Lower and Upper Solution Method for Singular System

by

Hina Bashir

A thesis submitted to the
Centre for Advanced Mathematics and Physics,
National University of Sciences and Technology, Islamabad,
Pakistan
for the degree of
Master of Philosophy

May 2013

Dedicated

To My Parents

Acknowledgements

I would like to express my sincere gratitude to my supervisor Dr. Naseer Ahmad Asif, who helped me a lot in research work and I came to know many new concepts in mathematics. This thesis would not have been possible without the help, support and patience of my supervisor. I have been extremely lucky to have a supervisor, who responded my questions and queries so promptly. I would also like to thank Dr. Mujeeb ur Rehman, Dr. Abid Ali Lashari and Dr. Muhammad Umer Farooq who have entirely assisted me with my thesis. I am indebted to them for their help and kind co-operation.

Secondly, I would also like to thank my parents, my brothers, my sisters and friends who helped me in the completion of my thesis within limited time, without their moral support I could not have made it untill the end. Special thanks goes to my mother, who raised me, supported me, and taught me in every field.

I would like to acknowledge the academic and technical support of National University of Sciences and Technology as well as the library and computer facilities in CAMP. I also express my gratitude to the administration and other staff of CAMP for their help during my studies.

I am making this thesis not only for marks but also to increase my knowledge.

May Allah bless you all (Ameen).

Hina Bashir

Preface

Singular boundary value problems (SBVPs) arise in various fields of Mathematics, Engineering and Physics, many problems can be modeled by SBVPs. In last couple of years a great work has been done about the existence of positive solution of SBVPs by different authors in different ways. This thesis is about the existence of positive solution of system of fourth order SBVPs.

In Chapter 1, we study several real world applications about SBVPs. In Section 1.1, we present some definitions, terminologies and results which will be used in the next chapter. Moreover, we used Arzela's Theorem and The Schauder's Fixed Point Theorem in the thesis. Also, we review some literature in Section 1.2.

In Chapter 2, we establish some results for the existence of positive solution to the system of fourth order BVPs,

$$\begin{aligned} x^{(4)}(t) &= f(t, x(t), y(t), -x''(t)), \quad t \in (0,1), \\ y^{(4)}(t) &= g(t, x(t), y(t), -y''(t)), \quad t \in (0,1), \\ x(0) &= y(0) = x''(0) = y''(0) = 0, \\ x(1) &= y(1) = x''(1) = y''(1) = 0, \end{aligned} \tag{0.0.1}$$

where $f, g : (0,1) \times (0,\infty)^3 \to [0,\infty)$ are continuous and singular at $t = 0$, $t = 1$, $x = 0$, $y = 0$, $x'' = 0$, $y'' = 0$. For this purpose, first of all we define lower and upper solutions of above BVP (0.0.1). We then consider a modified non-singular BVP and show the existence of positive solution for the modified problem. Moreover, we show that the positive solution of modified problem converges to the positive solution of SBVP (0.0.1). A descriptive example is also given at the end of the chapter.

Finally, the conclusion and future directions are also given in Chapter 3.

Contents

1 Preliminaries **1**

1.1 Some basic definitions and known results 4

1.2 Literature Review 9

1.2.1 Multiple positive solutions of singular second order boundary value problems 9

1.2.2 Existence of solution for a class of fourth order singular boundary value problems 12

1.2.3 Existence solution for the system of fourth order and second order boundary value problem 13

2 Existence Results for Fourth–order Singular System **16**

2.1 Introduction 16

2.2 Preliminaries 17

2.3 Solution existence for modified non-singular boundary value problem 22

2.4 Existence of at least one positive solution 26

2.5 An Example: Application of Theorem 2.4.1 33

3 Conclusion **35**

Bibliography **37**

Chapter 1

Preliminaries

The study about the singular boundary value problems (SBVPs) has become an important area of research in the existence theory, for instance, in the theory of viscous fluids [10], pseudo-plastic fluids [11], in the boundary layer theory, in shallow membrance caps theory [9], etc. Some applications of singular BVPs are as follows:

The Ekman boundary layer problem [18] on the interval $[0, \infty)$ is a system consists of fourth-order differential equations of the form

$$\begin{aligned} (-\partial^2 + \alpha^2)^2 y + (-\partial^2 + \alpha^2)(i\alpha RV - \lambda)y + i\alpha RV'' y + 2\partial z = 0, \\ (2\partial + i\alpha RV')y + (-\partial^2 + \alpha^2 + i\alpha RV)z - \lambda I z = 0. \end{aligned} \tag{1.0.1}$$

In the matrix form, we can write it as

$$\begin{pmatrix} (-\partial^2 + \alpha^2)^2 + i\alpha RV(-\partial^2 + \alpha^2) + i\alpha RV'' & 2\partial \\ 2\partial + i\alpha RV' & (-\partial^2 + \alpha^2) + i\alpha RV \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} =$$

$$\lambda \begin{pmatrix} -\partial^2 + \alpha^2 & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}$$

with boundary conditions

$$y(0) = y'(0) = z(0) = 0, \quad y(\infty) = y'(\infty) = z(\infty) = 0, \tag{1.0.2}$$

where $\partial = d/dx$ is the derivative with respect to x and $x \in [0, \infty)$. Reynolds number $R \geq 0$ and a wave number $\alpha \in \mathbb{R} \backslash \{0\}$. The parameter $\lambda = i\alpha Rc$ is a spectral parameter related to the exponential time dependence $\mathrm{e}^{i\alpha ct}$. It is assumed that the function U is differentiable, V is twice differentiable, and U', V, $V'' \in L^1[0, \infty) \cap L^\infty[0, \infty)$.

Again in [18] the linear stability of incompressible flow in a circular pipe, that is Hagen-Poiseuille flow, has concerned with non-axisymmetric disturbances and it is a spectral problem for a system of singular differential equations of the form

$$\begin{aligned} \mathcal{T}_r(r^2(k(r))^2\mathcal{T}_r)\Phi(r) + i\alpha R\mathcal{T}_r\Phi(r)(u(r)-c) + i\alpha R\frac{1}{r}(\frac{u'(r)}{k^2 r})' + 2\alpha n\mathcal{T}_r\Omega(r) = 0, \\ 2\alpha n\mathcal{T}_r\Phi(r) - inR\frac{u'(r)}{r}\Phi(r) + \mathcal{S}_r\Omega(r) + i\alpha R(k(r))^2 r^2\Omega(r)(u(r)-c) = 0, \end{aligned} \tag{1.0.3}$$

which can also be expressed in compact form as

$$\begin{pmatrix} \mathcal{T}_r(k(r)^2r^2\mathcal{T}_r) + i\alpha Ru(r)\mathcal{T}_r + \frac{i\alpha R}{r}\left(\frac{u'(r)}{(k(r))^2r}\right)' & 2\alpha n\mathcal{T}_r \\ 2\alpha n\mathcal{T}_r - inR\frac{u'(r)}{r} & \mathcal{S}_r + i\alpha Ru(r)(k(r))^2r^2 \end{pmatrix}\begin{pmatrix}\Phi \\ \Omega\end{pmatrix} =$$

$$i\alpha Rc\begin{pmatrix}\mathcal{T}_r & 0 \\ 0 & (k(r))^2r^2\end{pmatrix}\begin{pmatrix}\Phi \\ \Omega\end{pmatrix},$$

on an interval $(0,1]$. Here Reynolds number $R \geq 0$, $\alpha \in \mathbb{R}\setminus\{0\}$ is the stream wise wave number, $n \in \mathbb{Z}$ is the azimuthal number, and c is a complex wave speed which result from an exponential dependence on the axial, angular and time coordinate that is,

$$\mathrm{e}^{i(\alpha x + n\phi - \alpha ct)}$$

The function $u : [0,1] \to \mathbb{R}$ is the axial mean flow which is twice differential function bounded on $[0,1]$, and $u'(0) = 0$.

The function $k : (0,1] \to [0,\infty)$ is defined as

$$(k(r))^2 = \alpha^2 + \frac{n^2}{r^2}, \quad r \in (0,1].$$

Further, the differential operators $\mathcal{T}_r$ and $\mathcal{S}_r$ are of the forms

$$\mathcal{T}_r = \frac{1}{r^2} - \frac{1}{r}\frac{d}{dr}\left(\frac{1}{(k(r))^2 r}\frac{d}{dr}\right), \quad \mathcal{S}_r = (k(r))^4r^2 - \frac{1}{r}\frac{d}{dr}\left((k(r))^2r^3\frac{d}{dr}\right),$$

where the boundary conditions are given by

$$\begin{aligned} \lim_{r\to 0}\Phi(r) = \lim_{r\to 0}\Phi'(r) = \Phi(1) = \Phi'(1) = \Omega(1) = 0, \text{ if } n = 0, \\ \lim_{r\to 0}\Phi(r) = \lim_{r\to 0}\Omega(r) = \Phi(1) = \Phi'(1) = \Omega(1) = 0, \ \lim_{r\to 0}\Phi'(r) \text{ is finite, if } n = \pm 1, \\ \lim_{r\to 0}\Phi(r) = \lim_{r\to 0}\Phi'(r) = \lim_{r\to 0}\Omega(r) = \Phi(1) = \Phi'(1) = \Omega(1) = 0, \text{ if } |n| \geq 2. \end{aligned} \tag{1.0.4}$$

For further details on (1.0.3) and (1.0.4), see [18, 20].

Now, consider a model of simply supported plate relevant to thickness optimization problem. Assume that Ω is an open set having a sufficiently smooth boundary $\partial\Omega$. We minimize $\int_\Omega u(x)dx$. Consider a fourth-order boundary value problem

$$\begin{aligned} \Delta(u^3\Delta y) = f \quad \text{in } \Omega, \\ y = \Delta y = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.0.5}$$

$$0 < m \leq u(x) \leq M \quad a.e \text{ in } \Omega,\ y \in C. \tag{1.0.6}$$

Here $C \subset L^2(\Omega)$ is nonempty and closed. The dimension of Ω is arbitrary with the corresponding model plate to $\Omega \subset \mathbb{R}^2$ and the beam model to $\Omega \subset \mathbb{R}$. Here thickness is $u \in L^\infty(\Omega^+)$ and load is $f \in L^2(\Omega)$ and $y \in H^2(\Omega) \cap H_0^1$. Weak solutions represent the deflection of (1.0.5). So (1.0.5) can be written as

$$\begin{aligned} \Delta z &= f \text{ in } \Omega, \\ \Delta y &= z\ell \text{ in } \Omega, \\ z &= 0 \text{ on } \partial\Omega, \\ y &= 0 \text{ on } \partial\Omega, \end{aligned} \tag{1.0.7}$$

where $z \in H^2(\Omega) \cap H_0^1(\Omega)$ is completely determined by f and $\ell = u^{-3} \in L^\infty(\Omega^+)$. In view of the differential operators, the above system looks like the optimality conditions of some optimal control problem. So, formulation of such distributed control problem yields the integral

$$\min \left\{ \frac{1}{2} \int_\Omega \ell(x)(h(x))^2 dx \right\} \tag{1.0.8}$$

subject to

$$\begin{aligned} \Delta y &= \ell z + \ell h \text{ in } \Omega, \\ y &= 0 \text{ on } \partial\Omega. \end{aligned} \tag{1.0.9}$$

is minimum. No constraints are imposed to control $h \in L^2(\Omega)$ and z is defined by (1.0.7), $\ell = u^{-3}$, optimal control problem (1.0.8)-(1.0.9) has the trivial solution $h^* = 0$ on unique Ω and the optimal state y^* and z satisfies (1.0.7) and consequently (1.0.5). Equations (1.0.8)-(1.0.9) are directly equivalent to the minimization of the usual energy functional associated with (1.0.5). $h = \ell^{-1}\Delta y - z$ by (1.0.9) and we can write (1.0.8) as

$$\begin{aligned} &\min_{y \in H^2(\Omega) \cap H_0^1(\Omega)} \left\{ \frac{1}{2} \int_\Omega \frac{1}{\ell(x)} (\Delta y(x) - \ell(x) z(x))^2 dx \right\} \\ = &\min_{y \in H^2(\Omega) \cap H_0^1(\Omega)} \left\{ \frac{1}{2} \int_\Omega u^3(x)(\Delta y(x))^2 dx - \int_\Omega y(x) f(x) dx \right\} + \frac{1}{2} \int_\Omega \ell(x) z(x)^2 dx, \end{aligned}$$

and last integral does not depend on y. This example shows that the classical variational method for differential equations may be reformulated as a control problem. The above transformations allow us to reformulate it as follows

$$\min \left\{ \int_\Omega (\ell(x))^{-\frac{1}{3}} dx \right\}, \tag{1.0.10}$$

$$\begin{aligned} \Delta y &= z\ell \text{ in } \Omega, \\ y &= 0 \text{ on } \partial\Omega, \end{aligned} \tag{1.0.11}$$

$$0 < M^{-3} \leq \ell(x) < m^{-3} \;\; a.e \text{ in } \Omega,\ y \in C. \tag{1.0.12}$$

The integrand in (1.0.11) is strictly convex in the interval $[M^{-3}, m^{-3}]$. For more details see [21].

1.1 Some basic definitions and known results

In this section we present some definitions and known results from functional analysis [13–16].

Definition 1.1.1 (Compactness)**.** Let $X = (X, \|\cdot\|)$ be a normed space. A subset M of X is said to be compact if every sequence in M has a convergent subsequence whose limit is an element of M.

For example, a closed unit interval $[0, 1]$ is compact.

Definition 1.1.2 (Relatively Compact)**.** A subset M of a normed space X is relatively compact if and only if closure of M, denoted by $\overline{M}$ is compact.

For example, an interval $(0, 1]$ is relatively compact.

Definition 1.1.3 (Continuous Functions)**.** Let $M \subseteq \mathbb{R}$ and $a \in M$. A map $T : M \to \mathbb{R}$ is said to be continuous at a point $c \in M$, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x \in M$ we have $|f(x) - f(c)| < \varepsilon$, whenever $|x - c| < \delta$.

Definition 1.1.4 (Retraction)**.** Let X be a topological space and A is a subspace of X then a continuous mapping $T : X \to A$ is called a retraction if $T(x) = x$ for all $x \in A$.

Definition 1.1.5 (Uniformly Bounded)**.** A family of functions X defined over an interval $[a, b]$ is said to be uniformly bounded if there exist a constant M such that for each $x \in X$, we have

$$|x(t)| \leq M, \quad t \in [a, b].$$

Definition 1.1.6 (Equicontinuous)**.** A family of functions X defined over an interval $[a, b]$ is said to be equicontinuous if for every $\varepsilon > 0$ there exist a $\delta > 0$ such that

$$|x(t_1) - x(t_2)| < \varepsilon, \text{ for all } t_1, t_2 \in [a, b], \forall x \in X$$

such that $|t_1 - t_2| < \delta$.

Definition 1.1.7 (Sequentially Compact)**.** Let X be a Banach space, the subset M of X is said to be sequentially compact if and only if every sequence in M has a subsequence which converges to some point of M.

Definition 1.1.8 ($\varepsilon - net$)**.** Let X ba a Banach space, the subset M of X and $\varepsilon > 0$ then a subset A of X is said to be an $\varepsilon - net$ for M, if for every $m \in M$ there exist $a \in A$ such that $d(m, a) < \varepsilon$. Further, if A is finite then it is called finite ε-net.

Definition 1.1.9 (Totally Bounded Set)**.** A subset M of a Banach space X is said to be totally bounded if for every $\varepsilon > 0$, M have finite $\varepsilon - net$.

Definition 1.1.10 (Convex Set)**.** Let X be a Banach space. A subset M of X is convex if segment joining the points x and y is contained in M for all $x, y \in M$. For every $m_1, m_2, \cdots, m_n \in M$, a set M is called convex, such that $\sum_{i=1}^{n} \alpha_i m_i \in M$ where $\sum_{i=1}^{n} \alpha_i = 1$.

