Fixed point theorems in intuitionistic fuzzy metric spaces

by

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Dedicated to

My loving Mother
who has been there,
for supporting
me through
Everything!

Abstract

Overall analysis of fixed point theory in terms of fuzzy metric spaces and intuitionistic fuzzy metric spaces are presented. Some fixed point theorems are obtained for generalized contraction mapping in complete intuitionistic fuzzy metric spaces. These fixed point theorems are generalization of well known results Kannan's and Chatterjee's fixed point theorems, which are then generalized for a new Δ type contraction mappings.

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Hajira Khatoon

Introduction

The origin of fixed point theory was found in the earlier part of nineteenth century. Fixed point theory has been revealed as a burgeoning field of pure and applied mathematics. Fixed point techniques has been applied to establishing the existence and uniqueness of solution to nonlinear phenomena, differential and integral equations [1].

In 1922 S. Banach [23] make a great achievement in fixed point theory for contraction mapping by publishing most famous result Banach fixed point theorem which is also known as Banach contraction principle. Thereafter extension of these results were obtained by Edelstein [26], Reich [27], Kannan's [11] in 1968, and chatterjee's [12] in 1972.

The word fuzzy set was firstly investigated by L. A. Zadeh [5] in 1965 to handle the imprecise and uncertain data. Fuzzy set can be specified by membership degree of each element by allocating its value between 0 and 1. Fuzzy set theory is a generalization of classical set theory. It has tremendous applications in many fields like computer science, economics and mathematics etc. After the progressive development in the field of Fuzzy set theory, Intuitionistic fuzzy sets were presented in 1983 by K. T. Atanassov [[9], [10]] as a generalization of fuzzy set. Intuitionistic fuzzy set is a veritable tool, which has a wide range of applications in decision making [30], electoral system [28] medical diagnosis [29], etc. Intuitionistic fuzzy sets are more appropriate than fuzzy sets because it allocate both membership and non membership degree. An intuitionistic fuzzy set A can be expressed as

$$A = \{x,\ \mu_A(x),\ \nu_A(x), | x \in X\}$$

where $\mu_A(x)$, $\nu_A(x)$ both belongs to real unit interval $[0,1]$ and their sum also lies in this interval. Sets $\mu_A(x)$ and $\nu_A(x)$ characterizing membership and non membership degree respectively. Degree of hesitation or uncertainty can be defined as

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x),$$

where $0 \leq \pi_A(x) \leq 1$. Framework of fuzzy sets provide a new direction to many authors like Deng [24], Erceg [25], Kaleva & Seikkala [15] and Kramosil & Michalek [14] to extend these concepts into fuzzy metric spaces. Notion of intuitionistic fuzzy metric spaces after the systematic development of fuzzy metric spaces were initiated by Park [8], with the help of triangular norms.

A couple of results has been established for a variety of contraction mappings in Intuitionistic fuzzy metric spaces by M. Rafi & M. S. M. Noorani [19], A. Mohamad [2], W. Sintunavarat & P. Kumam [18], C. Alaca, D. Turkoglu & C. Yildiz [21], and T. K. Samanta, S. Mohinta & I. H. Jebril[7]. M. Rafi & M. S. M. Noorani's[19] contraction mappings[see definition 2.3.1] for complete intuitionistic fuzzy metric spaces were redefined by W. Sintunavarat & P. Kumam [18] and showed that their contractive constant is more general. Considering the techniques of W. Sintunavarat & P. Kumam [18] in intuitionistic fuzzy metric spaces known results Kannan's [11] and Chatterjee's [12] fixed point theorems are obtained in this thesis. It is interesting to note that Kannan's and Chatterjee's results are independent of Banach contraction principle and it also characterizes metric completeness , another reason for the importance of these theorems that it does not enforce a mapping to be continuous.

Thesis is organized as follows. Chapter **1** provides a brief description of fixed point theory and results involving fuzzy metric spaces. These concepts will helpful in understanding of next chapters. In chapter **2** section **I** and **II** contains some notions and results of Intuitionistic fuzzy metric spaces. A couple of fixed point theorems for different contractive mappings are discussed in section **III** by many authors. Some fixed point theorems are also obtained in section **IV** which generalize Kannan's [11], and Chattrejea's [12] results using the (TS-IF) contractive mapping[see definition 2.3.3]. These Kannan's and Chatterjee's type (TS-IF) results provides the partially answer of open problem discussed by T. K. Samanta, S. Mohinta & I. H. Jebril[7]. In chapter **3** we obtained the most general case of Kannan's and Chatterjee's results in Intuitionistic fuzzy metric followed by some techniques of W. Sintunavarat & P. Kumam[18] with new contractive constant Δ.

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Chapter 1

Preliminaries

This chapter is concerned to a concise introduction, elementary concepts, definitions and some essential theorems related to fixed point theory and fuzzy metric spaces. Fundamental concepts of cauchy sequences, contraction mapping and Banach fixed point theorems are extended in terms of fuzzy metric spaces. These definitions and theorems will be helpful in next chapters.

1.1 Fixed points of single valued mappings

1.1.1 Fixed point

Definition 1.1.1. [1] A fixed point of a mapping $T : X \to X$ of a set X into itself is an $x \in X$ which is mapped onto itself, that is,

$$Tx = x$$

the image Tx coincides with x.
A self mapping of a set can have no fixed points, a unique fixed point, and infinitely many fixed points. as illustrated by the following examples:

Example 1.1.1. [1] A translation has no fixed points.

Example 1.1.2. Let T: $\mathbb{N} \to \mathbb{N}$, be defined as $T(x) = x + 2$ for all $x \in \mathbb{N}$. Then T has no fixed point.

Example 1.1.3. [1] A rotation of the plane has a single fixed point(the center of rotation)

Example 1.1.4. Given a natural number n, Let T: $\mathbb{R} \to \mathbb{R}$ be defined as follows

$$T(x) = \begin{cases} x, & x \in \{1, 2, \cdots, n\} \\ \\ x^2 + 1, & x \notin \{1, 2, \cdots, n\} \end{cases}$$

Then T has exactly n fixed points.

1.2 Fixed point Theorems

Fixed point theorems has a enormous impact in the development of basic mathematical settings. Several fixed point theorems has been presented in many years dealing with various types of contractive mapping.

In this section Banach [1], Kannan's [11], Chatterjee's [12] fixed point theorems are presented which have a tremendous applications in many fields.

Definition 1.2.1. [1] Let $X = (X, d)$ be a metric space. A mapping $T : X \to X$ is called contraction on X if there is a positive real number $\alpha < 1$ such that for all $x,\ y\ \in X$

$$d(Tx,\ Ty) \leq \alpha d(x,\ y) \tag{1.2.1}$$

Lemma 1.2.1. *[1] A contraction T on a metric space X is a continuous mapping.*

Theorem 1.2.2. *[1]**Banach Fixed point theorem***
Consider a metric space $X = (X,\ d)$ where $X \neq \phi$. Suppose that X is complete and let $T : X \to X$ be a contraction on X. Then T has precisely one fixed point.

Corollary 1.2.3. *[1] From the conditions of Banach fixed point theorem following result holds:*

(i) Iterative sequence

$$x_0,\ x_1 = Tx_0,\ x_2 = Tx_1 = T^2x_0, \cdots,\ x_n = T^n x_0 \cdots$$

with $x_0 \in X$ converges to x(fixed point of T).

(ii) Prior estimate

$$d(x_m,\ x) \leq \frac{\alpha^m}{1-\alpha} d(x_0,\ x_1)$$

(iii) Posterior estimate

$$d(x_m,\ x) \leq \frac{\alpha}{1-\alpha} d(x_{m-1},\ x_m)$$

Theorem 1.2.4. *[11]**Kannan's fixed point theorem***
Let $(X,\ d)$ be a complete metric space and $T : X \to X$ is a contractive mapping with $k \in \left[0,\ \frac{1}{2}\right)$ such that for all $x,\ y \in X$

$$d(T(x),\ T(y)) \leq k[d(x,\ T(x)) + d(y,\ T(y))]$$

Then T has unique fixed point in X.

Theorem 1.2.5. *[12]**Chatterjee's fixed point theorem***
Let $(X,\ d)$ be a complete metric space and $T : X \to X$ is a contractive mapping with $k \in \left[0,\ \frac{1}{2}\right)$ such that for all $x,\ y \in X$

$$d(T(x),\ T(y)) \leq k[d(x,\ T(y)) + d(y,\ T(x)]$$

Then T has unique fixed point in X.

1.3 Fuzzy metric spaces

1.3.1 Fuzzy sets

The term fuzzy set was initiated by Zadeh [5] in 1965. Fuzzy set is an extension of classical notation of set. It consist of class of objects which characterized degree of membership of each element in close interval $[0, 1]$.

Definition 1.3.1. [5] let X be a space of points(objects), with a generic element of X denoted by x. Thus, $X = x$. A fuzzy set A in X is characterized by a membership function $f_A(x)$ which associates with each point in X a real number in the interval $[0, 1]$.

1.3.2 Fuzzy metric spaces

A number of authors have discussed the term fuzzy metric spaces in different ways [3], [13], [14], [15]. Structure of fuzzy metric spaces in this section is presented in the sense of George and Veeramani [3].

Definition 1.3.2. [3]
A binary operation $*$:$[0,\ 1]\times[0, 1] \rightarrow [0, 1]$ is continuous t-norm if $*$ satisfies the following conditions:

(a) $*$ is commutative and associative;

(b) $*$ is continuous;

(c) $a * 1 = a$ for all $a \in [0,\ 1]$

(d) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ $\forall$ a, b, c, $d \in [0,\ 1]$

Example 1.3.1. [3]

(1) $a * b = ab$

(2) $a * b$=min $(a,\ b)$

Definition 1.3.3. [3] (Fuzzy metric space)
A 3-tuple $(X,\ M,\ *)$ is said to be a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0,\ \infty)$ satisfying the following conditions;

(1) M$(x,\ y,\ t)> 0$,

(2) M$(x,\ y,\ t)$=1 $\Leftrightarrow x = y$,

(3) M$(x,\ y,\ t)$=M$(y,\ x,\ t)$

(4) M$(x,\ y,\ t)*$ M$(y,\ z,\ s)\leq$ M$(x,\ z,\ t+s)$

(5) M$(x,\ y,\ \cdot)$:$(0,\ \infty)\rightarrow [0,\ 1]$ is continuous $x,\ y,\ z \in X$ and $t,\ s > 0$

Example 1.3.2. [3] Let X=$\mathbb{R}$, Define $a*$b=$a\ b$ and

$$M(x,\ y,\ t) = \left[\exp\left(\frac{\mid x-y\mid}{t}\right)\right]^{-1}$$

for all $x,\ y \in X$ and t $\in (0,\ \infty)$. Then $(X,\ M,\ *)$ is a fuzzy metric space.

Proof. (1) $M(x,y,t) > 0$

(2) $M(x,y,t)$=1 $\Leftrightarrow x = y$

(3) $M(x,y,t)$=$\frac{1}{\left[\exp\left(\frac{|x-y|}{t}\right)\right]^{-1}} = \frac{1}{\left[\exp\left(\frac{|y-x|}{t}\right)\right]^{-1}}$

Hence M$(x,\ y,\ t)$=M$(y,\ x,\ t)$

(4) M$(x,\ y,\ t) *$ M$(y,\ z,\ s)\leq$ M$(x,\ z,\ t+s)$
we know that

$$\mid x-z\mid \leq \left(\frac{t+s}{t}\right) \mid x-y\mid + \left(\frac{t+s}{t}\right) \mid y-z\mid$$
$$\frac{\mid x-z\mid}{t+s} \leq \frac{\mid x-y\mid}{t} + \frac{\mid y-z\mid}{s}$$

therefore

$$\exp(\frac{\mid x-z \mid}{t+s}) \leq \exp(\frac{\mid x-y \mid}{t}) + \exp(\frac{\mid y-z \mid}{s})$$

Thus

$$M(x,\ y,\ t) * M(y,\ z,\ s) \leq M(x,\ z,\ t+s)$$

(5) M(x, y, $\cdot$) :(0, ∞) $\rightarrow$ [0, 1] is continuous.

