Generalized contractive mappings of integral type in spaces with two metrics

by

Akhtar Munir Khan

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Supervised by

Dr. Quanita Kiran

School of Natural Sciences(SNS)

National University of Sciences and Technology(NUST),

Islamabad, Pakistan.

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Examination Committee Members

1. Name: Dr. Matloob Anwar Signature: ____________
2. Name: Dr. Mujeeb ur Rehman Signature: ____________
3. Name: Dr. Yasir Ali Signature: ____________
4. Name: Prof. Akbar Azam Signature: ____________

Supervisor's Name: Dr. Quanita Kiran Signature: ____________

____
Head of Department

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____
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**Dedicated to my parents,
may Allah bless them
with healthy and long life
Ameen!**

Contents

Preface

The most important and well known result in fixed point theory is the Banach fixed point theorem [2] also known as "Contraction mapping principle" which guarantees the existence and uniqueness of a fixed point for a contraction mapping from a complete metric space to itself. A number of authors extended and generalized Banach contraction principle in many ways. One of the extension is in the abstract spaces in which more than one, in particular, when two metrics are defined. This work was initially due to Maia [13] and many generalizations were made by different authors(see [1], [10] and [12]). Hardy and Rogers [10] generalized and used these results in considering the solutions of differential equations in abstract spaces. These results were further generalized by Agarwal *et al.* [1] and Kiran *et al.* [12].

On the other hand Samet *et al.* [23] introduced the concept of α-ψ-contractive mapping and obtained some fixed point results which generalized Banach contraction principle. Branciari [8] generalized this notion by introducing integral version of α-ψ-contractive mapping. Various generalizations of α-ψ-contractive mapping and hence Banach principle were made by different authors(see [11], [24] and references therein). Furthermore, Salimi *et al.* [22] introduced twisted $(\alpha$-$\beta)$-ψ-contractive type mappings and established some fixed point results.

This thesis is organized as follows. Chapter 1 is devoted to fundamental notions which are used in understanding and development of fixed point theory. We illustrate the concepts by some examples. In Chapter 2, we discuss fixed point results for contraction mappings in spaces with two metrics in such a way that the underlying spaces are assumed to be complete with respect to one metric and satisfies a contractive condition with respect to another metric. Chapter 3 consists of two sections. In Section I, fixed point results related to the α-ψ-contractive mappings and generalizations of α-ψ-contractive mappings are discussed. In section II, fixed point theorems related to twisted (α, β)-ψ-contractive mappings are presented. In chapter 4, we establish some new fixed point theorems. This chapter consists of three sections. In section I, the established theorems generalize and extend the results of Samet *et al.* [23] and Karapinar *et al.* [11] related to α-ψ-contractive mappings and hence Banach principle. Moreover, this established theorems also generalize some of the results of Agarwal *et al.* [1] and Kiran *et al.* [12]. In the second section of this chapter, fixed point theorems are obtained which generalize and extend some of the results related to twisted (α, β)-ψ-contractive mappings of Salimi *et al.* [22]. At the end of this chapter, we also present some concluding remarks and future work.

Chapter 1

Basic Concepts and Notations

1.1 Fixed points and contraction mappings

Given a nonempty set X and a map $T : X \to X$, the problem of finding a point $x \in X$ such that $x = Tx$ is called Fixed Point Problem and the point $x \in X$ is known as Fixed Point of the map T. The term Metric Fixed Point Theory refers to those fixed point theoretic results in which geometric conditions on the underlying spaces and/or mappings play a crucial role. In this chapter some basic concepts of metric fixed point theory are presented which are used throughout this thesis. All the necessary notations and the terminologies used in the sequel are also introduced. Throughout this thesis, by X we denote a metric space with the metric d, unless stated otherwise. If X is non-empty set and T is a self map, then Tx denotes the image of x under T and $FixT$ denotes the set containing all the fixed points of a self map T.

Definition 1.1.1. Let $T : X \to X$ be a map then $x \in X$ is said to be a fixed point of T if $x = Tx$.

Example 1.1.1. *Let $X = [0, 1)$ and T be a self mapping from X defined by $Tx = x^2$. Then $x = 0$ is the fixed point of T.*

Not all the functions have fixed points. For example let $T_1, T_2 : \mathbb{R} \to \mathbb{R}$ defined by $T_1x = e^x$ and $T_2x = x + c$ where $0 \neq c \in \mathbb{R}$ then T_1 and T_2 have no fixed points. Furthermore, the fixed point of a map may not be unique. For example let $X = [-1, 1]$ and T_3 and T_4 are self mappings from X to X defined as $T_3x = x^2$ and $T_4x = x^3$. Then $FixT_3 = \{0, 1\}$ and $FixT_4 = \{-1, 0, 1\}$ respectively.

Definition 1.1.2. A function $T : X \to X$ is said to be *Lipschitzian* or *Lipschitz* function if there exists a positive real number $L \geq 0$ such that

$$d(Tx, Ty) \leq Ld(x, y); \qquad \forall\ x, y \in X.$$

Such a smallest $L \geq 0$, which satisfy the above inequality, is called *Lipschitz* constant.

Example 1.1.2. *All the linear functions are Lipschitzian.*

Example 1.1.3. *Let $X = [-2, 2]$ with usual metric $d(x, y) = |x-y|$; for all $x, y \in X$ and $T : X \to X$ defined by $Tx = x^2$ then T is Lipschitzian map with Lipschitz constant $L = 4$.*

The definition of Lipschitz function is due to the German mathematician Rudolph Lipschitz (1832-1903), who used this concept of continuity to prove existence and uniqueness of solutions to some important differential equations for example in Picard-Lindelöf theorem. In this theorem Lipschitz function plays an important role.

Proposition 1.1.1. *Let $T : [a, b] \subset \mathbb{R} \to \mathbb{R}$ be differentiable on (a, b). Suppose T' is continuous on $[a, b]$. Then T is a Lipchitz continuous.*

The following example shows that Lipchitzian mapping may not be differentiable.

Example 1.1.4. *Let $X = [-1, 1]$ and T a mapping from X to X defined as $Tx = |x|$. Then T is Lipchitzian mapping with $L = 1$ but T is not differentiable at $x = 0$.*

Remark 1.1.1. *Note that Lipschitzian map is uniformly continuous but the converse is not true.*

Proof. Let (X, d) be a metric space and $T : X \to X$ be a Lipschitzian mapping, then

$$d(Tx, Ty) \leq Ld(x, y).$$

Choosing an $\epsilon > 0$, then for all $x, y \in X$ there exists a $\delta = \delta(\epsilon) > 0$ such that

$$d(x, y) < \delta = \frac{\epsilon}{L},$$

which implies that

$$Ld(x, y) < \epsilon.$$

But since

$$d(Tx, Ty) \leq Ld(x, y),$$

thus

$$d(Tx, Ty) < \epsilon.$$

Which shows that T is uniform continuous(and hence continuous). □

The following two examples show that uniform continuity does not imply Lipchitz continuity.

Example 1.1.5. *Let* $X = [\frac{-1}{\pi}, \frac{1}{\pi}]$ *and* $T : X \to X$ *is defined by*

$$Tx = \begin{cases} 0, & \text{if } x = 0 \\ \frac{x}{2}\sin(\frac{1}{x}), & \text{if } x \neq 0. \end{cases}$$

Then T *is continuous(uniform continuous) but not Lipchitz continuous.*

Example 1.1.6. *Suppose that* $X = [0, 1]$ *and* $d(x, y) = |x - y|$. *Then* $T : X \to X$ *defined as* $Tx = \sqrt{x}$ *is uniform continuous by the uniform continuity criterion but is not Lipschitian since we cannot find a constant* $L \geq 0$ *such that the above inequality holds. Thus every uniform continuous function may not be Lipschitzian.*

Definition 1.1.3. A *Lipschitzian* mapping $T : X \to X$ with $L < 1$ is called *Contraction* mapping or simply *Contraction*. Geometrically this means that any two points in the range of T are closer than the corresponding pre-images in domain. More precisely, the ratio $d(Tx, Ty)/d(x, y)$ does not exceed a *Lipschitz* constant $L < 1$.

Example 1.1.7. *Let* $(X = [0, 1], d)$ *be the usual metric space and* T *a self map from* X *into* X *define by* $Tx = \frac{1}{2+x}$, *then*

$$d(Tx, Ty) = |\frac{1}{2 + x} - \frac{1}{2 + y}| = |\frac{y - x}{(2 + x)(2 + y)}|$$

$$= \frac{d(x, y)}{|2 + x||2 + y|} \leq \frac{1}{4}d(x, y).$$

Hence, T *is contraction with Lipschitz constant* $L = \frac{1}{4}$.

Definition 1.1.4. A function $T : X \to X$ is called *Contractive* if for all $x, y \in X$ and $x \neq y$,

$$d(Tx, Ty) < d(x, y).$$

Example 1.1.8. *Let* $T : X \to X$ *defined as* $Tx = x + \frac{1}{x}$ *with usual metric and* $X = [1, +\infty)$, *then*

$$|x + \frac{1}{x} - y - \frac{1}{y}| = |x - y||1 - \frac{1}{xy}| < |x - y|.$$

Hence T *is contractive.*

Remark 1.1.2. *Not all contraction(and hence contractive) mappings have fixed points.*

Example 1.1.9. *Let us consider* $X = (0,1]$ *with usual metric and* $T : X \to X$ *defined by* $Tx = \frac{x}{2}$*, then*

$$|Tx - Ty| = \frac{1}{2}|x - y|,$$

and hence T *is contraction(and hence contractive), has no fixed point because* X *is not complete.*

It is worth noting that every *Contraction* is *Contractive* but the converse is not true.

Theorem 1.1.1. *(Banach contraction principle [2]) Let* (X, d) *be a complete metric space and let* T *be a mapping from* X *into* X*. If there exists a real number* L *with* $0 \leq L < 1$ *satisfying*

$$d(Tx, Ty) \leq Ld(x, y)$$

for all $x, y \in X$*, then* T *has a unique fixed point* x^**. Moreover, for each* $x \in X$*:*

(i) *The iterative sequence* $(T^n x)$ *converges to* x^**;*

(ii) *For* $n \geq 1$ *the following apriori estimates holds;*

$$d(T^n x, x^*) \leq \frac{L^n}{1 - L} d(x, Tx);$$

(iii) *For* $n \geq 1$ *the following aposteriori estimates holds;*

$$d(T^{n+1} x, x^*) \leq d(T^{n+1} x, T^n x).$$

1.2 Some extensions of Banach contraction principle using gauge functions

Banach contraction principle was extended in many ways. One of the extension of this principle was to consider a gauge function. Here we mention some of the well known (related to our work) extensions.

Definition 1.2.1. A function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, where $\mathbb{R}_+$ is the set of all non-negative real numbers, is said to be a gauge function if one of the following conditions is satisfied:

(i) ψ is non-decreasing;

(ii) $\psi(t) < t, \quad \forall\ t > 0;$

(iii) $\psi(0) = 0;$

(iv) $\lim_{n\to+\infty}\psi^n(t)=0, \quad \forall\, t\geq 0$;

(v) $\sum_{n=0}^{+\infty}\psi^n(t)<+\infty, \quad \forall\, t>0$;

(vi) ψ is continuous;

(vii) $t-\psi(t)\to\infty$ as $t\to+\infty$;

(viii) ψ is sub-additive.

Definition 1.2.2. A function $\psi : I \to I$ where I is an interval of the form $[0,R]$, $[0,R)$ or $[0,+\infty)$ and $r\geq 1$ is said to be a gauge function of order r if it satisfies the following conditions:

(a) $\psi(\theta t)\leq \theta^r\psi(t),\ \theta\in(0,1),\ t\in I$;

(b) $\psi(t)<t;\ \forall\, t \in I/\{0\}$.

If (i) and (iv) are satisfied then ψ is called comparison function. If (i) and (v) are satisfied then ψ is called (c)-comparison function. Note that the last type of function is also known as Bianchini-Grandolfi [6] gauge function in the literature. Moreover, Ptak [18] observed that gauge functions satisfy the following functional equation:

$$\sigma(t)=\sigma(\psi(t))+t; \qquad \sigma(t)=\sum_{n=0}^{+\infty}\psi^n(t)<\infty,\ \forall\, t\in\mathbb{R}_+. \tag{1.1}$$

Example 1.2.1. *Let $\psi:\mathbb{R}_+\to\mathbb{R}_+$ defined by $\psi(t)=qt$, where $q\in(0,1)$ then ψ is gauge function.*

Example 1.2.2. *Let $\psi:\mathbb{R}_+\to\mathbb{R}_+$ be defined as $\psi(t)=\frac{t}{t+1}$; $t\in\mathbb{R}_+$ is a comparison function but not a (c)-comparison function.*

Example 1.2.3. *Let $\psi(t)=\frac{t}{2}$ when $t\in[0,1]$ and $\psi(t)=t-\frac{1}{2}$, when $t\in(1,+\infty)$ is a (c)-comparison function.*

One can easily deduce from the above examples that any (c)-comparison function is a comparison function but converse may not be true. Rakotch [19] generalized Banach contraction principle, using a gauge function, in the following way:

Theorem 1.2.1. [*19*] *Let X be a complete metric space and suppose that $T:X\to X$ satisfies*

$$d(Tx,Ty)\leq\psi(d(x,y))d(x,y),$$

for each $x,y\in X$ where $\psi:\mathbb{R}_+\to[0,1)$ is monotonically non decreasing. Then T has a unique fixed point, λ, and $(T^n(x))$ converges to λ for each $x\in X$.

A more general result was obtained by Boyd and Wong [7].

Theorem 1.2.2. [7] *Let X be a complete metric space and suppose that $T : X \to X$ satisfies*

$$d(Tx, Ty) \leq \psi(d(x, y)),$$

for each $x, y \in X$ where $\psi : \mathbb{R}_+ \to [0, +\infty)$ is upper semicontinuous from the right and satisfies $0 \leq \psi(t) < t$ for $t > 0$. Then T has a unique fixed point λ, and $(T^n(x))$ converges to λ for each $x \in X$.

The above result of Boyd and Wong was generalized by Browder [9] in the following way:

Theorem 1.2.3. [9] *Let X be a metric space and let D be a bounded subset of X. Suppose $T : D \to D$ satisfies*

$$d(Tx, Ty) \leq \psi(d(x, y)),$$

for each $x, y \in D$, where $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is a monotone non decreasing and continuous from the right, such that $\psi(t) < t$ for all $t > 0$. Then there is a unique element $\lambda \in D$ such that $(T^n x)$ converges to λ for each $x \in D$. Moreover, if κ is the diameter of D, then

$$d(T^n x, \lambda) \leq \psi(\kappa),$$

and $\psi(\kappa) \to 0$ as $n \to +\infty$.

The following variant is due to Matkowski [14] where the continuity condition on ψ is replaced with another condition.

Theorem 1.2.4. [14] *Let X be a complete metric space and suppose that $T : X \to X$ satisfies*

$$d(Tx, Ty) \leq \psi(d(x, y)),$$

for each $x, y \in X$, where $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is a monotone non decreasing and satisfies $\lim_{n \to +\infty} \psi^n(t) = 0$ for $t > 0$. Then T has a unique fixed point λ, and $\lim_{n \to} d(T^n(x), \lambda) = 0$ for every $x \in X$.

Banach principle was also extended using gauge function of higher order by many authors in the literature. For example Proinov [17] generalized the Banach contraction principle by using a gauge function of order greater or equal to 1.

1.3 α-admissible mappings

Samet *et al.* [23] on the other hand generalized Banach [2] contraction principle by introducing very interesting notion of α-admissible and $\alpha - \psi$-contractive mapping.

Definition 1.3.1. [23] Assume that $T : X \to X$ be given mapping and $\alpha : X \times X \to [0, +\infty)$, then T is called α-admissible mapping if, $\forall\ x, y \in X$ such that $\alpha(x, y) \geq 1$ implies that $\alpha(Tx, Ty) \geq 1$.

Example 1.3.1. *Let $T : X \to X$ with $X = [0, +\infty)$ and $Tx = \sqrt{x}$ and let us define α as $\alpha(x, y) = 2$; $\forall\ x, y \in X$. Then, since $\alpha(x, y) \geq 1$ so $\alpha(Tx, Ty) \geq 1$. Hence T is α-admissible.*

Remark 1.3.1. *Not all the maps are α-admissible and is shown in the below example.*

Example 1.3.2. *Let $X = [0, +\infty)$ and $T : X \to X$ defined by $Tx = \frac{1}{x+2}$ and*

$$\alpha(x, y) = \begin{cases} 0, & \text{if } x \geq y \\ e^{y-x}, & \text{if } x < y, \end{cases}$$

then $\alpha(x, y) \geq 1$ implies $x < y$ and since T is decreasing, so we get $Tx > Ty$. But from the definition of α, we deduce that

$$\alpha(Tx, Ty) = 0 \ngeq 1.$$

Thus, T is not α-admissible mapping.