Definition 1.1.11 (Convex Hull)**.** Let X be a Banach space and M is any subset of X, then the intersection of convex sets containing M is also a convex set which contains M and is contained in every convex set containing M, is called convex hull of M, which is defined as:

$$co(M) = \left\{ m : m = \sum_{i=1}^{k} \alpha_i m_i ; \sum_{i=1}^{k} \alpha_i = 1, \alpha_i \geq 0, m_i \in M, i = 1, 2, \cdots, k \right\}.$$

Definition 1.1.12 (Closed Convex Hull)**.** Convex hull of M is called closed convex hull, if convex sets containing a set M are closed.

Theorem 1.1.13 (Arzela's Theorem)**.** *A family of continuous functions defined on a closed interval* $[a, b]$ *is compact in* $C[a, b]$ *if and only if the family is uniformly bounded and equicontinuous over* $[a, b]$.

Proof. Let X be any set which is compact in $C[a, b]$. So, X is totally bounded as "every compact set is totally bounded". Since X is totally bounded so for each $\varepsilon > 0$ there exists a finite $\frac{\varepsilon}{3} - net$ such that $x_1, x_2, \cdots, x_k$ in X. Each of the functions x_i is bounded because of continuous function on closed interval, therefore there exist M_i such that

$$|x_i| \leq M_i, \quad i = 1, 2, \cdots, k.$$

Let $M = \max_{1 \leq i \leq k} M_i + \frac{\varepsilon}{3}$. Now by using the definition of $\frac{\varepsilon}{3} - net$, for every $x \in X$ there exist at least one x_i such that

$$d(x, x_i) = \max |x(t) - x_i(t)| < \frac{\varepsilon}{3}.$$

Consequently

$$|x| < |x_i| + \varepsilon/3 < M_i + \frac{\varepsilon}{3} < M.$$

So, X is uniformly bounded.

Since each of the functions x_i is continuous and also uniformly continuous on $[a, b]$, so for every $\frac{\varepsilon}{3} > 0$ there exists δ_i such that

$$|x_i(t_1) - x_i(t_2)| < \frac{\varepsilon}{3}, \quad |t_1 - t_2| < \delta_i.$$

Consider $\delta = \min_{1\leq i\leq k}\delta_i$. Then for any $x \in X$ and for $|t_1 - t_2| < \delta$ there exists x_i such that $d(x, x_i) < \frac{\varepsilon}{3}$, we have

$$\begin{aligned}|x(t_1) - x(t_2)| &= |x(t_1) - x_i(t_1) + x_i(t_1) - x_i(t_2) + x_i(t_2) - x(t_2)|,\\ &\leq |x(t_1) - x_i(t_1)| + |x_i(t_1) - x_i(t_2)| + |x_i(t_2) - x(t_2)|,\\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.\end{aligned}$$

Hence, X is equicontinuous.

Let X be a family of functions which is uniformly bounded and equicontinuous. We have to show that X is compact in $C[a, b]$, that is, for every $\varepsilon > 0$ there exists a finite $\varepsilon - net$ in $C[a, b]$. Consider for all $x \in X$, $|x| \leq M$ and also choose $\delta > 0$, then by the definition of equicontinuity, we have

$$|x(t_1) - x(t_2)| < \frac{\varepsilon}{5}, \qquad \text{for } |t_1 - t_2| < \delta; \quad t_1, t_2 \in [a, b], \forall x \in X.$$

Now we subdivide the segment $[a, b]$ on the $t - axis$ with intervals length less than δ such that $a = t_0, t_1, t_2, \cdots, t_n = b$ and also segment $[-M, M]$ on the $x - axis$ with intervals length $\frac{\varepsilon}{5}$ such that $-M = x_0, x_1, x_2, \cdots, x_m = M$. Construct vertical lines at the points of subdivision along $t - axis$ and horizontal lines at the points of subdivision along $x - axis$. Now we subdivide the rectangle $a \leq t \leq b$ and $-M \leq x \leq M$ into cells with horizontal sides of length less than δ and vertical sides of length $\frac{\varepsilon}{5}$. Now for every function $x \in X$ we assign a polygonal arc $z(t)$ with vertices at points(t_k, x_l), that is, at vertices of the constructed net and deviating at the points x_k from the function x by less than $\frac{\varepsilon}{5}$. Since

$$\begin{aligned}|x(t_k) - z(t_k)| &< \frac{\varepsilon}{5},\\ |x(t_{k+1}) - z(t_{k+1})| &< \frac{\varepsilon}{5},\end{aligned}$$

also

$$|x(t_k) - x(t_{k+1})| < \frac{\varepsilon}{5},$$

and

$$|z(t_k) - z(t_{k+1})| < \frac{3\varepsilon}{5}.$$

Hence, the function $z(t)$ is linear between t_k and t_{k+1}, we have

$$|z(t_k) - z(t)| < \frac{3\varepsilon}{5} \text{ for all } t_k \leq t \leq t_{k+1}.$$

Now let t be a point in closed interval $[a, b]$ and t_k is the subdivision point closed to t. Then

$$\begin{aligned}|x(t) - z(t)| &= |x(t) - x(t_k) + x(t_k) - z(t_k) + z(t_k) - z(t)|,\\ &\leq |x(t) - x(t_k)| + |x(t_k) - z(t_k)| + |z(t_k) - z(t)|,\\ &< \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{3\varepsilon}{5} = \varepsilon.\end{aligned}$$

So, polygonal arcs $z(t)$ form a $\varepsilon - net$ with respect to X. Finite numbers of polygonal arcs can be drawn through a finite number of points, so their number is finite. This implies that X is totally bounded. $\square$

Theorem 1.1.14 (The Schauder's Fixed Point Theorem)**.** *Let S be a nonempty compact and convex subset of a Banach space X and also let $T : S \to S$ ba a continuous map. Then T has a fixed point in S.*

Proof. Since S is compact and T is continuous. Therefore, $T(S)$ is also compact and hence $T(S)$ is totally bounded. So, for every $\varepsilon > 0$ there is a finite $\varepsilon - net$,

$$U = \left\{u_1, u_2, u_3, \cdots u_{n(\varepsilon)}\right\} \in T(S),$$

which implies that, for each $T(x) \in T(S)$ there exist $u_i \in U$, for some i, such that

$$\|T(x) - u_i\| < \varepsilon.$$

Now, we define

$$z_i : S \to [0, \infty)$$

by

$$z_i(x) = \max\left\{\varepsilon - \|T(x) - u_i\|, 0\right\}.$$

As T is continuous so each z_i is continuous, such that $z_i(x) \geq 0$ for all $x \in S$. Moreover, for each $x \in S$ there exist $i = \{1, 2, 3, \cdots, n(\varepsilon)\}$ such that $z_i(x) > 0$, and not all z_i are zero for a fixed value of x. Further, we define Schauder operator

$$p_\varepsilon : S \to U_\varepsilon$$

by

$$p_\varepsilon(x) = \frac{\sum_{i=1}^{n(\varepsilon)} z_i(x) u_i}{\sum_{i=1}^{n(\varepsilon)} z_i(x)},$$

and

$$U_\varepsilon = \overline{co(U)} = \overline{\left\{\sum_{i=1}^{n(\varepsilon)} \alpha_i u_i : \alpha_i \geq 0, \sum_{i=1}^{n(\varepsilon)} \alpha_i = 1\right\}}.$$

Since z_i is continuous, implies that each p_ε is continuous. If $z_i(x) > 0$, then we have

$$\|T(x) - u_i\| < \varepsilon.$$

So,

$$\begin{aligned}\|p_\varepsilon(x) - T(x)\| &= \left\| \frac{\sum_{i=1}^{n(\varepsilon)} z_i(x)u_i}{\sum_{i=1}^{n(\varepsilon)} z_i(x)} - T(x) \right\|, \\ &= \left\| \frac{\sum_{i=1}^{n(\varepsilon)} z_i(x)(u_i - T(x))}{\sum_{i=1}^{n(\varepsilon)} z_i(x)} \right\|, \\ &< \left\| \frac{\sum_{i=1}^{n(\varepsilon)} z_i(x)(\varepsilon)}{\sum_{i=1}^{n(\varepsilon)} z_i(x)} \right\| = \varepsilon,\end{aligned}$$

which gives

$$\|p_\varepsilon(x) - T(x)\| < \varepsilon.$$

Now, consider

$$T(S) \subseteq co(T(S)) \quad \because A \subseteq co(A)$$

which implies

$$U \subseteq co(T(S)) \quad \because U \subseteq T(S).$$

Now

$$U_\varepsilon = \overline{co(U)} \subseteq \overline{co(T(S))}.$$

So, S contains $\overline{co(T(S))}$ and S ia also compact and convex, hence S contains U_ε. This implies that U_ε is a closed subset of S, so U_ε is compact in S. Moreover, since the dimension of linear span of U is finite (say n) and linear span of U is a subspace of X. There exist $m \leq n$ that is we can relate linear span of S with $\mathbb{R}^m$. This also implies that U_ε is nonempty, convex and compact subset of $\mathbb{R}^m$. Let us define

$$T_\varepsilon : U_\varepsilon \to U_\varepsilon$$

by

$$T_\varepsilon = p_\varepsilon|_{U_\varepsilon},$$

where T_ε is a continuous map. Now by Brouwer's fixed point theorem, for every $\varepsilon > 0$ there exist $x_\varepsilon \in U_\varepsilon$ such that

$$T_\varepsilon(x_\varepsilon) = x_\varepsilon.$$

Since, S is a compact set and $\{x_\varepsilon\}$ is a sequence in S so by the definition of compactness there exist a subsequence $\{x_{\varepsilon_k}\}$ of x_ε such that $x_{\varepsilon_k} \to x$ as $\varepsilon_k \to 0$, where $x \in S$.

Now consider

$$\begin{aligned}\|x-T(x)\| &= \|x-x_{\varepsilon_k}+x_{\varepsilon_k}-T(x_{\varepsilon_k})+T(x_{\varepsilon_k})-T(x)\| \\ &\leq \|x-x_{\varepsilon_k}\|+\|x_{\varepsilon_k}-T(x_{\varepsilon_k})\|+\|T(x_{\varepsilon_k})-T(x)\| \\ &= \|x-x_{\varepsilon_k}\|+\|T_{\varepsilon_k}(x_{\varepsilon_k})-T(x_{\varepsilon_k})\|+\|T(x_{\varepsilon_k})-T(x)\| \\ &= \|x-x_{\varepsilon_k}\|+\|p_{\varepsilon_k}(x_{\varepsilon_k})-T(x_{\varepsilon_k})\|+\|T(x_{\varepsilon_k})-T(x)\| \\ &< \|x-x_{\varepsilon_k}\|+\varepsilon_k+\|T(x_{\varepsilon_k})-T(x)\| \to 0 \text{ as } \varepsilon_k \to 0.\end{aligned}$$

So

$$\|x-T(x)\| \to 0, \text{ as } \varepsilon_k \to 0.$$

Hence

$$T(x)=x,$$

that is, x is fixed point of T. □

1.2 Literature Review

1.2.1 Multiple positive solutions of singular second order boundary value problems

In paper [26], authors have described the existence of multiple positive solutions for the singular second order BVP

$$\begin{aligned} x''(t)+\Phi(t)f(t,x,x') &= 0 \quad t\in(0,1) \\ \alpha x(0)-\beta x'(0)=0, &\;\; x'(1)=0, \end{aligned} \tag{1.2.1}$$

where α, $\beta > 0$ and f is singular at $x=0$ and $x'=0$. First of all the authors have constructed a special cone and explained its properties. Then they have used the theory of fixed point index on a cone and presented the existence of multiple solutions of (1.2.1) when the nonlinearity has no singularities. f has sublinear or bounded in x or x' when f has a x' dependence, or (1.2.1) has pairs of upper and lower solutions. After this, they established the existence of multiple positive solutions to (1.2.1) when f is singular at $x'=0$ but not at $x=0$, and also when f is singular at $x'=0$, and $x=0$, and when f is singular at $x=0$ but not at $x'=0$.
Firstly, authors have given the following conditions for multiple positive solutions to (1.2.1) without singularity

$$\Phi\in C(0,1) \text{ with } \Phi(t)>0 \text{ for } t\in(0,1),$$

$$f\in C([0,1]\times\mathbb{R}^+\times\mathbb{R}^+,\mathbb{R}^+)$$

and there exist $g \in C(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$ such that

$$|f(t,x,y)| \le g(x,y), \;\; \forall (t,x,y) \in [0,1] \times \mathbb{R}^+ \times \mathbb{R}^+,$$

$$\sup_{c \in \mathbb{R}^+} \frac{c}{(1+\frac{\beta}{\alpha}) \int_0^1 \Phi(s) ds \max_{0 \le x \le c, 0 \le y \le c} g(x,y)} > 1,$$

and there also exists $g_1 \in C(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$ with

$$f(t,x,y) \ge g_1(x,y), \;\; \forall (t,x,y) \in [0,1] \times \mathbb{R}^+ \times \mathbb{R}^+,$$

such that $\lim_{x \to +\infty} \frac{g_1(x,y)}{x} = +\infty$, uniformally for $y \in (0,+\infty)$.
Further, they have described the existence of multiple solutions to (1.2.1) when f is singular at $x' = 0$ but not at $x = 0$.

$$\Phi \in C(0,1) \;\; \text{with } \Phi(t) > 0 \text{ for } t \in (0,1) \text{ and } \Phi \in L^1[0,1],$$

$$f : [0,1] \times \mathbb{R}^+ \times \mathbb{R}^+ \to +\infty \text{ is continuous with } f(t,x,y) > 0 \text{ for } (t,x,y) \in [0,1] \times \mathbb{R}^+ \times \mathbb{R}^+,$$

$$f(t,x,y) \le h(x)[g(y) + r(y)] \;\; \text{on} \;\; [0,1] \times \mathbb{R}^+ \times \mathbb{R}^+$$

with $g(y) > 0$ is continuous and nonincreasing on $(0,+\infty)$, and $h \ge 0$, $r \ge 0$ are continuous and nondecreasing on $[0,\infty)$,

$$\sup_{c \in \mathbb{R}^+} \frac{c}{(1+\frac{\beta}{\alpha}) I^{-1}\left(h(c) \int_0^1 \Phi(s) ds\right)} > 1,$$

where $I(z) = \int_0^z \frac{1}{g(u)+r(u)} du$, $z \in (0,+\infty)$ and $I(+\infty) = +\infty$ for a constant $H > 0$. Then there is a continuous function on $[0,1]$ which is Ψ_H and is positive on $(0,1)$, also $0 \le \delta \le 1$ is a constant with $f(t,x,y) \ge \Psi_H(t) x^\delta$ on $[0,1] \times [0,H] \times (0,H]$,

$$\int_0^1 \Phi(t) g\left(k_0 \int_t^1 \Phi(s) \Psi_H(s)\right) dt < +\infty$$

for any constant $k_0 \ge 0$, and there is $g_1 \in C(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$ with

$$f(t,x,y) \ge g_1(x,y), \;\; \forall (t,x,y) \in [0,1] \times \mathbb{R}^+ \times \mathbb{R}^+,$$

such that $\lim_{x \to +\infty} \frac{g_1(x,y)}{x} = +\infty$, uniformally for $y \in (0,+\infty)$.
Then, they have given the following conditions for the existence of multiple solutions for (1.2.1), when f is singular at $x' = 0$ and $x = 0$.