Hence $(X,\ M,\ *)$ is a fuzzy metric space. □

Remark 1.3.1. [3] We can replace $\mathbb{R}$ by any metric space X and $\mid x-y \mid$ by $d(x,\ y)$ in above example. Above example also hold with other t-norm $a * b$=min $(a,\ b)$.

Lemma 1.3.1. *[4] M(x, y, ·) is non decreasing for all* $x,\ y,\ z \in X$*.*

Example 1.3.3. [3] Let $(X,\ d)$ be a metric space. Define $a * b = ab$ and

$$M(x,\ y,\ t) = \frac{kt^n}{kt^n + md(x,\ y)},\ k,\ m,\ n \in\ \mathbb{R}^+$$

Then $(X,\ M,\ *)$ is a fuzzy metric space.

Remark 1.3.2. [3] Note that the above example holds even with the t-norm $a * b$=min$(a,\ b)$. In the above example by taking $k = m = n = 1$ we get

$$M(x,\ y,\ t) = \frac{t}{t + d(x,\ y)}$$

We call this fuzzy metric induced by a metric d the standard fuzzy metric.

Example 1.3.4. [3] Let $X = \mathbb{N}$. Define $a * b = ab$ and

$$M(x,\ y,\ t) = \begin{cases} \frac{x}{y}, & \text{if } x \leq y; \\ \\ \frac{y}{x}, & \text{if } y \leq x,\ \forall t > 0 \end{cases}$$

Then $(X,\ M,\ *)$ is a fuzzy metric space.

Remark 1.3.3. [3] It is interesting to note that there exist no metric on X satisfying

$$M(x,\ y,\ t) = \frac{t}{t + d(x,\ y)}$$

Where $M(x,\ y,\ t)$ is as defined in the above example. Also note that the above function M is not a fuzzy metric with the t-norm defined as $a * b$=min(a, b).

1.4 Cauchy sequences and contractive mappings in fuzzy metric spaces

Definition 1.4.1. [3] A sequence x_n in a fuzzy metric space $(X,\ M,\ *)$ is said to be cauchy sequence $\Leftrightarrow$

$$\lim_{n\to\infty} M(x_{n+p},\ x_n,\ t) = 1\ ,\ t > 0,\ p > 0.$$

Remark 1.4.1. [3] A fuzzy metric space in which every Cauchy sequence is convergent is called a complete Fuzzy metric space.

Example 1.4.1. [17] Let $X = \mathbb{R}^+$, with the metric d defined by $d(x,\ y) =\mid x - y \mid$, and t-norm $a * b$=min$(a,\ b)$, we define

$$M(x,\ y,\ t) = \frac{t}{t + d(x,\ y)},\ \forall\ x,\ y \in X,\ t > 0.$$

Clearly $(X,\ M,\ *)$ is a complete fuzzy metric space.

Remark 1.4.2. [3] We note that with the above definition, even $\mathbb{R}$ fails to be complete. For example, consider

$$S_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\ in\ (\mathbb{R},\ M,\ \cdot)$$

where

$$M(x,\ y,\ t) = \frac{t}{t + d(x,\ y)},\ d\ is\ metric\ on\ \mathbb{R}.$$

Now

$$M(S_{n+p},\ S_n,\ t) = \frac{t}{t+ \mid S_{n+p} - S_n \mid}$$
$$= \frac{t}{t + (\frac{1}{n} + 1) + (\frac{1}{n} + 2) + \cdots + (\frac{1}{n} + p)}$$

Therefore

$$\lim_{n\to\infty} M(x_{n+p},\ x_n,\ t) = 1.$$

Thus S_n is a cauchy sequence in the fuzzy metric space $\mathbb{R}$. If $\mathbb{R}$ is fuzzy complete then there exist $x \in \mathbb{R}$ such that

$$M(S_n,\ x,\ t) \to 1\ as\ n \to \infty.$$

From this it follows that

$$\frac{t}{t+ \mid S_n - x \mid} \to 1\ as\ n \to \infty.$$

Further,

$$\mid S_n - x \mid \to 0\ as\ n \to \infty$$

and so $S_n \to x$ in $\mathbb{R}$ which is not true.
Hence to make $\mathbb{R}$ complete fuzzy metric spaces we redefine Cauchy sequence as follows.

Definition 1.4.2. [3] A sequence x_n in a fuzzy metric space $(X, M, *)$ is a cauchy sequence iff for each $\epsilon > 0$, $t > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$M(x_n,\ x_m,\ t) > 1 - \epsilon\ \forall\ n,\ m\ \geq\ n_0.$$

Definition 1.4.3. [13] Let $(X,\ M,\ *)$ be a fuzzy metric space. We will say that mapping $T : X \to X$ is continuous if for given $r,\ t > 0,\ 0 < r < 1$ we can find $r_0 \in (0,\ 1)$, $t_0 > 0$, such that

$$M(x,\ y,\ t_0) > 1 - r_0, \Rightarrow M(Tx,\ Ty,\ \frac{t}{2}) > 1 - r$$

Definition 1.4.4. [6] Let $(X,\ M,\ *)$ be a fuzzy metric space. We will say that mapping $T : X \to X$ is t uniformly continuous if for each ϵ, with $0 < \epsilon < 1$, there exist $0 < r < 1$ such that

$$M(x,\ y,\ t) \geq 1 - r, \Rightarrow M(Tx,\ Ty,\ t) \geq 1 - \epsilon$$

for each $x, y, \in X$ and $t > 0$.

Proposition 1.4.1. *[6] Let $(X,\ M,\ *)$ be a fuzzy metric space. We will say that mapping $T : X \to X$ is t uniformly continuous if for each $\delta > 0$, there exist $\eta > 0$, such that*

$$\frac{1}{M(x,\ y,\ t)} - 1 \leq \eta, \ \Rightarrow \ \frac{1}{M(Tx,\ Ty,\ t)} - 1 \leq \delta$$

for each $x, y, \in X$, and $t > 0$.

Definition 1.4.5. [6] Let $(X,\ M,\ *)$ be a fuzzy metric space. We say that the mapping $T : X \to X$ is fuzzy contractive if there exist $k \in (0, 1)$ such that

$$\frac{1}{M(Tx,\ Ty,\ t)} - 1 \leq k\left(\frac{1}{M(x,\ y,\ t)} - 1\right) \tag{1.4.1}$$

for each $x,\ y, \in X$ and $t > 0$. Where k is called fuzzy contractive constant of T.

Definition 1.4.6. [6] Let $(X,\ M,\ *)$ be a fuzzy metric space. We will say that the sequence (x_n) in X is fuzzy contractive if there exist $k \in (0,\ 1)$ such that

$$\frac{1}{M(x_{n+1},\ Tx_{n+2},\ t)} - 1 \leq k[\frac{1}{M(x_n,\ x_{n+1},\ t)} - 1] \tag{1.4.2}$$

for all $t > 0$, $n \in N$.

Proposition 1.4.2. *[13] Every fuzzy contraction mapping on a fuzzy metric space is continuous.*

Remark 1.4.3. Converse of above proposition need not to be true. i-e a continuous mapping is not fuzzy contractive. It is illustrated by the following example.

Example 1.4.2. Let $X = \mathbb{R}$ and consider $(X,\ M,\ *)$ fuzzy metric space as following

$$M(x,\ y,\ t) = \left[\exp\left(\frac{\mid x - y \mid}{t}\right)\right]^{-1}$$

for all $x, y \in$ X and t $\in (0,\ \infty)$. Now define $T(x) = x^2$. Then

$$M(x,y,t) = \frac{1}{\left[\exp\left(\frac{|x-y|}{t}\right)\right]}$$

$$\Rightarrow M(x,\ y,\ t) \geq 1 - r_0$$

Where

$$\Rightarrow r_0 \geq \frac{\left[\exp\left(\frac{|x-y|}{t}\right)\right] - 1}{\left[\exp\left(\frac{|x-y|}{t}\right)\right]}$$

Then

$$M(Tx,\ Ty, \frac{t}{2}) = \frac{1}{\left[\exp\left(\frac{|x^2-y^2|}{\frac{t}{2}}\right)\right]} \geq 1 - r$$

where

$$r \geq \frac{\exp\left(\frac{|x^2-y^2|}{\frac{t}{2}}\right) - 1}{\exp\left(\frac{|x^2-y^2|}{\frac{t}{2}}\right)}$$

Hence by definition T is continuous.

But

$$M(Tx,\ Ty,\ t) = \left[\exp\left(\frac{\mid x^2 - y^2 \mid}{t}\right)\right]^{-1}$$

Now consider

$$\frac{1}{M(Tx,Ty,t)} - 1 = \left[\exp\left(\frac{\mid x^2 - y^2 \mid}{t}\right)\right] - 1 \tag{1.4.3}$$

$$k[\frac{1}{M(x,y,t)} - 1] = k\left[\exp\left(\frac{\mid x - y \mid}{t}\right) - 1\right] \tag{1.4.4}$$

From these two equations, we observe that definition 1.4.5 does not hold.

Hence T is not fuzzy contractive.

Proposition 1.4.3. *[6] Let $(X,\ M,\ *)$ be a fuzzy metric space. If mapping $T : X \to X$ is fuzzy contractive then T is uniformly continuous.*

1.5 Fixed point theorems in fuzzy metric spaces

A number of authors have made valuable contribution by extending Banach's [23] result in fuzzy metric spaces. George & Veeramani [3], Gregori & Sapena [6], S. Heilpern [20], Nadler [22], M. Grabiec [4] have been obtained these results for different contractive conditions in fuzzy metric spaces.
In this section Banach fixed point theorem are presented in sense of M. Grabiec [4] and Gregori & Sapena [6].

Theorem 1.5.1. *[4] Let $(X,\ M,\ *)$ be a complete fuzzy metric space such that*

$$\lim_{n\to\infty} M(x_{n+p},\ x_n,\ t) = 1,\ \forall\ x,\ y\ \in\ X.$$

Let $T : X \to X$ be a mapping satisfying

$$M(Tx,\ Ty,\ kt) \geq M(x,\ y,\ t)\ \forall\ x,\ y\ \in\ X,\ 0 < k < 1.$$

Then T has unique fixed point.

Theorem 1.5.2. *[6] Let $(X,\ M,\ *)$ be a complete fuzzy metric space in which fuzzy contractive sequences are Cauchy. Let $T : X \to X$ be a fuzzy contractive mapping such that*

$$\frac{1}{M(Tx,\ Ty,\ t)} - 1 \leq k\left(\frac{1}{M(x,\ y,\ t)} - 1\right)$$

where $k \in (0,1)$ is contractive constant. Then T has a unique fixed point.

Chapter 2

Intuitionistic fuzzy metric spaces

This chapter consist of three sections. In the first section we shall present fundamental concepts and notations of intuitionistic fuzzy metric spaces which is a generalization of fuzzy metric spaces. In the sections two we redefined the primitive results of cauchy sequences and contractive mappings. Last section consist of the fixed point theorems with variety of contractive mapping and Banach fixed point theorem is also proved with $TS - IF$ contractive mapping in intuitionistic fuzzy metric spaces. Intuitionistic fuzzy metric spaces are denoted by IFMS throughout this thesis.

2.1 Basic definitions and Notations

Definition 2.1.1. [2] A binary operation $\diamond$:[0, 1] $\times$ [0, 1] $\rightarrow$ [0, 1] is continuous t-conorm if $\diamond$ satisfies the following conditions:

(a) $\diamond$ is commutative and associative;

(b) $\diamond$ is continuous;

(c) $a\diamond 0=0$ for all $a \in [0, 1]$;

(d) $a \diamond$b$\leq c \diamond$d whenever $a \leq c$ and $b \leq d$

Example 2.1.1. [16]

(1) $a \diamond b$=max(a, b)

(2) $a \diamond b$=min$(1, a + b)$

Remark 2.1.1. [2]

(a) for any $r_1,\ r_2 \in (0,\ 1)$ with $r_1 > r_2$, there exists $r_3,\ r_4 \in (0, 1)$ such that $r_1 * r_3 \geq r_2$ and $r_1 \geq r_4 \diamond r_2$

(b) for any $r_5 \in (0, 1)$ there exists $r_6,\ r_7 \in (0, 1)$ such that $r_6 * r_6 \geq r_5$ and $r_7 \diamond r_7 \leq r_5$.

