# Continuity and Differentiability of Convex Functions

By

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Dedicated

***To My Parents***

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# Abstract

This dissertation deals with the continuity and differentiability of convex functions and quasiconvex functions. We emphasis on the Gâteaux and the Fréchet differentiability of convex and quasiconvex functions. This has been discussed earlier by Daryoush Behmardi, Encyeh Dehghan Nayeri, Oswaldo González Gaxiola and Jean-Pierre Crouzeix. We reviewed their work which shows some algebraic properties and the relation of Gâteaux and Fréchet differentiation.

# Contents

# Chapter 1

# Introduction

The study of convex functions begins in the context of real valued functions of a real variable. Here we find a rich variety of results with significant applications. More importantly they will serve as a model for deep generalizations in the setting of several variables.

This chapter is a brief introduction of convex functions on the Real line $\mathbb{R}$ and their continuity and differentiability.

We take our functions $f : I \to \mathbb{R}$ to be defined on some interval $I$ of real line $\mathbb{R}$. We mean to allow $I$ to be open, half open, or closed, bounded or unbounded, we even allow the possibility that $I$ may be a point.

## 1.1 Definitions

**Definition 1.1.1.** A function $f : I \to \mathbb{R}$ is called ***convex function*** if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \tag{1.1}$$

for all $x, y \in I$ and $\lambda$ is in open interval (0, 1). We could equivalently take $\lambda$ to be in closed interval [0, 1].

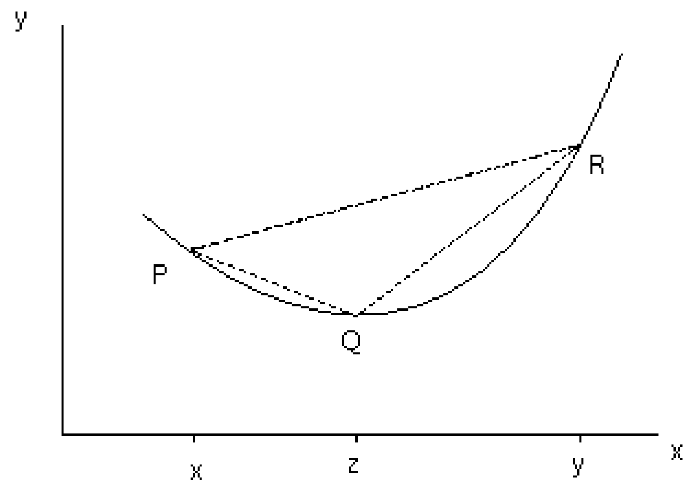


Figure 1.1

**Definition 1.1.2.** A function $f : I \to \mathbb{R}$ is called ***strictly convex function*** provided that the inequality (1.1) is strict for $x \neq y$.

Geometrically, (1.1) means that if $P$, $Q$ and $R$ are any three points on the graph of $f$ with $Q$ between $P$ and $R$, then $Q$ is on or below the chord $PR$ shown in the Figure 1.1. In terms of slopes, it is equivalent to

$$slope PQ \leqslant slope PR \leqslant slope QR \tag{1.2}$$

with strict inequalities when $f$ is strictly convex.

**Example 1.1.3.** Simple examples of convex functions are $f(x) = x^2$ on $(-\infty, \infty)$, $g(x) = \sin x$ on $[-\pi, 0]$, and $h(x) = |x|$ on $(-\infty, \infty)$. The first two are in fact strictly convex, the third one is not.

**Definition 1.1.4.** If $-f : I \to \mathbb{R}$ is a convex function, then we say that $f : I \to \mathbb{R}$ is a ***concave function***.

**Definition 1.1.5.** We say that $f : \mathbb{R} \to \mathbb{R}$ is ***linear*** if $f$ satisfies

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$$


Figure 1.2

for all $\alpha$, $\beta$, $x, y \in \mathbb{R}$.

It is known and is easy to show that $f$ is linear if and only if $f(x) = mx$ for some constant $m$.

**Definition 1.1.6.** We say that $f : I \to \mathbb{R}$ is an ***affine function*** if it is of the form $f(x) = mx + b$ on $I$. In the form of convexity, a function $f : I \to \mathbb{R}$ is called ***affine*** if it is convex and concave both, i.e.,

$$f(\lambda x + (1 - \lambda)y) = \lambda f(x) + (1 - \lambda)f(y).$$

**Definition 1.1.7.** The ***epigraph*** of a function $f : \mathbb{R}^n \to \mathbb{R}$ is the set of points in $\mathbb{R}^{n+1}$, lying on or above its graph and is written by:

$$\text{epi}f = \{(x, \mu) \,:\, x \in \mathbb{R}^n,\, \mu \in \mathbb{R},\, \mu \geq f(x)\} \subseteq \mathbb{R}^{n+1}.$$

The strict epigraph of the function is defined by:

$$\text{epi}_S f = \{(x, \mu) \,:\, x \in \mathbb{R}^n,\, \mu \in \mathbb{R},\, \mu > f(x)\} \subseteq \mathbb{R}^{n+1}.$$

**Theorem 1.1.8.** Let $f : I \to \mathbb{R}$ be a convex function on $I$ if and only if epigraph of $f$ is a convex set.

*Proof.* Let $f$ be convex and $(x_1, \mu_1)$ and $(x_2, \mu_2) \in \text{epi}f$, that is $f(x_1) \leq \mu_1$ and $f(x_2) \leq \mu_2$. Take $\lambda \in (0, 1)$. From the convexity of $f$ and the definition of the epigraph of $f$ we have:

$$\begin{aligned} f(\lambda x_1 + (1-\lambda)x_2) &\leq \lambda f(x_1) + (1-\lambda)f(x_2) \\ &\leq \lambda\alpha_1 + (1-\lambda)\alpha_2. \end{aligned}$$

This means that $(\lambda x_1 + (1-\lambda)x_2, \lambda\alpha_1 + (1-\lambda)\alpha_2) \in \text{epi}f$ or $\lambda(x_1, \alpha_1) + (1-\lambda)(x_2, \alpha_2) \in \text{epi}f$, hence $\text{epi}f$ is convex.

Conversely, let $\text{epi}f$ be convex and this implies that the $I$ is convex because it is the projection of epigraph on $\mathbb{R}$. It is sufficient to verify (1.1) over $I$. Thus, let us take $x_1, x_2 \in I$ and choose $a$ and $b$ such that $f(x_1) \leq a$ and $f(x_2) \leq b$ respectively. Since $(x_1, a), (x_2, b) \in \text{epi}f$. By the assumptions it follows that

$$\lambda(x_1, a) + (1-\lambda)(x_2, b) \in \text{ epi}f,$$

for all $\lambda \in (0, 1)$. This implies that

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda a + (1-\lambda)b.$$

If $f(x_1)$ and $f(x_2)$ are finite, we can take $a = f(x_1)$ and $b = f(x_2)$ to conclude the assertion (1.1). If either $f(x_1)$ or $f(x_2)$ is $-\infty$ we can let tend $a$ or $b$ to $-\infty$ and thus (1.1) is also fulfilled. $\square$

## 1.2 Continuity and Differentiability

A convex function which is finite on a closed interval $[a, b]$ is bounded from above by $M = \max\{f(a), f(b)\}$, since for any $z = \lambda a + (1-\lambda)b$ in interval $[a, b]$

$$f(z) \leqslant \lambda f(a) + (1-\lambda)f(b) \leqslant \lambda M + (1-\lambda)M = M.$$

It is also bounded from below since by writing an arbitrary point $x$ in the form $x = \frac{a+b}{2} + t$, with $\frac{a-b}{2} \leq t \leq \frac{b-a}{2}$, we see that $x$ lies in $[a, b]$. Thus

$$f\Big(\frac{a+b}{2}\Big) \leq \frac{1}{2}f\Big(\frac{a+b}{2} + t\Big) + \frac{1}{2}f\Big(\frac{a+b}{2} - t\Big)$$

or

$$f\Big(\frac{a+b}{2} + t\Big) \geq 2f\Big(\frac{a+b}{2}\Big) - f\Big(\frac{a+b}{2} - t\Big).$$

Using $M$ as an upper bound, $f[(a+b)/2 - t] \leq M$ or $-f[(a+b)/2 - t] \geq -M$, so

$$f\Big(\frac{a+b}{2} + t\Big) \geq 2f\Big(\frac{a+b}{2}\Big) - M = m.$$

Where $m$ denoting a lower bound of $f$.

**Definition 1.2.1.** For any closed subinterval [a, b] of the interior of the domain of the convex function $f : I \to \mathbb{R}$, there is a constant $K$ so that for any two points $x, y \in [a, b]$, we have

$$|f(x) - f(y)| \leq K|x - y|. \tag{1.3}$$

A function that satisfies (1.3) for some constant $K$ and all $x$ and $y$ in $[a, b]$ is said to satisfy ***Lipschitz condition*** (or to be Lipschitz) on the interval $[a, b]$.

**Definition 1.2.2.** A function $f : I \to \mathbb{R}$ is said to be ***absolutely continuous*** on $I$ if for every $\epsilon > 0$, there is a $\delta > 0$ such that whenever a finite sequence of pairwise disjoint sub-intervals $(x_k, y_k)$ of $I$ satisfies

$$\sum_k |y_k - x_k| < \delta,$$

then

$$\sum_k |f(y_k) - f(x_k)| < \epsilon.$$

In the following example we see the absolute continuity of a function defined on real line $\mathbb{R}$.

**Example 1.2.3.** Let $f : \mathbb{R} \to \mathbb{R}$ and $f(x) = 3x+7$. Then $f$ is absolutely continuous on $\mathbb{R}$.

*Proof.* Choose $\epsilon > 0$ and set $\delta = \epsilon/3$. Then a finite sequence of non-overlaping open subintervals $(a_k, b_k)$ of $\mathbb{R}$ $(1 \leq k \leq n)$, satisfies:

$$\sum_{k=1}^{n} |b_k - a_k| < \delta.$$

Then

$$\sum_{k=1}^{n} |f(b_k) - f(a_k)| = \sum_{k=1}^{n} |3b_k + 7 - 3a_k - 7|$$
$$= 3\sum_{k=1}^{n} |b_k - a_k| < 3\delta = \epsilon.$$

□

**Remark 1.2.4.** Every absolutely continuous function is continuous but the converse is not always true.

We illustrate it by the following example.

**Example 1.2.5.** Let $f : [0,1] \to \mathbb{R}$ be a function such that

$$f(x) = \begin{cases} x\sin(\frac{\pi}{x}) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Then $f$ is continuous on $[0,1]$ but not absolutely continuous.

*Proof.* Continuity of the given function can be obtain easily. But for disproving absolute continuity we have to take some steps. So let us suppose $\epsilon = 1$ then for each $\delta > 0$, we may pick some $M, N \in \mathbb{N}$, when $\frac{1}{\delta} < M < N$ such that:

$$\sum_{k=M}^{N} a_k > 1, \tag{1.4}$$

where

$$a_k = \frac{2}{4k+1}.$$

Furthermore we can take

$$b_k = \frac{2}{4k}.$$

Now we are able to consider the non-overlaping open subintervals $(a_k, b_k)$, with $M \leq k \leq N$, which satisfies:

$$\sum_{k=M}^{N} |b_k - a_k| = \sum_{k=M}^{N} \left| \frac{2}{4k} - \frac{2}{4k+1} \right|.$$

Now by simplifying and applying summation on later expression we get:

$$\sum_{k=M}^{N} |b_k - a_k| < \delta.$$

Then

$$\begin{aligned} \sum_{k=M}^{N} |f(b_k) - f(a_k)| &= \sum_{k=M}^{N} \left| b_k \sin\left(\frac{\pi}{b_k}\right) - a_k \sin\left(\frac{\pi}{a_k}\right) \right| \\ &= \sum_{k=M}^{N} \left| \frac{2}{4k} \sin(2\pi k) - \frac{2}{4k+1} \sin\left(\frac{\pi}{2}(4k+1)\right) \right|. \end{aligned}$$

Since $\sin(2\pi k) = 0$ and $\sin\left(\frac{\pi}{2}(2k+1)\right) = 1$ for all $k$, which implies that:

$$\sum_{k=M}^{N} |f(b_k) - f(a_k)| = \sum_{k=M}^{N} \left(\frac{2}{4k+1}\right) > 1 = \epsilon,$$

by our assumption (1.4). □

Now we can say that absolute continuity is stronger then continuity.

It is easily seen that a convex function may not be continuous at the boundary points of its domain. It may, in fact, have upward jumps there. On the interior, however, it is not only continuous, but it satisfies a stronger condition.

**Theorem 1.2.6.** [18] If $f : I \to \mathbb{R}$ is a convex function, then $f$ satisfies a Lipschitz condition on any closed interval $[a, b]$ contained in the interior $I^0$ of $I$. Consequently, $f$ is absolutely continuous on $[a, b]$ and continuous on $I^0$.

