(k\eta+1)}\Big] \\ &= q(t)\mathfrak{E}_\eta\Big[g(t)\mathfrak{d}(\eta)\Big(\frac{(t^\rho-a^\rho)}{\rho}\Big)^\eta\Big]. \end{aligned}$$

□

Next we will look at the dependence of the solution of a FDE on the initial values.

Theorem 3.3.2. *Assume that f is the solution of the IVP* (3.2.1) − (3.2.2) *and $\hat{f}$ is the solution of the following IVP*

$$ {}^{c}_{\sigma}\mathfrak{D}^{\eta,\rho}_{0}\hat{f}(t) = g(t, \hat{f}(t)), \quad t > 0, \tag{3.3.1}$$

$$ {}_{\sigma}\mathfrak{D}^{k,\rho}\hat{f}(0) = c_k, \quad k \in \{0, 1, 2, ..., m-1\}. \tag{3.3.2}$$

Let $\epsilon := \max_{k=0,1,\ldots,m-1} |b_k - c_k|$. If ϵ is sufficiently small, then $\exists$ some constant $h > 0$ such that f and $\hat{f}$ are defined on $[0, h]$, and

$$\sup_{0\le t\le h} |f(t) - \hat{f}(t)| = \mathcal{O}(\epsilon).$$

Proof. Let f and $\hat{f}$ be defined on $[0, h_1]$ and $[0, h_2]$, respectively. Take $h = \min\{h_1, h_2\}$, then both the functions f and $\hat{f}$, are at-least defined on the interval $[0, h]$. Define $\delta(t) := f(t) - \hat{f}(t)$, then δ is the solution of the following IVP

$$ {}^{c}_{\sigma}\mathfrak{D}^{\eta,\rho}_{0}\delta(t) = g(t, f(t)) - g(t, \hat{f}(t)), \quad t > 0, \tag{3.3.3}$$

$$ {}_{\sigma}\mathfrak{D}^{k,\rho}\delta(0) = b_k - c_k, \quad k \in \{0, 1, 2, ..., m-1\}. \tag{3.3.4}$$

The IVP (3.3.3) − (3.3.4) is equivalent to the Volterra equation

$$\delta(t) = e^{-\sigma t^{\rho}} \sum_{k=0}^{m-1} \frac{(b_k - c_k)}{\mathfrak{d}(k+1)} t^{\rho k} + \frac{\rho}{\mathfrak{d}(\eta)} \int_0^t \frac{e^{-\sigma(t^{\rho}-s^{\rho})}}{(t^{\rho}-s^{\rho})^{1-\eta}} s^{\rho-1}\Big(g(s, f(s)) - g(s, \hat{f}(s))\Big)ds.$$

Taking the absolute of above equation and using the triangle inequality and the Lipschitz condition on g, we get

$$\begin{aligned}
&|\delta(t)| \\
&= \left| e^{-\sigma t^{\rho}} \sum_{k=0}^{m-1} \frac{(b_k - c_k)}{\mathfrak{d}(k+1)} t^{\rho k} + \frac{\rho}{\mathfrak{d}(\eta)} \int_0^t \frac{e^{-\sigma(t^{\rho}-s^{\rho})}}{(t^{\rho}-s^{\rho})^{1-\eta}} s^{\rho-1}\Big(g(s, f(s)) - g(s, \hat{f}(s))\Big)ds \right| \\
&\le \left| e^{-\sigma t^{\rho}} \right| \sum_{k=0}^{m-1} \frac{t^{\rho k}}{\mathfrak{d}(k+1)} \Big| b_k - c_k \Big| + \frac{\rho}{\mathfrak{d}(\eta)} \int_0^t \frac{e^{-\sigma(t^{\rho}-s^{\rho})}}{(t^{\rho}-s^{\rho})^{1-\eta}} s^{\rho-1} \Big| g(s, f(s)) - g(s, \hat{f}(s)) \Big| ds \\
&\le \sum_{k=0}^{m-1} \frac{h^{\rho k}}{\mathfrak{d}(k+1)} \max_{k=0,1,\ldots,m-1} \Big| b_k - c_k \Big| + \frac{\rho L}{\mathfrak{d}(\eta)} \int_0^t \frac{e^{-\sigma(t^{\rho}-s^{\rho})}}{(t^{\rho}-s^{\rho})^{1-\eta}} s^{\rho-1} \Big| f(s) - \hat{f}(s) \Big| ds \\
&= m\epsilon \sum_{k=0}^{m-1} \frac{h^{\rho k}}{\mathfrak{d}(k+1)} + \frac{\rho L}{\mathfrak{d}(\eta)} \int_0^t \frac{e^{-\sigma(t^{\rho}-s^{\rho})}}{(t^{\rho}-s^{\rho})^{1-\eta}} s^{\rho-1} |\delta(s)| ds.
\end{aligned}$$

Taking $p(t) = |\delta(t)|$, $q(t) = m\epsilon \sum_{k=0}^{m-1} \frac{h^{\rho k}}{\mathfrak{d}(k+1)}$ and $g(t) = \frac{\rho^{\eta} L}{\mathfrak{d}(\eta)}$, and using Theorem 3.3.1, we find

$$|\delta(t)| \leq m\epsilon \sum_{k=0}^{m-1} \frac{h^{\rho k}}{\mathfrak{d}(k+1)} \mathfrak{E}_{\eta}(Lt^{\rho\eta}) \leq m\epsilon \sum_{k=0}^{m-1} \frac{h^{\rho k}}{\mathfrak{d}(k+1)} \mathfrak{E}_{\eta}(Lh^{\rho\eta}) = \mathcal{O}(\epsilon),$$

and this completes the proof. □

Now we present an example to verify the statement of Theorem 3.3.2.

Example 3.3.3. The unique analytical solutions of the following four IVPs

$${}_1^c\mathfrak{D}_0^{0.5,0.5} f_i(t) = 0.9 f_i(t), \quad f_1(0) = 1, \quad f_2(0) = 1.2, \quad f_3(0) = 1.4, \quad f_4(0) = 1.6,$$

are given by

$$f_i(t) = f_i(0) e^{-t^{0.5}} \mathfrak{E}_{0.5}(0.9 t^{0.25}), \quad 0 \leq t \leq h.$$

Plots of these solutions are given in Fig. 3.2.

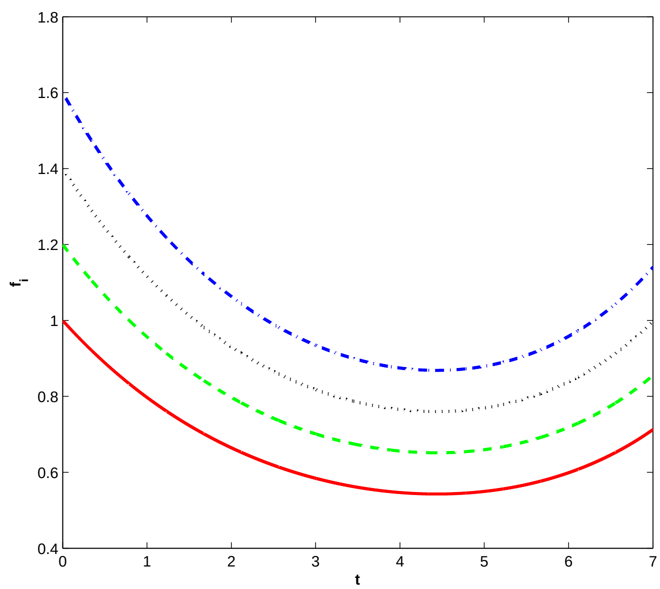


Figure 3.2: Graphs of solutions from Example 3.3.3.

From Fig. 3.2, we can see that the change in the solutions is bounded by the change in the initial conditions on the closed interval $[0, h]$. Thus, Example 3.3.3 verifies the statement of the Theorem 3.3.2.

In the next theorem, we analyze the dependence of the solution of the FDE on the force function g.

Theorem 3.3.4. *We assume that f is the solution of the IVP* $(3.2.1)-(3.2.2)$ *and $\hat{f}$ is the solution of the following IVP*

$$ {}^{c}_{\sigma}\mathfrak{D}^{\eta,\rho}_{0}\hat{f}(t) = \tilde{g}(t,\hat{f}(t)), \quad t>0, \tag{3.3.5} $$

$$ {}_{\sigma}\mathfrak{D}^{k,\rho}\hat{f}(0) = b_k, \quad k \in \{0,1,2,...,m-1\}, \tag{3.3.6} $$

where $\tilde{g}$ satisfies the same conditions as g. Let $\epsilon := \max_{(t,\hat{f}(t))\in H} |g(t,\hat{f}(t)) - \tilde{g}(t,\hat{f}(t))|$. If ϵ is sufficiently small, then $\exists$ some constant $h>0$ such that f and $\hat{f}$ are defined on $[0,h]$, and

$$ \sup_{0\le t\le h} |f(t)-\hat{f}(t)| = \mathcal{O}(\epsilon). $$

Proof. Let f and $\hat{f}$ be defined on $[0,h_1]$ and $[0,h_2]$, respectively. Take $h=\min\{h_1,h_2\}$, then both the functions f and $\hat{f}$, are at least defined on the interval $[0,h]$. Define $\delta(t) := f(t)-\hat{f}(t)$, then δ is the solution of the following IVP

$$ {}^{c}_{\sigma}\mathfrak{D}^{\eta,\rho}_{0}\delta(t) = g(t,f(t)) - \tilde{g}(t,\hat{f}(t)), \quad t>0, \tag{3.3.7} $$

$$ {}_{\sigma}\mathfrak{D}^{k,\rho}\delta(0) = 0, \quad k \in \{0,1,2,...,m-1\}. \tag{3.3.8} $$

The IVP $(3.3.7)-(3.3.8)$ is equivalent to the Volterra equation

$$ \delta(t) = \frac{\rho}{\mathfrak{d}(\eta)} \int_0^t \frac{e^{-\sigma(t^\rho - s^\rho)}}{(t^\rho - s^\rho)^{1-\eta}} s^{\rho-1} \Big(g(s,f(s)) - \tilde{g}(s,\hat{f}(s)) \Big) ds. $$

Taking the absolute of the above equation and using the Lipschitz condition on g, we get

$$ \begin{aligned} |\delta(t)| =& \Big| \frac{\rho}{\mathfrak{d}(\eta)} \int_0^t \frac{e^{-\sigma(t^\rho - s^\rho)}}{(t^\rho - s^\rho)^{1-\eta}} s^{\rho-1} \Big[\Big(g(s,f(s)) - f(s,\hat{f}(s)) \Big) \\ &+ \Big(g(s,\hat{f}(s)) - \tilde{g}(s,\hat{f}(s)) \Big) \Big] ds \Big| \\ \le& \frac{\rho}{\mathfrak{d}(\eta)} \Big\{ \int_0^t \frac{e^{-\sigma(t^\rho - s^\rho)}}{(t^\rho - s^\rho)^{1-\eta}} s^{\rho-1} \Big| g(s,f(s)) - f(s,\hat{f}(s)) \Big| ds \\ &+ \int_0^t \frac{e^{-\sigma(t^\rho - s^\rho)}}{(t^\rho - s^\rho)^{1-\eta}} s^{\rho-1} \Big| g(s,\hat{f}(s)) - \tilde{g}(s,\hat{f}(s)) \Big| ds \Big\} \end{aligned} $$

$$
\begin{aligned}
\leq & \frac{\rho}{\mathfrak{d}(\eta)} \int_0^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\eta}} \max_{(s,\hat{f}(s))\in H} \Big| g(s,\hat{f}(s)) - \tilde{g}(s,\hat{f}(s)) \Big| ds \\
& + \frac{\rho L}{\mathfrak{d}(\eta)} \int_0^t \frac{e^{-\sigma(t^\rho - s^\rho)}}{(t^\rho - s^\rho)^{1-\eta}} s^{\rho-1} \Big|\delta(s)\Big| ds \\
\leq & \frac{\rho L}{\mathfrak{d}(\eta)} \int_0^t \frac{e^{-\sigma(t^\rho - s^\rho)}}{(t^\rho - s^\rho)^{1-\eta}} s^{\rho-1} \Big|\delta(s)\Big| ds + \frac{\epsilon}{\mathfrak{d}(\eta+1)} t^{\rho\eta} \\
\leq & \frac{\epsilon h^{\rho\eta}}{\mathfrak{d}(\eta+1)} + \frac{\rho L}{\mathfrak{d}(\eta)} \int_0^t \frac{e^{-\sigma(t^\rho - s^\rho)}}{(t^\rho - s^\rho)^{1-\eta}} s^{\rho-1} \Big|\delta(s)\Big| ds.
\end{aligned}
$$

Taking $p(t) = |\delta(t)|$, $q(t) = \frac{\epsilon h^{\rho\eta}}{\mathfrak{d}(\eta+1)}$ and $g(t) = \frac{\rho^\eta L}{\mathfrak{d}(\eta)}$, and using the Theorem 3.3.1, we find

$$
|\delta(t)| \leq \frac{\epsilon h^{\rho\eta}}{\mathfrak{d}(\eta+1)} \mathfrak{E}_\eta(L t^{\rho\eta}) \leq \frac{\epsilon h^{\rho\eta}}{\mathfrak{d}(\eta+1)} \mathfrak{E}_\eta(L h^{\rho\eta}) = \mathcal{O}(\epsilon).
$$

Thus, the proof is complete. □

Finally, we explore the consequences of perturbing the order of the FDE.

Theorem 3.3.5. *Assume that f is the solution of the IVP (3.2.1) – (3.2.2) and $\hat{f}$ is the solution of the following IVP*

$$
{}_{\sigma}^{c}\mathfrak{D}_0^{\tilde{\eta},\rho} \hat{f}(t) = g(t, \hat{f}(t)), \quad t > 0, \tag{3.3.9}
$$

$$
{}_{\sigma}\mathfrak{D}^{k,\rho} \hat{f}(0) = b_k, \quad k \in \{0, 1, 2, ..., \tilde{m} - 1\}, \tag{3.3.10}
$$

where $\tilde{\eta} > \eta$ and $\tilde{m} := \lceil \tilde{\eta} \rceil$. Let $\epsilon := \tilde{\eta} - \eta$ and

$$
\tilde{\epsilon} := \begin{cases} 0 & \textit{if } \tilde{m} = m, \\ \max\Big\{ |b_k| : m \leq k \leq \tilde{m} - 1 \Big\} & \textit{otherwise.} \end{cases}
$$

If ϵ and $\tilde{\epsilon}$ are sufficiently small, then $\exists$ some constant $h > 0$ such that f and $\hat{f}$ are defined on $[0, h]$, and

$$
\sup_{0 \leq t \leq h} |f(t) - \hat{f}(t)| = \mathcal{O}(\epsilon) + \mathcal{O}(\tilde{\epsilon}).
$$

Proof. Let f and $\hat{f}$ be defined on $[0, h_1]$ and $[0, h_2]$, respectively. Take $h = \min\{h_1, h_2\}$, then both the functions f and $\hat{f}$, are at-least defined on the interval $[0, h]$. Define

$\delta(t) := f(t) - \hat{f}(t)$, then using Theorem 3.2.1

$$\begin{aligned}\delta(t) = -e^{-\sigma t^\rho}\sum_{k=m}^{\tilde{m}-1}\frac{b_k}{\mathfrak{d}(k+1)}t^{\rho k} + \frac{\rho}{\mathfrak{d}(\eta)}\int_0^t \frac{e^{-\sigma(t^\rho - s^\rho)}}{(t^\rho - s^\rho)^{1-\eta}}s^{\rho-1}g(s,f(s))ds \\ - \frac{\rho}{\mathfrak{d}(\tilde{\eta})}\int_0^t \frac{e^{-\sigma(t^\rho - s^\rho)}}{(t^\rho - s^\rho)^{1-\tilde{\eta}}}s^{\rho-1}g(s,\hat{f}(s))ds \\ = -e^{-\sigma t^\rho}\sum_{k=m}^{\tilde{m}-1}\frac{b_k}{\mathfrak{d}(k+1)}t^{\rho k} + \frac{\rho}{\mathfrak{d}(\eta)}\int_0^t \frac{e^{-\sigma(t^\rho - s^\rho)}}{(t^\rho - s^\rho)^{1-\eta}}s^{\rho-1}\Big(g(s,f(s)) - g(s,\hat{f}(s))\Big)ds \\ + \int_0^t \Big(\frac{\rho(t^\rho - s^\rho)^{\eta-1}}{\mathfrak{d}(\eta)} - \frac{\rho(t^\rho - s^\rho)^{\tilde{\eta}-1}}{\mathfrak{d}(\tilde{\eta})}\Big)e^{-\sigma(t^\rho - s^\rho)}s^{\rho-1}g(s,\hat{f}(s))ds.\end{aligned}$$

Taking the absolute of the above equation and using the Lipschitz condition on g, we get

$$\begin{aligned}|\delta(t)| &\leq \sum_{k=m}^{\tilde{m}-1}\frac{h^{\rho k}}{\mathfrak{d}(k+1)}\Big|b_k\Big| + \frac{\rho L}{\mathfrak{d}(\eta)}\int_0^t \frac{e^{-\sigma(t^\rho - s^\rho)}}{(t^\rho - s^\rho)^{1-\eta}}s^{\rho-1}|\delta(s)|ds \\ &\quad + \max_{(x,y)\in H}\Big|g(x,y)\Big|\int_0^t \Big|\frac{\rho(t^\rho - s^\rho)^{\eta-1}}{\mathfrak{d}(\eta)} - \frac{\rho(t^\rho - s^\rho)^{\tilde{\eta}-1}}{\mathfrak{d}(\tilde{\eta})}\Big|s^{\rho-1}ds \\ &\leq \mathcal{O}(\tilde{\epsilon}) + \frac{\rho L}{\mathfrak{d}(\eta)}\int_0^t \frac{e^{-\sigma(t^\rho - s^\rho)}}{(t^\rho - s^\rho)^{1-\eta}}s^{\rho-1}|\delta(s)|ds \\ &\quad + M\int_0^h \Big|\frac{(v)^{\eta-1}}{\mathfrak{d}(\eta)} - \frac{(v)^{\tilde{\eta}-1}}{\mathfrak{d}(\tilde{\eta})}\Big|dv.\end{aligned}$$

The zero of above integrand can be seen to be $v_0 = \left(\frac{\mathfrak{d}(\tilde{\eta})}{\mathfrak{d}(\eta)}\right)^{\frac{1}{\tilde{\eta}-\eta}}$. If $h \leq v_0$, then absolute value sign can be taken outside the integral. In the other case, the interval of integration must be separated at v_0, and each integral can be evaluated without difficulty. Thus in any case, we find that the integral is bounded by $\mathcal{O}(\tilde{\eta} - \eta) = \mathcal{O}(\epsilon)$. Hence, we have

$$|\delta(t)| \leq \mathcal{O}(\tilde{\epsilon}) + \mathcal{O}(\epsilon) + \frac{\rho L}{\mathfrak{d}(\eta)}\int_0^t \frac{e^{-\sigma(t^\rho - s^\rho)}}{(t^\rho - s^\rho)^{1-\eta}}s^{\rho-1}|\delta(s)|ds$$

and using Theorem the 3.3.1, we obtain the desired result. □

Chapter 4

Φ-Hadamard type fractional calculus

In this chapter, we present a generalization of the Hadamard type fractional calculus which has been named as the Φ-Hadamard type fractional calculus. Moreover, we discuss conditions for which the Φ-Hadamard type fractional integral is bounded in a generalized space $X^p_{\Phi,c}(a,b)$. We also prove sufficient conditions for the existence of Φ-Hadamard type fractional derivative. Finally, we give proofs of some basic properties and integration by parts formulas of fractional calculus.