Definition 2.1.2. [2] A 5-tuple $(X,\ M,\ N,\ *,\ \diamond)$ is said to be Intuitionistic Fuzzy metric space if X is an arbitrary set, $*$ is continuous t-norm $\diamond$ is continuous t-conorm and M, N are fuzzy sets on $X^2 \times (0,\ \infty)$ satisfying the following conditions: for all $x,\ y,\ z \in X,\ s,\ t > 0$;

(1) M$(x,\ y,\ t)$+ N$(x,\ y,\ t)\leq 1$;

(2) M$(x,\ y,\ t) > 0$;

(3) M$(x,\ y,\ t)$=1 $\Leftrightarrow x = y$;

(4) M$(x,\ y,\ t)$=M$(y,\ x,\ t)$;

(5) M$(x,\ y,\ t) *$ M$(y,\ z,\ s)\leq$ M$(x,\ z,\ t + s)$;

(6) M$(x,\ y,\ \cdot)$:$(0,\ \infty) \rightarrow [0,\ 1]$ is continuous;

(7) N$(x,\ y,\ t) > 0$;

(8) N$(x,\ y,\ t)$=0 $\Leftrightarrow x = y$;

(9) N(x, y, t)=N(y, x, t);

(10) N(x, y, t) $\diamond$ N(y, z, s) $\geq$ N(x, z, $t+s$);

(11) N(x, y, $\cdot$) :(0, ∞)$\rightarrow$ [0, 1] is continuous;

Then $(M,\ N)$ is called an intuitionistics fuzzy metric on X. The functions $M(x,\ y,\ t)$ and $N(x,\ y,\ t)$ denotes the degree of nearness and the degree of non nearness between x and y with respect to t, respectively.

Remark 2.1.2. [2] Every fuzzy metric space $(X, M, *)$ is an IFMS of the form $(X,\ M,\ 1-M,\ *,\ \diamond)$ such that t-norm $*$ and t-conorm $\diamond$ are associated. i.e.

$$x \diamond y = 1 - ((1-x) * (1-y))$$

for any $x, y \in$ X.

Remark 2.1.3. [2] In intuitionistic fuzzy metric space X, $M(x,\ y,\ \cdot)$ is non decreasing and $N(x,\ y,\ \cdot)$ is non increasing for all $x, y \in X$.

Example 2.1.2. [2] Let $(X,\ d)$ be a metric space. Denote $a * b$ =ab and $a \diamond b$=min$(1,\ a+b)$ for all $a, b \in [0,1]$ and let M_d and N_d be fuzzy sets on $X^2 \times (0, \infty)$ defined as follows:

$$M_d(x,\ y,\ t) = \frac{ht^n}{ht^n + md(x,\ y)}$$
$$N_d(x,\ y,\ t) = \frac{d(x,\ y)}{kt^n + md(x,\ y)}\ \forall\ h,\ k,\ n \in \mathbb{R}^+.$$

Then $(X,\ M_d,\ N_d,\ *,\ \diamond)$ is an IFMS.

Remark 2.1.4. [2] Above example also hold with t-norm $a * b$=min$(a,\ b)$ and the t-conorm $a \diamond b$=$max(a,\ b)$ and hence $(M,\ N)$ is an intuitionistic fuzzy metric with respect to any continuous t-norms and continuous t-conorm. In above example by taking $h = k = m = n = 1$, we get

$$M_d(x,\ y,\ t) = \frac{t}{t + d(x,\ y)}$$
$$N_d(x,\ y,\ t) = \frac{d(x,\ y)}{t + d(x,\ y)}$$

We call this intuitionistic fuzzy metric induced by a metric d the standard IFM.

Example 2.1.3. [2] Let $X = \mathbb{N}$. Define $a * b$=max$(0,\ a+b-1)$ and $a \diamond b$=$a+b-ab$ for all $a, b \in [0,\ 1]$ and let M and N be fuzzy sets on $X^2 \times (0,\ \infty)$ as follows:

$$M(x,\ y,\ t) = \begin{cases} \frac{x}{y}, & \text{if } x \le y; \\ \frac{y}{x}, & \text{if } y \le x; \end{cases}$$

$$N(x,\ y,\ t) = \begin{cases} \frac{y-x}{y}, & \text{if } x \le y; \\ \frac{x-y}{x}, & \text{if } y \le x; \end{cases}$$

for all $x,\ y \in X$ and $t > 0$. Then $(X,\ M,\ N,\ *,\ \diamond)$ is an IFMS.

Remark 2.1.5. [2] Note that in the above example, t-norms $*$ and t-conorms $\diamond$ are not associated. And there exists no metric d on X satisfying

$$M_d(x,\ y,\ t) = \frac{t}{t + d(x,\ y)}$$
$$N_d(x, y, t) = \frac{d(x,\ y)}{t + d(x,\ y)}$$

where $M(x,\ y,\ t)$ and $N(x,\ y,\ t)$ are as defined in above example.

Remark 2.1.6. [2] If we define $t-norms$ and $t-conorms$ as $a * b$=min$(a,\ b)$ and $a \diamond b$=max$(a,\ b)$ with

$$M_d(x,\ y,\ t) = \frac{t}{t + d(x,\ y)},\ N_d(x,\ y,\ t) = \frac{d(x,\ y)}{t + d(x,\ y)}$$

then $(M,\ N)$ is not an intuitionistic fuzzy metric space.

2.2 Cauchy sequences and contractive mappings in IFMS

Definition 2.2.1. [7] A sequence $(x_n)_n$ in an IFMS is said to be cauchy sequence $\Leftrightarrow$ for each $r \in (0,\ 1)$ and $t > 0$ there exist $n_0 \in N \ni$

$$M(x_n,\ x_m,\ t) > 1 - r \ and \ N(x_n,\ x_m,\ t) < r \ \forall\ n,\ m\ \geq n_0.$$

A sequence (x_n) in an IFMS is said to converge to $x \in X \Leftrightarrow$ for each $r \in (0, 1)$ and $t > 0$ there exist $n_0 \in N \ni$

$$M(x_n,\ x,\ t) > 1 - r \ and \ N(x_n,\ x,\ t) < r \ \forall\ n,\ m \geq\ n_0.$$

Definition 2.2.2. [7] A sequence $(x_n)_n$ in an intuitionistic fuzzy metric space is a cauchy sequence $\Leftrightarrow$

$$\lim_{n\to\infty} M(x_n, x_{n+p}, t) = 1 \ and \ \lim_{n\to\infty} N(x_n,\ x_{n+p},\ t) = 0$$

A sequence (x_n) in an intuitionistic fuzzy metric space is said to converge to $x \in X$

$$\Leftrightarrow \lim_{n\to\infty} M(x_n, x, t) = 1 \ and \ \lim_{n\to\infty} N(x_n,\ x,\ t) = 0$$

Example 2.2.1. [21] Let $X = \{\frac{1}{n} : n \in N\} \cup \{0\}$ with the metric d defined by

$$d(x,\ y) = \mid x - y \mid$$

for all $x, y \in X$ and $t \in [0,\ \infty)$ then define

$$M(x, y, t) = \begin{cases} 0, & \text{if } t = 0 \\ \frac{t}{t+|x-y|}, & \text{if } t > 0 \end{cases}$$

$$N(x,y,t) = \begin{cases} 1, & \text{if } t = 0 \\ \\ \frac{|x-y|}{kt+|x-y|}, & \text{if } t > 0, k > 0 \end{cases}$$

Clearly $(X,\ M,\ N,\ *,\ \diamond)$ is complete IFMS on X.

Definition 2.2.3. [7] Let $(X,\ M,\ N,\ *,\ \diamond)$ be a IFMS. We will say the mapping $T : X \to X$ is t uniformly continuous if for each ϵ with $0 < \epsilon < 1$, there exists $0 < r < 1$, such that

$$M(x,\ y,\ t) \geq 1 - r \text{ and } N(x,\ y,\ t) \leq r$$

$$\Rightarrow M(T(x),\ T(y),\ t) \geq 1 - \epsilon \text{ and } N(T(x),\ T(y),\ t) \leq \epsilon$$

for each $x, y \in X$ and $t > 0$.

Proposition 2.2.1. *[7] Let $(X,\ M,\ N,\ *,\ \diamond)$ be a IFMS. A mapping $T : X \to X$ is fuzzy contractive then T is uniformly continuous.*

Definition 2.2.4. [2] Let $(X,\ M,\ N,\ *,\ \diamond)$ be a IFMS. We will say that the sequence (x_n) in X is intuitionistic fuzzy contractive if there exists $k \in (0,\ 1)$ such that

$$\frac{1}{M(x_{n+1},\ x_{n+2},\ t)} - 1 \leq k \left(\frac{1}{M(x_n,\ x_{n+1},\ t)} - 1 \right)$$

$$\frac{1}{N(x_{n+1},\ x_{n+2},\ t)} - 1 \geq \frac{1}{k} \left(\frac{1}{N(x_n,\ x_{n+2},\ t)} - 1 \right)$$

$\forall\ t > 0, n \in N$

2.3 Fixed point theorems in IFMS

This section based on the results that are helpful in developing the fixed point theorems in IFMS. Comparison of different contractive conditions are presented by many authors [2, 19, 18, 7].

Lemma 2.3.1. *[7] Let* $(X,\ M,\ N,\ *,\ \diamond)$ *be IFMS and* $T : X \to X$ *is* t *uniformly continuous on* X*. If*

$$(x_n) \longrightarrow x \ as \ n \longrightarrow \infty.$$

then

$$T(x_n) \longrightarrow T(x) \ as \ n \longrightarrow \infty.$$

Lemma 2.3.2. *[7] Let* $(X,\ M,\ N,\ *,\ \diamond)$ *be IFMS. If*

$$(x_n) \longrightarrow x \ and \ (y_n) \longrightarrow y.$$

then

$$M(x_n,\ y_n,\ t) \longrightarrow M(x,\ y,\ t)$$

and

$$N(x_n,\ y_n,\ t) \longrightarrow N(x,\ y,\ t) \ as \ n \longrightarrow \infty.$$

for all $t > 0$ *in* $\mathbb{R}$*.*

Definition 2.3.1. [19] Let $(X, M, N, *, \diamond)$ be intuitionistic fuzzy metric space. We say that the mapping $T : X \longrightarrow X$ is intuitionistic fuzzy contractive if there exist $k \in (0,\ 1)$ such that

$$\frac{1}{M(Tx,\ Ty,\ t)} - 1 \leq k\left(\frac{1}{M(x,\ y,\ t) - 1}\right)$$

$$N(Tx,\ Ty,\ t) \leq kN(x,\ y,\ t)$$

for all $x, y \in X$ and $t > 0$.

Theorem 2.3.3. *[19] Let* $(X,\ M,\ N,\ *,\diamond)$ *be complete IFMS and* $T : X \to X$ *be intuitionistic fuzzy contractive mapping. Then* T *has a unique fixed point.*

Definition 2.3.2. [18] Let $(X,\ M,\ N,\ *,\ \diamond)$ be intuitionistic fuzzy metric space. A mapping $T : X \longrightarrow X$ is called an intuitionistic fuzzy contraction depend on Δ (IFC_{Δ}) if there exists a mapping $\Delta : X \to [0,\ 1)$ where $\Delta(Tx) \leq \Delta(x)$ such that

$$\frac{1}{M(Tx,\ Ty,\ t)} - 1 \leq \Delta(x)\left(\frac{1}{M(x,\ y,\ t) - 1}\right)$$

and

$$N(Tx,\ Ty,\ t) \leq \Delta(x)N(x,\ y,\ t)$$

for all $x, y \in X$ and $t > 0$.

Theorem 2.3.4. *[18] Let $(X,\ M,\ N,\ *,\diamond)$ be complete IFMS and $T : X \to X$ be (IFC_{Δ}) mapping, then T has a unique fixed point.*

Definition 2.3.3. [7] Let $(X,\ M,\ N,\ *,\ \diamond)$ be IFMS and $T : X \to X$. T is said to be $TS - IF$ contractive mapping if there exists $k \in (0, 1)$ such that

$$kM(T(x),\ T(y),\ t) \geq M(x,\ y,\ t) \tag{2.3.1}$$

and

$$\frac{1}{k}M(T(x),\ T(y),\ t) \leq M(x,\ y,\ t) \tag{2.3.2}$$

for all $t > 0$.