*Proof.* Choose $\epsilon > 0$ so that $a - \epsilon$ and $b + \epsilon$ remain in $I$, and let $m$ and $M$ be the lower and upper bounds for $f$ on $[a - \epsilon, b + \epsilon]$. Then by taking two distinct points $x$ and $y$ of $[a, b]$, set

$$z = y + \frac{\epsilon}{|y - x|}(y - x)$$

and

$$\lambda = \frac{|y - x|}{\epsilon + |y - x|}.$$

Then by the above definitions of $z$ and $\lambda$ we can see that $z \in [a - \epsilon, b + \epsilon]$ and $y = \lambda z + (1 - \lambda)x$. Also we have

$$f(y) \leq \lambda f(z) + (1 - \lambda)f(x) = \lambda[f(z) - f(x)] + f(x),$$

$$f(y) - f(x) \leq \lambda(M - m) < \frac{|y - x|}{\epsilon}(M - m) = K|y - x|,$$

where $K = (M - m)/\epsilon$. Since this is true for all $x, y \in [a, b]$, we conclude that

$$|f(y) - f(x)| \leq K|y - x| \tag{1.5}$$

as desired.

Now for the absolute continuity of $f$ on $[a, b]$, we choose $\delta = \epsilon/K$ and let $(a_i, b_i)$ is a finite sequence of pairwise disjoint sub-intervals of $[a, b]$, which satisfies

$$\sum_i |b_i - a_i| < \delta = \frac{\epsilon}{K}.$$

Then by using (1.5) we get:

$$\sum_i |f(b_i) - f(a_i)| \leq K \sum_i |b_i - a_i| < K\frac{\epsilon}{K} = \epsilon.$$

Finally, the continuity of $f$ on $I^0$ is a consequence of the arbitrariness of $[a, b]$. □

The derivative of a convex function is best studied in terms of the left and right derivatives defined by

$$f'_-(x) = \lim_{y \uparrow x} \frac{f(y) - f(x)}{y - x}$$

and

$$f'_+(x) = \lim_{y \downarrow x} \frac{f(y) - f(x)}{y - x},$$

respectively. Here $y \uparrow x$ means $y$ approaches to $x$ from left and $y \downarrow x$ means $y$ approaches to $x$ from right.

**Theorem 1.2.7.** [18] If $f : I \to \mathbb{R}$ is convex (strictly convex), then $f'_-(x)$ and $f'_+(x)$ exist and are increasing (strictly increasing) on $I^0$.

*Proof.* Consider four points $w < x < y < z$ in $I^0$ and let $P, Q, R,$ and $S$ be the corresponding points on the graph of $f$ shown in the Figure 1.3. Then the inequality (1.2) can be extended to four points which gives

$$slopePQ \leqslant slopePR \leqslant slopeQR \leqslant slopeQS \leqslant slopeRS, \tag{1.6}$$

with strict inequalities if $f$ is strictly convex. Since

$$slopeQR \leqslant slopeRS,$$

it is clear that the slope $QR$ increases as $x \uparrow y$ and similarly the slope $RS$ decreases as $z \downarrow y$. Thus the left side of the inequality

$$\frac{f(x) - f(y)}{x - y} \leqslant \frac{f(z) - f(y)}{z - y}$$

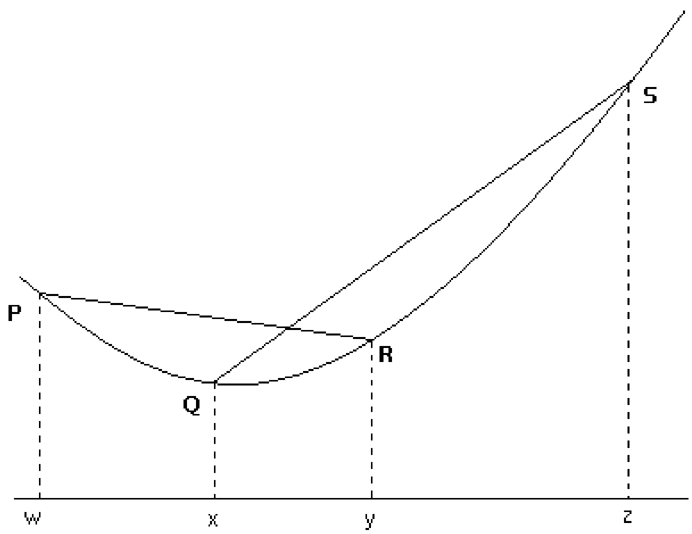


Figure 1.3

increases as $x \uparrow y$ and the right side decreases as $z \downarrow y$. This implies that $f'_-(x)$ and $f'_+(y)$ exist and satisfy

$$f'_-(x) \leqslant f'_+(y). \tag{1.7}$$

The result (1.7) holds for all $y \in I^0$. Moreover, using (1.6), we see that

$$f'_+(w) \leqslant \frac{f(x) - f(w)}{x - w} \leqslant \frac{f(y) - f(x)}{y - x} \leqslant f'_-(y) \tag{1.8}$$

with strict inequalities prevailing if $f$ is strictly convex. This combined with (1.7) yields

$$f'_-(w) \leqslant f'_+(w) \leqslant f'_-(y) \leqslant f'_+(y),$$

establishing the monotone nature of $f'_-$ and $f'_+$. □

In fact the results of Theorem 1.2.7 are valid for all of $I$, not just its interior. For example, if $I = (a, b]$, then $f'_-(b)$ exists at least in the infinite sense and $f'_-$ is increasing on $(a, b]$. We can restate the Theorem 1.2.7 for the case $I = [a, b]$ to describe the behavior of $f$ at the end points when $I = [a, b]$. In this case, $f'_+(a)$ and

$f'_-(b)$ exist at least in the infinite sense, $f'_+$ is increasing on $[a,b)$ and $f'_-$ is increasing on $(a,b]$.

There are a number of other important facts having to do with the continuity properties of $f'_+$ and $f'_-$. The monotone character of $f'_+$ means that the limit of $f'_+(x)$ exists as $x \downarrow w$. From the inequality (1.8) we have

$$f'_+(x) \leqslant \frac{f(y)-f(x)}{y-x}. \tag{1.9}$$

The inequality (1.9) along with the continuity of $f$ gives

$$\lim_{x \downarrow w} f'_+(x) \leqslant \lim_{x \downarrow w} \frac{f(y)-f(x)}{y-x} = \frac{f(y)-f(w)}{y-w}.$$

Thus

$$\lim_{x \downarrow w} f'_+(x) \leqslant \lim_{y \downarrow w} \frac{f(y)-f(w)}{y-w} = f'_+(w).$$

On the other hand, since $x > w$, monotonicity of $f'_+$ implies $f'_+(x) \geqslant f'_+(w)$. Thus

$$\lim_{x \downarrow w} f'_+(x) = f'_+(w). \tag{1.10}$$

Similar arguments show that

$$\lim_{x \uparrow w} f'_+(x) = f'_-(w). \tag{1.11}$$

Indeed (1.10) and (1.11) are also valid at the left and right endpoints of $I$, respectively, provided that $f$ is defined and continuous there. Finally, we remark that statements analogous to (1.10) and (1.11) hold for the left and right limits of $f'_-(x)$.

**Theorem 1.2.8.** [18] If $f : I \to \mathbb{R}$ is a convex function on the open interval $I$, then the set $E$ where $f'$ fails to exist is countable. Moreover, $f'$ is continuous on $I \setminus E$.

*Proof.* From (1.10) and (1.11), we conclude that $f'_+(w) = f'_-(w)$ if and only if $f'_+$ is continuous at $w$. Thus $E$ consists specifically of the discontinuities of the increasing

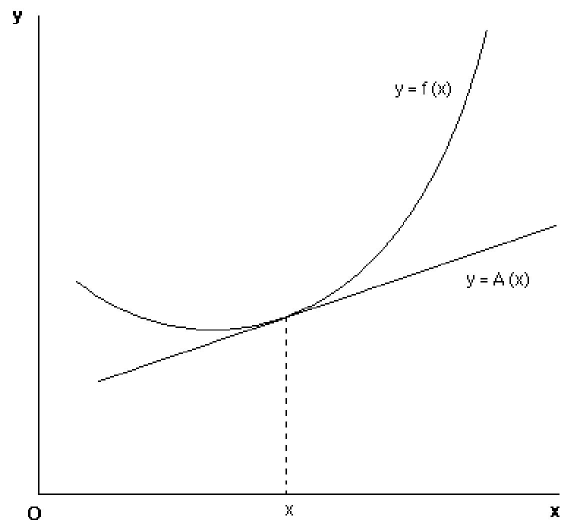


Figure 1.4

function $f'_+$, then by the well known result of real analysis i.e, the set of points of discontinuities of a monotonic function is at most countable [20], since $f'_+$ is monotonic function, then the set $E$ is countable. On $I \setminus E$, $f'_+$ is continuous, so that $f'$ which agrees with $f'_+$ on $I \setminus E$, is also continuous there. □

**Definition 1.2.9.** A function $f$ is defined on $I$ has a support at $x_0 \in I$ if there exists an affine function $A(x) = f(x_0) + m(x - x_0)$ such that $A(x) \leq f(x)$ for every $x \in I$. The graph of the support function A is called a ***line of support*** for $f$ at $x_0$ (see Figure 1.4).

**Theorem 1.2.10.** [18] A function $f : (a, b) \to \mathbb{R}$ is convex if and only if there is at least one line of support for $f$ at each $x_0 \in (a, b)$.

*Proof.* If $f$ is convex and $x_0 \in (a, b)$, choose $m \in [f'_-(x_0), f'_+(x_0)]$. Then as we saw before

$$\frac{f(x) - f(x_0)}{x - x_0} \geq m$$

or

$$\frac{f(x) - f(x_0)}{x - x_0} \leq m$$

according as $x > x_0$ or $x < x_0$. In either case,

$$f(x) - f(x_0) \geq m(x - x_0),$$

that is,

$$f(x) \geq f(x_0) + m(x - x_0).$$

Conversely, suppose that $f$ has a line of support at each point of $(a, b)$. Let $x, y \in (a, b)$. If $x_0 = \lambda x + (1 - \lambda)y$ for $\lambda \in [0, 1]$ and let $A(x) = f(x_0) + m(x - x_0)$ be the support function for $f$ at $x_0$. Then

$$f(x_0) = A(x_0) = \lambda A(x) + (1 - \lambda)A(y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

as desired. □

**Theorem 1.2.11.** [18] Let $f : (a, b) \to \mathbb{R}$ be a convex function. Then $f$ is differentiable at $x_0$ if and only if the line of support for $f$ at $x_0$ is unique. In this case, $A(x) = f(x_0) + f'(x_0)(x - x_0)$ provides this unique support.

*Proof.* It is clear from the proof of the above theorem that complimentary to each $m \in [f'_-(x_0), f'_+(x_0)]$, there exist a line of support for $f$ at $x_0$. Accordingly the uniqueness of the line means $f'_-(x_0) = f'_+(x_0)$, that is, $f'(x_0)$ exists.

On the other hand, suppose $f'(x_0)$ exists. Any line of support $A(x) = f(x_0) + m(x - x_0)$ gives us

$$f(x) - f(x_0) \geq m(x - x_0).$$

Then, for $x_1 < x_0 < x_2$, we have

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0} \leq m \leq \frac{f(x_2) - f(x_0)}{x_2 - x_0}.$$

On taking limits as $x_1 \uparrow x_0$ and $x_2 \downarrow x_0$ we have

$$f'_-(x_0) \le m \le f'_+(x_0). \tag{1.12}$$

Since $f$ is differentiable at $x_0$, thus inequality (1.12) infers that $m$ is unique. Hence the support $A$ at $x_0$ is unique. □

**Example 1.2.12.** We can see that for the function $f : \mathbb{R} \to \mathbb{R}_+$ defined by

$$f(x) = |x|,$$

the line of support is not unique at $x_0 = 0$ so $f$ is not differentiable at $x_0 = 0$.

# Chapter 2

# Convex Functions On Real Banach Spaces

## 2.1 Introduction

In this chapter we study the continuity and differentiability of real valued convex functions defined on an open convex set in a Banach space. Note that in finite-dimensional spaces such functions are always locally bounded. In the first section we recall some useful definitions. The second section presents the continuity of convex functions on real Banach spaces. In third section of this chapter we shall study the Gâteaux differentiability of convex functions in first part and the Fréchet differentiability in second part. Then we will end up by discussing the algebraic properties of both differentials and conditions on which both coincide. The similar results for the quasiconvex functions will be discussed in the next chapter.

The letter $E$ will always denote a real Banach space, $D$ will be a nonempty open convex subset of $E$ and $f$ will be a convex function on $D$ throughout this chapter.

The definition of a convex function has a very natural generalization to a real-valued function defined on an arbitrary real Banach Space $E$. We merely require that the domain $D$ of $f$ be convex. This assures us that for $x, y \in D$, $t \in (0,1)$, $f$ will always be defined at $tx + (1-t)y$.

**Definition 2.1.1.** Let $f : D \to \mathbb{R}$ be a function defined on a nonempty open convex set $D \subseteq E$ is called ***convex*** on $D$ if, for each $x, y \in D$ and $t \in (0,1)$

$$f[tx + (1-t)y] \leqslant tf(x) + (1-t)f(y). \tag{2.1}$$

We assume convex functions to be finite valued and defined on convex sets.

If the equality in (2.1) always holds then $f$ is said to be **affine**. A function $f : D \to R$ is said to be **concave** if $-f$ is convex.