1.4 Twisted (α,β)-admissible mappings

Salimi *et al.* [22] generalized the $\alpha - \psi$-notion introduced by Samet *et al.* [23] and hence Banach [2] principle by introducing twisted (α, β)-admissible and twisted (α, β)-ψ-contractive mappings.

Definition 1.4.1. [22] Let $T : X \to X$ be given mapping and $\alpha, \beta : X \times X \to [0, +\infty)$. Then T is said to be twisted (α,β)-admissible mapping if,

$\forall\ x, y \in X$ and $\alpha(x, y) \geq 1$ and $\beta(x, y) \geq 1$ then $\alpha(Ty, Tx) \geq 1$ and $\beta(Ty, Tx) \geq 1$.

Note that chapter 3 is devoted to the fixed point results related to $\alpha - \psi$ and twisted (α, β)-ψ-contractive type mappings.

Chapter 2

Generalized contractions on spaces with two metrics

The aim of this chapter is, to discuss the techniques used in the extension of Banach principle [2] on spaces with two metrics. Banach fixed point theorem is extended and generalized in different ways. One of these extensions, is in those abstract spaces upon which there is more than one, in particular, when two metrics are defined. Banach principle [2] was first extended on the spaces with two metrics by Maia [13] which was further generalized by Precup [16] in order to find the solution of differential equation in abstract spaces. These results were later generalized by Agarwal *et al.* [1] and Kiran *et al.* [12].

2.1 Fixed point results on the spaces with two metrics

Definition 2.1.1. Let (X, d) be a metric space and d' be another metric on X which is complete and $x_0 \in X,\ r > 0$ then by $S(x_0, r)$, we denote the open ball centered at x_0 with radius r defined as:

$$S(x_0, r) = \{x \in X : d(x, x_0) < r\},$$

and by $\overline{S(x_0, r)^d}$ and $\overline{S(x_0, r)^{d'}}$ the d-closure and d'-closure of $S(x_0, r)$ respectively.

Agarwal *et al.* [1] established some fixed point theorems in spaces with two metrics in the following way.

Theorem 2.1.1. [1] *Let (X, d') be a complete metric space and d be another metric on X, $x_0 \in X$, $r > 0$ and $T : \overline{S(x_0, r)^{d'}} \to X$. Suppose there exists $q \in (0, 1)$ such that, for $x, y \in \overline{S(x_0, r)^{d'}}$ we have*

$$d(Tx, Ty) \leq q \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{[d(x, Ty) + d(y, Tx)]}{2}\right\}.$$

In addition assume the following three properties hold:

$$d(x_0, Tx_0) < (1-q)r. \tag{2.1}$$

If $d \ngeq d'$ *assume* T *is uniformly continuous from* $(S(x_0,r), d)$ *into* (X, d');
If $d \neq d'$ *assume* T *is continuous from* $(\overline{S(x_0,r)^{d'}}, d')$ *into* (X, d');
Then T *has a fixed point. That is, there exists* $x \in \overline{S(x_0,r)^{d}}$ *with* $x = Tx$.

We take in account the special case of the above theorem when $d = d'$.

Theorem 2.1.2. [1] *Let* (X, d) *be a complete metric space,* $x_0 \in X$, $r > 0$ *and* $T : \overline{S(x_0,r)^{d}} \to X$. *Suppose there exists* $q \in (0,1)$ *such that for* $x, y \in \overline{S(x_0,r)^{d}}$ *we have*

$$d(Tx,Ty) \leq q \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{[d(x,Ty)+d(y,Tx)]}{2}\right\}.$$

In addition assume that the following the property holds;

$$d(x_0, Tx_0) < (1-q)r.$$

Then there exists $x \in \overline{S(x_0,r)^{d'}}$ *with* $x = Tx$.

Proof. Let $x_1 = Tx_0$. Now from (2.1) we have
$d(x_0,x_1) < (1-q)r < r$ and so we have $x_1 \in S(x_0,r)$.
Next $x_2 = Tx_1$ then

$$\begin{aligned} d(x_1,x_2) &= d(Tx_0,Tx_1) \\ &\leq q\max\left\{d(x_0,x_1), d(x_1,x_2), \frac{[d(x_0,x_2)]}{2}\right\} \\ &\leq q\max\left\{d(x_0,x_1), d(x_1,x_2), \frac{d(x_0,x_1)+d(x_1,x_2)}{2}\right\} \end{aligned}$$

We claim that $\max\left\{d(x_0,x_1), d(x_1,x_2), \frac{d(x_0,x_1)+d(x_1,x_2)}{2}\right\} = d(x_0,x_1)$. Otherwise, assume that

$$\max\left\{d(x_0,x_1), d(x_1,x_2), \frac{d(x_0,x_1)+d(x_1,x_2)}{2}\right\} = \frac{d(x_0,x_1)+d(x_1,x_2)}{2}.$$

Then, we have

$$\begin{aligned} d(x_1,x_2) &\leq q\left\{\frac{d(x_0,x_1)+d(x_1,x_2)}{2}\right\} \\ (1-\frac{q}{2})d(x_1,x_2) &\leq \frac{q}{2}d(x_0,x_1) \\ d(x_1,x_2) &\leq \frac{q}{2-q}d(x_0,x_1) \leq qd(x_0,x_1). \end{aligned}$$

The other cases are simple. Thus

$$\begin{aligned} d(x_1, x_2) &\leq qd(x_0, x_1) \\ &< q(1-q)r. \end{aligned}$$

Notice that $x_2 \in S(x_0, r)$ since from triangular inequality and (2.1) we have

$$\begin{aligned} d(x_0, x_2) &\leq d(x_0, x_1) + d(x_1, x_2) \\ &\leq (1-q)r + q(1-q)r \\ &= (1-q)r[1+q] \\ &\leq (1-q)r[1+q+q^2+\ldots] = r. \end{aligned}$$

Proceeding inductively, we obtain $x_{n+1} = Tx_n$ with

$$\begin{aligned} d(x_{n+1}, x_n) &\leq qd(x_n, x_{n-1}) \\ &\leq q^n d(x_0, x_1) \\ &< q^n(1-q)r. \end{aligned} \tag{2.2}$$

Since $q \in (0,1)$, so $q^n \to 0$ as $n \to +\infty$ which further shows that $\lim_{n\to+\infty} d(x_{n+1}, x_n) = 0$ and $x_{n+1} \in S(x_0, r)$. Now, we show that (x_n) is a Cauchy sequence. Let $m, n \in \mathbb{N}$ such that $m > n$ then, from triangular inequality and (2.2) we infer that,

$$\begin{aligned} d(x_n, x_m) &\leq \sum_{i=n+1}^{m} d(x_{i-1}, x_i) \\ &\leq [q^n + q^{n+1} + \ldots + q^{m-n-1}]d(x_0, x_1) \\ &= q^n[\frac{1-q^{m-n-1}}{1-q}] \\ &< \frac{q^n}{1-q} d(x_0, x_1). \end{aligned}$$

Letting $n \to +\infty$ in the above inequality, we get that (x_n) is Cauchy sequence. Thus from the completeness of X we have $x_n \to x$ and also (x_n) is a sequence in $S(x_0, r)$ so $x \in \overline{S(x_0, r)}$. Finally we claim that $x = Tx$, then

$$\begin{aligned} d(x, Tx) &\leq d(x, x_n) + d(Tx_{n-1}, Tx) \\ &\leq d(x, x_n) + q \max \Big\{ d(x, x_{n-1}), d(x, Tx), d(x_{n-1}, Tx_{n-1}), \\ &\quad \frac{d(x, Tx_{n-1}) + d(x_{n-1}, Tx)}{2} \Big\}. \end{aligned} \tag{2.3}$$

Taking limit as $n \to +\infty$ in the above inequality, we deduce that $x = Tx$. □

The following global result can easily be deduced from Theorem 2.1.1.

Theorem 2.1.3. [1] *Let (X, d') be a complete metric space and d be another metric on X and $T : X \to X$. Suppose there exists $q \in (0, 1)$ such that for $x, y \in X$ we have*

$$d(Tx, Ty) \leq q \max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{[d(x, Ty) + d(y, Tx)]}{2}. \right\}$$

In addition assume that the following three properties hold:

if $d \ngeq d'$ assume T is uniformly continuous from (X, d) into (X, d'), and

if $d \neq d'$ assume T is continuous from (X, d') into (X, d').

Then T has a fixed point. That is, there exists $x \in X$ with $x = Tx$.

Now we consider some of the results of Kiran *et al.* [12] which generalize and extend the above results of Agarwal *et al.* [1].

Theorem 2.1.4. [12] *Let (X, d') be complete metric space and d be another metric on X, $x_0 \in X$, $r > 0$ and $T : S(x_0, r) \to X$ satisfies;*

$$d(Tx, T^2x) \leq \psi(d(x, Tx));\ \forall\ x, Tx \in S(x_0, r) \text{ with } d(x, Tx) \in J, \tag{2.4}$$

where ψ is a Bianchini-Grandolfi gauge function on the interval $J = [0, \infty)$. Then starting from x_0 the iterative sequence

$$x_{n+1} = Tx_n;\ \forall\ n \geq 0, \tag{2.5}$$

converges to a fixed point $\lambda \in \overline{S(x_0, r)^{d'}}$. Which will be the fixed point of T if the following conditions are satisfied:

(i) $d(x_0, Tx_0) < \delta$,

where $\delta > 0$ is such that $\sigma(\delta) \leq r$.

(ii) If $d \ngeq d'$ assume T is uniformly continuous from $(S(x_0, r), d)$ into (X, d')

(iii) If $d \neq d'$ then T is continuous from $(\overline{S(x_0, r)^{d'}}, d')$ into (X, d').

(iv) If $d = d'$ then T is continuous at λ.

Proof. Let $x_1 = Tx_0$. Then from (i)

$$d(x_0, Tx_0) < \delta \leq \sigma(\delta) \leq r.$$

Next let $x_2 = Tx_1$. Then from (2.4) we have

$$d(x_1, x_2) = d(Tx_0, Tx_1) = d(Tx_0, T^2x_0) \leq \psi(d(x_0, Tx_0)$$

Note that $d(x_1, x_2) \in J$. Further, $x_2 \in \overline{S(x_0, r)^d}$ since from triangular inequality and using (1.1) and (i), we have

$$\begin{aligned} d(x_0, x_2) &\leq d(x_0, x_1) + d(x_1, x_2) \\ &\leq d(x_0, x_1) + \psi(d(x_0, x_1)) \\ &< \delta + \sigma(\psi(\delta)) \\ &= \sigma(\delta) \leq r. \end{aligned}$$

Let $x_3 = Tx_2$. Then again using (2.4), we have

$$\begin{aligned} d(x_2, x_3) &= d(Tx_1, Tx_2) = d(Tx_1, T^2x_1) \\ &\leq \psi(d(x_1, Tx_1)) \\ &= \psi(d(Tx_0, T^2x_0)) \\ &\leq \psi^2(d(x_0, x_1)). \end{aligned}$$

Note that $d(x_2, x_3) \in J$. Further $x_3 \in \overline{S(x_0, r)^d}$, since

$$\begin{aligned} d(x_0, x_3) &\leq d(x_0, x_1) + d(x_1, x_2) + d(x_2, x_3) \\ &\leq d(x_0, x_1) + \psi(d(x_0, x_1)) + \psi^2(d(x_0, x_1)) \\ &< \sum_{k=0}^{\infty} \psi^k(\delta) \\ &= \sigma(\delta) \leq r. \end{aligned}$$

Proceeding in the same way, we obtain a sequence (x_n) in $\overline{S(x_0, r)^d}$ such that

$$d(x_n, x_{n+1}) \leq \psi^n(d(x_0, x_1)). \tag{2.6}$$

We now show that (x_n) is Cauchy sequence with respective d. Let $n, p \in \mathbb{N}$ then from (2.6) together with triangular inequality we infer that,

$$\begin{aligned} d(x_{n+p}, x_n) &\leq d(x_{n+p}, x_{n+p-1}) + ... + d(x_{n+1}, x_n) \\ &\leq \psi^{n+p-1}\left(d(x_0, x_1) + ... + \psi^n(d(x_0, x_1)\right) \\ &\leq \sum_{k=n}^{\infty} \psi^k\left(d(x_0, x_1)\right) \to 0. \end{aligned}$$

Thus, there is an $N \in \mathbb{N}$ satisfying

$$d(x_n, x_m) < \xi \qquad \text{whenever} \qquad n, m \geq N. \tag{2.7}$$

We claim that (x_n) is Cauchy sequence with respect to d'. If $d \geq d'$ then our claim is trivially true. If $d \ngeq d'$ and let $\epsilon > 0$ then (ii) guarantees that, there is an $\xi > 0$ such that

$$d'(Tx, Ty) < \epsilon \qquad \text{whenever} \qquad x, y \in S(x_0, r), d(x, y) < \xi. \tag{2.8}$$

Now (2.7) and (2.8) imply that

$$d'(x_{n+1}, x_{m+1}) = d'(Tx_n, Tx_m) < \epsilon \qquad \text{whenever} \qquad n, m \geq N,$$

which proves our claim. Since (x, d') is complete, so there exists $\lambda \in \overline{S(x_0, r)^{d'}}$ with $d(x_n, \lambda) \to 0$ as $n \to \infty$. We further claim that $\lambda = T\lambda$. First considering the case when $d \neq d'$

$$d'(\lambda, T\lambda) \leq d'(\lambda, x_n) + d'(x_n, T\lambda) = d'(\lambda, x_n) + d'(Tx_{n-1}, T\lambda). \tag{2.9}$$

Letting $n \to \infty$ and the continuity of T follows from (iii) insures that $d'(\lambda, x_n) \to 0$ implies $d'(Tx_{n-1}, T\lambda) \to 0$ which further shows that $\lambda = T\lambda$. Next assume the case, when $d = d'$ then

$$d(\lambda, Tx_n) \leq d(\lambda, x_n) + d(x_n, Tx_n) = d(\lambda, x_n) + d(x_n, x_{n+1}).$$

Taking limit $n \to \infty$ we get,

$$\lim_{n \to \infty} d(\lambda, Tx_n) \leq 0.$$

From (iv) since T is continuous at λ so we have $d(\lambda, T\lambda) = 0$ which implies that $\lambda = T\lambda$. □

Remark 2.1.1. *Theorem* 2.1.4 *remains true if ψ is a gauge function of order $r \geq 1$.*

The following global results can easily be deduced from Theorem 2.1.4 and remark 2.1.1.

Theorem 2.1.5. [*12*] *Let (X, d') be a complete metric space, d be another metric on X and $T : X \to X$ is an operator satisfying* (2.4) *with gauge function ψ of order $r \geq 1$ on an interval $J = [0, \infty)$. Then T has a fixed point provided that, the following three conditions are satisfied:*

(a) If $d \ngeq d'$ assume that T is uniformly continuous from (X, d) into (X, d');

(b) If $d \neq d'$ then T is continuous from (X, d') into (X, d');

(c) If $d = d'$ then T is continuous at λ.