4.1 Generalization of the Hadamard type fractional operators

Motivated by definitions of the Φ-RL, Φ-Caputo and Hadamard type fractional operators, we present a modified construction of the general case of the Hadamard type fractional operators. We introduce new definitions of the Hadamard type fractional operators by generalizing these operators. The new generalization is based on the observation that for $m \in \mathbb{N}$, the Φ-Hadamard type fractional integral, of order m, is given by the mth iteration of the integral ${}_H\mathfrak{J}^{1,\Phi}_{a,\mu}$ as below:

$$\begin{aligned} {}_H\mathfrak{J}^{m,\Phi}_{a,\mu} f(t) &= \{\Phi(t)\}^{-\mu} \int_a^t \frac{\Phi'(s_1)}{\Phi(s_1)} ds_1 \int_a^{s_1} \frac{\Phi'(s_2)}{\Phi(s_2)} ds_2 \cdots \int_a^{s_{m-1}} \{\Phi(s_m)\}^{\mu} f(s_m) \frac{\Phi'(s_m)}{\Phi(s_m)} ds_m \\ &= \frac{1}{\mathfrak{d}(m)} \int_a^t \left(\frac{\Phi(s)}{\Phi(t)}\right)^{\mu} \left(\log \frac{\Phi(t)}{\Phi(s)}\right)^{m-1} f(s) \frac{\Phi'(s) ds}{\Phi(s)}. \end{aligned} \tag{4.1.1}$$

Thus, the fractional version of (4.1.1) is given below

$$ {}_H\mathfrak{J}^{\eta,\Phi}_{a,\mu}f(t)=\frac{1}{\mathfrak{d}(\eta)}\int_a^t\left(\frac{\Phi(s)}{\Phi(t)}\right)^{\mu}\left(\log\frac{\Phi(t)}{\Phi(s)}\right)^{\eta-1}f(s)\frac{\Phi'(s)ds}{\Phi(s)},\quad \text{where } \eta\in\mathbb{R}. \tag{4.1.2}$$

We next define the Φ-Hadamard type fractional operators in the following definitions.

Definition 4.1.1. Let η be a real number such that $\eta>0$, $m-1<\eta\le m$, $-\infty\le a<b\le\infty$, f be an integrable function defined on $[a,b]$ and $\Phi\in C^1([a,b])$ be an increasing function such that $\Phi'(t)\neq 0$ for all $t\in[a,b]$. Then, the left and right-sided Φ-Hadamard type fractional integrals of a function f of order η are defined as

$$ {}_H\mathfrak{J}^{\eta,\Phi}_{a^+,\mu}f(t)=\frac{1}{\mathfrak{d}(\eta)}\int_a^t\left(\frac{\Phi(s)}{\Phi(t)}\right)^{\mu}\left(\log\frac{\Phi(t)}{\Phi(s)}\right)^{\eta-1}f(s)\frac{\Phi'(s)ds}{\Phi(s)},\quad \text{for } t>a, \tag{4.1.3}$$

and

$$ {}_H\mathfrak{J}^{\eta,\Phi}_{b^-,\mu}f(t)=\frac{1}{\mathfrak{d}(\eta)}\int_t^b\left(\frac{\Phi(t)}{\Phi(s)}\right)^{\mu}\left(\log\frac{\Phi(s)}{\Phi(t)}\right)^{\eta-1}f(s)\frac{\Phi'(s)ds}{\Phi(s)},\quad \text{for } t<b, \tag{4.1.4}$$

respectively.

It is noteworthy that for $\Phi(t)\to t$, ${}_H\mathfrak{J}^{\eta,\Phi}_{a,\mu}\to{}_H\mathfrak{J}^{\eta}_{a,\mu}$ which is the Hadamard type fractional integral. Moreover, for $\Phi(t)\to t$ and $\mu=0$, we obtain ${}_H\mathfrak{J}^{\eta,\Phi}_{a,\mu}\to{}_H\mathfrak{J}^{\eta}_{a}$ which is the classical Hadamard fractional integral.

Definition 4.1.2. Let η be a real number such that $\eta>0$, $m-1<\eta\le m$, $-\infty\le a<b\le\infty$, f be an integrable function defined on $[a,b]$ and $\Phi\in C^1([a,b])$ be an increasing function such that $\Phi'(t)\neq 0$ for all $t\in[a,b]$. Then, the left and right-sided Φ-Hadamard type fractional derivatives of a function f of order η are defined as

$$ {}_H\mathfrak{D}^{\eta,\Phi}_{a^+,\mu}f(t)={}_H\mathfrak{D}^{m,\Phi}_{a^+,\mu}\,{}_H\mathfrak{J}^{m-\eta,\Phi}_{a^+,\mu}f(t),\quad \text{for } t>a, \tag{4.1.5}$$

and

$$ {}_H\mathfrak{D}^{\eta,\Phi}_{b^-,\mu}f(t)={}_H\mathfrak{D}^{m,\Phi}_{b^-,\mu}\,{}_H\mathfrak{J}^{m-\eta,\Phi}_{b^-,\mu}f(t),\quad \text{for } t<b, \tag{4.1.6}$$

respectively, where

$$ {}_H\mathfrak{D}^{m,\Phi}_{a^+,\mu}=\{\Phi(t)\}^{-\mu}\left(\frac{\Phi(t)}{\Phi'(t)}\frac{d}{dt}\right)^{m}\{\Phi(t)\}^{\mu} \tag{4.1.7}$$

and

$$ {}_{H}\mathfrak{D}^{m,\Phi}_{b^-,\mu} = (-1)^m \{\Phi(t)\}^{\mu} \left(\frac{\Phi(t)}{\Phi'(t)} \frac{d}{dt} \right)^m \{\Phi(t)\}^{-\mu}. \tag{4.1.8} $$

In 1967, the definition of the RL fractional derivative was reformulated by Caputo in such a way that he switched the order of the ordinary derivative with the fractional integral operator [7]. By motivation of this reformulation, we present the following definition.

Definition 4.1.3. Assume that $\eta > 0$, $m-1 < \eta \leq m$, I is the interval $-\infty < a < b < \infty$, $f, \Phi \in \mathbb{C}^m(I)$ two functions such that Φ is increasing and $\Phi'(t) \neq 0$ for all $t \in I$. Then, the left and right-sided Caputo Φ-Hadamard type fractional derivatives of order η are defined as

$$ {}^{C}_{H}\mathfrak{D}^{\eta,\Phi}_{a^+,\mu} f(t) = {}_{H}\mathfrak{J}^{m-\eta,\Phi}_{a^+,\mu} \, {}_{H}\mathfrak{D}^{m,\Phi}_{a^+,\mu} f(t), \quad \text{for } t > a, \tag{4.1.9} $$

and

$$ {}^{C}_{H}\mathfrak{D}^{\eta,\Phi}_{b^-,\mu} f(t) = {}_{H}\mathfrak{J}^{m-\eta,\Phi}_{b^-,\mu} \, {}_{H}\mathfrak{D}^{m,\Phi}_{b^-,\mu} f(t), \quad \text{for } t < b, \tag{4.1.10} $$

respectively.

Lemma 4.1.1. *For the Φ-Hadamard type fractional integral ${}_{H}\mathfrak{J}^{\eta,\Phi}_{a^+,\mu}$ and fractional derivative ${}_{H}\mathfrak{D}^{\eta,\Phi}_{a^+,\mu}$, the following properties hold*

(a) $\lim_{\eta \to 1} {}_{H}\mathfrak{J}^{\eta,\Phi}_{a^+,\mu} f(t) = \int_a^t \left(\frac{\Phi(s)}{\Phi(t)} \right)^{\mu} f(s) \frac{\Phi'(s) ds}{\Phi(s)}$,

(b) $\lim_{\eta \to 0^+} {}_{H}\mathfrak{J}^{\eta,\Phi}_{a^+,\mu} f(t) = f(t)$,

(c) $\lim_{\eta \to 0^+} {}_{H}\mathfrak{D}^{\eta,\Phi}_{a^+,\mu} f(t) = f(t)$,

(d) $\lim_{\eta \to (m-1)^+} {}_{H}\mathfrak{D}^{\eta,\Phi}_{a^+,\mu} f(t) = \{\Phi(t)\}^{-\mu} \delta^{m-1,\Phi} \{\Phi(t)\}^{\mu} f(t)$,

(e) $\lim_{\eta \to m^-} {}_{H}\mathfrak{D}^{\eta,\Phi}_{a^+,\mu} f(t) = \{\Phi(t)\}^{-\mu} \delta^{m,\Phi} \{\Phi(t)\}^{\mu} f(t)$,

where $\delta^{m,\Phi} = \left(\frac{\Phi(t)}{\Phi'(t)} \frac{d}{dt} \right)^m$.

Proof. For (a), taking $\eta = 1$ in the definition of Φ-Hadamard type fractional integral ${}_H\mathfrak{J}^{\eta,\Phi}_{a^+,\mu}$, the desired result is obtained.

For (b), considering the definition of ${}_H\mathfrak{J}^{\eta,\Phi}_{a^+,\mu}$ and applying integration by parts, we get

$$\begin{aligned}
{}_H\mathfrak{J}^{\eta,\Phi}_{a^+,\mu} f(t) =& \frac{1}{\mathfrak{d}(\eta)} \int_a^t \left(\frac{\Phi(s)}{\Phi(t)}\right)^{\mu} \left(\log \frac{\Phi(t)}{\Phi(s)}\right)^{\eta-1} f(s) \frac{\Phi'(s)ds}{\Phi(s)} \\
=& -\frac{1}{\mathfrak{d}(\eta+1)} \left\{ \left(\frac{\Phi(s)}{\Phi(t)}\right)^{\mu} \left(\log \frac{\Phi(t)}{\Phi(s)}\right)^{\eta} f(s)\Big|_a^t \right. \\
& \left. - \int_a^t \left(\log \frac{\Phi(t)}{\Phi(s)}\right)^{\eta} d\left[\left(\frac{\Phi(s)}{\Phi(t)}\right)^{\mu} f(s)\right] \right\} \\
=& \frac{1}{\mathfrak{d}(\eta+1)} \left\{ \left(\frac{\Phi(a)}{\Phi(t)}\right)^{\mu} \left(\log \frac{\Phi(t)}{\Phi(a)}\right)^{\eta} f(a) \right. \\
& \left. + \int_a^t \left(\log \frac{\Phi(t)}{\Phi(s)}\right)^{\eta} d\left[\left(\frac{\Phi(s)}{\Phi(t)}\right)^{\mu} f(s)\right] \right\}.
\end{aligned}$$

Taking the limit as $\eta \to 0^+$, we obtain

$$\lim_{\eta \to 0^+} {}_H\mathfrak{J}^{\eta,\Phi}_{a^+,\mu} f(t) = \left(\frac{\Phi(a)}{\Phi(t)}\right)^{\mu} f(a) + f(t) - \left(\frac{\Phi(a)}{\Phi(t)}\right)^{\mu} f(a) = f(t).$$

For (c), consider the definition of ${}_H\mathfrak{D}^{\eta,\Phi}_{a^+,\mu}$

$$\begin{aligned}
{}_H\mathfrak{D}^{\eta,\Phi}_{a^+,\mu} f(t) = \{\Phi(t)\}^{-\mu} \left(\frac{\Phi(t)}{\Phi'(t)} \frac{d}{dt}\right) \{\Phi(t)\}^{\mu} \left\{ \frac{1}{\mathfrak{d}(1-\eta)} \int_a^t \left(\frac{\Phi(s)}{\Phi(t)}\right)^{\mu} \right. \\
\left. \times \left(\log \frac{\Phi(t)}{\Phi(s)}\right)^{-\eta} f(s) \frac{\Phi'(s)ds}{\Phi(s)} \right\}.
\end{aligned}$$

Taking the limit as $\eta \to 0^+$ and applying the Leibniz rule, we get

$$\lim_{\eta \to 0^+} {}_H\mathfrak{D}^{\eta,\Phi}_{a^+,\mu} f(t) = \{\Phi(t)\}^{-\mu} \left(\frac{\Phi(t)}{\Phi'(t)} \frac{d}{dt}\right) \{\Phi(t)\}^{\mu} \left\{ \int_a^t \left(\frac{\Phi(s)}{\Phi(t)}\right)^{\mu} f(s) \frac{\Phi'(s)ds}{\Phi(s)} \right\} = f(t).$$

Similarly, one can prove (d) and (e). Therefore, we omit the proofs here. Hence, the Lemma 4.1.1 is proved. □

Lemma 4.1.2. *If $0 < \eta < 1$ and $\zeta > 0$, then the following relations hold*

$$ {}_{H}\mathfrak{J}^{\eta,\Phi}_{a^+,\mu}\left\{\{\Phi(t)\}^{-\mu}\left(\log\frac{\Phi(t)}{\Phi(a)}\right)^{\zeta-1}\right\} = \frac{\mathfrak{d}(\zeta)}{\mathfrak{d}(\eta+\zeta)}\{\Phi(t)\}^{-\mu}\left(\log\frac{\Phi(t)}{\Phi(a)}\right)^{\zeta+\eta-1}, \quad (4.1.11) $$

$$ {}_{H}\mathfrak{D}^{\eta,\Phi}_{a^+,\mu}\left\{\{\Phi(t)\}^{-\mu}\left(\log\frac{\Phi(t)}{\Phi(a)}\right)^{\zeta-1}\right\} = \frac{\mathfrak{d}(\zeta)}{\mathfrak{d}(\zeta-\eta)}\{\Phi(t)\}^{-\mu}\left(\log\frac{\Phi(t)}{\Phi(a)}\right)^{\zeta-\eta-1}. \quad (4.1.12) $$

Proof. First we prove (4.1.11). Using the definition of ${}_{H}\mathfrak{J}^{\eta,\Phi}_{a^+,\mu}$, we have

$$ {}_{H}\mathfrak{J}^{\eta,\Phi}_{a^+,\mu}\left\{\{\Phi(t)\}^{-\mu}\left(\log\frac{\Phi(t)}{\Phi(a)}\right)^{\zeta-1}\right\} = \frac{1}{\mathfrak{d}(\eta)}\{\Phi(t)\}^{-\mu}\int_a^t\left(\log\frac{\Phi(t)}{\Phi(s)}\right)^{\eta-1} \times\left(\log\frac{\Phi(s)}{\Phi(a)}\right)^{\zeta-1}\frac{\Phi'(s)ds}{\Phi(s)}. $$

Substituting $y = \frac{\log\left(\frac{\Phi(s)}{\Phi(a)}\right)}{\log\left(\frac{\Phi(t)}{\Phi(a)}\right)}$ and using the relation between the Beta function and the Gamma function, we obtain

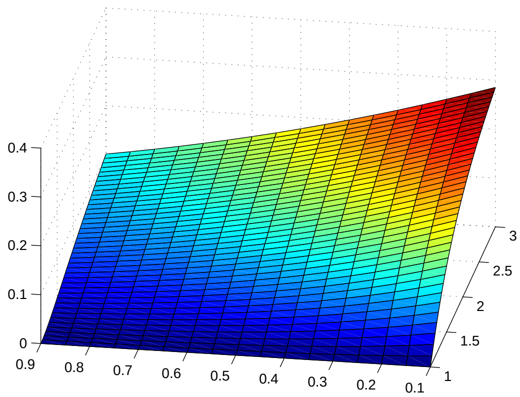
$$ \begin{aligned} {}_{H}\mathfrak{J}^{\eta,\Phi}_{a^+,\mu}\left\{\{\Phi(t)\}^{-\mu}\left(\log\frac{\Phi(t)}{\Phi(a)}\right)^{\zeta-1}\right\} =& \frac{1}{\mathfrak{d}(\eta)}\{\Phi(t)\}^{-\mu}\left(\log\frac{\Phi(t)}{\Phi(a)}\right)^{\zeta+\eta-1} \\ &\times\int_0^1(1-y)^{\eta-1}y^{\zeta-1}dy \\ =&\frac{\mathfrak{d}(\zeta)}{\mathfrak{d}(\eta+\zeta)}\{\Phi(t)\}^{-\mu}\left(\log\frac{\Phi(t)}{\Phi(a)}\right)^{\zeta+\eta-1}. \end{aligned} $$

Thus, we have proved our lemma for the integral. The proof for the relation (4.1.12) can be done in a similar manner as above. □

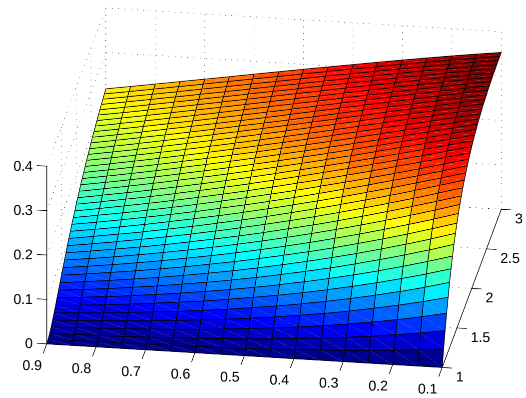
Fractional integrals and derivatives of $f(t)$, for $\zeta = 2$, $a = 1$, $\mu = 1$ and different functions $\Phi(t)$ are illustrated graphically in Fig. 4.1 and Fig. 4.2, respectively.

