Lorentzian Wormholes in Minimally and Non-minimally Coupled $f(R)$ Modified Gravity

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بسم الله الرحمن الرحيم

In the name of Allah, the most beneficent, the most merciful.

This Dissertation is dedicated to

my mother who brought me in this world and to my father who taught me how to deal with it and to my siblings who always uplifted my morale whenever I needed!

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Abstract

Wormholes are hypothetical tunnel in spacetime, possibly through which observers may freely traverse. In classical general relativity, wormholes are supported by exotic matter, which involves a stress-energy tensor that violates the null energy condition (NEC). The NEC forbids the existence, within general relativity, of throats in space, both static and time dependent. Such a throat could join asymptotically flat regions of space, forming a Lorentzian wormhole. Alternatively, it could serve as a bridge between a large but finite region of space and asymptotically flat region. $f(R)$ gravity modifies and generalizes the Einstein's general relativity by introducing a new function of Ricci scalar. In this dissertation, traversable wormhole geometries in the context of $f(R)$ modified gravity is constructed by imposing the condition that matter threading wormhole satisfies the energy conditions. The possibility that wormhole geometries be constructed by considering an explicit coupling between an arbitrary function of the scalar curvature and the Lagrangian density of matter. The coupling between the matter and the higher derivative curvature terms describes an exchange of energy and momentum, which is responsible for supporting the wormhole geometries.

Notations

In this thesis, signature convention used for the metric will be $(-,+,+,+)$ and the following list of acronyms and notations are used:

SR:	Special Relativity
GR:	General Relativity
$f(R)$:	Function of Ricci Scalar
$b(r)$:	Shape Function
$\Phi(r)$:	Redshift Function
$\rho(r)$:	Energy Density
$p_r(r)$:	Radial Pressure
$p_t(r)$:	Transverse Pressure
$g_{\mu\nu}$:	Metric Tensor
$T_{\mu\nu}$:	Stress-Energy Tensor
$G_{\mu\nu}$:	Einstein Tensor
$R_{\mu\nu}$:	Ricci Tensor
R:	Ricci Scalar
G:	Gravitational Constant
c:	Speed of Light
NEC:	Null Energy Condition
WEC:	Weak Energy Condition
SEC:	Strong Energy Condition
EFEs:	Einstein's Field Equations

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Introduction

Relativity is about the consequences for the laws of physics that follow from the principle that it is impossible to detect a state of uniform motion of an isolated system. It is only the relative motion of two systems that is observable. This is the Principle of Relativity. Galileo Galilei (1564 - 1642) was the first who published a formulation of a principle of relativity.

A hundred years ago physicists were tackling the unsolved problems of classical physics. A major one concerned the reconciliation of the foundations of classical mechanics with the very successful theory of electromagnetism developed by James Maxwell. Among many distinguished physicists involved in this endeavor, the names George Fitzgerald, Joseph Larmor, Henri Poincare, Hendrik Lorentz are prominent. But it took the genius of Einstein to bring simplicity where previously there had been strained hypothesis. He modified classical mechanics in a manner which was to be essential to the development of the modern physics. Thus was Special Relativity (SR) born.

Einstein solved the difficulties of reconciling electromagnetism and Galilean relativity with a new principle of relativity which later became known as SR. The effects the theory describes are particularly important for bodies moving with speeds that are a significant fraction of the speed of light. Such speeds are sometimes labeled as relativistic. Before SR, the theory of light was based on the presumed existence of the ether, the medium classical physicists needed to support electromagnetic waves. Experiments designed to detect the existence of this ether had failed. Einstein's relativity explained this failure and solved the problems associated with electromagnetism, without requiring an ether of any kind.

The origins of General Relativity (GR) can be traced to the conceptual revolution that follows Einstein's introduction of SR in 1905. In GR the rigid spacetime structure of the special theory of relativity is generalized. It is a physical theory which links gravity, space and time. GR has been accurately tested in the solar system. It underlies our understanding of the universe on the largest distance scales and is central to the explanation of such frontier astrophysical phenomena as gravitational collapse, black holes, X-ray sources, neutron stars, active galactic nuclei, gravitational waves and the big bang. GR is the intellectual origin of many ideas in contemporary elementary particle physics and is a necessary requirement to understanding theories of the unification of all forces. It is also concerned with the minute departures of the orbits of the planets from the laws of Newton and is a necessary ingredient in the operation of the Global Positioning System (GPS) used every day [1].

In 1935, physicists Albert Einstein and Nathan Rosen used the theory of GR to propose the existence of "bridges" through spacetime. These paths are called Einstein-Rosen bridges or wormholes. Wormholes are solutions to the Einstein Field Equations (EFEs) for gravity that act as "tunnels" connecting points in spacetime in such a way that the trip between the points through the wormholes could take much less time than the trip through normal space [2]. Wormholes have excited interest amongst the general public and researchers alike. For general public, they provide excellent fodder for imagination by allowing for faster than light travel, time machines and gateways to other universes. For academia, they provide interesting tests for general relativity, cosmology, quantum gravity etc.

Researchers have no observational evidence for wormholes, but the equations of the theory of GR have valid solutions that contain wormholes. Because of its robust theoretical strength, a wormhole is one of the great physics metaphors for teaching general relativity. The first type of wormhole solution discovered was the Schwarzschild wormhole, which would be present in the Schwarzschild metric describing black hole, but it was found that this type of wormhole would collapse too quickly for anything to cross from one end to the other. Wormholes which could actually be crossed in both directions, known as traversable wormholes, would only be possible if exotic matter with negative energy density could be used to stabilize them [2].

Cosmology is said to be thriving in a golden age, where a central theme is the perplexing fact that the Universe is undergoing an accelerating expansion. The late-time cosmic accelerated expansion is one of the most important and challenging current problems in cosmology, and represents a new imbalance in the governing gravitational equations. $f(R)$ gravity is a type of modified gravity which generalizes Einstein's general relativity and it was first proposed in 1970 by Buchdahl. It is actually a family of models, each one defined by a different function of the Ricci scalar. A renaissance of $f(R)$ modified theories of gravity has been verified in an attempt to explain the late-time accelerated expansion of the Universe [3]. There has been a recent stimulus in the study of alternative theories of gravity lately, mostly triggered from combined motivation coming from cosmology/astrophysics and high energy physics. Among the proposed theories, one that has attracted much attention is $f(R)$ gravity [4]. This modified gravity is achieved by replacing Ricci scalar R by a general function of $f(R)$ in the Einstein-Hilbert action.

This thesis is organized in the following manner:

Chapter 1 - Review of general relativity

In this chapter, basic knowledge of GR is provided.

Chapter 2 - Basics of traversable wormholes

This chapter starts with a brief concept of wormholes, which then leads to the discussion of criteria for construction and mathematical details of traversable wormhole. Basic concept of energy conditions is also discussed.

Chapter 3 - Wormhole geometries in $f(R)$ theory of gravity

This chapter deals with the geometry and stability of static and stationary wormholes in $f(R)$ gravity. Wormhole geometries in curvature-matter coupling modified gravity are explored. Wormhole solutions are obtained by assuming various forms of equations of state and viable shape functions. The energy conditions are satisfied in the desired range of radial coordinate. In this chapter, two papers titled as " wormhole geometries in $f(R)$ modified theories of gravity [3]" and "wormhole geometries supported by a nonminimal curvature-matter coupling [5]" are reviewed.

Chapter 1

REVIEW OF GENERAL RELATIVITY

In 1916, the physicist Albert Einstein published the geometric theory of gravitation which is known as general theory of relativity (or GR). Currently this theory describes gravitation in modern physics. Basically GR generalizes Newton's law of universal gravitation and SR, by giving a unified description of gravity as a geometric property of space and time, known as **spacetime**, which is a four dimensional set (t, x, y, z) with elements labeled by three dimensions of space and one of time. The individual point in spacetime is called an **event**. The path of a particle is a curve through spacetime, a parameterized one-dimensional set of events, called the **worldline**. The set of all possible vectors traced for a point P in spacetime is said to be the **tangent space** T_P at P. A collection of vectors (objects) that can be linearly added and multiplied together by real numbers is called real **vector space** [6]. For any two vectors U and V and real numbers m and n, we have

$$(m+n)(U+V) = mU + nU + mV + nV.$$

1.1 Tensors

A **tensor** is a generalization of vectors and dual vectors. A tensor T of type (k, l) is a multilinear map from a collection of dual vectors and vectors to R

$$T : \underbrace{T_p^* \times \cdots \times T_p^*}_{(k\ times)} \times \underbrace{T_p \times \cdots \times T_p}_{l\ times} \to \mathbf{R}, \tag{1.1}$$

where " $\times$ " denotes the cartesian product [6]. A vector is a type $(1,0)$ tensor and a dual vector is a type $(0,1)$ tensor which is linear map from vectors to R. A multilinear function is a function which is linear in each of its arguments [7]. For instance,

$$\mathbf{T}(a\omega + b\xi, cU + dV) = a\mathbf{T}(\omega, cU + dV) + b\mathbf{T}(\xi, cU + dV)$$

$$= ac\mathbf{T}(\omega, U) + ad\mathbf{T}(\omega, V) + bc\mathbf{T}(\xi, U) + bd\mathbf{T}(\xi, V), \tag{1.2}$$

where U and V are two vectors, ω and ξ are dual vectors and a, b, c and d are real numbers. The number of indices carried by a tensor is called the **rank**. A tensor with rank r has n^r components in an n-dimensional space. A tensor with zero rank has no indices and a tensor which has only one compon ent known as scalar. A scalar is a type $(0,0)$ tensor. If all the indices are in the subscript position then the tensor is known as **covariant tensor** and a tensor is known as **contravariant tensor**, if the indices are in superscript. A **mixed tensor** is that which carries indices both in subscript and superscript.

Any object A^μ which transform as

$$A'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} A^\nu.$$

contains a contravariant vector [8] or called a **contravariant tensor** of rank one. The transformation law of a covariant vector or **covariant tensor** of rank one is

$$A'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} A_\nu.$$

One can use the abbreviations,

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} \qquad \textit{and} \qquad \partial^\mu \equiv \frac{\partial}{\partial x_\mu}.$$

A tensor of rank two is said to be **symmetric** regarding to two covariant (or two contravariant) indices if on an interchange of the two indices, the components of tensor remain unchanged. For example, the tensor $T^{\mu\nu}$ is symmetric if

$$T^{\mu\nu} = T^{\nu\mu}.$$

A tensor of rank two is said to be **anti-symmetric** regarding to two covariant (or two contravariant) indices if the sign of components of tensor change on interchange of the two indices. For example, the tensor $T^{\mu\nu}$ is symmetric if

$$T^{\mu\nu} = -T^{\nu\mu}.$$

If a second rank tensor $T_{\mu\nu}$ is symmetric in n dimensions then it has $\frac{1}{2}n(n+1)$ independent components and anti-symmetric tensor has $\frac{1}{2}n(n-1)$ independent components. An arbitrary second rank tensor can always be splitted into the sum of its symmetric and anti-symmetric parts i.e.,

$$T_{\mu\nu} = \frac{1}{2}(T_{\mu\nu} + T_{\nu\mu}) + \frac{1}{2}(T_{\mu\nu} - T_{\nu\mu}). \tag{1.3}$$

A notation used to denote the components of the symmetric and antisymmetric part is

$$T_{(\mu\nu)} = \frac{1}{2}(T_{\mu\nu} + T_{\nu\mu}),$$

and

$$T_{[\mu\nu]} = \frac{1}{2}(T_{\mu\nu} - T_{\nu\mu}),$$

respectively.

A general tensor of rank m can be splitted into symmetric and anti-symmetric parts as

$$T_{(\mu,\nu...\lambda)} = \frac{1}{m!}(\text{sum of all permutations of the indices } \mu, \nu...\lambda),$$

and

$$T_{[\mu,\nu...\lambda]} = \frac{1}{m!}(\text{alternating sum over all permutation of the indices } \mu, \nu...\lambda).$$

The **metric tensor** is a second rank tensor which is used to define the inner product between two vectors u and v i.e.,

$$g(u, v) = u \cdot v. \tag{1.4}$$

Its covariant and contravariant components are given by

$$g_{\mu\nu} = g(e_\mu, e_\nu) = e_\mu \cdot e_\nu,$$

and

$$g^{\mu\nu} = g(e^\mu, e^\nu) = e^\mu \cdot e^\nu.$$

The metric and its inverse are symmetric tensors and can be used to raise and lower indices. The mixed components of metric tensor is given by

$$g^{\mu\lambda}\, g_{\lambda\nu} = \delta^{\mu}_{\nu}, \tag{1.5}$$

where δ^{μ}_{ν} is Kronecker delta and given by

$$\begin{aligned} \delta^{\mu}_{\nu} &= 1 \quad \text{for } \mu = \nu \\ &= 0 \quad \text{for } \mu \neq \nu. \end{aligned}$$

Defining the **line element** ds^2 with signature $(-, +, +, +)$ for infinitesimal dx^μ which are components of a vector as:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu.$$

If

$ds^2 < 0$	timelike separation,
$ds^2 = 0$	lightlike or null separation,
$ds^2 > 0$	spacelike separation,

where the set of all points which lies inside the past and future light cones are called **timelike separated**, where as those points which lies on the cone are **null separated** and the set of those points which lies outside from the light cone are called **spacelike separated**. The **light cone** is a set of all those points that are all connected to a single event. The cone is naturally divided into past and future (see Fig. 1.1).