We note immediately that for three points $x_1, x_2, x_3 \in D$ and three positive numbers $t_1, t_2$ and $t_3$ such that $t_1 + t_2 + t_3 = 1$, a convex function satisfies

$$\begin{aligned} f(t_1x_1 + t_2x_2 + t_3x_3) =& f\left(t_1x_1 + (t_2+t_3)\left(\frac{t_2}{t_2+t_3}x_2 + \frac{t_3}{t_2+t_3}x_3\right)\right) \\ \leq& t_1(f(x_1)) + (t_2+t_3)f\left(\frac{t_2}{t_2+t_3}x_2 + \frac{t_3}{t_2+t_3}x_3\right) \\ \leq& t_1(f(x_1)) + t_2(f(x_2)) + t_3(f(x_3)). \end{aligned} \tag{2.2}$$

Following the same pattern, one easily establishes inductively the inequality (2.3).

**Definition 2.1.2.** For $n$ points in $D$ and $n$ positive numbers $t_i$ with $\sum_1^n t_i = 1$, a convex funtion satisfies

$$f\left(\sum_1^n t_i x_i\right) \leq \sum_1^n t_i f(x_i). \tag{2.3}$$

The relation (2.3) is known as ***Jensen's inequality***. Sometimes (2.3) is taken as the definition of the convex function.

**Definition 2.1.3.** A function $f : E \to \mathbb{R}$ is called **Sublinear funtional** if it satisfies the following two conditions:

$$f(x+y) \leqslant f(x) + f(y),$$

$$f(tx) = tf(x)$$

whenever $t \geqslant 0$.

**Example 2.1.4.** There are some examples of convex functions defined on real Banach space.

1. The norm function $f(x) = \|x\|$ is a simple example. More generally if $S$ is a nonempty convex subset of $E$, then we can define a distance function on $S$ as:

   $$d_S(x) = \inf \{\|x-y\| : y \in S\},$$

   $x \in D$, which is continuous and convex on $D \subseteq E$. (Note that $d_S(x) = \|x\|$ if $S = \{0\}$.)

   *Proof.* Let $z = tx_1 + (1-t)x_2 \in D$, as $x_1, x_2 \in D$ and $t \in (0,1)$, then

   $$\begin{aligned} d_S(z) &= \inf \{\|z-y\| : y \in S\} \\ &= \inf \{\|tx_1 + (1-t)x_2 - ty - (1-t)y\|\} \\ &\leq \inf \{\|tx_1 - ty\| + \|(1-t)x_2 - (1-t)y\|\} \\ &= \inf\{\|tx_1 - ty\|\} + \inf\{\|(1-t)x_2 - (1-t)y\|\} \\ &= t\ \inf\{\|x_1 - y\| : y \in S\} + (i-t)\ \inf\{\|x_2 - y\| : y \in S\} \\ &= t\ d_S(x_1) + (1-t)\ d_S(x_2) \end{aligned}$$

   □

2. The Minkowski gauge functional is another generalization of the norm function. This is also a good example of convex function. Suppose $G$ is a convex subset

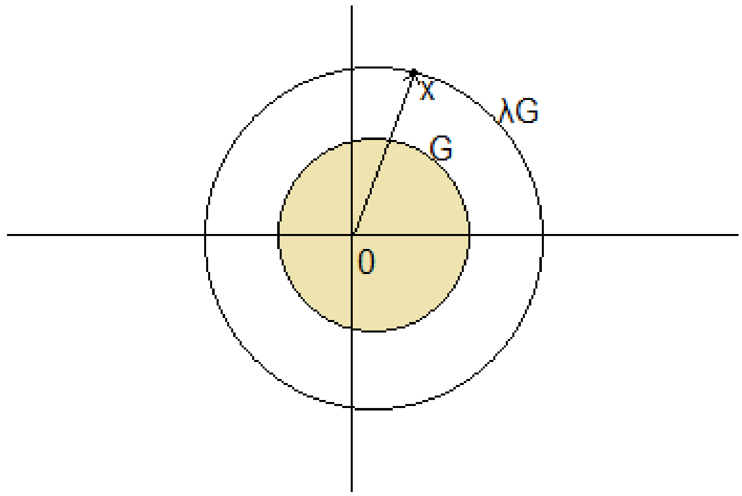


Figure 2.1

of $E$, with $0 \in intG$, defined by:

$$p_G(x) = \inf\{\lambda > 0 : x \in \lambda G\},$$

when $x \in E$. In other words, $p_G(x)$ is the smallest factor by which the set $G$ must enlarged to contain the point $x$, which can be seen in Figure 2.1. The functional $p_G$ is sublinear and non-negative.

*Proof.* Clearly $p_G$ is non-negative and positively homogeneous, let $t > 0$, then for all $x, y \in E$ we can write:

$$\begin{aligned} p_G(tx) &= \inf\{\lambda > 0 : tx \in \lambda G\} \\ &= t\ \inf\{\lambda > 0 : x \in \lambda G\}. \end{aligned} \tag{2.4}$$

Now let $\alpha, \beta > 0$ satisfy $x \in \alpha G$ and $y \in \beta G$, which implies that

$$x + y \in \alpha G + \beta G = (\alpha + \beta)G.$$

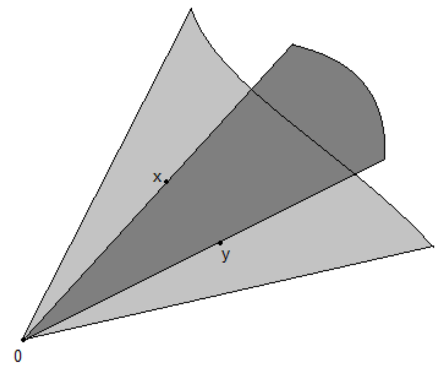


Figure 2.2

Then the subadditive property of $p_G$ can be seen as:

$$\begin{aligned} p_G(x+y) &= \inf\{\alpha+\beta>0 : x+y \in (\alpha+\beta)G\} \\ &\le \inf\{\alpha+\beta>0 : x \in \alpha G + y \in \beta G\} \\ &= \inf\{\alpha>0 : x \in \alpha G\} + \inf\{\beta>0 : y \in \beta G\} \\ p_G(x+y) &\le p_G(x) + p_G(y). \end{aligned} \tag{2.5}$$

So the convexity immediately follows from (2.4) and (2.5). □

**Definition 2.1.5.** A subset $C$ of $\mathbb{R}^n$ is called ***cone*** if it is closed under positive scalar multiplication, i.e.,

$$\lambda x \in C \text{ when } x \in K \text{ and } \lambda > 0.$$

Such a set is a union of half lines emanating from origin. The origin itself may or may not be included.

**Definition 2.1.6.** Cone defined on a subset $C$ of $\mathbb{R}^n$ is called ***convex cone*** if $C$ is a convex set. Algebraically, convex cone is defined as follows: For any positive scalars $\alpha$, $\beta$ and any $x$, $y \in C$,

$$\alpha x + \beta y \in C.$$

As an example we can see in the Figure 2.5 that, a cone (light one). Inside of it (dark one) convex cone consists of all points $\alpha x + \beta y$ with $\alpha > 0$ and $\beta > 0$, for the depicted $x$ and $y$. The curves on the upper right symbolize that the regions are infinite in extent.

One of the first thing we learned about the real valued convex functions on an open interval is that they are continuous. This is not generally true on an infinite-dimensional Banach space $E$, but a convex function defined on an open set $D \subseteq E$ when $E$ is finite-dimensional, is continuous. In section 2.2 we prove this fact and explore related ideas.

## 2.2 Continuity of Convex Functions

There are two directions to go for the continuity of convex functions on real Banach spaces. One can ask what additional conditions need to be put on a convex function in order to guarantee its continuity. Or, one can ask what further restrictions must be placed on $E$ in order to guarantee that a function convex on $D$ will be continuous there. We will take some steps in both directions.

We have seen in first chapter that in order to prove the continuity of a real valued convex function $f$ on $(a, b) \subseteq \mathbb{R}$ we have to establish boundedness of $f$ on each closed subinterval of $(a, b)$. Then we are able to establish a Lipschitz condition and thus to conclude that $f$ is continuous. It turns out that it is sufficient here to take $f$ to be bounded in a neighborhood of atleast one point of $D$. Which finally leads us to Lipschitz condition and then continuity.

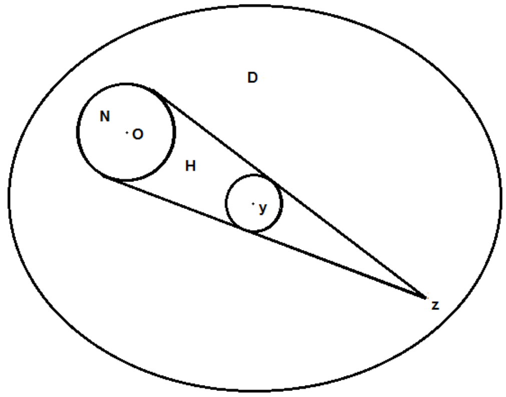


Figure 2.3

**Theorem 2.2.1.** [18] Let $f$ be a convex function on $D$. If $f$ is bounded from above in a neighborhood of one point $x_0 \in D$, then it is locally bounded; that is, each $x \in D$ has a neighborhood on which $f$ is bounded.

*Proof.* We first show that if $f$ is bounded above in an $\epsilon$-neighborhood of some point, it is bounded below in the same neighborhood. Taking the point to be 0 (origion) for convenience, suppose $f$ is bounded above by $M$ in a neighborhood $N_\epsilon$ of the origion. Since

$$O = \frac{1}{2}x + \frac{1}{2}(-x)$$
$$f(O) \leq \frac{1}{2}f(x) + \frac{1}{2}f(-x),$$

and therefore

$$f(x) \geq 2f(O) - f(-x).$$

Now $\|x\| < \epsilon$ implies $\|-x\| < \epsilon$, so $-f(-x) \geq -M$ and $f(x) \geq 2f(O) - M$, this shows that $f$ is bounded from below. We now return to our theorem, we take $f$ to be bounded from above by $M$ on an $\epsilon$-neighborhood $N_\epsilon$ of the origin. We will show $f$ to be bounded in neighborhood of $y \in D$, $y_0 = 0$. Choose $\rho > 1$ so that $z = \rho y \in D$ and let $\lambda = \frac{1}{\rho}$. Then

$$H = \{v \in E : v = (1-\lambda)x + \lambda z, x \in N_\epsilon\}$$

is a neighborhood of $\lambda z = y$ with radius $(1-\lambda)\epsilon$. Moreover

$$f(v) \leq (1-\lambda)f(x) + \lambda f(z) \leq M + f(z)$$

that is, $f$ is bounded above on $H$; and by the first remark of this proof, $f$ is also bounded below on $H$. □

**Definition 2.2.2.** A function $f$ defined on $D$ is said to be ***locally Lipschitz*** if at each $x \in D$, there is neighborhood $N_\epsilon(x)$ and a constant $K(x)$ such that $y, z \in N_\epsilon$, then

$$|f(y) - f(z)| \leq K\|y - z\|.$$

If this inequality holds on a set $V \subseteq D$ with $K$ independent of $x$, then we say that $f$ is ***Lipschitz*** on $V$.

**Theorem 2.2.3.** [18] Let $f$ be a convex function on $D$. If $f$ is bounded from above in an neighborhood of one point $x_0 \in D$, then $f$ is locally Lipschitz in $D$.

*Proof.* By Theorem 2.2.1, $f$ is locally bounded, so given $x_0$ we may find a neighborhood $N_{2\epsilon} \subseteq D$ on which $f$ is bounded, say by $M$. Then $f$ satisfies the stated Lipschitz condition on $N_\epsilon(x_0)$, for if it does not, we may choose $x_1, x_2 \in N_\epsilon(x_0)$ such that

$$\frac{f(x_2) - f(x_1)}{\|x_2 - x_1\|} > \frac{2M}{\epsilon}.$$

Then we may choose $t > 0$ so that $x_3 = x_2 + t(x_2 - x_1)$ is in $N_{2\epsilon}(x_0)$ and such that $\|x_3 - x_2\| = \epsilon$. Because $f$ is convex on the line through $x_1$, $x_2$ and $x_3$, we may use what we know about functions convex on a line (1.2) to write

$$\frac{f(x_3) - f(x_2)}{\|x_3 - x_2\|} \geq \frac{f(x_2) - f(x_1)}{\|x_2 - x_1\|} > \frac{2M}{\epsilon}$$

this says $f(x_3) - f(x_2) > 2M$, contradicting the fact that $|f| \leq M$. □

**Theorem 2.2.4.** [18] Let $f$ be a convex function on $D$. If $f$ is bounded from above in an neighborhood of one point $x_0 \in D$, then $f$ is continuous on $D$.

*Proof.* We have proved in previous theorem that if a convex function $f$ on $D$ is bounded from above in a neighborhood of one point $x_0 \in D$ then $f$ is locally Lipschiz in D. That is for $x_0 \in D$, there exist $N_\delta(x_0)$ neighborhood and a constant $K(x_0)$, then for all $x_1 and x_2 \in N_\delta(x_0)$,

$$|f(x_2) - f(x_1)| \leq K\|x_2 - x_1\|. \tag{2.6}$$

Since continuity implies immediately from (2.6), that is for all $\epsilon > 0$, let $\delta = \epsilon/K$. Hence for all $x_1, x_2 \in D$ such that

$$\|x_2 - x_1\| < \delta,$$

satisfies (2.6). Thus

$$\begin{aligned} |f(x_2) - f(x_1)| &\leq K\|x_2 - x_1\| \\ &< K\left(\frac{\epsilon}{K}\right) = \epsilon. \end{aligned}$$

□

## 2.3 Differentiability of Convex Functions

Starting from the following elementary lemma which is fundamental to the study of differentiability of convex functions.