Proof. Fix $x_0 \in X$ and choose $\delta > 0$ such that $d(x_0, Tx_0) < \delta$, and take $r = \sigma(\delta)$. Now Theorem 2.1.4 guarantees that there exists $\lambda \in \overline{S(x_0, r)}^{d'}$ such that $\lambda = T\lambda$. $\square$

Theorem 2.1.6. [*12*] *Let (X, d') be a complete metric space, d be another metric on X, $x_0 \in X$, $r > 0$ and $T : S(x_0, r) \to X$ is an operator satisfying*

$$d(Tx, Ty) \leq \psi(M(x, y)); \qquad \forall\ x, y, Tx, Ty \in S(x_0, r), \tag{2.10}$$

where ψ is a gauge function of order $r \geq 0$ on an interval $J = [0, \infty)$ and

$$M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right\}. \tag{2.11}$$

Then the iterative sequence (2.5) *converges to a unique fixed point λ of T provided that (i) – (iv) of Theorem* 2.1.4 *hold. Moreover, if ψ is continuous, then continuity of T in (iv) of Theorem* 2.1.4 *can be omitted.*

Proof. Let $x_1 = Tx_0$. Then from (i)

$$d(x_0, Tx_0) < \delta \leq r.$$

Next let $x_2 = Tx_1$ then from (2.10), we have

$$d(x_1, x_2) = d(Tx_0, Tx_1) \leq \psi(M(x_0, x_1))$$

where

$$\begin{aligned} M(x_0, x_1) &= \max\left\{d(x_0, x_1), d(x_0, Tx_0), d(x_1, Tx_1), \frac{d(x_0, Tx_1) + d(x_1, Tx_0)}{2}\right\} \\ &= \max\left\{d(x_0, x_1), d(x_0, x_1), d(x_1, x_2), \frac{d(x_0, x_2) + d(x_1, x_1)}{2}\right\} \\ &= \max\left\{d(x_0, x_1), d(x_1, x_2), \frac{d(x_0, x_2)}{2}\right\}. \end{aligned}$$

We claim that

$$d(x_1, x_2) \leq \psi(d(x_0, x_1)). \tag{2.12}$$

<u>CaseI.</u> If $M(x_0, x_1) = d(x_0, x_1)$ our claim is trivially true.
<u>CaseII.</u> If $M(x_0, x_1) = d(x_1, x_2)$. Then we have, $d(x_1, x_2) \leq \psi(d(x_1, x_2) < d(x_1, x_2)$ since $\psi(t) < t$ leads to a contradiction.
<u>CaseIII.</u> Finally suppose $M(x_0, x_1) = \frac{d(x_0, x-2)}{2}$. Then we have, from the triangular

inequality and considering the property of gauge function $\psi(t) < t$ that

$$\begin{aligned}
d(x_1, x_2) &\leq \psi(\frac{d(x_0, x_2)}{2}) \\
&< \frac{d(x_0, x_2)}{2} \\
&\leq \frac{d(x_0, x_1) + d(x_1, x_2)}{2} \\
d(x_1, x_2) - \frac{d(x_1, x_2)}{2} &< \frac{d(x_0, x_1)}{2} \\
d(x_1, x_2) &< d(x_0, x_1).
\end{aligned}$$

Which further leads to the conclusion that,

$$\begin{aligned}
M(x_0, x_1) &= \frac{d(x_0, x_2)}{2} \\
&\leq \frac{d(x_0, x_1) + d(x_1, x_2)}{2} \\
&< d(x_0, x_1),
\end{aligned}$$

which is not true since $M(x_0, x_1)$ is maximum and this proves our claim. Proceeding in a same way as in Theorem 2.1.4 we obtain the iterative sequence (2.5) converges to the fixed point λ of T. Now we show that this fixed point λ is unique. Let γ be another fixed point such that $\lambda \neq \gamma$, then $d(\lambda, \gamma) \neq 0$ then from (2.10) and (2.11) we have $M(\lambda, \gamma) = d(\lambda, \gamma)$ since $T\lambda = \lambda$ and $T\gamma = \gamma$ and

$$\begin{aligned}
d(\lambda, \gamma) &= d(T\lambda, T\gamma) \\
&\leq \psi(M(\lambda, \gamma) \\
&= \psi(d(\lambda, \gamma)) \\
&< d(\lambda, \gamma)
\end{aligned}$$

which is not possible. This contradiction arises due to our wrong supposition that $\lambda \neq \gamma$. Hence $\lambda = \gamma$. Finally, suppose that $d = d'$ and ψ is continuous then it follows from (2.10), that

$$\begin{aligned}
d(x_{n+1}, T\lambda) &= d(Tx_n, T\lambda) \\
&\leq \psi(M(x_n, \lambda)) \\
&= \psi\left(\max\left\{d(x_n, \lambda), d(x_n, Tx_n), d(\lambda, T\lambda), \frac{d(x_n, T\lambda) + d(\lambda, Tx_n)}{2}\right\}\right).
\end{aligned}$$

Taking limit as $n \to \infty$ in the above inequality we obtain $d(\lambda, T\lambda) \leq \psi(d(\lambda, T\lambda))$ which is possible only if $\lambda = T\lambda$. □

The following global result can be easily deduced from Theorem 2.1.6.

Theorem 2.1.7. [*12*] *Let (X, d') be a complete metric space on X and $T : X \to X$ is an operator satisfying* (2.10) *with gauge function ψ of order $r \geq 1$ on an interval $J = [0, \infty)$ and $M(x, y)$ is defined in* (2.11). *Then T has a unique fixed point provided that the following conditions are satisfied;*
(I) If d $\ngeq$ d' assume T is uniformly continuous from (X, d) into (X, d').
(II) If d $\neq$ d' then T is continuous from (X, d') into (X, d').
(III) If d $=$ d' then T is continuous at λ.
Moreover, if ψ is continuous, then continuity of T in (III) can be omitted.

Proof. Fix $x_0 \in X$ and choose $\delta > 0$ such that $d(x_0, Tx_0) < \delta$ and take $r = \sigma(\delta)$. Then from Theorem 2.1.6, we deduce that there exists $\lambda \in \overline{S(x_0, r)}^{d'}$ such that $\lambda = T\lambda$. □

Chapter 3

α-ψ-contractive and twisted (α,β)-ψ-contractive type mappings

Banach contraction principle [2] is the simplest and one of the most versatile elementary result in nonlinear analysis especially in fixed point theory. It produces approximations of any required accuracy, and moreover, even the number of iterations needed to get a specified accuracy can be determined. This principle has various applications and has been extended by many authors in different ways. Samet *et al.* [23] introduced a new concept of α-ψ-contractive type mapping and established various fixed point theorems for such a mapping in context of complete metric spaces. These theorems extend, generalize and improve many existing results in literature, in particular, the Banach contraction principle and the results of Ran and Reurings [20], Nieto and Rogriguez-Lopez [15] and Bhasker and Lakshmikantham [5]. Recently, Shahi *et al.* [24] gave the integral version of α-ψ-contractive type mapping and proved some related fixed point theorems. As a consequence of main results of Shahi *et al.* [24] the well known Branciari [8] fixed point theorem, in which the mapping was considered to satisfy the integral version of contraction condition and hence Banach contraction principle were obtained. Very recently, Karapinar *et al.* [11] introduced two new classes of generalized α-ψ-contractive mappings of integral type and obtained some fixed point results. Salimi *et al.* [22] introduced the concept of twisted (α,β)-ψ-contractive mappings and obtained some fixed point results. The results of Salimi *et al.* [22] also generalize and extend the results of Samet *et al.* [23] and hence Banach contraction principle [2]. The basic aim of this chapter is, to review some important fixed point results related to α-ψ and twisted (α,β)-ψ-notions. This chapter consists on two sections. In section one, fixed point results related to $\alpha-\psi$-notion are discussed. In second section of this chapter, fixed point results of twisted (α,β)-ψ-contractive mappings are discussed.

3.1 α-ψ-contractive type mappings

Definition 3.1.1. [23] Let (X, d) be a metric space and $T : X \to X$ be given mapping. We say that T is α-ψ-contractive mapping if there exist two functions $\alpha : X \times X \to [0, +\infty)$ and $\psi \in \Psi$ such that

$$\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y)),\ \forall\ x, y \in X.$$

Example 3.1.1. *Let $X = \mathbb{R}$ is a metric space with standard metric $d(x, y) = |x - y|$; $\forall\ x, y \in X$. Define the self map T from X by*

$$Tx = \begin{cases} 2x - \frac{3}{2}, & \text{if } x > 1 \\ \frac{x}{2}, & \text{if } 0 \leq x \leq 1 \\ 0, & \text{if } x < 0. \end{cases}$$

Then T is not contraction, since $d(T1, T2) > d(1, 2)$ and let α be defined as;

$$\alpha(x, y) = \begin{cases} 1, & \text{if } x, y \in [0, 1] \\ 0, & \text{otherwise.} \end{cases}$$

If $\alpha(x, y) = 0$ then T is α-ψ-contractive mapping. Now let $\alpha(x, y) = 1$ then $x, y \in [0, 1]$ and hence $Tx = \frac{x}{2}$ and $Ty = \frac{y}{2}$ from which we have

$$d(Tx, Ty) = \frac{1}{2}|x - y| = \psi(d(x, y),\ \text{with } \psi(t) = \frac{1}{2}t,$$

and hence we get

$$\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y)),\ \forall\ x, y \in X = \mathbb{R}.$$

Theorem 3.1.1. *[23] Let (X, d) be a complete metric space and $T : X \to X$ be an α-ψ-contractive mapping satisfying the following conditions:*
(i) T is α-admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;
(iii) T is continuous.
Then T has a fixed point, that is, there exists $x \in X$ such that $Tx = x$.

Example 3.1.2. *Let $X = (-\infty, +\infty)$ endowed with usual metric $d(x, y) = |x - y|$, $\forall\ x, y \in X$. Defining the map $T : X \to X$ by;*

$$Tx = \begin{cases} -x^2, & \text{if } x < 0, \\ \frac{x}{2}, & \text{if } 0 \leq x \leq 1, \\ 2x - \frac{3}{2}, & \text{if } x > 1. \end{cases}$$

Here, we can observe that the mapping T is continuous but we cannot apply Banach contraction principle since for $x = -1$ and $y = -2$ in $\mathbb{R}$, we have

$$d(T_{-1}, T_{-2}) = 3 > 1 = d(-1, -2).$$

Although, the mapping T is not a contraction but T is an α-ψ-contractive type mapping and taking α as defined below:

$$\alpha(x, y) = \begin{cases} 1, & \text{if } x, y \in [0, 1] \\ 0, & \text{otherwise,} \end{cases}$$

together with $\psi(t) = \frac{t}{2}$ for all $t \geq 0$. As we have, for $x, y \in [0, 1]$,

$$\alpha(x, y)d(Tx, Ty) \leq \frac{1}{2}d(x, y).$$

Other case is trivial. Finally $x_0 = 1 \in X$ such that $\alpha(x_0, Tx_0) = 1$. Since all the hypotheses of Theorem 3.1.1 *are satisfied. Thus T has at least one fixed point. Hence $FixT = \{-1, 0, \frac{3}{2}\}$.*

Notice that the above theorem does not guarantee the uniqueness of the fixed point. To assure the uniqueness of fixed point, the authors in [23] added the following condition to Theorem 3.1.1.
(U): For all $x, y \in X$, there exists $z \in X$ such that $\alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$.

Theorem 3.1.2. [*23*] *Adding condition (U) to the hypotheses of Theorem* 3.1.1*, we obtain uniqueness of the fixed point of T.*

Karapinar *et al.* [11] introduced two new classes of generalized α-ψ-contractive mappings of integral type and established some fixed point theorems in the following way:

Definition 3.1.2. Define $\Phi = \{\phi \mid \phi : \mathbb{R}_+ \to \mathbb{R}\}$ such that ϕ is nonnegative, Lebesgue integrable and satisfies

$$\int_0^\epsilon \phi(t)dt > 0; \quad \forall\, \epsilon > 0.$$

Definition 3.1.3. [4] Let $N \in \mathbb{N}$, we say that α is N-transitive (on X) if

$$\begin{aligned} x_o, x_1, ..., x_{N+1} : \alpha(x_i, x_{i+1}) &\geq 1; \quad \forall\, i \in \{0, 1, ..., N\} \\ \text{then } \alpha(x_o, x_{N+1}) &\geq 1. \end{aligned}$$

In particular, we say that α is transitive if it is 1-transitive that is
if $x, y, z \in X : \alpha(x, y) \geq 1$ and $\alpha(y, z) \geq 1$ then $\alpha(x, z) \geq 1$

Remark 3.1.1. *If α is transitive, then it is N-transitive for all $N \in \mathbb{N}$.*

Remark 3.1.2. *If α is N-transitive, then it is not necessarily transitive for all $N \in \mathbb{N}$.*

Definition 3.1.4. [11] Let $T : X \to X$ be given mapping, we say that T is generalized α-ψ-contractive mapping of integral type I if

$$\alpha(x,y) \int_0^{d(Tx,Ty)} \phi(t)dt \leq \psi\bigg(\int_0^{M(x,y)} \phi(t)dt \bigg), \tag{3.1}$$

where

$$M(x,y) = \max\left\{ d(x,y), d(x,Tx), d(y,Ty), \frac{[d(x,Ty) + d(y,Tx)]}{2} \right\};$$

$\forall\ x, y \in X$, $\psi \in \Psi$ and $\phi \in \Phi$.

Definition 3.1.5. [11] Let $T : X \to X$ be given mapping , we say that T is generalized α-ψ-contractive mapping of integral type II if

$$\alpha(x,y) \int_0^{d(Tx,Ty)} \phi(t)dt \leq \psi\bigg(\int_0^{M(x,y)} \phi(t)dt \bigg), \tag{3.2}$$

$$M(x,y) = \max\left\{ d(x,y), \frac{[d(x,Tx) + d(y,Ty)]}{2}, \frac{[d(x,Ty) + d(y,Tx)]}{2} \right\};$$

$\forall\ x, y \in X$, $\psi \in \Psi$ and $\phi \in \Phi$.

Theorem 3.1.3. *[11] Let (X,d) be a complete metric space and $\alpha : X \times X \to [0,+\infty)$ be a transitive mapping. Suppose that $T : X \to X$ is a generalized α-ψ-contractive mapping of integral type I and satisfies the following conditions :*
(i) T is α-admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;
(iii) T is continuous.
Then T has a fixed point, that is, there exists $z \in X$ such that $Tz = z$.

Proof. Let x_0 be an arbitrary element of X such that $\alpha(x_0, Tx_0) \geq 1$. We construct an iterative sequence $(x_n) \in X$ in the following way

$$x_{n+1} = Tx_n; \quad \forall\ n \in \mathbb{N} \cup \{0\}.$$

If there is a natural number $n_0 \in \mathbb{N}$ such that $x_{n_0} = Tx_{n_0}$ then x_{n_0} is a fixed point of T and we are finished. Assume otherwise, let $x_n \neq Tx_n$ for all n. Then since T is α-admissible, we find that

$$\alpha(x_0, Tx_0) = \alpha(x_0, x_1) \geq 1.$$

Proceeding inductively we get

$$\alpha(x_n, x_{n+1}) \geq 1; \quad \forall\, n \in \mathbb{N}. \tag{3.3}$$

Taking $x = x_{n-1}$ and $y = x_n$ in inequality (3.1) and using (3.3), we infer that

$$\begin{aligned}\int_0^{d(x_n,x_{n+1})} \phi(t)dt &= \int_0^{d(Tx_{n-1},Tx_n)} \phi(t)dt \\ &\leq \alpha(x_{n-1}, x_n) \int_0^{d(Tx_{n-1},Tx_)} \phi(t)dt \\ &\leq \psi(\int_0^{M(x_{n-1},x_n)} \phi(t)dt),\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}M(x_{n-1}, x_n) &= \max\Big\{d(x_{n-1}, x_n), d(x_{n-1}, x_n) \\ &\qquad d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_{n+1}) + d(x_n, x_n)}{2}\Big\} \\ &\leq \max\Big\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_{n+1})}{2}\Big\} \\ &\leq \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}.\end{aligned} \tag{3.5}$$

By using (3.5) and regarding the property of gauge function ψ, we derive from (3.4) that

$$\begin{aligned}\int_0^{d(x_n,x_{n+1})} \phi(t)dt &= \int_0^{d(Tx_{n-1},Tx_n)} \phi(t)dt \\ &\leq \alpha(x_{n-1}, x_n) \int_0^{d(Txn-1,Tx_n)} \phi(t)dt \\ &\leq \psi\left(\int_0^{\max\{d(x_{n-1},x_n),d(x_n,x_{n+1})\}} \phi(t)dt\right) \\ &\leq \psi\left(\max\left\{\int_0^{d(x_{n-1},x_n)} \phi(t)dt, \int_0^{d(x_n,x_{n+1})} \phi(t)dt\right\}\right).\end{aligned}$$

We claim that $M(x_{n-1}, x_n) = d(x_{n-1}, x_n)$. Assume otherwise, then since $\psi(t) < t$ for all $t > 0$, we deduce from the above inequality that

$$\int_0^{d(x_n,x_{n+1})} \phi(t)dt \leq \psi\Big(\int_0^{d(x_n,x_{n+1})} \phi(t)dt\Big) < \int_0^{d(x_n,x_{n+1})} \phi(t)dt,$$

which is not possible and so our claim is true and we get

$$\int_0^{d(x_n,x_{n+1})} \phi(t)dt \leq \psi\left(\int_0^{d(x_n,x_{n-1})} \phi(t)dt\right). \tag{3.6}$$