$$\Phi \in C[0,1] \;\; \text{with } \Phi(t) > 0 \text{ for } t \in (0,1),$$

$$f : [0,1] \times \mathbb{R}^+ \times \mathbb{R}^+ \to +\infty \text{ is continuous with } f(t,x,y) > 0 \text{ for } (t,x,y) \in [0,1] \times \mathbb{R}^+ \times \mathbb{R}^+,$$

$$f(t,x,y) \le [h(x) + w(x)][g(y) + r(y)]$$

on $[0,1] \times \mathbb{R}^+ \times \mathbb{R}^+$ with $g(y) > 0$ and $w(x) > 0$ are continuous and nonincreasing on $(0,+\infty)$. Also $h \geq 0$ and $r \geq 0$ are continuous and nondecreasing on $[0,\infty)$,

$$\sup_{c\in\mathbb{R}^+} \frac{c}{(1+\frac{\beta}{\alpha})I^{-1}\left(ch(c)\|\Phi\|_1 + \|\Phi\|_1 \int_0^c w(s)ds\right)} > 1,$$

where $I(z) = \int_0^z \frac{u}{g(u)+r(u)}du$, $z \in (0,+\infty)$ and $I(+\infty) = +\infty$. Moreover, $\int_0^a w(s)ds < +\infty$, for a constant $H > 0$ and there is a continuous function on interval $[0,1]$. That is Ψ_H and positive on $(0,1)$, with $f(t,x,y) \geq \Psi_H(t)$ on $[0,1] \times [0,H] \times (0,H]$,

$$\int_0^1 \Phi(t) g\left(\int_t^1 \Phi(s)\Psi_H(s)ds\right) dt < +\infty$$

for any constant $k_0 > 0$, and there is $g_1 \in C(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$ with

$$f(t,x,y) \geq g_1(x,y), \ \ \forall(t,x,y) \in [0,1] \times \mathbb{R}^+ \times \mathbb{R}^+,$$

such that $\lim_{x\to+\infty} \frac{g_1(x,y)}{x} = +\infty$, uniformally for $y \in (0,+\infty)$.
Moreover, they have discussed the following conditions for multiple solutions of (1.2.1), when f is singular at $x = 0$ but not at $x' = 0$,

$$\Phi \in C[0,1] \ \text{ with } \Phi(t) > 0 \text{ for } t \in (0,1),$$

and $f : [0,1] \times \mathbb{R}^+ \times \mathbb{R}^+ \to +\infty$ is continuous with $f(t,x,y) > 0$ for $(t,x,y) \in [0,1] \times \mathbb{R}^+ \times \mathbb{R}^+$,

$$f(t,x,y) \leq [h(x) + w(x)]r(y)$$

on $[0,1] \times \mathbb{R}^+ \times \mathbb{R}^+$ with $w > 0$ are continuous and nonincreasing on $(0,+\infty)$, also $h \geq 0$ and $r \geq 0$ are continuous and nondecreasing on $[0,\infty)$,

$$\sup_{c\in\mathbb{R}^+} \frac{c}{I^{-1}\left(ch(c)\|\Phi\|_1 + \|\Phi\|_1 \int_0^c w(s)ds\right)} > 1,$$

where $I(z) = \int_0^z \frac{u}{r(u)}du$, $z \in (0,+\infty)$ and $I(+\infty) = +\infty$. Further, $\int_0^a w(s)ds < +\infty$, for a constant $H > 0$ and there is a continuous function ψ_H on interval $[0,1]$, and $0 \leq \delta \leq 1$ is also a constant with $f(t,x,y) \geq \psi_H(t)y^\delta$ on $[0,1] \times [0,H] \times (0,H]$, and there is $g_1 \in C(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$ with

$$f(t,x,y) \geq g_1(x,y), \ \ \forall(t,x,y) \in [0,1] \times \mathbb{R}^+ \times \mathbb{R}^+,$$

such that $\lim_{x\to+\infty} \frac{g_1(x,y)}{x} = +\infty$, uniformally for $y \in (0,+\infty)$.
Also in [27], authors have studied the existence of multiple solutions of the above BVP with $\alpha = 1$ and $\beta = 0$ by using the fixed point index in a cone of an ordered Banach space in similar way.

1.2.2 Existence of solution for a class of fourth order singular boundary value problems

In paper [24], the author has investigated the existence of positive solutions of a class of fourth order singular sublinear BVP

$$
\begin{aligned}
x^{(4)}(t) &= f(t, x(t), -x''(t)), \quad t \in (0,1), \\
x(0) &= ax''(0) - bx'''(0) = 0, \\
x(1) &= c\, x''(1) + dx'''(1) = 0,
\end{aligned}
\tag{1.2.2}
$$

where $a, b, c, d \geq 0$, $a+b, c+d > 0$, $ac+ad+bc > 0$ and $f : (0,1) \times \mathbb{R} \setminus \{0\} \times \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is continuous with $f(t, t(1-t), 1) \neq 0$ for $t \in (0,1)$. Also, there exist constants λ_1, μ_1, λ_2, μ_2 such that $-\infty < \lambda_i \leq 0 \leq \mu_i$ $(i = 1, 2)$, $\mu_2 < 1$ such that $\mu_1 + \mu_2 < 1$, and

$$
c^{\mu_1} f(t,x,y) \leq f(t,cx,y) \leq c^{\lambda_1} f(t,x,y) \quad \text{for } (t,x,y) \in (0,1) \times (0,\infty)^2,\ 0 < c \leq 1,
\tag{1.2.3}
$$

$$
c^{\mu_2} f(t,x,y) \leq f(t,x,cy) \leq c^{\lambda_2} f(t,x,y) \quad \text{for } (t,x,y) \in (0,1) \times (0,\infty)^2,\ 0 < c \leq 1.
\tag{1.2.4}
$$

(1.2.3) implies

$$
c^{\lambda_1} f(t,x,y) \leq f(t,cx,y) \leq c^{\mu_1} f(t,x,y) \ \text{ if } c \geq 1;
\tag{1.2.5}
$$

and (1.2.4) implies

$$
c^{\lambda_2} f(t,x,y) \leq f(t,x,cy) \leq c^{\mu_2} f(t,x,y) \ \text{ if } c \geq 1.
\tag{1.2.6}
$$

Conversally, (1.2.5) implies (1.2.3), and (1.2.6) implies (1.2.4). The function f is singular at $t = 0$ and $t = 1$, which means that f is allowed to be unbounded at the end points $t = 0$ and $t = 1$. A function $x(t) \in C^2[0,1] \cap C^4(0,1)$ is called a $C^2[0,1]$ positive solution of (1.2.2) if it satisfies (1.2.2) and $x(t) > 0$, $x''(t) < 0$ for $t \in (0,1)$. A $C^2[0,1]$ positive solution of (1.2.2) is called a $C^3[0,1]$ positive solution if $x^3(0^+)$ and $x^3(1^-)$ exist ($x(t) > 0$, $x''(t) < 0$ for $t \in (0,1)$). A sufficient condition for the existence of solutions of the singular problem (1.2.2) was given by O' Regan in [19] with a topological transversal theorem. In the case of $b = d = 0$, a sufficient and necessary condition for the existence of $C^2[0,1]$ as well as $C^3[0,1]$ positive solutions of the singular problem (1.2.2) was given by Wei in [23] with the method of lower and upper solutions. In this paper, author has given a sufficient and necessary condition for the existence of $C^2[0,1]$ as well as $C^3[0,1]$ positive solutions of the singular problem (1.2.2) by constructing lower and upper solutions and with the maximal theorem "if $x \in C^2[a_n, b_n] \cap C^4(a_n, b_n)$, $x(a_n) \geq 0$, $x(b_n) \geq 0$, $ax''(a_n) - bx'''(a_n) \leq 0$, $cx''(b_n) + dx'''(b_n) \leq 0$ such that $x^{(4)}(t) \geq 0$ for $t \in (a_n, b_n)$, then $x(t) \geq 0$, $x''(t) \leq 0$, $t \in [a_n, b_n]$". Then the author has introduced the following Green's functions

$$
G_1(t,s) = \begin{cases} s(1-t), & s < t, \\ t(1-s), & t \leq s, \end{cases}
\tag{1.2.7}
$$

$$G_2(t,s) = \frac{1}{c+d} \begin{cases} s[c(1-t)+d], & s < t, \\ t[c(1-s)+d], & t \le s, \end{cases} \tag{1.2.8}$$

$$G_3(t,s) = \frac{1}{a+b} \begin{cases} (b+as)(1-t), & s < t, \\ (b+at)(1-s), & t \le s, \end{cases} \tag{1.2.9}$$

$$G_4(t,s) = \frac{1}{ac+ad+bc} \begin{cases} (b+as)[c(1-t)+d], & s < t, \\ (b+at)[c(1-s)+d], & t \le s, \end{cases} \tag{1.2.10}$$

For $b = d = 0$, the necessary and sufficient condition for problem (1.2.2) to have $C^2[0,1]$ positive solutions is that the following integral conditions hold:

$$0 < \int_0^1 t(1-t)f(t, t(1-t), 1)dt < \infty,$$

also

$$\lim_{t \to 0^+} t \int_t^1 (1-s)f(s, s(1-s), 1)ds = 0 \text{ if } \int_0^1 (1-s)f(s, s(1-s), 1)ds = \infty,$$

and

$$\lim_{t \to 1^-} (1-t) \int_0^t sf(s, s(1-s), 1)ds = 0 \text{ if } \int_0^1 sf(s, s(1-s), 1)ds = \infty.$$

When $b = 0$, $d > 0$, then a necessary and sufficient condition for problem (1.2.2) to have $C^3(0,1]$ positive solutions is that the following integral conditions hold:

$$0 < \int_0^1 tf(t, t(1-t), 1)dt < \infty,$$

also

$$\lim_{t \to 0^+} t \int_t^1 f(s, s(1-s), 1)ds = 0 \text{ if } \int_0^1 f(s, s(1-s), 1)ds = \infty.$$

If $b > 0$, $d = 0$ then a necessary and sufficient condition for (1.2.2) to have $C^3[0,1)$ positive solutions is that the following integral conditions hold:

$$0 < \int_0^1 (1-t)f(t, t(1-t), 1)dt < \infty,$$

$$\lim_{t \to 1^-} (1-t) \int_0^t f(s, s(1-s), 1)ds = 0 \text{ if } \int_0^1 f(s, s(1-s), 1)ds = \infty.$$

1.2.3 Existence solution for the system of fourth order and second order boundary value problem

In [25], authors have considered BVPs of singular nonlinear system of fourth-order and second-order ordinary differential equations of the form

$$\begin{aligned} x^{(4)}(t) &= f(t,y), \quad t \in (0,1), \\ -y''(t) &= g(t,x), \quad t \in (0,1), \\ x(0) &= x''(0) = y(0) = 0, \\ x(1) &= x''(1) = y(1) = 0, \end{aligned} \tag{1.2.11}$$

where $f, g \in C((0,1) \times [0,\infty), [0,\infty))$, f and g are singular at $t = 0$ and $t = 1$. Moreover, $f(t,0) \equiv 0$ and $g(t,0) \equiv 0$ and $(x,y) \in C^4(0,1) \cap C^2[0,1] \times C^2(0,1) \cap C[0,1]$ is solution of singular BVP (1.2.11) if (x,y) satisfies (1.2.11). Moreover, authors have shown that (x,y) is a positive solution of singular BVP (1.2.11) if $x(t) > 0$, $y(t) > 0$, for $t \in (0,1)$. First of all they have discussed the following assumptions

(a) There exist $q_i \in C([0,\infty), [0,\infty))$, $p_i \in C((0,1), [0,\infty))$ such that $f(t,u)$, $g(t,u) \leq p_i(t)q_i(u)$ and

$$0 < \int_0^1 t(1-t)p_i(t)dt < +\infty, \quad (i = 1,2).$$

(b) There exists $\alpha \in (0,1]$, $0 < a < b < 1$ such that

$$\lim_{u \to +\infty} \inf \frac{f(t,u)}{u^\alpha} > 0, \qquad \lim_{u \to +\infty} \inf \frac{g(t,u)}{u^{1/\alpha}} = +\infty$$

uniformally on $t \in [a,b]$.

(c) There exists $\beta \in (0,+\infty)$, such that

$$\lim_{u \to 0^+} \sup \frac{f(t,u)}{u^\beta} < +\infty, \qquad \lim_{u \to 0^+} \sup \frac{g(t,u)}{u^{1/\beta}} = 0$$

uniformally on $t \in (0,1)$.

(d) There exists $\gamma \in (0,1]$, $0 < a < b < 1$ such that

$$\lim_{u \to 0^+} \inf \frac{f(t,u)}{u^\gamma} > 0, \qquad \lim_{u \to 0^+} \inf \frac{g(t,u)}{u^{1/\gamma}} = +\infty$$

uniformally on $t \in [a,b]$.

(e) There exists $R > 0$ such that $q_1[0,N] \int_0^1 t(1-t)p_1(t)dt < R$, where $N = q_2[0,R] \int_0^1 t(1-t)p_2(t)dt$, $q_i[0,d] = \sup\{q_i(u) : u \in [0,d]\}$ and $i = (1,2)$.

After this, authors have showed that if assumptions (a), (b) and (c) hold, then singular BVP (1.2.11) has at least one positive solution and if (a), (d) and (e) hold then also BVP (1.2.11) has at least one positive solution. Moreover, when (a), (b), (d) and (e) hold then BVP (1.2.11) has at least two positive solutions.

In same paper, authors have considered BVPs of nonlinear system of fourth-order and second-order ordinary differential equations for the continuous case,

$$\begin{aligned} x^{(4)}(t) &= f(t,y), \quad t \in [0,1], \\ -y''(t) &= g(t,x), \quad t \in [0,1], \\ x(0) &= x''(0) = y(0) = 0, \\ x(1) &= x''(1) = y(1) = 0, \end{aligned} \tag{1.2.12}$$

where $f, g \in C([0,1] \times [0,\infty), [0,\infty))$. Moreover, $f(t,0) \equiv 0$ and $g(t,0) \equiv 0$ and $(x,y) \in C^4[0,1] \times C^2[0,1]$ is a solution of BVP (1.2.12) if (x,y) satisfies (1.2.12). Further, authors have shown that (x,y) is a positive solution of BVP (1.2.12) if $x(t) > 0$, $y(t) > 0$, for $t \in (0,1)$.

So, they have considered the following assumptions:
(f) There exists $\tau \in (0, +\infty)$ such that

$$\lim_{u \to +\infty} \sup \frac{f(t, u)}{u^{\tau}} < +\infty, \quad \lim_{u \to +\infty} \sup \frac{g(t, u)}{u^{1/\tau}} = 0$$

uniformally on $t \in [0, 1]$.
(g) There exists $\beta \in (0, +\infty)$ such that

$$\lim_{u \to 0^+} \sup \frac{f(t, u)}{u^{\beta}} < +\infty, \quad \lim_{u \to 0^+} \inf \frac{g(t, u)}{u^{1/\beta}} = 0$$

uniformally on $t \in [0, 1]$.
(h) There exist $q_i \in C([0, \infty), [0, \infty))$, $p_i \in C([0, 1], [0, \infty))$ such that $f(t, u)$, $g(t, u) \leq p_i(t)q_i(u)$ and there exists $R > 0$ such that $q_1[0, N] \int_0^1 t(1 - t)p_1(t)dt < R$, where $N = q_2[0, R] \int_0^1 t(1 - t)p_2(t)dt$, $q_i[0, d] = \sup\{q_i(u) : u \in [0, d]\}$ and $i = (1, 2)$.
Then, they have showed that the BVP (1.2.12) has at least one positive solution if assumptions (d) and (f) hold or when (b) and (g) hold or (d) and (h) hold. Further, when (b), (d) and (h) hold then BVP (1.2.12) has at least two positive solutions.

In next chapter, we will discuss some new results about the existence of positive solution for the system of fourth order SBVP by determining lower and upper solutions, of the form

$$\begin{aligned} x^{(4)}(t) &= f(t, x(t), y(t), -x''(t)), \quad t \in (0, 1), \\ y^{(4)}(t) &= g(t, x(t), y(t), -y''(t)), \quad t \in (0, 1), \\ x(0) &= y(0) = x''(0) = y''(0) = 0, \\ x(1) &= y(1) = x''(1) = y''(1) = 0, \end{aligned}$$

where $f, g : (0, 1) \times (0, \infty)^3 \to [0, \infty)$ are continuous and singular at $t = 0$, $t = 1$, $x = 0$, $y = 0$, $x'' = 0$, $y'' = 0$. Then, we consider a modified non-singular BVP of above SBVP over $[a_n, b_n]$, and show the solution existence for modified non–singular boundary value problem by defining a map which has a fixed point property. Moreover, we discuss that the positive solution of modified non–singular BVP converges to the positive solution of SBVP.