4.2 Φ-Hadamard type fractional integral operator in the space $X^p_{\Phi,c}(a,b)$

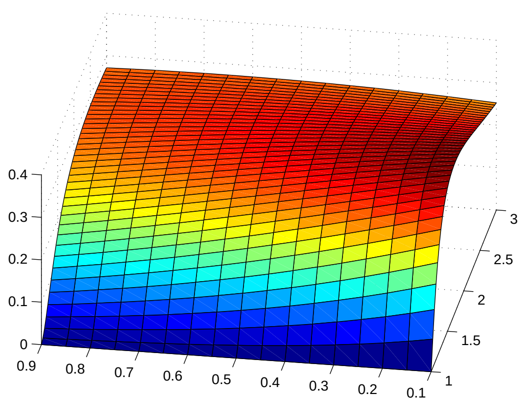
In this section, we discuss the conditions under which the Φ-Hadamard type fractional integral operator ${}_{H}\mathfrak{J}^{\eta,\Phi}_{a^+,\mu}$ is bounded in the space $X^p_{\Phi,c}(a,b)$ ($c \in \mathbb{R}$, $1 \le p \le \infty$) of those



(a) $\Phi(t) = \sqrt{t}$, $0.1 \leq \eta \leq 0.9$.



(b) $\Phi(t) = t$, $0.1 \leq \eta \leq 0.9$.



(c) $\Phi(t) = t^2$, $0.1 \leq \eta \leq 0.9$.

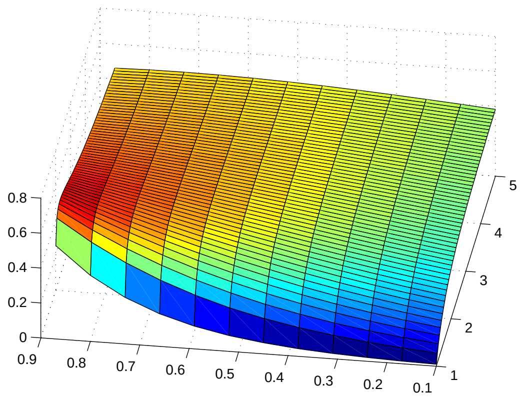
Figure 4.1: Fractional integrals ${}_H\mathfrak{I}^{\eta,\Phi}_{a^+,\mu}$ of $f(t) = \{\Phi(t)\}^{-\mu}\left(\log\frac{\Phi(t)}{\Phi(a)}\right)^{\zeta-1}$.

complex-valued Lebesgue measurable functions f on $[a,b]$ for which $||f||_{X^p_{\Phi,c}} < \infty$, where

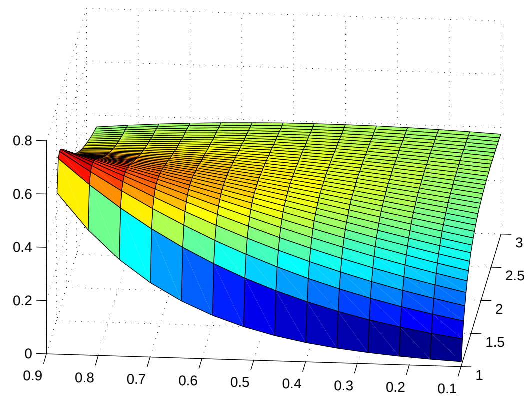
$$||f||_{X^p_{\Phi,c}} = \left(\int_a^b \left|\left\{\Phi(t)\right\}^c f(t)\right|^p \frac{\Phi'(t)}{\Phi(t)}dt\right)^{\frac{1}{p}}, \quad \text{for } c \in \mathbb{R}, 1 \leq p < \infty \tag{4.2.1}$$

and

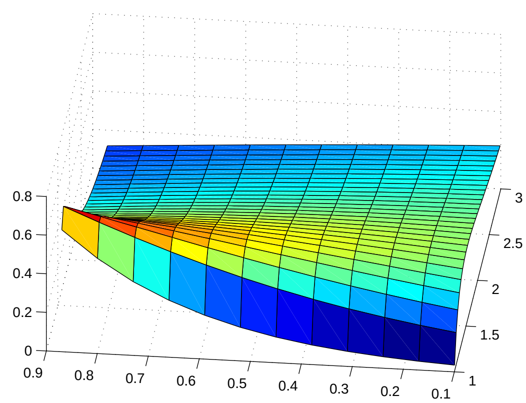
$$||f||_{X^\infty_{\Phi,c}} = \operatorname*{ess\,sup}_{a \leq t \leq b}\left(\left\{\Phi(t)\right\}^c \left|f(t)\right|\right), \quad \text{for } c \in \mathbb{R}. \tag{4.2.2}$$



(a) $\Phi(t) = \sqrt{t}$, $0.1 \leq \eta \leq 0.9$.



(b) $\Phi(t) = t$, $0.1 \leq \eta \leq 0.9$.



(c) $\Phi(t) = t^2$, $0.1 \leq \eta \leq 0.9$.

Figure 4.2: Fractional derivatives ${}_H\mathfrak{D}^{\eta,\Phi}_{a^+,\mu}$ of $f(t) = \{\Phi(t)\}^{-\mu}\left(\log \frac{\Phi(t)}{\Phi(a)}\right)^{\zeta-1}$.

If we consider $c = \frac{1}{p}$ and $\Phi(t) = t$, then the space $X^p_{\Phi,c}(a,b)$ coincides with the space $L^p(a,b)$ with

$$||f||_p = \left(\int_a^b \left|f(t)\right|^p dt\right)^{\frac{1}{p}} \quad \text{for } 1 \leq p < \infty$$

and

$$||f||_\infty = \operatorname*{ess\,sup}_{a \leq t \leq b} \left|f(t)\right| \quad \text{for } c \in \mathbb{R}.$$

In the theorem below, for a non-negative increasing function Φ and $\mu \geq c$, we prove that the Φ-Hadamard type fractional integral operator ${}_H\mathfrak{J}^{\eta,\Phi}_{a^+,\mu}$ is well-defined on the space $X^p_{\Phi,c}(a,b)$.

Theorem 4.2.1. *Let $\eta > 0$, $1 \le p \le \infty$, $0 < a < b < \infty$, Φ be a non-negative increasing function and $\mu, c \in \mathbb{R}$ be such that $\mu \ge c$. Then the operator ${}_H\mathfrak{J}^{\eta,\Phi}_{a^+,\mu}$ is bounded in $X^p_{\Phi,c}(a,b)$ and*

$$||{}_H\mathfrak{J}^{\eta,\Phi}_{a^+,\mu} f||_{X^p_{\Phi,c}} \le K||f||_{X^p_{\Phi,c}} \tag{4.2.3}$$

where

$$K = \frac{1}{\mathfrak{d}(\eta+1)}\left(\log\frac{\Phi(b)}{\Phi(a)}\right)^{\eta} \quad \text{for } \mu = c \tag{4.2.4}$$

while

$$K = \frac{1}{\mathfrak{d}(\eta)}(\mu-c)^{-\eta}\gamma\left(\eta, (\mu-c)\log\frac{\Phi(b)}{\Phi(a)}\right) \quad \text{for } \mu > c. \tag{4.2.5}$$

Proof. First we discuss the case $1 \le p < \infty$. Using the definition of ${}_H\mathfrak{J}^{\eta,\Phi}_{a^+,\mu}$ and Eq. (4.2.1), we find

$$||{}_H\mathfrak{J}^{\eta,\Phi}_{a^+,\mu} f||_{X^p_{\Phi,c}} = \left(\int_a^b \{\Phi(t)\}^{cp}\left|\frac{1}{\mathfrak{d}(\eta)}\int_a^t \left(\frac{\Phi(s)}{\Phi(t)}\right)^{\mu}\left(\log\frac{\Phi(t)}{\Phi(s)}\right)^{\eta-1} \times f(s)\frac{\Phi'(s)ds}{\Phi(s)}\right|^p \frac{\Phi'(t)dt}{\Phi(t)}\right)^{\frac{1}{p}}.$$

Making the substitution $\Phi(s) = \frac{\Phi(t)}{\Phi(u)}$, we get

$$||{}_H\mathfrak{J}^{\eta,\Phi}_{a^+,\mu} f||_{X^p_{\Phi,c}} = \left(\int_a^b \left|\int_{\Phi^{-1}(1)}^{\Phi^{-1}\left(\frac{\Phi(b)}{\Phi(a)}\right)} \frac{\{\Phi(u)\}^{-\mu}}{\mathfrak{d}(\eta)}\{\log\Phi(u)\}^{\eta-1}\frac{\left\{\Phi(t)\right\}^{1-\frac{1}{p}}}{\left\{\Phi'(t)\right\}^{-\frac{1}{p}}} \times f\left(\Phi^{-1}\left(\frac{\Phi(t)}{\Phi(u)}\right)\right)\frac{\Phi'(u)du}{\Phi(u)}\right|^p dt\right)^{\frac{1}{p}}.$$

Since $f(t) \in X^p_{\Phi,c}(a,b)$, thus $\frac{\left\{\Phi(t)\right\}^{1-\frac{1}{p}}}{\left\{\Phi'(t)\right\}^{-\frac{1}{p}}} f(t) \in L^p(a,b)$ and by applying the generalized Minkowsky inequality, we have

$$||{}_H\mathfrak{J}^{\eta,\Phi}_{a^+,\mu} f||_{X^p_{\Phi,c}} \le \frac{1}{\mathfrak{d}(\eta)}\int_{\Phi^{-1}(1)}^{\Phi^{-1}\left(\frac{\Phi(b)}{\Phi(a)}\right)} \{\Phi(u)\}^{-\mu-1}\{\log\Phi(u)\}^{\eta-1}\Phi'(u)$$

$$\times \left(\int_b^{\Phi^{-1}\left(\Phi(u)\Phi(a)\right)} \{\Phi(t)\}^{cp} \left| f\left(\Phi^{-1}\left(\frac{\Phi(t)}{\Phi(u)} \right) \right) \right|^p \frac{\Phi'(t)dt}{\Phi(t)} \right)^{\frac{1}{p}} du$$

$$= \frac{1}{\mathfrak{d}(\eta)} \int_{\Phi^{-1}(1)}^{\Phi^{-1}\left(\frac{\Phi(b)}{\Phi(a)}\right)} \left(\int_a^{\Phi^{-1}\left(\Phi(b)\Phi(u)\right)} \left|\{\Phi(s)\}^c f(s)\right|^p \frac{\Phi'(s)ds}{\Phi(s)} \right)^{\frac{1}{p}}$$
$$\times \{\Phi(u)\}^{c-\mu-1}\{\log \Phi(u)\}^{\eta-1}\Phi'(u)du$$

and hence

$$||{}_H\mathfrak{J}^{\eta,\Phi}_{a^+,\mu} f||_{X^p_{\Phi,c}} \leq M||f||_{X^p_{\Phi,c}}$$

where

$$M = \frac{1}{\mathfrak{d}(\eta)} \int_{\Phi^{-1}(1)}^{\Phi^{-1}\left(\frac{\Phi(b)}{\Phi(a)}\right)} \{\Phi(u)\}^{c-\mu-1}\{\log \Phi(u)\}^{\eta-1}\Phi'(u)du. \tag{4.2.6}$$

When $\mu = c$, then we have

$$M = \frac{1}{\mathfrak{d}(\eta)} \int_{\Phi^{-1}(1)}^{\Phi^{-1}\left(\frac{\Phi(b)}{\Phi(a)}\right)} \{\log \Phi(u)\}^{\eta-1} \frac{\Phi'(u)du}{\Phi(u)} = \frac{1}{\mathfrak{d}(\eta+1)} \left(\log \frac{\Phi(b)}{\Phi(a)} \right)^{\eta}.$$

If $\mu > c$, then making the substitution $s = (\mu - c)\log \Phi(u)$ in Eq. (4.2.6) and by using the definition of the incomplete Gamma function, we have

$$M = \frac{(\mu-c)^{-\eta}}{\mathfrak{d}(\eta)} \int_0^{(\mu-c)\log\left(\frac{\Phi(b)}{\Phi(a)}\right)} e^{-s}s^{\eta-1}ds = \frac{(\mu-c)^{-\eta}}{\mathfrak{d}(\eta)} \gamma\left(\eta, (\mu-c)\log\frac{\Phi(b)}{\Phi(a)}\right).$$

Thus the result is proved for $1 \leq p < \infty$.

Next we assume that $p = \infty$. Then by using the definition of ${}_H\mathfrak{J}^{\eta,\Phi}_{a^+,\mu}$ and Eq. (4.2.2), we have

$$\begin{aligned} \left|\{\Phi(t)\}^c {}_H\mathfrak{J}^{\eta,\Phi}_{a^+,\mu} f(t)\right| &\leq \frac{1}{\mathfrak{d}(\eta)} \int_a^t \left(\frac{\Phi(s)}{\Phi(t)}\right)^{\mu-c} \left(\log\frac{\Phi(t)}{\Phi(s)}\right)^{\eta-1} \left|\{\Phi(s)\}^c f(s)\right| \frac{\Phi'(s)ds}{\Phi(s)} \\ &\leq K(t)||f||_{X^\infty_{\Phi,c}} \end{aligned} \tag{4.2.7}$$

where

$$K(t) = \frac{1}{\mathfrak{d}(\eta)} \int_a^t \left(\frac{\Phi(s)}{\Phi(t)}\right)^{\mu-c} \left(\log\frac{\Phi(t)}{\Phi(s)}\right)^{\eta-1} \frac{\Phi'(s)ds}{\Phi(s)}.$$

Substituting $\Phi(u) = \frac{\Phi(t)}{\Phi(s)}$ in the above equation, we get

$$K(t) = \frac{1}{\mathfrak{d}(\eta)} \int_{\Phi^{-1}(1)}^{\Phi^{-1}\left(\frac{\Phi(t)}{\Phi(a)}\right)} \{\Phi(u)\}^{c-\mu} \{\log \Phi(u)\}^{\eta-1} \frac{\Phi'(u)du}{\Phi(u)}.$$

If $\mu = c$. then for any $a \leq t \leq b$

$$K(t) = \frac{1}{\mathfrak{d}(\eta+1)} \left(\log \frac{\Phi(t)}{\Phi(a)}\right)^{\eta} \leq \frac{1}{\mathfrak{d}(\eta+1)} \left(\log(\frac{\Phi(b)}{\Phi(a)}\right)^{\eta}. \tag{4.2.8}$$

If $\mu > c$, then making the substitution $s = (\mu - c)\log \Phi(u)$ and by using the definition of the incomplete Gamma function, we find

$$K(t) = \frac{1}{\mathfrak{d}(\eta)} (\mu - c)^{-\eta} \gamma \left(\eta, (\mu - c)^{\eta-1} \log \frac{\Phi(t)}{\Phi(a)}\right).$$

Since $\gamma(v, t)$ is an increasing function, thus for any $a \leq t \leq b$ it follows that

$$K(t) \leq \frac{1}{\mathfrak{d}(\eta)} (\mu - c)^{-\eta} \gamma \left(\eta, (\mu - c)^{\eta-1} \log \frac{\Phi(b)}{\Phi(a)}\right). \tag{4.2.9}$$

Hence, from Eqs. (4.2.7)-(4.2.9), we see that for any $a \leq t \leq b$

$$\left| \{\Phi(t)\}^c {}_H\mathfrak{J}^{\eta,\Phi}_{a^+,\mu} f(t) \right| \leq K ||f||_{X^{\infty}_{\Phi,c}}$$

where K is given by Eqs. (4.2.4) and (4.2.5) when $\mu = c$ and $\mu > c$, respectively. Thus we have proved the result for $p = \infty$. □

Remark 4.2.2. The result for Φ-Hadamard type fractional integral operator in Theorem 4.2.1 is analogous to the classical RL fractional integral operator [38]. Moreover, considering the case that $\Phi(t) = t$, we have an analogous conclusion for Hadamard type fractional integral operator. Taking into consideration $\Phi(t) = t$ and $\mu = 0$, Theorem 4.2.1 holds true in the settings of Hadamard fractional integral operator [24].

4.3 Existence of the Φ-Hadamard type fractional derivative

In this section, we present sufficient conditions for the existence of the Φ-Hadamard type fractional derivative ${}_H\mathfrak{D}^{\eta,\Phi}_{a^+,\mu}$ in the space

$$AC^m_{\delta^{\Phi},\mu}[a,b] := \left\{ h : [a,b] \to \mathbb{C} : \delta^{m-1,\Phi} \left\{ \{\Phi(t)\}^{\mu} h(t) \right\} \in AC[a,b] \right\} \tag{4.3.1}$$

where $\mu \in \mathbb{R}$ and $\delta^{k,\Phi} = \left(\frac{\Phi(t)}{\Phi'(t)}\frac{d}{dt}\right)^k$. Here $AC[a,b]$ is the set of absolutely continuous functions on $[a,b]$ which coincides with the space of primitives of Lebesgue measurable functions i.e.

$$h(t) \in AC[a,b] \iff h(t) = \int_a^t \hat{h}(s)dt + c \tag{4.3.2}$$

where $\hat{h}(s) \in L_1(a,b)$.

In the following theorem, we characterize the space $AC^m_{\delta^\Phi,\mu}[a,b]$.

Theorem 4.3.1. *The space $AC^m_{\delta^\Phi,\mu}[a,b]$ consists of those and only those functions $g(t)$, which are represented in the form*

$$g(t) = \left\{\Phi(t)\right\}^{-\mu} \left\{ \frac{1}{(m-1)!}\int_a^t \left(\log\frac{\Phi(t)}{\Phi(s)}\right)^{m-1} \hat{h}(s)ds + \sum_{k=0}^{m-1} c_k \left(\log\frac{\Phi(t)}{\Phi(a)}\right)^k \right\}. \tag{4.3.3}$$

Proof. Let $g(t) \in AC^m_{\delta^\Phi,\mu}[a,b]$. Then by Eqs. (4.3.1) and (4.3.2), we have

$$\delta^{m-1,\Phi}\Big\{ \left\{\Phi(t)\right\}^\mu g(t)\Big\} = \int_a^t \hat{h}(s)ds + c_{m-1} \tag{4.3.4}$$

i.e.

$$\frac{d}{dt}\left[\delta^{m-2,\Phi}\Big\{ \left\{\Phi(t)\right\}^\mu g(t)\Big\}\right] = \frac{\Phi'(t)}{\Phi(t)}\int_a^t \hat{h}(s)ds + \frac{\Phi'(t)}{\Phi(t)}c_{m-1}.$$

Therefore, we have

$$\delta^{m-2,\Phi}\Big\{ \left\{\Phi(t)\right\}^\mu g(t)\Big\} = \int_a^t \log\frac{\Phi(t)}{\Phi(s)}\hat{h}(s)ds + c_{m-2} + c_{m-1}\log\frac{\Phi(t)}{\Phi(a)}.$$

Repeating this procedure i times, we get

$$\delta^{m-i,\Phi}\Big\{ \left\{\Phi(t)\right\}^\mu g(t)\Big\} = \frac{1}{(i-1)!}\int_a^t \left(\log\frac{\Phi(t)}{\Phi(s)}\right)^{i-1} \hat{h}(s)ds + \sum_{k=0}^{i-1} c_{m+k-i}\left(\log\frac{\Phi(t)}{\Phi(a)}\right)^k. \tag{4.3.5}$$

Substituting $i = m$ in Eq. (4.3.5), we get Eq. (4.3.3).