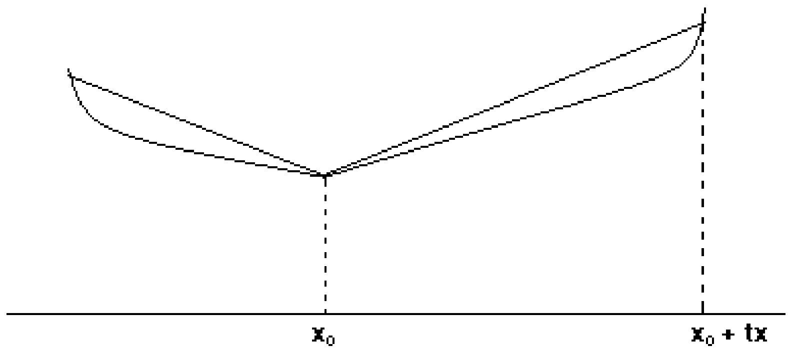


Figure 2.4

**Lemma 2.3.1.** [17] If $x_0 \in D$, then for each $x \in E$ the right handed directional derivative

$$f_+(x_0)(x) = \lim_{t \downarrow 0} \frac{f(x_0 + tx) - f(x_0)}{t}$$

exists and defines a sublinear functional on $E$.

*Proof.* Recall that $D$ is an open convex subset of $E$. $f(x_0 + tx)$ is defined for sufficiently small $t > 0$. Figure 2.4 shows why $f_+(x_0)$ exists; but the left and right handed directional derivatives are not equal. The difference quotient is nonincreasing as $t \downarrow 0$, and bounded below, by the corresponding difference quotient from the left, i.e,

$$f(x_0 - tx) + f(x_0) < f(x_0 + tx) - f(x_0).$$

To prove this, we can assume that $x_0 = 0$ and $f(x_0) = 0$. If $0 < t < s$ then by convexity:

$$f(tx) = f\Big(\frac{t}{s}(sx) + \frac{s-t}{s}(0)\Big) \leqslant \frac{t}{s} f(sx) + \frac{s-t}{s} f(0) = \frac{t}{s} f(sx),$$

so for $tx < sx$ we have $f(tx) \leq f(sx)$ which proves monotonicity. Then by taking $-x$ in place of $x$, we see that

$$-\frac{[f(x_0 - tx) - f(x_0)]}{t}$$

is nondecreasing as $t \downarrow 0$. Then again by convexity, for $t > 0$

$$
\begin{aligned}
f(x_0) &= f\left(\frac{2x_0 - 2tx + 2tx}{2}\right) \\
&= f\left(\frac{x_0 - 2tx + x_0 + 2tx}{2}\right) \\
&\leqslant \frac{1}{2} f(x_0 - 2tx) + \frac{1}{2} f(x_0 + 2tx).
\end{aligned}
$$

So we can write:

$$2f(x_0) \leqslant f(x_0 - 2tx) + f(x_0 + 2tx)$$

$$
\begin{aligned}
f(x_0) - f(x_0 - 2tx) &\leqslant f(x_0 + 2tx) - f(x_0) \\
-\frac{[f(x_0 - 2tx) - f(x_0)]}{2t} &\leqslant \frac{f(x_0 + 2tx) - f(x_0)}{2t}
\end{aligned}
$$

which shows that the right side is bounded below and the left side is bounded above. Thus the both limits exist, left one is $-f_+(x_0)(-x)$ and we have

$$-f_+(x_0)(-x) \leqslant f_+(x_0)(x).$$

It is also obvious that $f_+(x_0)(x)$ is positively homogeneous. To see that it is subadditive, use convexity again: for $t > 0$

$$
\begin{aligned}
f_+(x_0)(u+v) &\leq \frac{f(x_0 + t(u+v)) - f(x_0)}{t} \\
&= f\left(\frac{\frac{1}{2}(x_0 + 2tu) + \frac{1}{2}(x_0 + 2tv) - f(x_0)}{t}\right) \\
&\leq \frac{1}{t}\left\{\frac{1}{2} f(x_0 + 2tu) + \frac{1}{2} f(x_0 + 2tv) - \frac{1}{2} f(x_0) - \frac{1}{2} f(x_0)\right\} \\
&= \frac{f(x_0 + 2tu) - f(x_0)}{2t} + \frac{f(x_0 + 2tv) - f(x_0)}{2t}.
\end{aligned}
$$

Now by taking limit $t \downarrow 0$ we get

$$f_+(x_0)(u+v) = f_+(x_0)(u) + f_+(x_0)(v).$$

□

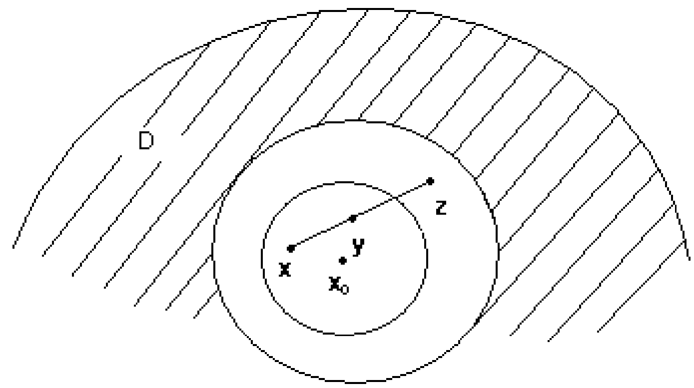


Figure 2.5

### 2.3.1 Gâteaux Derivative

**Definition 2.3.2.** A function $f$ is said to be ***Gâteaux differentiable*** at $x_0 \in D$ provided the limit

$$df(x_0)(x) = \lim_{t \to 0} \frac{f(x_0 + tx) - f(x_0)}{t}$$

exists for each $x \in E$. The function $df(x_0)$ is called the ***Gâteaux derivative*** (or ***Gâteaux differential***) of $f$ at $x_0$.

**Notation 2.3.3.** If $x \in E$ and $r > 0$, then the closed ball with radius $r$ and center at $x$ is denoted by $B(x; r) = \{y \in E : \|x - y\| \leq r\}$.

**Proposition 2.3.4.** [17] If the convex function $f$ is continuous at $x_0 \in D$, then it is locally Lipschitzian at $x_0$, that is, there exist $M > 0$ and $\delta > 0$ such that $B(x_0; \delta) \subset D$ and

$$|f(x) - f(y)| \leq M\|x - y\|$$

whenever $x, y \in B(x_0; \delta)$.

*Proof.* Since $f$ is continuous at $x_0$, it is locally bounded there; that is there exist $M_1 > 0$ and $\delta > 0$ such that $|f| \leq M_1$ on $B(x_0; \delta) \subset D$. If $x, y$ are distinct points in

$B(x_0, \delta)$, let $\alpha = \|x - y\|$ and let

$$z = y + \left(\frac{\delta}{\alpha}\right)(y - x),$$

see Figure (2.5). Note that $z \in B(x_0; \delta)$. Since

$$y = \left(\frac{\alpha}{\alpha + \delta}\right) z + \left(\frac{\delta}{\alpha + \delta}\right) x$$

is the convex combination (lying in $B(x_0; 2\delta)$), we have

$$f(y) \leq \left(\frac{\alpha}{\alpha + \delta}\right) f(z) + \left(\frac{\delta}{\alpha + \delta}\right) f(x)$$

so

$$f(y) - f(x) \leq \left(\frac{\alpha}{\alpha + \delta}\right) \{f(z) - f(x)\} \leq \left(\frac{\alpha}{\delta}\right) .2M_1 = \left(\frac{2M_1}{\delta}\right) \|x - y\|.$$

Interchanging $x$ and $y$ gives the desired result, with $M = \frac{2M_1}{\delta}$. □

**Corollary 2.3.5.** [17] If a convex function $f$ is continuous at $x_0 \in D$, then $d_+ f(x_0)$ is a continuous sublinear functional on $E$, and hence $df(x_0)$ (when it exists) is a continuous linear functional.

*Proof.* It is given that $f$ is continuous on $x_0 \in D$, then by the previous Proposition it is locally lipschitz on $x_0$. That is there exists a neighborhood $B$ of $x_0$ and $M > 0$ such that, if $x \in E$. Then

$$|f(x_0 + tx) - f(x_0)| \leq Mt\|x\|,$$

provided $t > 0$ is sufficiently small so that $x_0 + tx \in B$. Thus, for all points $x \in E$, we have $d_+ f(x_0)(x) \leq M\|x\|$. Since $d_+ f(x_0)(x)$ is a sublinear functional proved in Lemma 2.2.3 (a sublinear function is trivially a convex function), which implies that $d_+ f(x_0)$ is continuous (by Theorem 2.2.4). □

**Lemma 2.3.6.** [13] Let $X$ and $Y$ be Banach spaces. Let $f_n : X \to Y$ be Gâteaux differentiable mapping, for all $n$. Assume that $(\Sigma fn)$ converge pointwise on $X$, and that there exists a constant $K > 0$ so that for all $x$,

$$\sum_{n\geq 1} \sup_{x_0 \in X} \left\| \frac{\partial f_n}{\partial h}(x_0) \right\| \leq K\|x\|.$$

Then the mapping

$$f = \sum_{n\geq 1} f_n$$

is Gâteaux differentiable on $X$ for all $x_0$ and

$$df(x_0)(x) = \sum_{n\geq 1} df_n(x_0)(x).$$

*Proof.* First we have to check the Gâteaux differentiability of

$$f = \sum_{n\geq 1} f_n,$$

i.e:

$$\begin{aligned} df(x_0)(x) &= \lim_{t\to 0} \frac{f(x_0+tx) - f(x)}{t} \\ &= \lim_{t\to 0} \frac{\sum_{n\geq 1} f_n(x_0+tx) - \sum_{n\geq 1} f_n(x)}{t} \\ &= \lim_{t\to 0} \left\{ \frac{f_1(x_0+tx) - f_1(x_0)}{t} + \frac{f_2(x_0+tx) - f_2(x_0)}{t} + \cdots \right\}. \end{aligned} \tag{2.7}$$

By utilizing Weirstrass M-test we show that the infinite series on right hand side of (2.7) converges.( For convenience we recall W. M-test which stated that: Suppose $f_n : X \to Y$ are functions such that there exist constants $M_n$ with $\|f_n(x)\| \leq M_n$ for all $n \geq 1$ and $x \in X$, and the series $\sum_{n=1}^{\infty} M_n$ converges then $\sum_{n\geq 1} f_n$ converges uniformly (and absolutely).)

As it is given that $\sum f_n$ converges pointwise, so we can write:

$$\left\| \frac{\partial f_n}{\partial x}(x_0) \right\| = \left\| \frac{f_n(x_0+tx) - f_n(x_0)}{t} \right\| \leq \sup_{x_0\in X} \left\| \frac{\partial f_n}{\partial x}(x_0) \right\|.$$

By applying summation we get:

$$\sum_{n\geq 1}\left\|\frac{\partial f_n}{\partial x}(x_0)\right\| \leq \sum_{n\geq 1}\sup_{x_0\in X}\left\|\frac{\partial f_n}{\partial x}(x_0)\right\| \leq K\|x\|.$$

Thus by W. M-test

$$\sum_{n\geq 1}\sup_{x_0\in X}\left\|\frac{\partial f_n}{\partial x}(x_0)\right\| \text{ converges.}$$

Hence

$$\sum_{n\geq 1}\left\|\frac{\partial f_n}{\partial x}(x_0)\right\| \text{ converges uniformly.}$$

Since we can say that (2.7) is summable and we can write it as:

$$\begin{aligned}\lim_{t\to 0}\frac{f(x_0+tx)-f(x_0)}{t} =& \lim_{t\to 0}\left(\frac{f_1(x_0+tx)-f_1(x_0)}{t}\right)\\ &+\lim_{t\to 0}\left(\frac{f_2(x_0+tx)-f_2(x_0)}{t}\right)+\cdots,\end{aligned}$$

or

$$df(x_0)(x) = df_1(x_0)(x) + df_2(x_0)(x) + \cdots = \sum df_n(x_0)(x).$$

Finally this implies that $f(x_0)$ is Gâteaux differentiable, and

$$df(x_0)(x) = \sum df_n(x_0)(x).$$

□

**Definition 2.3.7.** If $f$ is a convex function defined on the convex set $C$ and $x \in C$, we define the *subdifferential* of $f$ at $x$ to be the set $\partial f(x)$ of all $x^* \in E^*$ satisfying

$$\langle x^*, y-x\rangle \leq f(y) - f(x) \text{ for all } y \in C.$$

Note that this is the same as satisfying that the affine function $x^* + \alpha$, where $\alpha = f(x) - \langle x^*, x\rangle$ is dominated by $f$ and is equal to it at $y = x$, as indicated in the Figure 2.6 .