Chapter 2

Existence Results for Fourth–order Singular System

2.1 Introduction

The singular boundary value problems to differential equations have been studied widely in recent years [1–8, 12, 17, 22, 25, 26]. In this chapter we discuss some new results about the existence of positive solution for the system of fourth–order boundary value problem (BVP):

$$\begin{aligned} x^{(4)}(t) &= f(t, x(t), y(t), -x''(t)), \quad t \in (0,1),\\ y^{(4)}(t) &= g(t, x(t), y(t), -y''(t)), \quad t \in (0,1),\\ x(0) &= y(0) = x''(0) = y''(0) = 0,\\ x(1) &= y(1) = x''(1) = y''(1) = 0, \end{aligned} \tag{2.1.1}$$

where $f, g : (0,1) \times (0,\infty)^3 \to [0,\infty)$ are continuous and singular at $t = 0$, $t = 1$, $x = 0$, $y = 0$, $x'' = 0$, $y'' = 0$.

In paper [24], the author has investigated the existence of positive solutions of a class of fourth order singular sublinear BVP

$$\begin{aligned} x^{(4)}(t) &= f(t, x(t), -x''(t)), \quad t \in (0,1),\\ x(0) &= ax''(0) - bx'''(0) = 0,\\ x(1) &= c\, x''(1) + dx'''(1) = 0, \end{aligned} \tag{2.1.2}$$

where $a, b, c, d \geq 0$, $a+b, c+d > 0$, $ac+ad+bc > 0$ and $f : (0,1) \times \mathbb{R} \setminus \{0\} \times \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is continuous with $f(t, t(1-t), 1) \neq 0$ for $t \in (0,1)$. Also, there exist constants λ_1, μ_1, λ_2, μ_2 such that $-\infty < \lambda_i \leq 0 \leq \mu_i$ $(i = 1,2)$, $\mu_2 < 1$ such that $\mu_1 + \mu_2 < 1$, and

$$\begin{aligned} c^{\mu_1} f(t,x,y) \leq f(t,cx,y) \leq c^{\lambda_1} f(t,x,y) \quad &\text{for } (t,x,y) \in (0,1) \times (0,\infty)^2,\\ c^{\mu_2} f(t,x,y) \leq f(t,x,cy) \leq c^{\lambda_2} f(t,x,y) \quad &\text{for } (t,x,y) \in (0,1) \times (0,\infty)^2, \end{aligned} \tag{2.1.3}$$

for $0 < c \leq 1$. The function f is singular at $t = 0$, $t = 1$, $x = 0$ and $y = 0$.

In [25], authors have considered BVPs of singular nonlinear system of fourth-order and second-order ordinary differential equations of the form

$$\begin{aligned} x^{(4)}(t) &= f(t, y), \quad t \in (0, 1), \\ -y''(t) &= g(t, x), \quad t \in (0, 1), \\ x(0) &= x''(0) = y(0) = 0, \\ x(1) &= x''(1) = y(1) = 0, \end{aligned} \tag{2.1.4}$$

where $f, g \in C((0, 1) \times [0, \infty), [0, \infty))$, f and g are singular at $t = 0$ and $t = 1$. Moreover, $f(t, 0) \equiv 0$ and $g(t, 0) \equiv 0$. They proved the existence of positive solution of above BVP (2.1.4) under certain conditions on nonlinear functions f and g.

Furthermore, B. Yan *et al.* [26] described the existence of multiple positive solutions for the singular second order BVP

$$\begin{aligned} x''(t) + \Phi(t) f(t, x, x') = 0 \quad t \in (0, 1) \\ \alpha x(0) - \beta x'(0) = 0, \ \ x'(1) = 0, \end{aligned}$$

where α, $\beta > 0$ and f is singular at $x = 0$ and $x' = 0$. Also in [27], authors have studied the existence of multiple solutions of the above BVP with $\alpha = 1$ and $\beta = 0$ by using the fixed point index in a cone of an ordered Banach space.

The main feature of this chapter is that we proved existence of at least one positive solution of (2.1.1) by using only lower and upper solutions in an ordered Banach space $C^2[0, 1]$. For this work we use Schauder's fixed point theorem by defining a completely continuous map. An example is also worked out to show the applicability of our results.

2.2 Preliminaries

We need following terminologies and lemmas in the subsequent discussion.

Definition 2.2.1. By singularity of a function $h(t, x, y, z)$ we mean that h is allowed to be unbounded at $t = 0$, $t = 1$, $x = 0$, $y = 0$ and $z = 0$.

Definition 2.2.2. $(\alpha_1, \alpha_2) \in (C^2[0, 1] \cap C^4(0, 1)) \times (C^2[0, 1] \cap C^4(0, 1))$ is called a lower solution of (2.1.1), if it satisfies

$$\begin{aligned} \alpha_1^{(4)}(t) &\leq f(t, \alpha_1(t), \alpha_2(t), -\alpha_1''(t)), \quad t \in (0, 1), \\ \alpha_2^{(4)}(t) &\leq g(t, \alpha_1(t), \alpha_2(t), -\alpha_2''(t)), \quad t \in (0, 1), \\ &\alpha_1(0) \leq 0, \ \alpha_1(1) \leq 0, \ -\alpha_1''(0) \leq 0, \ -\alpha_1''(1) \leq 0, \\ &\alpha_2(0) \leq 0, \ \alpha_2(1) \leq 0, \ -\alpha_2''(0) \leq 0, \ -\alpha_2''(1) \leq 0. \end{aligned} \tag{2.2.1}$$

Definition 2.2.3. $(\beta_1, \beta_2) \in (C^2[0,1] \cap C^4(0,1)) \times (C^2[0,1] \cap C^4(0,1))$ is called an upper solution of (2.1.1), if it satisfies

$$\begin{aligned} \beta_1^{(4)}(t) &\geq f(t, \beta_1(t), \beta_2(t), -\beta_1''(t)), \quad t \in (0,1), \\ \beta_2^{(4)}(t) &\geq g(t, \beta_1(t), \beta_2(t), -\beta_2''(t)), \quad t \in (0,1), \\ \beta_1(0) &\geq 0,\ \beta_1(1) \geq 0,\ -\beta_1''(0) \geq 0,\ -\beta_1''(1) \geq 0, \\ \beta_2(0) &\geq 0,\ \beta_2(1) \geq 0,\ -\beta_2''(0) \geq 0,\ -\beta_2''(1) \geq 0. \end{aligned} \tag{2.2.2}$$

We choose sequences of real numbers $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ with $0 < \cdots < a_{n+1} < a_n < \cdots < a_1 < \frac{1}{2} < b_1 < \cdots < b_n < b_{n+1} < \cdots < 1$ such that $a_n \to 0$ and $b_n \to 1$ as $n \to \infty$. Further, we choose sequences of real constants $\{\xi_{ij}^{(n)}\}_{n=1}^{\infty}$ and $\{\eta_{ij}^{(n)}\}_{n=1}^{\infty}$ $(i,j = 1,2)$ with $\xi_{ij}^{(n)} \to 0$ and $\eta_{ij}^{(n)} \to 0$ as $n \to \infty$, such that

$$\begin{aligned} \alpha_1(a_n) \leq \xi_{11}^{(n)} &\leq \beta_1(a_n), \qquad \alpha_2(a_n) \leq \xi_{21}^{(n)} \leq \beta_2(a_n), \\ \alpha_1(b_n) \leq \xi_{12}^{(n)} &\leq \beta_1(b_n), \qquad \alpha_2(b_n) \leq \xi_{22}^{(n)} \leq \beta_2(b_n), \\ -\alpha_1''(a_n) \leq \eta_{11}^{(n)} &\leq -\beta_1''(a_n), \quad -\alpha_2''(a_n) \leq \eta_{21}^{(n)} \leq -\beta_2''(a_n), \\ -\alpha_1''(b_n) \leq \eta_{12}^{(n)} &\leq -\beta_1''(b_n), \quad -\alpha_2''(b_n) \leq \eta_{22}^{(n)} \leq -\beta_2''(b_n). \end{aligned}$$

For $x \in C^2[a_n, b_n]$, we write $\|x\| = \max\{|x(t)| : t \in [a_n, b_n]\}$ and $\|x\|_1 = \max\{|x''(t)| : t \in [a_n, b_n]\}$. Further, for $(x,y) \in C^2[a_n, b_n] \times C^2[a_n, b_n]$ we write $\|(x,y)\|_2 = \|x\|_1 + \|y\|_1$. Clearly, $(C^2[a_n, b_n] \times C^2[a_n, b_n], \|\cdot\|_2)$ is a Banach space. We define partial order $\preceq$ in $C^2[a_n, b_n]$ by $x \preceq y$ if and only if $x(t) \leq y(t)$ and $-x''(t) \leq -y''(t)$ for $t \in [a_n, b_n]$.

Lemma 2.2.4. *Suppose that $0 \leq a_n < b_n$, if $x \in C^2[a_n, b_n] \cap C^4(a_n, b_n)$, $x(a_n) \geq 0$, $x(b_n) \geq 0$, $x''(a_n) \leq 0$, $x''(b_n) \leq 0$, such that $x^{(4)}(t) \geq 0$ for $t \in (a_n, b_n)$, then*

$$x(t) \geq 0, \;\; x''(t) \leq 0, \quad t \in [a_n, b_n].$$

Proof. Let

$$x^{(4)}(t) = \sigma(t), \quad t \in (a_n, b_n), \tag{2.2.3}$$

$$x(a_n) = \xi_{11}^{(n)}, \quad x(b_n) = \xi_{12}^{(n)}, \quad x''(a_n) = \eta_{11}^{(n)}, \quad x''(b_n) = \eta_{12}^{(n)}, \tag{2.2.4}$$

then

$$\xi_{11}^{(n)} \geq 0, \quad \xi_{12}^{(n)} \geq 0, \quad \eta_{11}^{(n)} \leq 0, \quad \eta_{12}^{(n)} \leq 0, \quad \sigma(t) \geq 0, \quad t \in (a_n, b_n). \tag{2.2.5}$$

Suppose that

$$y(t) = x(t) - \left(\frac{t - a_n}{b_n - a_n} \xi_{12}^{(n)} + \frac{b_n - t}{b_n - a_n} \xi_{11}^{(n)} + \int_{a_n}^{b_n} H_n(t,\tau)(-R(\tau)) d\tau \right), \quad t \in [a_n, b_n], \tag{2.2.6}$$

where

$$H_n(t,s) = \frac{1}{b_n - a_n} \begin{cases} (b_n - t)(s - a_n), & a_n \le s \le t \le b_n, \\ (b_n - s)(t - a_n), & a_n \le t \le s \le b_n, \end{cases} \tag{2.2.7}$$

$$R(t) = \frac{1}{b_n - a_n}[(b_n - t)\eta_{11}^{(n)} + (t - a_n)\eta_{12}^{(n)}], \tag{2.2.8}$$

$$b_n - a_n > 0. \tag{2.2.9}$$

Then $y(t) \in C^2[a_n, b_n] \cap C^4(a_n, b_n)$, and differentiate (2.2.6) with respect to t,

$$\begin{aligned} y'(t) &= x'(t) - \left(\frac{1}{b_n - a_n}\xi_{12}^{(n)} + \frac{-1}{b_n - a_n}\xi_{11}^{(n)} + \int_{a_n}^{b_n} \frac{\partial H_n(t,\tau)}{\partial t}(-R(\tau))d\tau \right), \\ &= x'(t) - \left(\frac{\xi_{12}^{(n)} - \xi_{11}^{(n)}}{b_n - a_n} - \int_{a_n}^{t} \frac{-(\tau - a_n)}{b_n - a_n}(R(\tau))d\tau - \int_{t}^{b_n} \frac{(b_n - \tau)}{b_n - a_n}(R(\tau))d\tau \right) \\ &= x'(t) - \frac{\xi_{12}^{(n)} - \xi_{11}(n)}{b_n - a_n} - \int_{a_n}^{t} \frac{(\tau - a_n)}{b_n - a_n}(R(\tau))d\tau + \int_{t}^{b_n} \frac{(b_n - \tau)}{b_n - a_n}(R(\tau))d\tau. \end{aligned} \tag{2.2.10}$$

Again differentiate (2.2.10), we have

$$\begin{aligned} y''(t) &= x''(t) - 0 - \frac{(\tau - a_n)}{b_n - a_n}(R(\tau))|_{a_n}^{t} + \frac{(b_n - \tau)}{b_n - a_n}(R(\tau))|_{t}^{b_n}, \\ &= x''(t) - \frac{t - a_n}{b_n - a_n}R(t) - \frac{b_n - t}{b_n - a_n}R(t), \end{aligned}$$

or

$$y''(t) = x''(t) - R(t). \tag{2.2.11}$$

Differentiating (2.2.11) once again, we have

$$y^{(3)}(t) = x^{(3)}(t) - \frac{1}{b_n - a_n}\left[-\eta_{11}^{(n)} + \eta_{12}^{(n)} \right].$$

One more differentiation yields

$$y^{(4)}(t) = x^{(4)}(t).$$

Now the boundary conditions become

$$\begin{aligned} y(t) &= x(t) - \frac{t - a_n}{b_n - a_n}\xi_{12}^{(n)} - \frac{b_n - t}{b_n - a_n}\xi_{11}^{(n)} + \int_{a_n}^{t} \frac{(b_n - t)(\tau - a_n)}{b_n - a_n}R(\tau)d\tau \\ &\quad + \int_{t}^{b_n} \frac{(b_n - \tau)(t - a_n)}{b_n - a_n}R(\tau)d\tau, \\ y(a_n) &= x(a_n) - 0 - \xi_{11}^{(n)} + \int_{a_n}^{a_n} \frac{(b_n - a_n)(\tau - a_n)}{b_n - a_n}R(\tau)d\tau + 0, \\ y(a_n) &= \xi_{11}^{(n)} - \xi_{11}^{(n)} = 0. \end{aligned}$$

Similarly

$$y(b_n) = \xi_{12}^{(n)} - \xi_{12}^{(n)} = 0. \tag{2.2.12}$$

For other two boundary conditions, consider (2.2.11)

$$\begin{aligned}
y''(a_n) &= x''(a_n) - R(a_n), \\
y''(a_n) &= \eta_{11}^{(n)} - \frac{1}{b_n - a_n}\left[(b_n - a_n)\eta_{11}^{(n)} + (a_n - a_n)\eta_{12}^{(n)}\right], \\
y''(a_n) &= \eta_{11}^{(n)} - \eta_{11}^{(n)} = 0.
\end{aligned}$$

Similarly

$$y''(b_n) = \eta_{12}^{(n)} - \eta_{12}^{(n)} = 0.$$

Finally, we get

$$y^{(4)}(t) = \sigma(t), \quad t \in (a_n, b_n), \tag{2.2.13}$$

while the boundary conditions take the form

$$y(a_n) = y(b_n) = 0, \quad y''(a_n) = y''(b_n) = 0. \tag{2.2.14}$$

Integrate twice (2.2.13) and then using (2.2.14), we have

$$-y''(t) = \int_{a_n}^{b_n} H_n(t,s)\sigma(s)ds, \tag{2.2.15}$$

where $H_n(t,s)$ is defined in (2.2.7). By twice integration of (2.2.15) and employing (2.2.14), we get

$$y(t) = \int_{a_n}^{b_n} H_n(t,\tau)\int_{a_n}^{b_n} H_n(\tau,s)\sigma(s)dsd\tau, \quad t \in [a_n, b_n],$$

that is

$$\begin{aligned}
x(t) = &\left(\frac{t - a_n}{b_n - a_n}\xi_{12}^{(n)} + \frac{b_n - t}{b_n - a_n}\xi_{11}^{(n)} + \int_{a_n}^{b_n} H_n(t,\tau)(-R(\tau))d\tau\right) \\
&+ \int_{a_n}^{b_n} H_n(t,\tau)\int_{a_n}^{b_n} H_n(\tau,s)\sigma(s)dsd\tau, \quad t \in [a_n, b_n].
\end{aligned} \tag{2.2.16}$$