"Connection" relates vectors in the tangent spaces of nearby points. There is a unique connection that can be constructed from the metric called **Christoffel symbol** and given by

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\alpha}(g_{\nu\alpha,\mu} + g_{\mu\alpha,\nu} - g_{\mu\nu,\alpha}), \tag{1.6}$$

where $\alpha, \mu, \nu, \lambda = t, r, \theta, \phi$.

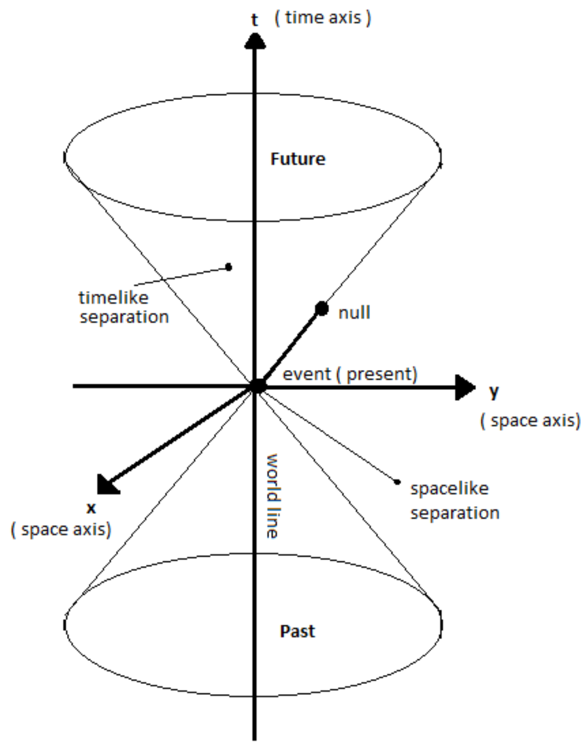

Figure 1.1: A light cone, portrayed on a spacetime diagram.

The fundamental use of a Christoffel symbol is to take a **covariant derivative** ∇_μ. The covariant derivative of a covariant vector B_ν is described as

$$\nabla_\mu B_\nu = B_{\nu;\mu} \equiv \partial_\mu B_\nu - \Gamma^\lambda_{\mu\nu} B_\lambda. \tag{1.7}$$

The covariant derivative of a contravariant vector B^ν is described as

$$\nabla_\mu B^\nu = B^\nu_{;\mu} \equiv \partial_\mu B^\nu + \Gamma^\nu_{\mu\lambda} B^\lambda. \tag{1.8}$$

The covariant derivative of a metric tensor is

$$\nabla_\lambda g_{\mu\nu} = g_{\mu\nu;\lambda} = g_{\mu\nu,\lambda} - \Gamma^\rho_{\mu\lambda} g_{\rho\nu} - \Gamma^\rho_{\nu\lambda} g_{\rho\mu}.$$

Properties of covariant derivative

Some properties of covariant derivative are given as follows [6]

1. It commutes with contractions

$$\nabla_\mu (T^\alpha{}_{\beta\gamma}) = (\nabla T^\alpha{}_{\beta\gamma})_\mu. \tag{1.9}$$

2. It reduces to the partial derivative on scalars $\phi(x^\mu)$ that is

$$\nabla_\mu \phi = \partial_\mu \phi. \tag{1.10}$$

3. Metric compatibility

$$\nabla_\lambda g^{\mu\nu} = 0. \tag{1.11}$$

Spacetime is globally curved by massive objects in the universe. Thus, a coordinate system that satisfies the following conditions at a point P which lies in the four-dimensional spacetime is known as **locally inertial coordinates**, where the Christoffel symbol and derivative of the metric vanish i.e.

$$g_{\mu\nu}(P) = \eta_{\mu\nu}, \qquad \partial_\sigma g_{\mu\nu}(P) = 0, \qquad \Gamma^\lambda_{\mu\nu}(P) = 0. \tag{1.12}$$

1.1.1 The Riemann, Ricci and Einstein Tensor

The connection also appears in the definition of the **Riemann tensor** which is the technical expression of curvature and is given by

$$R^\lambda{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\lambda_{\nu\sigma} - \partial_\nu \Gamma^\lambda_{\mu\sigma} + \Gamma^\lambda_{\mu\rho}\Gamma^\rho_{\nu\sigma} - \Gamma^\lambda_{\nu\rho}\Gamma^\rho_{\mu\sigma}. \tag{1.13}$$

The Riemann tensor can also be written as

$$R_{\lambda\sigma\mu\nu} = g_{\lambda\rho} R^\rho{}_{\sigma\mu\nu}. \tag{1.14}$$

Properties of the Riemann tensor

The Riemann tensor has following properties:

1. It is antisymmetric in the first two indices

$$R_{\lambda\sigma\mu\nu} = -R_{\sigma\lambda\mu\nu}. \tag{1.15}$$

2. It is antisymmetric in last two indices

$$R_{\lambda\sigma\mu\nu} = -R_{\lambda\sigma\nu\mu}. \tag{1.16}$$

3. It is invariant if the first pair of indices is interchanged with the second pair

$$R_{\lambda\sigma\mu\nu} = R_{\mu\nu\lambda\sigma}. \tag{1.17}$$

4. The sum of cyclic permutations of the last three indices vanishes

$$R_{\lambda\sigma\mu\nu} + R_{\lambda\mu\nu\sigma} + R_{\lambda\nu\sigma\mu} = 0, \tag{1.18}$$

which is known as the **first Bianchi identity**.

5. Contraction of the Riemann tensor given by (1.13) provides the **Ricci tensor**

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}. \tag{1.19}$$

6. Trace of the Ricci tensor is curvature scalar or the **Ricci scalar** R

$$R = R^{\mu}_{\mu} = g^{\mu\nu} R_{\mu\nu}. \tag{1.20}$$

The **Einstein tensor** is defined as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R. \tag{1.21}$$

1.1.2 The Einstein Tensor is Divergence Free

The **second Bianchi identity** states that

$$R_{\mu\nu\lambda\sigma;\eta} + R_{\mu\nu\sigma\eta;\lambda} + R_{\mu\nu\eta\lambda;\sigma} = 0. \tag{1.22}$$

Multiplying both sides of (1.22) by $g^{\mu\sigma} g^{\nu\lambda}$ we get

$$g^{\mu\sigma} g^{\nu\lambda} R_{\mu\nu\lambda\sigma;\eta} + g^{\mu\sigma} g^{\nu\lambda} R_{\mu\nu\sigma\eta;\lambda} + g^{\mu\sigma} g^{\nu\lambda} R_{\mu\nu\eta\lambda;\sigma} = 0,$$

or

$$g^{\nu\lambda} R_{\nu\lambda;\eta} + g^{\mu\sigma} g^{\nu\lambda} (-R_{\mu\nu\eta\sigma;\lambda}) + g^{\mu\sigma} g^{\nu\lambda} (-R_{\nu\mu\eta\lambda;\sigma}) = 0,$$

or

$$R_{;\eta} - g^{\nu\lambda} R_{\nu\eta;\lambda} - g^{\mu\sigma} R_{\mu\eta;\sigma} = 0,$$

or

$$R_{;\eta} - 2R^{\lambda}_{\eta;\lambda} = 0,$$

or

$$\left(R^{\lambda}_{\eta} - \frac{1}{2}\delta^{\lambda}_{\eta} R\right)_{;\lambda} = 0,$$

or

$$G^{\lambda}_{\eta;\lambda} = 0, \tag{1.23}$$

which shows that the Einstein tensor is divergence free.

1.1.3 Stress-Energy Tensor

The term pressure implies force per unit area. The generalization of pressure gives **stress**. The concept of pressure is applicable if only magnitude is relevant and directions do not matter. A medium for which this requirement holds is called **isotropic**. For example, gas and water are isotropic mediums. For **anisotropic** medium, consider the example of helical spring. The energy stored in it can be released by motion in one direction only, not orthogonal to it. A single momentum four-vector field is insufficient to describe the energy and momentum of a fluid, so one may define **energy-momentum tensor** or sometime known as **stress-energy tensor** $T_{\mu\nu}$. This tensor tells about the energy-like aspects of a system such as stress, pressure, energy density and so forth. A **perfect fluid** is a medium in which pressure is isotropic in the rest frame. For perfect fluid, the general form of isotropic stress-energy tensor in rest frame is

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + p g_{\mu\nu}, \tag{1.24}$$

where $U_\mu = (1, 0, 0, 0)$ is four-velocity vector, ρ is energy density and p is pressure. In component form (1.24) gives

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}.$$

For anisotropic medium, the stress-energy tensor is expressed as

$$T_{\mu\nu} = (\rho + p_r)U_\mu U_\nu + p_t g_{\mu\nu} + (p_r - p_t)\eta_\mu \eta_\nu, \tag{1.25}$$

where η_μ is a spacelike vector, p_t and p_r are transverse and radial pressures respectively. In component form (1.25) gives

$$\begin{pmatrix} \rho(r) & 0 & 0 & 0 \\ 0 & p_r(r) & 0 & 0 \\ 0 & 0 & p_t(r) & 0 \\ 0 & 0 & 0 & p_t(r) \end{pmatrix}.$$

The **Einstein-Hilbert Action** is defined as

$$S_H = \int \sqrt{-g} R d^4x, \tag{1.26}$$

where R is the Ricci scalar and $g = det(g_{\mu\nu})$ denotes the determinant of the metric tensor.

1.2 Einstein's Field Equations (EFEs)

Maxwell's equations describes the response of electric and magnetic fields to charges and currents, same as EFEs describes the response of metric to energy and momentum. The fundamental equations of GR are the EFEs.

Consider the Einstein-Hilbert action by adding a matter term $\mathcal{L}_m$ as following

$$S = \int \left(\frac{1}{2\kappa} R + \mathcal{L}_m \right) \sqrt{-g}\ d^4x, \tag{1.27}$$

where Lagrangian density $\mathcal{L}_m$ defines any matter field appears in the theory, κ is a constant, determined under the condition when the field equations reduce to Newton's law in the weak field limit.

Applying variation 'δ ', we get

$$\delta S = \int \left(\frac{1}{2\kappa} \delta(R\sqrt{-g}) + \delta(\sqrt{-g}\mathcal{L}_m) \right)\ d^4x,$$

or

$$\delta S = \int \left(\frac{1}{2\kappa} \left[\sqrt{-g}\delta R + R\delta\sqrt{-g} \right] + \delta(\sqrt{-g}\mathcal{L}_m) \right)\ d^4x. \tag{1.28}$$

To calculate δR, we introduce a local coordinate system as defined in (1.12) with vanishing Christoffel symbols in infinitesimal region, the components of Ricci tensor reduce to

$$R_{\mu\nu} = \Gamma^\lambda_{\mu\nu,\lambda} - \Gamma^\lambda_{\mu\lambda,\nu},$$

$$\delta R_{\mu\nu} = \delta\Gamma^{\lambda}_{\mu\nu,\lambda} - \delta\Gamma^{\lambda}_{\mu\lambda,\nu}.$$

The variation commutes with the partial derivatives, so

$$\delta R_{\mu\nu} = (\delta\Gamma^{\lambda}_{\mu\nu})_{,\lambda} - (\delta\Gamma^{\lambda}_{\mu\lambda})_{,\nu}. \tag{1.29}$$

By using the definition of Ricci scalar (1.20), we have

$$\delta R = R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}.$$

Using (1.29) in the above equation, we get

$$\delta R = R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}[(\delta\Gamma^{\lambda}_{\mu\nu})_{,\lambda} - (\delta\Gamma^{\lambda}_{\mu\lambda})_{,\nu}], \tag{1.30}$$

or

$$\delta R = R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}(\delta\Gamma^{\lambda}_{\mu\nu})_{,\lambda} - g^{\mu\lambda}(\delta\Gamma^{\nu}_{\mu\nu})_{,\lambda},$$

or

$$\delta R = R_{\mu\nu}\delta g^{\mu\nu} + (g^{\mu\nu}\delta\Gamma^{\lambda}_{\mu\nu} - g^{\mu\lambda}\delta\Gamma^{\nu}_{\mu\nu})_{,\lambda},$$

or

$$\sqrt{-g}\delta R = \sqrt{-g}R_{\mu\nu}\delta g^{\mu\nu} + \sqrt{-g}A^{\lambda}{}_{,\lambda}, \tag{1.31}$$

where vector A^{λ} is defined as

$$A^{\lambda} = g^{\mu\nu}\delta\Gamma^{\lambda}_{\mu\nu} - g^{\mu\lambda}\delta\Gamma^{\nu}_{\mu\nu}.$$

Now consider

$$\delta\sqrt{-g} = -\frac{1}{2\sqrt{-g}}\delta g,$$

or

$$\delta\sqrt{-g} = -\frac{1}{2\sqrt{-g}}(\frac{\partial g}{\partial g_{\mu\nu}})\delta g_{\mu\nu}. \tag{1.32}$$

To calculate $\partial g/\partial g_{\mu\nu}$ we use the formula

$$g = \sum_{\mu} g_{\mu\nu}Cofac(g^{\mu\nu}), \tag{1.33}$$

where $Cofac(g^{\mu\nu})$ is the cofactor matrix of $g_{\mu\nu}$ in the matrix made of the components of the metric tensor.