By using mathematical induction, we get, for all $n \in \mathbb{N}$

$$\int_0^{d(x_n,x_{n+1})} \phi(t)dt \leq \psi^n\left(\int_0^{d(x_0,x_1)} \phi(t)dt\right) = \psi^n(\varsigma), \tag{3.7}$$

where $\varsigma = \int_0^{d(x_0,x_1)} \phi(t)dt$. Taking limit as $n \to +\infty$ in (3.7) together with the property of ψ as a gauge function we get that

$$\lim_{n\to+\infty} \int_0^{d(x_n,x_{n+1})} \phi(t)dt = 0. \tag{3.8}$$

Using the definition of $\phi \in \Phi$, implies that

$$d(x_n, x_{n+1}) \to 0 \quad \text{whenever} \quad n \to +\infty. \tag{3.9}$$

Note that (3.9) is not sufficient condition for the sequence (x_n) to be a Cauchy sequence in general (see the definition of Cauchy sequence). We now show that (x_n) is a Cauchy sequence. Assume on contrary, that (x_n) is not Cauchy. Then there exists an $\epsilon > 0$ and subsequences $(m(k))$ and $(n(k))$ with the property that $m(k) < n(k) < m(k+1)$ with

$$d(x_{m(k)}, x_{n(k)}) \geq \epsilon, \quad d(x_{m(k)}, x_{n(k)-1}) < \epsilon. \tag{3.10}$$

Using the definition of $M(x,y)$ with $x = x_{m(k)-1}$ and $y = x_{n(k)-1}$ we have that

$$M(x_{m(k)-1}, x_{n(k)-1}) = \max\Bigg\{d(x_{m(k)-1}, x_{n(k)-1}), d(x_{m(k)-1}, x_{m(k)}), d(x_{n(k)-1}, x_{n(k)}) \\ \frac{d(x_{m(k)-1}, x_{n(k)}) + d(x_{n(k)-1}, x_{m(k)})}{2}\Bigg\}. \tag{3.11}$$

From (3.9) it follows that

$$\lim_{k\to+\infty} \int_0^{d(x_{m(k)-1},x_{m}(k)} \phi(t)dt = \lim_{n\to+\infty} \int_0^{d(x_{n(k)-1},n(k)} \phi(t)dt = 0 \tag{3.12}$$

from the triangular inequality and using (3.10), we infer that

$$\begin{aligned} d(x_{m(k)-1}, x_{n(k)-1}) &\leq d(x_{m(k)-1}, x_{m(k)}) + d(x_{m(k)}, x_{n(k)-1}) \\ &\leq \epsilon + d(x_{m(k)-1}, x_{m(k)}). \end{aligned}$$

Taking limit as $k \to +\infty$ we get

$$\lim_{k\to+\infty} \int_0^{d(x_{m(k)-1},x_{n(k)-1})} \phi(t)dt \le \int_0^{\epsilon} \phi(t)dt. \tag{3.13}$$

For $x_0, x_1, \dots x_{m(k)-1}, x_{m(k)}, \dots, x_{n(k)-2}, x_{n(k)-1}, \dots$ together with (3.3) and using the transitivity of α, we deduce that

$$\alpha(x_0, x_1) \ge 1, \dots, \alpha(x_{m(k)-1}, x_{m(k)}) \ge 1, \dots, \alpha(x_{n(k)-2}, x_{n(k)-1}) \ge 1, \dots$$

then

$$\alpha(x_{m(k)-1}, x_{n(k)-1}) \ge 1. \tag{3.14}$$

From (3.1) and (3.13), we get

$$\begin{aligned} \int_0^{d(x_{m(k)},x_{n(k)})} \phi(t)dt &= \int_0^{d(Tx_{m(k)-1},Tx_{n(k)-1})} \phi(t)dt \\ &\le \alpha(x_{m(k)-1}, x_{n(k)-1}) \int_0^{d(Tx_{m(k)-1},Tx_{n(k)-1})} \phi(t)dt \\ &\le \psi\left(\int_0^{M(x_{m(k)-1},x_{n(k)-1})} \phi(t)dt\right). \end{aligned} \tag{3.15}$$

Regarding (3.10) and from the triangular inequality, we get

$$\omega(m(k), n(k)) = \frac{d(x_{m(k)-1}, x_{n(k)}) + d(x_{n(k)-1}, x_{m(k)})}{2} \tag{3.16}$$

$$\begin{aligned} &\le \frac{d(x_{m(k)-1}, x_{m(k)}) + d(x_{m(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{n(k)}) + d(x_{m(k)}, x_{n(k)-1})}{2} \\ &= \frac{d(x_{m(k)-1}, x_{m(k)}) + 2d(x_{m(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{n(k)})}{2} \\ &\le \frac{d(x_{m(k)-1}, x_{m(k)}) + d(x_{n(k)-1}, x_{n(k)})}{2} + \epsilon. \end{aligned} \tag{3.17}$$

Letting $k \to +\infty$ in (3.17) and regarding (3.9), we deduce that

$$\lim_{k\to+\infty} \omega(m(k), n(k)) < \epsilon,$$

which gives

$$\lim_{k\to+\infty} \int_0^{\omega} \phi(t)dt \le \int_0^{\epsilon} \phi(t)dt. \tag{3.18}$$

From (3.1), (3.10)-(3.14) and (3.17) we deduce that

$$\begin{aligned}\int_0^{\epsilon}\phi(t)dt &\leq \int_0^{d(x_{m(k)},x_{n(k)})}\phi(t)dt\\ &\leq \alpha(x_{m(k)-1},x_{n(k)-1})\int_0^{d(Tx_{m(k)-1},Tx_{n(k)-1})}\phi(t)dt\\ &\leq \psi\left(\int_0^{M(x_{m(k)-1},x_{n(k)-1})}\phi(t)dt\right).\end{aligned}$$

Now since ψ is nondecreasing, so we infer from the above inequality that,

$$\int_0^{\epsilon}\phi(t)dt \leq \psi\left(\int_0^{\epsilon}\phi(t)dt\right), \tag{3.19}$$

which is a contradiction to the property of ψ as a gauge function that is $\psi(t) < t$. Hence (x_n) is a Cauchy sequence in X. Now as (X, d) is complete, so there is z in X such that $x_n \to z$. From the continuity of T it follows that $Tx_n \to Tz$ that is $x_{n+1} \to Tz$ but (x_n) is convergent in X and as limit of convergent sequence is unique so we get $z = Tz$ and hence z is the fixed point of T. □

Theorem 3.1.4. *[11] Let (X, d) be a complete metric space and $\alpha : X \times X \to [0, +\infty)$ be a transitive mapping. Suppose that $T : X \to X$ is a generalized α-ψ-contractive mapping of integral type II and satisfies the following conditions:*
(i) T is α-admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;
(iii) T is continuous.
Then there exists $z \in X$ such that $Tz = z$.

Proof. Proof of this theorem runs on the same line as the proof of the above theorem. □

Notice that the above theorems do not assure the uniqueness of the fixed point of T. To assure the uniqueness of the fixed point, the following theorem is proved.

Theorem 3.1.5. *Adding condition (U) to the hypotheses of Theorem* 3.1.4 *(resp. Theorem* 3.1.3 *guarantees uniqueness of the fixed point x of T.*

Proof. We prove the theorem by contradiction method. Assume that, x is not the only fixed point of T. That is there is also y in X such that $Ty = y$ and $x \neq y$. From hypothesis (U), we obtain that there exists $\mu \in X$ so that

$$\alpha(x, \mu) \geq 1, \quad \alpha(y, \mu) \geq 1. \tag{3.20}$$

Taking α-admissibility of T in account we deduce from (3.20) that, for all $n \in \mathbb{N}$

$$\alpha(T^n x, T^n \mu) \geq 1, \quad \alpha(T^n y, T^n \mu) \geq 1.$$

But since x and y are fixed points, so we have

$$\alpha(x, T^n \mu) \geq 1, \quad \alpha(y, T^n \mu) \geq 1. \tag{3.21}$$

Considering a sequence (μ_n) in X defined by $\mu_{n+1} = T\mu_n;\ \forall\ n\ \in \mathbb{N} \cup \{0\}$ with $\mu_0 = \mu$, then from (3.2) and (3.21), we have

$$\begin{aligned} \int_0^{d(x,\mu_{n+1})} \phi(t)dt &\leq \alpha(x, \mu_n) \int_0^{d(Tx,T\mu_n)} \phi(t)dt \\ &\leq \psi\Big(\int_0^{M(x,\mu_n)} \phi(t)dt \Big). \end{aligned} \tag{3.22}$$

where

$$\begin{aligned} M(x, \mu_n) &= \max\left\{ d(x, \mu_n), \frac{d(x, T\mu) + d(\mu_n, T\mu_n)}{2}, \frac{d(x, T\mu_n) + d(\mu_n, Tx)}{2} \right\} \\ &= \max\left\{ d(x, \mu_n), d(\mu_n, T\mu_n), \frac{d(x, T\mu_n) + d(\mu_n, Tx)}{2} \right\} \\ &\leq \left\{ d(x, \mu_n), d(\mu_n, \mu_{n+1}), d(x, \mu_{n+1}) \right\}. \end{aligned} \tag{3.23}$$

Since ψ is nondecreasing monotone, it follows from (3.22) and (3.23) that

$$\begin{aligned} \int_0^{d(x,\mu_{n+1})} \phi(t) &\leq \psi\left(\int_0^{M(x,\mu_n)} \phi(t)dt \right) \\ &\leq \psi\left(\int_0^{\max\{d(x,\mu_n),d(\mu_n,\mu_{n+1}),d(x,\mu_{n+1})\}} \phi(t)dt \right) \\ &\leq \psi\left(\max\left\{ \int_0^{d(x,\mu_n)} \phi(t)dt, \int_0^{d(\mu_n,\mu_{n+1})} \phi(t)dt, \int_0^{d(x,\mu_{n+1})} \phi(t)dt \right\} \right). \end{aligned} \tag{3.24}$$

We examine the following three possibilities in (3.24) and since ψ is nondecreasing together with $\psi(t) < t$ and let us assume for simplicity that

$$A(x, \mu_n) = \max\left\{ \int_0^{d(x,\mu_n)} \phi(t)dt, \int_0^{d(\mu_n,\mu_{n+1})} \phi(t)dt, \int_0^{d(x,\mu_{n+1})} \phi(t)dt \right\}.$$

Case 1 : If $A(x, \mu_n) = \int_0^{d(x,\mu_{n+1})} \phi(t)dt$, then

$$\int_0^{d(x,\mu_{n+1})} \phi(t)dt \leq \psi\Big(\int_0^{d(x,\mu_{n+1})} \phi(t)dt \Big) < \int_0^{d(x,\mu_{n+1})} \phi(t)dt,$$

is a contradiction due to the property of ψ that $\psi(t) < t$.
Case 2 : If $A(x,\mu_n) = \int_0^{d(x,\mu_n)} \phi(t)$, then

$$\int_0^{d(x,\mu_n)} \phi(t)dt \leq \psi\bigg(\int_0^{d(x,\mu_n)} \phi(t)dt\bigg),$$

which further shows that

$$\int_0^{d(x,\mu_{n+1})} \phi(t)dt \leq \psi^n\bigg(\int_0^{d(x,\mu_0)} \phi(t)dt\bigg); \quad \forall\, n \in \mathbb{N}\cup\{0\}. \tag{3.25}$$

Taking limit $n \to +\infty$ in (3.25) and regarding the properties of $\phi \in \Phi$ and $\psi \in \Psi$, we get that

$$\lim_{n\to+\infty} d(x,\mu_n) = 0 \tag{3.26}$$

Case 3: If $A(x,\mu_n) = \int_0^{d(\mu_n,\mu_{n+1})} \phi(t)dt$ then utilizing the triangular inequality we get

$$\begin{aligned} d(\mu_n,\mu_{n+1}) &\leq d(\mu_n,x) + d(x,\mu_{n+1}) \\ &\leq 2\max\{d(x,\mu_n), d(x,\mu_{n+1})\}. \end{aligned} \tag{3.27}$$

Notice that

$$d(x,\mu_{n+1}) \leq d(x,\mu_n),$$

so the inequality (3.27) will reduce to the case

$$d(\mu_n,\mu_{n+1}) \leq 2d(x,\mu_n). \tag{3.28}$$

Otherwise, let $d(x,\mu_n) \leq d(x,\mu_{n+1})$, then we have a contradiction as in Case 1 and hence from (3.24) and (3.28) we infer that

$$\int_0^{d(x,\mu_{n+1})} \phi(t)dt \leq \psi\bigg(\int_0^{d(\mu_n,\mu_{n+1})} \phi(t)dt\bigg).$$

Which from triangular inequality the monotone property of ψ implies that

$$\begin{aligned} \int_0^{d(x,\mu_{n+1})} \phi(t)dt &\leq \psi\bigg(\int_0^{d(\mu_n,x)+d(x,\mu_{n+1})} \phi(t)dt\bigg) \\ &\leq \psi\left(\int_0^{2\max\{d(\mu_n,x)+d(x,\mu_{n+1})\}} \phi(t)dt\right) \\ &\leq \psi\left(\int_0^{2d(x,\mu_n)} \phi(t)dt\right);\ \forall\, n \in \mathbb{N}. \end{aligned}$$

Consequently, we find that

$$\int_0^{d(x,\mu_{n+1})} \phi(t)dt \leq \psi^n \left(\int_0^{2d(x,\mu_0)} \phi(t)dt \right); \quad \forall\, n \in \mathbb{N}. \tag{3.29}$$

Taking limit in the above inequality we obtain

$$\lim_{n\to+\infty} \int_0^{d(x,\mu_{n+1})} \phi(t)dt = 0, \tag{3.30}$$

which from the definition of ϕ, implies that

$$\lim_{n\to+\infty} d(x,\mu_n) = 0. \tag{3.31}$$

Similar is the case considering y, we will obtain

$$\lim_{n\to+\infty} d(y,\mu_n) = 0. \tag{3.32}$$

Which further from the uniqueness of the limit of convergent sequence gives that $x = y$. Hence T has a unique fixed point. □

3.2 Twisted (α,β)-ψ-contractive type mappings

Let $T : X \to X$ be a contraction mapping from a nonempty complete metric space X to itself then Banach contraction principle [2] states that there must be a unique element $x \in X$ such that $x = Tx$. Although, this principle is very powerful tool in nonlinear analysis especially in metric fixed point theory, but what will happen if T is not contraction or is not continuous? To give answer to this question many authors generalized this result for the sake to make the contraction condition on T weakened and so on. One of this generalization was the introduction of α-ψ-notion by Samet *et al.* [23], which was recently extended by Salimi *et al.* [22], by introducing three new twisted (α,β)-ψ-contractive type mappings. In this section, some results regarding twisted (α,β)-ψ-contractive type mappings by Salimi *et al.* [22] are presented, which will be used in the last chapter.