From (2.2.6), (2.2.8) and (2.2.15),we have

$$x''(t) = \frac{1}{b_n - a_n}[(b_n - t)\eta_{11}^{(n)} + (t - a_n)\eta_{12}^{(n)}] - \int_{a_n}^{b_n} H_n(t,s)\sigma(s)ds, \quad t \in [a_n, b_n]. \tag{2.2.17}$$

Note that (2.2.5) and $-R(t) \geq 0, \quad t \in [a_n, b_n], \quad H_n(t,s) \geq 0, \quad (t,s) \in [a_n, b_n] \times [a_n, b_n]$, and by (2.2.16) and (2.2.17), we have $x(t) \geq 0, x''(t) \leq 0, t \in [a_n, b_n]$. □

Lemma 2.2.5. *For $a, b \in \mathbb{R}$ and $x \in C^2[a,b]$, we write $\|x\|_1 = \max_{t\in[a,b]} |x''(t)|$. Then norm space $(C^2[a,b], \|\cdot\|_1)$ is a Banach space.*

Proof. Let $\{x_m\}$ be a cauchy sequence in $C^2[a,b]$, then for every $\varepsilon > 0$, there exists a positive integer n_0 such that

$$\|x_m - x_n\|_1 < \varepsilon \text{ for all } m, n \geq n_0$$

which implies that

$$\|x_m - x_n\|_1 = \max_{t\in[a,b]} |x''_m(t) - x''_n(t)| < \varepsilon \text{ for all } m, n \geq n_0. \tag{2.2.18}$$

Thus for any fixed $t = t_0 \in [a, b]$, we have

$$|x''_m(t_0) - x''_n(t_0)| < \varepsilon \text{ for all } m, n \geq n_0,$$

which shows that $\{x''_m(t_0)\}$ is a cauchy sequence of real numbers. Since $\mathbb{R}$ is complete, so there exist $x''(t_0) \in \mathbb{R}$ such that $x''_m(t_0) \to x''(t_0)$ as $m \to \infty$. In such way we can associate with each $t \in [a, b]$ a unique real number $x''(t)$ which defines pointwise a function x'' on $[a, b]$. Now we have to show that $x'' \in C[a, b]$ and $x''_m \to x''$, from (2.2.18) with $n \to \infty$, we have

$$\max_{t\in[a,b]} |x''_m(t) - x''(t)| < \varepsilon \text{ for all } m \geq n_0.$$

Hence for every $t \in [a, b]$

$$|x''_m(t) - x''(t)| < \varepsilon \text{ for all } m \geq n_0.$$

This implies that $x''_m(t) \to x''(t)$ uniformly on $[a, b]$. Since $x''_m(t)$ is continuous on $[a, b]$ and converges uniformly, so the limit function x'' is continuous on $[a, b]$ as "if a sequence $\{x_m\}$ of continuous function on $[a, b]$ converges on $[a, b]$ and also convergence is uniform, then limit function x is continuous on $[a, b]$." Hence space $C^2[a, b]$ is complete. □

Lemma 2.2.6. *For $a, b \in \mathbb{R}$ with $a < b$ and $v \in C[a, b]$. The boundary value problem*

$$\begin{aligned} u''(t) &= v(t), \quad t \in [a, b], \\ u(a) &= u_a,\ u(b) = u_b \end{aligned} \tag{2.2.19}$$

has integral representation

$$u(t) = \frac{b-t}{b-a}u_a + \frac{t-a}{b-a}u_b - \int_a^b H(t, s)v(s)ds, \tag{2.2.20}$$

where

$$H(t, s) = \frac{1}{b-a}\begin{cases} (b-t)(s-a), & a \leq s \leq t \leq b, \\ (b-s)(t-a), & a \leq t \leq s \leq b. \end{cases} \tag{2.2.21}$$

Proof. Integrating (2.2.19) from a to t, we obtain

$$u'(t) = C_1 + \int_a^t v(s)ds. \tag{2.2.22}$$

Again integrating (2.2.22) from a to t, we get

$$u(t) = C_2 + (t-a)C_1 + \int_a^t \int_a^s v(\tau)d\tau ds,$$

where C_1 and C_2 are constants and can be determined by imposing boundary conditions (2.2.19), which can be written as

$$u(t) = C_2 + (t-a)C_1 + \int_a^t (t-s)v(s)ds. \tag{2.2.23}$$

Now after imposing boundary conditions (2.2.19)

$$\begin{aligned} C_1 &= \frac{u_b - u_a}{b-a} - \frac{1}{b-a}\int_a^b (b-s)v(s)ds, \\ C_2 &= u_a. \end{aligned} \tag{2.2.24}$$

Using the values of C_1 and C_2 in (2.2.23), we have

$$\begin{aligned} u(t) &= \frac{b-t}{b-a}u_a + \frac{t-a}{b-a}u_b - \frac{t-a}{b-a}\int_a^b (b-s)v(s)ds + \int_a^t (t-s)v(s)ds, \\ &= \frac{b-t}{b-a}u_a + \frac{t-a}{b-a}u_b - \frac{1}{b-a}\int_t^b (t-a)(b-s)v(s)ds - \frac{1}{b-a}\int_a^t (b-t)(s-a)v(s)ds, \end{aligned}$$

which is equivalent to (2.2.20). □

Lemma 2.2.7. *The Green's function* (2.2.21) *satisfies*

$$H(t,s) \le \frac{1}{b-a}(s-a)(b-s), \quad t,s \in [a,b]. \tag{2.2.25}$$

Proof. Case i: For $s \le t$, using (2.2.21)

$$H(t,s) = \frac{1}{b-a}(b-t)(s-a),$$

which implies that

$$H(t,s) \le \frac{1}{b-a}(b-s)(s-a). \tag{2.2.26}$$

Case ii: Now for $t \le s$, in view of (2.2.21), we have

$$H(t,s) = \frac{1}{b-a}(b-s)(t-a),$$

which again implies that

$$H(t,s) \le \frac{1}{b-a}(b-s)(s-a). \tag{2.2.27}$$

□

2.3 Solution existence for modified non-singular boundary value problem

For each $n \in \{1, 2, 3, \cdots\}$ consider the modified non-singular boundary value problem

$$\begin{aligned} &x^{(4)}(t) = f_*(t, x(t), y(t), -x''(t)), \quad t \in [a_n, b_n], \\ &y^{(4)}(t) = g_*(t, x(t), y(t), -y''(t)), \quad t \in [a_n, b_n], \\ &x(a_n) = \xi_{11}^{(n)}, y(a_n) = \xi_{21}^{(n)}, -x''(a_n) = \eta_{11}^{(n)}, -y''(a_n) = \eta_{21}^{(n)}, \\ &x(b_n) = \xi_{12}^{(n)}, y(b_n) = \xi_{22}^{(n)}, -x''(b_n) = \eta_{12}^{(n)}, -y''(b_n) = \eta_{22}^{(n)}, \end{aligned} \tag{2.3.1}$$

where the modified functions f_* and g_* are defined as follows:

$$f_*(t,x(t),y(t),-x''(t)) = \begin{cases} f(t,\alpha_1(t),\alpha_2(t),-\alpha_1''(t)), & \text{if } (\alpha_1,\alpha_2) \npreceq (x,y), \\ f(t,x(t),y(t),-x''(t)), & \text{if } (\alpha_1,\alpha_2) \preceq (x,y) \preceq (\beta_1,\beta_2), \\ f(t,\beta_1(t),\beta_2(t),-\beta_1''(t)), & \text{if } (x,y) \npreceq (\beta_1,\beta_2), \end{cases} \tag{2.3.2}$$

$$g_*(t,x(t),y(t),-y''(t)) = \begin{cases} g(t,\alpha_1(t),\alpha_2(t),-\alpha_2''(t)), & \text{if } (\alpha_1,\alpha_2) \npreceq (x,y), \\ g(t,x(t),y(t),-y''(t)), & \text{if } (\alpha_1,\alpha_2) \preceq (x,y) \preceq (\beta_1,\beta_2), \\ g(t,\beta_1(t),\beta_2(t),-\beta_2''(t)), & \text{if } (x,y) \npreceq (\beta_1,\beta_2). \end{cases} \tag{2.3.3}$$

Employing Lemma 2.2.6, the boundary value problem (2.3.1) becomes

$$\begin{aligned} x''(t) &= \frac{b_n-t}{b_n-a_n}\eta_{11}^{(n)} + \frac{t-a_n}{b_n-a_n}\eta_{12}^{(n)} - \int_{a_n}^{b_n} H_n(t,s) f_*(s,x(s),y(s),-x''(s))ds, \quad t \in [a_n,b_n], \\ y''(t) &= \frac{b_n-t}{b_n-a_n}\eta_{21}^{(n)} + \frac{t-a_n}{b_n-a_n}\eta_{22}^{(n)} - \int_{a_n}^{b_n} H_n(t,s) g_*(s,x(s),y(s),-y''(s))ds, \quad t \in [a_n,b_n], \\ x(a_n) &= \xi_{11}^{(n)}, y(a_n) = \xi_{21}^{(n)}, x(b_n) = \xi_{12}^{(n)}, y(b_n) = \xi_{22}^{(n)}, \end{aligned} \tag{2.3.4}$$

where

$$H_n(t,s) = \frac{1}{b_n-a_n} \begin{cases} (b_n-t)(s-a_n), & s \le t, \\ (b_n-s)(t-a_n), & t \le s. \end{cases} \tag{2.3.5}$$

Again employing Lemma 2.2.6, the integro-differential boundary value problem (2.3.4) becomes

$$\begin{aligned} x(t) &= \frac{b_n-t}{b_n-a_n}\xi_{11}^{(n)} + \frac{t-a_n}{b_n-a_n}\xi_{12}^{(n)} - \int_{a_n}^{b_n} H_n(t,s)\left(\frac{b_n-s}{b_n-a_n}\eta_{11}^{(n)} + \frac{s-a_n}{b_n-a_n}\eta_{12}^{(n)}\right)ds \\ &\quad + \int_{a_n}^{b_n} H_n(t,s)\int_{a_n}^{b_n} H_n(s,\tau) f_*(\tau,x(\tau),y(\tau),-x''(\tau))d\tau ds, \quad t \in [a_n,b_n], \\ y(t) &= \frac{b_n-t}{b_n-a_n}\xi_{21}^{(n)} + \frac{t-a_n}{b_n-a_n}\xi_{22}^{(n)} - \int_{a_n}^{b_n} H_n(t,s)\left(\frac{b_n-s}{b_n-a_n}\eta_{21}^{(n)} + \frac{s-a_n}{b_n-a_n}\eta_{22}^{(n)}\right)ds \\ &\quad + \int_{a_n}^{b_n} H_n(t,s)\int_{a_n}^{b_n} H_n(s,\tau) g_*(\tau,x(\tau),y(\tau),-y''(\tau))d\tau ds, \quad t \in [a_n,b_n]. \end{aligned} \tag{2.3.6}$$

Define map $T_n : C^2[a_n,b_n] \times C^2[a_n,b_n] \to C^2[a_n,b_n] \times C^2[a_n,b_n]$ by

$$T_n = (A_n, B_n), \tag{2.3.7}$$

where the maps $A_n, B_n : C^2[a_n, b_n] \times C^2[a_n, b_n] \to C^2[a_n, b_n]$ are defined as

$$\begin{aligned}
A_n(x,y)(t) &= \frac{b_n - t}{b_n - a_n}\xi_{11}^{(n)} + \frac{t - a_n}{b_n - a_n}\xi_{12}^{(n)} - \int_{a_n}^{b_n} H_n(t,s)\left(\frac{b_n - s}{b_n - a_n}\eta_{11}^{(n)} + \frac{s - a_n}{b_n - a_n}\eta_{12}^{(n)}\right) ds \\
&\quad + \int_{a_n}^{b_n} H_n(t,s)\int_{a_n}^{b_n} H_n(s,\tau) f_*(\tau, x(\tau), y(\tau), -x''(\tau)) d\tau ds, \quad t \in [a_n, b_n], \\
B_n(x,y)(t) &= \frac{b_n - t}{b_n - a_n}\xi_{21}^{(n)} + \frac{t - a_n}{b_n - a_n}\xi_{22}^{(n)} - \int_{a_n}^{b_n} H_n(t,s)\left(\frac{b_n - s}{b_n - a_n}\eta_{21}^{(n)} + \frac{s - a_n}{b_n - a_n}\eta_{22}^{(n)}\right) ds \\
&\quad + \int_{a_n}^{b_n} H_n(t,s)\int_{a_n}^{b_n} H_n(s,\tau) g_*(\tau, x(\tau), y(\tau), -y''(\tau)) d\tau ds, \quad t \in [a_n, b_n].
\end{aligned} \tag{2.3.8}$$

Clearly if (x_n, y_n) is a fixed point of T_n then (x_n, y_n) is a solution of (2.3.1).

Lemma 2.3.1. *Assume that* $0 < \alpha_i(t) \le \beta_i(t)$ *and* $0 < -\alpha_i''(t) \le -\beta_i''(t)$ *for* $t \in (0,1)$, $i = 1, 2$. *Further,* $\alpha_i(0) = \alpha_i(1) = \alpha_i''(0) = \alpha_i''(1) = 0$, $i = 1, 2$. *Then the map* T_n *defined by* (2.3.7) *is completely continuous.*

Proof. First we show that $T_n(C^2[a_n, b_n] \times C^2[a_n, b_n])$ is uniformly bounded. Differentiating (2.3.8) twice with respect to t, we have

$$A_n(x,y)''(t) = \frac{b_n - t}{b_n - a_n}\eta_{11}^{(n)} + \frac{t - a_n}{b_n - a_n}\eta_{12}^{(n)} - \int_{a_n}^{b_n} H_n(t,s) f_*(s, x(s), y(s), -x''(s)) ds, \tag{2.3.9}$$

which implies

$$|A_n(x,y)''(t)| \le |\eta_{11}^{(n)}| + |\eta_{12}^{(n)}| + \int_{a_n}^{b_n} H_n(t,s) f_*(s, x(s), y(s), -x''(s)) ds, \quad t \in [a_n, b_n],$$

which in view of Lemma (2.2.7) becomes

$$|A_n(x,y)''(t)| \le |\eta_{11}^{(n)}| + |\eta_{12}^{(n)}| + \frac{1}{b_n - a_n}\int_{a_n}^{b_n} (s - a_n)(b_n - s) f_*(s, x(s), y(s), -x''(s)) ds,$$

which can be written as

$$\|A_n(x,y)\|_1 \le |\eta_{11}^{(n)}| + |\eta_{12}^{(n)}| + \frac{1}{b_n - a_n}\int_{a_n}^{b_n} (s - a_n)(b_n - s) f_*(s, x(s), y(s), -x''(s)) ds. \tag{2.3.10}$$

Similarly, we can show that

$$\|B_n(x,y)\|_1 \le |\eta_{21}^{(n)}| + |\eta_{22}^{(n)}| + \frac{1}{b_n - a_n}\int_{a_n}^{b_n} (s - a_n)(b_n - s) g_*(s, x(s), y(s), -y''(s)) ds. \tag{2.3.11}$$

From (2.3.10) and (2.3.11), it follows that $T_n(C^2[a_n, b_n] \times C^2[a_n, b_n])$ is uniformly bounded.