Conversely, let $g(t)$ be represented by the Eq. (4.3.3), i.e.

$$\left\{\Phi(t)\right\}^\mu g(t) = \frac{1}{(m-1)!}\int_a^t \left(\log\frac{\Phi(t)}{\Phi(s)}\right)^{m-1} \hat{h}(s)ds + \sum_{k=0}^{m-1} c_k \left(\log\frac{\Phi(t)}{\Phi(a)}\right)^k.$$

Applying $\delta^{i,\Phi}$ on both sides of the above equation, we get

$$\begin{aligned}\delta^{i,\Phi}\Big\{\left\{\Phi(t)\right\}^{\mu} g(t)\Big\} =& \frac{1}{(m-i-1)!}\int_a^t \left(\log\frac{\Phi(t)}{\Phi(s)}\right)^{m-i-1}\hat{h}(s)ds \\ &+\sum_{k=i}^{m-1}\frac{k!c_k}{(k-i)!}\left(\log\frac{\Phi(t)}{\Phi(a)}\right)^{k-i}.\end{aligned}$$

For $i = m-1$, we obtain

$$\delta^{m-1,\Phi}\Big\{\left\{\Phi(t)\right\}^{\mu} g(t)\Big\} = \int_a^t \hat{h}(s)ds + c$$

where $c = (m-1)!c_{m-1}$ and hence, according to Eq. (4.3.2), we deduce that $g(t) \in AC^m_{\delta^{\Phi},\mu}[a,b]$. Thus, the proof of the theorem is complete. □

Remark 4.3.2. From Eq. (4.3.4), we have $\hat{h}(s) = g'_{m-1}(s)$. Furthermore, it follows from our proof that $c_k = \frac{g_k(a)}{k!}$ where $k = 0,1,...,m-1$ and $g_k(t) = \delta^{k,\Phi}\Big\{(\Phi(t))^{\mu}g(t)\Big\}$. Hence, $g(t)$ can be represented as

$$g(t) = \left\{\Phi(t)\right\}^{-\mu}\left\{\int_a^t \left(\log\frac{\Phi(t)}{\Phi(s)}\right)^{m-1}\frac{g'_{m-1}(s)}{(m-1)!}ds + \sum_{k=0}^{m-1}\frac{g_k(a)}{k!}\left(\log\frac{\Phi(t)}{\Phi(a)}\right)^{k}\right\}. \tag{4.3.6}$$

Now we give proof of a result giving sufficient conditions for the existence of the Φ-Hadamard type fractional derivative.

Theorem 4.3.3. *Let $\eta > 0$, $m = [\eta]+1$, $\mu \in \mathbb{R}$ and $g(t) \in AC^m_{\delta^{\Phi},\mu}[a,b]$. Then the Φ-Hadamard type fractional derivative ${}_H\mathfrak{D}^{\eta,\Phi}_{a^+,\mu}g$ exists almost everywhere on $[a,b]$ and may be represented in the form*

$$\begin{aligned}{}_H\mathfrak{D}^{\eta,\Phi}_{a^+,\mu}g(t) = \left\{\Phi(t)\right\}^{-\mu}\Bigg\{&\int_a^t \left(\log\frac{\Phi(t)}{\Phi(s)}\right)^{m-\eta-1}\frac{g'_{m-1}(s)}{\mathfrak{d}(m-\eta)}ds \\ &+\sum_{k=0}^{m-1}\frac{g_k(a)}{\mathfrak{d}(k-\eta+1)}\left(\log\frac{\Phi(t)}{\Phi(a)}\right)^{k-\eta}\Bigg\}.\end{aligned} \tag{4.3.7}$$

Proof. Since $g(t) \in AC^m_{\delta^\Phi,\mu}[a,b]$, by substituting the form for $g(t)$ given in Eq. (4.3.6) into the definition of the ${}_H\mathfrak{D}^{\eta,\Phi}_{a^+,\mu}$, we get

$$
{}_H\mathfrak{D}^{\eta,\Phi}_{a^+,\mu} g(t) = \{\Phi(t)\}^{-\mu} \delta^{m,\Phi} \frac{1}{\mathfrak{d}(m-\eta)} \int_a^t \left(\log \frac{\Phi(t)}{\Phi(s)}\right)^{m-\eta-1} \left\{ \sum_{k=0}^{m-1} \frac{g_k(a)}{k!} \left(\log \frac{\Phi(t)}{\Phi(a)}\right)^k \right.
$$

$$
\left. + \int_a^s \left(\log \frac{\Phi(s)}{\Phi(u)}\right)^{m-1} \frac{g'_{m-1}(u)}{(m-1)!} du \right\} \frac{\Phi'(s)}{\Phi(s)} ds. \tag{4.3.8}
$$

Considering the term involving the double integral in the above equation and by using the Dirichlet formula for the change of order of the integration, we obtain

$$
\int_a^t \left(\log \frac{\Phi(t)}{\Phi(s)}\right)^{m-\eta-1} \int_a^s \left(\log \frac{\Phi(s)}{\Phi(u)}\right)^{m-1} \frac{g'_{m-1}(u)}{(m-1)!} du \frac{\Phi'(s)}{\Phi(s)} ds
$$

$$
= \int_a^t \frac{g'_{m-1}(u)}{(m-1)!} \int_u^t \left(\log \frac{\Phi(t)}{\Phi(s)}\right)^{m-\eta-1} \left(\log \frac{\Phi(s)}{\Phi(u)}\right)^{m-1} \frac{\Phi'(s)}{\Phi(s)} ds du.
$$

Evaluating the inner integral by making the substitution $y = \frac{log\left(\frac{\Phi(s)}{\Phi(u)}\right)}{log\left(\frac{\Phi(t)}{\Phi(u)}\right)}$ and by using the definition of the Beta function, we find

$$
\int_a^t \left(\log \frac{\Phi(t)}{\Phi(s)}\right)^{m-\eta-1} \int_a^s \left(\log \frac{\Phi(s)}{\Phi(u)}\right)^{m-1} \frac{g'_{m-1}(u)}{(m-1)!} du \frac{\Phi'(s)}{\Phi(s)} ds
$$

$$
= \frac{\mathfrak{d}(m)\mathfrak{d}(m-\eta)}{\mathfrak{d}(2m-\eta)} \int_a^t \frac{g'_{m-1}(u)}{(m-1)!} \left(\log \frac{\Phi(t)}{\Phi(u)}\right)^{2m-\eta-1} du.
$$

By substituting this relation into Eq. (4.3.8) and taking $\delta^{m,\Phi}$-differentiation, we obtain the required result. □

Remark 4.3.4. The result for Φ-Hadamard type fractional differential operator in Theorem 4.3.3 is the analogue of the classical RL fractional differential operator [38]. Moreover, when we consider the case that $\Phi(t) = t$, we can make an analogous conclusion for the Hadamard type fractional differential operator. Considering $\Phi(t) = t$ and $\mu = 0$, Theorem 4.3.3 can be seen to hold true in the settings of Hadamard fractional differential operator [24].

4.4 Semi-group and reciprocal properties of Φ-Hadamard type fractional operators

In this section, we give proof of the semi-group and reciprocal properties of the Φ-Hadamard type fractional integral operators.

Theorem 4.4.1. *Let $\eta > 0$, $\zeta > 0$, $1 \leq p \leq \infty$, $0 < a < b < \infty$, Φ be a non-negative increasing function and $\mu, c \in \mathbb{R}$ be such that $\mu \geq c$. Then for $f \in X^p_{\Phi,c}(a,b)$, the following property holds*

$$ {}_H\mathfrak{J}^{\eta,\Phi}_{a^+,\mu}\, {}_H\mathfrak{J}^{\zeta,\Phi}_{a^+,\mu} f(t) = {}_H\mathfrak{J}^{\eta+\zeta,\Phi}_{a^+,\mu} f(t). \tag{4.4.1}$$

Proof. Using the definition of Φ-Hadamard type fractional integral ${}_H\mathfrak{J}^{\eta,\Phi}_{a^+,\mu}$ and the Dirichlet formula, we obtain

$$\begin{aligned} {}_H\mathfrak{J}^{\eta,\Phi}_{a^+,\mu}\, {}_H\mathfrak{J}^{\zeta,\Phi}_{a^+,\mu} f(t) =& \frac{1}{\mathfrak{d}(\eta)\mathfrak{d}(\zeta)} \int_a^t \left(\frac{\Phi(s)}{\Phi(t)}\right)^{\mu} \left(\log\frac{\Phi(t)}{\Phi(s)}\right)^{\eta-1} \\ &\times \left\{ \int_a^s \left(\frac{\Phi(u)}{\Phi(s)}\right)^{\mu} \left(\log\frac{\Phi(s)}{\Phi(u)}\right)^{\zeta-1} f(u)\frac{\Phi'(u)du}{\Phi(u)} \right\} \frac{\Phi'(s)ds}{\Phi(s)} \\ =& \frac{1}{\mathfrak{d}(\eta)\mathfrak{d}(\zeta)} \int_a^t \left(\frac{\Phi(u)}{\Phi(t)}\right)^{\mu} f(u) \left\{ \int_u^t \left(\log\frac{\Phi(t)}{\Phi(s)}\right)^{\eta-1} \right. \\ &\left. \times \left(\log\frac{\Phi(s)}{\Phi(u)}\right)^{\zeta-1} \frac{\Phi'(s)ds}{\Phi(s)} \right\} \frac{\Phi'(u)du}{\Phi(u)}. \end{aligned}$$

Evaluating the inner integral by making the substitution $y = \frac{\log\left(\frac{\Phi(s)}{\Phi(u)}\right)}{\log\left(\frac{\Phi(t)}{\Phi(u)}\right)}$ and using the definition of the Beta function, we get

$$\begin{aligned} {}_H\mathfrak{J}^{\eta,\Phi}_{a^+,\mu}\, {}_H\mathfrak{J}^{\zeta,\Phi}_{a^+,\mu} f(t) =& \frac{1}{\mathfrak{d}(\eta)\mathfrak{d}(\zeta)} \int_a^t \left(\frac{\Phi(u)}{\Phi(t)}\right)^{\mu} f(u) \left\{ \left(\log\frac{\Phi(t)}{\Phi(u)}\right)^{\eta+\zeta-1} \right. \\ &\left. \times \int_0^1 y^{\zeta-1}(1-y)^{\eta-1}dy \right\} \frac{\Phi'(u)du}{\Phi(u)} \\ =& \frac{1}{\mathfrak{d}(\eta)\mathfrak{d}(\zeta)} \int_a^t \left(\frac{\Phi(u)}{\Phi(t)}\right)^{\mu} f(u) \left\{ \frac{\mathfrak{d}(\eta)\mathfrak{d}(\zeta)}{\mathfrak{d}(\eta+\zeta)} \left(\log\frac{\Phi(t)}{\Phi(u)}\right)^{\eta+\zeta-1} \right\} \frac{\Phi'(u)du}{\Phi(u)} \\ =& \frac{1}{\mathfrak{d}(\eta+\zeta)} \int_a^t \left(\frac{\Phi(u)}{\Phi(t)}\right)^{\mu} \left(\log\frac{\Phi(t)}{\Phi(u)}\right)^{\eta+\zeta-1} f(u)\frac{\Phi'(u)du}{\Phi(u)} \\ =& {}_H\mathfrak{J}^{\eta,\Phi}_{a^+,\mu}\, {}_H\mathfrak{J}^{\zeta,\Phi}_{a^+,\mu} f(t). \end{aligned}$$

Hence, the proof of the Theorem 4.4.1 is complete. $\square$

Now we consider the composition between the Φ-Hadamard type fractional derivative ${}_H\mathfrak{D}^{\zeta,\Phi}_{a^+,\mu}$ of order ζ and fractional integral ${}_H\mathfrak{J}^{\eta,\Phi}_{a^+,\mu}$ of order η.

Theorem 4.4.2. *Let $\eta > \zeta > 0$, $1 \leq p \leq \infty$, $0 < a < b < \infty$, Φ be a non-negative increasing function and $\mu, c \in \mathbb{R}$ be such that $\mu \geq c$. Then for $f \in X^p_{\Phi,c}(a,b)$, the following property holds. That is,*

$$ {}_H\mathfrak{D}^{\zeta,\Phi}_{a^+,\mu}\,{}_H\mathfrak{J}^{\eta,\Phi}_{a^+,\mu} f(t) = {}_H\mathfrak{J}^{\eta-\zeta,\Phi}_{a^+,\mu} f(t). \tag{4.4.2} $$

Particularly, if $\zeta = n \in \mathbb{N}$, then

$$ {}_H\mathfrak{D}^{n,\Phi}_{a^+,\mu}\,{}_H\mathfrak{J}^{\eta,\Phi}_{a^+,\mu} f(t) = {}_H\mathfrak{J}^{\eta-n,\Phi}_{a^+,\mu} f(t). \tag{4.4.3} $$

Proof. Let $n-1 < \zeta \leq n$, such that $n \in \mathbb{N}$. If $\zeta = n$, then

$$ {}_H\mathfrak{D}^{n,\Phi}_{a^+,\mu} g(t) = \{\Phi(t)\}^{-\mu} \left(\frac{\Phi(t)}{\Phi'(t)}\frac{d}{dt}\right)^n \{\Phi(t)\}^{\mu} g(t). \tag{4.4.4} $$

Using the definition of ${}_H\mathfrak{J}^{\eta,\Phi}_{a^+,\mu}$ and Eq. (4.4.4), we obtain

$$ {}_H\mathfrak{D}^{n,\Phi}_{a^+,\mu}\,{}_H\mathfrak{J}^{\eta,\Phi}_{a^+,\mu} f(t) = \{\Phi(t)\}^{-\mu} \left(\frac{\Phi(t)}{\Phi'(t)}\frac{d}{dt}\right)^{n-1} \left(\frac{\Phi(t)}{\Phi'(t)}\frac{d}{dt}\right) \frac{1}{\mathfrak{d}(\eta)} \int_a^t \{\Phi(u)\}^{\mu} \times \left(\log\frac{\Phi(t)}{\Phi(u)}\right)^{\eta-1} f(u) \frac{\Phi'(u)du}{\Phi(u)}. $$

By application of the Leibniz rule and by using the relation $\mathfrak{d}(\eta+1) = \eta\mathfrak{d}(\eta)$, we have

$$ \begin{aligned} {}_H\mathfrak{D}^{n,\Phi}_{a^+,\mu}\,{}_H\mathfrak{J}^{\eta,\Phi}_{a^+,\mu} f(t) =& \{\Phi(t)\}^{-\mu} \left(\frac{\Phi(t)}{\Phi'(t)}\frac{d}{dt}\right)^{n-1} \frac{\{\Phi(t)\}^{\mu}}{\mathfrak{d}(\eta-1)} \int_a^t \left(\frac{\Phi(u)}{\Phi(t)}\right)^{\mu} \left(\log\frac{\Phi(t)}{\Phi(u)}\right)^{\eta-2} \\ & \times f(u)\frac{\Phi'(u)du}{\Phi(u)} \\ =& \{\Phi(t)\}^{-\mu} \left(\frac{\Phi(t)}{\Phi'(t)}\frac{d}{dt}\right)^{n-1} \{\Phi(t)\}^{\mu}\,{}_H\mathfrak{J}^{\eta-1,\Phi}_{a^+,\mu} f(t). \end{aligned} $$

Repeating this procedure k times, where $1 \leq k \leq n$, we find

$$ {}_H\mathfrak{D}^{n,\Phi}_{a^+,\mu}\,{}_H\mathfrak{J}^{\eta,\Phi}_{a^+,\mu} f(t) = \{\Phi(t)\}^{-\mu} \left(\frac{\Phi(t)}{\Phi'(t)}\frac{d}{dt}\right)^{n-k} \{\Phi(t)\}^{\mu}\,{}_H\mathfrak{J}^{\eta-k,\Phi}_{a^+,\mu} f(t) $$

and the particular case follows for $k = n$.