Theorem 2.3.5. *[7] Let $(X,\ M,\ N,\ *,\ \diamond)$ be complete IFMS and $T : X \to X$ be $TS - IF$ contractive mapping with k its contractive constant. Then T has a unique fixed point.*

Proof. Let $x \in X$ and $x_n = T^n(x)$ for all $n \in \mathbb{N}$. Now for each $t > 0$

$$\begin{aligned} kM(x_2,\ x_1,\ t) &= kM(T(x_1),\ T(x),\ t) \\ &\geq M(x_1,\ x,\ t) \\ i-e.,\quad kM(x_2,\ x_1,\ t) &\geq M(x_1,\ x,\ t) \\ \frac{1}{k}N(x_2,\ x_1,\ t) &= \frac{1}{k}N(T(x_1),\ T(x),\ t) \\ &\leq N(x_1,\ x,\ t) \\ i-e.,\quad \frac{1}{k}N(x_2,\ x_1,\ t) &\leq N(x_1,\ x,\ t) \end{aligned}$$

Again

$$\begin{aligned} kM(x_3,\ x_2,\ t) &= kM(T(x_2),\ T(x_1),\ t) \\ &\geq M(x_2,\ x_1,\ t) \\ \Rightarrow k^2M(x_3,\ x_2,\ t) &\geq kM(x_2,\ x_1,\ t) \geq M(x_1,\ x,\ t) \\ i-e.,\quad k^2M(x_3,\ x_2,\ t) &\geq M(x_1,\ x,\ t) \\ \frac{1}{k}N(x_3,\ x_2,\ t) &= \frac{1}{k}N(T(x_2),\ T(x_1),\ t) \\ &\leq N(x_2,\ x_1,\ t) \\ \Rightarrow \frac{1}{k^2}N(x_3,\ x_2,\ t) &\leq \frac{1}{k}N(x_2,\ x_1,\ t) \leq N(x_1,\ x,\ t) \\ i-e.,\quad \frac{1}{k^2}N(x_3,\ x_2,\ t) &\leq N(x_1,\ x,\ t) \end{aligned}$$

By mathematical induction, we have

$$\begin{aligned} &k^nM(x_{n+1},\ x_n,\ t) \geq M(x_1,\ x,\ t)\ and \\ &\frac{1}{k^n}N(x_{n+1},\ x_n,\ t) \leq N(x_1,\ x,\ t),\ for\ all\ t > 0. \end{aligned} \tag{2.3.3}$$

We now verify that $\{x_n\}$ is a cauchy sequence in $(X,\ M,\ N,\ *,\ \diamond)$. Let $t_1 = \frac{t}{p}$ and from equations 2.3.3

$$\begin{aligned}
M(x_n,\ x_{n+p},\ t) &\geq M(x_n,\ x_{n+1},\ t_1) * M(x_{n+1},\ x_{n+2},\ t_1) * \cdots * M(x_{n+p-1},\ x_{n+p},\ t_1) \\
&= \left(\frac{1}{k^n} k^n M(x_n,\ x_{n+1},\ t_1)\right) * \left(\frac{1}{k^{n+1}} k^{n+1} M(x_{n+1},\ x_{n+2},\ t_1)\right) * \cdots \\
&\quad * \left(\frac{1}{k^{n+p-1}} k^{n+p-1} M(x_{n+p-1},\ x_{n+p},\ t_1)\right) \\
&\geq \left(\frac{1}{k^n} M(x,\ x_1,\ t_1)\right) * \left(\frac{1}{k^{n+1}} M(x,\ x_1,\ t_1)\right) * \cdots * \left(\frac{1}{k^{n+p-1}} M(x,\ x_1,\ t_1)\right) \\
&\geq \left(\frac{1}{k^n} M(x,\ x_1,\ t)\right) * \cdots * \left(\frac{1}{k^n} M(x,\ x_1,\ t)\right) \\
&= \left(\frac{1}{k^n} M(x,\ x_1,\ t)\right)
\end{aligned}$$

$$\Rightarrow 1 < \lim_{n\to\infty} \left(\frac{1}{k^n} M(x,\ x_1,\ t)\right) \leq \lim_{n\to\infty} M(x_n,\ x_{n+p},\ t) \leq 1$$

$$\Rightarrow \lim_{n\to\infty} M(x_n,\ x_{n+p},\ t) = 1 \tag{2.3.4}$$

$$\begin{aligned}
N(x_n,\ x_{n+p},\ t) &\leq N(x_n,\ x_{n+1},\ t_1) \diamond N(x_{n+1},\ x_{n+2},\ t_1) \diamond \cdots \diamond N(x_{n+p-1},\ x_{n+p},\ t_1) \\
&\leq \left(\frac{1}{k^n} k^n N(x_n,\ x_{n+1},\ t_1)\right) \diamond \left(\frac{1}{k^{n+1}} k^{n+1} N(x_{n+1},\ x_{n+2},\ t_1)\right) \diamond \cdots \\
&\quad \diamond \left(\frac{1}{k^{n+p-1}} k^{n+p-1} N(x_{n+p-1},\ x_{n+p},\ t_1)\right) \\
&\leq \left(\frac{1}{\beta^n} N(x,\ x_1,\ t_1)\right) \diamond \left(\frac{1}{k^{n+1}} N(x,\ x_1,\ t_1)\right) \diamond \cdots \diamond \left(\frac{1}{k^{n+p-1}} N(x,\ x_1,\ t_1)\right) \\
&\leq \left(\frac{1}{k^n} N(x,\ x_1,\ t)\right) \diamond \cdots \diamond \left(\frac{1}{k^n} N(x,\ x_1,\ t)\right) \\
&= \left(\frac{1}{k^n} N(x,\ x_1,\ t)\right)
\end{aligned}$$

$$\Rightarrow 0 > \lim_{n\to\infty} \left(\frac{1}{k^n} N(x,\ x_1,\ t)\right) \geq \lim_{n\to\infty} N(x_n,\ x_{n+p},\ t) \geq 0$$

$$\Rightarrow \lim_{n\to\infty} N(x_n,\ x_{n+p},\ t) = 0 \tag{2.3.5}$$

Hence from equations 2.3.4, and 2.3.5 $\{x_n\}_n$ is a cauchy sequence. So, there exist $y \in X$ such that $x_n \longrightarrow y$ as $n \to \infty$.

Now,

$$\begin{aligned} kM(T(x_n,\ T(y),\ t) &\geq M(x_n),\ y,\ t) \\ i-e.,\quad M(T(x_n),\ T(y),\ t) &\geq \frac{1}{k}M(x_n),\ y,\ t) \\ \Rightarrow \lim_{n\to\infty} M(T(x_n),\ T(y),\ t) &\geq \lim_{n\to\infty} \frac{1}{k}M(x_n),\ y,\ t) = \frac{1}{k} = 1 \end{aligned}$$

$$\begin{aligned} \Rightarrow 1 < \lim_{n\to\infty} M(T(x_n),\ T(y),\ t) \leq 1 \\ \Rightarrow M(T(x_n),\ T(y),\ t) = 1 \end{aligned}$$

Again

$$\begin{aligned} N(T(x_n),\ T(y),\ t) &\leq kN(x_n),\ y,\ t) \\ \Rightarrow \lim_{n\to\infty} N(T(x_n),\ T(y),\ t) &\leq \lim_{n\to\infty} kN(x_n),\ y,\ t) = 0 \\ \Rightarrow \lim_{n\to\infty} N(T(x_n),\ T(y),\ t) &= 0 \end{aligned}$$

Thus applying the definition of IFMS i-e. , $M(x, y, t) = 1$ and $N(x, y, t) = 0 \Leftrightarrow x = y$

$$\lim_{n\to\infty} M(T(x_n),\ T(y),\ t) = 1\ and\ \lim_{n\to\infty} N(T(x_n),\ T(y),\ t) = 0,\ for\ all\ t > 0.$$

$$\Rightarrow \lim_{n\to\infty} T(x_n) = T(y) \Rightarrow \lim_{n\to\infty} x_{n+1} = T(y)$$

i-e. , $y = T(y)$, $\Rightarrow$ y is a fixed point of T.

To prove the uniqueness, assume $T(z) = z$ for some $z \in X$ then for $t > 0$, we have

$$\begin{aligned}
M(y,\ z,\ t) &= M(T(y),\ T(z),\ t) \\
&\geq \frac{1}{k} M(y,\ z,\ t) \\
&= \frac{1}{k} M(T(y),\ T(z),\ t) \\
&\geq \frac{1}{k^2} M(y,\ z,\ t) \\
&\vdots \\
&\geq \frac{1}{k^n} M(y,\ z,\ t) \longrightarrow \infty \ as\ n \longrightarrow\ \infty
\end{aligned}$$

$$\Rightarrow 1 < \lim_{n\to\infty} \frac{1}{k^n} M(y,\ z,\ t) \leq M(y,\ z,\ t) \leq 1 \Rightarrow M(y,\ z,\ t) = 1$$

$$\begin{aligned}
N(y,\ z,\ t) &= N(T(y),\ T(z),\ t) \\
&\leq kN(y,\ z,\ t) \\
&= kN(T(y),\ T(z),\ t) \\
&\leq k^2 N(y,\ z,\ t) \\
&\vdots \\
&\leq k^n N(y,\ z,\ t) \longrightarrow 0 \ as\ n \longrightarrow\ \infty
\end{aligned}$$

$$\Rightarrow 0 < \lim_{n\to\infty} \frac{1}{k^n} N(y,\ z,\ t) \leq N(y,\ z,\ t) \leq 0 \Rightarrow N(y,\ z,\ t) = 0$$

Hence by definition of IFMS i-e. , $M(x, y, t) = 1$ and $N(x, y, t) = 0 \Leftrightarrow x = y$
$\Rightarrow y = z$.
This completes the proof. □

Chapter 3

Fixed point theorems for generalized contraction mapping in IFMS

In 2012 Sintunavarat and Kumam [18] redefined the contractive constant of Rafi and Noorani's theorem in 2.3.3. New defined contractive constant is more general and easier to find.

Motivated by the work of Sintunavarat and Kumam [18]in this chapter we have develop some new contractive mapping and thereafter we proved well known kannan's [11] and chatterjee's [12] type $(TS - IF_{\Delta})$ Banach fixed point theorem for these mappings.

3.1 Main Results

Definition 3.1.1. Let $(X, M, N, *, \diamond)$ be IFMS and $T : X \to X$. T is said to be $TS - IF$ contractive mapping depend on Δ $(TS - IF_{\Delta})$ if there exists a mapping

$\Delta : X \to (0,\ 1)$ where

$$\Delta(Tx) \leq \Delta(x) \tag{3.1.1}$$

such that

$$\Delta(x)M(T(x),\ T(y),\ t) \geq M(x,\ y,\ t) \tag{3.1.2}$$

and

$$\frac{1}{\Delta(x)}N(T(x),\ T(y),\ t) \leq N(x,\ y,\ t) \tag{3.1.3}$$

for all $t > 0$.