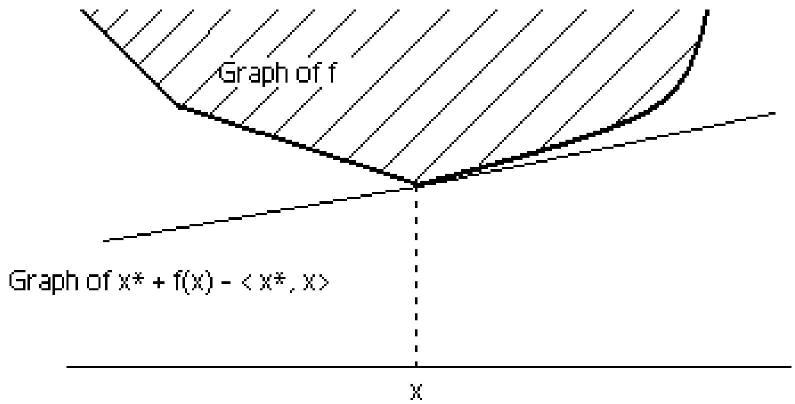


Figure 2.6

### 2.3.2 Fréchet Derivative

**Definition 2.3.8.** Suppose that $E$ and $F$ are real Banach spaces and $\varphi : D \to F$ is a continuous function. We can extend the definition of *Gâteaux differentiability* as follows: Let $\varphi$ is Gâteaux differentiable at the point $x_0 \in D$ provided there exists a continuous linear map from $E$ to $F$ (denoted by $d\varphi(x_0)$) such that

$$d\varphi(x_0)(x) = \lim_{t\to 0} \frac{\varphi(x_0 + tx) - \varphi(x_0)}{t} \text{ for each } x \in E. \tag{2.8}$$

Another way of stating this is to say that $\varphi$ has directional derivatives at $x_0$ in every direction $x$ and the resulting function of $x$ is continuous and linear.

We say that $\varphi$ is ***Fréchet differentiable*** at $x_0 \in D$ provided there exists a continuous linear map from $E$ to $F$ (denoted by $\varphi'(x_0)$) such that

$$\varphi(x_0 + x) - \varphi(x_0) = \varphi'(x_0)(x) + r(x) \text{ for all } \|x\| < \epsilon \tag{2.9}$$

with some $\epsilon > 0$, where $\frac{\|r(x)\|}{\|x\|} \to 0$ as $\|x\| \to 0$. The operator $\varphi'(x_0)$ is called ***Fréchet differential*** (or Fréchet derivative) of $\varphi$ at $(x_0)$.

**Theorem 2.3.9.** [14] Let $f$ be a convex function defined on $D \subseteq E$, that is continuous at $x_0 \in D$. Then $f$ is fréchet differentiable at $x_0$ iff

$$\lim_{t\to 0} \frac{f(x_0+tx)+f(x_0-tx)-2f(x_0)}{t} = 0 \tag{2.10}$$

exists uniformly for all $x \in N_{x_0}$ (neighborhood of $x_0$).

*Proof.* Given $f$ is fréchet differentiable at $x_0 \in D$ then by the definition of fréchet differentiablity

$$\lim_{t\to 0} \frac{f(x_0+tx)-f(x_0)}{t} = f'(x_0)(x) \tag{2.11}$$

is uniform for every $x \in N_{x_0}$. Similarly we can write for $-x$ :

$$\lim_{t\to 0} \frac{f(x)-f(x_0-tx)}{t} = f'(x_0)(-x), \tag{2.12}$$

it is also uniform for every $x \in N_{x_0}$. By equating (3.1) and (2.12) we get

$$\lim_{t\to 0} \frac{f(x_0+tx)-f(x_0)}{t} = \lim_{t\to 0} \frac{f(x)-f(x_0-tx)}{t},$$

or we can write it as:

$$\lim_{t\to 0} \frac{f(x_0+tx)+f(x_0-tx)-2f(x_0)}{t} = 0.$$

This exists uniformly for every $x \in N_{x_0}$ by (3.1) and (2.12).

Conversely, if the limit in (2.10) exist uniformly for every $x \in N_{x_0}$ then we can write

$$\lim_{t\to 0} \frac{f(x)-f(x_0-tx)}{t} + \lim_{t\to 0} \frac{f(x_0+tx)-f(x_0)}{t} = 0. \tag{2.13}$$

Therefore both limits exists uniformly so we can say that $f$ is Fréchet differentiable. (2.13) can be written as

$$f'(x_0)(x) + f'(x_0)(-x) = 0$$

. □

### 2.3.3 Properties for G$\hat{a}$teaux and Fr$\acute{e}$chet Differentiable Functions

Many of the properties of ordinary derivatives carry over to the Gâteaux and Fréchet derivatives. So we are discussing some algebraic properties of Fréchet and Gâteaux differentiable functions here.

1. Scalar multiplication:

   (a) Let $\alpha \in \mathbb{R}$ and $f : D \to F$ is a Fréchet differentiable function at $x_0$, then the definition of $f$ allows us to write:

   $$(\alpha f(x_0))' = \alpha f'(x_0).$$

   (b) Similarly for a function $f : D \to F$ which is Gâteaux differentiable at $x_0$ and $\alpha \in \mathbb{R}$ we can write:

   $$\begin{aligned} d(\alpha f(x_0)(x)) &= \lim_{t \to 0} \frac{\alpha f(x_0 + tx) - \alpha f(x_0)}{t} \\ &= \alpha \lim_{t \to 0} \frac{f(x_0 + tx) - f(x_0)}{t} \\ &= \alpha \; df(x_0)(x). \end{aligned}$$

2. Additive Property:

   (a)Sum of two Fréchet differentiable functions is Fréchet differentiable.

   *Proof.* Let $f_1$ and $f_2$ are two Fréchet differentiable functions on an open subset $D$ of a Banach space $X$, then for any $x \in D$ let

   $$f_3(x) = (f_1 + f_2)(x) = f_1(x) + f_2(x),$$

   here $f_1$ and $f_2$ are continuous and therefore locally bounded by some constants $M_1$ and $M_2$ respectively that is

   $$\|f_1\| \leqslant M_1, \qquad \|f_2\| \leqslant M_2. \tag{2.14}$$

Then $f_3(x) = f_1(x) + f_2(x)$ will also be continuous and locally bounded for all $x \in S_x$, where $S_x$ is the neighborhood of $x$. Then

$$\|f_3(x)\| = \|f_1(x) + f_2(x)\| \leqslant \|f_1(x)\| + \|f_2(x)\| \leq M_1 + M_2.$$

By the definition of Fréchet differentiable functions we have:

$$f_1(x_0 + x) - f_1(x_0) = f_1'(x_0)(x) + r_1(x), \tag{2.15}$$

where $\frac{\|r_1(x)\|}{\|x\|} \to 0$ as $\|x\| \to 0$. And

$$f_2(x_0 + x) - f_2(x_0) = f_2'(x_0)(x) + r_2(x), \tag{2.16}$$

where $\frac{\|r_2(x)\|}{\|x\|} \to 0$ as $\|x\| \to 0$. Hence by using (2.15) and (2.16) we get:

$$\begin{aligned} f_3(x_0 + x) - f_3(x_0) &= (f_1(x_0 + x) - f_1(x_0)) + (f_2(x_0 + x) - f_2(x_0)) \\ &= f_1'(x_0)(x) + r_1(x) + f_2'(x_0)(x) + r_2(x) \\ &= (f_1'(x_0) + f_2'(x_0))(x) + (r_1(x) + r_2(x)), \end{aligned}$$

where

$$\frac{\|r_3(x)\|}{\|x\|} = \frac{\|r_1(x) + r_2(x)\|}{\|x\|} \leq \frac{\|r_1(x)\|}{\|x\|} + \frac{\|r_2(x)\|}{\|x\|} \to 0$$

as $\|x\| \to 0$. Thus $df_1(x_0)(x)$ and $df_2(x_0)(x)$ exist uniformly for all $x \in S_x$, which implies that $df_3(x_0)(x)$ exist uniformly on every $x \in S_x$, i.e;

$$f_3'(x_0)(x) = (f_1'(x_0) + f_2'(x_0))(x).$$

□

(b) Sum of two Gâteaux differentiable functions is Gâteaux differentiable.

*Proof.* Let $f_1$ and $f_2$ are two Gâteaux differentiable functions on an open subset $D$ of a Banach space $X$, then for any $x \in D$ let

$$f_3(x) = (f_1 + f_2)(x) = f_1(x) + f_2(x),$$

Now by the definition of Gâteaux derivative we can write:

$$\begin{aligned} df_3(x_0)(x) &= \lim_{t\to 0} \frac{f_3(x_0+tx)-f_3(x_0)}{t} \\ &= \lim_{t\to 0} \frac{f_1(x_0+tx)+f_2(x_0+tx)-f_1(x_0)-f_2(x_0)}{t} \\ &= \lim_{t\to 0} \frac{f_1(x_0+tx)-f_1(x_0)}{t} + \lim_{t\to 0} \frac{f_2(x_0+tx)-f_2(x_0)}{t} \\ &= df_1(x_0)(x)+df_2(x_0)(x), \end{aligned}$$

so that we can write it as

$$df_3(x_0)(x) = df_1(x_0)(x) + df_2(x_0)(x). \tag{2.17}$$

□

3. The product of two Fréchet differentiable functions is Fréchet differentiable.

*Proof.* Now let $f_3(x) = f_1(x).f_2(x)$ (showing dot product). Again $f_3$ will be continuous and locally bounded for all $x \in S_x$. Then by using (2.15) and (2.16) we can get

$f_3(x_0+x) - f_3(x_0)$

$= f_1(x_0+x).f_2(x_0+x) - f_1(x_0).f_2(x_0)$

$= f_1(x_0+x).f_2(x_0+x) - f_1(x).f_2(x_0+x) + f_2(x_0+x).f_1(x_0) - f_1(x_0).f_2(x_0)$

$= (f_1(x_0+x) - f_1(x)).f_2(x_0+x) + (f_2(x_0+x) - f_2(x_0)).f_1(x_0)$

$= f_1'(x_0)(x).f_2(x_0+x) + f_2'(x_0)(x).f_1(x_0) + (r_1(x)f_2(x_0+x) + r_2(x)f_1(x_0)).$

Thus by (2.14), (2.15) and (2.16) we have:

$$\frac{\|r_3(x)\|}{\|x\|} = \frac{\|r_1(x)f_2(x_0+x)+r_2(x)f_1(x_0)\|}{\|x\|} \leq M_2\frac{\|r_1(x)\|}{\|x\|} + M_1\frac{\|r_2(x)\|}{\|x\|} \to 0$$

as $\|x\| \to 0$. Finally we get:

$$f_3'(x_0)(x) = f_1'(x_0)(x).f_2(x_0) + f_2'(x_0)(x).f_1(x_0).$$

□

**Remark 2.3.10.** The product of two Gâteaux differentiable functions is not Gâteaux differentiable necessary.

**Lemma 2.3.11** ( [2], [12]). Suppose that $X$, $Y$ and $Z$ are Banach spaces. If $g : X \to Y$ is Fréchet differentiable at $x_0 \in X$ and $f : Y \to Z$ is Fréchet differentiable at $g(x_0) = y_0 \in Y$, then their composition $f \circ g$ has a Fréchet derivative at $y_0 \in Y$ and the derivative of the composition is given by the chain rule

$$(f \circ g)'(x) = f'(g(x))(g'(x)).$$

*Proof.* Given that $g : X \to Y$ is Fréchet differentiable at $x_0 \in X$ and $g(x_0) = y_0$, which implies that $g'(x)$ exists. Hence by the definition of Fréchet differentiability we have:

$$y - y_0 = g(x) - g(x_0) = g'(x_0)(x - x_0) + r_1(x - x_0) \tag{2.18}$$

Therefore $f : Y \to Z$ is also Fréchet differentiable, i.e:

$$f(y) - f(y_0) = f'(y_0)(y - y_0) + r(y - y_0).$$

By using (2.18) we get:

$$f(y) - f(y_0) = f'(y_0)(g'(x_0)(x - x_0)) + f'(y_0)r_1(x - x_0) + r(y - y_0),$$

where

$$\frac{\|r_1(x - x_0)\|}{\|x - x_0\|} \to 0 \text{ as } \|x - x_0\| \to 0$$

and

$$\frac{\|r(y - y_0)\|}{\|y - y_0\|} \to 0 \text{ as } \|y - y_0\| \to 0.$$

This implies that $f'(y_0)g'(x_0)$ is the Fréchet derivative of composition $f(g(x_0))$ and can be written in the form of chain rule as:

$$(f \circ g)'(x_0) = f'(g(x_0))(g'(x_0)).$$

□

**Remark 2.3.12.** The theorem on differentiation of composite function is usually invalid for the Gâteaux derivative.

We illustrate it by the following example:

**Example 2.3.13.** [2] Let $f : \mathbb{R} \to \mathbb{R}^2$ and $f(x) = (x, x^2)$; $g : \mathbb{R}^2 \to \mathbb{R}$ with

$$g(x,y) = \begin{cases} x & x = y^2, \\ 0 & otherwise. \end{cases}$$

By calculating the composition $g \circ f$ on $x$ we get:

$$(g \circ f)(x) = x,$$

and its Gâteaux differential on 0 is

$$(g \circ f)'(0) = 1.$$

Therefore $f(0) = (0,0) = 0$; $g'(0)$ and $f'(0)$ can be calculated as

$$\begin{aligned} g'(0) &= \lim_{t\to 0} \frac{g(th, th)}{t} = \lim_{t\to 0} \frac{0}{t} = 0 \\ f'(0) &= \lim_{t\to 0} \frac{f(th)}{t} = \lim_{t\to 0} \left(\frac{th}{t}, \frac{t^2h^2}{t}\right) \\ &= \lim_{t\to 0} (h, th^2) = (h, 0). \end{aligned}$$

So that we get the differential of $(g \circ f)(x)$:

$$g'(f(0)).f'(0) = 0.(h,0) = 0.$$

This implies that

$$(g \circ f)'(0) \neq g'(f(0)).f'(0).$$

By the above discussion we also reveal another intrusting fact that a function which is Fr$\acute{e}$chet differentiable at a point, is continuous there. This is not the case for G$\hat{a}$teaux differentiable functions even in finite dimensions. It can be seen in the following example.