If $n-1<\zeta<n$, then Eq. (4.4.2) follows from Eqs. (4.4.1) and (4.4.3):

$$ {}_H\mathfrak{D}^{\zeta,\Phi}_{a^+,\mu}\,{}_H\mathfrak{I}^{\eta,\Phi}_{a^+,\mu}f(t) = {}_H\mathfrak{D}^{n,\Phi}_{a^+,\mu}\,{}_H\mathfrak{I}^{n-\zeta,\Phi}_{a^+,\mu}\;{}_H\mathfrak{I}^{\eta,\Phi}_{a^+,\mu}f(t) = {}_H\mathfrak{D}^{n,\Phi}_{a^+,\mu}\,{}_H\mathfrak{I}^{n+\eta-\zeta,\Phi}_{a^+,\mu} = {}_H\mathfrak{I}^{\eta-\zeta,\Phi}_{a^+,\mu}f(t). \tag{4.4.5} $$

Hence, the theorem has been proved. □

Theorem 4.4.3. *Let $\eta>0$, $m-1<\eta\le m$, $0<a<b<\infty$, $\mu\in\mathbb{R}$, Φ be a non-negative increasing function. Assuming f such that ${}_H\mathfrak{I}^{m-\eta,\Phi}_{a^+,\mu}f\in AC^m_{\delta^\Phi,\mu}[a,b]$. Then,*

$$ {}_H\mathfrak{I}^{\eta,\Phi}_{a^+,\mu}\,{}_H\mathfrak{D}^{\eta,\Phi}_{a^+,\mu}f(t) = f(t) - \left(\frac{\Phi(a)}{\Phi(t)}\right)^{\mu}\sum_{k=1}^{m}\frac{1}{\mathfrak{d}(\eta-k+1)}\left(\log\frac{\Phi(t)}{\Phi(a)}\right)^{\eta-k}\lim_{s\to a^+}{}_H\mathfrak{D}^{\eta-k,\Phi}_{a^+,\mu}f(s). $$

Particularly, for $0<\eta<1$ we have

$$ {}_H\mathfrak{I}^{\eta,\Phi}_{a^+,\mu}\,{}_H\mathfrak{D}^{\eta,\Phi}_{a^+,\mu}f(t) = f(t) - \frac{1}{\mathfrak{d}(\eta)}\left(\frac{\Phi(a)}{\Phi(t)}\right)^{\mu}\left(\log\frac{\Phi(t)}{\Phi(a)}\right)^{\eta-1}\lim_{s\to a^+}{}_H\mathfrak{I}^{1-\eta,\Phi}_{a^+,\mu}f(s). $$

Proof. Using the Leibniz rule, the following relation can be established.

$$ \begin{aligned} &\{\Phi(t)\}^{-\mu}\,\delta^{1,\Phi}\,\{\Phi(t)\}^{\mu}\int_a^t\left(\frac{\Phi(s)}{\Phi(t)}\right)^{\mu}\left(\log\frac{\Phi(t)}{\Phi(s)}\right)^{\eta}{}_H\mathfrak{D}^{\eta,\Phi}_{a^+,\mu}f(t)\frac{\Phi'(s)ds}{\Phi(s)} \\ &\qquad= \eta\int_a^t\left(\frac{\Phi(s)}{\Phi(t)}\right)^{\mu}\left(\log\frac{\Phi(t)}{\Phi(s)}\right)^{\eta-1}{}_H\mathfrak{D}^{\eta,\Phi}_{a^+,\mu}f(s)\frac{\Phi'(s)ds}{\Phi(s)}. \end{aligned} \tag{4.4.6} $$

By the definition of ${}_H\mathfrak{I}^{\eta,\Phi}_{a^+,\mu}$, we have

$$ {}_H\mathfrak{I}^{\eta,\Phi}_{a^+,\mu}\,{}_H\mathfrak{D}^{\eta,\Phi}_{a^+,\mu}f(t) = \frac{1}{\mathfrak{d}(\eta)}\int_a^t\left(\frac{\Phi(s)}{\Phi(t)}\right)^{\mu}\left(\log\frac{\Phi(t)}{\Phi(s)}\right)^{\eta-1}{}_H\mathfrak{D}^{\eta,\Phi}_{a^+,\mu}f(s)\frac{\Phi'(s)ds}{\Phi(s)}. \tag{4.4.7} $$

From Eq. (4.4.6) and (4.4.7), we get

$$ \begin{aligned} {}_H\mathfrak{I}^{\eta,\Phi}_{a^+,\mu}\,{}_H\mathfrak{D}^{\eta,\Phi}_{a^+,\mu}f(t) =& \frac{1}{\mathfrak{d}(\eta+1)}\{\Phi(t)\}^{-\mu}\,\delta^{1,\Phi}\,\{\Phi(t)\}^{\mu}\int_a^t\left(\frac{\Phi(s)}{\Phi(t)}\right)^{\mu} \\ &\times\left(\log\frac{\Phi(t)}{\Phi(s)}\right)^{\eta}{}_H\mathfrak{D}^{\eta,\Phi}_{a^+,\mu}f(s)\frac{\Phi'(s)ds}{\Phi(s)}. \end{aligned} \tag{4.4.8} $$

From the definition of ${}_H\mathfrak{D}^{\eta,\Phi}_{a^+,\mu}$ and Eq. (4.4.8), we find

$$ \begin{aligned} {}_H\mathfrak{I}^{\eta,\Phi}_{a^+,\mu}\,{}_H\mathfrak{D}^{\eta,\Phi}_{a^+,\mu}f(t) =& \frac{1}{\mathfrak{d}(\eta+1)}\{\Phi(t)\}^{-\mu}\,\delta^{1,\Phi}\,\{\Phi(t)\}^{\mu}\int_a^t\left(\frac{\Phi(s)}{\Phi(t)}\right)^{\mu}\left(\log\frac{\Phi(t)}{\Phi(s)}\right)^{\eta} \\ &\times\{\Phi(t)\}^{-\mu}\,\delta^{1,\Phi}\,\{\Phi(t)\}^{\mu}\,{}_H\mathfrak{D}^{m-1,\Phi}_{a^+,\mu}\,{}_H\mathfrak{I}^{m-\eta,\Phi}_{a^+,\mu}f(s)\frac{\Phi'(s)ds}{\Phi(s)}. \end{aligned} $$

Applying integration by parts, we have

$$
\begin{aligned}
{}_{H}\mathfrak{J}^{\eta,\Phi}_{a^+,\mu}\,{}_{H}\mathfrak{D}^{\eta,\Phi}_{a^+,\mu}f(t) =& -\frac{1}{\mathfrak{d}(\eta)}\left(\frac{\Phi(a)}{\Phi(t)}\right)^{\mu}\left(\log\frac{\Phi(t)}{\Phi(a)}\right)^{\eta-1}\lim_{s\to a^+}{}_{H}\mathfrak{D}^{m-1,\Phi}_{a^+,\mu}\,{}_{H}\mathfrak{J}^{m-\eta,\Phi}_{a^+,\mu}f(s)\\
&+\frac{1}{\mathfrak{d}(\eta)}\left\{\Phi(t)\right\}^{-\mu}\delta^{1,\Phi}\left\{\Phi(t)\right\}^{\mu}\int_a^t\left(\frac{\Phi(s)}{\Phi(t)}\right)^{\mu}\left(\log\frac{\Phi(t)}{\Phi(s)}\right)^{\eta-1}\\
&\times\left\{\Phi(t)\right\}^{-\mu}\delta^{1,\Phi}\left\{\Phi(t)\right\}^{\mu}\,{}_{H}\mathfrak{D}^{m-2,\Phi}_{a^+,\mu}\,{}_{H}\mathfrak{J}^{m-\eta,\Phi}_{a^+,\mu}f(s)\frac{\Phi'(s)ds}{\Phi(s)}.
\end{aligned}
$$

By continuation in this manner, we get

$$
\begin{aligned}
{}_{H}\mathfrak{J}^{\eta,\Phi}_{a^+,\mu}\,{}_{H}\mathfrak{D}^{\eta,\Phi}_{a^+,\mu}f(t) =& -\left(\frac{\Phi(a)}{\Phi(t)}\right)^{\mu}\sum_{k=1}^{m}\frac{1}{\mathfrak{d}(\eta-k+1)}\left(\log\frac{\Phi(t)}{\Phi(a)}\right)^{\eta-k}\lim_{s\to a^+}{}_{H}\mathfrak{D}^{\eta-k,\Phi}_{a^+,\mu}f(s)\\
&+\frac{1}{\mathfrak{d}(\eta-m+1)}\left\{\Phi(t)\right\}^{-\mu}\delta^{1,\Phi}\left\{\Phi(t)\right\}^{\mu}\int_a^t\left(\frac{\Phi(s)}{\Phi(t)}\right)^{\mu}\\
&\times\left(\log\frac{\Phi(t)}{\Phi(s)}\right)^{\eta-m}\,{}_{H}\mathfrak{J}^{m-\eta,\Phi}_{a^+,\mu}f(s)\frac{\Phi'(s)ds}{\Phi(s)}
\end{aligned}
$$

where

$$
\begin{aligned}
f(t) =&\frac{1}{\mathfrak{d}(\eta-m+1)}\left\{\Phi(t)\right\}^{-\mu}\delta^{1,\Phi}\left\{\Phi(t)\right\}^{\mu}\\
&\times\int_a^t\left(\frac{\Phi(s)}{\Phi(t)}\right)^{\mu}\left(\log\frac{\Phi(t)}{\Phi(s)}\right)^{\eta-m}\,{}_{H}\mathfrak{J}^{m-\eta,\Phi}_{a^+,\mu}f(s)\frac{\Phi'(s)ds}{\Phi(s)}.
\end{aligned}
$$

Hence, we obtain our desired result. □

Remark 4.4.4. Taking into consideration the case that $\Phi(t)=t$, we have analogous conclusions for the Hadamard type fractional operators. Furthermore, considering $\Phi(t)=t$ and $\mu=0$, all of the theorems in this section are seen to hold true in the settings of the Hadamard fractional operators [26, 27].

4.5 Fractional integration by parts formulas

In the proofs of Lemma 4.1.1 and Theorem 4.4.3, we use one of the most important techniques of classical calculus: integration by parts. In this section we derive several formulas of integration by parts in the settings of Φ-Hadamard type fractional operators.

Lemma 4.5.1. *Let $\eta > 0$, $m-1 < \eta \le m$, $p \ge 1$, $q \ge 1$, and $\frac{1}{p}+\frac{1}{q} \le 1+\eta$ ($p \ne 1$ and $q \ne 1$ in the case when $\frac{1}{p}+\frac{1}{q} = 1+\eta$). If $f(t) \in X^p_{\Phi,c}(a,b)$ and $g(t) \in X^q_{\Phi,c}(a,b)$, then*

$$\int_a^b \frac{\Phi'(t)}{\Phi(t)} f(t) {}_H\mathfrak{J}^{\eta,\Phi}_{a^+,\mu} g(t)dt = \int_a^b \frac{\Phi'(t)}{\Phi(t)} g(t) {}_H\mathfrak{J}^{\eta,\Phi}_{b^-,\mu} f(t)dt. \tag{4.5.1}$$

Proof. Using the definition of ${}_H\mathfrak{J}^{\eta,\Phi}_{a^+,\mu}$ and the Dirichlet formula, we have

$$\begin{aligned}\int_a^b \frac{\Phi'(t)}{\Phi(t)} f(t) {}_H\mathfrak{J}^{\eta,\Phi}_{a^+,\mu} g(t)dt =& \frac{1}{\mathfrak{d}(\eta)} \int_a^b \frac{\Phi'(t)}{\Phi(t)} f(t) \int_a^t \left(\frac{\Phi(s)}{\Phi(t)}\right)^{\mu} \left(\log\frac{\Phi(t)}{\Phi(s)}\right)^{\eta-1} \\ & \times g(s)\frac{\Phi'(s)ds}{\Phi(s)}dt \\ =& \frac{1}{\mathfrak{d}(\eta)} \int_a^b \frac{\Phi'(s)}{\Phi(s)} g(s) \int_s^b \left(\frac{\Phi(s)}{\Phi(t)}\right)^{\mu} \left(\log\frac{\Phi(t)}{\Phi(s)}\right)^{\eta-1} \\ & \times f(t)\frac{\Phi'(t)dt}{\Phi(t)}ds \\ =& \int_a^b \frac{\Phi'(t)}{\Phi(t)} g(t) {}_H\mathfrak{J}^{\eta,\Phi}_{b^-,\mu} f(t)dt.\end{aligned}$$

Hence, our result is proved. □

Theorem 4.5.1. *Assume that $\eta > 0$, $m-1 < \eta \le m$, $f(t) \in AC^m_{\delta^\Phi,\mu}[a,b]$ and $g(t) \in X^p_{\Phi,c}(a,b)$ where $1 \le p \le \infty$. Then, the relation below holds*

$$\begin{aligned}\int_a^b f(t) {}_H\mathfrak{D}^{\eta,\Phi}_{a^+,\mu} g(t)dt =& \int_a^b \frac{\Phi'(t)}{\Phi(t)} g(t) {}^C_H\mathfrak{D}^{\eta,\Phi}_{b^-,\mu} \left\{\frac{\Phi(t)}{\Phi'(t)} f(t)\right\} dt \\ & + \sum_{k=0}^{m-1} {}_H\mathfrak{D}^{k,\Phi}_{b^-,\mu} \left\{\frac{\Phi(t)}{\Phi'(t)} f(t)\right\} {}_H\mathfrak{J}^{k-\eta+1,\Phi}_{a^+,\mu} g(t)\Big|_a^b.\end{aligned}$$

Proof. Using the definition of ${}_H\mathfrak{D}^{\eta,\Phi}_{a^+,\mu}$, we have

$$\begin{aligned}\int_a^b f(t) {}_H\mathfrak{D}^{\eta,\Phi}_{a^+,\mu} g(t)dt =& \int_a^b f(t) {}_H\mathfrak{D}^{m,\Phi}_{a^+,\mu} {}_H\mathfrak{J}^{m-\eta,\Phi}_{a^+,\mu} g(t)dt \\ =& \int_a^b f(t) \left\{\Phi(t)\right\}^{-\mu} \frac{\Phi(t)}{\Phi'(t)}\frac{d}{dt}\Bigg\{ \left(\frac{\Phi(t)}{\Phi'(t)}\frac{d}{dt}\right)^{m-1} \\ & \times \left\{\Phi(t)\right\}^{\mu} {}_H\mathfrak{J}^{m-\eta,\Phi}_{a^+,\mu} g(t)\Bigg\}dt\end{aligned}$$

Using integration by parts, we find

$$\begin{aligned}\int_a^b f(t)\,{}_H\mathfrak{D}^{\eta,\Phi}_{a^+,\mu}g(t)dt &= f(t)\left\{\Phi(t)\right\}^{-\mu}\frac{\Phi(t)}{\Phi'(t)}\left(\frac{\Phi(t)}{\Phi'(t)}\frac{d}{dt}\right)^{m-1}\left\{\left\{\Phi(t)\right\}^{\mu}\,{}_H\mathfrak{J}^{m-\eta,\Phi}_{a^+,\mu}g(t)\right\}\Big|_a^b \\ &\quad-\int_a^b\left\{\left(\frac{\Phi(t)}{\Phi'(t)}\frac{d}{dt}\right)^{m-1}\left\{\Phi(t)\right\}^{\mu}\,{}_H\mathfrak{J}^{m-\eta,\Phi}_{a^+,\mu}g(t)\right\} \\ &\quad\times\frac{d}{dt}\left\{f(t)\left\{\Phi(t)\right\}^{-\mu}\frac{\Phi(t)}{\Phi'(t)}\right\}dt \\ &= \frac{\Phi(t)}{\Phi'(t)}f(t)\,{}_H\mathfrak{D}^{m-1,\Phi}_{a^+,\mu}\,{}_H\mathfrak{J}^{m-\eta,\Phi}_{a^+,\mu}g(t)\Big|_a^b \\ &\quad+\int_a^b\frac{\Phi'(t)}{\Phi(t)}\,{}_H\mathfrak{D}^{m-1,\Phi}_{a^+,\mu}\,{}_H\mathfrak{J}^{m-\eta,\Phi}_{a^+,\mu}g(t)\,{}_H\mathfrak{D}^{1,\Phi}_{b^-,\mu}\left\{\frac{\Phi(t)}{\Phi'(t)}f(t)\right\}dt.\end{aligned}$$

Again by application of integration by parts, we get

$$\begin{aligned}\int_a^b f(t)\,{}_H\mathfrak{D}^{\eta,\Phi}_{a^+,\mu}g(t)dt =&\frac{\Phi(t)}{\Phi'(t)}f(t)\,{}_H\mathfrak{D}^{m-1,\Phi}_{a^+,\mu}\,{}_H\mathfrak{J}^{m-\eta,\Phi}_{a^+,\mu}g(t)\Big|_a^b \\ &+{}_H\mathfrak{D}^{1,\Phi}_{b^-,\mu}\left\{\frac{\Phi(t)}{\Phi'(t)}f(t)\right\}{}_H\mathfrak{D}^{m-2,\Phi}_{a^+,\mu}\,{}_H\mathfrak{J}^{m-\eta,\Phi}_{a^+,\mu}g(t)\Big|_a^b \\ &+\int_a^b\frac{\Phi'(t)}{\Phi(t)}\,{}_H\mathfrak{D}^{m-2,\Phi}_{a^+,\mu}\,{}_H\mathfrak{J}^{m-\eta,\Phi}_{a^+,\mu}g(t)\,{}_H\mathfrak{D}^{2,\Phi}_{b^-,\mu}\left\{\frac{\Phi(t)}{\Phi'(t)}f(t)\right\}dt.\end{aligned}$$

Continuing in this manner, we get

$$\begin{aligned}\int_a^b f(t)\,{}_H\mathfrak{D}^{\eta,\Phi}_{a^+,\mu}g(t)dt =&\sum_{k=0}^{m-1}{}_H\mathfrak{D}^{k,\Phi}_{b^-,\mu}\left\{\frac{\Phi(t)}{\Phi'(t)}f(t)\right\}{}_H\mathfrak{J}^{k-\eta+1,\Phi}_{a^+,\mu}g(t)\Big|_a^b \\ &+\int_a^b\frac{\Phi'(t)}{\Phi(t)}\,{}_H\mathfrak{J}^{m-\eta,\Phi}_{a^+,\mu}g(t)\,{}_H\mathfrak{D}^{m,\Phi}_{b^-,\mu}\left\{\frac{\Phi(t)}{\Phi'(t)}f(t)\right\}dt.\end{aligned}$$

Using Lemma 4.5.1, we have

$$\begin{aligned}\int_a^b f(t)\,{}_H\mathfrak{D}^{\eta,\Phi}_{a^+,\mu}g(t)dt =&\sum_{k=0}^{m-1}{}_H\mathfrak{D}^{k,\Phi}_{b^-,\mu}\left\{\frac{\Phi(t)}{\Phi'(t)}f(t)\right\}{}_H\mathfrak{J}^{k-\eta+1,\Phi}_{a^+,\mu}g(t)\Big|_a^b \\ &+\int_a^b\frac{\Phi'(t)}{\Phi(t)}g(t)\,{}_H\mathfrak{J}^{m-\eta,\Phi}_{b^-,\mu}\,{}_H\mathfrak{D}^{m,\Phi}_{b^-,\mu}\left\{\frac{\Phi(t)}{\Phi'(t)}f(t)\right\}dt.\end{aligned}$$

Finally by using the definition of ${}^C_H\mathfrak{D}^{\eta,\Phi}_{b^-,\mu}$, we get the required result. □

Theorem 4.5.2. *Assume that $\eta > 0$, $m-1 < \eta \leq m$, $f(t) \in AC^m_{\delta^\Phi,\mu}[a,b]$ and $g(t) \in X^p_{\Phi,c}(a,b)$ where $1 \leq p \leq \infty$. Then, the following relation holds*

$$\int_a^b f(t)\,{}_H\mathfrak{D}^{\eta,\Phi}_{b^-,\mu} g(t)dt = (-1)^m \int_a^b \frac{\Phi'(t)}{\Phi(t)} g(t)\,{}^C_H\mathfrak{D}^{\eta,\Phi}_{a^+,\mu} \left\{ \frac{\Phi(t)}{\Phi'(t)} f(t) \right\} dt$$
$$- \sum_{k=0}^{m-1} {}_H\mathfrak{D}^{k,\Phi}_{a^+,\mu} \left\{ \frac{\Phi(t)}{\Phi'(t)} f(t) \right\} {}_H\mathfrak{I}^{k-\eta+1,\Phi}_{b^-,\mu} g(t) \Big|_a^b.$$

Proof. It is easy to derive the required result by using the technique demonstrated in the previous result. So we have omitted the straightforward but lengthy details. □

Chapter 5

Φ-Laplace transform method and its applications to Φ-FDEs

In this chapter, we present some new properties and uniqueness of the Φ-Laplace transform. Moreover, we discuss the effectiveness of this generalized transform and make use of it for solving the ordinary and partial FDEs in the settings of Φ-RL, Φ-Caputo and Φ-Hilfer fractional derivatives.