Theorem 3.1.1. *Let $(X,\ M,\ N,\ *,\diamond)$ be complete IFMS and $T : X \to X$ be $(TS - IF_\Delta)$ mapping with Δ its contractive constant. Then T has a unique fixed point.*

Proof. Let x_0 be an arbitrary point in X. We can construct the sequence x_m in X by

$$x_m = T^m x_0 = Tx_{m-1}$$

for $m = 1, 2, \cdots$ Now for each $t > 0$ and using equations 3.1.1 and 3.1.2 We obtain

$$\begin{aligned}
M(x_{m+1},\ x_m,\ t) &= M(Tx_m,\ Tx_{m-1},\ t) \geq \frac{1}{\Delta(x_m)}M(x_m,\ x_{m-1},\ t)\\
&= \frac{1}{\Delta(Tx_{m-1})}M(Tx_{m-1},\ Tx_{m-2},\ t) \geq \frac{1}{\Delta(x_{m-1})}\frac{1}{\Delta(x_{m-1})}M(x_{m-1},\ x_{m-2},\ t)\\
&= \frac{1}{\Delta(Tx_{m-2})}\frac{1}{\Delta(Tx_{m-2})}M(Tx_{m-2},\ Tx_{m-3},\ t)\\
&\geq \frac{1}{\Delta(x_{m-2})}\frac{1}{\Delta(x_{m-2})}\frac{1}{\Delta(x_{m-2})}M(x_{m-2},\ x_{m-3},\ t)\\
&\ \vdots\\
&\geq \frac{1}{(\Delta(x_1))^m}M(x_1,\ x_0, t)
\end{aligned}$$

Now to verify that x_m is a cauchy sequence for positive integer m and n such that $n \geq m$ let $t_1 = \frac{t}{p}$ and applying triangular inequality

$$\begin{aligned}
M(x_m,\ x_{n+p},\ t) &\geq M(x_m,\ x_{m+1},\ t_1) * M(x_{m+1},\ x_{m+2},\ t_1) * \cdots * M(x_{n+p-1},\ x_{n+p},\ t_1) \\
&= \left(\frac{1}{(\Delta(x_1))^m} (\Delta(x_1))^m M(x_m,\ x_{m+1},\ t_1) \right) \\
&\quad * \left(\frac{1}{(\Delta(x_1))^{m+1}} (\Delta(x_1))^{m+1} M(x_{m+1},\ x_{m+2},\ t_1) \right) * \cdots \\
&\quad * \left(\frac{1}{(\Delta(x_1))^{m+p-1}} (\Delta(x_1))^{m+p-1} M(x_{n+p-1},\ x_{n+p},\ t_1) \right) \\
&\geq \left(\frac{1}{(\Delta(x_1))^m} M(x_1,\ x_0, t_1) \right) * \cdots * \left(\frac{1}{(\Delta(x_1))^{m+p-1}} M(x_1,\ x_0, t_1) \right) \\
&\geq \left(\frac{1}{(\Delta(x_1))^m} M(x_1,\ x_0, t_1) \right) * \cdots * \left(\frac{1}{(\Delta(x_1))^m} M(x_1,\ x_0, t_1) \right) \\
&= \left(\frac{1}{(\Delta(x_1))^m} M(x_1,\ x_0, t_1) \right)
\end{aligned}$$

$$\Rightarrow 1 \leq \lim_{m \to \infty} \left(\frac{1}{\Delta(x_1)^m} M(x_1,\ x_0, t_1) \right) \leq \lim_{m \to \infty} M(x_m,\ x_{n+p},\ t) \leq 1$$

$$\Rightarrow \lim_{m,\ n \to \infty} M(x_m,\ x_{n+p},\ t) = 1$$

Again for positive integer m and $t > 0$ and using equations 3.1.1 and 3.1.3. We have

$$\begin{aligned}
N(x_{m+1},\ x_m,\ t) &= N(Tx_m,\ Tx_{m-1},\ t) \\
&\leq \Delta(x_m) N(x_m,\ x_{m-1},\ t) \\
&= \Delta(Tx_{m-1}) N(Tx_{m-1},\ Tx_{m-1},\ t) \\
&\leq \Delta(x_{m-1}) \Delta(x_{m-1}) N(x_{m-1},\ x_{m-1},\ t) \\
&= \Delta(Tx_{m-2}) \Delta(Tx_{m-2}) N(Tx_{m-2},\ Tx_{m-2},\ t) \\
&\leq \Delta(x_{m-2}) \Delta(x_{m-2}) \Delta(x_{m-2}) N(x_{m-2},\ x_{m-2},\ t) \\
&\ \vdots \\
&\leq (\Delta(x_1))^m N(x_1,\ x_0,\ t)
\end{aligned}$$

for $n \geq m$ let $t_1 = \frac{t}{p}$ and applying triangular inequality

$$\begin{aligned}
N(x_m,\ x_{n+p},\ t) &\leq N(x_m,\ x_{m+1},\ t_1) \diamond N(x_{m+1},\ x_{m+2},\ t_1) \diamond \cdots \diamond N(x_{n+p-1},\ x_{n+p},\ t_1) \\
&= \left(\frac{1}{(\Delta(x_1))^m} (\Delta(x_1))^m N(x_m,\ x_{m+1},\ t_1) \right) \\
&\quad \diamond \left(\frac{1}{(\Delta(x_1))^{m+1}} (\Delta(x_1))^{m+1} N(x_{m+1},\ x_{m+2},\ t_1) \right) \diamond \cdots \\
&\quad \diamond \left(\frac{1}{(\Delta(x_1))^{m+p-1}} (\Delta(x_1))^{m+p-1} N(x_{n+p-1},\ x_{n+p},\ t_1) \right) \\
&\leq \left(\frac{1}{(\Delta(x_1))^m} N(x_1,\ x_0, t_1) \right) \diamond \cdots \diamond \left(\frac{1}{(\Delta(x_1))^{m+p-1}} N(x_1,\ x_0, t_1) \right) \\
&\leq \left(\frac{1}{(\Delta(x_1))^m} N(x_1,\ x_0, t_1) \right) \diamond \cdots \diamond \left(\frac{1}{(\Delta(x_1))^m} N(x_1,\ x_0, t_1) \right) \\
&= \left(\frac{1}{(\Delta(x_1))^m} N(x_1,\ x_0, t_1) \right)
\end{aligned}$$

$$\Rightarrow 0 \geq \lim_{m \to \infty} \left(\frac{1}{\Delta(x_1)^m} N(x_1,\ x_0, t_1) \right) \geq \lim_{m \to \infty} N(x_m,\ x_{n+p},\ t) \geq 0$$

$$\Rightarrow \lim_{m,\ n \to \infty} N(x_m,\ x_{n+p},\ t) = 0$$

Hence x_m is a cauchy sequence in IFMS X. As X is a complete, there exists a point $z \in X$ such that $x_m \longrightarrow z$ as $m \longrightarrow \infty$, which implies that

$$M(x_m,\ z,\ t) \longrightarrow 1 \ \textit{and} \ N(x_m,\ z,\ t) \longrightarrow 0 \ \textit{as} \ m \longrightarrow \infty$$

Now we show that z is a fixed point of T. Since T is (IFC_Δ) for all $m \in N$ we get

$$M(Tz,\ Tx_m,\ t) \geq \frac{1}{\Delta(z)} M(z,\ x_m,\ t)$$

$$\lim_{m \to \infty} M(Tz,\ Tx_m,\ t) \geq \lim_{m \to \infty} \frac{1}{\Delta(z)} M(z,\ x_m,\ t) = \frac{1}{\Delta(z)} > 1$$

$$\Rightarrow 1 < \lim_{m \to \infty} M(Tz,\ Tx_m,\ t) \leq 1$$

$$\Rightarrow \lim_{m \to \infty} M(Tz,\ Tx_m,\ t) = 1$$

Again using the fact that T is (IFC_Δ), We have

$$N(Tz,\ Tx_m,\ t) \le \Delta(z) N(z,\ x_m,\ t)$$

Now for $t > 0$ and for all $m \in N$. Taking $m \longrightarrow \infty$, We get

$$N(Tz,\ Tx_m,\ t) \longrightarrow 0$$

In both case, it can be concluded that $Tx_m \longrightarrow Tz$. Therefore, $x_{m+1} \longrightarrow Tz$ and then $z = Tz$.

Now we show that z is a unique fixed point of T. Assume $T(z_1) = z_1$ for $z_1 \in X$. We use the notation of (IFC_Δ), for $t > 0$, and using equations 3.1.1 and 3.1.2. We have

$$\begin{aligned}
M(z,\ z_1,\ t) &= M(Tz,\ Tz_1,\ t) \\
&\ge \frac{1}{\Delta(z)} M(z,\ z_1,\ t) \\
&= \frac{1}{\Delta(Tz)} M(Tz,\ Tz_1,\ t) \\
&\ge \frac{1}{(\Delta(z))^2} M(z,\ z_1,\ t) \\
&\vdots \\
&\ge \frac{1}{(\Delta(z))^m} M(z,\ z_1,\ t) \longrightarrow 1\ as\ m\ \longrightarrow \infty
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow 1 < \lim_{m\to\infty} \frac{1}{(\Delta(z))^m} M(z,\ z_1,\ t) \le M(z,\ z_1,\ t) \le 1 \\
&\Rightarrow M(z,\ z_1,\ t) = 1
\end{aligned}$$

Similarly using equations 3.1.1 and 3.1.3. We have

$$
\begin{aligned}
N(z,\ z_1,\ t) &= N(Tz,\ Tz_1,\ t)\\
&\le \Delta(z)N(z,\ z_1,\ t)\\
&= \Delta(Tz)N(Tz,\ Tz_1,\ t)\\
&\le (\Delta(z))^2 N(z,\ z_1,\ t)\\
&\vdots\\
&\le (\Delta(z))^m N(z,\ z_1,\ t) \longrightarrow 0\ as\ m\ \longrightarrow \infty
\end{aligned}
$$

$$
\begin{aligned}
&\Rightarrow 0 \le N(z,\ z_1,\ t) \le \lim_{m\to\infty} (\Delta(z))^m N(z,\ z_1,\ t) < 0\\
&\Rightarrow N(z,\ z_1,\ t) = 0
\end{aligned}
$$

This shows that $z = z_1$. Hence z is a unique fixed point of T.
This completes the proof. □

Remark 3.1.1. Note that if we take $\Delta(x) = k$ in above Theorem 3.1.1 then these results provide the generalization of Theorem 2.3.5. Theorem 3.1.1 results illustrated by the example given below.

Example 3.1.1. Let $(X,\ M,\ N,\ *,\ \diamond)$ be complete IFMS defined as

$$M(x,\ y,\ t) = \frac{t}{t+ \mid x-y \mid},\quad N(x,\ y,\ t) = \frac{\mid x-y \mid}{t+ \mid x-y \mid},\quad for\ all\ t \in (0,\ \infty)$$

Take a mapping $T : X \longrightarrow X$ with $\alpha(x) = \frac{2}{x^2}$ such that

$$T(x) = \begin{cases} \frac{1}{x}, & 0 < x \le 1\\ 0, & x = 0 \end{cases}$$

Then mapping T is $(TS - IF_\Delta)$ contractive mapping.

Proof. Case I: When $x \in (0,1]$ and y=0

$$M(Tx,\ Ty,\ t) = \frac{t}{t+ \mid Tx - Ty \mid} = \frac{t}{t+ \mid \frac{1}{x} - 0 \mid} = \frac{xt}{xt+1} = \frac{x^2t}{x^2t+x} \qquad (3.1.4)$$

since we can write

$$x \leq 1 \Rightarrow x^2 t + x \leq t + x \Rightarrow \frac{t}{x^2 t + x} \geq \frac{t}{t + x} \tag{3.1.5}$$

from equations 3.3.1 and 3.3.2

$$M(Tx,\ Ty,\ t) = x^2 \frac{t}{x^2 t + x} \geq x^2 \frac{t}{t + x} \geq \frac{x^2}{2} \frac{t}{t + x}$$

$$M(T(x),\ T(y),\ t) \geq \frac{1}{\Delta(x)} M(x,\ y,\ t)$$

now

$$x^2 \leq 1 \Rightarrow xt + x^2 \leq xt + 1 \Rightarrow \frac{1}{xt + 1} \leq \frac{1}{xt + x^2} \tag{3.1.6}$$

from equation 3.1.6

$$N(Tx,\ Ty,\ t) = \frac{\mid Tx - Ty \mid}{t + \mid Tx - Ty \mid} = \frac{1}{xt + 1} \leq \frac{1}{xt + x^2} = \frac{1}{x^2} \frac{x}{t + x} \leq \frac{2}{x^2} \frac{x}{t + x}$$

$$N(Tx,\ Ty,\ t) \leq \Delta(x) N(x,\ y,\ t)$$

Case II: When x=0 and $y \in (0, 1]$

It can be proved similar approach of case I by replacing $x = y$

Case III: When $x \in (0, 1]$ and $y \in (0, 1]$

$$M(Tx,\ Ty,\ t) = \frac{t}{t + \mid \frac{1}{x} - \frac{1}{y} \mid} = \frac{xyt}{xyt + \mid x - y \mid} \tag{3.1.7}$$