Differentiating (1.33) w.r.t. $g_{\mu\nu}$, we obtain

$$\frac{\partial g}{\partial g_{\mu\nu}} = gg^{\mu\nu}. \tag{1.34}$$

Using (1.34) in (1.32), we get

$$\delta\sqrt{-g} = -\frac{1}{2\sqrt{-g}} g g^{\mu\nu}\delta g_{\mu\nu},$$

or

$$\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g} g^{\mu\nu}\delta g_{\mu\nu}. \tag{1.35}$$

Using equations (1.31) and (1.35) in (1.28), we get

$$\delta S = \int \left[\frac{1}{2\kappa}\left(\sqrt{-g}R_{\mu\nu}\delta g^{\mu\nu} + \sqrt{-g}A^{\lambda}{}_{,\lambda} + \frac{1}{2}R(\sqrt{-g}g^{\mu\nu}\delta g_{\mu\nu})\right) + \delta(\sqrt{-g}\mathcal{L}_m)\right]\ d^4x. \tag{1.36}$$

According to Stoke's Theorem, "the surface integral of the function over any surface bounded by a closed path is equal to the line integral of a particular vector function round the path". So the integral of term $\sqrt{-g}A^{\lambda}{}_{,\lambda}$ only contributes with a boundary term. As on the boundary, the metric and its derivative vanishes (or $\sqrt{-g}A^{\lambda}{}_{,\lambda}$ term does not contribute when the variation of the metric $\delta g^{\mu\nu}$ vanishes at boundary).

$$\int \sqrt{-g}A^{\lambda}{}_{,\lambda} d^4x = 0.$$

Also,

$$g^{\mu\nu}\delta g_{\mu\nu} = -g_{\mu\nu}\delta g^{\mu\nu}.$$

Using above values in (1.36), which takes the form

$$\delta S = \int \left[\frac{1}{2\kappa}\left(\sqrt{-g}R_{\mu\nu} - \frac{1}{2}\sqrt{-g}Rg_{\mu\nu}\right) + \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}}\right]\ \delta g^{\mu\nu} d^4x. \tag{1.37}$$

Variation of the action with respect to the metric $g_{\mu\nu}$, gives $\delta S = 0$. Then (1.37) can be written as

$$\frac{\delta S}{\delta g^{\mu\nu}} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -\frac{2\kappa}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}},$$

or

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa T_{\mu\nu}, \tag{1.38}$$

where $\kappa = 8\pi G/c^4$, c is the speed of light in vacuum, G denotes Newton's gravitational constant and $T_{\mu\nu}$ is stress-energy momentum tensor given by

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}}. \tag{1.39}$$

Equation (1.38) can also be written as

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}, \tag{1.40}$$

which are EFEs. These equations are system of ten non-linear partial differential equations (PDEs) which gives the relationship between matter and field. EFEs are independent of the choice of coordinates.

Chapter 2

BASICS OF TRAVERSABLE WORMHOLES

The existence of "bridges" in spacetime was proposed by Albert Einstein and Nathan Rosen using theory of GR in 1935. These bridges were named as wormholes or Einstein-Rosen bridges. Wormholes serve as the solution to the Einstein field equations (EFEs) for gravity. These wormholes act as tunnels that connect points in spacetime. The characteristics of these tunnel is that the trip time between the points through tunnels is much less as compared to trip time in normal space. A Lorentzain wormhole is actually a shortcut through spacetime [2].

The concept of a Lorentzain wormhole is essentially synonymous with that of a spacewarp, which interpret as a warping, bending or folding of space [9]. Lorentzian wormholes have at least two categories:

1. **Inter-universe wormhole**, which connects two different universes.
2. **Intra-universe wormhole**, which connects two distant regions of same universe with each other.

2.1 The Schwarzchild Wormholes

The equations of the theory of GR have valid solutions that contain wormholes but researchers have no observational indication for existence of wormholes. Schwarzchild discovered the first ever solutions for wormhole, which would be present in the Schwarzschild metric describing a black hole. A **black hole** is a region of spacetime from which noth-

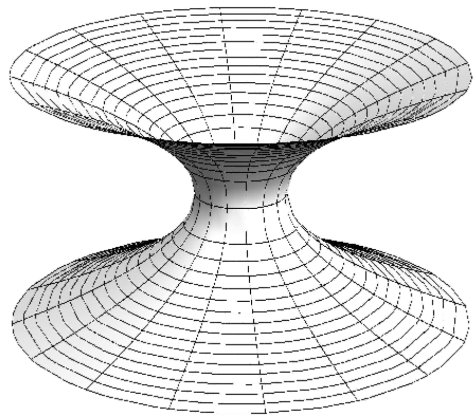

Figure 2.1: Inter-universe wormhole: wormhole connecting two universes [9].

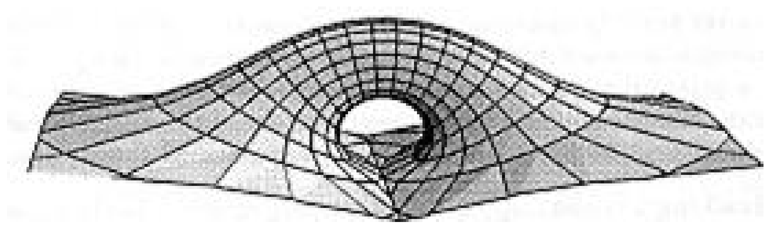

Figure 2.2: Intra-universe wormhole: wormhole connecting two distant regions of same universe [9].

ing can escape, not even light. But it was found that this type of wormhole would collapse instantaneously if anything crosses from one end to the other. The traveling through a Schwarzschild wormhole is possible in only one direction [2].

Objections on the Schwarzschild wormholes

The objections raised on the Schwarzschild wormholes are as following [10]:

(1). Tidal gravitational forces at the throat of a Schwarzschild wormhole are of the same magnitude.

(2). A Schwarzschild wormhole is actually dynamic, not static. As time passes, it expands from zero throat circumference (two disconnected universes) to a maximum circumference, and they recontract to zero circumference (the universe disconnect). This expansion and recontraction is so fast that even traveling with speed of light one cannot easily pass all through the wormhole without being caught in the crunch of recontraction and killed by tidal gravity.

(3). A Schwarzschild wormhole possesses a past horizon ("Anti-horizon"). There is a mathematically described surface around a black hole which identifies the point of no return is known an **event horizon**. Basically, in spacetime event horizons defines a boundary between the events that can communicate with distinct observers and those which are not able to communicate.

2.2 Traversable Wormholes

In 1988, the great change in physicist's willingness to consider wormhole systems came with the realization by Morris and Thorne that "traversable" wormhole spacetime was possible. The word "traversable" means that anything can move in both directions through wormhole in reasonable time [9]. The type of traversable wormhole proposed by them are held open by a spherical shell of exotic matter, which are known as a Morris-Thorne wormhole. Later, other types of traversable wormholes were discovered as allowable solutions to the equations of general relativity, including a variety analyzed in a 1989 paper by Matt Visser, in which a path through the wormhole can be made where the traversing path does not pass through a region of exotic matter [2].

2.2.1 Construction Criteria for Traversable Wormholes

To construct the traversable wormholes, following properties are required [10]:

- To keep calculations simple, metric must be both spherically symmetric and static.
- Solution should obey the Einstein field equations, which assumes the correctness of GR.
- There must be no horizon. It is necessary because it will put off two way travel through the wormhole.
- Solution should have a throat that connects two regions of spacetime which are asymptotically flat.
- Tidal gravitational forces must be reasonably small which are practiced by a traveler.
- Traveler should be able to cross all the way through the wormhole in a finite and acceptably small proper time (e.g less than a year) as measured by both traveler and observer who await outside the wormhole.
- Physically reasonable stress-energy tensor.
- Solution must be stable under small perturbation.
- It should be possible to assemble the wormhole i.e. assembly should require much less than both the total mass of the universe and the age of the universe.

2.2.2 Metric

To keep simplicity, assume the traversable wormhole to be time independent, spherically symmetric and nonrotating bridges between two universes. Thus, our manifold should possess two asymptotically flat regions that are static and spherically symmetric spacetime. Let us start with

$$ds^2 = -e^{2\Phi(l)}dt^2 + dl^2 + r^2(l)[d\theta^2 + \sin^2\theta d\phi^2], \tag{2.1}$$

where l denotes the proper radial distance. Some key features are listed below [9].

- $l \, \epsilon \, (-\infty, +\infty)$.
- Assume absence of event horizons $\rightarrow \Phi(l)$ must be everywhere finite.
- Assume two asymptotically flat regions at $l \approx \pm\infty$.
- For spatial geometry to approach to a suitable asymptotically flat limit, impose

$$\lim_{l \to \pm\infty} [r(l)/l] = 1.$$

- Spacetime geometry tends to an appropriate asymptotically flat limit, if

$$\lim_{l \to \pm\infty} [\Phi(l)] = \Phi_{\pm},$$

is be finite.

- At throat, radius of the wormhole is identified by

$$r_0 = min[r(l)].$$

To simplify, we assume there is only one such minimum and it occurs at $l = 0$.

- Components of metric are at least twice differentiable with respect to l.

We could use this to calculate the Riemann, Ricci and Einstein tensors using this coordinate system, but it is much easier to use Schwarzschild coordinates. We write in (t, r, θ, ϕ) as [9]

$$ds^2 = -e^{2\Phi_{\pm}(r)}dt^2 + \frac{dr^2}{1 - b_{\pm}(r)/r} + r^2[d\theta^2 + \sin^2\theta d\phi^2], \tag{2.2}$$

where we introduced **"redshift function"** denoted by $\Phi(r)$ that determines magnitude of the gravitational redshift and the **"shape function"** denoted by $b(r)$ that determines the spatial shape of wormhole. The gravitational redshift is the reduction in the frequency that a photon will experience when it climbs out of a gravitational potential well in order to escape to infinity. Some key features are

1. Spatial coordinate r has a geometrical significance. The throat circumference is $2\pi r$. Also, r decreases from $+\infty$ to some minimum radius r_0 as one moves through the lower universe, then increases from r_0 to $+\infty$ moving out of the throat and into the upper universe.

2. For convenience, demand t coordinate to be continuous across the throat, so that $\Phi_+(r_0) = \Phi_-(r_0)$.

3. l relates the r coordinate by

$$l(r) = \pm \int_{r_0}^{r} \frac{dr'}{\sqrt{1 - b_\pm(r')/r'}}. \tag{2.3}$$

For spatial geometry which tends to an appropriate asymptotically flat limit, we require both limits

$$\lim_{r\to\infty} [b_\pm(r)] = b_\pm,$$

to be finite.

For spacetime geometry to tend to an appropriate asymptotically flat limit, we require both limits

$$\lim_{r\to\infty} [\Phi_\pm(r)] = \Phi_\pm,$$

to be finite.

Since at the throat, $dr/dl = 0$ (throat is at minimum of $r(l)$), we have $dl/dr \to \infty$. Since

$$\frac{dl}{dr} = \pm\frac{1}{\sqrt{1 - b_\pm(r)/r}}.$$

This implies $b_\pm(r) = r_0$ at the throat.

Metric components should be at least twice differentiable with respect to r. We can simplify things and symmetry should be assumed under exchange of asymptotically flat regions, $\pm \leftrightarrow \mp$ or $\Phi_+(r) = \Phi_-(r)$ and $b_+(r) = b_-(r)$. This is not a requirement, just for convenience. So, metric (2.2) can be written as

$$ds^2 = -e^{2\Phi(r)}dt^2 + \frac{dr^2}{1 - b(r)/r} + r^2[d\theta^2 + \sin^2\theta d\phi^2], \tag{2.4}$$

2.2.3 Tensor Calculations

The Christoffel symbols and the components of the Riemann curvature tensor are calculated by using standard formulas given by (1.6) and (1.13) respectively.