5.1 The Φ-Laplace transform

In this section, we discuss a generalized integral transform, that has been introduced by Jarad and Abdeljawad [19], and which can be used to solve linear FDEs in the frame of Φ-RL, Φ-Caputo and Φ-Hilfer fractional derivatives. This new integral transform is the obvious generalization of classical Laplace transform. Throughout this thesis, we call it the Φ-Laplace transform. In the settings of Φ-fractional calculus, some new properties and uniqueness of the Φ-Laplace transform constitute part of this section.

Definition 5.1.1. Let $f : [0, \infty) \to \mathbb{R}$ be a real valued function and Φ be a non-negative increasing function such that $\Phi(0) = 0$. Then the Φ-Laplace transform of f is denoted by $\mathfrak{L}_\Phi \{f\}$ and is defined by

$$F(\nu) = \mathfrak{L}_\Phi \{f(t)\} = \int_0^\infty e^{-\nu\Phi(t)}\Phi'(t)f(t)dt \tag{5.1.1}$$

for all ν.

Definition 5.1.2. A function $f : [0, \infty) \to \mathbb{R}$ is of Φ-exponential order $c > 0$ if there exists a positive constant M such that for all $t > T$

$$\Big|f(t)\Big| \leq Me^{c\Phi(t)}.$$

Symbolically, we write

$$f(t) = \mathcal{O}(e^{c\Phi(t)}) \qquad \text{as} \quad t \to \infty.$$

Example 5.1.3. If $f(t) = 1$ for $t > 0$, then

$$\mathfrak{L}_\Phi \{1\} = \int_0^\infty e^{-\nu\Phi(t)}\Phi'(t)dt = \frac{1}{\nu}, \qquad \text{for} \quad \nu > 0. \tag{5.1.2}$$

Example 5.1.4. If $f(t) = e^{a\Phi(t)}$ where a is a constant, then

$$\mathfrak{L}_\Phi \left\{e^{a\Phi(t)}\right\} = \int_0^\infty e^{-(\nu-a)\Phi(t)}\Phi'(t)dt = \frac{1}{\nu - a}, \qquad \text{for} \quad \nu > a.$$

Example 5.1.5. If $f(t) = (\Phi(t))^n$ where $n \in \mathbb{N}$, then

$$\mathfrak{L}_\Phi \left\{(\Phi(t))^n\right\} = \frac{n!}{\nu^{n+1}}, \qquad \text{for} \quad \nu > 0.$$

Differentiating (5.1.2) with respect to ν, we get

$$\mathfrak{L}_\Phi \left\{\Phi(t)\right\} = \int_0^\infty e^{-\nu\Phi(t)}\Phi'(t)\Phi(t)dt = \frac{1}{\nu^2}. \tag{5.1.3}$$

Differentiation of (5.1.3) with respect to ν yields

$$\mathfrak{L}_\Phi \left\{(\Phi(t))^2\right\} = \int_0^\infty e^{-\nu\Phi(t)}\Phi'(t)(\Phi(t))^2dt = \frac{2}{\nu^3}.$$

Similarly, differentiating (5.1.2) with respect to ν, n times gives

$$\mathfrak{L}_\Phi \left\{(\Phi(t))^n\right\} = \frac{n!}{\nu^{n+1}}.$$

Remark 5.1.6. If $\eta > -1$ is a real number, then

$$\mathfrak{L}_\Phi \left\{(\Phi(t))^\eta\right\} = \frac{\mathfrak{d}(\eta + 1)}{\nu^{\eta+1}},$$

which can easily be shown by making a suitable substitution and afterwards using the definition of Gamma function.

In the following examples we calculate the Φ-Laplace transform of some special functions.

Example 5.1.7. Assume that $\mathrm{Re}(\eta) > 0$ and $\left|\frac{\lambda}{\nu^\eta}\right| < 1$. If $f(t) = \mathfrak{E}_\eta\Big(\lambda(\Phi(t))^\eta\Big)$ where $\mathfrak{E}_\eta$ denotes the Mittag-Leffler function (2.6.2), then

$$\begin{aligned}\mathfrak{L}_\Phi\left\{\mathfrak{E}_\eta\Big(\lambda(\Phi(t))^\eta\Big)\right\} &= \mathfrak{L}_\Phi\left\{\sum_{i=0}^{\infty}\frac{\lambda^i}{\mathfrak{d}(i\eta+1)}(\Phi(t))^{i\eta}\right\}\\ &= \sum_{i=0}^{\infty}\frac{\lambda^i}{\mathfrak{d}(i\eta+1)}\mathfrak{L}_\Phi\left\{(\Phi(t))^{i\eta}\right\}\\ &= \sum_{i=0}^{\infty}\frac{\lambda^i}{\mathfrak{d}(i\eta+1)}\frac{\mathfrak{d}(i\eta+1)}{\nu^{i\eta+1}}\\ &= \frac{1}{s}\sum_{i=0}^{\infty}\left(\frac{\lambda}{\nu^\eta}\right)^i = \frac{\nu^{\eta-1}}{\nu^\eta-\lambda}.\end{aligned}$$

Example 5.1.8. Assume that $\mathrm{Re}(\eta) > 0$ and $\left|\frac{\lambda}{\nu^\eta}\right| < 1$. If $f(t) = (\Phi(t))^{\eta-1}\mathfrak{E}_{\eta,\eta}\Big(\lambda(\Phi(t))^\eta\Big)$ where $\mathfrak{E}_{\eta,\eta}$ denotes the Wiman function or two parameter Mittag-Leffler function (2.6.3), then

$$\begin{aligned}\mathfrak{L}_\Phi\left\{(\Phi(t))^{\eta-1}\mathfrak{E}_{\eta,\eta}\Big(\lambda(\Phi(t))^\eta\Big)\right\} &= \mathfrak{L}_\Phi\left\{\sum_{i=0}^{\infty}\frac{\lambda^i}{\mathfrak{d}(i\eta+\eta)}(\Phi(t))^{i\eta+\eta-1}\right\}\\ &= \sum_{i=0}^{\infty}\frac{\lambda^i}{\mathfrak{d}(i\eta+\eta)}\mathfrak{L}_\Phi\left\{(\Phi(t))^{i\eta+\eta-1}\right\}\\ &= \sum_{i=0}^{\infty}\frac{\lambda^i}{\mathfrak{d}(i\eta+\eta)}\frac{\mathfrak{d}(i\eta+\eta)}{\nu^{i\eta+\eta}}\\ &= \frac{1}{\nu^\eta}\sum_{i=0}^{\infty}\left(\frac{\lambda}{\nu^\eta}\right)^i = \frac{1}{\nu^\eta-\lambda}.\end{aligned}$$

Example 5.1.9. Assume that $\mathrm{Re}(\eta) > 0$ and $\left|\frac{\lambda}{\nu^\eta}\right| < 1$. If $f(t) = (\Phi(t))^{\zeta-1}\mathfrak{E}^\gamma_{\eta,\zeta}\Big(\lambda(\Phi(t))^\eta\Big)$ where $\mathfrak{E}^\gamma_{\eta,\zeta}$ denotes the Prabhakar function (2.6.4), then by Definition 5.1.1 and the Binomial series, we have

$$\mathfrak{L}_\Phi\left\{(\Phi(t))^{\zeta-1}\mathfrak{E}^\gamma_{\eta,\zeta}\Big(\lambda(\Phi(t))^\eta\Big)\right\} = \mathfrak{L}_\Phi\left\{\sum_{i=0}^{\infty}\frac{\lambda^i\mathfrak{d}(\gamma+i)}{i!\mathfrak{d}(\eta i+\zeta)}(\Phi(t))^{\eta i+\zeta-1}\right\}$$

$$= \sum_{i=0}^{\infty} \frac{\lambda^i \mathfrak{d}(\gamma+i)}{i!\mathfrak{d}(\eta i+\zeta)} \mathfrak{L}_\Phi \left\{ (\Phi(t))^{\eta i+\zeta-1} \right\}$$
$$= \sum_{i=0}^{\infty} \frac{\lambda^i \mathfrak{d}(\gamma+i)}{i!\mathfrak{d}(\eta i+\zeta)} \frac{\mathfrak{d}(\eta i+\zeta)}{\nu^{\eta i+\zeta}}$$
$$= \frac{1}{\nu^\zeta} \sum_{i=0}^{\infty} \frac{\mathfrak{d}(\gamma+i)}{i!} \left(\frac{\lambda}{\nu^\eta}\right)^i = \frac{\nu^{\eta\gamma-\zeta}}{(\nu^\eta-\lambda)^\gamma}.$$

Next, we state sufficient conditions for the existence of Φ-Laplace transform of a function.

Theorem 5.1.10. *If $f : [0,\infty) \to \mathbb{R}$ is a piecewise continuous function and is of Φ-exponential order, then the Φ-Laplace transform of f exists for $\nu > c$.*

Proof. We have

$$\left|\mathfrak{L}_\Phi \left\{ f(t) \right\}\right| = \left| \int_0^\infty e^{-\nu\Phi(t)} \Phi'(t) f(t) dt \right| \le \int_0^\infty e^{-\nu\Phi(t)} \Phi'(t) \left| f(t) \right| dt \quad (5.1.4)$$
$$\le M \int_0^\infty e^{-\nu\Phi(t)} \Phi'(t) e^{c\Phi(t)} dt$$
$$= \frac{M}{\nu - c}, \text{ for } \nu > c.$$

Thus, the proof of Theorem 5.1.10 is complete. □

Remark 5.1.11. From Eq. (5.1.4), it follows that $\lim_{\nu\to\infty} \left| \mathfrak{L}_\Phi \left\{ f(t) \right\} (\nu) \right| = 0$, i.e., $\lim_{\nu\to\infty} \mathfrak{L}_\Phi \left\{ f(t) \right\} (\nu) = 0$. This property can be named as the limiting property of the Φ-Laplace transform.

In the following Theorems, we state the Φ-Laplace transforms of the Φ-RL and Φ-Caputo fractional operators [19].

Theorem 5.1.12. *Let $\eta > 0$ and f be of Φ-exponential order, piecewise continuous function over each finite interval $[0, T]$. Then*

$$\mathfrak{L}_\Phi \left\{ (\mathfrak{I}_0^{\eta,\Phi} f)(t) \right\} = \nu^{-\eta} \mathfrak{L}_\Phi \left\{ f(t) \right\}. \quad (5.1.5)$$

Theorem 5.1.13. *If $\eta > 0$, $m = [\eta] + 1$, and $f(t)$, $\mathfrak{I}_0^{m-\eta,\Phi} f(t)$, $\mathfrak{D}^{1,\Phi}\mathfrak{I}_0^{m-\eta,\Phi} f(t), \dots,$ $\mathfrak{D}^{m-1,\Phi}\mathfrak{I}_0^{m-\eta,\Phi} f(t)$ where $\mathfrak{D}^{j,\Phi} = \left(\frac{1}{\Phi'(t)}\frac{d}{dt}\right)^j$, are continuous on $(0,\infty)$ and of Φ-exponential order, while $\mathfrak{D}_0^{\eta,\Phi} f(t)$ is piecewise continuous on $[0,\infty)$. Then*

$$\mathfrak{L}_\Phi\left\{\mathfrak{D}_0^{\eta,\Phi} f(t)\right\} = \nu^\eta \mathfrak{L}_\Phi\left\{f(t)\right\} - \sum_{i=0}^{m-1} \nu^{m-i-1}(\mathfrak{I}_0^{m-i-\eta,\Phi} f)(0).$$

Theorem 5.1.14. *If $\eta > 0$, $m = [\eta] + 1$, and $f(t)$, $\mathfrak{D}^{1,\Phi} f(t)$, $\mathfrak{D}^{2,\Phi} f(t), \dots, \mathfrak{D}^{m-1,\Phi} f(t)$ are continuous on $[0,\infty)$ and of Φ-exponential order, while ${}^C\mathfrak{D}_0^{\eta,\Phi} f(t)$ is piecewise continuous on $[0,\infty)$. Then*

$$\mathfrak{L}_\Phi\left\{{}^C\mathfrak{D}_0^{\eta,\Phi} f(t)\right\} = \nu^\eta \mathfrak{L}_\Phi\left\{f(t)\right\} - \sum_{i=0}^{m-1} \nu^{\eta-i-1}(\mathfrak{D}^{i,\Phi} f)(0).$$

Definition 5.1.15. [19] Let f and g be of Φ-exponential order, piecewise continuous functions over each finite interval $[0,T]$. Then, we define the Φ-convolution of f and g by

$$(f *_\Phi g)(t) = \int_0^{t=\Phi^{-1}(\Phi(t))} f\Big(\Phi^{-1}(\Phi(t) - \Phi(\tau))\Big) g(\tau)\Phi'(\tau)d\tau. \tag{5.1.6}$$

In the following theorem, we discuss the commutativity, associativity and distributivity of the Φ-convolution of two functions.

Theorem 5.1.16. *Let f and g be of Φ-exponential order, piecewise continuous functions over each finite interval $[0,T]$. Then*

(a) $f *_\Phi g = g *_\Phi f$.

(b) $(f *_\Phi g) *_\Phi h = f *_\Phi (g *_\Phi h)$.

(c) $f *_\Phi (ag + bh) = af *_\Phi g + bf *_\Phi h$.

Proof. The proof of (a) can be seen in [19]. For (b), consider the left-hand side and using (5.1.6) we have

$$\{f *_\Phi g\}(t) *_\Phi h(t) = \int_0^{t=\Phi^{-1}(\Phi(t))} (f *_\Phi g)(s) h\Big(\Phi^{-1}(\Phi(t) - \Phi(s))\Big)\Phi'(s)ds$$

$$
\begin{aligned}
&= \int_0^t \left(\int_0^s f(u) g\Big(\Phi^{-1}(\Phi(s) - \Phi(u))\Big) \Phi'(u) du \right) \\
&\quad \times h\Big(\Phi^{-1}(\Phi(t) - \Phi(s))\Big) \Phi'(s) ds \\
&= \int_0^t \int_u^t f(u) g\Big(\Phi^{-1}(\Phi(s) - \Phi(u))\Big) \Phi'(u) \\
&\quad \times h\Big(\Phi^{-1}(\Phi(t) - \Phi(s))\Big) \Phi'(s) ds du.
\end{aligned}
$$

By setting $v = \Phi^{-1}(\Phi(s) - \Phi(u))$, we get

$$
\begin{aligned}
\{f *_\Phi g\}(t) *_\Phi h(t) = & \int_0^t f(u)\Phi'(u) \int_0^{\Phi^{-1}(\Phi(t)-\Phi(u))} g(v) h\Big(\Phi^{-1}(\Phi(t) - \Phi(u) - \Phi(v))\Big) \\
& \times \Phi'(v) dv du \\
= & \int_0^t f(u)\Phi'(u) \{f *_\Phi g\}(t) du \\
= & f(t) *_\Phi \{g *_\Phi h\}(t).
\end{aligned}
$$

The proof of (c) is easy. So we omit the straightforward details. □

Remark 5.1.17. Consider a set A of all Φ-Laplace transformable functions then A forms a commutative semi-group with respect to the binary operation $*_\Phi$. Moreover, A does not form a group because $f^{-1} *_\Phi g$ is not Φ-Laplace transformable, generally speaking.

In the following theorem, we prove the uniqueness of the Φ-Laplace transform.

Theorem 5.1.18. *Assume that f and g are piecewise continuous functions on $[0, \infty)$ and of Φ-exponential order $c > 0$. If $F(\nu) = G(\nu)$ for $\nu > a$, then $f(t) = g(t)$ for all $t \geq 0$.*

Proof. Since $F(\nu) = G(\nu)$, so $\mathfrak{L}_\Phi \{f - g\} = 0$. Thus, we will prove that if $\mathfrak{L}_\Phi \big\{f(t)\big\}(\nu) = 0$ for all $\nu > a$ then $f(t) = 0$ for all $t \geq 0$.
Fixing $\nu_0 > a$ and making the substitution $u = e^{-\Phi(t)}$ in (5.1.1), then for $\nu = \nu_0 + n + 1$ we get

$$
0 = F(\nu) = \int_0^\infty e^{-\nu_0 \Phi(t)} e^{-n\Phi(t)} e^{-\Phi(t)} \Phi'(t) f(t) dt = \int_0^1 u^n \Big\{u^{\nu_0} f\left(\Phi^{-1}(-\ln u)\right)\Big\} du \tag{5.1.7}
$$

where $n = 0, 1, 2, \ldots$ Assume that $r(u) = u^{\nu_0} f\left(\Phi^{-1}(-\ln u)\right)$ which is a piecewise continuous function on $(0, 1]$ and

$$\lim_{u \to 0} r(u) = \lim_{t \to \infty} e^{-\nu_0 \Phi(t)} f(t) = 0.$$

If we consider $r(0) = 0$, then h is a piecewise continuous function satisfying

$$\int_0^1 p(u) r(u) du = 0 \tag{5.1.8}$$

where p is any polynomial. Thus, if $\hat{r}$ has a power series expansion which converges uniformly on $[0, 1]$, then Eq. (5.1.8) can be rewritten as

$$\int_0^1 \hat{r}(u) r(u) du = 0. \tag{5.1.9}$$

On the contrary, suppose that r is not a zero function then we can find a point $u_0 \in (0, 1)$, an interval $I = [u_0 - c_0, u_0 + c_0] \subset [0, 1]$ and a constant c such that $r(u) \geq c > 0$ for all $u \in I$. If we set $\hat{r}(u) = e^{-(u-u_0)^2}$, then clearly Eq. (5.1.9) holds. Thus for $x = u - u_0$, we have

$$J_1 = \int_{u_0 - c_0}^{u_0 + c_0} \hat{r}(u) du = \int_{-c_0}^{c_0} e^{-x^2} dx$$

and

$$J_2 = \int_{u_0 + c_0}^{1} \hat{r}(u) du = \int_{c_0}^{1 - u_0} e^{-x^2} dx$$

and

$$J_3 = \int_0^{u_0 - c_0} \hat{r}(u) du = \int_{-u_0}^{-c_0} e^{-x^2} dx.$$

If we set $l = \int_{-\infty}^{\infty} e^{-x^2} dx$, then clearly $l > 0$ and for a given $\epsilon > 0$, we deduce

$$J_1 \geq \frac{l}{2}, \quad 0 \leq J_2 \leq \epsilon, \quad 0 \leq J_3 \leq \epsilon.$$

Since $r(u) \geq c > 0$ for all $u \in I$ and $|h| < n_0$ where $n_0 \in \mathbb{N}$, we have

$$\int_I \hat{r}(u) r(u) du \geq \frac{lc}{2} > 0, \qquad \left| \int_{[0,1] \setminus I} \hat{r}(u) r(u) du \right| \leq 2 n_0 \epsilon$$

and hence

$$\int_0^1 \hat{r}(u) r(u) du \geq \frac{lc}{2} - 2 n_0 \epsilon > 0$$

provided $\epsilon < \frac{lc}{4n_0}$, contradicting Eq. (5.1.9). Thus, r is the zero function which implies that f is the zero function and thus, this completes the proof. □

5.2 The Φ-Laplace transform of the Φ-Hilfer fractional derivative

In this section, we compute the Φ-Laplace transform of the Φ-Hilfer fractional derivative.