Since

$$xy \leq 1 \Rightarrow xyt + \mid x - y \mid \leq t + \mid x - y \mid \Rightarrow \frac{xyt}{xyt + \mid x - y \mid} \geq \frac{xyt}{t + \mid x - y \mid} \tag{3.1.8}$$

from equation 3.1.7 and 3.1.8

$$M(Tx,\ Ty,\ t) = \frac{xyt}{xyt + \mid x - y \mid} \geq \frac{xyt}{t + \mid x - y \mid} \geq \frac{x^2}{2} \frac{t}{t + \mid x - y \mid} \tag{3.1.9}$$

$$M(T(x),\ T(y),\ t) \geq \frac{1}{\Delta(x)} M(x,\ y,\ t)$$

$$N(Tx,\ Ty,\ t) = \frac{\mid \frac{1}{x} - \frac{1}{y} \mid}{t + \mid \frac{1}{x} - \frac{1}{y} \mid} = \frac{\mid x - y \mid}{xyt + \mid x - y \mid} \tag{3.1.10}$$

since for our convenience we take $x \leq y$

$$x^2 \leq xy \Rightarrow xyt + \mid x - y \mid \geq x^2 t + \mid x - y \mid \tag{3.1.11}$$

also we can write

$$x^2 \leq 1 \Rightarrow x^2 t + \mid x - y \mid \geq x^2 \mid x - y \mid + x^2 t \tag{3.1.12}$$

from equations 3.1.11 and 3.1.12

$$xyt + \mid x - y \mid \geq x^2 (t + \mid x - y \mid)$$

$$N(Tx,\ Ty,\ t) = \frac{\mid x - y \mid}{xyt + \mid x - y \mid} \leq \frac{1}{x^2} \frac{\mid x - y \mid}{t + \mid x - y \mid} \leq \frac{2}{x^2} \left(\frac{\mid x - y \mid}{t + \mid x - y \mid} \right)$$

which can be written as

$$N(Tx,\ Ty,\ t) \leq \Delta(x) N(x,\ y,\ t)$$

Case IV: When $x = 0$ and $y = 0$
It's trivial case. Thus all conditions of Theorem 3.1.1 are satisfied so T have the fixed point 1. □

3.2 Fixed point theorems for Kannan's and Chatterjee's type $(TS - IF_\Delta)$ contractive mapping

Generalization of classical results Kannan's[11] and Chatterjee's[12] type $(TS - IF_\Delta)$ are obtained followed by new contractive constant Δ.

Definition 3.2.1. Let $(X,\ M,\ N,\ *,\ \diamond)$ be IFMS and $T : X \to X$. T is said to be $TS-IF$ contractive mapping depend on Δ $(TS-IF_{K\Delta})$ if there exists a mapping $\Delta : X \to (0,\ \frac{1}{2})$ where

$$\Delta(Tx) \leq \Delta(x) \tag{3.2.1}$$

such that

$$\Delta(x)M(T(x),\ T(y),\ t) \geq [M(x,\ Tx,\ t) + M(y,\ Ty,\ t)] \tag{3.2.2}$$

and

$$\frac{1}{\Delta(x)}N(T(x),\ T(y),\ t) \leq [N(x,\ Tx,\ t) + N(y,\ Ty, t)] \tag{3.2.3}$$

for all $t > 0$.

Theorem 3.2.1. *Let $(X,\ M,\ N,\ *,\ \diamond)$ is a complete IFMS and $T : X \to X$ is an $(TS-IF_{K\Delta})$ mapping Then T has a unique fixed point.*

Proof. Let x_0 be an arbitrary point in X. We can construct the sequence x_n in X by

$$x_n = T^n x_0 = Tx_{n-1}$$

for all $n \in N$. Now for each $t > 0$, and using equations 3.2.1 and 3.2.2 We have

$$\begin{aligned}
\Delta(x_1)M(x_1,\ x_2,\ t) &= \Delta(x_1)M(T(x_0),\ T(x_1),\ t) \\
&\geq [M(x_0,\ T(x_0),\ t) + M(x_1,\ T(x_1),\ t)] \\
&\geq M(x_1,\ T(x_1),\ t) \\
M(x_1,\ x_2,\ t) &\geq \frac{1}{\Delta(Tx_0)}M(Tx_0,\ T(x_1),\ t) \\
&\geq \frac{1}{\Delta(x_0)}[M(x_0,\ T(x_0),\ t) + M(x_1,\ Tx_1, t)] \\
&\geq \frac{1}{\Delta(x_0)}M(x_0,\ T(x_0),\ t) \\
&= \frac{1}{\Delta(x_0)}M(x_0,\ x_1,\ t)
\end{aligned}$$

$$M(x_1,\ x_2,\ t) \geq \frac{1}{\Delta(x_0)} M(x_0,\ x_1,\ t) \tag{3.2.4}$$

Now again,

$$\begin{aligned}
\Delta(x_2)M(x_2,\ x_3,\ t) &= \Delta(x_2)M(T(x_1),\ T(x_2),\ t) \\
&\geq [M(x_1,\ T(x_1),\ t) + M(x_2,\ T(x_2),\ t)] \\
M(x_2,\ x_3,\ t) &\geq \frac{1}{\Delta(x_2)} M(x_2,\ T(x_2),\ t) \\
&= \frac{1}{\Delta(Tx_1)} M(Tx_1,\ T(x_2),\ t) \\
&\geq \frac{1}{\Delta(x_1)} [M(x_1,\ T(x_1),\ t) + M(x_2,\ Tx_2, t)] \\
&\geq \frac{1}{\Delta(x_1)} M(x_1,\ T(x_1),\ t) \\
&= \frac{1}{\Delta(Tx_0)} M(Tx_0,\ Tx_1),\ t) \\
&\geq \frac{1}{\Delta(x_0)} [M(x_0,\ T(x_0),\ t) + M(x_1,\ Tx_1, t)]
\end{aligned}$$

$$\geq \frac{1}{\Delta(x_0)} M(x_1,\ T(x_1),\ t)$$

$$= \frac{1}{\Delta(x_0)} M(x_1,\ (x_2),\ t)$$

from equation 3.2.4. We have

$$M(x_2,\ x_3,\ t) \geq \frac{1}{(\Delta(x_0))^2} M(x_0,\ x_1,\ t) \cdots \tag{3.2.5}$$

By Mathematical induction. We get

$$M(x_n,\ x_{n+1},\ t) \geq \frac{1}{(\Delta(x_0))^n} M(x,\ x_1,\ t)$$

Again Let $x \in X$ and for each $t > 0$, and using equations 3.2.1 and 3.2.3

$$\begin{aligned}
\frac{1}{\Delta(x_1)} N(x_1,\ x_2,\ t) &= \frac{1}{\Delta(x_1)} N(T(x_0),\ T(x_1),\ t) \\
&\leq [N(x_0,\ T(x_0),\ t) + N(x_1,\ T(x_1),\ t)] \\
&\leq [N(x_0,\ x_1,\ t) + N(x_1,\ x_2,\ t)] \\
N(x_1,\ x_2,\ t)\left(\frac{1}{\Delta(x_1)} - 1\right) &\leq N(x_0,\ x_1,\ t) \\
N(x_1,\ x_2,\ t) &\leq \left(\frac{\Delta(x_1)}{1 - \Delta(x_1)}\right) N(x_0,\ x_1,\ t)
\end{aligned}$$

Let $\beta = \left(\frac{\Delta(x_1)}{1-\Delta(x_1)}\right)$ where $0 < \left(\frac{\Delta(x_1)}{1-\Delta(x_1)}\right) \leq 1$ for $\Delta(x_1) \in (0,\ \frac{1}{2})$ We can write above equation as

$$N(x_1,\ x_2,\ t) \leq \beta N(x_0,\ x_1,\ t) \tag{3.2.6}$$

Now again,

$$\begin{aligned}
\frac{1}{\Delta(x_2)} N(x_2,\ x_3,\ t) &= \frac{1}{\Delta(x_2)} N(T(x_1),\ T(x_2),\ t) \\
&\leq [N(x_1,\ T(x_1),\ t) + N(x_2,\ T(x_2),\ t)] \\
&\leq [N(x_1,\ x_2,\ t) + N(x_2,\ x_3,\ t)] \\
N(x_2,\ x_3,\ t)\left(\frac{1}{\Delta(x_2)} - 1\right) &\leq N(x_1,\ x_2,\ t) \\
N(x_1,\ x_2,\ t) &\leq \left(\frac{\Delta(x_2)}{1 - \Delta(x_2)}\right) N(x_1,\ x_2,\ t) \\
N(x_1,\ x_2,\ t) &\leq \beta N(x_1,\ x_2,\ t)
\end{aligned}$$

Where $\beta = \left(\frac{\Delta(x_2)}{1-\Delta(x_2)}\right)$
From equation 3.2.6. We have

$$N(x_1,\ x_2,\ t) \leq (\beta)^2 N(x_0,\ x_1,\ t) \cdots$$

By Mathematical induction, We have,

$$N(x_n,\ x_{n+1},\ t) \leq (\beta)^n\, N(x_0,\ x_1,\ t)$$

We now verify that x_n is a cauchy sequence in $(X,\ M,\ N,\ *,\ \diamond)$.
Let $t_1 = \frac{t}{p}$

$$\begin{aligned}
M(x_n,\ x_{n+p},\ t) &\geq M(x_n,\ x_{n+1},\ t_1) * M(x_{n+1},\ x_{n+2},\ t_1) * \cdots * M(x_{n+p-1},\ x_{n+p},\ t_1) \\
&= \left(\frac{1}{(\Delta(x_0))^n}\,(\Delta(x_0))^n\, M(x_n,\ x_{n+1},\ t_1)\right) \\
&\quad * \left(\frac{1}{(\Delta(x_0))^{n+1}}\,(\Delta(x_0))^{n+1}\, M(x_{n+1},\ x_{n+2},\ t_1)\right) * \cdots \\
&\quad * \left(\frac{1}{(\Delta(x_0))^{n+p-1}}\,(\Delta(x_0))^{n+p-1}\, M(x_{n+p-1},\ x_{n+p},\ t_1)\right) \\
&\geq \left(\frac{1}{(\Delta(x_0))^n} M(x_0,\ x_1,\ t_1)\right) * \left(\frac{1}{(\Delta(x_0))^{n+1}} M(x_0,\ x_1,\ t_1)\right) * \cdots \\
&\quad * \left(\frac{1}{(\Delta(x_0))^{n+p-1}} M(x_0,\ x_1,\ t_1)\right)
\end{aligned}$$

$$\begin{aligned}
&\geq \left(\frac{1}{(\Delta(x_0))^n} M(x_0,\ x_1,\ t)\right) * \cdots * \left(\frac{1}{(\Delta(x_0))^n} M(x_0,\ x_1,\ t)\right) \\
&= \left(\frac{1}{(\Delta(x_0))^n} M(x_0,\ x_1,\ t)\right)
\end{aligned}$$

$$\Rightarrow 1 < \lim_{n\to\infty} \left(\frac{1}{(\Delta(x_0))^n} M(x_0,\ x_1,\ t)\right) \leq \lim_{n\to\infty} M(x_n,\ x_{n+p},\ t) \leq 1$$

$$\Rightarrow \lim_{n\to\infty} M(x_n,\ x_{n+p},\ t) = 1$$

Now again Let $t_1 = \frac{t}{p}$

$$\begin{aligned}
N(x_n,\ x_{n+p},\ t) &\leq N(x_n,\ x_{n+1},\ t_1) \diamond N(x_{n+1},\ x_{n+2},\ t_1) \diamond \cdots \diamond N(x_{n+p-1},\ x_{n+p},\ t_1) \\
&\leq \left(\frac{1}{(\beta)^n} (\beta)^n N(x_n,\ x_{n+1},\ t_1) \right) \\
&\quad \diamond \left(\frac{1}{(\beta)^{n+1}} (\beta)^{n+1} N(x_{n+1},\ x_{n+2},\ t_1) \right) \diamond \cdots \\
&\quad \diamond \left(\frac{1}{(\beta))^{n+p-1}} (\beta)^{n+p-1} N(x_{n+p-1},\ x_{n+p},\ t_1) \right) \\
&\leq \left(\frac{1}{(\beta)^n} N(x_0,\ x_1,\ t_1) \right) \diamond \left(\frac{1}{(\beta)^{n+1}} N(x_0,\ x_1,\ t_1) \right) \diamond \cdots \\
&\quad \diamond \left(\frac{1}{(\beta)^{n+p-1}} N(x_0,\ x_1,\ t_1) \right) \\
&\leq \left(\frac{1}{(\beta)^n} N(x_0,\ x_1,\ t) \right) \diamond \cdots \diamond \left(\frac{1}{(\beta)^n} N(x_0,\ x_1,\ t) \right) \\
&= \left(\frac{1}{(\beta)^n} N(x_0,\ x_1,\ t) \right)
\end{aligned}$$

$$\begin{aligned}
\Rightarrow 0 > \lim_{n\to\infty} \left(\frac{1}{(\beta)^n} N(x_0,\ x_1,\ t) \right) \geq \lim_{n\to\infty} N(x_n,\ x_{n+p},\ t) \geq 0 \\
\Rightarrow \lim_{n\to\infty} N(x_n,\ x_{n+p},\ t) = 0
\end{aligned}$$