**Example 2.3.14.** [2]

The function $f : R^2 \to R$ defined by

$$f(x,y) = \begin{cases} 0 & (x,y) = (0,0), \\ \frac{x^4 y}{x^6+y^3} & x^2+y^2 > 0. \end{cases}$$

The G$\hat{a}$teaux derivative of $f$ at origin is:

$$\begin{aligned} df(0)(x) &= \lim_{t\to 0} \frac{f(0+tx)-f(0)}{t} \\ &= \lim_{t\to 0} \frac{f(tx)}{t}. \end{aligned} \tag{2.19}$$

By applying (2.19) on $f(x,y)$, we get

$$\begin{aligned} df(0,0)(x) &= \lim_{t\to 0} \frac{t^5 x_1^4 x_2}{t(t^6 x_1 + t^3 x_2)} \\ &= \lim_{t\to 0} \frac{t^5 x_1^4 x_2}{t^4(t^3 x_1 + x_2)} \\ &= \frac{0}{0+1} = 0. \end{aligned}$$

But fails to be continuous at origin:

Since the limit of $f(x)$ at origin can be calculated as

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^4 y}{x^6+y^3}. \tag{2.20}$$

Now by considering the path $y = mx$, (2.20) can be rewritten as:

$$\begin{aligned} \lim_{(x,y)\to(0,0)} f(x,y) &= \lim_{(x,y)\to(0,0)} \frac{mx^5}{x^6+mx^3} \\ &= \lim_{(x,y)\to(0,0)} \frac{m}{x+\frac{m^3}{x^2}} \\ &= \frac{m}{0+1} \\ &= m. \end{aligned}$$

It can be seen that the limit of $f(x)$ at $(0,0)$ varies according to $m$, so the limit of $f(x)$ does not exist at $(0,0)$. This implies that $f(x)$ is not continuous at origin.

**Theorem 2.3.15** ( [12], [17])**.** If $f : D \to F$ be a function which is Fréchet differentiable at $x_0 \in D$, then it is Gâteaux differentiable at $x_0$ and the Gâteaux derivative coincide with the Fréchet derivative.

*Proof.* By putting $tx$ in place of $x$ in (2.9) we get:

$$\varphi(x_0 + tx) - \varphi(x_0) = \varphi'(x_0)(tx) + r(tx).$$

It follows that

$$\varphi'(x_0)(x) = \lim_{t \to 0} \frac{\varphi(x_0 + tx) - \varphi(x_0)}{t},$$

since $\frac{\|r(tx)\|}{\|tx\|} \to 0$ as $t \to 0$. This implies that $\varphi'(x_0)$ is Gâteaux derivative of $f(x)$ at $x_0$. □

**Remark 2.3.16.** [17] Note that $\varphi$ is Fréchet differentiable at $x_0$ if it is Gâteaux differentiable there and if the limit in (2.7) exists *uniformly* for $\|x\| \leq 1$ as $t \downarrow 0$.

So we can say that every Fréchet differentiable function is Gâteaux differentiable, but the converse is not true. We illustrate it by the following examples.

**Example 2.3.17.** [12] Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function defined by:

$$f(x,y) = \begin{cases} \frac{x^3 y}{x^4 + y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases}$$

Then $f$ is Gâteaux differentiable at $x_0 = 0$ but it is not Fréchet differentiable at $x_0 = 0$.

*Proof.* Gâteaux derivative at $x_0 = 0$ gives:

$$\begin{aligned} df(0)(x) &= \lim_{t\to 0} \frac{f(tx,ty)}{t} \\ &= \lim_{t\to 0} \frac{t^4(x^3y)}{t^5x^4 + t^3y^2} \\ &= \lim_{t\to 0} \frac{t(x^3y)}{t^2x^4 + y^2} \\ &= \frac{0}{y^2} = 0. \end{aligned}$$

But $f$ is not Fréchet differentiable at $x_0 = 0$. Since

$$\frac{\|f(x,y)\|}{\|(x,y)\|} = \left|\frac{x^3y}{x^4+y^2}\right| \cdot \frac{1}{\sqrt{x^2+y^2}}.$$

If we move along the path $y = x^2$, we see that:

$$\frac{\|f(x,y)\|}{\|(x,y)\|} = \frac{1}{2\sqrt{1+x^2}} \to \frac{1}{2}, \text{ as } x \to 0.$$

□

**Example 2.3.18.** [17] Canonical norm of $l_1$ is nowhere Fréchet differentiable and is Gâteaux differentiable at $x = (x_i)$ iff $x_i \neq 0$ for every $i$ $(1 \geq i \geq n < \infty)$.

*Proof.* If $x \in l_1$ and $x_i = 0$ for some $i$, let

$$\delta_i = (0, 0, \ldots, 0, 1, 0. \ldots),$$

be the sequence whose only nonzero term is 1 in the $i - th$ place. It follows that

$$\|x + t\delta_i\|_1 - \|x\|_1 = |t|,$$

then by dividing both sides by $t$ and taking limit $t \to 0$, so the (two sided) limit does not exist.

Now suppose on the other hand, for every $i$, $x_i \neq 0$ that $\epsilon > 0$ and $y \in l_1$ we can choose $N > 0$ such that

$$\sum_{i>N} |y_i| < \frac{\epsilon}{2},$$

therefore $y = (y_1, y_2, \ldots, y_N, y_{N+1}, y_{N+2}, \ldots) \in l_1$ so $\sum_i |y_i|$ converges.

Now for sufficiently small $\delta > 0$

$$sgn(x_i + ty_i) = sgnx_i, \qquad \text{if } 1 \leq i \leq N; \qquad |t| < \delta.$$

Therefore $sgn(x)$ is defined as:

$$sgn(x) = \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ 1 & x < 0 \end{cases}$$

as $x_i \neq 0$ for all $i$, it may be greater then 0 or less then 0, thus the small increment or decrement does not change the value of $sgn(x_i)$.

Consequently,

$$\begin{aligned} \left| \frac{\|x+ty\|_1 - \|x\|_1}{t} - \sum y_i sgn(x_i) \right| &\leq \left| \sum_{i=1}^{N} \left( \frac{|x_i + ty_i| - |x_i| - ty_i sgnx_i}{t} \right) \right| + 2 \sum_{i>N} |y_i| \\ &\leq \left| \sum_{i=1}^{N} \left( \frac{|x_i| + |ty_i| - |x_i| - ty_i|1|}{t} \right) \right| + 2\frac{\epsilon}{2} \\ &< \epsilon, \end{aligned}$$

provided $|t| < \delta$. □

As an immediate corollary, we see that Gâteaux differentiability and Fréchet differentiability coincide for locally Lipschitz functions on $E$ when it is finite dimentional. This can be seen in the following result, which has been reviewed from [2], [15], [18] and [11].

**Corollary 2.3.19.** [17] Let $f$ be a locally Lipschitz function on an open subset $D$ of a finite dimensional normed linear space $E$, if $f$ is Gâteaux differentiable at $x$ then it is Fréchet differentiable at $x$ as well.

*Proof.* Consider the unit sphere $S$ in $E$ and for any given $\epsilon > 0$ consider a cover by open balls having centers $y_k \in X$ such that $\|y_k\| = 1$ with radius $\epsilon$. Since $X$ is finite dimensional, it is isomorphic to some $\mathbb{R}^n$ (with standard topology). Then by Heine - Borel theorem the unit sphere $S$ is closed and bounded, which implies that $S$ is compact, so there exists a finite subcover of $S$ by such balls with centers $y_1, y_2, \ldots, y_m$. Since $f$ is Gâteaux differentiable at $x$, given $y_k$, where $k \in \{1, 2, ..., m\}$, there exists a $\delta_k(\epsilon, y_k) > 0$ such that

$$\left| \frac{f(x + ty_k) - f(x)}{t} - f'(x)(y_k) \right| < \epsilon \qquad \forall \qquad 0 < t < \delta_k.$$

Since $f$ is locally Lipschitz there exists a $K > 0$ and a $\sigma(x) > 0$ such that

$$|f(x + ty) - f(x + ty_k)| \leq K|t|\|y - y_k\|$$

for all $y_k$, $k \in 1, 2, ..., m$ and $0 < |t| < \sigma$. Therefore, given $\|y\| = 1$,

$$\begin{aligned} \left| \frac{f(x + ty) - f(x)}{t} - f'(x)(y) \right| &\leq \left| \frac{f(x + ty) - f(x)}{t} - \frac{f(x + ty_k) - f(x)}{t} \right| \\ &\quad + \left| \frac{f(x + ty_k) - f(x)}{t} - f'(x)(y_k) \right| \\ &\quad + |f'(x)(y_k) - f'(x)(y)| \\ &\leq K\|y - y_k\| + \epsilon + \|f'(x)\|\|y_k - y\|, \end{aligned}$$

when $0 < |t| < \min\{\sigma, \delta_k\}$. Since we can write it as:

$$\left| \frac{f(x + ty) - f(x)}{t} - f'(x)(y) \right| < (K + \|f'(x)\| + 1)\epsilon, \tag{2.21}$$

for $y_k$ chosen such that $\|y - y_k\| < \epsilon$. Hence (2.21) holds for all $\|y\| = 1$ where $0 < |t| < min\{\sigma, \delta_1, ..., \delta_m\}$. That is, $f$ is Fréchet differentiable at $x$. □

**Theorem 2.3.20.** [17] If $f$ is a convex function on an open interval $D \subset \mathbb{R}$, then $f'(x)$ exists for all but (at most) countably many points of $D$.

*Proof.* As we have proved the Theorem 1.2.8, in the same manner first we will show that $d_+f(x_0)(x)$ is a nondecreasing function of $x$. Without loss of generality, we may assume that $x = 1$, i.e, $d_+f(x_0)(1)$. For simplicity, we will write $d_+f(x_0)$. Then for $x_1, x_2 \in D$, suppose that $x_1 < x_2$, we want $d_+f(x_1) \leq d_+f(x_2)$. We have shown the monotonicity of convex function $f$ in the proof of Lemma 2.3.2, that is for $x_1 < x_2$, we have $f(x_1) \leq f(x_2)$. Now for $t > 0$ we may assume

$$x_2 = x_1 + 2t. \tag{2.22}$$

Therefore,

$$\begin{aligned} x_2 &= \frac{2x_2 - 2t + 2t}{2} \\ &= \frac{x_2 - 2t + x_2 + 2t}{2}. \end{aligned} \tag{2.23}$$

By applying convexity on (2.23) we get:

$$\begin{aligned} f(x_2) &\leq \frac{1}{2}f(x_2 - 2t) + \frac{1}{2}f(x_2 + 2t) \\ 2f(x_2) &\leq f(x_2 - 2t) + f(x_2 + 2t) \\ f(x_2) - f(x_2 - 2t) &\leq f(x_2 + 2t) - f(x_2) \\ \frac{f(x_2) - f(x_2 - 2t)}{2t} &\leq \frac{f(x_2 + 2t) - f(x_2)}{2t}. \end{aligned}$$

Hence by (2.22) $x_2 = x_1 + 2t$ and $x_1 = x_2 - 2t$, by putting these values in above expression,

$$\frac{f(x_1 + 2t) - f(x_1)}{2t} \leq \frac{f(x_2 + 2t) - f(x_2)}{2t}.$$

By applying limit $t \downarrow 0$, we get:

$$d_+f(x_1) \leq d_+f(x_2).$$

A geometrical proof of the monotonicity of $d_+f$ can be obtained from the Figure 2.7. It is clear in the the graph of $f$ the various cords naming $PQ$, $QR$, etc., have