Theorem 5.2.1. *If $\eta > 0$, $m = [\eta] + 1$, $0 \leq \zeta \leq 1$, and $f(t)$, $\mathfrak{D}^{j,\Phi}\mathfrak{I}_0^{(1-\zeta)(m-\eta),\Phi} f(t) \in C[0,\infty)$ and of Φ-exponential order for $j = 0, 1, 2, \cdots, m-1$, while $\mathfrak{D}_0^{\eta,\zeta,\Phi} f(t)$ is piecewise continuous on $[0,\infty)$. Then*

$$\mathfrak{L}_\Phi \left\{ \mathfrak{D}_0^{\eta,\zeta,\Phi} f(t) \right\} = \nu^\eta \mathfrak{L}_\Phi \left\{ f(t) \right\} - \sum_{i=0}^{m-1} \nu^{m(1-\zeta)+\eta\zeta-i-1} (\mathfrak{I}_0^{(1-\zeta)(m-\eta)-i,\Phi} f)(0).$$

Proof. From the definition of the integral operator $\mathfrak{D}_0^{\eta,\zeta,\Phi} f$ and (5.1.1), we have

$$\mathfrak{L}_\Phi \left\{ (\mathfrak{D}_0^{\eta,\zeta,\Phi} f)(t) \right\} = \mathfrak{L}_\Phi \left\{ \mathfrak{I}_a^{\zeta(m-\eta),\Phi} \left(\frac{1}{\Phi'(t)} \frac{d}{dt} \right)^m \mathfrak{I}_a^{(1-\zeta)(m-\eta),\Phi} f(t) \right\}.$$

Using Theorem 5.1.13 and 5.1.12, we get

$$\begin{aligned}
\mathfrak{L}_\Phi \left\{ (\mathfrak{D}_0^{\eta,\zeta,\Phi} f)(t) \right\} =& \nu^{-\zeta(m-\eta)} \mathfrak{L}_\Phi \left\{ \left(\frac{1}{\Phi'(t)} \frac{d}{dt} \right)^m \mathfrak{I}_a^{(1-\zeta)(m-\eta),\Phi} f(t) \right\} \\
=& \nu^{-\zeta(m-\eta)} \Big[\nu^m \mathfrak{L}_\Phi \left\{ (\mathfrak{I}_0^{(1-\zeta)(m-\eta),\Phi} f)(t) \right\} \\
& - \sum_{i=0}^{m-1} \nu^{m-i-1} (\mathfrak{D}^{i,\Phi} \mathfrak{I}_0^{(1-\zeta)(m-\eta),\Phi} f)(0) \Big] \\
=& \nu^{-\zeta(m-\eta)} \Big[\nu^m \nu^{-(1-\zeta)(m-\eta)} \mathfrak{L}_\Phi \left\{ (\mathfrak{I}_0^{(1-\zeta)(m-\eta),\Phi} f)(t) \right\} \\
& - \sum_{i=0}^{m-1} \nu^{m-i-1} (\mathfrak{I}_0^{(1-\zeta)(m-\eta)-i,\Phi} f)(0) \Big] \\
=& \nu^\eta \mathfrak{L}_\Phi \left\{ f(t) \right\} - \sum_{i=0}^{m-1} \nu^{m(1-\zeta)+\eta\zeta-i-1} (\mathfrak{I}_0^{(1-\zeta)(m-\eta)-i,\Phi} f)(0).
\end{aligned}$$

Thus, we have completed the proof. □

5.3 Effectiveness of the Φ-Laplace transform method for solving fractional-order differential equations

In this section, we examine and investigate the effectiveness of the Φ-Laplace transform method for solving fractional-order differential equations of the following type

$$ {}^{C}\mathfrak{D}_{0}^{\eta,\Phi} y(t) = Ay(t) + g(t), \quad 0 < \eta < 1, \quad t \geq 0, \tag{5.3.1}$$

$$ y(0) = \tilde{y}_0, \tag{5.3.2}$$

where ${}^{C}\mathfrak{D}_{0}^{\eta,\Phi}$ is the Φ-Caputo fractional differential operator, A is a $n \times n$ constant matrix and $g(t)$ is an n-dimensional continuous function.

Theorem 5.3.1. *Let* (5.3.1)-(5.3.2) *has a unique and continuous solution* $y(t)$*. Assume that* $g(t)$ *is continuous on* $[0, \infty)$ *and* Φ*-exponentially bounded, then* $y(t)$ *and* ${}^{C}\mathfrak{D}_{0}^{\eta,\Phi} y(t)$ *are both* Φ*-exponentially bounded.*

Proof. It can be noticed that (5.3.1)-(5.3.2) is equivalent to the Volterra equation given below

$$ y(t) = \tilde{y}_0 + \frac{1}{\mathfrak{d}(\eta)} \int_0^t \left(\Phi(t) - \Phi(\tau)\right)^{\eta-1} \Phi'(\tau) \left\{Ay(\tau) + g(\tau)\right\} d\tau, \quad 0 \leq t < \infty. \tag{5.3.3}$$

By assumption, $g(t)$ is Φ-exponentially bounded, so there exist positive constants c, M and large enough T such that $||g(t)|| \leq Me^{c\Phi(t)}$ for all $t \geq T$. For $t \geq T$, (5.3.3) can be written as

$$ \begin{aligned} y(t) = \tilde{y}_0 &+ \frac{1}{\mathfrak{d}(\eta)} \int_0^T \left(\Phi(t) - \Phi(\tau)\right)^{\eta-1} \Phi'(\tau) \left\{Ay(\tau) + g(\tau)\right\} d\tau \\ &+ \frac{1}{\mathfrak{d}(\eta)} \int_T^t \left(\Phi(t) - \Phi(\tau)\right)^{\eta-1} \Phi'(\tau) \left\{Ay(\tau) + g(\tau)\right\} d\tau. \end{aligned}$$

Since, $y(t)$ is a unique and continuous solution of $(5.3.1) - (5.3.2)$ on $[0, \infty)$, thus $Ay(t) + g(t)$ is bounded on $[0, T]$ that is there exists a constant $l > 0$ such that

$||Ay(t)+g(t)||<l$. So, we get

$$||y(t)|| \leq ||\tilde{y_0}|| + \frac{l}{\mathfrak{d}(\eta)} \int_0^T \left(\Phi(t)-\Phi(\tau)\right)^{\eta-1} \Phi'(\tau) d\tau$$
$$+ \frac{1}{\mathfrak{d}(\eta)} \int_T^t \left(\Phi(t)-\Phi(\tau)\right)^{\eta-1} \Phi'(\tau) ||A||\, ||y(\tau)|| d\tau$$
$$+ \frac{1}{\mathfrak{d}(\eta)} \int_T^t \left(\Phi(t)-\Phi(\tau)\right)^{\eta-1} \Phi'(\tau) ||g(\tau)|| d\tau.$$

Using $e^{-c\Phi(t)} \leq e^{-c\Phi(T)}$, $e^{-c\Phi(t)} \leq e^{-c\Phi(\tau)}$, $||g(t)|| \leq Me^{c\Phi(t)}$ and multiplying the above inequality by $e^{-c\Phi(t)}$, we find

$$||y(t)||e^{-c\Phi(t)} \leq ||\tilde{y_0}||e^{-c\Phi(t)} + \frac{le^{-c\Phi(t)}}{\mathfrak{d}(\eta)} \int_0^T \left(\Phi(t)-\Phi(\tau)\right)^{\eta-1} \Phi'(\tau) d\tau$$
$$+ \frac{e^{-c\Phi(t)}}{\mathfrak{d}(\eta)} \int_T^t \left(\Phi(t)-\Phi(\tau)\right)^{\eta-1} \Phi'(\tau) ||A||\, ||y(\tau)|| d\tau$$
$$+ \frac{e^{-c\Phi(t)}}{\mathfrak{d}(\eta)} \int_T^t \left(\Phi(t)-\Phi(\tau)\right)^{\eta-1} \Phi'(\tau) ||g(\tau)|| d\tau$$
$$\leq ||\tilde{y_0}||e^{-c\Phi(T)} + \frac{le^{-c\Phi(T)}}{\eta\mathfrak{d}(\eta)} \Big((\Phi(t))^\eta - (\Phi(t)-\Phi(T))^\eta \Big)$$
$$+ \frac{||A||}{\mathfrak{d}(\eta)} \int_0^t \left(\Phi(t)-\Phi(\tau)\right)^{\eta-1} \Phi'(\tau)\ ||y(\tau)||e^{-c\Phi(\tau)} d\tau$$
$$+ \frac{M}{\mathfrak{d}(\eta)} \int_0^t \left(\Phi(t)-\Phi(\tau)\right)^{\eta-1} \Phi'(\tau) e^{c(\Phi(\tau)-\Phi(t))} d\tau$$
$$\leq ||\tilde{y_0}||e^{-c\Phi(T)} + \frac{l(\Phi(T))^\eta e^{-c\Phi(T)}}{\eta\mathfrak{d}(\eta)} + \frac{M}{\mathfrak{d}(\eta)} \int_0^\infty e^{-cs} \nu^{\eta-1} ds$$
$$+ \frac{||A||}{\mathfrak{d}(\eta)} \int_0^t \left(\Phi(t)-\Phi(\tau)\right)^{\eta-1} \Phi'(\tau)\ ||y(\tau)||e^{-c\Phi(\tau)} d\tau$$
$$\leq ||\tilde{y_0}||e^{-c\Phi(T)} + \frac{l(\Phi(T))^\eta e^{-c\Phi(T)}}{\eta\mathfrak{d}(\eta)} + \frac{M}{c^\eta}$$
$$+ \frac{||A||}{\mathfrak{d}(\eta)} \int_0^t \left(\Phi(t)-\Phi(\tau)\right)^{\eta-1} \Phi'(\tau)\ ||y(\tau)||e^{-c\Phi(\tau)} d\tau.$$

Assume that

$$a = ||\tilde{y_0}||e^{-c\Phi(T)} + \frac{l(\Phi(T))^\eta e^{-c\Phi(T)}}{\eta\mathfrak{d}(\eta)} + \frac{M}{c^\eta},\ b = \frac{||A||}{\mathfrak{d}(\eta)},\ r(t) = ||y(t)||e^{-c\Phi(t)},$$

then, we have

$$r(t) \leq a + b\int_0^t \left(\Phi(t)-\Phi(\tau)\right)^{\eta-1} \Phi'(\tau)\ ||y(\tau)||e^{-c\Phi(\tau)} d\tau.$$

Using the Gronwall-inequality [42], we deduce that

$$r(t) \leq a\mathfrak{E}_\eta\Big(||A||\left(\Phi(t)-\Phi(\tau)\right)^\eta\Big) \leq a\mathfrak{E}_\eta\Big(||A||\left(\Phi(t)\right)^\eta\Big). \tag{5.3.4}$$

For $0<\eta<1$, $u>0$, $t\geq 0$, the following inequality can be easily proved

$$\mathfrak{E}_\eta\Big(u\left(\Phi(t)\right)^\eta\Big) \leq Ce^{u^{1/\eta}\Phi(t)}, \text{ where } C>0. \tag{5.3.5}$$

From (5.3.4) and (5.3.5), we have

$$r(t) \leq aCe^{(||A||)^{1/\eta}\Phi(t)},$$

and finally, we get

$$||y(t)|| \leq aCe^{\left\{(||A||)^{1/\eta}+c\right\}\Phi(t)}.$$

Thus, $y(t)$ is Φ-exponentially bounded.

Furthermore, from Eq. (5.3.1), we have

$$\begin{aligned} ||{}^C\mathfrak{D}_0^{\eta,\Phi}y(t)|| &\leq ||A||\ ||y(t)|| + ||g(t)|| \\ &\leq a||A||Ce^{\left\{(||A||)^{1/\eta}+c\right\}\Phi(t)} + Me^{c\Phi(t)} \\ &\leq \Bigg(a||A||C+M\Bigg)e^{\left\{(||A||)^{1/\eta}+c\right\}\Phi(t)}. \end{aligned}$$

Thus, ${}^C\mathfrak{D}_0^{\eta,\Phi}y(t)$ is also Φ-exponentially bounded and this completes the proof. □

Similar results can be proved for fractional-order differential equations in the settings of Φ-RL and Φ-Hilfer fractional derivatives.

5.4 Applications

In this section, by using the Φ-Laplace transformation method, and in the settings of Φ-RL, Φ-Caputo and Φ-Hilfer fractional derivatives, we state and find solutions of different classes of linear FDEs with constant coefficients. We now divide this section into the following subsections:

5.4.1 Solutions of some non-homogeneous linear Φ-RL and Φ-Caputo FDEs

In this subsection, we use the Φ-Laplace transformation method to solve the FDEs in the frame of Φ-RL and Φ-Caputo fractional derivatives.

Theorem 5.4.1. *[19] The FDE*

$$\mathfrak{D}_0^{\eta,\Phi}y(t) - \lambda y(t) = f(t), \quad 0 < \eta \leq 1, \quad \lambda \in \mathbb{R}, \tag{5.4.1}$$

with initial condition

$$(\mathfrak{I}_0^{1-\eta,\Phi})y(0) = c, \quad c \in \mathbb{R}, \tag{5.4.2}$$

has the solution

$$y(t) = c(\Phi(t))^{\eta-1}\mathfrak{E}_{\eta,\eta}\Big(\lambda(\Phi(t))^{\eta}\Big) + (\Phi(t))^{\eta-1}\mathfrak{E}_{\eta,\eta}\Big(\lambda(\Phi(t))^{\eta}\Big) *_{\Phi} f(t).$$

Theorem 5.4.2. *[19] The FDE*

$$^{C}\mathfrak{D}_0^{\eta,\Phi}y(t) - \lambda y(t) = f(t), \quad 0 < \eta \leq 1, \quad \lambda \in \mathbb{R}, \tag{5.4.3}$$

with initial condition

$$y(0) = c, \quad c \in \mathbb{R}, \tag{5.4.4}$$

has the solution

$$y(t) = c\mathfrak{E}_{\eta}\Big(\lambda(\Phi(t))^{\eta}\Big) + (\Phi(t))^{\eta-1}\mathfrak{E}_{\eta,\eta}\Big(\lambda(\Phi(t))^{\eta}\Big) *_{\Phi} f(t). \tag{5.4.5}$$

Remark 5.4.3. Sometimes natural states are more adequately modeled by FDEs. As an example, if we consider $\Phi(t) = t$ and $f(t) = 0$ in (5.4.3) then the resulting FDE is more appropriate for modeling the population growth than the ordinary differential equation [4]. Moving a step forward, Almeida [2] showed that by considering different Φ's, a population growth model can be reproduced with more accuracy.

Corollary 5.4.4. *Consider a special case of IVP* (5.4.3)-(5.4.4)

$$^{C}\mathfrak{D}_0^{\eta,\Phi}y(t) - y(t) = 1, \quad 0 < \eta \leq 1, \tag{5.4.6}$$

$$y(0) = 1. \tag{5.4.7}$$

Then

(a) $y(t) = \mathfrak{E}_\eta(t^{\frac{\eta}{2}}) + t^{\frac{\eta}{2}}\mathfrak{E}_{\eta,\eta+1}(t^{\frac{\eta}{2}}), \quad for \quad \Phi(t) = \sqrt{t}.$

(b) $y(t) = \mathfrak{E}_\eta(t^\eta) + t^\eta\mathfrak{E}_{\eta,\eta+1}(t^\eta), \quad for \quad \Phi(t) = t.$

(c) $y(t) = \mathfrak{E}_\eta(t^{2\eta}) + t^{2\eta}\mathfrak{E}_{\eta,\eta+1}(t^{2\eta}), \quad for \quad \Phi(t) = t^2.$

Proof. (b) From (5.1.6) and (5.4.5), we have

$$\begin{aligned} y(t) &= \mathfrak{E}_\eta(t^\eta) + \int_0^t \tau^{\eta-1}\mathfrak{E}_{\eta,\eta}(\tau^\eta)d\tau = \mathfrak{E}_\eta(t^\eta) + \int_0^t \sum_{k=0}^{\infty} \frac{\tau^{\eta k+\eta-1}}{\mathfrak{d}(\eta k+\eta)}d\tau \\ &= \mathfrak{E}_\eta(t^\eta) + \sum_{k=0}^{\infty} \frac{t^{\eta k+\eta}}{\mathfrak{d}(\eta k+\eta+1)} = \mathfrak{E}_\eta(t^\eta) + t^\eta\mathfrak{E}_{\eta,\eta+1}(t^\eta). \end{aligned}$$

Similarly, one can prove part (a) and (c). Plots of solutions (a), (b) and (c) are given in Fig.5.1 (a), (b) and (c), respectively. □

Theorem 5.4.5. *The fractional diffusion equation*

$$\frac{\partial^\eta u}{\partial t^\eta} = \kappa\frac{\partial^2 u}{\partial x^2}, \quad where \quad 0 < \eta \leq 1, \tag{5.4.8}$$

with initial and boundary conditions

$$u(x,t) \to 0 \ as \ |x| \to \infty, \tag{5.4.9}$$

$$(\mathfrak{I}_0^{1-\eta,\Phi})u(x,t)\Big|_{t=0} = f(x), \quad x \in \mathbb{R}, \tag{5.4.10}$$

has the solution

$$u(x,t) = \int_{-\infty}^{\infty} G(x-\tilde{x},t)f(\tilde{x})d\tilde{x},$$

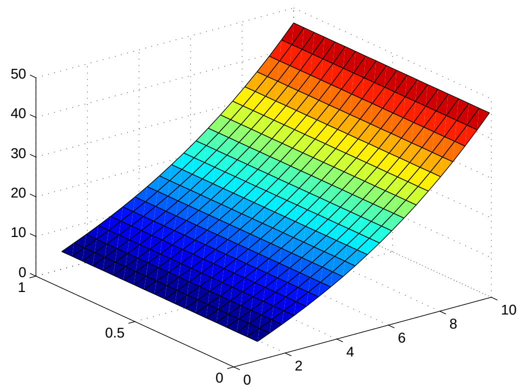
where

$$G(x,t) = \frac{1}{2\sqrt{\kappa}}(\Phi(t))^{\frac{\eta}{2}-1}W\Big(-\frac{|x|}{\sqrt{\kappa}(\Phi(t))^{\frac{\eta}{2}}}, -\frac{\eta}{2}, \frac{\eta}{2}\Big).$$

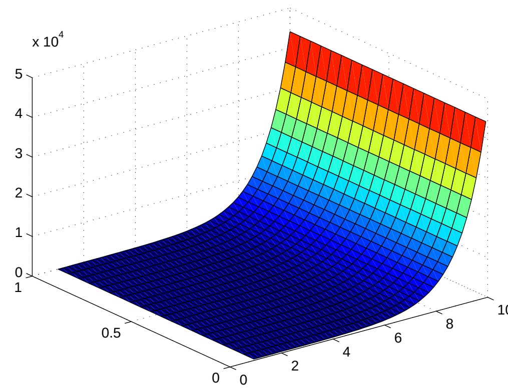
Proof. By application of the Fourier transform to both sides of (5.4.8) and (5.4.10) with respect to x, and by using (5.4.9), we have

$$\mathfrak{D}_0^{\eta,\Phi}\tilde{u}(k,t) = -\kappa k^2\tilde{u}(k,t), \tag{5.4.11}$$

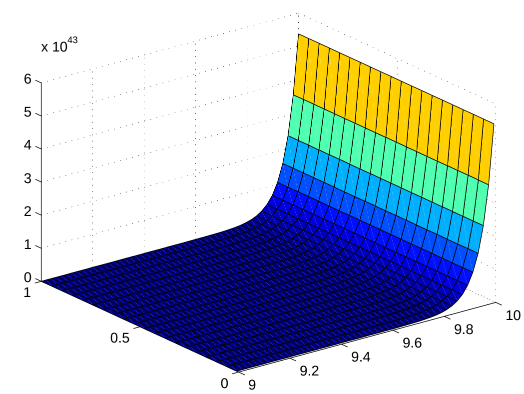
$$(\mathfrak{I}_0^{1-\eta,\Phi})\tilde{u}(k,t)\Big|_{t=0} = \tilde{f}(k). \tag{5.4.12}$$



(a) $1 \le t \le 10, 0.01 \le \alpha \le 1$, $\Phi(t) = \sqrt{t}$.