Now we show that the map T_n is equicontinuous. Differentiating (2.3.9) with respect to t, we obtain

$$A_n(x,y)'''(t) = \frac{\eta_{12}^{(n)} - \eta_{11}^{(n)}}{b_n - a_n} - \int_{a_n}^{b_n} \frac{\partial}{\partial t} H_n(t,s) f_*(s, x(s), y(s), -x''(s)) ds.$$

But

$$\frac{\partial}{\partial t}H_n(t,s) = \frac{1}{b_n - a_n}\begin{cases} -(s-a_n), & s \le t, \\ (b_n - s), & t \le s. \end{cases}$$

So

$$\begin{aligned} A_n(x,y)'''(t) &= \frac{\eta_{12}^{(n)} - \eta_{11}^{(n)}}{b_n - a_n} + \frac{1}{b_n - a_n}\int_{a_n}^{t}(s-a_n)f_*(s,x(s),y(s),-x''(s))ds \\ &\quad - \frac{1}{b_n - a_n}\int_{t}^{b_n}(b_n - s)f_*(s,x(s),y(s),-x''(s))ds, \quad t \in [a_n, b_n], \end{aligned}$$

which implies that

$$\begin{aligned} |A_n(x,y)'''(t)| &= \left| \frac{\eta_{12}^{(n)} - \eta_{11}^{(n)}}{b_n - a_n} + \frac{1}{b_n - a_n}\int_{a_n}^{t}(s-a_n)f_*(s,x(s),y(s),-x''(s))ds \right. \\ &\quad \left. - \frac{1}{b_n - a_n}\int_{t}^{b_n}(b_n - s)f_*(s,x(s),y(s),-x''(s))ds \right|, \\ &\le \frac{|\eta_{12}^{(n)} - \eta_{11}^{(n)}|}{b_n - a_n} + \frac{1}{b_n - a_n}\int_{a_n}^{t}(s-a_n)f_*(s,x(s),y(s),-x''(s))ds \\ &\quad + \frac{1}{b_n - a_n}\int_{t}^{b_n}(b_n - s)f_*(s,x(s),y(s),-x''(s))ds, \\ &\le \frac{|\eta_{12}^{(n)} - \eta_{11}^{(n)}|}{b_n - a_n} + \frac{1}{b_n - a_n}\int_{a_n}^{b_n}(s-a_n)f_*(s,x(s),y(s),-x''(s))ds \\ &\quad + \frac{1}{b_n - a_n}\int_{a_n}^{b_n}(b_n - s)f_*(s,x(s),y(s),-x''(s))ds, \\ &= \frac{|\eta_{12}^{(n)} - \eta_{11}^{(n)}|}{b_n - a_n} + \frac{1}{b_n - a_n}\int_{a_n}^{b_n}(b_n-a_n)f_*(s,x(s),y(s),-x''(s))ds. \end{aligned}$$

Thus

$$|A_n(x,y)'''(t)| \le \frac{|\eta_{12}^{(n)} - \eta_{11}^{(n)}|}{b_n - a_n} + \int_{a_n}^{b_n} f_*(s,x(s),y(s),-x''(s))ds,$$

which shows that

$$\|A_n(x,y)'\|_1 \le \frac{|\eta_{12}^{(n)} - \eta_{11}^{(n)}|}{b_n - a_n} + \int_{a_n}^{b_n} f_*(s,x(s),y(s),-x''(s))ds. \tag{2.3.12}$$

Similarly, we can show that

$$\|B_n(x,y)'\|_1 \le \frac{|\eta_{22}^{(n)} - \eta_{21}^{(n)}|}{b_n - a_n} + \int_{a_n}^{b_n} g_*(s,x(s),y(s),-y''(s))ds. \tag{2.3.13}$$

Consequently, from (2.3.12) and (2.3.13), $T_n(C^2[a_n, b_n] \times C^2[a_n, b_n])$ is equicontinuous.

Next we show that T_n is continuous. Let $(x_m, y_m), (x, y) \in C^2[a_n, b_n] \times C^2[a_n, b_n]$ such that $\|(x_m, y_m) - (x, y)\| \to 0$ as $m \to \infty$. In view of (2.3.8), we have

$$|A_n(x_m, y_m)(t) - A_n(x, y)(t)| = \bigg| \int_{a_n}^{b_n} H_n(t, s) \int_{a_n}^{b_n} H_n(s, \tau) f_*(\tau, x_m(\tau), y_m(\tau), -x_m''(\tau)) d\tau ds$$
$$- \int_{a_n}^{b_n} H_n(t, s) \int_{a_n}^{b_n} H_n(s, \tau) f_*(\tau, x(\tau), y(\tau), -x''(\tau)) d\tau ds \bigg|, \quad t \in [a_n, b_n],$$

which implies that

$$|A_n(x_m, y_m)(t) - A_n(x, y)(t)| = \int_{a_n}^{b_n} H_n(t, s) \int_{a_n}^{b_n} H_n(s, \tau) |f_*(\tau, x_m(\tau), y_m(\tau), -x_m''(\tau))$$
$$- f_*(\tau, x(\tau), y(\tau), -x''(\tau))| d\tau ds, \quad t \in [a_n, b_n].$$

Now using Lemma 2.2.7, we get

$$|A_n(x_m, y_m)(t) - A_n(x, y)(t)| \leq \frac{1}{(b_n - a_n)^2} \int_{a_n}^{b_n} (s - a_n)(b_n - s) ds \int_{a_n}^{b_n} (\tau - a_n)(b_n - \tau)$$
$$|f_*(\tau, x_m(\tau), y_m(\tau), -x_m''(\tau)) - f_*(\tau, x(\tau), y(\tau), -x''(\tau))| d\tau, \quad t \in [a_n, b_n].$$

This yields $\|A_n(x_m, y_m) - A_n(x, y)\| \to 0$ as $m \to \infty$. Similarly, we can show that $\|B_n(x_m, y_m) - B_n(x, y)\| \to 0$ as $m \to \infty$. Consequently T_n is continuous, which together with the compactness of T_n, implies that T_n is completely continuous. □

2.4 Existence of at least one positive solution

Theorem 2.4.1. *Assume that* (2.1.1) *has a lower solution* (α_1, α_2) *and an upper solution* (β_1, β_2) *such that* $0 < \alpha_i(t) \leq \beta_i(t)$ *and* $0 < -\alpha_i''(t) \leq -\beta_i''(t)$ *for* $t \in (0, 1)$, $i = 1, 2$. *Further,* $\alpha_i(0) = \alpha_i(1) = \alpha_i''(0) = \alpha_i''(1) = 0$, $i = 1, 2$. *Then the boundary value problem* (2.1.1) *has a positive solution* $(u, v) \in (C^2[0, 1] \cap C^4(0, 1)) \times (C^2[0, 1] \cap C^4(0, 1))$ *such that* $(\alpha_1, \alpha_2) \preceq (u, v) \preceq (\beta_1, \beta_2)$.

Proof. Since T_n is completely continuous and f_*, g_* are bounded on $[a_n, b_n] \times \mathbb{R}^3$, by Schauder's fixed point theorem 1.1.14, T_n has a fixed point $(x_n, y_n) \in C^4[a_n, b_n] \times C^4[a_n, b_n]$. We claim that $(\alpha_1, \alpha_2) \preceq (x_n, y_n) \preceq (\beta_1, \beta_2)$, that is

$$\begin{aligned} \alpha_1(t) &\leq x_n(t) \leq \beta_1(t), \quad t \in [a_n, b_n], \\ \alpha_2(t) &\leq y_n(t) \leq \beta_2(t), \quad t \in [a_n, b_n], \\ -\alpha_1''(t) &\leq -x_n''(t) \leq -\beta_1''(t), \quad t \in [a_n, b_n], \\ -\alpha_2''(t) &\leq -y_n''(t) \leq -\beta_2''(t), \quad t \in [a_n, b_n]. \end{aligned} \tag{2.4.1}$$

Suppose $(x_n, y_n) \not\leq (\beta_1, \beta_2)$. By the definition of f_* and g_*, we have

$$f_*(t, x_n(t), y_n(t), -x_n^{''}(t)) = f(t, \beta_1(t), \beta_2(t), -\beta_1^{''}(t)), \quad t \in [a_n, b_n],$$
$$g_*(t, x_n(t), y_n(t), -y_n^{''}(t)) = g(t, \beta_1(t), \beta_2(t), -\beta_2^{''}(t)), \quad t \in [a_n, b_n].$$

Therefore

$$x_n^{(4)}(t) = f(t, \beta_1(t), \beta_2(t), -\beta_1^{''}(t)), \quad t \in [a_n, b_n],$$
$$y_n^{(4)}(t) = g(t, \beta_1(t), \beta_2(t), -\beta_2^{''}(t)), \quad t \in [a_n, b_n].$$

On the other hand, since (β_1, β_2) is an upper solution of (2.1.1), we also have

$$\beta_1^{(4)}(t) \geq f(t, \beta_1(t), \beta_2(t), -\beta_1^{''}(t)), \quad t \in [a_n, b_n],$$
$$\beta_2^{(4)}(t) \geq g(t, \beta_1(t), \beta_2(t), -\beta_2^{''}(t)), \quad t \in [a_n, b_n],$$
$$\beta_1(a_n) \geq \xi_{11}^{(n)}, \beta_2(a_n) \geq \xi_{21}^{(n)}, -\beta_1''(a_n) \geq \eta_{11}^{(n)}, -\beta_2''(a_n) \geq \eta_{21}^{(n)},$$
$$\beta_1(b_n) \geq \eta_{12}^{(n)}, \beta_2(b_n) \geq \xi_{22}^{(n)}, -\beta_1''(b_n) \geq \eta_{12}^{(n)}, -\beta_2''(b_n) \geq \eta_{22}^{(n)}.$$

Let

$$p_1(t) = \beta_1(t) - x_n(t), \quad t \in [a_n, b_n],$$
$$p_2(t) = \beta_2(t) - y_n(t), \quad t \in [a_n, b_n].$$

Now consider

$$p_1^{(4)}(t) = \beta_1^{(4)}(t) - x_n^{(4)}(t) \geq 0, \quad t \in (a_n, b_n),$$
$$p_2^{(4)}(t) = \beta_2^{(4)}(t) - y_n^{(4)}(t) \geq 0, \quad t \in (a_n, b_n).$$

Also

$$p_1(a_n) = \beta_1(a_n) - x_n(a_n) \geq \xi_{11}^{(n)} - \xi_{11}^{(n)} = 0,$$
$$p_2(a_n) = \beta_2(a_n) - y_n(a_n) \geq \xi_{21}^{(n)} - \xi_{21}^{(n)} = 0,$$
$$p_1(b_n) = \beta_1(b_n) - x_n(b_n) \geq \xi_{12}^{(n)} - \xi_{12}^{(n)} = 0,$$
$$p_2(b_n) = \beta_2(a_n) - y_n(b_n) \geq \xi_{22}^{(n)} - \xi_{22}^{(n)} = 0.$$

Moreover

$$-p_1^{''}(a_n) = -\beta_1^{''}(a_n) + x_n^{''}(a_n) \geq \eta_{11}^{(n)} - \eta_{11}^{(n)} = 0,$$
$$-p_2^{''}(a_n) = -\beta_2^{''}(a_n) + y_n^{''}(a_n) \geq \eta_{21}^{(n)} - \eta_{21}^{(n)} = 0,$$
$$-p_1^{''}(b_n) = -\beta_1^{''}(b_n) + x_n^{''}(b_n) \geq \eta_{12}^{(n)} - \eta_{12}^{(n)} = 0,$$
$$-p_2^{''}(b_n) = -\beta_2^{''}(b_n) + y_n^{''}(b_n) \geq \eta_{22}^{(n)} - \eta_{22}^{(n)} = 0.$$

By Lemma 2.2.4, we conclude that

$$p_i(t) \geq 0, \quad -p_i''(t) \geq 0, \quad t \in [a_n, b_n],\ i = 1, 2,$$

which is a contradiction to our assumption that $(x_n, y_n) \not\preceq (\beta_1, \beta_2)$. Therefore, $(x_n, y_n) \preceq (\beta_1, \beta_2)$.

Similarly we will show that $(\alpha_1, \alpha_2) \preceq (x_n, y_n)$. For this again suppose on contrary that $(\alpha_1, \alpha_2) \not\preceq (x_n, y_n)$. By definition of f_* and g_*, we have

$$\begin{aligned} f_*(t, x_n(t), y_n(t), -x_n''(t)) &= f(t, \alpha_1(t), \alpha_2(t), -\alpha_1''(t)), \quad t \in [a_n, b_n], \\ g_*(t, x_n(t), y_n(t), -y_n''(t)) &= g(t, \alpha_1(t), \alpha_2(t), -\alpha_2''(t)), \quad t \in [a_n, b_n], \end{aligned}$$

therefore

$$\begin{aligned} x_n^{(4)}(t) &= f(t, \alpha_1(t), \alpha_2(t), -\alpha_1''(t)), \quad t \in [a_n, b_n], \\ y_n^{(4)}(t) &= g(t, \alpha_1(t), \alpha_2(t), -\alpha_2''(t)), \quad t \in [a_n, b_n]. \end{aligned}$$

Since (α_1, α_2) is a lower solution of (2.1.1), so

$$\begin{aligned} &\alpha_1^{(4)}(t) \leq f(t, \alpha_1(t), \alpha_2(t), -\alpha_1''(t)), \quad t \in [a_n, b_n], \\ &\alpha_2^{(4)}(t) \leq g(t, \alpha_1(t), \alpha_2(t), -\alpha_2''(t)), \quad t \in [a_n, b_n], \\ &\alpha_1(a_n) \leq \xi_{11}^{(n)}, \alpha_2(a_n) \leq \xi_{21}^{(n)}, -\alpha_1''(a_n) \leq \eta_{11}^{(n)}, -\alpha_2''(a_n) \leq \eta_{21}^{(n)}, \\ &\alpha_1(b_n) \leq \xi_{12}^{(n)}, \alpha_2(b_n) \leq \xi_{22}^{(n)}, -\alpha_1''(b_n) \leq \eta_{12}^{(n)}, -\alpha_2''(b_n) \leq \eta_{22}^{(n)}. \end{aligned}$$

Let

$$\begin{aligned} q_1(t) &= x_n(t) - \alpha_1(t), \quad t \in [a_n, b_n], \\ q_2(t) &= y_n(t) - \alpha_2(t), \quad t \in [a_n, b_n], \end{aligned}$$

then

$$\begin{aligned} q_1^{(4)}(t) &= x_n^{(4)}(t) - \alpha_1^{(4)}(t) \geq 0, \quad t \in (a_n, b_n), \\ q_2^{(4)}(t) &= y_n^{(4)}(t) - \alpha_2^{(4)}(t) \geq 0, \quad t \in (a_n, b_n). \end{aligned}$$

Also

$$\begin{aligned} q_1(a_n) &= x_n(a_n) - \alpha_1(a_n) \geq \xi_{11}^{(n)} - \xi_{11}^{(n)} = 0, \\ q_2(a_n) &= y_n(a_n) - \alpha_2(a_n) \geq \xi_{21}^{(n)} - \xi_{21}^{(n)} = 0, \\ q_1(b_n) &= x_n(b_n) - \alpha_1(b_n) \geq \xi_{12}^{(n)} - \xi_{12}^{(n)} = 0, \\ q_2(b_n) &= y_n(b_n) - \alpha_2(a_n) \geq \xi_{22}^{(n)} - \xi_{22}^{(n)} = 0. \end{aligned}$$

Further

$$\begin{aligned} -q_1''(a_n) &= -x_n''(a_n) + \alpha_1''(a_n) \geq \eta_{11}^{(n)} - \eta_{11}^{(n)} = 0, \\ -q_2''(a_n) &= -y_n''(a_n) + \alpha_2''(a_n) \geq \eta_{21}^{(n)} - \eta_{21}^{(n)} = 0, \\ -q_1''(b_n) &= -x_n''(b_n) + \alpha_1''(b_n) \geq \eta_{12}^{(n)} - \eta_{12}^{(n)} = 0, \\ -q_2''(b_n) &= -y_n''(b_n) + \alpha_2''(b_n) \geq \eta_{22}^{(n)} - \eta_{22}^{(n)} = 0. \end{aligned}$$