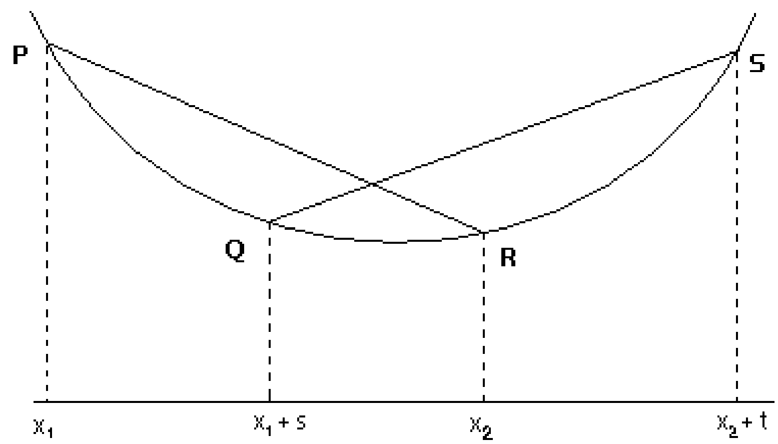


Figure 2.7

some relation which is

$$slopePQ \leq slopePR \leq slopeQR \leq slopeQS \leq slopeRS.$$

Expressing the first and last of these in terms of $f$, we see that (for any $s > 0$ such that $x_1 + s < x_2$, and any $t > 0$)

$$\frac{f(x_1 + s) - f(x_1)}{s} \leq \frac{f(x_1 + s) - f(x_1)}{t},$$

which shows that $d_+ f(x_1) \leq d_+ f(x_2)$. We next show that any point where $f$ fails to be differentiable is a point where the monotone function $x \to d_+ f(x)$ has a jump. There are, of course, at most countably many such points. Now, if $f'(x_0)$ fails to exist, then

$$-d_+ f(x_0)(-1) < d_+ f(x_0)(1),$$

so it suffices to show that the latter inequality implies that $d_+ f(x)$ has a jump at $x = x_0$, that is,

$$\lim_{x \downarrow x_0} d_+ f(x) \leq \lim_{x \uparrow x_0} d_+ f(x).$$

Since the right side of this expression dominates $d_+f(x_0)$, it suffices to show that the left side is dominated by $-d_+f(x_0)(-1)$, that is, if $x < x_0$, then $d_+f(x_0)(1) < -d_+f(x_0)(-1)$. In view of the monotonicity of the limits which define these two quantities, we only need to show that, letting

$$t_0 = \left(\frac{1}{2}\right)(x_0 - x)$$

we get

$$\frac{[f(x + t_0) - f(x)]}{t_0} \leq \frac{[f(x_0 - t_0) - f(x_0)]}{t_0}.$$

But this is easily seen to be equivalent to the convexity inequality

$$f\left[\frac{1}{2}(x + x_0)\right] \leq \frac{1}{2}[f(x) + f(x_0)]$$

and the proof is complete. □

# Chapter 3

# Quasiconvex Functions

## 3.1 Introduction

A detailed discussion of convex functions has been given in previous chapters, which shows that a convex function of one real variable admits right hand and left hand derivatives at any interior point of its domain. Furthermore it is continuous at any point interior point of its domain. On the other hand a convex function $f$ defined on a real Banach space $E$ is continuous at $x_0 \in E$ if it is bounded in a neighborhood of $x_0$. If, in addition, if $E = \mathbb{R}^n$ and $x$ belongs to the interior of the domain of $f$, then $f$ is continuous at $x_0$, the Gâteaux derivatives $df(x_0)(x)$ of $f$ at $x_0$ with respect to the directions $x$ are well defined. Furthermore, $df(x_0)(x)$ is convex in $x$, and the subdifferential $\partial f(x_0)$ of $f$ at $x_0$ is defined as the closed convex set such that

$$\partial f(x_0) = \{x^* : \langle d, x^* \rangle \leq f'(x_0, x) \text{ for all } x\}.$$

It is important to notice that all these properties are due to the geometrical structures induced by the convexity of $f$. Indeed, the epigraph of $f$ is convex in $E \times \mathbb{R}$.

Let us define the level set of a function $f$, that is for $\lambda \in (-\infty, +\infty)$, we have

$$S_\lambda(f) = \{x : f(x) \leq \lambda\}.$$

Similarly the strict level set is defined by

$$\tilde{S}_\lambda(f) = \{x : f(x) < \lambda\}.$$

Clearly, for $\lambda < \mu$,

$$S_\lambda(f) \subseteq \tilde{S}_\lambda(f) \subseteq S_\mu(f) \subseteq \tilde{S}_\mu(f).$$

It is also easily seen that

$$S_\lambda(f) = \cap_{\mu>\lambda} \tilde{S}_\mu(f) = \cap_{\mu>\lambda} S_\mu(f).$$

The function $f$ can be recovered from its level sets, certainly:

$$f(x) = inf[\lambda : x \in S_\lambda(f)] = inf[\lambda : x \in \tilde{S}_\lambda(f)].$$

**Notation 3.1.1.** Therefore $\overline{\mathbb{R}}$ is the notation for extended real line, which is also can be written as $\mathbb{R} \cup \{-\infty, +\infty\}$.

**Definition 3.1.2.** Let $E$ is a real Banach space and $D \subset E$ is convex.

(i) A function $f : D \to \overline{\mathbb{R}}$ is said to be ***quasiconvex*** if for $x, y \in D$ and $0 < t < 1$ we have

$$f(tx + (1-t)y) \leq max\{f(x), f(y)\}.$$

(ii) A function $f : D \to \mathbb{R}$ is said to be **strictly quasiconvex** if for $x, y \in D$, $x \neq y$ and $0 < t < 1$ we have

$$f(tx + (1-t)y) < max\{f(x), f(y)\}.$$

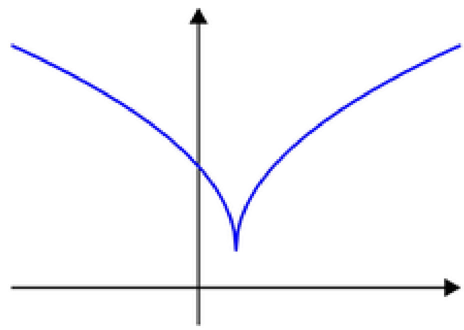

Figure 3.1: A quasiconvex function that is not convex.

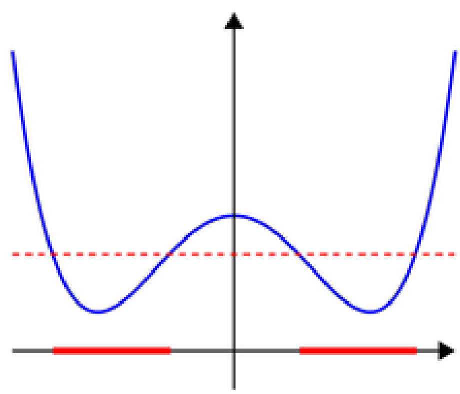

Figure 3.2: A function that is not quasiconvex: the set '$S_\lambda(f)$' of points in the domain of the function for which the function values are below the dashed line is the union of the two intervals, which is not a convex set.

In words, if $f$ is a function such that it does not attain a higher value on a point directly between two other points of the function, than the value of the function on both end points, then $f$ is quasiconvex. See Figures 3.1 and 3.2.

All convex functions are also quasiconvex. But the converse is not always true. So quasiconvexity is a generalization of convexity. By the geometrical structure of quasiconvex functions one can extract that the epigraph of a quasiconvex function may not be a convex set. Although there is an alternative way of defining a quasiconvex function $f$, that is for each $\lambda \in \mathbb{R}$ the level set of $f$

$$S_\lambda = \{x : f(x) \leq \lambda\},$$

is convex. Comprehensively, one can invoke the following relation:

$$f \text{ quasiconvex} \Leftrightarrow S_\lambda(f) \text{ convex } \forall\lambda \in \mathbb{R} \Leftrightarrow \tilde{S}_\lambda(f) \text{ convex } \forall\lambda \in \mathbb{R}.$$

Now we recall some definitions:

**Definition 3.1.3.** The function $f : E \to \overline{\mathbb{R}}$ is said to be ***lower semicontinuous*** at $x_0 \in E$ if for $\epsilon > 0$ there exists a neighborhood $U(x_0)$ of $x_0$ such that $\epsilon < f(x) - f(x_0)$ for all $x \in U(x_0)$.

Or we can say that: the function $f : E \to \overline{\mathbb{R}}$ is said to be ***lower semicontinuous*** at $x_0 \in E$ if for each $k \in \mathbb{R}$, $k < f(x_0)$ there exists a neighborhood $U(x_0)$ of $x_0$ such that

$$f(u) > k \quad \forall\ u\ \in\ U(x_0).$$

**Definition 3.1.4.** The function $f : E \to \overline{\mathbb{R}}$ is said to be ***upper semicontinuous*** at $x_0 \in E$ if for $\epsilon > 0$ there exists a neighborhood $U(x_0)$ of $x_0$ such that $\epsilon > f(x) - f(x_0)$ for all $x \in U(x_0)$.

Or we can say that: the function $f : E \to \overline{\mathbb{R}}$ is said to be ***upper semicontinuous*** at $x_0 \in E$ if for each $k \in \mathbb{R}$, $k > f(x_0)$ there exists a neighborhood $U(x_0)$ of $x_0$ such that

$$f(u) < k \text{ for all } u \in U(x_0).$$

Therefore we have:

$$f \text{ lower semi-continuous} \Leftrightarrow S_\lambda(f) \text{ closed } \forall\ \lambda\ \in\ \mathbb{R}$$

and

$$f \text{ upper semi-continuous} \Leftrightarrow S_\lambda(f) \text{ open } \forall\ \lambda\ \in\ \mathbb{R}.$$

## 3.2 Quasiconvexity and Monotonicity

Let us first consider quasiconvex functions of one real variable. On the one hand nondecreasing and nonincreasing functions are quasiconvex and on the other hand the domain of a quasiconvex function can be partitioned in two intervals, the function being nonincreasing on the first and nondecreasing on the second.

**Remark 3.2.1.** [8] Let $f : I \to \overline{\mathbb{R}}$, therefore $I \subseteq \mathbb{R}$. Then $f$ is quasiconvex if and only if there exists $x_0 \in \overline{\mathbb{R}}$ so that:

1. Either $f$ is nonincreasing on $(-\infty, x_0] \cap I$ and nondecreasing on $(x_0, +\infty) \cap I$.

2. Or $f$ is nonincreasing on $(-\infty, x_0) \cap I$ and nondecreasing on $[x_0, +\infty) \cap I$.

Thus,the nondecreasing functions of one real variable are the simplest examples of quasiconvex functions. It results that, unlike convex functions, quasiconvex functions are not continuous in the interior of their domain. By inference, the directional derivatives are not necessarily defined. Still, nondecreasing functions of one real variable are almost everywhere continuous and differentiable, hence quasiconvex functions of one real variable are also almost everywhere continuous and differentiable on the interior of their domains.

Quasiconvexity of the functions of several variables is also connected with monotonicity. Let $f : E \to \overline{\mathbb{R}}$ be quasiconvex function and $K$ be a convex cone containing in $E$, then $f$ is said to be nondecreasing with respect to $K$ if

$$x, y \in E, y - x \in K \Rightarrow f(x) \leq f(y).$$

**Theorem 3.2.2** ( [6], [5])**.** Let $f : E \to \overline{\mathbb{R}}$, $f$ is quasiconvex, $\lambda \in \mathbb{R}$ and $a \in E$ such that $int(S_\lambda(f)) \neq \emptyset$ and $a$ does not belong to $cl(S_\lambda(f))$. Then there exists an open convex neighbourhood V of $a$ and a nonempty open convex cone $K$ so that

$$x, y \in V, y - x \in K \Rightarrow f(x) \leq f(y)$$

Furthermore, if $f$ is strictly quasiconvex

$$x, y \in V, y - x \in K, x \neq y \Rightarrow f(x) < f(y).$$

*Proof.* Let $b \in int(S_\lambda(f))$, $r > 0$ and $R > 0$ be such that $B(b, r) \subseteq S_\lambda(f)$ and $S_\lambda(f) \cap B(a, R) = \emptyset$. Let some $\alpha > 0$. Set

$$c = a + \alpha(a - b)$$

and

$$K = \{d : c - td \in B(b, r) \text{ for some } t > 0\}.$$

Then $K$ is a nonempty open convex cone. Hence $y - K \subseteq c - K$ for all $y \in c - K$. Set

$$V = (c - K) \cap B(a, R).$$

Take $x, y \in V$ with $y - x \in K$. Then there exists $t > 1$ such that

$$z = y + t(x - y) \in B(b, r). \tag{3.1}$$

Notice that $f(z) \le \lambda < f(y)$. By using quasiconvexity of $f$ and (3.1), we get:

$$x = \left(\frac{t-1}{t}\right) y + \left(\frac{1}{t}\right) z,$$

$$f(x) \le f(y).$$

Second result follows from the strict quasiconvexity of the function. □

Particularly for $E = \mathbb{R}^n$, we have the following result.

**Corollary 3.2.3** ( [6], [5]). Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a quasiconvex function, $\lambda \in \mathbb{R}$ and $a \in \mathbb{R}^n$ such that $int(S_\lambda(f)) \neq \emptyset$ and $a$ does not belongs to $cl(S_\lambda(f))$. Then there exist an open convex neighborhood $V$ of $a$ and $v_1, v_2, \cdots, v_n$ are $n$ linearly independent vectors such that

$$x, x + \sum t_i v_i \in V, t_1, t_2, \cdots, t_n \ge 0 \Rightarrow f(x) \le f(x + \sum t_i v_i).$$

Furthermore, if $f$ is strictly quasiconvex and $\sum t_i > 0$, then the inequality is strict.