(b) $1 \le t \le 10, 0.01 \le \alpha \le 1$, $\Phi(t) = t$.



(c) $9 \le t \le 10, 0.01 \le \alpha \le 1$, $\Phi(t) = t^2$.

Figure 5.1: Solutions of IVP (5.4.6)-(5.4.7).

Applying the Φ-Laplace transform to both sides of (5.4.11) with respect to t, and using (5.4.12), we get

$$\begin{aligned}\mathfrak{L}_\Phi \left\{\tilde{u}(k,t)\right\} &= \frac{\tilde{f}(k)}{(\nu^\eta + \kappa k^2)} \\ &= \mathfrak{L}_\Phi \left\{ \tilde{f}(k)(\Phi(t))^{\eta-1}\mathfrak{E}_{\eta,\eta}\Big(-\kappa k^2 (\Phi(t))^\eta \Big) \right\},\end{aligned}$$

and from the above equality, we find

$$\tilde{u}(k,t) = \tilde{f}(k)(\Phi(t))^{\eta-1}\mathfrak{E}_{\eta,\eta}\Big(-\kappa k^2 (\Phi(t))^\eta \Big). \tag{5.4.13}$$

The inverse Fourier transform of (5.4.13) gives

$$u(x,t) = \int_{-\infty}^{\infty} G(x-\tilde{x},t) f(\tilde{x}) d\tilde{x} \tag{5.4.14}$$

where

$$G(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} (\Phi(t))^{\eta-1}\mathfrak{E}_{\eta,\eta}\Big(-\kappa k^2(\Phi(t))^{\eta}\Big) cos(kx) dk.$$

The above integral can be evaluated by using the Φ-Laplace transform of $G(x,t)$ with respect to t as

$$\begin{aligned}
\mathfrak{L}_{\Phi}\left\{G(x,t)\right\} &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{cos(kx)}{(\nu^{\eta}+\kappa k^2)} dk \\
&= \frac{1}{2\sqrt{\kappa}}\nu^{-\frac{\eta}{2}} e^{-\frac{|x|}{\sqrt{\kappa}}\nu^{\frac{\eta}{2}}} \\
&= \mathfrak{L}_{\Phi}\left\{\frac{1}{2\sqrt{\kappa}}(\Phi(t))^{\frac{\eta}{2}-1} W\Big(-\frac{|x|}{\sqrt{\kappa}(\Phi(t))^{\frac{\eta}{2}}}, -\frac{\eta}{2}, \frac{\eta}{2}\Big)\right\}.
\end{aligned}$$

Finally,

$$G(x,t) = \frac{1}{2\sqrt{\kappa}}(\Phi(t))^{\frac{\eta}{2}-1} W\Big(-\frac{|x|}{\sqrt{\kappa}(\Phi(t))^{\frac{\eta}{2}}}, -\frac{\eta}{2}, \frac{\eta}{2}\Big).$$

Thus, this completes the proof. □

It is noteworthy that for $\Phi(t) = t$ and $\eta = 1$, the Cauchy problem (5.4.8)-(5.4.10) reduces to the classical diffusion problem, and solution (5.4.14) reduces to the classical fundamental solution.

5.4.2 Solutions of some general Φ-Hilfer FDEs

Assume that

$$0 < \eta_1 \leq \eta_2 < 1,\ 0 \leq \zeta_j \leq 1,\ a_j \in \mathbb{R} \quad \text{for } j = 1, 2.$$

Consider the Φ-Hilfer FDE

$$a_1\mathfrak{D}_0^{\eta_1,\zeta_1,\Phi} y(t) + a_2\mathfrak{D}_0^{\eta_2,\zeta_2,\Phi} y(t) + a_3 y(t) = f(t), \tag{5.4.15}$$

with initial conditions

$$(\mathfrak{I}_0^{(1-\zeta_j)(1-\eta_j),\Phi}) y(0) = b_j \quad \text{for } j = 1, 2. \tag{5.4.16}$$

For dielectric relaxation in glasses, Hilfer introduced an equation of the form (5.4.15) in [16]. Ž. Tomovski et al. [44] obtained the solution of a particular case of Cauchy problem (5.4.15)-(5.4.16) when $\Phi(t) = t$, in the space of Lebesgue integrable functions.

In the following Theorem, using Φ-Laplace transform we find the general solution of the IVP (5.4.15)-(5.4.16).

Theorem 5.4.6. *The IVP* (5.4.15)-(5.4.16) *has the solution*

$$
\begin{aligned}
y(t) =& \frac{1}{a_2}\sum_{i=0}^{\infty}\Big(-\frac{a_1}{a_2}\Big)^i \Big[(\Phi(t))^{(\eta_2-\eta_1)i+\eta_2-1}\mathfrak{E}^{i+1}_{\eta_2,(\eta_2-\eta_1)i+\eta_2}\Big(-\frac{a_3}{a_2}(\Phi(t))^{\eta_2}\Big) *_\Phi f(t) \\
&+ a_2 b_2 (\Phi(t))^{(\eta_2-\eta_1)i+\eta_2+\zeta_2(1-\eta_2)-1}\mathfrak{E}^{i+1}_{\eta_2,(\eta_2-\eta_1)i+\eta_2+\zeta_2(1-\eta_2)}\Big(-\frac{a_3}{a_2}(\Phi(t))^{\eta_2}\Big) \\
&+ a_1 b_1 (\Phi(t))^{(\eta_2-\eta_1)i+\eta_2+\zeta_1(1-\eta_1)-1}\mathfrak{E}^{i+1}_{\eta_2,(\eta_2-\eta_1)i+\eta_2+\zeta_1(1-\eta_1)}\Big(-\frac{a_3}{a_2}(\Phi(t))^{\eta_2}\Big)\Big].
\end{aligned}
$$

Proof. Applying Φ-Laplace transform to both sides of (5.4.15) and using the initial conditions (5.4.16), we have

$$
\begin{aligned}
\mathfrak{L}_\Phi\left\{y(t)\right\} =& \frac{\mathfrak{L}_\Phi\left\{f(t)\right\}}{a_1\nu^{\eta_1}+a_2\nu^{\eta_2}+a_3} + a_2 b_2 \frac{\nu^{\zeta_2(\eta_2-1)}}{a_1\nu^{\eta_1}+a_2\nu^{\eta_2}+a_3} \\
&+ a_1 b_1 \frac{\nu^{\zeta_1(\eta_1-1)}}{a_1\nu^{\eta_1}+a_2\nu^{\eta_2}+a_3}.
\end{aligned}
\tag{5.4.17}
$$

Furthermore, for $j = 1, 2$ we have

$$
\begin{aligned}
\frac{\nu^{\zeta_j(\eta_j-1)}}{a_1\nu^{\eta_1}+a_2\nu^{\eta_2}+a_3} =& \frac{1}{a_2}\left(\frac{\nu^{\zeta_j(\eta_j-1)}}{\nu^{\eta_2}+\frac{a_3}{a_2}}\right)\left(\frac{1}{1+\frac{a_1}{a_2}\left(\frac{\nu^{\eta_1}}{\nu^{\eta_2}+\frac{a_3}{a_2}}\right)}\right) \\
=& \frac{1}{a_2}\sum_{i=0}^{\infty}\left(-\frac{a_1}{a_2}\right)^i \frac{\nu^{\eta_1 i+\zeta_j\eta_j-\zeta_j}}{\left(\nu^{\eta_2}+\frac{a_3}{a_2}\right)^{i+1}} \\
=& \mathfrak{L}_\Phi\left[\frac{1}{a_2}\sum_{i=0}^{\infty}\Big(-\frac{a_1}{a_2}\Big)^i(\Phi(t))^{(\eta_2-\eta_1)i+\eta_2+\zeta_j(1-\eta_j)-1}\right. \\
&\left.\times\ \mathfrak{E}^{i+1}_{\eta_2,(\eta_2-\eta_1)i+\eta_2+\zeta_j(1-\eta_j)}\Big(-\frac{a_3}{a_2}(\Phi(t))^{\eta_2}\Big)\right]
\end{aligned}
$$

and

$$\frac{\mathfrak{L}_{\Phi}\left\{f(t)\right\}}{a_1\nu^{\eta_1}+a_2\nu^{\eta_2}+a_3} = \frac{1}{a_2}\sum_{i=0}^{\infty}\left(-\frac{a_1}{a_2}\right)^i \frac{\nu^{\eta_1 i}}{\left(\nu^{\eta_2}+\frac{a_3}{a_2}\right)^{i+1}}\mathfrak{L}_{\Phi}\left\{f(t)\right\}$$

$$= \mathfrak{L}_{\Phi}\left[\frac{1}{a_2}\sum_{i=0}^{\infty}\Big(-\frac{a_1}{a_2}\Big)^i(\Phi(t))^{(\eta_2-\eta_1)i+\eta_2-1} \times \mathfrak{E}^{i+1}_{\eta_2,(\eta_2-\eta_1)i+\eta_2}\Big(-\frac{a_3}{a_2}(\Phi(t))^{\eta_2}\Big) *_{\Phi} f(t)\right].$$

Thus, from Equation (5.4.17) we find

$$\begin{aligned}Y(s) =& \frac{1}{a_2}\sum_{i=0}^{\infty}\Big(-\frac{a_1}{a_2}\Big)^i\mathfrak{L}_{\Phi}\Big[(\Phi(t))^{(\eta_2-\eta_1)i+\eta_2-1}\mathfrak{E}^{i+1}_{\eta_2,(\eta_2-\eta_1)i+\eta_2}\Big(-\frac{a_3}{a_2}(\Phi(t))^{\eta_2}\Big) *_{\Phi} f(t)\\
&+a_2b_2(\Phi(t))^{(\eta_2-\eta_1)i+\eta_2+\zeta_2(1-\eta_2)-1}\mathfrak{E}^{i+1}_{\eta_2,(\eta_2-\eta_1)i+\eta_2+\zeta_2(1-\eta_2)}\Big(-\frac{a_3}{a_2}(\Phi(t))^{\eta_2}\Big)\\
&+a_1b_1(\Phi(t))^{(\eta_2-\eta_1)i+\eta_2+\zeta_1(1-\eta_1)-1}\mathfrak{E}^{i+1}_{\eta_2,(\eta_2-\eta_1)i+\eta_2+\zeta_1(1-\eta_1)}\Big(-\frac{a_3}{a_2}(\Phi(t))^{\eta_2}\Big)\Big]\end{aligned}$$

and finally we have

$$\begin{aligned}y(t) =& \frac{1}{a_2}\sum_{i=0}^{\infty}\Big(-\frac{a_1}{a_2}\Big)^i\Big[(\Phi(t))^{(\eta_2-\eta_1)i+\eta_2-1}\mathfrak{E}^{i+1}_{\eta_2,(\eta_2-\eta_1)i+\eta_2}\Big(-\frac{a_3}{a_2}(\Phi(t))^{\eta_2}\Big) *_{\Phi} f(t)\\
&+a_2b_2(\Phi(t))^{(\eta_2-\eta_1)i+\eta_2+\zeta_2(1-\eta_2)-1}\mathfrak{E}^{i+1}_{\eta_2,(\eta_2-\eta_1)i+\eta_2+\zeta_2(1-\eta_2)}\Big(-\frac{a_3}{a_2}(\Phi(t))^{\eta_2}\Big)\\
&+a_1b_1(\Phi(t))^{(\eta_2-\eta_1)i+\eta_2+\zeta_1(1-\eta_1)-1}\mathfrak{E}^{i+1}_{\eta_2,(\eta_2-\eta_1)i+\eta_2+\zeta_1(1-\eta_1)}\Big(-\frac{a_3}{a_2}(\Phi(t))^{\eta_2}\Big)\Big].\end{aligned}$$

□

Theorem 5.4.7. *Assume that* $0<\eta_1\le\eta_2\le\eta_3<1,\ 0\le\zeta_j\le 1$ *and* $a_j\in\mathbb{R}$ *for* $j=1,2,3$. *Then the IVP*

$$a_1\mathfrak{D}_0^{\eta_1,\zeta_1,\Phi}y(t)+a_2\mathfrak{D}_0^{\eta_2,\zeta_2,\Phi}y(t)+a_3\mathfrak{D}_0^{\eta_3,\zeta_3,\Phi}y(t)+a_4y(t)=f(t), \tag{5.4.18}$$

$$(\mathfrak{I}_0^{(1-\zeta_j)(1-\eta_j),\Phi})y(0)=b_j \quad \textit{for } j=1,2,3. \tag{5.4.19}$$

has the solution

$$
\begin{aligned}
y(t) = \sum_{i=0}^{\infty} \frac{(-1)^i}{{a_3}^{i+1}} \sum_{k=0}^{i} \binom{i}{k} {a_1}^k {a_2}^{i-k} (\Phi(t))^{(\eta_3-\eta_2)i+(\eta_2-\eta_1)k+\eta_3-1} \\
\times \Big[a_1 b_1 (\Phi(t))^{\zeta_1(1-\eta_1)} \mathfrak{E}_{\eta_3,(\eta_3-\eta_2)i+(\eta_2-\eta_1)k+\eta_3+\zeta_1(1-\eta_1)} \Big(-\frac{a_4}{a_3}(\Phi(t))^{\eta_3} \Big) \\
+ a_2 b_2 (\Phi(t))^{\zeta_2(1-\eta_2)} \mathfrak{E}_{\eta_3,(\eta_3-\eta_2)i+(\eta_2-\eta_1)k+\eta_3+\zeta_2(1-\eta_2)} \Big(-\frac{a_4}{a_3}(\Phi(t))^{\eta_3} \Big) \\
+ a_3 b_3 (\Phi(t))^{\zeta_3(1-\eta_3)} \mathfrak{E}_{\eta_3,(\eta_3-\eta_2)i+(\eta_2-\eta_1)k+\eta_3+\zeta_3(1-\eta_3)} \Big(-\frac{a_4}{a_3}(\Phi(t))^{\eta_3} \Big) \\
+ \mathfrak{E}^{i+1}_{\eta_3,(\eta_3-\eta_2)i+(\eta_2-\eta_1)k+\eta_3} \Big(-\frac{a_4}{a_3}(\Phi(t))^{\eta_3} \Big) *_{\Phi} f(t) \Big].
\end{aligned}
$$

Proof. Using the technique demonstrated in the previous result, it is simple to produce the derivation of the solution. So here we omit the straightforward but tedious details. □

Remark 5.4.8. For dielectric relaxation in glycerol over 12 decades in frequency, Hilfer introduced an equation of the form (5.4.18) in [16]. Ž. Tomovski et al. [44] obtained the solution of a particular case of Cauchy problem (5.4.18)-(5.4.19) when

$$\Phi(t) = t, \quad a_4 = 1, \quad f(t) = 0, \quad \eta_j = \zeta_j = 1 \text{ for } j = 1, 2, 3.$$

Chapter 6

Conclusion

In the first part of this thesis a generalized version of substantial fractional operators has been presented, which we have named as the generalized substantial fractional operator. Some basic properties of fractional calculus in the settings of this operator have been proved. By taking into consideration a class of FDEs in the settings of generalized substantial fractional derivative, we have discussed the existence, uniqueness and continuous dependence of its solutions on initial data.

We have presented, in the second part of this thesis, a generalized form for the Hadamard type fractional operators named as the Φ-Hadamard type fractional operators. We have given proofs for the important properties of the new generalized operators. Conditions have been given under which the Φ-Hadamard type fractional integral is bounded in a generalized space and sufficient conditions for the existence of the Φ-Hadamard type fractional derivative have been established. We have proved the semi-group and reciprocal properties for the generalized operators. Finally, we have derived the fractional integration by parts formulas in the frame of the Φ-Hadamard type fractional operators.

In the third and last part of this thesis, we have given proof for the several important properties and uniqueness of the Φ-Laplace transform. We have used this generalized transform for solving linear FDEs in the settings of Φ-Hilfer fractional derivative.

In future work, we will introduce the Φ-type generalizations of the generalized substantial fractional operators. Furthermore, we will introduce some other generalized integral transforms which may help to solve FDEs in the frame of the generalized substantial fractional operators and Φ-Hadamard type fractional operators.

However, we as researchers, are well aware that we are still unable to geometrically and physically interpret fractional integration and differentiation, generally speaking, as compared to being able to simply interpret the integer-order integrals and derivatives. Thus, further work and research can be done to give physical and geometric meaning to fractional order integrals and derivatives, so that we may be able to aptly apply these operators in the physical world.

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