The nonzero components of the Christoffel symbols for metric given by (2.4) are

$$\Gamma^t{}_{tr} = \Gamma^t{}_{rt} = \Phi'(r),$$

$$\Gamma^r{}_{tt} = -\frac{(b-r)e^{2\Phi}\Phi'}{r}, \qquad \Gamma^r{}_{rr} = -\frac{b'r-b}{2r(b-r)},$$

$$\Gamma^r{}_{\theta\theta} = b-r, \qquad \Gamma^r{}_{\phi\phi} = (b-r)\sin^2\theta, \tag{2.5}$$

$$\Gamma^\theta{}_{r\theta} = \Gamma^\theta{}_{\theta r} = \frac{1}{r}, \qquad \Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta,$$

$$\Gamma^\phi{}_{\theta\phi} = \Gamma^\phi{}_{\phi\theta} = \cot\theta, \qquad \Gamma^\phi{}_{r\phi} = \Gamma^\phi{}_{\phi r} = \tfrac{1}{r}.$$

Components of Riemann Tensor

The nonzero components of the Riemann tensor for metric given by (2.4) are

$$\begin{aligned} R^t{}_{rtr} &= -R^t{}_{rrt}, \\ &= (1-b/r)^{-1}e^{-2\Phi}R^r{}_{ttr}, \\ &= -(1-b/r)^{-1}e^{-2\Phi}R^r{}_{trt}, \\ &= -\Phi'' + \tfrac{(b'r-b)\Phi'}{2r(r-b)} - (\Phi')^2. \end{aligned}$$

$$\begin{aligned} R^t{}_{\theta t\theta} &= -R^t{}_{\theta\theta t}, \\ &= r^2e^{-2\Phi}R^\theta{}_{tt\theta}, \\ &= -r^2e^{-2\Phi}R^\theta{}_{t\theta t}, \\ &= -r\Phi'(1-b/r). \end{aligned}$$

$$\begin{aligned} R^t{}_{\phi t\phi} &= -R^t{}_{\phi\phi t}, \\ &= r^2e^{-2\Phi}\sin^2\theta R^\phi{}_{tt\phi}, \\ &= -r^2e^{-2\Phi}\sin^2\theta R^\phi{}_{t\phi t}, \\ &= -r\Phi'(1-b/r)\sin^2\theta. \end{aligned}$$

$$\begin{aligned} R^r{}_{\theta r\theta} &= -R^r{}_{\theta\theta r}, \\ &= -r^2(1-b/r)R^\theta{}_{rr\theta}, \\ &= -r^2(1-b/r)R^\theta{}_{r\theta r}, \\ &= (b'r-b)/2r. \end{aligned} \tag{2.6}$$

$$
\begin{aligned}
R^{r}{}_{\phi r\phi} &= -R^{r}{}_{\phi\phi r}, \\
&= -r^2(1-b/r)\sin^2\theta R^{\phi}{}_{rr\phi}, \\
&= -r^2(1-b/r)\sin^2\theta R^{\phi}{}_{r\phi r}, \\
&= (b'r-b)\sin^2\theta/2r.
\end{aligned}
$$

$$
\begin{aligned}
R^{\theta}{}_{\phi\theta\phi} &= -R^{\theta}{}_{\phi\phi\theta}, \\
&= \sin^2\theta R^{\phi}{}_{\theta\phi\theta}, \\
&= -\sin^2\theta R^{\phi}{}_{\theta\theta\phi}, \\
&= (b/r)\sin^2\theta.
\end{aligned}
$$

The basis vectors being used are those $(e_t, e_r, e_\theta, e_\phi)$ associated with coordinate system (t, r, θ, ϕ). The details of physical interpretations will be simplified by switching to a set of orthonormal basis vectors, such as in rest frame (i.e. r, θ, ϕ *constant*) [10]

$$
e_{\hat{t}} = e^{-\Phi}e_t, \qquad e_{\hat{r}} = (1-b/r)^{1/2}e_r,
$$

$$
e_{\hat{\theta}} = r^{-1}e_\theta, \qquad e_{\hat{\phi}} = (r\sin\theta)^{-1}e_\phi.
$$

In this basis the metric coefficients are expressed as [10]

$$
g_{\hat{\alpha}\hat{\beta}} = e_{\hat{\alpha}}.e_{\hat{\beta}} = \eta_{\hat{\alpha}\hat{\beta}} \equiv \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

and the nonzero components of the Riemann curvature tensor takes the following form

$$
\begin{aligned}
R^{\hat{t}}{}_{\widehat{r t r}} = -R^{\hat{t}}{}_{\widehat{r r t}} = R^{\hat{r}}{}_{\widehat{t t r}} = R^{\hat{r}}{}_{\widehat{t r t}}, \\
= (1-b/r)\left(-\Phi'' + \frac{(b'r-b)}{2r(r-b)}\Phi' - (\Phi')^2\right).
\end{aligned}
$$

$$
\begin{aligned}
R^{\hat{t}}{}_{\widehat{\theta t\theta}} = -R^{\hat{t}}{}_{\widehat{\theta\theta t}} = R^{\hat{\theta}}{}_{\widehat{t t\theta}} = -R^{\hat{\theta}}{}_{\widehat{t\theta t}}, \\
= -(1-b/r)\Phi'/r.
\end{aligned}
$$

$$
\begin{aligned}
R^{\hat{t}}{}_{\widehat{\phi t\phi}} = -R^{\hat{t}}{}_{\widehat{\phi\phi t}} = R^{\hat{\phi}}{}_{\widehat{t t\phi}} = R^{\hat{\phi}}{}_{\widehat{t\phi t}}, \\
= -(1-b/r)\Phi'/r.
\end{aligned} \tag{2.7}
$$

$$R^{\hat{r}}{}_{\hat{\theta}\hat{r}\hat{\theta}} = -R^{\hat{r}}{}_{\hat{\theta}\hat{\theta}\hat{r}} = R^{\hat{\theta}}{}_{\hat{r}\hat{r}\hat{\theta}} = R^{\hat{\theta}}{}_{\hat{r}\hat{\theta}\hat{r}},$$
$$= (b'r - b)/2r^3.$$

$$R^{\hat{r}}{}_{\hat{\phi}\hat{r}\hat{\phi}} = -R^{\hat{r}}{}_{\hat{\phi}\hat{\phi}\hat{r}} = R^{\hat{\phi}}{}_{\hat{r}\hat{r}\hat{\phi}} = R^{\hat{\phi}}{}_{\hat{r}\hat{\phi}\hat{r}},$$
$$= (b'r - b)/2r^3.$$

$$R^{\hat{\theta}}{}_{\hat{\phi}\hat{\theta}\hat{\phi}} = -R^{\hat{\theta}}{}_{\hat{\phi}\hat{\phi}\hat{\theta}} = R^{\hat{\phi}}{}_{\hat{\theta}\hat{\phi}\hat{\theta}} = R^{\hat{\phi}}{}_{\hat{\theta}\hat{\theta}\hat{\phi}},$$

$$= b/r^3.$$

Components of Einstein Tensor

The above computation provides the following nonzero components of Einstein tensor [10]

$$\begin{aligned} G_{\hat{t}\hat{t}} &= \frac{b'}{r^2}, \\ G_{\hat{r}\hat{r}} &= -\frac{b}{r^3} + 2(1 - b/r)\Phi'/r, \\ G_{\hat{\theta}\hat{\theta}} = G_{\hat{\phi}\hat{\phi}} &= \left(1 - \frac{b}{r}\right)\left(\Phi'' - \frac{b'r - b}{2r(r-b)}\Phi' + (\Phi')^2 + \frac{\Phi'}{r} - \frac{b'r - b}{2r^2(r-b)}\right). \end{aligned} \tag{2.8}$$

Components of Stress-Energy Tensor

Non-vanishing stress-energy tensor components can be expressed as following relationships [10]

$$\begin{aligned} T_{\hat{t}\hat{t}} &= \rho(r), \\ T_{\hat{r}\hat{r}} &= -\tau(r), \\ T_{\hat{\theta}\hat{\theta}} = T_{\hat{\phi}\hat{\phi}} &= p(r), \end{aligned} \tag{2.9}$$

where $\rho(r)$ denotes the total energy density, $\tau(r)$ denotes the radial tension per unit area and $p(r)$ denotes the pressure (radial or transverse pressure).

The Einstein Field Equations

The Einstein field equations are defined as

$$G_{\hat{\alpha}\hat{\beta}} = 8\pi G T_{\hat{\alpha}\hat{\beta}}.$$

Using (2.8) and (2.9) in above relation, we get the following form [10]

$$b'(r) = 8\pi G r^2 \rho, \tag{2.10}$$

$$\Phi'(r) = (-8\pi G \tau r^3 + b)/[2r(r-b)], \tag{2.11}$$

$$\tau'(r) = (\rho - r)\Phi' - 2(p+\tau)/r. \tag{2.12}$$

Equations (2.10) and (2.11) show the temporal and radial parts of the field equations respectively. It is also convenient to rewrite (2.10)-(2.12) in slightly different form

$$\rho(r) = \frac{b'}{8\pi G r^2}, \tag{2.13}$$

$$\tau(r) = \frac{b/r - 2(r-b)\Phi'}{8\pi G r^2}, \tag{2.14}$$

$$p(r) = \frac{r}{2}[(\rho - \tau)\Phi' - \tau'] - \tau. \tag{2.15}$$

By using (2.13) - (2.15), one may able to solve for $\rho(r)$ and $\tau(r)$ by suitable choice of redshift function $\Phi(r)$ and shape function $b(r)$ and using ρ and τ, together $p(r)$ can easily be determined.

2.3 Energy Conditions

Wormhole construction depends on the existence of exotic matter. In wormhole physics, a fundamental fact is energy conditions violation. Energy conditions are coordinate-invariant restrictions on the stress-energy tensor. Let us use the Hawking-Ellis type I stress- energy tensor to discuss the energy conditions in an orthonormal frame, which is expressed in the following manner

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p_1 & 0 & 0 \\ 0 & 0 & p_2 & 0 \\ 0 & 0 & 0 & p_3 \end{pmatrix}. \tag{2.16}$$

2.3.1 Weak Energy Condition

For all timelike vectors υ^{ν}, the weak energy condition (WEC) states that

$$T_{\mu\nu}\upsilon^{\mu}\upsilon^{\nu} \geq 0.$$

According to this condition, a positive energy density of matter is measured by each observer in his rest frame or equivalently that

$$\rho \geq 0,$$

and

$$\rho + p_i \geq 0. \qquad i = 1, 2, 3$$

2.3.2 Null Energy Condition

For all null vectors κ^{μ}, the null energy condition (NEC) states that

$$T_{\mu\nu}\kappa^{\mu}\kappa^{\nu} \geq 0,$$

or equivalently that

$$\rho + p_i \geq 0. \qquad i = 1, 2, 3$$

It is a special case of WEC, in which timelike vector is replaced by a null vector. In this case, as long as there is compensating positive pressure, energy density ρ may be negative.

2.3.3 Strong Energy Condition

For all timelike vectors υ^{ν}, the strong energy condition (SEC) states that

$$\left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right)\upsilon^{\mu}\upsilon^{\nu} \geq 0,$$

where $T = T^{\mu}_{\mu} = g^{\mu\nu}T_{\mu\nu}$ describes the trace of the stress energy tensor or equivalently

$$\rho + p_i \geq 0,$$

and

$$\rho + 3p_i \geq 0.$$

It implies the NEC along with excluding large negative pressure. According to SEC, gravitation is attractive.

2.3.4 Dominant Energy Condition

For all timelike vectors υ^ν, the dominant energy condition (DEC) includes weak energy condition

$$T_{\mu\nu}\upsilon^\mu\upsilon^\nu \geq 0,$$

and

$$T_{\mu\nu}\upsilon_\mu \text{ is a nonspacelike vector.}$$

This condition says that the locally measured energy density is always positive and the energy flux is timelike or null, that is

$$\rho \geq 0.$$

2.3.5 Null Dominant Energy Condition

The null dominant energy condition (NDEC) is the dominant energy condition for null vectors only. For any null vectors κ^μ,

$$T_{\mu\nu}\kappa^\mu\kappa^\nu \geq 0,$$

and

$$T_{\mu\nu}\kappa_\mu \text{ is a nonspacelike vector.}$$

Chapter 3

WORMHOLE GEOMETRY IN MODIFIED GRAVITY

In this chapter, the analysis of spherically symmetric and static spacetime is extended and traversable wormhole geometries in $f(R)$ modified gravity is also constructed. Also, wormhole geometries in curvature-matter coupled modified gravity are explored, by taking into consideration an explicit nonminimial coupling between an arbitrary function of the Ricci scalar R, and the Lagrangian density of matter. In particular, exact solutions are found for $f(R)$ by taking viable shape functions and a linear R nonminimial curvature-matter coupling and by the choice of an energy density function which is monotonically decreasing. Energy conditions are also checked for specific cases. In this chapter, I will review two papers [3] and [5].