Again by Lemma 2.2.4, we conclude that

$$q_i(t) \geq 0, \quad -q_i''(t) \geq 0, \quad t \in [a_n, b_n],\ i = 1, 2,$$

which is a contradiction to our assumption $(\alpha_1, \alpha_2) \not\preceq (x_n, y_n)$. Therefore, $(\alpha_1, \alpha_2) \preceq (x_n, y_n)$. Hence, we have shown that (2.4.1) hold. Consequently, $(x_n, y_n) \in C^4[a_n, b_n] \times C^4[a_n, b_n]$ is a solution of the following boundary value problem

$$\begin{aligned} x^{(4)}(t) &= f(t, x(t), y(t), -x''(t)), \quad t \in [a_n, b_n], \\ y^{(4)}(t) &= g(t, x(t), y(t), -y''(t)), \quad t \in [a_n, b_n], \\ x(a_n) &= \xi_{11}^{(n)}, y(a_n) = \xi_{21}^{(n)}, -x''(a_n) = \eta_{11}^{(n)}, -y''(a_n) = \eta_{21}^{(n)}, \\ x(b_n) &= \xi_{12}^{(n)}, y(b_n) = \xi_{22}^{(n)}, -x''(b_n) = \eta_{12}^{(n)}, -y''(b_n) = \eta_{22}^{(n)}. \end{aligned} \tag{2.4.2}$$

Let

$$\begin{aligned} M &= \max\left\{ \max_{t\in[0,1]} \beta_1(t), \max_{t\in[0,1]} \beta_2(t), \max_{t\in[0,1]} -\beta_1''(t), \max_{t\in[0,1]} -\beta_2''(t) \right\}, \\ \gamma_1^{(n)} &= \min\{\alpha_1(a_n), \alpha_1(b_n)\}, \qquad \gamma_2^{(n)} = \min\{\alpha_2(a_n), \alpha_2(b_n)\}, \\ \delta_1^{(n)} &= \min\{-\alpha_1''(a_n), -\alpha_1''(b_n)\},\ \delta_2^{(n)} = \min\{-\alpha_2''(a_n), -\alpha_2''(b_n)\}. \end{aligned}$$

Then

$$\begin{aligned} &\gamma_1^{(n)} \leq x_n(t) \leq M, \qquad \gamma_2^{(n)} \leq y_n(t) \leq M, \qquad t \in [a_n, b_n], \\ &\delta_1^{(n)} \leq -x_n''(t) \leq M, \quad \delta_2^{(n)} \leq -y_n''(t) \leq M, \quad t \in [a_n, b_n]. \end{aligned} \tag{2.4.3}$$

In view of (2.4.3), the sequence $\{(x_m, y_m)\}_{m=n}^{\infty}$ satisfies

$$\begin{aligned} &\gamma_1^{(n)} \leq x_m(t) \leq M, \qquad \gamma_2^{(n)} \leq y_m(t) \leq M, \qquad t \in [a_n, b_n], \\ &\delta_1^{(n)} \leq -x_m''(t) \leq M, \quad \delta_2^{(n)} \leq -y_m''(t) \leq M, \quad t \in [a_n, b_n]. \end{aligned} \tag{2.4.4}$$

Moreover, the sequence $\{(x_m, y_m)\}_{m=n}^{\infty}$ satisfies the integral equations

$$\begin{aligned} x_m(t) &= \frac{b_n - t}{b_n - a_n} x_m(a_n) + \frac{t - a_n}{b_n - a_n} x_m(b_n) - \int_{a_n}^{b_n} H_n(t,s) \left(\frac{b_n - s}{b_n - a_n} x_m''(a_n) + \frac{s - a_n}{b_n - a_n} x_m''(b_n) \right) ds \\ &\quad + \int_{a_n}^{b_n} H_n(t,s) \int_{a_n}^{b_n} H_n(s,\tau) f_*(\tau, x_m(\tau), y_m(\tau), -x_m''(\tau)) d\tau ds, \quad t \in [a_n, b_n], \\ y_m(t) &= \frac{b_n - t}{b_n - a_n} y_m(a_n) + \frac{t - a_n}{b_n - a_n} y_m(b_n) - \int_{a_n}^{b_n} H_n(t,s) \left(\frac{b_n - s}{b_n - a_n} y_m''(a_n) + \frac{s - a_n}{b_n - a_n} y_m''(b_n) \right) ds \\ &\quad + \int_{a_n}^{b_n} H_n(t,s) \int_{a_n}^{b_n} H_n(s,\tau) g_*(\tau, x_m(\tau), y_m(\tau), -y_m''(\tau)) d\tau ds, \quad t \in [a_n, b_n]. \end{aligned}$$

Differentiating three times with respect to t, we have

$$x_m'''(t) = \frac{x_m''(b_n) - x_m''(a_n)}{b_n - a_n} + \frac{1}{b_n - a_n}\int_{a_n}^{t}(s-a_n)f(s, x_m(s), y_m(s), -x_m''(s))ds$$
$$- \frac{1}{b_n - a_n}\int_{t}^{b_n}(b_n - s)f(s, x_m(s), y_m(s), -x_m''(s))ds, \quad t \in [a_n, b_n],$$
$$y_m'''(t) = \frac{y_m''(b_n) - y_m''(a_n)}{b_n - a_n} + \frac{1}{b_n - a_n}\int_{a_n}^{t}(s-a_n)g(s, x_m(s), y_m(s), -y_m''(s))ds$$
$$- \frac{1}{b_n - a_n}\int_{t}^{b_n}(b_n - s)g(s, x_m(s), y_m(s), -y_m''(s))ds, \quad t \in [a_n, b_n],$$

which implies that

$$|x_m'''(t)| \leq \frac{|x_m''(b_n) - x_m''(a_n)|}{b_n - a_n} + \frac{1}{b_n - a_n}\int_{a_n}^{t}(s-a_n)f(s, x_m(s), y_m(s), -x_m''(s))ds$$
$$+ \frac{1}{b_n - a_n}\int_{t}^{b_n}(b_n - s)f(s, x_m(s), y_m(s), -x_m''(s))ds, \quad t \in [a_n, b_n],$$
$$|y_m'''(t)| \leq \frac{|y_m''(b_n) - y_m''(a_n)|}{b_n - a_n} + \frac{1}{b_n - a_n}\int_{a_n}^{t}(s-a_n)g(s, x_m(s), y_m(s), -y_m''(s))ds$$
$$+ \frac{1}{b_n - a_n}\int_{t}^{b_n}(b_n - s)g(s, x_m(s), y_m(s), -y_m''(s))ds, \quad t \in [a_n, b_n].$$

Hence

$$|x_m'''(t)| \leq \frac{|x_m''(b_n) - x_m''(a_n)|}{b_n - a_n} + \int_{a_n}^{b_n} f(s, x_m(s), y_m(s), -x_m''(s))ds, \quad t \in [a_n, b_n],$$
$$|y_m'''(t)| \leq \frac{|y_m''(b_n) - y_m''(a_n)|}{b_n - a_n} + \int_{a_n}^{b_n} g(s, x_m(s), y_m(s), -y_m''(s))ds, \quad t \in [a_n, b_n].$$

Let

$$M_1 = \max\left\{f(t,x,y,z) : (t,x,y,z) \in [a_n, b_n] \times [\gamma_1^{(n)}, M] \times [\gamma_2^{(n)}, M] \times [\delta_1^{(n)}, M]\right\},$$
$$M_2 = \max\left\{g(t,x,y,z) : (t,x,y,z) \in [a_n, b_n] \times [\gamma_1^{(n)}, M] \times [\gamma_2^{(n)}, M] \times [\delta_2^{(n)}, M]\right\},$$

with this above inequalities become

$$|x_m'''(t)| \leq \frac{|x_m''(b_n) - x_m''(a_n)|}{b_n - a_n} + (b_n - a_n)M_1, \quad t \in [a_n, b_n],$$
$$|y_m'''(t)| \leq \frac{|y_m''(b_n) - y_m''(a_n)|}{b_n - a_n} + (b_n - a_n)M_2, \quad t \in [a_n, b_n],$$

and

$$\|x_m'\|_1 \leq \frac{2M}{b_n - a_n} + (b_n - a_n)M_1,$$
$$\|y_m'\|_1 \leq \frac{2M}{b_n - a_n} + (b_n - a_n)M_2, \tag{2.4.5}$$

Thus in view of (2.4.4) and (2.4.5) the sequence $\{(x_m, y_m)\}_{m=n}^{\infty}$ is uniformly bounded and equicontinuous on $[a_n, b_n]$. Define constant extension $\{(u_m, v_m)\}_{m=n}^{\infty}$ of $\{(x_m, y_m)\}_{m=n}^{\infty}$ by

$$u_m(t) = \begin{cases} x_m(a_n), & 0 \le t \le a_n, \\ x_m(t), & a_n \le t \le b_n, \\ x_m(b_n), & b_n \le t \le 1, \end{cases} \quad v_m(t) = \begin{cases} y_m(a_n), & 0 \le t \le a_n, \\ y_m(t), & a_n \le t \le b_n, \\ y_m(b_n), & b_n \le t \le 1. \end{cases} \tag{2.4.6}$$

Clearly $\{(u_m, v_m)\}_{m=n}^{\infty}$, being constant extension of $\{(x_m, y_m)\}_{m=n}^{\infty}$ over the interval $[0, 1]$, is uniformly bounded and equicontinuous on $[0, 1]$. Thus, there exists a subsequence $\{(u_{m_k}, v_{m_k})\}$ of $\{(u_m, v_m)\}$ converging uniformly to $(u, v) \in C^2[0, 1] \times C^2[0, 1]$. Further, the sequence $\{(u_{m_k}, v_{m_k})\}$ satisfies the integral equations

$$\begin{aligned}
u_{m_k}(t) = {} & \frac{b_n - t}{b_n - a_n} u_{m_k}(a_n) + \frac{t - a_n}{b_n - a_n} u_{m_k}(b_n) \\
& - \int_{a_n}^{b_n} H_n(t, s) \left(\frac{b_n - s}{b_n - a_n} u''_{m_k}(a_n) + \frac{s - a_n}{b_n - a_n} u''_{m_k}(b_n) \right) ds \\
& + \int_{a_n}^{b_n} H_n(t, s) \int_{a_n}^{b_n} H_n(s, \tau) f(\tau, u_{m_k}(\tau), v_{m_k}(\tau), -u''_{m_k}(\tau)) d\tau ds, \quad t \in [a_n, b_n], \\
v_{m_k}(t) = {} & \frac{b_n - t}{b_n - a_n} v_{m_k}(a_n) + \frac{t - a_n}{b_n - a_n} v_{m_k}(b_n) \\
& - \int_{a_n}^{b_n} H_n(t, s) \left(\frac{b_n - s}{b_n - a_n} v''_{m_k}(a_n) + \frac{s - a_n}{b_n - a_n} v''_{m_k}(b_n) \right) ds \\
& + \int_{a_n}^{b_n} H_n(t, s) \int_{a_n}^{b_n} H_n(s, \tau) g(\tau, u_{m_k}(\tau), v_{m_k}(\tau), -v''_{m_k}(\tau)) d\tau ds, \quad t \in [a_n, b_n].
\end{aligned}$$

Letting $m_k \to \infty$, we have

$$\begin{aligned}
u(t) = {} & \frac{b_n - t}{b_n - a_n} u(a_n) + \frac{t - a_n}{b_n - a_n} u(b_n) - \int_{a_n}^{b_n} H_n(t, s) \left(\frac{b_n - s}{b_n - a_n} u''(a_n) + \frac{s - a_n}{b_n - a_n} u''(b_n) \right) ds \\
& + \int_{a_n}^{b_n} H_n(t, s) \int_{a_n}^{b_n} H_n(s, \tau) f(\tau, u(\tau), v(\tau), -u''(\tau)) d\tau ds, \quad t \in [a_n, b_n], \\
v(t) = {} & \frac{b_n - t}{b_n - a_n} v(a_n) + \frac{t - a_n}{b_n - a_n} v(b_n) - \int_{a_n}^{b_n} H_n(t, s) \left(\frac{b_n - s}{b_n - a_n} v''(a_n) + \frac{s - a_n}{b_n - a_n} v''(b_n) \right) ds \\
& + \int_{a_n}^{b_n} H_n(t, s) \int_{a_n}^{b_n} H_n(s, \tau) g(\tau, u(\tau), v(\tau), -v''(\tau)) d\tau ds, \quad t \in [a_n, b_n].
\end{aligned}$$

Differentiating four times with respect to t, we have

$$\begin{aligned}
u^{(4)}(t) &= f(t, u(t), v(t), -u''(t)), \quad t \in [a_n, b_n], \\
v^{(4)}(t) &= g(t, u(t), v(t), -v''(t)), \quad t \in [a_n, b_n].
\end{aligned}$$

Now taking $\lim_{n\to\infty}$, we have

$$\begin{aligned}
u^{(4)}(t) &= f(t, u(t), v(t), -u''(t)), \quad t \in (0, 1), \\
v^{(4)}(t) &= g(t, u(t), v(t), -v''(t)), \quad t \in (0, 1),
\end{aligned}$$

which shows that $(u, v) \in C^4(0,1) \times C^4(0,1)$. Now, we show that (u, v) also satisfies the boundary conditions. As

$$\begin{aligned}
u(0) &= \lim_{m_k \to \infty} u(a_{m_k}) = \lim_{m_k \to \infty} u_{m_k}(a_{m_k}) = \lim_{m_k \to \infty} \xi_{11}^{(m_k)} = 0, \\
u(1) &= \lim_{m_k \to \infty} u(b_{m_k}) = \lim_{m_k \to \infty} u_{m_k}(b_{m_k}) = \lim_{m_k \to \infty} \xi_{12}^{(m_k)} = 0, \\
u''(0) &= \lim_{m_k \to \infty} u''(a_{m_k}) = \lim_{m_k \to \infty} u''_{m_k}(a_{m_k}) = \lim_{m_k \to \infty} -\eta_{11}^{(m_k)} = 0, \\
u''(1) &= \lim_{m_k \to \infty} u''(b_{m_k}) = \lim_{m_k \to \infty} u''_{m_k}(b_{m_k}) = \lim_{m_k \to \infty} -\eta_{12}^{(m_k)} = 0.
\end{aligned}$$

Similarly

$$v(0) = v(1) = v''(0) = v''(1) = 0.$$

Furthermore, the sequence $\{(u_{m_k}, v_{m_k})\}$ satisfies

$$\begin{aligned}
\alpha_1(t) \le u_{m_k}(t) \le \beta_1(t), \quad & t \in [a_n, b_n], \\
\alpha_2(t) \le v_{m_k}(t) \le \beta_2(t), \quad & t \in [a_n, b_n], \\
-\alpha_1''(t) \le -u_{m_k}''(t) \le -\beta_1''(t), \quad & t \in [a_n, b_n], \\
-\alpha_2''(t) \le -v_{m_k}''(t) \le -\beta_2''(t), \quad & t \in [a_n, b_n].
\end{aligned} \tag{2.4.7}$$

Allowing $m_k \to \infty$, we have

$$\begin{aligned}
\alpha_1(t) \le u(t) \le \beta_1(t), \quad & t \in [a_n, b_n], \\
\alpha_2(t) \le v(t) \le \beta_2(t), \quad & t \in [a_n, b_n], \\
-\alpha_1''(t) \le -u''(t) \le -\beta_1''(t), \quad & t \in [a_n, b_n], \\
-\alpha_2''(t) \le -v''(t) \le -\beta_2''(t), \quad & t \in [a_n, b_n],
\end{aligned} \tag{2.4.8}$$

and finally $\lim_{n \to \infty}$, gives

$$\begin{aligned}
\alpha_1(t) \le u(t) \le \beta_1(t), \quad & t \in [0, 1], \\
\alpha_2(t) \le v(t) \le \beta_2(t), \quad & t \in [0, 1], \\
-\alpha_1''(t) \le -u''(t) \le -\beta_1''(t), \quad & t \in [0, 1], \\
-\alpha_2''(t) \le -v''(t) \le -\beta_2''(t), \quad & t \in [0, 1].
\end{aligned} \tag{2.4.9}$$

Hence, $(u, v) \in (C^2[0,1] \cap C^4(0,1)) \times (C^2[0,1] \cap C^4(0,1))$ is a positive solution of the system (2.1.1) and satisfies $(\alpha_1, \alpha_2) \preceq (u, v) \preceq (\beta_1, \beta_2)$. □

2.5 An Example: Application of Theorem 2.4.1

Example 2.5.1. Consider the following system of singular BVPs

$$
\begin{aligned}
x^{(4)}(t) &= \frac{|6y + 3t^2 - 3t - 6|}{t(t^3 - 2t^2 + 1)}\left[\frac{1}{x(t)} + \frac{1}{y(t)} - \frac{1}{x''(t)}\right],\\
y^{(4)}(t) &= \frac{|6x + 3t^2 - 3t - 6|}{t(t^3 - 2t^2 + 1)}\left[\frac{1}{x(t)} + \frac{1}{y(t)} - \frac{1}{y''(t)}\right],\\
&x(0) = y(0) = x''(0) = y''(0) = 0,\\
&x(1) = y(1) = x''(1) = y''(1) = 0.
\end{aligned}
\tag{2.5.1}
$$

We choose

$$
\begin{aligned}
\alpha_1(t) = \alpha_2(t) &= \frac{t}{6}(t^3 - 2t^2 + 1),\\
\beta_1(t) = \beta_2(t) &= \frac{1}{6}t^4 - \frac{1}{3}t^3 - \frac{1}{2}t^2 + \frac{2}{3}t + 1.
\end{aligned}
$$

Clearly

$$
\begin{aligned}
&\alpha_i(0) = \alpha_i(1) = \alpha_i''(0) = \alpha_i''(1) = 0, \quad i = 1, 2,\\
&\beta_i(0) = \beta_i(1) = -\beta_i''(0) = -\beta_i''(1) = 1, \quad i = 1, 2.
\end{aligned}
$$

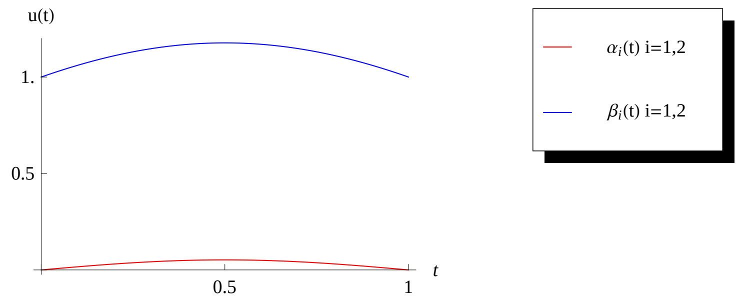

Figure 2.1:

Moreover, from Figure 2.1 it is clear that $0 < \alpha_i(t) \leq \beta_i(t)$, $i = 1, 2$.