Hence $(x_n)_n$ is a cauchy sequence in IFMS X. As X is complete there exist a point $y \in X$ such that $x_n \longrightarrow y$ as $n \to \infty$. Which implies that

$$M(x_n,\ y,\ t) \to 1 \ and \ N(x_n,\ y,\ t) \to 0 \ as \ n \to \infty$$

Now we show that y is a fixed point of T.

$$\begin{aligned}
\Delta(y) M(T(y),\ T(x_n),\ t) &\geq [M(y,\ T(y),\ t) + M(x_n,\ T(x_n),\ t)] \\
M(T(y),\ T(x_n),\ t) &\geq \frac{1}{\Delta(y)} [M(y,\ T(y),\ t) + M(x_n,\ T(x_n),\ t)]
\end{aligned}$$

$$\Rightarrow \lim_{n\to\infty} M(T(y),\ T(x_n),\ t) \geq \lim_{n\to\infty} \frac{1}{\Delta(y)}[M(y,\ T(y),\ t) + M(x_n,\ T(x_n),\ t)]$$

$$\Rightarrow \lim_{n\to\infty} M(T(y),\ T(x_n),\ t) \geq \frac{1}{\Delta(y)}[M(y,\ T(y),\ t)] + \lim_{n\to\infty} \frac{1}{\Delta(y)}[M(x_n,\ T(x_n),\ t)]$$

$$\Rightarrow \lim_{n\to\infty} M(T(y),\ T(x_n),\ t) \geq \frac{1}{\Delta(y)}[M(y,\ T(y),\ t)] + \frac{1}{\Delta(y)}[M(y,\ T(y),\ t)]$$

$$\Rightarrow \lim_{n\to\infty} M(T(y),\ T(x_n),\ t) \geq \frac{2}{\Delta(y)}[M(y,\ T(y),\ t)]$$

$$\Rightarrow \lim_{n\to\infty} M(T(y),\ T(x_n),\ t) \geq \frac{2}{\Delta(y)} > 1$$

$$\Rightarrow 1 < \lim_{n\to\infty} M(T(y),\ T(x_n),\ t) \leq 1$$

$$\Rightarrow \lim_{n\to\infty} M(T(y),\ T(x_n),\ t) = 1$$

Again,

$$N(T(y),\ T(x_n),\ t) \leq \Delta(y)[N(y,\ T(y),\ t) + N(x_n,\ T(x_n),\ t)]$$

$$\Rightarrow \lim_{n\to\infty} N(T(y),\ T(x_n),\ t) \leq \lim_{n\to\infty} \Delta(y)[N(y,\ T(y),\ t) + N(x_n,\ T(x_n),\ t)]$$

$$\Rightarrow \lim_{n\to\infty} N(T(y),\ T(x_n),\ t) \leq 2\Delta(y)N(y,\ T(y),\ t)$$

$$\Rightarrow \lim_{n\to\infty} N(T(y),\ T(x_n),\ t) \leq 2\Delta(y)$$

$$\Rightarrow \lim_{n\to\infty} N(T(y),\ T(x_n),\ t) < 0$$

$$\Rightarrow 0 \geq \lim_{n\to\infty} N(T(y),\ T(x_n),\ t) > 0$$

$$\Rightarrow \lim_{n\to\infty} N(T(y),\ T(x_n),\ t) = 0$$

Hence we can see that

$$\lim_{n\to\infty} M(T(y),\ T(x_n),\ t) = 1 \ and \ \lim_{n\to\infty} N(T(y),\ T(x_n),\ t) = 0 \ \forall\ t > 0$$

$$\Rightarrow \lim_{n\to\infty} T(x_n) = T(y) \Longrightarrow \lim_{n\to\infty} x_{n+1} = T(y)$$

$$i - e,\ y = T(y)$$

$\Rightarrow y$ is a fixed point of T. Now to prove uniqueness, assume that $T(z) = z$ for some $z \in X$.

Then for $t > 0$, and from equation 3.2.2 We have

$$\begin{aligned}
1 \geq M(y,\ z,\ t) &= M(T(y),\ T(z),\ t) \\
&\geq \frac{1}{\Delta(y)}[M(y,\ T(y),\ t) + M(z,\ T(z),\ t)] \\
&\geq \frac{1}{\Delta(y)}[M(y,\ y,\ t) + M(z,\ z,\ t)] \\
&\geq \frac{1}{\Delta(y)}[1+1] \\
&\geq \frac{2}{\Delta(y)} \\
&> 1 \\
\Rightarrow M(y,\ z,\ t) &= 1
\end{aligned}$$

Similarly, from equation 3.2.3

$$\begin{aligned}
0 \leq N(y,\ Z,\ t) &= N(T(y),\ T(z),\ t) \\
&\leq \Delta(y)[N(y,\ T(y),\ t) + N(z,\ T(Z),\ t)] \\
&\leq \Delta(y)[N(y,\ y,\ t) + N(z,\ z,\ t)] \\
&\leq \Delta(y)[0+0] \\
&\leq 0 \\
\Rightarrow N(y,\ z,\ t) &= 0
\end{aligned}$$

This shows that $y = z$. Hence y is a unique fixed point of T.
This completes the proof. □

Definition 3.2.2. Let $(X,\ M,\ N,\ *,\ \diamond)$ be IFMS and $T : X \to X$. T is said to be $TS - IF$ contractive mapping depend on Δ $(TS - IF_{C\Delta})$ if there exists a mapping $\Delta : X \to (0,\ \frac{1}{2})$ where

$$\Delta(Tx) \leq \Delta(x) \tag{3.2.7}$$

such that

$$\Delta(x)M(T(x),\ T(y),\ t) \geq [M(x,\ Ty,\ t) + M(y,\ Tx,\ t)] \tag{3.2.8}$$

and

$$\frac{1}{\Delta(x)}N(T(x),\ T(y),\ t) \leq [N(x,\ Ty,\ t) + M(y,\ Tx,\ t)] \tag{3.2.9}$$

for all $t > 0$.

Theorem 3.2.2. *Let $(X,\ M,\ N,\ *,\ \diamond)$ is a complete IFMS and $T : X \to X$ be $(TS - IF_{C\Delta})$ mapping Then T has a unique fixed point.*

Proof. Let x_0 be an arbitrary point in X. We can construct the sequence x_n in X by

$$x_n = T^n x_0 = Tx_{n-1}$$

for all $n \in N$. Now for each $t > 0$, and from equation 3.2.8 We have

$$\begin{aligned}
\Delta(x_1)M(x_1,\ x_2,\ t) &= \Delta(x_1)M(T(x_0),\ T(x_1),\ t) \\
&\geq [M(x_0,\ T(x_1),\ t) + M(x_1,\ T(x_0),\ t)] \\
&\geq [M(x_0,\ x_2,\ t) + M(x_1,\ x_1,\ t)] \\
&\geq M(x_0,\ x_2,\ t) + 1 \\
&\geq M(x_0,\ x_2,\ t)
\end{aligned}$$

$$M(x_1,\ x_2,\ t) \geq \frac{1}{\Delta(x_1)}M(x_0,\ x_2,\ t) \tag{3.2.10}$$

Now again for $t > 0$. and from equations 3.2.7, 3.2.8 We have

$$\begin{aligned}
\Delta(x_2)M(x_2,\ x_3,\ t) &= \Delta(x_2)M(Tx_1,\ Tx_2,\ t) \\
&\geq [M(x_1,\ Tx_2,\ t) + M(x_2,\ Tx_1.\ t)] \\
&\geq [M(x_1,\ x_3,\ t) + M(x_2,\ x_2.\ t)] \\
&\geq [M(x_1,\ x_3,\ t) + 1] \\
&\geq \frac{1}{\Delta(x_2)}M(x_1,\ x_3,\ t) \\
&= \frac{1}{\Delta(Tx_1)}M(Tx_0,\ Tx_2,\ t) \\
&\geq \frac{1}{\Delta(x_1)}[M(x_0,\ Tx_2,\ t) + M(x_2,\ Tx_0,\ t)] \\
&= \frac{1}{\Delta(x_1)}[M(x_0,\ x_3,\ t) + M(x_2,\ x_1,\ t)] \\
&\geq \frac{1}{\Delta(x_1)}M(x_2,\ x_1,\ t)
\end{aligned}$$

From equation 3.3.3, We get

$$M(x_2,\ x_3,\ t) \geq \frac{1}{(\Delta(x_1))^2}M(x_0,\ x_2,\ t)$$

By Mathematical induction. We get

$$M(x_n,\ x_{n+1},\ t) \geq \frac{1}{(\Delta(x_1))^n}M(x_0,\ x_2,\ t)$$

Again let for $x \in X$ and for each $t > 0$

$$\begin{aligned}
\frac{1}{\Delta(x_1)}N(x_1,\ x_2,\ t) &= \frac{1}{\Delta(x_1)}N(T(x_0),\ T(x_1),\ t) \\
&\leq [N(x_0,\ T(x_1),\ t) + N(x_1,\ T(x_0),\ t)] \\
&= [N(x_0,\ x_2,\ t) + N(x_1,\ x_1,\ t)] \\
&= N(x_0,\ x_2,\ t) + 0 \\
&\leq N(x_0,\ x_2,\ t)
\end{aligned}$$

$$N(x_1,\ x_2,\ t) \leq \Delta(x_1) N(x_0,\ x_2,\ t) \tag{3.2.11}$$

Again for $t > 0$, and from 3.2.7 and 3.2.8 We have

$$\begin{aligned}
\Delta(x_2) N(x_2,\ x_3,\ t) &= \Delta(x_2) N(Tx_1,\ Tx_2,\ t) \\
&\leq [N(x_1,\ Tx_2,\ t) + N(x_2,\ Tx_1,\ t)] \\
&= [N(x_1,\ x_3,\ t) + N(x_2,\ x_2,\ t)] = [N(x_1,\ x_3,\ t) + 0] \\
&\leq \Delta(x_2) N(x_1,\ x_3,\ t) \\
&= \Delta(Tx_1) N(Tx_0,\ Tx_2,\ t) \\
&\leq \Delta(x_1)[N(x_0,\ Tx_2,\ t) + N(x_2,\ Tx_0,\ t)] \\
&\leq \Delta(x_1) N(x_2,\ x_1,\ t)
\end{aligned}$$