3.1 $f(R)$ Gravity

Modifications of GR by including higher order curvature invariants in the gravitational action have a long history. Recently, a new stimulus appeared in higher-order theories of gravity. This time the motivation came from cosmological and astrophysical observations. The latest data sets coming from different sources, such as the Cosmic Microwave Background Radiation (CMBR) and supernovae surveys, seem to indicate that the energy budget of the universe is 4 percent ordinary baryonic matter, 20 percent dark matter and 76 percent dark energy [11]-[14]. The term **dark matter** refers to an unknown form of matter, which has the clustering properties of ordinary matter but has not yet been detected in the laboratory. The term **dark energy** is reserved for an unknown form of energy which not only has not been detected directly, but also does not cluster as ordinary matter does [4].

The rapid development of observational cosmology which started from 1990s shows that the universe has undergone two phases of cosmic acceleration i.e. the universe appears to be expanding at an increasing rate. The first one is called the inflation, which occurred prior to the radiation domination. This phase is required to solve the flatness and horizon problems plagued in big-bang cosmology. The second acceleration phase has started after the matter domination. Dark energy gives rise to the late time cosmic acceleration. Dark energy corresponds to a modification of the stress-energy tensor in Einstein field equations and this corresponds to the modified gravity in which gravitational theory is modified as compared to GR [16].

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}R. \tag{3.1}$$

For a general function of R, (3.1) takes the form as

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}f(R). \tag{3.2}$$

3.2 Spacetime Metric, Ricci and Einstein Tensors

Consider the static and spherically symmetric metric that describes the wormhole geometry and given by [10]

$$ds^2 = -e^{2\Phi(r)}dt^2 + \frac{dr^2}{1-\frac{b(r)}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{3.3}$$

where two arbitrary functions, "redshift function" denoted by $\Phi(r)$ that determines magnitude of the gravitational redshift and the "shape function" denoted by $b(r)$ that determines the spatial shape of wormhole. Both functions $b(r)$ and $\Phi(r)$ depend on radial coordinate only. The radial coordinate r is nonmonotonic. It decreases from infinity to a minimum value r_0, which locates the throat of the wormhole and then it increases from minimum value r_0 back to infinity [3].

Properties of Wormhole

The following properties of wormhole should be satisfied:

- In spacetime, the radial distance given in Eq.(2.3) is finite everywhere i.e.,

$$1 - \frac{b(r)}{r} > 0, \tag{3.4}$$

- Flaring out condition of the throat is imposed, which is a fundamental property of the wormhole and given by [10]

$$\frac{(b - b'r)}{b^2} > 0. \tag{3.5}$$

- At the throat, (3.4) can be written as

$$b'(r_0) < 1. \tag{3.6}$$

  The above condition is imposed to have wormhole solutions because it indicates the violation of null energy condition.

For traversable wormhole, there must be no horizons. Because of this, $\Phi(r)$ should be finite everywhere. In the analysis below, to make calculations simple, the redshift function is considered as constant, which implies $\Phi' = 0$ and provide interesting exact solutions for wormhole.

Using above conditions $\Phi' = 0$, the nonzero components of the Riemann tensor for metric given by (3.3) are given as

$$\begin{aligned}
R^{r}{}_{\theta r\theta} = -R^{r}{}_{\theta\theta r} = R^{\theta}{}_{rr\theta} = R^{\theta}{}_{r\theta r} \\
= (b'r - b)/2r^3, \\
\\
R^{r}{}_{\phi r\phi} = -R^{r}{}_{\phi\phi r} = R^{\phi}{}_{rr\phi} = R^{\phi}{}_{r\phi r} \\
= (b'r - b)/2r^3, \\
\\
R^{\theta}{}_{\phi\theta\phi} = -R^{\theta}{}_{\phi\phi\theta} = R^{\phi}{}_{\theta\phi\theta} = R^{\phi}{}_{\theta\theta\phi} \\
\\
= b/r^3.
\end{aligned} \tag{3.7}$$

The curvature scalar R for metric given by (3.3), is given as

$$R = \frac{2b'}{r^2}. \tag{3.8}$$

This yields the nonzero components of the Einstein tensor for metric given by (3.3), is given as

$$\begin{aligned} G_{tt} &= \frac{b'}{r^2}, \\ G_{rr} &= -\frac{b}{r^3}, \\ G_{\theta\theta} = G_{\phi\phi} &= -\frac{b'r - b}{2r^3}. \end{aligned} \tag{3.9}$$

3.3 $f(R)$ Modified Theories of Gravity

In this section, i will review the paper " wormhole geometries in $f(R)$ modified theories of gravity [3]". A fundamental property in classical GR is that the wormholes are supported by exotic matter, which involves a stress energy tensor $T_{\mu\nu}$. This tensor $T_{\mu\nu}$ violates the null energy condition which is given by $T_{\mu\nu}k^\mu k^\nu \geq 0$, where k^μ is any null vector. Effective stress-energy tensor may be interpreted as a gravitational fluid which contains higher order curvature derivatives and responsible for the violation of NEC. In this section, we imposed that the matter threading the wormholes satisfies the energy conditions. In particular, by considering specific shape functions and equations of state, the exact solutions for $f(R)$ are deduced.

Consider the action for $f(R)$ modified theories of gravity, which is given by [3]

$$S = \frac{1}{2\kappa}\int f(R)\sqrt{-g}\ d^4x + S_M(g^{\mu\nu}, \psi), \tag{3.10}$$

where $f(R)$ is arbitrary function of curvature scalar, $\kappa = 8\pi G/c^4$ and $g = \det g_{\mu\nu}$ is the determinant of the metric tensor. $S_M(g^{\mu\nu}, \psi)$ is matter action, which is defined as

$$S_M(g^{\mu\nu}, \psi) = \int \mathcal{L}_m(g^{\mu\nu}, \psi)\sqrt{-g}\ d^4x, \tag{3.11}$$

where $\mathcal{L}_m$ denotes the Lagrangian density of matter and ψ is the matter field.
So (3.10) can be written as

$$S = \int \left[\frac{1}{2\kappa} f(R)\sqrt{-g} + \mathcal{L}_m(g^{\mu\nu}, \psi)\sqrt{-g}\right]\ d^4x. \tag{3.12}$$

Applying 'δ ' on both sides of (3.12), we get

$$\delta S = \int \left[\frac{1}{2\kappa}(\sqrt{-g}\delta f(R) + f(R)\delta\sqrt{-g}) + \delta(\sqrt{-g}\mathcal{L}_m) \ \right] d^4x,$$

or

$$= \int \left[\frac{1}{2\kappa}(\sqrt{-g}\frac{\partial f}{\partial R}\delta R + f(R)\delta\sqrt{-g}) + \delta(\sqrt{-g}\mathcal{L}_m) \ \right] d^4x,$$

or

$$= \int \left[\frac{1}{2\kappa}(F\sqrt{-g}\delta R + f(R)\delta\sqrt{-g}) + \delta(\sqrt{-g}\mathcal{L}_m) \ \right] d^4x, \qquad (3.13)$$

where $\partial f/\partial R \equiv F(R)$. Consider (1.30), which is given as

$$\delta R = R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}[(\delta\Gamma^{\lambda}_{\mu\nu})_{,\lambda} - (\delta\Gamma^{\lambda}_{\mu\lambda})_{,\nu}],$$

or

$$= R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}[\nabla_\lambda\delta\Gamma^{\lambda}_{\mu\nu} - \nabla_\nu\delta\Gamma^{\lambda}_{\mu\lambda}]. \qquad (3.14)$$

Using the definition of Christoffel symbol (1.6), consider

$$\delta\Gamma^{\lambda}_{\mu\nu} \qquad = \delta\left[\frac{1}{2}g^{\lambda\alpha}(g_{\nu\alpha,\mu} + g_{\mu\alpha,\nu} - g_{\mu\nu,\alpha})\right],$$

or

$$= \frac{1}{2}\left[\delta g^{\lambda\alpha}(g_{\nu\alpha,\mu} + g_{\mu\alpha,\nu} - g_{\mu\nu,\alpha}) + g^{\lambda\alpha}\delta(g_{\nu\alpha,\mu} + g_{\mu\alpha,\nu} - g_{\mu\nu,\alpha})\right],$$

or

$$= \frac{1}{2}g^{\lambda\alpha}(\nabla_\mu\delta g_{\nu\alpha} + \nabla_\nu\delta g_{\mu\alpha} - \nabla_\alpha\delta g_{\mu\nu}),$$

Now applying covariant derivative on both sides of above equation, we get

$$\nabla_\lambda\delta\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\alpha}(\nabla_\lambda\nabla_\mu\delta g_{\nu\alpha} + \nabla_\lambda\nabla_\nu\delta g_{\mu\alpha} - \nabla_\lambda\nabla_\alpha\delta g_{\mu\nu}), \qquad (\because \quad \nabla_\lambda g^{\lambda\alpha} = 0)$$

or

$$= \frac{1}{2}(\nabla^\alpha\nabla_\mu\delta g_{\nu\alpha} + \nabla^\alpha\nabla_\nu\delta g_{\mu\alpha} - \nabla^\alpha\nabla_\alpha\delta g_{\mu\nu}),$$

or

$$\nabla_\lambda\delta\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}(\nabla^\alpha\nabla_\mu\delta g_{\nu\alpha} + \nabla^\alpha\nabla_\nu\delta g_{\mu\alpha} - \Box\delta g_{\mu\nu}), \qquad (3.15)$$

where $\Box = \nabla^\alpha\nabla_\alpha$. Similarly,

$$\nabla_\nu\delta\Gamma^{\lambda}_{\mu\lambda} = \frac{1}{2}(\nabla_\nu\nabla^\alpha\delta g_{\mu\alpha} + g^{\lambda\alpha}\nabla_\nu\nabla_\mu\delta g_{\lambda\alpha} - \nabla_\nu\nabla^\lambda\delta g_{\mu\lambda}). \qquad (3.16)$$

Using (3.15) and (3.16) in (3.14), we get the following expression

$$\delta R = R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\Big[\tfrac{1}{2}(\nabla^{a}\nabla_{\mu}\delta g_{\nu a} + \nabla^{a}\nabla_{\nu}\delta g_{\mu a} - \Box\delta g_{\mu\nu}) - \tfrac{1}{2}(\nabla_{\nu}\nabla^{a}\delta g_{\mu a} + g^{\lambda a}\nabla_{\nu}\nabla_{\mu}\delta g_{\lambda a} - \nabla_{\nu}\nabla^{\lambda}\delta g_{\mu\lambda})\Big],$$

$$= R_{\mu\nu}\delta g^{\mu\nu} + \frac{1}{2}\Big[\nabla^{a}\nabla^{\nu}\delta g_{\nu a} + \nabla^{a}\nabla^{\mu}\delta g_{\mu a} - g^{\mu\nu}\Box\delta g_{\mu\nu} - \nabla^{\mu}\nabla^{a}\delta g_{\mu a} - g^{\lambda a}\Box\delta g_{\lambda a} + \nabla^{\mu}\nabla^{\lambda}\delta g_{\mu\lambda}\Big],$$

where $\Box = g^{\mu\nu}\nabla_{\nu}\nabla_{\mu}$. Replacing α and λ by μ and ν respectively and simplifying, then above equation takes the form

$$\delta R = R_{\mu\nu}\delta g^{\mu\nu} + \nabla^{\mu}\nabla^{\nu}\delta g_{\mu\nu} - g^{\mu\nu}\Box\delta g_{\mu\nu},$$

or

$$\delta R = R_{\mu\nu}\delta g^{\mu\nu} - \nabla_{\mu}\nabla_{\nu}\delta g^{\mu\nu} + g_{\mu\nu}\Box\delta g^{\mu\nu}, \qquad (3.17)$$

where $\nabla^{\mu}\nabla^{\nu}\delta g_{\mu\nu} = -\nabla_{\mu}\nabla_{\nu}\delta g_{\mu\nu}$. Using values of $\delta\sqrt{-g}$ and δR from (1.35) and (3.17) respectively in (3.13), we get

$$\delta S = \int \Big[\frac{1}{2\kappa}\Big(F\sqrt{-g}(R_{\mu\nu}\delta g^{\mu\nu} - \nabla_{\mu}\nabla_{\nu}\delta g^{\mu\nu} + g_{\mu\nu}\Box\delta g^{\mu\nu}) + f(R)(-\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu})\Big) + \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}}\delta g^{\mu\nu}\Big] d^4x,$$

or

$$\delta S = \int \Big[\frac{1}{2\kappa}\Big(FR_{\mu\nu} + (g_{\mu\nu}\Box - \nabla_{\mu}\nabla_{\nu})F(R) - \frac{1}{2}f(R)g_{\mu\nu}\Big) + \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}}\Big]\sqrt{-g}\delta g^{\mu\nu} d^4x. \qquad (3.18)$$

Varying the action with respect to the metric $g_{\mu\nu}$ provides $\delta S = 0$, then (3.18) takes the following form

$$FR_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - \nabla_{\mu}\nabla_{\nu}F + g_{\mu\nu}\Box F = \frac{-2\delta(\mathcal{L}_m\sqrt{-g})}{\sqrt{-g}\delta g^{\mu\nu}},$$

or

$$FR_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - \nabla_{\mu}\nabla_{\nu}F + g_{\mu\nu}\Box F = T^{(m)}_{\mu\nu}, \qquad (3.19)$$

where

$$T^{(m)}_{\mu\nu} = \frac{-2\delta(\mathcal{L}_m\sqrt{-g})}{\sqrt{-g}\delta g^{\mu\nu}},$$

is stress-energy tensor and $\kappa = 1$.