Definition 3.2.1. [22] Let (X, d) be a metric space and $T : X \to X$ be twisted (α, β)-admissible mapping. Then T is said to be
(a) twisted (α, β)-ψ-contractive mapping of type I, if

$$\alpha(x, y)\beta(x, y)d(Tx, Ty) \leq \psi(d(x, y)),\ \forall\ x, y \in X \text{ and } \psi \in \Psi, \tag{3.33}$$

(b) twisted (α, β)-ψ-contractive mapping of type II, if there is $0 < r \leq 1$ such that

$$\Big(\alpha(x, y)\beta(x, y) + r\Big)^{d(Tx,Ty)} \leq \Big(1 + r\Big)^{\psi(d(x,y))},\ \forall\ x, y \in X \text{ and } \psi \in \Psi, \tag{3.34}$$

(c) twisted (α, β)-ψ-contractive mapping of type III, if there is $r \geq 1$ such that

$$\Big(d(Tx, Ty) + r\Big)^{\alpha(x,y)\beta(x,y)} \leq \psi\big(d(x,y)\big) + r, \ \forall \ x, y \in X \text{ and } \psi \in \Psi. \qquad (3.35)$$

Theorem 3.2.1. *[22] Let (X, d) be a complete metric space and let $T : X \to X$ be a continuous twisted (α,β)-ψ-contractive mapping of type I, II or III. If there is $u_0 \in X$ such that $\alpha(u_0, Tu_0) \geq 1$ and $\beta(u_0, Tu_0) \geq 1$. Then there exists $\lambda \in X$ such that $\lambda = T\lambda$.*

Proof. As $u_0 \in X$ such that $\alpha(u_0, Tu_0) \geq 1$ and $\beta(u_0, Tu_0) \geq 1$. Defining an iterative sequence (u_n) starting from u_0 in X by $u_{n+1} = Tu_n$; $\forall \ n \in \mathbb{N} \cup \{0\}$. Now the twisted (α,β)-admissibility of T implies that $\alpha(u_0, Tu_0) = \alpha(u_0, u_1) \geq 1$ and $\beta(u_0, Tu_0) = \beta(u_0, u_1) \geq 1$ then $\alpha(Tu_1, Tu_0) \geq 1 = \alpha(u_2, u_1) \geq 1$ and $\beta(Tu_1, Tu_0) = \beta(u_2, u_1) \geq 1$ which further implies $\alpha(u_2, u_3) \geq 1$ and $\beta(u_2, u_3) \geq 1$. Proceeding inductively, we get

$$\alpha(u_{2n}, u_{2n-1}) \geq 1 \qquad \text{and} \qquad \alpha(u_{2n}, u_{2n+1}) \geq 1 \qquad (3.36)$$

$$\beta(u_{2n}, u_{2n-1}) \geq 1 \qquad \text{and} \qquad \beta(u_{2n}, u_{2n+1}) \geq 1. \qquad (3.37)$$

1) Taking T to be a twisted (α,β)-ψ-contractive of type I and considering $x = u_{2n}$, $y = u_{2n+1}$ in (3.33) and utilizing (3.36) and (3.37) we get

$$\begin{aligned}
d(Tu_{2n}, Tu_{2n+1}) &= d(u_{2n+1}, u_{2n+2}) \\
&\leq \alpha(u_{2n}, u_{2n+1})\beta(u_{2n}, u_{2n+1})d(u_{2n+1}, u_{2n+2}) \\
&\leq \psi(d(u_{2n}, u_{2n+1})) \\
d(u_{2n+1}, u_{2n+2}) &\leq \psi(d(u_{2n}, u_{2n+1}).
\end{aligned} \qquad (3.38)$$

Taking $x = u_{2n}, y = u_{2n-1}$ in (3.33) and utilizing (3.36) and (3.37) again, we get

$$\begin{aligned}
d(Tu_{2n}, Tu_{2n-1}) &= d(u_{2n+1}, u_{2n}) \\
&\leq \alpha(u_{2n}, u_{2n-1})\beta(u_{2n}, u_{2n-1})d(u_{2n+1}, u_{2n}) \\
&\leq \psi(d(u_{2n}, u_{2n-1})) \\
d(u_{2n+1}, u_{2n}) &\leq \psi(d(u_{2n}, u_{2n-1})).
\end{aligned} \qquad (3.39)$$

From (3.38) and (3.39), we have

$$d(u_n, u_{n+1}) \leq \psi^n(d(u_0, u_1)); \qquad \forall \ n \in \mathbb{N}. \qquad (3.40)$$

2) Assume that T be a twisted (α,β)-ψ-contractive mapping of type II. Then taking $x = u_{2n}$ and $y = u_{2n+1}$ in (3.34) together with (3.36) and (3.37), we get

$$\begin{aligned}
\big(1 + r\big)^{d(u_{2n+1}, u_{2n+2})} &= (1 + r)^{d(Tu_{2n}, Tu_{2n+1})} \\
&\leq (\alpha(u_{2n}, u_{2n+1})\beta(u_{2n}, u_{2n+1}) + r)^{d(u_{2n+1}, u_{2n+2})} \\
&\leq (1 + r)^{\psi(d(u_{2n}, u_{2n+1}))}
\end{aligned}$$

$$\big(1+r\big)^{d(u_{2n+1},u_{2n+2})} \leq (1+r)^{\psi(d(u_{2n},u_{2n+1}))}. \tag{3.41}$$

Taking $x = u_{2n}$, and $y = u_{2n-1}$ in (3.34) and using (3.36) and (3.37) again, we get

$$\begin{aligned} \big(1+r\big)^{d(u_{2n},u_{2n+1})} &= \big(1+r\big)^{d(Tu_{2n-1},Tu_{2n})} \\ &\leq (\alpha(u_{2n-1},u_{2n})\beta(u_{2n-1},u_{2n})+r)^{d(u_{2n},u_{2n+1})} \\ &\leq \big(1+r\big)^{\psi(d(u_{2n-1},u_{2n}))} \end{aligned}$$

which implies that

$$\big(1+r\big)^{d(u_{2n},u_{2n+1})} \leq \big(1+r\big)^{\psi(d(u_{2n-1},u_{2n}))}. \tag{3.42}$$

From (3.41) and (3.42) we get, since $r \geq 0$ is constant and proceeding inductively that,

$$d(u_n,u_{n+1}) \leq \psi^n(d(x_0,x_1)). \tag{3.43}$$

3) Finally let T be twisted (α,β)-ψ-contractive of type III. Then taking $x = u_{2n}$ and $y = u_{2n+1}$ in (3.35), we get

$$\begin{aligned} d(u_{2n+1},u_{2n+2})+r &= d(Tu_{2n},u_{2n+1})+r \\ &\leq (d(u_{2n+1},u_{2n+2})+r)^{\alpha(u_{2n},u_{2n+1})\beta(u_{2n},u_{2n+1})} \end{aligned}$$

$$d(u_{2n+1},u_{2n+2}) \leq \psi((d(u_{2n},u_{2n+1})+r), \tag{3.44}$$

similarly considering $x = u_{2n}$ and $y = u_{2n-1}$ in (3.35) we have

$$d(u_{2n+1},u_{2n}) \leq \psi((d(u_{2n},u_{2n-1})+r). \tag{3.45}$$

Hence from (3.44) and (3.45), we get

$$d(u_n,u_{n+1}) \leq \psi^n(d(x_0,x_1)). \tag{3.46}$$

From all the above three cases, we infer that

$$d(u_n,u_{n+1}) \leq \psi^n(d(x_0,x_1)).$$

Taking limit as $n \to +\infty$ in above inequality we get that

$$\lim_{n\to+\infty} d(u_n,u_{n+1}) = 0$$

Now, we show that (u_n) is a Cauchy sequence. Let $\epsilon > 0$ be fixed then there exists $n_0 \in \mathbb{N}$ such that

$$\sum_{n\geq n_0} \psi^n(d(x_0,x_1)) < \epsilon.$$

Let $m, n \in \mathbb{N}$ with $m > n > n_0$. Then from the triangular inequality we deduce that

$$\begin{aligned} d(u_n, u_m) &\leq \sum_{p=n}^{m-1} d(u_p, u_{p+1}) \\ &\leq \sum_{n \geq n_0} \psi^n(d(u_0, u_1)) < \epsilon. \end{aligned}$$

Which implies that (u_n) is Cauchy sequence. Now since (X, d) is complete. Hence there exists $\lambda \in X$ such that $u_n \to \lambda$ as $n \to +\infty$ and also since T is continuous, so $u_{n+1} = Tu_n \to T\lambda$ and from the uniqueness of the limit of convergent sequence, we have that $\lambda = T\lambda$ that is λ is the fixed point of T. □

Notice that if $T : X \to X$ is α-β-admissible then it is a special case of twisted (α,β)-admissibility of T. But the converse is not true. In the following theorem continuity condition on T has been replaced by another parallel condition.

Theorem 3.2.2. *[22] Let $T : X \to X$ be twisted (α,β)-contractive mapping of type I, II or III and assume that the following conditions are satisfied*
(i) there exists $u_0 \in X$ such that $\alpha(u_0, Tu_0) \geq 1$ and $\beta(u_0, Tu_0) \geq 1$;
(ii) (u_n) is a sequence in X such that $\alpha(u_{2n}, u_{2n+1}) \geq 1$ and $\beta(u_{2n}, u_{2n+1}) \geq 1$, $\forall$ $\mathbb{N}$ and $u_n \to \lambda$ as $n \to +\infty$, then $\alpha(u_{2n}, \lambda) \geq 1$ and $\beta(u_{2n}, \lambda) \geq 1$ for all n.
Then T has a fixed point.

To assure the uniqueness of the fixed point, the authors in [22] added the following condition to the hypotheses of Theorem 3.2.2 (resp. Theorem 3.2.1).
(H): For all $x, y \in X$ and $x \neq y$, there exists $\nu \in X$ such that $\alpha(x, \nu) \geq 1$ and $\alpha(y, \nu) \geq 1$, $\beta(x, \nu) \geq 1$ and $\beta(y, \nu) \geq 1$.

Theorem 3.2.3. *Assume that all the conditions of Theorem* 3.2.2 *(resp. Theorem* 3.2.1*) together with (H) are satisfied. Then T has a unique fixed point.*

Example 3.2.1. *Let $X = \mathbb{R}$ with usual metric $d(u, v) = |u - v|$, for all $u, v \in X$ and T is defined as*

$$T(u) = \begin{cases} -\frac{1}{4}u, & \text{if } -1 \leq u \leq 1 \\ \sqrt[3]{\frac{u+1}{u^2+1}}, & \text{otherwise,} \end{cases}$$

and $\psi(t) = \frac{1}{2}$, for all $t \geq 0$.
Defining α and β as

$$\alpha(u, v) = \beta(u, v) = \begin{cases} 1, & \text{if } u \in [0, 1] \quad \text{and} \quad v \in [-1, 0] \\ 0, & \text{otherwise.} \end{cases}$$

If $\alpha(u,v) \geq 1$ for $u, v \in X$. Then we have, $u \in [0,1]$ and $v \in [-1,0]$, which implies that, $Tv \in [0,1]$ and $Tu \in [-1,0]$. If $\beta(u,v) \geq 1$ for $u, v \in X$. Then $u \in [0,1]$ and $v \in [-1,0]$ and hence again $Tv \in [-1,0]$ and $Tu \in [0,1]$. Let $u_0 = 0 \in [0,1]$ then $Tu_0 = 0 \in [-1,0]$ and so $\alpha(u_0, Tu_0) \geq 1$ and $\beta(u_0, Tu_0) \geq 1$. Now, let (u_n) be a sequence in X such that $\alpha(u_{2n}, u_{2n+1}) \geq 1$ and $\beta(u_{2n}, u_{2n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $u_n \to \lambda$ as $n \to +\infty$. This implies that (u_{2n+1}) is a sequence in $[-1,0]$ and (u_{2n}) is a sequence in $[0,1]$. Thus $\lambda = 0$ and so $\alpha(u_{2n}, \lambda) \geq 1$ and $\beta(u_{2n}, \lambda) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. Moreover, for $u \in [0,1]$ and $v \in [-1,0]$ we have

$$\begin{aligned}\alpha(u,v)\beta(u,v)d(Tu,Tv) &= |Tu - Tv| \\ &= \frac{1}{4}|u-v| \\ &\leq \frac{1}{2}|u-v| = \psi(d(u,v)).\end{aligned}$$

Otherwise

$$\alpha(u,v)\beta(u,v) = 0$$

and (3.33) is trivially true. Then T is twisted (α, β)-ψ-contractive mapping of type I and from Theorem 3.2.2, T has a fixed point.

Example 3.2.2. *Let (X, d) and α and β be defined as in the above example and $T : X \to X$ is defined as;*

$$T(u) = \begin{cases} -\frac{1}{4\pi}(u + u^2), & \text{if } u \in [-1,1] \\ \frac{u^2 - \cos(u^5)}{2 + \sin(u)}, & \text{otherwise.} \end{cases}$$

Defining $\psi(t) = \frac{1}{4}t$. Let $\alpha(u,v) \geq 1$ then $u \in [0,1]$ and $v \in [-1,0]$ and so we have $Tu \in [-1,0]$ and $Tv \in [0,1]$ and so $\alpha(Tv, Tu) \geq 1$, similarly $\beta(Tv, Tu) \geq 1$. Moreover, there exists $u_0 = 0 \in [0,1]$ and $Tu_0 = 0 \in [-1,0]$ such that by the definition of α and β we have $\alpha(u_0, Tu_0 \geq 1$ and $\beta(u_0, Tu_0) \geq 1$. Similarly the condition (ii) of Theorem 3.2.2 also holds. Furthermore, let $u \in [0,1]$ and $v \in [-1,0]$ together with $0 \leq r \leq 1$ then we have

$$(\alpha(u,v)\beta(u,v) + r)^{d(Tu,Tv)} = (1+r)^{d(Tu,Tv)}$$

where

$$\begin{aligned}d(Tu,Tv) &= \frac{1}{4\pi}|u-v||u+v+1| \\ &\leq \frac{3}{4\pi}|u-v| \\ &\leq \frac{1}{4}|u-v|\end{aligned}$$

and so

$$\begin{aligned}(\alpha(u,v)\beta(u,v)+r)^{d(Tu,Tv)} &\leq (1+r)^{\frac{1}{4}|u-v|}\\ &= (1+r)^{\psi(d(u,v))},\end{aligned}$$

and otherwise $\alpha(u,v)=\beta(u,v)=0$ *and so* (3.34) *is true for all* $u,v \in X$*. Hence all the hypotheses of Theorem* 3.2.2 *are satisfied, so* T *has a fixed point.*

Example 3.2.3. *Let* $X=[0,+\infty)$ *with usual metric* $d(u,v)=|u-v|$ *for all* $u,v \in X$ *and* $T: X \to X$ *be defined as*

$$T(u)=\begin{cases}\frac{1}{8}u^4, & \text{if } u \in [0,1]\\ \frac{1}{u}-\frac{1}{1+u}, & \text{if } u \in (1,+\infty).\end{cases}$$

Defining α *and* β *as;*

$$\alpha(u,v)=\beta(u,v)=\begin{cases}1, & \text{if } u,v \in [0,1]\\ 0, & \text{otherwise},\end{cases}$$

and $\psi:[0,+\infty)\to[0,+\infty)$ *by* $\psi(t)=\frac{1}{2}t$ *for all* t*. As from the above two examples it is easy to show that* T *is (*α*,*β*)-admissible and conditions (i) and (ii) of Theorem* 3.2.2 *also hold. Furthermore,* $u,v \in [0,1]$ *and* $r \geq 1$ *then* $\alpha(u,v)=\beta(u,v)=1$ *and* T *is twisted (*α*,*β*)-*ψ*-contractive mapping of type III and has a fixed point.*

$$\begin{aligned}(d(Tu,Tv)+r)^{\alpha(u,v)\beta(u,v)} &= \frac{1}{8}|u^4-v^4|+r\\ &= \frac{1}{8}|u-v||u+v||u^2+v^2|+r\\ &\leq \frac{1}{2}|u-v|+r.\end{aligned}$$

Thus T *has a fixed point.*

Note that in the last example, since uniqueness hypothesis does not satisfied, so the fixed points of T are not unique.

Chapter 4

Generalized α-ψ and twisted (α, β)-ψ-contractive mappings of integral type in spaces with two metrics

This chapter consists on two sections. The motivation behind the first section is, to investigate fixed point results related to the generalized contractive mappings of integral type in spaces with two metrics. In the second section of this chapter, some new generalization of integral types of twisted (α, β)-ψ-contractive mappings are obtained in the spaces with two metrics.

4.1 Generalized α-ψ-contractive mappings of integral type in spaces with two metrics

In this section, some new fixed point results are presented which generalize and extend many existing results in literature.

Theorem 4.1.1. *Let (X, d') be a complete metric space and d be another metric on X. Let $T : X \to X$ and $\alpha : X \times X \to [0, +\infty)$ be such that α is transitive and T is generalized α-ψ-contractive map of integral type I with respect to d and satisfies the following conditions:*
(i) T is α-admissible and there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;
(ii) If $d \ngeq d'$ assume T is uniformly continuous from (X, d) to (X, d');
(iii) If $d \neq d'$ assume T is continuous from (X, d') into (X, d');
(iv) If $d = d'$ then T is simply continuous.
Then there exists $\lambda \in X$ such that $\lambda = T\lambda$.