*Proof.* Choose for vectors $v_i$ $n$ linearly independent vectors in $K$. □

In $\mathbb{R}^n$, locally Lipschitz functions are also strongly connected to monotonicity. Let us take $f$ to be locally Lipschitz function in a neighborhood of $x$, i.e., there exist $a > 0$ and $L > 0$ such that

$$y_i, z_i \in [x_i - a, x_i + a] \text{ for all } i,$$

which implies that

$$|f(z) - f(y)| \le L \sum |zi - yi|.$$

Let $g(x) = f(x) + L\sum y_i$. Then

$$x_i - a \leq y_i \leq z_i \leq x_i + a$$

for all $i$. This implies that

$$g(x) \leq g(y).$$

## 3.3 Continuity of Quasiconvex Functions

Let $f : E \to \overline{\mathbb{R}}$, $a \in E$ with $f(a)$ to be finite valued, that is

$$-\infty < f(a) < +\infty.$$

We can define the function of one real variable in direction $d \in E$ such that

$$f_{a,d}(t) = f(a + td).$$

Hence the first result is regarding to nondecreasing functions.

**Proposition 3.3.1** ( [6], [9])**.** Let $f(a)$ is finite, $K$ is a convex cone with nonempty interior, $f$ is nondecreasing with respect to $K$ and $d \in int(K)$. Then $f$ is lower semicontinuous (upper semicontinuous) at $a$ if and only if $f_{a,d}$ is lower semicontinuous (upper semicontinuous) at 0.

*Proof.* Take $f_{a,d}$ is lower semicontinuous at 0, which implies that for each $\alpha \in \mathbb{R}$

$$\alpha < f_{a,d}(0) = f(a + 0d) = f(a).$$

Then there exists a $t_- < 0$ such that $\alpha < f(a + td)$ for all $t \geq t_-$. Take $V = (a + t_- d) + K$. Where V is a neighborhood of $a$. Since $f$ is nondecreasing with respect to $K$ we have

$$(a + t_- d),\ x \in V,\ x - (a + t_- d) \in K \Rightarrow f(a + t_- d) \leq f(x)$$

for all $x \in V$. This implies that $\alpha < f(x)$, hence $f$ is lower semicontinuous at $a$.

Now take $f_{a,d}$ is upper semicontinuous at 0. which implies that for each $\beta \in \mathbb{R}$

$$\beta > f_{a,d}(0) = f(a + 0d) = f(a).$$

Then there exists a $t_+ > 0$ such that $\beta > f(a + td)$ for all $t \leq t_+$. Take $V = (a + t_+d) - K$. Where V is a neighborhood of $a$. Since $f$ is nondecreasing with respect to $K$ we have

$$(a + t_-d),\ x \in V,\ (a + t_-d) - x \in K \Rightarrow f(a + t_+d) \geq f(x)$$

for all $x \in V$. This implies that $\beta > f(x)$, hence $f$ is upper semicontinuous at $a$. □

By Theorem 3.2.2 we can say that quasiconvex functions can be considered locally nondecreasing with respect to some open convex cone K. Hence above proposition can be applied. However, a stronger result holds. Let $f$ be a quasiconvex function and $f(a)$ is finite valued. We can define a cone

$$\tilde{K}(a) = \{d : f(a + td) < f(a) \text{ for some } t > 0\},$$

which is convex. Its interior is nonempty as soon as $int(\tilde{S}_{f(a)}(f)) \neq \emptyset$. Notice that $\tilde{K}(a)$ contains the cones $K$ of Theorem 3.2.2. Then, we have:

**Proposition 3.3.2.** [6] Let $f$ is a quasiconvex function, $f(a)$ is finite and $d \in int(\tilde{K}(a))$. Then $f$ is lower semicontinuous (upper semicontinuous) at $a$ if and only if $f_{a,d}$ is lower semicontinuous (upper semicontinuous) at 0.

*Proof.* If $int(\tilde{K}(a) = \emptyset$, there is nothing to prove. But if $int(\tilde{K}(a)) \neq \emptyset$ then take $f_{a,d}$ is lower semicontinuous at 0, which implies that for each $\alpha \in \mathbb{R}$

$$\alpha < f_{a,d}(0) = f(a + 0d) = f(a).$$

Then there exists a $t_- < 0$ such that $\alpha < f(a+td)$ for all $t \geq t_-$. Let $\tilde{S}_{f(a)}(f) = \{x : f(x) < f(a)\}$, hence $int(\tilde{S}_{f(a)}(f)) \neq \emptyset$ and $a$ does not belong to $cl(S_{f(a)}(f))$. Now take $V = (a + t_-d) + K$. Where V is a neighborhood of $a$. Thus by applying Theorem 3.2.2 we get:

$$(a + t_-d),\ x \in V,\ x - (a + t_-d) \in \tilde{K}(a) \Rightarrow f(a + t_-d) \leq f(x)$$

for all $x \in V$. This implies that $\alpha < f(x)$, hence $f$ is lower semicontinuous at $a$.

Now take $f_{a,d}$ is upper semicontinuous at 0. which implies that for each $\beta \in \mathbb{R}$

$$\beta > f_{a,d}(0) = f(a + 0d) = f(a).$$

Then there exists a $t_+ > 0$ such that $\beta > f(a+td)$ for all $t \leq t_+$. Take $V = (a + t_+d) - K$. Again by applying Theorem 3.2.2 we get:

$$(a + t_-d),\ x \in V,\ (a + t_-d) - x \in \tilde{K}(a) \Rightarrow f(a + t_+d) \geq f(x)$$

for all $x \in V$. This implies that $\beta > f(x)$, hence $f$ is upper semicontinuous at $a$.

□

In Propositions 3.3.1 and 3.3.2, the continuity has been considered only in one direction $d$ which belongs to a specific cone. Next result associates with the continuity in all directions since it is weaker, but because of its very simple formulation it deserves to be stated.

**Theorem 3.3.3** ( [7], [11]). Assume that $f$ is quasiconvex on $\mathbb{R}^n$ and $f(a)$ is finite. Then $f$ is lower semicontinuous (upper semicontinuous) at $a$ if and only if, for all $d \in \mathbb{R}^n$ , the function $f_{a,d}$ is lower semicontinuous (upper semicontinuous) at 0.

*Proof.* Let $f_{a,d}$ is lower semicontinuous at 0 for all $d \in E$. We have to prove that $f$ is lower semicontinuous at $a$. If $\tilde{S}_{f(a)}(f) = \emptyset$ there is nothing to prove. Otherwise we can take some $d$ in the relative interior of $\tilde{K}(a)$ and the proof can be obtain by adapting the proof of the Proposition 3.3.1. Now, take $f_{a,d}$ is upper semicontinuous at 0 for all $d$. Let $\lambda > f(a)$. Take $d = e_i$ be the $i - th$ vector of the canonical basis of $\mathbb{R}^n$. There is $t_i > 0$ such that $f(a + td_i) < \lambda$ for all $t \in [-t_i, t_i]$. Take the convex hull of the $2n$ points $a \pm t_i d_i$ for $V$, which is a neigborhood of $a$ and $V \subseteq S_\lambda(f)$. □

It can be extract from the above result that, a quasiconvex function in $\mathbb{R}^n$ is continuous at a point $x$ if it is continuous along all the lines at $x$. This result is not true for nondecreasing functions. Also, it does not hold for an infinite dimensional Banach space.

## 3.4 Differentiability of Quasiconvex Functions

Let $f(x_0)$ be a finite valued function and $x \in E$. Then the upper and the lower Dini-derivative of $f$ at $x_0$ in direction $x$ are respectively, defined by

$$f'_+(x_0, x) = \lim_{t \to 0_+} \sup \frac{f(x_0 + tx) - f(x_0)}{t},$$

$$f'_-(x_0, x) = \lim_{t \to 0_+} \inf \frac{f(x_0 + tx) - f(x_0)}{t}.$$

If $-\infty < f'_-(x_0, x) = f'_+(x_0, x) < +\infty$ then the directional derivative of $f$ with respect to the direction $x$ exists and is defined by

$$f'(x_0, x) = f'_-(x_0, x) = f'_+(x_0, x).$$

**Theorem 3.4.1.** [5] Let $f : X \to \overline{\mathbb{R}}$, $K$ be a nonempty open convex cone. If $f$ is nondecreasing with respect to $K$ and $f$ is Gâteaux differentiable at $x$. Then $f$ is Fré́chet differentiable at $x$.

*Proof.* Take $f$ to be Gâteaux differentiable but not Fréchet differentiable at $x$. Then for some $\epsilon > 0$ and sequence $\{h_n\}_n$ converges to 0 for all $n$ such that:

$$\epsilon < \frac{|f(x+h_n) - f(x) - df(x)(h_n)}{\|h_n\|}. \tag{3.2}$$

Set $t_n = \frac{1}{\|h_n\|}h_n$. Without loss of generality, we can assume that all the sequence $\{t_n\}_n$ converges to some $\bar{t}$. Let $e \in int(K)$. Then $\mu > 0$ exists so that

$$df(x)(t - \bar{t}) < \epsilon \quad \forall\, t \ \in \ V, \tag{3.3}$$

where

$$V = \{t : \bar{t} - \mu e = t_- \leq t \leq t_+ = \bar{t} + \mu e\}.$$

Then $V$ is a neighborhood of $\bar{t}$. For $n$ large enough, $t_n \in V$ and therefore

$$f(x + \|h_n\|t_-) - f(x) \leq f(x+h_n) - f(x) \leq f(x + \|h_n\|t_+) - f(x). \tag{3.4}$$

Since $f$ is Gâteaux differentiable at $x$, for $n$ large enough

$$\frac{|f(x+\|h_n\|t_-) - f(x) - df(x)(\|h_n\|t_-)}{\|h_n\|} < \epsilon, \tag{3.5}$$

and

$$\frac{|f(x+\|h_n\|t_+) - f(x) - df(x)(\|h_n\|t_+)}{\|h_n\|} < \epsilon. \tag{3.6}$$

Now by doing some algebraic operations with (3.3) and (3.4), we get

$$\frac{|f(x+\|h_n\|t_-) - f(x) - df(x)(\|h_n\|t_-)}{\|h_n\|} \leq \frac{|f(x+h_n) - f(x) - df(x)(h_n)}{\|h_n\|}$$
$$\leq \frac{|f(x+\|h_n\|t_+) - f(x) - df(x)(\|h_n\|t_+)}{\|h_n\|}.$$

The contradiction occurs from Equations (3.2), (3.5) and (3.6). □

Since the above Theorem can be applied to quasiconvex functions.

**Theorem 3.4.2.** [6] Let $f : X \to \overline{\mathbb{R}}$ be a quasiconvex function. If $f$ is Gâteaux differentiable at $x$, then $f$ is Fréchet differentiable at $x$ as well.

*Proof.* Take $B \subset X$ such that $B = \{y : f(y) < f(x), y \in X\}$.

CASE 1: If $int(B) \neq \emptyset$, there is $\lambda \in \mathbb{R}$ such that $int(S_\lambda(f)) \neq \emptyset$ and $x$ does not belong to $cl(S_\lambda(f))$. Then by applying Theorems 3.2.2 and 3.4.1 we can obtain the proof.

CASE 2: If $B = \emptyset$, then $f(y) \geq f(x)$ for all $y \in X$. Since $df(x) = 0$. Let $(e_1, e_2, \cdots, e_n)$ be the canonical basis of $X$. By setting $e_{i+n} = -e_i$ for all $i = 1, \ldots, n$. Now for $h \in X$ we can take $r_i = max[0, h_i]$ and $r_{i+n} = max[0, -h_i]$. Then

$$x + h = x + \sum_{i=1}^{i=2n} r_i e_i = \sum_{i=1}^{i=2n} \left( x + \frac{r_i}{\|h\|} \|h\| e_i \right)$$

where

$$\|h\| = \sum_{i=1}^{i=n} |h_i| = \sum_{i=1}^{i=2n} r_i.$$

Then, since $f$ is quasiconvex

$$0 \leq \frac{f(x + th) - f(x)}{\|h\|} \leq max_{i=1,\cdots,2n} \frac{f(x + \|h\| e_i) - f(x)}{\|h\|}$$

and the result follows again.

CASE 3: If $int(B) = \emptyset$ but $B \neq \emptyset$. Here again $df(x) = 0$. We can obtain the proof by working on the affine set generated by $B$ and using the same proof as above. □

# Conclusion

In this thesis we have discussed the continuity and differentiability of convex functions. In [17] and [18], it has been shown that a convex function of one real variable continuous and admits left hand and right hand derivatives at any interior point of its domain. On the other hand a convex function $f$ defined on a real Banach space $E$ is continuous at any point in $E$ if it is bounded in the neighborhood of that point.

In the differentiability of convex functions defined on real Banach spaces we have studied the Fr$\acute{e}$chet and G$\hat{a}$teaux derivatives from [13], [14], [17] and [18], the Fr$\acute{e}$chet derivative is defined on Banach spaces is the generalization of the concept of total derivative and the G$\hat{a}$teaux derivative is the generalization of the concept of directional derivative in differential calculus.

This thesis also contains a comparative study of the G$\hat{a}$teaux and Fr$\acute{e}$chet derivatives and their algebraic properties also has been discussed.

Let $f$ be a function defined as $f : D \to F$ where $D \subseteq E$, here $E$ and $F$ are real Banach spaces. Generally we know that if $f$ is Fr$\acute{e}$chet differentiable then it is G$\hat{a}$teaux differentiable which has been discussed in [2], [3], [16] and [17], but the reciprocal is not true in general as seen in some examples shown in [1], [2], [16] and [19]. So we review that what conditions do we impose on convex functions and quasiconvex functions in order to guarantee the coincidence of both differentials.

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