Trace Equation

The expression for trace equation can be obtained by contraction of field equation (3.19)

$$g^{\sigma\mu}[FR_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - \nabla_\mu\nabla_\nu F + g_{\mu\nu}\Box F] = g^{\sigma\mu}T^{(m)}_{\mu\nu},$$

$$FR^{\sigma}_{\nu} - \frac{1}{2}f\delta^{\sigma}_{\nu} - \nabla^{\sigma}\nabla_\nu F + \delta^{\sigma}_{\nu}\Box F = T^{\sigma}_{\nu}.$$

Replace $\sigma = \nu$ then $R^{\sigma}_{\sigma} = R$, $T^{\sigma}_{\sigma} = T$ and $\delta^{\sigma}_{\sigma} = 4$, so last equation takes the form

$$FR - 2f + 3\Box F = T. \tag{3.20}$$

Einstein Tensor

Now compute Einstein tensor by substituting (3.20) in (3.19), we obtained

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = T^{\text{eff}}_{\mu\nu}, \tag{3.21}$$

where $G_{\mu\nu}$ is the Einstein tensor and $T^{\text{eff}}_{\mu\nu}$ is the effective stress-energy tensor, which is given by

$$T^{\text{eff}}_{\mu\nu} = T^{(c)}_{\mu\nu} + \widetilde{T}^{(m)}_{\mu\nu}, \tag{3.22}$$

where the term $\widetilde{T}^{(m)}_{\mu\nu} = T^{(m)}_{\mu\nu}/F$ and $T^{(c)}_{\mu\nu}$ is the curvature stress-energy tensor, defined as

$$T^{(c)}_{\mu\nu} = \frac{1}{F}\left[\nabla_\mu\nabla_\nu F - \frac{1}{4}g_{\mu\nu}(FR + \Box F + T)\right]. \tag{3.23}$$

Equation (3.22) can be written as

$$T^{\text{eff}}_{\mu\nu} = T^{(c)}_{\mu\nu} + \frac{T^{(m)}_{\mu\nu}}{F}, \tag{3.24}$$

Conservation law

Using (3.24) in (3.21), we get

$$G_{\mu\nu} = T^{(c)}_{\mu\nu} + \frac{T^{(m)}_{\mu\nu}}{F}. \tag{3.25}$$

Taking covariant derivative of the above equation, we get

$$\nabla^\mu G_{\mu\nu} = \nabla^\mu T^{(c)}_{\mu\nu} + \nabla^\mu \frac{T^{(m)}_{\mu\nu}}{F},$$

$$\nabla^\mu G_{\mu\nu} = \nabla^\mu T^{(c)}_{\mu\nu} + \frac{1}{F}\nabla^\mu T^{(m)}_{\mu\nu} - \frac{1}{F^2}(\nabla^\mu F)T^{(m)}_{\mu\nu}.$$

Taking into account $\nabla^\mu G_{\mu\nu} = 0$ and $\nabla^\mu T^{(m)}_{\mu\nu} = 0$, the above effective EFEs provides the following conservation law

$$\nabla^\mu T^{(c)}_{\mu\nu} = \frac{1}{F^2}(\nabla^\mu F)T^{(m)}_{\mu\nu}. \tag{3.26}$$

Stress-energy tensor

The stress-energy tensor for an anisotropic distribution of matter threading the wormholes is given by

$$T_{\mu\nu} = (\rho + p_t)U_\mu U_\nu + p_t g_{\mu\nu} + (p_r - p_t)\chi_\mu\chi_\nu, \tag{3.27}$$

where U^μ is the four-velocity, $\chi^\mu = \sqrt{1 - b/r}\,\delta^\mu_r$ is the unit spacelike vector in the radial direction. $\rho(r)$ denotes the energy density, $p_r(r)$ denotes the radial pressure measured in the direction of χ^μ, and $p_t(r)$ is the transverse pressure measured in the orthogonal direction to χ^μ. The stress energy tensor given by (3.27) in component form is given as

$$T^\mu{}_\nu = \begin{pmatrix} -\rho(r) & 0 & 0 & 0 \\ 0 & p_r(r) & 0 & 0 \\ 0 & 0 & p_t(r) & 0 \\ 0 & 0 & 0 & p_t(r) \end{pmatrix}. \tag{3.28}$$

3.3.1 Gravitational Field Equations

For notational simplicity, let us define a term

$$H(r) = \frac{1}{4}(FR + \Box F + T), \tag{3.29}$$

where $\Box F = g^{\mu\nu}\nabla_\mu\nabla_\nu F(r)$ is calculated by using metric given by (3.3). The nonzero components are given by

$$\Box F(r) = g^{tt}\nabla_t\nabla_t F(r) + g^{rr}\nabla_r\nabla_r F(r) + g^{\theta\theta}\nabla_\theta\nabla_\theta F(r) + g^{\phi\phi}\nabla_\phi\nabla_\phi F(r),$$

$$= [-e^{-2\Phi}(F(r)_{,tt} - \Gamma^{r}{}_{tt}F(r)_{,r})] + [(1-b/r)(F(r)_{,rr} - \Gamma^{r}{}_{rr}F(r)_{,r})]$$
$$+ [(1/r^2)(F(r)_{,\theta\theta} - \Gamma^{r}{}_{\theta\theta}F(r)_{,r})] + [(1/r^2\sin^2\theta)(F(r)_{,\phi\phi} - \Gamma^{r}{}_{\phi\phi}F(r)_{,r})]. \tag{3.30}$$

Substituting the values from (2.5) in (3.30), we get

$$\Box F = \left(1-\frac{b}{r}\right)\left(F'' - \frac{b'r-b}{2r^2(1-b/r)}F'\right) + \left(1-\frac{b}{r}\right)\frac{F'}{r} + \left(1-\frac{b}{r}\right)\frac{F'}{r},$$

$$= \left(1-\frac{b}{r}\right)\left(F'' - \frac{b'r-b}{2r^2(1-\frac{b}{r})}F' + \frac{2F'}{r}\right), \tag{3.31}$$

where prime denotes the radial derivative. Using (3.29) in (3.23), we get

$$T^{(c)}_{\mu\nu} = \frac{1}{F}\left[\nabla_\mu\nabla_\nu F - g_{\mu\nu}H\right]. \tag{3.32}$$

Using (3.9), (3.28) and (3.32) in (3.25), we get the following relationships [3]

$$\frac{b'}{r^2} = \frac{\rho}{F} + \frac{H}{F}, \tag{3.33}$$

$$-\frac{b}{r^3} = \frac{p_r}{F} + \frac{1}{F}\left[\left(1-\frac{b}{r}\right)\left(F'' - F'\frac{b'r-b}{2r^2(1-\frac{b}{r})}\right) - H\right], \tag{3.34}$$

$$-\frac{b'r-b}{2r^3} = \frac{p_t}{F} + \frac{1}{F}\left[\left(1-\frac{b}{r}\right)\frac{F'}{r} - H\right]. \tag{3.35}$$

Equations (3.33) - (3.35) can be expressed as the following field equations [3]

$$\rho = \frac{Fb'}{r^2}, \tag{3.36}$$

$$p_r = -\frac{bF}{r^3} + \frac{F'}{2r^2}(b'r-b) - F''\left(1-\frac{b}{r}\right), \tag{3.37}$$

$$p_t = -\frac{F'}{r}\left(1-\frac{b}{r}\right) + \frac{F}{2r^3}(b-b'r), \tag{3.38}$$

which are the general expressions of the matter threading the wormholes, with shape function $b(r)$ and the specific form of $F(r)$.

3.3.2 Energy Conditions

In view of a radial null vector, the NEC violation i.e. $T^{\text{eff}}_{\mu\nu}k^{\mu}k^{\nu} < 0$ have the following form by using (3.24)

$$\rho^{\text{eff}} + p_r^{\text{eff}} = \frac{\rho + p_r}{F} + \frac{1}{F}\left(1 - \frac{b}{r}\right)\left[F'' - F'\frac{b'r - b}{2r^2(1 - \frac{b}{r})}\right], \tag{3.39}$$

where $\rho^{\text{eff}} + p_r^{\text{eff}} < 0$. Using the field equations (3.36)-(3.38), (3.39) have the form

$$\rho^{\text{eff}} + p_r^{\text{eff}} = \frac{b'r - b}{r^3},$$

which is negative by considering the flare out condition i.e. $(b'r - b)/b^2 < 0$.

Let us assume that the energy conditions are obeyed by the matter threading the wormhole. For this fact, imposing the weak energy condition (WEC), which states that $\rho \geq 0$ and $\rho + p_r \geq 0$, then (3.36) and (3.37) provides the following inequalities [3]

$$\frac{Fb'}{r^2} \geq 0, \tag{3.40}$$

and

$$\frac{(2F + rF')(b'r - b)}{2r^2} - F''\left(1 - \frac{b}{r}\right) \geq 0, \tag{3.41}$$

respectively. Thus, $f(R)$ must obey (3.39), (3.40) and (3.41), while finding the wormhole solutions.

Specific Solutions

In this section, $F(r)$ is obtained from gravitational field equations given by (3.36) - (3.38) by using specific equations of state and by considering different specific shape functions $b(r)$, which yields the parametric form of curvature scalar R given in (3.8). Then, gravitational field equations (3.36) - (3.38) provides the stress-energy tensor threading the wormhole, expressed as a function of energy density $\rho(r)$, transverse pressure $p_t(r)$ and radial pressure $p_r(r)$. Finally, one may obtain specific form of $f(R)$ from (3.20).

3.4 Traceless Stress-Energy Tensor

The traceless stress-energy tensor T is a specific form of equation of state and is given by

$$T = -\rho + p_r + 2p_t = 0. \tag{3.42}$$

Using (3.36) - (3.38) in above equation (3.42), to get the following differential equation

$$\left(1 - \frac{b}{r}\right) F'' - \frac{b'r + b - 2r}{2r^2} F' - \frac{b'r - b}{2r^3} F = 0. \tag{3.43}$$

Let us consider the specific shape function given by

$$b(r) = \frac{r_0^2}{r},$$

where r_0 is radius of wormhole throat and radial derivative of shape function is given as

$$b'(r) = -\frac{r_0^2}{r^2}.$$

Using the above values of $b(r)$ and $b'(r)$, the equation (3.43) provides the following solution

$$F(r) = C_1 \sinh\left[\sqrt{2} \arctan\left(\frac{r_0}{\sqrt{r^2 - r_0^2}}\right)\right] + C_2 \cosh\left[\sqrt{2} \arctan\left(\frac{r_0}{\sqrt{r^2 - r_0^2}}\right)\right]. \tag{3.44}$$

Differentiating $F(r)$ with respect to r, we get

$$F'(r) = \frac{\sqrt{2} r_0}{r^2 \sqrt{r^2 - r_0^2}} \left(C_1 \cosh\left[\sqrt{2} \arctan\left(\frac{r_0}{\sqrt{r^2 - r_0^2}}\right)\right] + C_2 \sinh\left[\sqrt{2} \arctan\left(\frac{r_0}{\sqrt{r^2 - r_0^2}}\right)\right]\right), \tag{3.45}$$

Using values of F and F' from (3.44)and (3.45) respectively and specific shape function $b(r) = r_0^2/r$ into gravitational field equations (3.36) - (3.38), which gives the following relationships

$$\rho(r) = -\frac{r_0^2}{r^4} \left(C_1 \sinh\left[\sqrt{2} \arctan\left(\frac{r_0}{\sqrt{r^2 - r_0^2}}\right)\right] + C_2 \cosh\left[\sqrt{2} \arctan\left(\frac{r_0}{\sqrt{r^2 - r_0^2}}\right)\right]\right), \tag{3.46}$$

$$p_r(r) = -\frac{r_0}{r^4}\Bigg((2C_2\sqrt{2(r^2-r_0^2)}+3r_0C_1)\sinh\left[\sqrt{2}\arctan\left(\frac{r_0}{\sqrt{r^2-r_0^2}}\right)\right]$$
$$+(2C_1\sqrt{2(r^2-r_0^2)}+3r_0C_2)\cosh\left[\sqrt{2}\arctan\left(\frac{r_0}{\sqrt{r^2-r_0^2}}\right)\right]\Bigg), \qquad (3.47)$$

$$p_t(r) = -\frac{r_0}{r^4}\Bigg((C_2\sqrt{2(r^2-r_0^2)}+r_0C_1)\sinh\left[\sqrt{2}\arctan\left(\frac{r_0}{\sqrt{r^2-r_0^2}}\right)\right]$$
$$+(C_1\sqrt{2(r^2-r_0^2)}+r_0C_2)\cosh\left[\sqrt{2}\arctan\left(\frac{r_0}{\sqrt{r^2-r_0^2}}\right)\right]\Bigg). \qquad (3.48)$$

In Figs.(3.1), (3.2) and (3.3), the graphs are drawn for the choice of the parameters $C_1 = 0$, $C_2 = -1$ and for the shape function $b(r) = r_0^2/r$. Fig.(3.1) shows the energy density $\rho(r)$, radial pressure $p_r(r)$ and transverse pressure $p_t(r)$. Fig.(3.2) shows that the stress-energy tensor obeys the WEC, for the specific case of equation of state which is the traceless energy tensor and from graph it is clear that WEC is satisfied, where WEC1 and WEC2 shows that $\rho \geq 0$ and $\rho + p_r \geq 0$ respectively. Fig.(3.2) shows the specific form of $f(R)$ for the traceless energy tensor.