From equation 3.3.4, We get

$$N(x_2,\ x_3,\ t) \leq (\Delta(x_1))^2 N(x_0,\ x_2,\ t)$$

By Mathematical induction, We have,

$$N(x_n,\ x_{n+1},\ t) \leq (\Delta(x_1))^n N(x_0,\ x_2,\ t)$$

We now verify that x_n is a cauchy sequence in $(X,\ M,\ N,\ *,\ \diamond)$.
Let $t_1 = \frac{t}{p}$

$$\begin{aligned}
M(x_n,\ x_{n+p},\ t) \geq\ & M(x_n,\ x_{n+1},\ t_1) * M(x_{n+1},\ x_{n+2},\ t_1) * \cdots * M(x_{n+p-1},\ x_{n+p},\ t_1) \\
=\ & \left(\frac{1}{(\Delta(x_1))^n} (\Delta(x_1))^n M(x_n,\ x_{n+1},\ t_1) \right) \\
& * \left(\frac{1}{(\Delta(x_1))^{n+1}} (\Delta(x_1))^{n+1} M(x_{n+1},\ x_{n+2},\ t_1) \right) * \cdots \\
& * \left(\frac{1}{(\Delta(x_1))^{n+p-1}} (\Delta(x_1))^{n+p-1} M(x_{n+p-1},\ x_{n+p},\ t_1) \right)
\end{aligned}$$

$$\geq \left(\frac{1}{(\Delta(x_1))^n} M(x_0,\ x_2,\ t_1)\right) * \left(\frac{1}{(\Delta(x_1))^{n+1}} M(x_0,\ x_2,\ t_1)\right) * \cdots$$
$$* \left(\frac{1}{(\Delta(x_1))^{n+p-1}} M(x_0,\ x_2,\ t_1)\right)$$
$$\geq \left(\frac{1}{(\Delta(x_1))^n} M(x_0,\ x_2,\ t)\right) * \cdots * \left(\frac{1}{(\Delta(x_1))^n} M(x_0,\ x_2,\ t)\right)$$
$$= \left(\frac{1}{(\Delta(x_1))^n} M(x_0,\ x_2,\ t)\right)$$

$$\Rightarrow 1 < \lim_{n\to\infty} \left(\frac{1}{(\Delta(x_1))^n} M(x_0,\ x_2,\ t)\right) \leq \lim_{n\to\infty} M(x_n,\ x_{n+p},\ t) \leq 1$$

$$\Rightarrow \lim_{n\to\infty} M(x_n,\ x_{n+p},\ t) = 1 \tag{3.2.12}$$

Now again Let $t_1 = \frac{t}{p}$

$$N(x_n,\ x_{n+p},\ t) \leq N(x_n,\ x_{n+1},\ t_1) \diamond N(x_{n+1},\ x_{n+2},\ t_1) \diamond \cdots \diamond N(x_{n+p-1},\ x_{n+p},\ t_1)$$
$$\leq \left(\frac{1}{(\Delta(x_1))^n} (\Delta(x_1))^n N(x_n,\ x_{n+1},\ t_1)\right)$$
$$\diamond \left(\frac{1}{(\Delta(x_1))^{n+1}} (\Delta(x_1))^{n+1} N(x_{n+1},\ x_{n+2},\ t_1)\right) \diamond \cdots$$
$$\diamond \left(\frac{1}{(\Delta(x_1))^{n+p-1}} (\Delta(x_1))^{n+p-1} N(x_{n+p-1},\ x_{n+p},\ t_1)\right)$$

$$\leq \left(\frac{1}{(\Delta(x_1))^n} N(x_0,\ x_2,\ t_1)\right) \diamond \left(\frac{1}{(\Delta(x_1))^{n+1}} N(x_0,\ x_2,\ t_1)\right) \diamond \cdots$$
$$\diamond \left(\frac{1}{(\Delta(x_1))^{n+p-1}} N(x_0,\ x_2,\ t_1)\right)$$
$$\leq \left(\frac{1}{(\Delta(x_1))^n} N(x_0,\ x_2,\ t)\right) \diamond \cdots \diamond \left(\frac{1}{(\Delta(x_1))^n} N(x_0,\ x_2,\ t)\right)$$
$$= \left(\frac{1}{(\Delta(x_1))^n} N(x_0,\ x_2,\ t)\right)$$

$$\Rightarrow 0 > \lim_{n\to\infty}\left(\frac{1}{(\Delta(x_1))^n}N(x_0,\ x_2,\ t)\right) \geq \lim_{n\to\infty} N(x_n,\ x_{n+p},\ t) \geq 0$$

$$\Rightarrow \lim_{n\to\infty} N(x_n,\ x_{n+p},\ t) = 0 \tag{3.2.13}$$

Hence from equation 3.2.12 and 3.2.13, $(x_n)_n$ is a cauchy sequence in IFMS X. As X is complete there exist a point $y \in X$ such that $x_n \longrightarrow y$ as $n \to \infty$. Which implies that

$$M(x_n,\ y,\ t) \to 1 \ and\ N(x_n,\ y,\ t) \to 0\ as\ n \to \infty$$

Now we show that y is a fixed point of T.

$$\begin{aligned}
\Delta(y)M(T(y),\ T(x_n),\ t) &\geq [M(y,\ T(x_n),\ t) + M(x_n,\ T(y),\ t)] \\
M(T(y),\ T(x_n),\ t) &\geq \frac{1}{\Delta(y)}[M(y,\ T(x_n),\ t) + M(x_n,\ T(y),\ t)]
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \lim_{n\to\infty} M(T(y),\ T(x_n),\ t) &\geq \lim_{n\to\infty}\frac{1}{\Delta(y)}[M(y,\ T(x_n),\ t) + M(x_n,\ T(y),\ t)] \\
&\geq \lim_{n\to\infty}\frac{1}{\Delta(y)}[M(y,\ T(x_n),\ t)] + \lim_{n\to\infty}\frac{1}{\Delta(y)}[M(x_n,\ T(y),\ t)] \\
&\geq \frac{2}{\Delta(y)}[M(y,\ T(y),\ t)] \\
&\geq \frac{2}{\Delta(y)} > 1
\end{aligned}$$

$$\begin{aligned}
\Rightarrow 1 < \lim_{n\to\infty} M(T(y),\ T(x_n),\ t) \leq 1 \\
\Rightarrow \lim_{n\to\infty} M(T(y),\ T(x_n),\ t) = 1
\end{aligned}$$

Again,

$$N(T(y),\ T(x_n),\ t) \leq \Delta(y)[N(y),\ T(x_n),\ t) + N(x_n,\ T(x_n),\ t)]$$
$$\Rightarrow \lim_{n\to\infty} N(T(y),\ T(x_n),\ t) \leq \lim_{n\to\infty} \Delta(y)[N(y),\ T(x_n),\ t) + N(x_n,\ T(x_n),\ t)]$$
$$\Rightarrow \lim_{n\to\infty} N(T(x_n),\ T(y),\ t) \leq \lim_{n\to\infty} \Delta(y)[N(y),\ T(x_n),\ t)] + \lim_{n\to\infty} \Delta(y)N(x_n,\ T(x_n),\ t)]$$
$$\Rightarrow \lim_{n\to\infty} N(T(y),\ T(x_n),\ t) \leq 2\Delta(y)N(y,\ T(y),\ t)$$
$$\Rightarrow \lim_{n\to\infty} N(T(y),\ T(x_n),\ t) \leq 2\Delta(y)$$
$$\Rightarrow \lim_{n\to\infty} N(T(y),\ T(x_n),\ t) < 0$$

$$\Rightarrow 0 \geq \lim_{n\to\infty} N(T(y),\ T(x_n),\ t) > 0$$
$$\Rightarrow \lim_{n\to\infty} N(T(y),\ T(x_n),\ t) = 0$$

Hence we can see that

$$\lim_{n\to\infty} M(T(y),\ T(x_n),\ t) = 1 \ and \ \lim_{n\to\infty} N(T(y),\ T(x_n),\ t) = 0 \ \forall\ t > 0$$
$$\Rightarrow \lim_{n\to\infty} T(x_n) = T(y) \Longrightarrow \lim_{n\to\infty} x_{n+1} = T(y)$$
$$i - e,\ y = T(y)$$

$\Rightarrow y$ is a fixed point of T.

To prove uniqueness, assume $T(z) = z$ for some $z \in X$.

Then for $t > 0$, and from equation 3.2.8 We have

$$\begin{aligned}
1 \geq M(y,\ z,\ t) &= M(T(y),\ T(z),\ t) \\
&\geq \frac{1}{\Delta(y)}[M(y,\ T(z),\ t) + M(z,\ T(y),\ t)] \\
&\geq \frac{1}{\Delta(y)}[M(y,\ z,\ t) + M(z,\ y,\ t)] \\
&\geq \frac{1}{\Delta(y)}[2M(y,\ z,\ t)] \\
&\geq \frac{2}{\Delta(y)}M(y,\ z,\ t) \\
&= \frac{2}{\Delta(y)}M(T(y),\ T(z),\ t) \\
&\geq \frac{2}{(\Delta(y))^2}[M(y,\ T(z),\ t) + M(z,\ T(y),\ t)] \\
&\geq \left(\frac{2}{\Delta(y)}\right)^2[M(y,\ z,\ t)] \\
&\vdots \\
&\geq \left(\frac{2}{\Delta(y)}\right)^n[M(y,\ z,\ t)] \longrightarrow \infty \ as\ n \longrightarrow \infty
\end{aligned}$$

$$\Rightarrow 1 < \lim_{n\to\infty}\left(\frac{2}{\Delta(y)}\right)^n M(y,\ z,\ t) \leq M(y,\ z,\ t) \leq 1$$

$$\Rightarrow M(y,\ z,\ t) = 1$$

Similarly, from equation 3.2.9. We have

$$
\begin{aligned}
0 \leq N(y,\ Z,\ t) &= N(T(y),\ T(z),\ t) \\
&\leq \Delta(y)[N(y,\ T(z),\ t) + N(z,\ T(y),\ t)] \\
&\leq \Delta(y)[N(y,\ z,\ t) + N(z,\ y,\ t)] \\
&\leq \Delta(y)[2N(y,\ z,\ t)] \\
&\leq 2\Delta(y)[N(y,\ z,\ t)] \\
&= 2\Delta(y)[N(T(y),\ T(z),\ t)] \\
&\leq 2\left(\Delta(y)\right)^2 [N(y,\ T(z),\ t) + N(z,\ T(y),\ t)] \\
&\leq \left(2\Delta(y)\right)^2 [N(y,\ z,\ t)] \\
&\vdots \\
&\leq \left(2\Delta(y)\right)^n [N(y,\ z,\ t)] \longrightarrow 0\ as\ n \longrightarrow \infty
\end{aligned}
$$

$$
\begin{aligned}
&\Rightarrow 0 \leq [N(y,\ z,\ t)] \leq \lim_{n\to\infty} \left(2\Delta(y)\right)^n [N(y,\ z,\ t)] < 0 \\
&\Rightarrow N(y,\ z,\ t) = 0
\end{aligned}
$$

Hence $y = z$. y is a unique fixed point of T. This completes the proof. □

3.3 Kannan's and Chatterjee's Fixed point theorem in IFMS

In T. K. Samanta, S. Mohinta & I. H. Jebril paper [7] the authors have posed some open problem that Kannan's and Chatterjee's fixed point theorems can be proved with the help of $TS-IF$ contractive mapping. We have given this answer in **Section 3.2** as a generalization of these results. Now by taking $\Delta(x) = k$ in Theorem 3.2.2 and Theorem 3.2.1 with $TS - IF$ contractive mapping results hold as a following corollaries.

Corollary 3.3.1. ***Kannan's fixed point theorem***

Suppose $(X,\ M,\ N,\ *,\ \diamond)$ *is a complete IFMS and* $T : X \rightarrow X$ *be TS-IF contractive mapping such that*

$$kM(T(x),\ T(y),\ t) \geq [M(x,\ T(x),\ t) + M(y,\ T(y),\ t)] \qquad (3.3.1)$$

$$N(T(x),\ T(y),\ t) \leq k[N(x,\ T(x),\ t) + N(y,\ T(y),\ t)] \qquad (3.3.2)$$

where k *is contractive constant and* $k \in (0,\ 1/2)$*. Then* T *has a unique fixed point.*

Corollary 3.3.2. ***Chatterjee's fixed point theorem***

Suppose $(X,\ M,\ N,\ *,\ \diamond)$ *is a complete IFMS and* $T : X \rightarrow X$ *be TS-IF contractive mapping such that*

$$kM(T(x),\ T(y),\ t) \geq [M(x,\ T(y),\ t) + M(y,\ T(x),\ t)] \qquad (3.3.3)$$

$$N(T(x),\ T(y),\ t) \leq k[N(x,\ T(y),\ t) + N(y,\ T(x),\ t)] \qquad (3.3.4)$$

where k *is contractive constant and* $k \in (0, 1/2)$*. Then* T *has a unique fixed point.*

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