Proof. Let x_0 be an arbitrary point of X such that $\alpha(x_0, Tx_0) \geq 1$. We construct an iterative sequence (x_n) in X as $x_{n+1} = Tx_n$; for all $n \in \mathbb{N} \cup \{0\}$. If $x_{n_0+1} = x_{n_0}$ for some $n_0 \in \mathbb{N}$ then $Tx_{n_0} = x_{n_0}$ and hence $x' = x_{n_0}$ is the fixed point of T. Now, we will consider that $x_{n+1} \neq x_n$. Then from (i) and the admissibility of T, we infer that

$$\alpha(x_0, Tx_0) \geq 1 \text{ implies } \alpha(Tx_0, T^2x_0) \geq 1.$$

Proceeding inductively, we get

$$\alpha(x_n, x_{n+1}) \geq 1; \ \forall\, n \in \mathbb{N} \cup \{0\}. \tag{4.1}$$

By taking $x = x_n$ and $y = x_{n+1}$, we deduce from inequality (3.33) that

$$\begin{aligned}
\int_0^{d(x_n,x_{n+1})} \phi(t)dt &= \int_0^{d(Tx_{n-1},Tx_n)} \phi(t)dt \\
&\leq \alpha(x_{n-1}, x_n) \int_0^{d(Tx_{n-1},x_n)} \phi(t)dt \\
&\leq \psi\Big(\int_0^{M(x_{n-1},x_n)} \phi(t)dt\Big),
\end{aligned} \tag{4.2}$$

where

$$\begin{aligned}
M(x_{n-1}, x_n) &= \max\left\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{[d(x_{n-1}, x_{n+1}) + d(x_n, x_n)]}{2}\right\} \\
&\leq \max\left\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_{n+1})}{2}\right\} \\
&\leq \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}.
\end{aligned}$$

By using (4.1) and regarding the property of ψ together with $M(x_{n-1}, x_n)$, we deduce from (3.1) that

$$\begin{aligned}
\int_0^{d(x_n,x_{n+1})} \phi(t)dt &= \int_0^{d(Tx_{n-1},Tx_n)} \phi(t)dt \\
&\leq \alpha(x_{n-1}, x_n) \int_0^{d(Tx_{n-1},Tx_n)} \phi(t)dt \\
&\leq \psi\left(\int_0^{\max\{d(x_{n-1},x_n),d(x_n,x_{n+1})\}} \phi(t)dt\right) \\
&\leq \psi\left(\max\left\{\int_0^{d(x_{n-1},x_n)} \phi(t)dt, \int_0^{d(x_n,x_{n+1})} \phi(t)dt\right\}\right) \\
&\leq \psi\left(\int_0^{d(x_{n-1},x_n)} \phi(t)dt\right).
\end{aligned} \tag{4.3}$$

Notice that the case

$$\begin{aligned}\int_0^{d(x_n,x_{n+1})}\phi(t)dt &\leq \psi\bigg(\int_0^{d(x_n,x_{n+1})}\phi(t)dt\bigg)\\ &< \int_0^{d(x_n,x_{n+1})}\phi(t)dt,\end{aligned}$$

is not possible because $\psi(t) < t;\ \forall\ t > 0$. Proceeding inductively we get, for $n \in \mathbb{N}\cup\{0\}$

$$\int_0^{d(x_n,x_{n+1})}\phi(t)dt \leq \psi^n\left(\int_0^{d(x_o,x_1)}\phi(t)dt\right). \tag{4.4}$$

Taking limit as $n \to +\infty$ in (4.4) and taking the property of ψ as gauge function. Also, since every gauge function is Bianchini-Grandolfi gauge function so, we get $\int_0^{d(x_n,x_{n+1})}\phi(t)dt = 0$ as $n \to +\infty$. Now, since ϕ is non negative. This implies that

$$d(x_n, x_{n+1}) \to 0 \text{ as } n \to +\infty. \tag{4.5}$$

Which further, show that (x_n) is a Cauchy sequence. Suppose, on the contrary, that (x_n) is not Cauchy and for some $\epsilon > 0$ and $m, p \in \mathbb{N}$ the following inequality holds:

$$d(x_m, x_{m+p}) \geq \epsilon,\ \ d(x_{m-1}, x_{m+p}) < \epsilon. \tag{4.6}$$

Then from (4.6 and triangular inequality we get

$$d(x_{m-1}, x_{m+p-1}) \leq d(x_{m+p-1}, x_{m+p}) + d(x_{m-1}, x_{m+p}). \tag{4.7}$$

Letting $p \to +\infty$ in (4.7) and from (4.5) and (4.6), we deduce that

$$\lim_{p\to+\infty} d(x_{m-1}, x_{m+p-1}) < \epsilon. \tag{4.8}$$

From transitivity of α and (4.1), it then follows

$$\alpha(x_{m+p-1}, x_{m-1}) \geq 1. \tag{4.9}$$

From (4.8), we deduce that

$$\lim_{p}\int_0^{d(x_{m-1},x_{m+p-1})}\phi(t)dt < \int_0^{\epsilon}\phi(t)dt. \tag{4.10}$$

From (4.7) and triangular inequality we infer that, let

$$\begin{aligned}\Theta(m,p) &= \frac{d(x_{m-1}, x_{m+p}) + d(x_{m+p-1}, x_m)}{2}\\ &\leq \frac{d(x_{m-1}, x_m) + d(x_m, x_{m+p-1}) + d(x_{m+p-1}, x_{m+p}) + d(x_{m+p-1}, x_m)}{2}\\ &\leq \frac{d(x_{m-1}, x_m) + 2d(x_m, x_{m+p-1}) + d(x_{m+p-1}, x_m)}{2}\\ &< \frac{d(x_{m-1}, x_m) + d(x_{m+p-1}, x_{m+p})}{2} + \epsilon.\end{aligned}$$

Therefore, we deduce that

$$\lim_p \int_0^{\Theta(m,p)} \phi(t)dt \leq \int_0^{\epsilon} \phi(t)dt.$$

From (4.8)-(4.10), it follows that

$$\begin{aligned} \int_0^{\epsilon} \phi(t) &\leq \int_0^{d(x_m,x_{m+p})} \phi(t)dt \\ &\leq \alpha(x_{m-1}, x_{m+p-1}) \int_0^{d(x_{m-1},x_{m+p-1})} \phi(t)dt \\ &\leq \psi(\int_0^{M(x_{m-1},x_{m+p-1})} \phi(t)dt \\ &\leq \int_0^{\epsilon} \phi(t)dt. \end{aligned}$$

Which is a contradiction. Hence (x_n) is Cauchy sequence with respect to d. Thus by definition of Cauchy sequence for each $\delta > 0$, there exists $N_0 \in \mathbb{N}$ depending on δ such that

$$d(x_n, x_m) < \delta \text{ whenever } n, m \geq N_0.$$

We claim that (x_n) is Cauchy with respect to d'. If $d \geq d'$ then our claim is trivially true. Next assuming $d \ngeq d'$ then (ii) guarantees that for each $\rho > 0$ there exists $\delta > 0$ such that

$$d'(Tx, Ty) < \rho \text{ whenever } d(x, y) < \delta.$$

Now, taking $x = x_n$ and $y = x_m$ in above inequality, we deduce that

$$d'(x_{n+1}, x_{m+1}) = d'(Tx_n, Tx_m) < \rho \text{ whenever } n, m \geq N_0.$$

This shows that our claim is true. Now since (X, d') is complete, then there exists $\lambda \in X$ endowed with the metric d' such that $d'(x_n, \lambda) \to 0$ as $n \to +\infty$. We claim that λ is a fixed point of T, that is $T\lambda = \lambda$. First consider the case when $d \neq d'$ and taking triangular inequality in account we have

$$d'(\lambda, T\lambda) \leq d'(\lambda, x_n) + d'(x_n, T\lambda) = d'(\lambda, x_n) + d'(Tx_{n-1}, T\lambda).$$

Let $n \to +\infty$ then (iii) insures that $d'(\lambda, x_n) \to 0$ implies $d'(Tx_{n-1}, T\lambda) \to 0$ and so $\lambda = T\lambda$. Next assume that $d = d'$ then

$$d(\lambda, Tx_n) \leq d(\lambda, x_n) + d(x_n, Tx_n) = d(\lambda, x_n) + d(x_n, x_{n+1}).$$

Letting $n \to +\infty$ we get

$$\lim_{n \to +\infty} d(\lambda, Tx_n) \leq 0.$$

From (iv) as T is continuous at λ so we have $d(\lambda, T\lambda) = 0$ which means $\lambda = T\lambda$. Thus λ is the fixed point of T. □

Theorem 4.1.2. *Let (X, d') be a complete metric space and d be another metric on X. Suppose that $\alpha : X \times X \to [0, +\infty)$ be a transitive mapping and T is generalized α-ψ-contractive mapping of integral type II with respect to d and satisfies the following conditions;*
(i) T is α-admissible and there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;
(ii) If $d \geq d'$ assume T is uniformly continuous from (X, d) to (X, d');
(iii) If $d \neq d'$ assume that T is continuous from (X, d) to (X, d');
(iv) If $d = d'$ then T is simply continuous.
Then there exists $\lambda \in X$ such that $\lambda = T\lambda$.

Proof. Let $x_0 \in X$ be such that $\alpha(x_0, Tx_0) \geq 1$ and (x_n) be the iterative sequence $x_{n+1} = Tx_n = T^n x_0$; for all $n \in \mathbb{N} \cup \{0\}$. Then (i) and admissibility of T imply that

$$\alpha(x_0, Tx_0) \geq 1 \text{ then } \alpha(Tx_0, T^2x_0) \geq 1,$$

which further implies that

$$\alpha(x_n, x_{n+1}) \geq 1; \ \forall\, n \in \mathbb{N} \cup \{0\}. \tag{4.11}$$

Taking $x = x_n$ and $y = x_{n+1}$, we infer from inequality (3.2) that

$$\begin{aligned} \int_0^{d(x_n,x_{n+1})} \phi(t)dt &= \int_0^{d(Tx_{n-1},Tx_n)} \phi(t)dt \\ &\leq \alpha(x_{n-1}, x_n) \int_0^{d(Tx_{n-1},Tx_n)} \phi(t)dt \\ &\leq \psi\left(\int_0^{M(x_{n-1},x_n)} \phi(t)\right) dt. \end{aligned} \tag{4.12}$$

Where

$$\begin{aligned} & M(x_{n-1}, x_n) = \\ & \max\left\{d(x_{n-1}, x_n), \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{2}, \frac{d(x_{n-1}, x_{n+1}) + d(x - n, x_n)}{2}\right\} \\ = \ & \max\left\{d(x_{n-1}, x_n), \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{2}, \frac{d(x_{n-1}, x_{n+1})}{2}\right\}. \end{aligned} \tag{4.13}$$

We claim that $M(x_{n-1}, x_n) = d(x_{n-1}, x_n)$ then consider the following three cases:

Case(I). If $M(x_{n-1}, x_n) = d(x_{n-1}, x_n)$, then our claim is trivially true.

Case(II). If $M(x_{n-1}, x_n) = \left[\frac{d(x_{n-1},x_n)+d(x_n,x_{n+1})}{2}\right]$, then from (4.12), we deduce that

$$\begin{aligned}\int_0^{d(x_n,x_{n+1})}\phi(t)dt &\leq \psi\left(\int_0^{\frac{[d(x_{n-1},x_n)+d(x_n,x_{n+1})]}{2}}\phi(t)dt\right)\\ &< \int_0^{\frac{[d(x_{n-1},x_n)+d(x_n,x_{n+1})]}{2}}\phi(t)dt.\end{aligned}$$

Since ϕ is non-negative, we have

$$\begin{aligned}d(x_n,x_{n+1}) &< \frac{[d(x_{n-1},x_n)+d(x_n,x_{n+1})]}{2}\\ \frac{d(x_n,x_{n+1})}{2} &< \frac{d(x_{n-1},x_n)}{2}.\end{aligned}$$

So we get

$$M(x_{n-1},x_n) = [\frac{d(x_{n-1},x_n)+d(x_n,x_{n+1})}{2}] < d(x_{n-1},x_n),$$

this implies that

$$M(x_{n-1},x_n) < d(x_{n-1},x_n),$$

which is contradiction to the definition of $M(x_{n-1},x_n)$.
Case(III). Let $M(x_{n-1},x_n) = \frac{d(x_n,x_{n+1})}{2}$, then from (4.12) we have,

$$\begin{aligned}\int_0^{d(x_n,x_{n+1})}\phi(t)dt &\leq \psi\left(\int_0^{\frac{d(x_n,x_{n+1})}{2}}\phi(t)dt\right)\\ &< \int_0^{\frac{d(x_n,x_{n+1})}{2}}\phi(t)dt.\end{aligned}$$

Which is contradiction due to $\psi(t) < t;\ \forall\ t > 0$. Hence, our claim is true and $M(x_{n-1},x_n) = d(x_{n-1},x_n)$ and proceeding in the same way as in the above theorem, we get that, (x_n) is Cauchy sequence with respect to d and also with respect to d'. But d' is complete, hence $x_n \to \lambda \in X$ from which we have that $\lambda = T\lambda$ that is λ is the fixed point of T. □

To assure the uniqueness of the fixed point of T in the above theorems, we consider the following result.

Theorem 4.1.3. *Adding condition (U) to the hypotheses of Theorem* 4.1.1 *(resp. Theorem* 4.1.2*), we obtain the unique fixed point of T.*

Proof. Let us suppose, to the contrary that, λ and λ^* be two distinct fixed points of T and consider a sequence (z_n) in X defined by $T^n z = z_{n+1}$. From (U), there exists $z \in X$ such that $\alpha(\lambda, z) \geq 1$ and $\alpha(\lambda^*, z) \geq 1$. Since T is α-admissible, so proceeding inductively, we get

$$\alpha(\lambda, z_{n+1}) \geq 1 \text{ and } \alpha(\lambda^*, z_{n+1}) \geq 1.$$

Taking $x = \lambda$ and $y = z_{n+1}$ and considering T to be generalized contractive mapping of integral type I, we infer that

$$\begin{aligned} \int_0^{d(\lambda, z_{n+1})} \phi(t)dt &\leq \alpha(\lambda, z_n) \int_0^{d(\lambda, z_{n+1})} \phi(t)dt \\ &\leq \psi\left(\int_0^{M(\lambda, z_n)} \phi(t)dt\right). \end{aligned} \tag{4.14}$$

Where $M(\lambda, z_n) = \max\left\{d(\lambda, z_n), d(z_n, z_{n+1}), \frac{[d(\lambda, z_{n+1}) + d(\lambda, z_n)]}{2}\right\}$.
We claim that $M(\lambda, z_n) = d(\lambda, z_n)$ and consider the following three cases:
(i') If $M(\lambda, z_n) = d(\lambda, z_n)$, then our claim is trivially true.

(ii') If $M(\lambda, z_n) = \frac{d(z_n, z_{n+1})}{2}$, then using triangular inequality, we get

$$\frac{d(z_n, z_{n+1})}{2} \leq \frac{d(\lambda, z_n) + d(\lambda, z_{n+1})}{2}. \tag{4.15}$$

If $d(\lambda, z_{n+1}) \leq d(\lambda, z_n)$, then we have

$$\frac{d(z_n, z_{n+1})}{2} < d(\lambda, z_n).$$

Which is a contradiction to the definition of $M(\lambda, z_n)$. Now Let $d(\lambda, z_n) \leq d(\lambda, z_{n+1})$, then from (4.14), it follows that $\frac{d(z_n, z_{n+1})}{2} < d(\lambda, z_{n+1})$ and hence we infer that

$$\begin{aligned} \int_0^{d(\lambda, z_{n+1})} \phi(t)dt &\leq \psi\left(\int_0^{\frac{d(z_n, z_{n+1})}{2}} \phi(t)dt\right) \\ &\leq \psi\left(\int_o^{d(\lambda, z_{n+1})} \phi(t)dt\right) \\ &< \int_0^{d(\lambda, z_{n+1})} \phi(t)dt. \end{aligned}$$

Which is again a contradiction due to the property of gauge function $\psi(t) < t$.
(iii') Let assume that, $M(\lambda, z_n) = \frac{d(\lambda,z_{n+1})+d(\lambda,z_n)}{2}$, then from (4.14) it follows that

$$\begin{aligned}\int_0^{d(\lambda,z_{n+1})} \phi(t)dt &\leq \psi\left(\int_0^{\frac{d(\lambda,z_{n+1})+d(\lambda,z_n)}{2}} \phi(t)dt\right) \\ &\leq \int_0^{\frac{d(\lambda,z_{n+1})+d(\lambda,z_n)}{2}} \phi(t)dt.\end{aligned}$$

Since ϕ is nonnegative, thus we get from the above inequality that

$$\begin{aligned}d(\lambda, z_{n+1}) &< \frac{d(\lambda, z_{n+1}) + d(\lambda, z_n)}{2} \\ d(\lambda, z_{n+1}) &< d(\lambda, z_n) \\ M(\lambda, z_n) &= \frac{d(\lambda, z_{n+1}) + d(\lambda, z_n)}{2} \\ &< d(\lambda, z_n).\end{aligned}$$

Which is again a contradiction to the definition of $M(\lambda, z_n)$. Hence our claim is true and from (4.14), it follows that

$$\begin{aligned}\int_0^{d(\lambda,z_{n+1})} \phi(t)dt &\leq \psi\Big(\int_0^{M(\lambda,z_n)} \phi(t)dt\Big) \\ &= \psi\Big(\int_0^{d(\lambda,z_n)} \phi(t)dt\Big).\end{aligned}$$

Proceeding inductively, we get

$$\int_0^{d(\lambda,z_{n+1})} \phi(t)dt \leq \psi^n\Big(\int_0^{d(\lambda,z_0)} \phi(t)dt\Big).$$

Taking $n \to +\infty$ in the above inequality and since ψ is gauge function, so we get $\lim_{n\to+\infty} \int_0^{d(\lambda,z_{n+1})} \phi(t)dt = 0$ and taking the property of ϕ on account, we deduce that

$$\lim_{n\to+\infty} d(\lambda, z_{n+1}) = \lim_{n\to+\infty} d(\lambda, Tz_n) = 0. \tag{4.16}$$

Similarly for $x = \lambda^*$ and $y = z_{n+1}$, we have

$$\lim_{n\to+\infty} d(\lambda^*, z_{n+1}) = d(\lambda^*, Tz_n) = 0 \tag{4.17}$$

Thus from (4.16) and (4.17) we deduce that $\lambda = \lambda^*$, that is λ is the unique fixed point of T. □

4.2 Generalized twisted (α, β)-ψ-contractive mappings of integral type in spaces with two metrics

In this section, we establish three new fixed point results for twisted (α, β)-ψ-contractive mappings of integral type as a generalization of many existing results in the literature especially results of Salimi *et al.* [22].