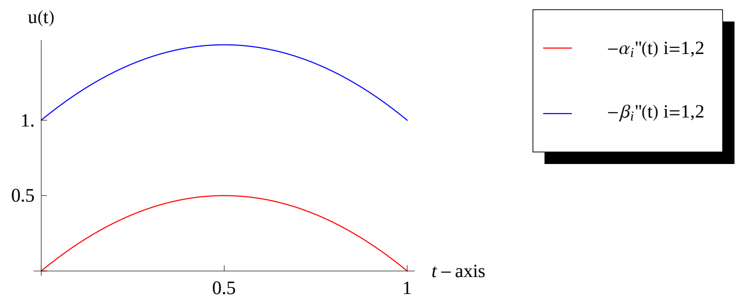

Figure 2.2:

Also, from Figure 2.2 it is evident that $0 < -\alpha_i''(t) \leq -\beta_i''(t)$ for $i = 1, 2$.
Let

$$\omega_1(t) = \frac{|6\alpha_2(t) + 3t^2 - 3t - 6|}{t(t^3 - 2t^2 + 1)} \left[\frac{1}{\alpha_1(t)} + \frac{1}{\alpha_2(t)} - \frac{1}{\alpha_1''(t)}\right] - \alpha_1^{(4)}(t),$$
$$\omega_2(t) = \frac{|6\alpha_1(t) + 3t^2 - 3t - 6|}{t(t^3 - 2t^2 + 1)} \left[\frac{1}{\alpha_1(t)} + \frac{1}{\alpha_2(t)} - \frac{1}{\alpha_2''(t)}\right] - \alpha_2^{(4)}(t),$$
$$\psi_1(t) = \beta_1^{(4)}(t) - \frac{|6\beta_2(t) + 3t^2 - 3t - 6|}{t(t^3 - 2t^2 + 1)} \left[\frac{1}{\beta_1(t)} + \frac{1}{\beta_2(t)} - \frac{1}{\beta_1''(t)}\right],$$
$$\psi_2(t) = \beta_2^{(4)}(t) - \frac{|6\beta_1(t) + 3t^2 - 3t - 6|}{t(t^3 - 2t^2 + 1)} \left[\frac{1}{\beta_1(t)} + \frac{1}{\beta_2(t)} - \frac{1}{\beta_2''(t)}\right].$$

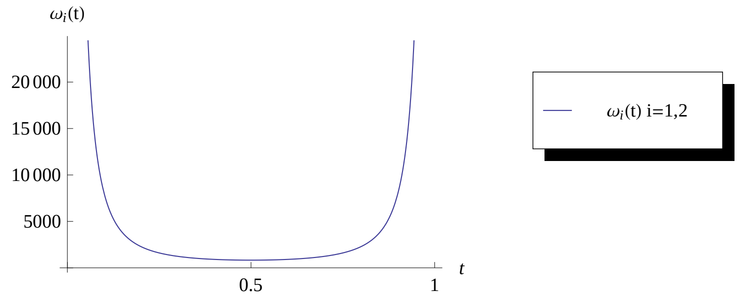

Figure 2.3:

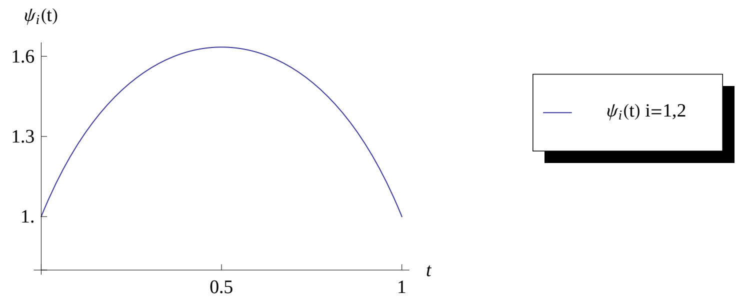

Figure 2.4:

Figures 2.3 and 2.4, respectively, shows that $\omega_i(t) \geq 0$ and $\psi_i(t) \geq 0$ for $i = 1, 2$. Which shows that, order pairs (α_1, α_2) and (β_1, β_2) are lower and upper solutions of singular system (2.5.1). Hence by Theorem 2.4.1, the system (2.5.1) has a positive solution $(u, v) \in C^2[0, 1] \cap C^4(0, 1)$ such that $(\alpha_1, \alpha_2) \preceq (u, v) \preceq (\beta_1, \beta_2)$.

Chapter 3

Conclusion

In this thesis, we have studied some results about the existence of positive solution for the system of fourth order SBVP

$$\begin{aligned} x^{(4)}(t) &= f(t, x(t), y(t), -x''(t)), \quad t \in (0,1), \\ y^{(4)}(t) &= g(t, x(t), y(t), -y''(t)), \quad t \in (0,1), \\ x(0) &= y(0) = x''(0) = y''(0) = 0, \\ x(1) &= y(1) = x''(1) = y''(1) = 0, \end{aligned}$$

where $f, g : (0,1) \times (0,\infty)^3 \to [0,\infty)$ are continuous and singular at $t = 0$, $t = 1$, $x = 0$, $y = 0$, $x'' = 0$, $y'' = 0$. For this, in Chapter 2, first of all we determine lower solution (α_1, α_2) and upper solution (β_1, β_2) of BVP (2.1.1). Also, we describe some lemmas relevant to our work. Then, we consider a modified non-singular BVP of SBVP (2.1.1) over $[a_n, b_n]$, which is of form

$$\begin{aligned} x^{(4)}(t) &= f_*(t, x(t), y(t), -x''(t)), \quad t \in [a_n, b_n], \\ y^{(4)}(t) &= g_*(t, x(t), y(t), -y''(t)), \quad t \in [a_n, b_n], \\ x(a_n) &= \xi_{11}^{(n)}, y(a_n) = \xi_{21}^{(n)}, -x''(a_n) = \eta_{11}^{(n)}, -y''(a_n) = \eta_{21}^{(n)}, \\ x(b_n) &= \xi_{12}^{(n)}, y(b_n) = \xi_{22}^{(n)}, -x''(b_n) = \eta_{12}^{(n)}, -y''(b_n) = \eta_{22}^{(n)}, \end{aligned}$$

where $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are the sequences of real numbers and $\{\xi_{ij}^{(n)}\}_{n=1}^{\infty}$ and $\{\eta_{ij}^{(n)}\}_{n=1}^{\infty}$ $(i, j = 1, 2)$ are the sequences of real constants. Moreover, we define a map $T_n : C^2[a_n, b_n] \times C^2[a_n, b_n] \to C^2[a_n, b_n] \times C^2[a_n, b_n]$ and showed that T_n has a fixed point $(x_n, y_n) \in C^4[a_n, b_n] \times C^4[a_n, b_n]$ which is the solution of modified BVP (2.3.1). After this we take a sequence of functions $\{(x_m, y_m)\}_{m=n}^{\infty}$ and define a constant extension $\{(u_m, v_m)\}_{m=n}^{\infty}$ of $\{(x_m, y_m)\}_{m=n}^{\infty}$ over the interval $[0, 1]$. Further, we have showed that a subsequence $\{(u_{m_k}, v_{m_k})\}$ of $\{(u_m, v_m)\}_{m=n}^{\infty}$ converges uniformally $(u, v) \in C^2[0, 1] \times C^2[0, 1]$ and (u, v) satisfies the boundary conditions as well. In Theorem (2.4.1), we have established that SBVP (2.1.1) has a positive solution $(u, v) \in (C^2[0, 1] \cap C^4(0, 1)) \times (C^2[0, 1] \cap C^4(0, 1))$ such that $(\alpha_1, \alpha_2) \preceq (u, v) \preceq (\beta_1, \beta_2)$ and also satisfies, $0 < \alpha_i(t) \leq \beta_i(t)$ and $0 < -\alpha_i''(t) \leq -\beta_i''(t)$ for $t \in (0,1)$, $i = 1, 2$ with $\alpha_i(0) = \alpha_i(1) = \alpha_i''(0) = \alpha_i''(1) = 0$, $i = 1, 2$. All of this is also verified by example given at the end of the Chapter 2.

For the future work, we can extend this work for some other systems of fourth order BVP. One of those systems is given bellow

$$\begin{aligned} x^{(4)}(t) &= f(t, x(t), y(t), x'(t), -x''(t), -x'''(t)), \quad t \in (0,1), \\ y^{(4)}(t) &= g(t, x(t), y(t), y'(t), -y''(t), -y'''(t)), \quad t \in (0,1), \\ x(0) &= y(0) = x''(0) = y''(0) = 0, \\ x'(1) &= y'(1) = x'''(1) = y'''(1) = 0, \end{aligned} \tag{3.0.1}$$

where f, g are singular at $t = 0$, $t = 1$, $x = 0$, $y = 0$, $x' = 0$, $y' = 0$, $x'' = 0$, $y'' = 0$, $x''' = 0$, $y''' = 0$. For the boundary conditions given in BVP (2.1.1), the singularity at x' and x''' is not possible. But if we take singularity at x' and x''' then the boundary conditions will be in form of boundary conditions that is given in BVP (3.0.1).

Bibliography

[1] R. P. Agarwal and D. O' Regan, Nonlinear superlinear singular and nonsingular second order boundary value problems, *J. Differential Equations.*, **143** (1998) 60–95.

[2] R. P. Agarwal, F. Wong and W. Lian, Positive solutions for nonlinear singular boundary value problems, *Appl. Math. Lett.*, **12** (1999) 115–120.

[3] R. P. Agarwal and D. O' Regan, Singular Differential and Integral Equations with Applications, *Kluwer. Acad. Publi.*, (2003).

[4] N. A. Asif and R. A. Khan, Positive solutions for a class of coupled system of singular threepoint boundary value problems, *Boundary. Val. Prob.*, (2009) Article ID 273063, 18 pages.

[5] N. A. Asif and R. A. Khan, Multiplicity results for positive solutions of a coupled system of singular boundary value problems, *Communication. Appl. Nonlinear Anal.*, **17** (2010) 53–68.

[6] N. A. Asif, P. W. Eloe and R. A. Khan, Positive solutions for a system of singular second order nonlocal boundary value problems, *J. Korean Math. Soc.*, **47(5)** (2010) 985-1000.

[7] N. A. Asif, R. A. Khan and J. Henderson, Existence of positive solutions to a system of singular boundary value problems, *Dynam. Systems Appl.*, **19** (2010) 395-404.

[8] O. A. Arqub, Z. Abo-Hammour, S. Momani and N. Shawagfeh, Solving singular two-point boundary value problems using continuous genetic algorithm, *Abs. Appl. Anal.*, (2012) Article ID 205391 25 pages.

[9] J. V. Baxley and S. B. Robinson, Nonlinear boundary value problems for shallow membrane caps, II, *J. Computational and Appl. Math.*, **88** (1998) 203–224.

[10] A. Callegari and Fiedman, An analytic solution of a nonlinear singular boundary value problem in the theory of viscous fluids, *J. Math. Anal. Appl.*, **21** (1968) 510–529.

[11] A. Callegari and A. Nachman, A nonlinear singular boundary value problem in the theory of pseudoplastic fluids, *SIAM J. Appl. Math.*, **38** (1980) 275–281.

[12] A. M. Fink and J. A. Gatica, Positive solutions of second order systems of boundary value problems, *J. Math. Anal. Appl.*, **180** (1993) 93–108.

[13] A. N. Kolomogorov and S. V. Fomin, Elements of the theory of functions and functional analysis, *Dover Publications. Inc. Mineola. New York.*, (1999).

[14] W. A. Kirk and B. Sims, Handbook of metric fixed point theory, *Kluwer. Acad. Publi.*, (2001).

[15] E. Kreyszig, Introductory functional analysis with applications, *John Wiley and Sons. Inc.*, (1978).

[16] S. Lipschutz, Theory and problems of general topology, *Schaum Publishing Co. New York*, (1965).

[17] B. Liu, L. Liu and Y. Wu, Positive solutions for singular systems of threepoint boundary value problems, *Comput. Math. Appl.*, **53** (2007) 1429–1438.

[18] M. Marletta and C. Tretter, Essential spectra of coupled systems of differential equations and applications in hydrodynamics, *J. Differential Equations.*, **243** (2007) 36–69.

[19] D. O' Regan, Solvability of some fourth (and higher) order singular boundary value problems, *J. Math. Anal. Appl.*, **161** (1991) 78-116.

[20] P. J. Schmid and D. S. Henningson, Optimal energy density growth in Hagen-Poiseuille flow, *J. Fluid Mech.*, **277** (1994) 197–225.

[21] J. Sprekels and D. Tiba, Optimization problems for thin elastic structures, *Int. Series of Numerical Math.* **158** (2000) 255–273.

[22] S. Stanek, The existence of positive solutions of singular fractional boundary value problems, *Computers and Mathematics with Appl.*, **62** (2011) 1379–1388.

[23] Z. Wei, Positive solutions of singular boundary value problems of fourth order differential equations, *Math. Sin.*, **42** (4) (1999) 715-722 (in Chinese).

[24] Z. Wei, A class of fourth order singular boundary value problems, *Appl. Math. Comput.*, **153** (3) (2004) 865–884.

[25] S. Xie and J. Zhu, Positive solutions of boundary value problems for system of nonlinear fourth order differential equations, *Boundary. Val. Prob.*, (2007) **A**rticle ID 76493 12 pages.

[26] B. Yan, D. O' Regan and R. P. Agarwal, Multiple positive solutions of singular second-order boundary value problems with derivative dependence, *Aequa. Math.*, **74** (2007) 62–89.

[27] B. Yan, D. O' Regan and R. P. Agarwal, Multiple positive solutions via index theory for singular boundary value problems with derivative dependence, *Positivity* **11** (2007) 687–720.