For traceless energy tensor $T = 0$, (3.20) takes the following form

$$f = \frac{1}{2}(FR + 3\Box F). \qquad (3.49)$$

Substituting shape function $b(r) = r_0^2/r$ in the Ricci scalar given by (3.8), provides

$$R = -2r_0^2/r^4,$$

which can be written as

$$r = (-2r_0^2/R)^{1/4}.$$

At the throat, the Ricci scalar given by (3.8) is expressed as

$$R_0 = -2/r_0^2,$$

and is converted to get

$$r_0 = (-2/R_0)^{1/2}.$$

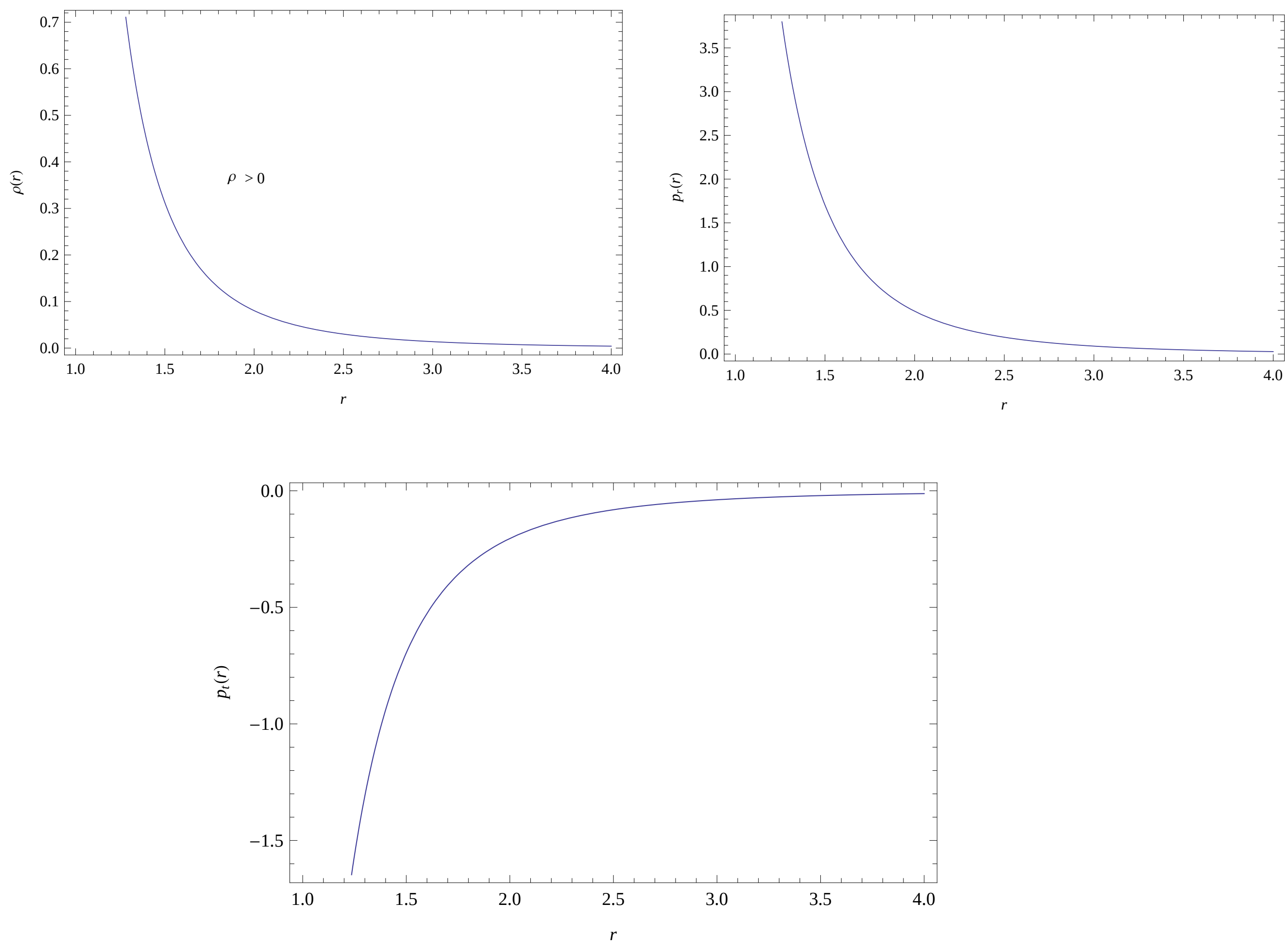

Figure 3.1: Energy density $\rho(r)$, radial pressure $p_r(r)$ and transverse pressure $p_t(r)$ for traceless stress-energy tensor.

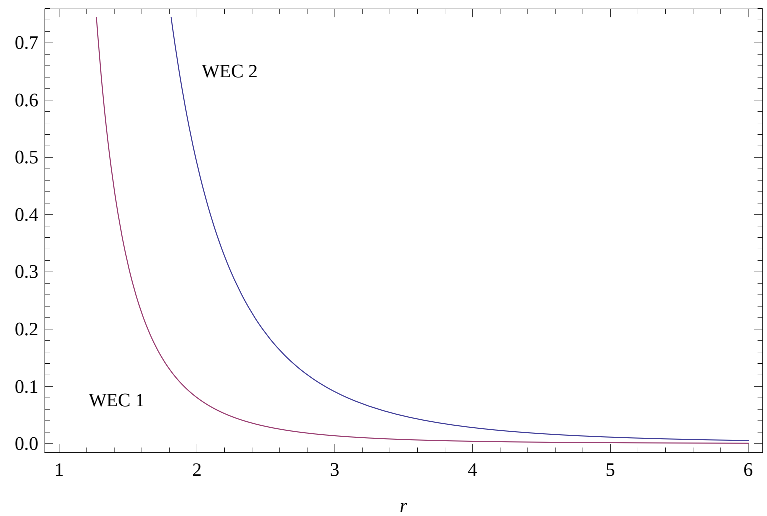

Figure 3.2: Weak energy condition for traceless stress-energy tensor having $b(r) = r_0^2/r$. WEC 1 presents ρ against r and WEC 2 presents $(\rho + p_r)$ against r.

Substituting these relationships and values of $\Box F$ and F from (3.31) and (3.44) respectively in (3.49), we get the specific form $f(R)$ to be

$$f(R) = -R\left[C_1 \sinh\left[\sqrt{2}\arctan\left(\frac{1}{\sqrt{(R_0/R)^{1/2} - 1}}\right)\right] + C_2 \cosh\left[\sqrt{2}\arctan\left(\frac{1}{\sqrt{(R_0/R)^{1/2} - 1}}\right)\right]\right]. \tag{3.50}$$

3.5 Equation of State: $p_t = \alpha\rho$

Consider the equation of state

$$p_t = \alpha\rho, \tag{3.51}$$

where α is a constant. Substituting the values of p_t and ρ from (3.36) and (3.38) respectively in (3.51) to get the following differential equation [3]

$$F'\left(1 - \frac{b}{r}\right) - \frac{F}{2r^2}[b - b'r(1 + 2\alpha)] = 0. \tag{3.52}$$

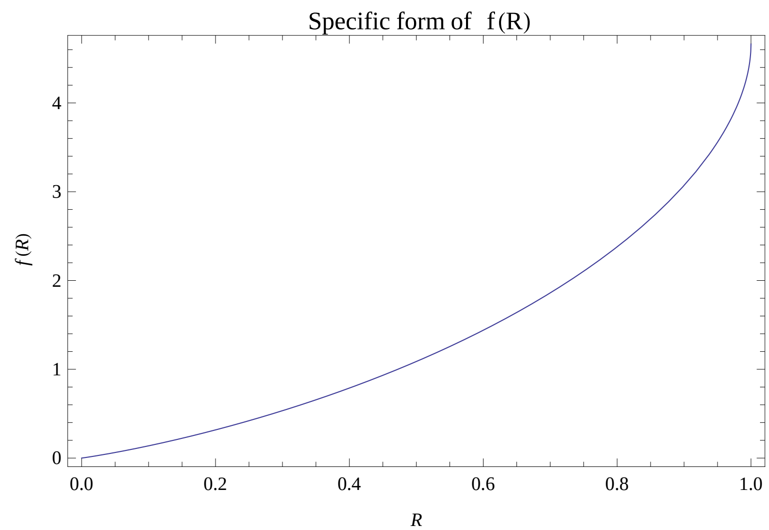

Figure 3.3: The specific form of $f(R)$ for the traceless stress-energy tensor having $b(r) = r_0^2/r$.

3.5.1 Specific Shape Function: $b(r) = \frac{r_0^2}{r}$

Consider specific shape function $b(r) = r_0^2/r$ and its derivative is given as $b'(r) = -r_0^2/r^2$, so that (3.52) yields the following solution

$$F(r) = C_1\left(1 - \frac{r_0^2}{r^2}\right)^{(1+\alpha)/2}. \tag{3.53}$$

Taking radial derivatives of (3.53), we get

$$F'(r) = C_1(1+\alpha)\frac{r_0^2}{r^3}\left(1 - \frac{r_0^2}{r^2}\right)^{(\alpha-1)/2}, \tag{3.54}$$

$$F''(r) = C_1(1+\alpha)\frac{r_0^2}{r^4}\left(1 - \frac{r_0^2}{r^2}\right)^{(\alpha-1)/2}[(\alpha+2)r_0^2 - 3r^2]. \tag{3.55}$$

Using shape function $b(r) = r_0^2/r$, (3.53), (3.54) and (3.55) into gravitational field equations (3.36) - (3.38), which gives the following relationships

$$\rho(r) = -\frac{C_1 r_0^2}{r^4}\left(1 - \frac{r_0^2}{r^2}\right)^{(1+\alpha)/2}, \tag{3.56}$$

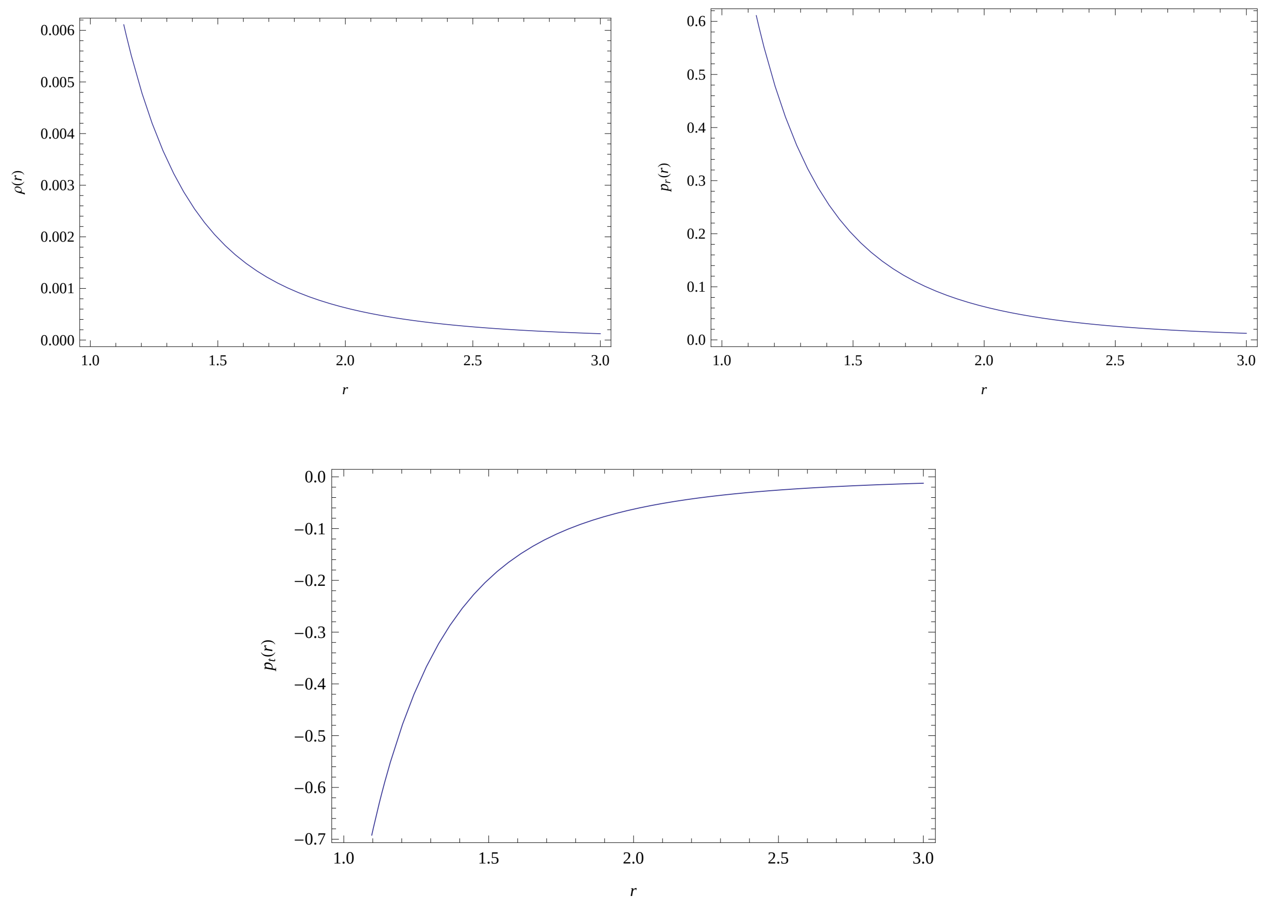

Figure 3.4: Energy density $\rho(r)$, radial pressure $p_r(r)$ and transverse pressure $p_t(r)$ for $p_t = \alpha\rho$.