Definition 4.2.1. Let (X, d) be a metric space and $T : X \to X$ be twisted (α, β)-admissible mapping. Then T is said to be a
(a') generalized twisted (α, β)-ψ-contractive mapping of integral type I, if

$$\alpha(x, y)\beta(x, y) \int_0^{d(Tx,Ty)} \phi(t)dt \leq \psi\Big(\int_0^{d(x,y)} \phi(t)dt \Big), \tag{4.18}$$

for all $x, y \in X$, $\psi \in \Psi$ and $\phi \in \Phi$,
(b') generalized twisted (α, β)-ψ-contractive mapping of integral type II, if there is $0 < r \leq 1$ such that

$$\Big(\alpha(x, y)\beta(x, y) + r\Big)^{\int_0^{d(Tx,Ty)} \phi(t)dt} \leq (1 + r)^{\psi\left(\int_0^{d(x,y)} \phi(t)dt\right)}, \tag{4.19}$$

for all $x, y \in X$, $\psi \in \Psi$ and $\phi \in \Phi$,
(c') generalized twisted (α, β)-ψ-contractive mapping of integral type III, if there is $r \geq 1$ such that

$$\left(\int_0^{d(Tx,Ty)} \phi(t)dt + r \right)^{\alpha(x,y)\beta(x,y)} \leq \psi \left(\int_0^{d(x,y)} \phi(t)dt \right) + r, \tag{4.20}$$

for all $x, y \in X$, $\psi \in \Psi$ and $\phi \in \Phi$.

Theorem 4.2.1. *Let (X, d') be a complete metric space and d be another metric on X. Let $T : X \to X$ be a continuous generalized twisted (α, β)-ψ-contractive mapping of integral type I, type II or type III with respect to d and α, β are transitive. Then T has a fixed point if the following conditions are satisfied:*
(i) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\beta(x_0, Tx_0) \geq 1$;
(ii) if $d \not\geq d'$ assume T is uniformly continuous from (X, d) to (X, d').

Proof. Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\beta(x_0, Tx_0) \geq 1$. Defining a sequence (x_n) by $x_n = T^n x_0 = Tx_{n-1}$; for all $n \in \mathbb{N}$. Then twisted (α, β)-admissibility of T implies that if $\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1$ then $\alpha(x_2, x_1) = \alpha(Tx_1, Tx_0) \geq 1$ which implies $\alpha(x_2, x_3) = \alpha(Tx_1, Tx_2) \geq 1$. Proceeding inductively, we get $\alpha(x_{2n}, x_{2n+1}) \geq$

1 and $\alpha(x_{2n}, x_{2n-1}) \geq 1$, for all $n \in \mathbb{N}$. In the same way we have, $\beta(x_{2n}, x_{2n+1}) \geq 1$ and $\beta(x_{2n}, x_{2n-1}) \geq 1$, for all $n \in \mathbb{N}$. Now we consider the following three cases:
(a) Let T be twisted (α, β)-ψ-contractive mapping of type I. Then by (4.18) with $x = x_{2n}$ and $y = x_{2n+1}$, we get

$$\begin{aligned}\int_0^{d(x_{2n+1},x_{2n+2})} \phi(t)dt &\leq \alpha(x_{2n}, x_{2n+1})\beta(x_{2n}, x_{2n+1}) \int_0^{d(x_{2n+1},x_{2n+2})} \phi(t)dt \\ &\leq \psi\left(\int_0^{d(x_{2n},x_{2n+1})} \phi(t)dt\right).\end{aligned}$$

Similarly from (4.18) with $x = x_{2n}$ and $y = x_{2n-1}$, we have

$$\begin{aligned}\int_0^{d(Tx_{2n},Tx_{2n-1})} \phi(t)dt &= \int_0^{d(x_{2n+1},x_{2n})} \phi(t)dt \\ &\leq \alpha(x_{2n}, x_{2n-1})\beta(x_{2n}, x_{2n-1}) \int_0^{d(x_{2n+1},x_{2n})} \phi(t)dt \\ &\leq \psi\left(\int_0^{d(x_{2n},x_{2n-1})} \phi(t)dt\right); \ \forall\ n \in \mathbb{N}.\end{aligned}$$

We get from the above inequalities and using mathematical induction, that

$$\int_0^{d(x_n,x_{n+1})} \phi(t)dt \leq \psi^n\left(\int_0^{d(x_0,x_1)} \phi(t)dt\right).$$

(b) Let T be a generalized twisted (α, β)-ψ-contractive mapping of integral type II. Then with $x = x_{2n}$ and $y = x_{2n+1}$ we have, from (4.19) that

$$\begin{aligned}\Big(1 + r\Big)^{\int_0^{d(x_{2n+1},x_{2n+2})} \phi(t)dt} &\leq \Big(\alpha(x_{2n}, x_{2n+1})\beta(x_{2n}, x_{2n+1}) + r\Big)^{\int_0^{d(x_{2n+1},x_{2n+2})} \phi(t)dt} \\ &\leq \Big(1 + r\Big)^{\psi(\int_0^{d(x_{2n},x_{2n+1})} \phi(t)dt)}.\end{aligned}$$

Similarly from (4.19) with $x = x_{2n}$ and $y = x_{2n-1}$, we have

$$\int_0^{d(x_{2n+1},x_{2n})} \phi(t)dt \leq \psi\left(\int_0^{d(x_{2n},x_{2n-1})} \phi(t)dt\right).$$

Again for all $n \in \mathbb{N}$ we infer, from the above inequalities and using mathematical induction that

$$\int_0^{d(x_{n+1},x_n)} \phi(t)dt \leq \psi^n\left(\int_0^{d(x_0,x_1)} \phi(t)dt\right). \tag{4.21}$$

(c) Let T be a twisted (α,β)-ψ-contractive mapping of integral type III . Then by (4.20) with $x = x_{2n}$ and $y = x_{2n-1}$, we have

$$\begin{aligned}
\int_0^{d(x_{2n+1},x_{2n+2})} \phi(t)dt + r &\leq \left(\int_0^{d(x_{2n+1},x_{2n+1})} \phi(t)dt + r\right)^{\alpha(x_{2n},x_{2n+1})\beta(x_{2n},x_{2n+1})} \\
&\leq \psi\left(\int_0^{d(x_{2n},x_{2n+1})} \phi(t)dt\right) + r.
\end{aligned}$$

Then

$$\int_0^{d(x_{2n+1},x_{2n+2})} \phi(t)dt \leq \psi\left(\int_0^{d(x_{2n},x_{2n+1})} \phi(t)dt\right).$$

Similarly by (4.20) with $x = x_{2n}$ and $y = x_{2n-1}$, we get

$$\int_0^{d(x_{2n+1},x_{2n})} \phi(t)dt \leq \psi\left(\int_0^{d(x_{2n},x_{2n-1})} \phi(t)dt\right).$$

Thus in all the cases we have, for all $n \in \mathbb{N}$;

$$\int_0^{d(x_{n+1},x_n)} \phi(t)dt \leq \psi^n\left(\int_0^{d(x_0,x_1)} \phi(t)dt\right).$$

Taking $\lim_{n\to+\infty}$ in the above equation and we get

$$\lim_{n\to+\infty} \int_0^{d(x_{n+1},x_n)} \phi(t)dt = 0.$$

Which from the definition of ϕ, implies that

$$\lim_{n\to+\infty} d(x_n, x_{n+1}) = 0.$$

We now prove that (x_n) is a Cauchy sequence. Suppose, on the contrary, that there exist $\epsilon > 0$ and subsequences$\{m(k)\}$ and $\{n(k)\}$ such that $k < m(k) < n(k)$ with

$$d(x_{m(k)}, x_{n(k)}) \geq \epsilon, \; d(x_{m(k)}, x_{n(k)-1}) < \epsilon. \tag{4.22}$$

From the above and triangular inequalities, we deduce that

$$\begin{aligned}
d(x_{m(k)-1}, x_{n(k)-1}) &\leq d(x_{m(k)-1}, x_{m(k)}) + d(x_{m(k)}, x_{n(k)-1}) \\
&< \epsilon + d(x_{m(k)-1}, x_{m(k)}).
\end{aligned}$$

Taking limit as $k \to +\infty$ in above inequality, we infer that

$$\begin{aligned}\lim_{k\to+\infty} d(x_{m(k)-1}, x_{n(k)-1}) &< \epsilon,\\ \lim_{k} \int_0^{d(x_{m(k)-1}, x_{n(k)-1})} \phi(t)dt &\leq \int_0^{\epsilon} \phi(t)dt.\end{aligned}$$

(a) Consider the first case:

$$\begin{aligned}\int_0^{d(x_{m(k)}, x_{n(k)})} \phi(t)dt &= \int_0^{d(Tx_{m(k)-1}, Tx_{n(k)-1})} \phi(t)dt\\ &\leq \alpha(x_{m(k)-1}, x_{n(k)-1})\beta(x_{m(k)-1}, x_{n(k)-1})\\ &\quad \int_0^{d(Tx_{m(k)-1}, Tx_{n(k)-1})} \phi(t)dt\\ &\leq \psi\left(\int_0^{d(x_{m(k)-1}, x_{n(k)-1})} \phi(t)dt\right)\\ \int_0^{d(x_{m(k)-1}, x_{n(k)-1})} \phi(t)dt &\leq \psi\left(\int_0^{\epsilon} \phi(t)dt\right). \qquad (4.23)\end{aligned}$$

From (4.23), it then follows

$$\begin{aligned}\int_0^{\epsilon} \phi(t)dt &\leq \int_0^{d(x_{m(k)}, x_{n(k)})} \phi(t)dt\\ &\leq \psi\left(\int_0^{\epsilon} \phi(t)dt\right).\end{aligned}$$

Which is contradiction to the definition of ψ as a gauge function.
(b) From (4.19), we deduce that

$$\begin{aligned}\left(1+r\right)^{\int_0^{\epsilon} \phi(t)dt} &\leq \left(1+r\right)^{\int_0^{d(x_{m(k)}, x_{n(k)})} \phi(t)dt}\\ &= \left(1+r\right)^{\int_0^{d(Tx_{m(k)-1}, Tx_{n(k)-1})} \phi(t)dt}\\ &\leq \left(\alpha(x_{m(k)-1}, x_{n(k)-1})\beta(x_{m(k)-1}, x_{n(k)-1}) + r\right)^{\int_0^{d(Tx_{m(k)-1}, Tx_{n(k)-1})} \phi(t)dt}\\ &\leq \left(1+r\right)^{\int_0^{d(x_{m(k)-1}, x_{n(k)-1})} \phi(t)dt}\\ &\leq \left(1+r\right)^{\int_0^{\epsilon} \phi(t)dt},\end{aligned}$$

which again leads to the contradiction that $\psi(t) < t$. Considering the third case will also leads to a contradiction. Hence from all the cases we infer that (x_n) is a Cauchy sequence in (X, d). Thus for each $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$d(x_n, x_m) < \epsilon \text{ whenever } n, m \geq N. \tag{4.24}$$

We claim that (x_n) is also Cauchy with respect to d'. If $d \geq d'$ then our claim is trivially true. If $d \ngeq d'$ then from the uniform continuity of T we have, for all $\eta > 0$ there exists an $\epsilon > 0$ such that

$$d'(Tx, Ty) < \eta; \ \forall \ d(x, y) < \epsilon. \tag{4.25}$$

Considering (4.24) and from (4.25), with $x = x_n$ and $y = x_m$ we get

$$d'(x_{n+1}, x_{m+1}) = d'(Tx_n, Tx_m) < \eta \ ; \ \forall \ n, m \geq N.$$

Hence (x_n) is Cauchy with respect to d'. Since (X, d') is complete so there exists $\lambda \in X$ such that $x_n \to \lambda$. Finally since T is continuous so $x_{n+1} = Tx_n \to T\lambda$. Which gives $\lambda = T\lambda$. □

In the following result, we omit the continuity condition on T in the case when $d = d'$.

Theorem 4.2.2. *Let (X, d') be a complete metric space and d be another metric on X. Let $T : X \to X$ be a generalized twisted (α, β)-ψ-contractive map of integral type I, type II or type III with respect to d and α,β are transitive. Also suppose that the following conditions hold;*
(i) T is twisted (α, β)-admissible and there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\beta(x_0, Tx_0) \geq 1$;
(ii) if $d' > d$ assume that $T : (X, d) \to (X, d')$ is uniformly continuous;
(iii) if $d' \neq d$ assume that $T : (X, d) \to (X, d')$ is continuous;
(iv) if (x_n) is a sequence in X such that $\alpha(x_{2n}, x_{2n+1}) \geq 1$ and $\beta(x_{2n}, x_{2n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $n \to +\infty$ then $\alpha(x_{2n}, x)$ and $\beta(x_{2n}, x)$, for all $n \in \mathbb{N} \cup \{0\}$.
Then T has a fixed point.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\beta(x_0, Tx_0) \geq 1$. Proceeding as in the Theorem 4.2.1 we have that the iterative sequence $x_n = T^n x_0 = Tx_{n-1}$ converges to $\lambda \in X$ as $n \to +\infty$ and also $\alpha(x_{2n}, x_{2n+1}) \geq 1$ and $\beta(x_{2n}, x_{2n+1}) \geq 1$. We shall prove that $\lambda = T\lambda$. Suppose, on the contrary, that $\lambda \neq T\lambda$ and assume that $d' = d$ then from (iv) we have $\alpha(x_{2n}, \lambda) \geq 1$ and $\beta(x_{2n}, \lambda) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. Now we consider the following three cases:
(a) Let T be twisted (α, β)-ψ-contractive mapping of integral type I. Then by (4.18)

with $x = x_{2n}$ and $y = \lambda$, we infer that

$$\begin{aligned}\int_0^{d(Tx_{2n},T\lambda)} \phi(t)dt &\leq \alpha(x_{2n},\lambda)\beta(x_{2n},\lambda)\int_0^{d(Tx_{2n},T\lambda)} \phi(t)dt \\ &\leq \psi\left(\int_0^{d(x_{2n},\lambda)} \phi(t)dt\right).\end{aligned}$$

(b) Let T be a twisted (α,β)-ψ-contractive mapping of integral type II. Then by (4.19) with $x = x_{2n}$ and $y = \lambda$, we have

$$\begin{aligned}\Big(1+r\Big)^{\int_0^{d(Tx_{2n},T\lambda)} \phi(t)dt} &\leq \Big(\alpha(x_{2n},\lambda)\beta(x_{2n},\lambda)+r\Big)^{\int_0^{d(Tx_{2n},T\lambda)} \phi(t)dt} \\ &\leq \Big(1+r\Big)^{\psi\left(\int_0^{d(x_{2n},\lambda)} \phi(t)dt\right)}.\end{aligned}$$

Last inequality further gives

$$\int_0^{d(Tx_{2n},T\lambda)} \phi(t)dt \leq \psi\left(\int_0^{d(x_{2n},\lambda)} \phi(t)dt\right).$$

(c) Let T be a twisted (α,β)-ψ-contractive mapping of integral type III. Then by (4.20) with $x = x_{2n}$ and $y = \lambda$, we have

$$\begin{aligned}\left(\int_0^{d(Tx_{2n},T\lambda)} \phi(t)dt + r\right) &\leq \left(\int_0^{d(Tx_{2n},T\lambda)} \phi(t)dt + r\right)^{\alpha(x_{2n},\lambda)\beta(x_{2n},\lambda)} \\ &\leq \psi\left(\int_0^{d(x_{2n},\lambda)} \phi(t)dt\right) + r,\end{aligned}$$

which gives

$$\int_0^{d(Tx_{2n},T\lambda)} \phi(t)dt \leq \psi\left(\int_0^{d(x_{2n},\lambda)} \phi(t)dt\right).$$

Therefore in all the cases we get

$$\int_0^{d(Tx_{2n},T\lambda)} \phi(t)dt \leq \psi\left(\int_0^{d(x_{2n},\lambda)} \phi(t)dt\right).$$

Taking limit as $n \to +\infty$ in the last inequality we infer that

$$\int_0^{d(\lambda,T\lambda)} \phi(t)dt \leq \psi(0),$$

which is possible only if $d(\lambda, T\lambda) = 0$ that is $\lambda = T\lambda$. Next consider $d' \neq d$, then from triangular inequality, we have

$$d'(\lambda, T\lambda) \leq d'(\lambda, x_n) + d'(x_n, T\lambda) = d'(\lambda, x_n) + d'(Tx_{n-1}, T\lambda),$$

taking limit as $n \to +\infty$ and using the continuity assumption in (iii), we obtain from the above inequality that $d'(\lambda, T\lambda) = 0$ that is $\lambda = T\lambda$. □

Theorem 4.2.3. *Assume that all the hypotheses of Theorem* 4.2.2*(resp. Theorem* 4.2.1*) hold together with condition (H), then there exists a unique fixed point of* T*.*

Proof. Suppose that λ and λ^* are two fixed point such that $\lambda \neq \lambda^*$ that is $d(\lambda, \lambda^*) \neq 0$. From condition (H), there exists $\nu \in X$ such that $\alpha(\lambda, \nu) \geq 1$, $\alpha(\lambda^*, \nu) \geq 1$ and $\beta(\lambda, \nu) \geq 1$, $\beta(\lambda^*, \nu) \geq 1$. Now since T is twisted (α,β)-admissible mapping, we deduce that $\alpha(T^{2n}\lambda, T^{2n}\nu) \geq 1$, $\alpha(T^{2n-1}\nu, T^{2n-1}) \geq 1$ and $\alpha(T^{2n}\lambda^*, T^{2n}\nu) \geq 1$, $\alpha(T^{2n-1}\nu, T^{2n-1}\lambda^*) \geq 1$. Similar is the case for β. Since T is twisted (α,β)-ψ-contractive of integral type I, II, or III, we consider the very first case, then taking $x = T^{2n}\lambda$ and $y = T^{2n}\nu$, in (4.18), we get

$$\begin{aligned} \int_0^{d(T(T^{2n}\lambda),T(T^{2n}\nu))} \phi(t)dt &\leq \alpha(T^{2n}\lambda, T^{2n}\nu)\beta(T^{2n}\lambda, T^{2n}\nu) \int_0^{d(T(T^{2n}\lambda),T(T^{2n}\nu))} \phi(t)dt \\ &\leq \psi\left(\int_0^{d((T^{2n}\lambda),(T^{2n}\nu))} \phi(t)dt\right). \end{aligned} \tag{4.26}$$

Similarly from (4.18) with $x = T^{2n-1}\nu$ and $y = T^{2n-1}\lambda$, we infer that

$$\begin{aligned} \int_0^{d(T(T^{2n-1}\lambda),T(T^{2n-1}\nu))} \phi(t)dt &\leq \alpha(T^{2n-1}\lambda, T^{2n-1}\nu)\beta(T^{2n-1}\lambda, T^{2n-1}\nu) \\ &\quad \int_0^{d(T(T^{2n-1}\lambda),T(T^{2n-1}\nu))} \phi(t)dt \\ &\leq \psi\left(\int_0^{d((T^{2n-1}\lambda),(T^{2n-1}\nu))} \phi(t)dt\right). \end{aligned} \tag{4.27}$$

Hence, from (4.26) and (4.27) we get that, for all $n \in \mathbb{N}$

$$\int_0^{d(T(T^n\lambda),T(T^n\nu))} \phi(t)dt \leq \psi\left(\int_0^{d(T^n\lambda,T^n\nu)} \phi(t)dt\right).$$

Proceeding inductively, we get

$$\int_0^{d(T^{n+1}\lambda,T^{n+1}\nu)} \phi(t)dt \leq \psi^n\left(\int_0^{d(\lambda,\nu)} \phi(t)dt\right).$$

But since λ is the fixed point of T, so we have

$$\int_0^{d(\lambda,T^{n+1}\nu)} \phi(t)dt \leq \psi^n \left(\int_0^{d(\lambda,\nu)} \phi(t)dt\right).$$

Taking limit as $n \to +\infty$ in the above inequality, and since ψ is a gauge function and using definition $\phi \in \Phi$, we came across the conclusion that

$$\lim_{n\to+\infty} d(\lambda, T^{n+1}\nu) = 0. \tag{4.28}$$

Similar is the case for λ^*, hence we have,

$$\lim_{n\to+\infty} d(\lambda^*, T^{n+1}\nu) = 0 \tag{4.29}$$

from (4.28) and (4.29) together with triangular inequality, we have

$$d(\lambda, \lambda^*) \leq \lim n \to +\infty \left[d(\lambda, T^{n+1}\nu) + d(T^{n+1}\nu, \lambda^*)\right].$$

Thus $d(\lambda, \lambda^*) \leq 0$ and $d(\lambda, \lambda^*) \geq 0$ imply that $\lambda = \lambda^*$ that is T has a unique fixed point λ. □

Example 4.2.1. *Let $X = [0, +\infty)$ and $T : X \to X$ be defined as $Tx = \frac{x}{4} + 1$ and let X be endowed with usual metric $d = d'$. Let $\psi(t) = \frac{t}{8}$, and $\alpha = \beta = 1$ then T is not twisted (α, β)-ψ-contractive mapping of type I, II or III since*

$$\alpha(x, y)\beta(x, y)d(Tx, Ty) = \frac{1}{4}|x - y|,$$

and

$$\psi(d(x, y)) = \frac{1}{8}|x - y|.$$

But T is generalized twisted (α, β)-ψ-contractive mapping of integral type I, II or III with $\phi(t) = t$. since $\alpha = \beta = 1$, so we have

$$\alpha(x, y)\beta(x, y) \int_0^{d(Tx,Ty)} tdt = \frac{1}{32}|x - y|^2, \tag{4.30}$$

and

$$\psi\left(\int_0^{d(x,y)} tdt\right) = \psi\left(\frac{d(x, y)^2}{2}\right) = \frac{1}{16}|x - y|^2. \tag{4.31}$$

From (4.30) and (4.31), we get that T is generalized twisted (α, β)-ψ-contractive of integral type and also T satisfy all the conditions of Theorem 4.2.1 with the uniqueness hypothesis (H) and so we have that $x = \frac{4}{3}$ is the unique fixed point of T.

Now, we shall list some of the existing results in the literature that can be deduced easily from our theorems in Chapter 4.

Corollary 4.2.1. *(Banach [2]) Let (X,d) be a complete metric space and $T : X \to X$ be given mapping satisfying*

$$d(Tx,Ty) \leq \gamma d(x,y); \quad \forall\ x,y \in X,$$

where $\gamma \in [0,1)$. Then T has a unique fixed point.

Proof. Let $\alpha(x,y) = 1,\ M(x,y) = d(x,y), d = d', \phi(t) = 1$ and $\psi(t) = \gamma t$ where $\gamma \in [0,1)$ in Theorem 4.1.1. Then all the conditions of Theorem 4.1.1 are satisfied and so T has a unique fixed point. □

Notice that condition (U) immediately follows from the fact that $\alpha = 1$ which guarantees the uniqueness of fixed point.

Corollary 4.2.2. *(See Agarwal and O'Regan [1])*

Theorem 2.1.1 of Agarwal *et al.* [1] can be easily obtained by $\alpha(x,y) = 1, \psi(t) = qt; q \in (0,1), \phi(t) = 1$. It is important to note that our results (Theorem 4.1.1 and Theorem 4.1.2) together with the uniqueness condition (U) gives the unique fixed point of T.

Corollary 4.2.3. *(See Kiran et al. [12])*

Theorem 2.1.3, Theorem 2.1.4 and more importantly Theorem 2.1.5 of Kiran *et al.* [12] can easily be obtained from Theorem 4.1.1 by just taking $\phi(t) = 1$.

Corollary 4.2.4. *(See Karapinar et al. [11])*

Theorem 2.2(resp. Theorem 2.3) are special cases of our result Theorem 4.1.1 (resp. Theorem 4.1.2) taking $d = d'$.

Corollary 4.2.5. *(See Samet et al. [23]) Let (X,d) be a complete metric space and $T : X \to X$ be an α-ψ-contractive mapping satisfying the following conditions together with hypothesis (U):*
(i) T is α-admissible;
(ii) there is $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;
(iii) T is continuous.
Then T has a unique fixed point. That is, there is an $x^* \in X$ such that $Tx^* = x^*$ and such an x^* is unique.

This result can be easily deduced from our Theorem 4.1.1 by simply taking $\phi(t) = 1;\ \forall\ t \geq 0$ and $M(x,y) = d(x,y)$ and $d = d'$.

Corollary 4.2.6. *(See Branciari [8]) Let (X,d) be a complete metric space, $\kappa \in [0,1)$, and let $T : X \to X$ be a mapping such that for each $x, y \in X$,*

$$\int_0^{d(Tx,Ty)} \phi(t)dt \leq \kappa \int_0^{d(x,y)} \phi(t)dt.$$

This result can be easily deduced from Theorem 4.1.1 by considering $\alpha(x,y) = 1$, $\psi(t) = \kappa t$ and $d = d'$, $M(x,y) = d(x,y)$.
Notice that, since $\alpha(x,y) = 1; \forall\ x, y \in X$, so condition (U) holds and fixed point is unique.

Corollary 4.2.7. *(See Rhoades and Abbas [21]) Let T be a self mapping of a complete metric space (X,d) satisfying;*

$$\int_0^{d(Tx,Tx^2)} \phi(t)dt \leq \xi \int_0^{d(x,Tx)} \phi(t)dt$$

then T has a unique fixed point.

This result follows from Theorem 4.1.1 and also from Theorem 4.1.2 taking $M(x,y) = d(x,Tx)$, $d = d'$, $\alpha(x,Tx) = 1;\ \forall\ x, y \in X$ and $\psi(t) = \xi t$ and $\phi \in \Phi$, where $\xi \in [0,1)$ and $t \in [0,+\infty)$.

Corollary 4.2.8. *(See Berinde [3]) Let (X,d) be a complete metric space and $T : X \to X$ be a given mapping satisfying*

$$d(Tx,Ty) \leq \lambda d(x,y)$$

for all $x, y \in X$ where $\lambda \in [0,1)$. Then T has unique fixed point.

This result can be easily deduced from our result Theorem 4.1.1 considering $d = d'$, $M(x,y) = d(x,y)$, $\psi(t) = \lambda t;\ t \in [0,\infty),\ \lambda \in [0,1)$ *and* $\phi(t) = 1$.

Corollary 4.2.9. *(See Salimi et al. [22])*

In the main results of Salimi *et al.* [22], Theorem 2.1, Theorem 2.2 and Theorem 2.3 can be easily obtained from our results Theorem 4.2.1, Theorem 4.2.2 and Theorem 4.2.3 respectively by taking $\phi(t) = 1$ corresponding to the fact that $\phi(0) = 0$.

Corollary 4.2.10. *(See [7]) Let (X,d) be a complete metric space let $T : X \to X$ is continuous mapping. If there exists $\psi \in \Psi$ such that*

$$d(Tx,Ty) \leq \psi(d(x,y));\ \forall\ x, y \in X.$$

Then T has a unique fixed point.

The above result of Boyd and Wong [7] can be easily obtained from our result Theorem 4.2.1 by taking $\phi(t) = 1$, $M(x,y) = d(x,y)$ and $\alpha = \beta = 1$.

4.3 Some concluding remarks and future work

In this dissertation, an attempt has been made to investigate generalized $\alpha - \psi$-contractive mappings of integral type in those abstract spaces upon which two metrics are defined. Furthermore, the Integral type generalizations of twisted (α,β)-ψ-contractive type mappings are also discussed. Hopefully this discussion is a stimulus and simplify some results in Fixed Point Theory. In this thesis, we were just able to extend some theorems regarding contractive type mappings. We invite the readers to investigate the followings:

◇ to introduce multi-valued contractive type mappings on spaces with two metrics

◇ to introduce α-ψ-contractive multi-valued mappings of integral types

◇ to replace one of the two metrics by generalized metric (b-metric)

◇ finally to check the validation of the above results in best proximity fixed point.

Bibliography

[1] R. P. Agarwal, D. O'Regan, *"Fixed point theory for generalized contractions on spaces with two metrics"*, J. Math. Anal. Appl., 248 (2000), 402-414.

[2] S. Banach, *"Sur les opérations dans les ensembles abstraits et leur application aux equations integrales"*, Fund. Math., 3 (1922), 133-181.

[3] V. Berinde, *"Iterative Approximation of Fixed Points"*, Editura Efemeride, Baia Mare., (2002).

[4] M. Berzig, M. Rus, *"Fixed point theorems for α-contractive mappings of Meir-Keeler type and applications"*, arXiv:1303.5798 [math.GN].

[5] T. G. Bhaskar, V. Lakshmikanthum, *"Fixed point theory in partially ordered metric spaces and applications"*, Nonlinear Anal., 65 (2006), 1379-1393.

[6] R. M. Bianchini, M. Grandolfi, *"Transformazioni di tipo contracttivo generalizzato in uno spazio metrico"*, Atti Accad. Naz. Lincei Rend. Cl. Sci.Fiz. Math. Natur., 45 (1968), 212-216.

[7] D. W. Boyd, J. S. W. Wong, *"On nonlinear contractions"*, Proceedings of the American Mathematical Society 20 (1969), 1-18.

[8] A. Branciari *"A fixed point theorem for mappings satisfying a general contractive condition of integral type"*, Int. J. Math. Sci., 29 (9) (2002), 531-536.

[9] F. E. Browder, *"On the convergence of successive approximations for nonlinear functional equations"*, Nedrel. Akad. Wetench. Math., 30 (1968), 27-35.

[10] G. E. Hardy, T. D. Rogers, *"A generalization of a fixed point theorem of Reich"*, Canad. Math. Bull., 16 (2), 1973.

[11] E. Karapinar, P. Shahi, K. Tas, *"Generalized α-ψ-contractive type mappings of integral type and related fixed point theorems"*, Journal of Inequalities and Applications., 160 (2014). doi:10.1186/1029-242X-2014-160.

[12] Q. Kiran, T. Kamran, *"Fixed point and homotopy results for generalized contractions on spaces with two metrics"*, Dem. Math., 43 (2010), 151-160.

[13] M. G. Maia, *"Un'Observazione sulle contrazioni metriche"*, Rend. Sem. Mat. Univ. Padova., 40 (1968), 139-143.

[14] J. Matkowski, *"Integrable solutions of functional equations"*, Diss. Math. Warsaw., 127 (1975).

[15] J. J. Nieto, R. L. Pouso, R. Rodriguez-lopez, *"Contractive mapping theorems in partially ordered sets and applications to the ordinary differential equations"*, Acta. Math.Sin.(Engl. Ser)., 23 (2007), 2205-2212.

[16] D. O'Regan and Precup, *"Theorems of Leray-Schauder Type and Applications"*, Gordon and Breach, New York, 3 (2001).

[17] P. D. Proinov, *A generalization of Banach contraction principle with high order of convergence of successive approximations"*, Nonlinear Analysis., 67 (2007), 2361-2369.

[18] V. Ptak, *"The rate of convergence of Newton's process"*, Number. Math., 25 (1976), 279-285.

[19] E. Rakotch, *"A note on contractive mappings"*, Proc. A.M.S., 13 (1965), 459-465.

[20] A. C. M. Ran, M. C. B. Reurings, *"A fixed point theorm in partially ordered sets and some applications to matrix equations"*, Proc. Amer. Math. Soc., 132 (2003), 1435-1443.

[21] B. E. Rhoades, M. Abbas, *"Maps satisying generalized contractive conditions of integral type for which $F(T) = F(T^n)$"*, Int. J. Pure Appl. Math., 45 (2) (2008), 225-231.

[22] P. Salimi, C. Vetro, P. Vetro, *"Fixed point theorems for twisted (α, β)-ψ-contractive mappings and applications"*, Filomat., 27 (4) (2013), 605-615, DOI 10.2298/FIL1304605S.

[23] B. Samet, C. Vetro, P. Vetro, *"Fixed point theorems for α-ψ-contractive type mappings"*, Nonlinear Analysis., (2011), doi:10.1016/j.na.2011.10.014.

[24] P. Shahi, J. Kaur, S. S. Bhatia, *"Fixed point theorems for α-ψ-contractive type mappings of integral type with applications"*, J. Nonlinear Convex Anal. (accepted).