$$p_r(r) = \frac{C_1 r_0^2}{r^6}\left(1 - \frac{r_0^2}{r^2}\right)^{(-1+\alpha)/2} [2(r^2 - r_0^2) + 3\alpha r^2 - 4r_0^2\alpha - r_0^2\alpha^2], \tag{3.57}$$

$$p_t(r) = -\frac{C_1 r_0^2 \alpha}{r^4}\left(1 - \frac{r_0^2}{r^2}\right)^{(1+\alpha)/2}. \tag{3.58}$$

Using shape function $b(r) = \frac{r_0^2}{r}$ in the Ricci scalar given by (3.8), gives

$$R = -2r_0^2/r^4,$$

and is converted to get

$$r = (-2r_0^2/R)^{1/4}. \tag{3.59}$$

The Ricci scalar given by (3.8) at the throat is given as

$$R_0 = -2/r_0^2,$$

and is converted to get

$$r_0 = (-2/R_0)^{1/2}.$$

Substituting (3.31), (3.53) and (3.59) in (3.20), we get the specific form $f(R)$ which is given as follows

$$f(R) = C_1 R\left(1 - \sqrt{\frac{R}{R_0}}\right)^{(\alpha-1)/2}\left[\sqrt{\frac{R}{R_0}}(\alpha^2 + 2\alpha + 2) + (\alpha + 2)\right]. \tag{3.60}$$

In Figs.(3.4), (3.5) and (3.6), the graphs are drawn for the choice of the parameters $C_1 = 0$, $\alpha = -1$ and for the shape function $b(r) = r_0^2/r$. Fig.(3.4) shows the energy density $\rho(r)$, radial pressure $p_r(r)$ and transverse pressure $p_t(r)$. Fig.(3.5) shows that the stress-energy tensor obeys the WEC for the specific case of equation of state which is $p_t = \alpha\rho$ and from graph it is clear that WEC is satisfied, where WEC1 and WEC2 which presents the $\rho \geq 0$ and $\rho + p_r \geq 0$ respectively. Fig.(3.6) shows the specific form of $f(R)$ for the equation of state $p_t = \alpha\rho$. The (3.60) is depicted in Fig.(3.6) which shows the specific form of $f(R)$.

3.5.2 Specific Shape Function: $b(r) = \sqrt{r_0 r}$

Consider specific shape function $b(r) = \sqrt{r_0 r}$ and its derivative $b'(r) = \frac{1}{2}\sqrt{\frac{r_0}{r}}$, so that (3.52) yields the following solution [3]

$$F(r) = C_1\left(1 - \sqrt{\frac{r_0}{r}}\right)^{(1/2)-\alpha}. \tag{3.61}$$

Differentiating last equation with respect to r, we have

$$F'(r) = \frac{C_1(1-2\alpha)\sqrt{r r_0}}{4r^2}\left(1 - \sqrt{\frac{r_0}{r}}\right)^{-(1/2+\alpha)}, \tag{3.62}$$

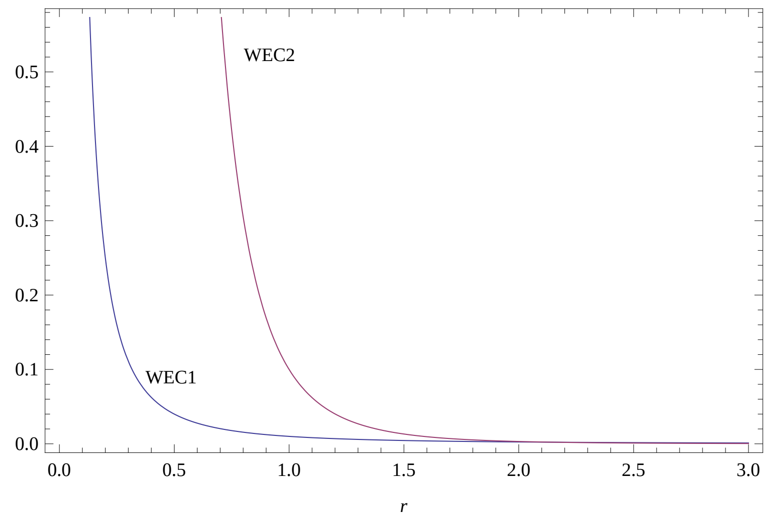

Figure 3.5: The stress-energy tensor satisfying the WEC, for specific case of the equation of state $p_t = \alpha\rho$ having $b(r) = r_0^2/r$. WEC 1 shows ρ v/s r and WEC 2 shows $(\rho + p_r)$ v/s r.

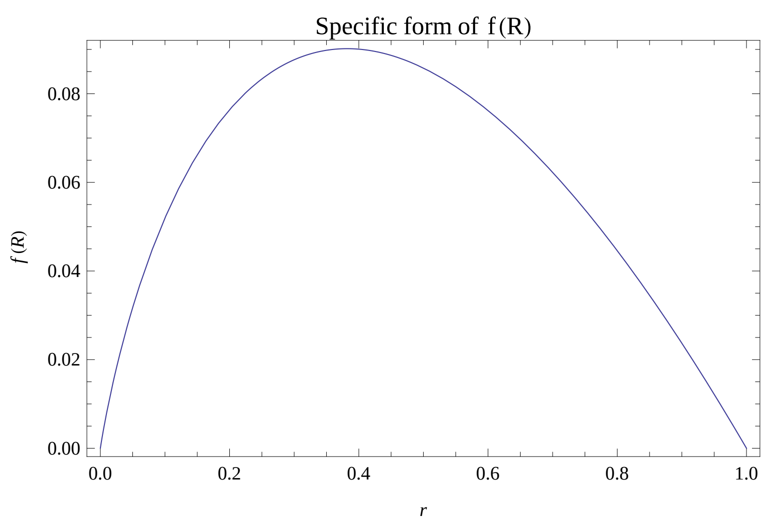

Figure 3.6: The specific form of $f(R)$, for specific case of equation of state $p_t = \alpha\rho$ having $b(r) = r_0^2/r$.

$$F''(r) = \frac{C_1(1-2\alpha)\sqrt{rr_0}}{16r^2}\left(1-\sqrt{\frac{r_0}{r}}\right)^{-(\alpha+\frac{3}{2})}\left(2\alpha\sqrt{\frac{r_0}{r}}-5\sqrt{\frac{r_0}{r}}+6\right). \quad (3.63)$$

Substituting values of F, F', and F'' from (3.61), (3.62) and (3.63) respectively and shape function $b(r) = \sqrt{r_0 r}$ into gravitational field equations (3.36) - (3.38), which gives the following relationships

$$\rho(r) = \frac{C_1}{2r^2\sqrt{r_0/r}}\left(1-\sqrt{\frac{r_0}{r}}\right)^{(1/2)-\alpha}, \quad (3.64)$$

$$p_t(r) = \frac{C_1\alpha}{2r^2\sqrt{r_0/r}}\left(1-\sqrt{\frac{r_0}{r}}\right)^{(1/2)-\alpha}, \quad (3.65)$$

$$p_r(r) = -\frac{C_1 r_0}{16r^3}\left(1-\sqrt{\frac{r_0}{r}}\right)^{-(\alpha+\frac{3}{2})}\left[10\sqrt{\frac{r}{r_0}}+\sqrt{\frac{r_0}{r}}(14\alpha+10)+(4\alpha^2-26\alpha+5)\right]. \quad (3.66)$$

Using value of shape function $b(r) = \sqrt{r_0 r}$ in the Ricci scalar given by (3.8) to get

$$R = \frac{1}{r^2}\sqrt{\frac{r_0}{r}},$$

and is converted to get

$$r = (\sqrt{r_0}/R)^{2/5}. \quad (3.67)$$

The Ricci scalar given by (3.8) at the throat is given as

$$R_0 = 1/r_0^2,$$

which can be written as

$$r_0 = 1/\sqrt{R_0}.$$

Substituting (3.31), (3.61) and (3.67) in (3.20), we get the specific form $f(R)$ which is expressed in following manner

$$f(R) = -\frac{1}{8}\left(\frac{C_1}{R^{2/5}-2(RR_0)^{1/5}+R_0^{2/5}}\right)\times[R_0^{1/5}-R^{1/5}]^{(1/2)-\alpha}R^{(3-2\alpha)/10}R_0^{(-21+10\alpha)/40}$$
$$[-8RR_0^{2/5}+(11+10\alpha)R^{4/5}R_0^{3/5}+(2-22\alpha+4\alpha^2)R^{3/5}R_0^{4/5}$$
$$+(-5+12\alpha-4\alpha^2)R^{2/5}R_0]. \quad (3.68)$$

3.5.3 Specific Shape Function: $b(r) = r_0 + \gamma^2 r_0(1 - r_0/r)$

Consider specific shape function $b(r) = r_0 + \gamma^2 r_0(1 - r_0/r)$ and its radial derivative is $b'(r) = \gamma^2 r_0^2/r^2$ (where $0 < \gamma < 1$), so that (3.52) yields the following solution [3]

$$F(r) = C_1(r - \gamma^2 r_0)^{(1/2)[(\gamma^2-2\alpha-1)/(\gamma^2-1)]} r^{-(\alpha+1)}(r - r_0)^{(1/2)[(\gamma^2(1+2\alpha)-1)/(\gamma^2-1)]}. \quad (3.69)$$

Let

$$X = r - \gamma^2 r_0, \quad Y = r - r_0,$$
$$u = \frac{\gamma^2 - 2\alpha - 1}{2(\gamma^2 - 1)}, \quad v = \frac{\gamma^2(1 + 2\alpha) - 1}{2(\gamma^2 - 1)}.$$

Then (3.69) can be written as

$$F(r) = C_1 X^u r^{-(\alpha+1)} Y^v. \quad (3.70)$$

Taking radial derivative of (3.70), we get

$$F'(r) = C_1[uX^{u-1}r^{-(\alpha+1)}Y^v + vX^uY^{v-1}r^{-(\alpha+1)} - (\alpha+1)X^uY^vr^{-(\alpha+2)}], \quad (3.71)$$

$$F''(r) = C_1[u(u-1)X^{u-2}Y^vr^{-(\alpha+1)} + v(v-1)X^uY^{v-2}r^{-(\alpha+1)} + 2uvX^{u-1}Y^{v-1}r^{-(\alpha+1)}$$
$$- u(\alpha+1)X^{u-1}Y^vr^{-(\alpha+1)}(1 + 1/r) - v(\alpha+1)X^uY^{v-1}r^{-(\alpha+1)}(1 + 1/r)$$
$$+ (\alpha+1)(\alpha+2)X^uY^vr^{-(\alpha+3)}]. \quad (3.72)$$

Substituting (3.70), (3.71) and (3.72) into gravitational field equations (3.36) - (3.38), which gives the following relationships of energy density, radial and transverse pressures, for specific shape function $b(r) = r_0 + \gamma^2 r_0(1 - r_0/r)$

$$\rho(r) = C_1\gamma^2 r_0^2 X^u r^{-(5+\alpha)} Y^v, \quad (3.73)$$

$$p_t(r) = C_1\alpha\gamma^2 r_0^2 X^u r^{-(5+\alpha)} Y^v, \quad (3.74)$$

$$p_r(r) = \frac{C_1}{2r^3} \times X^uY^v[r^{-\alpha}(2\alpha^2 + 6\alpha + 4) + r^{-(1+\alpha)}r_0(-7\alpha + 2\alpha^2\gamma^2 - 3\gamma^2 - 7\alpha\gamma^2 - 2\alpha^2 - 3)$$
$$+ r^{-(2+\alpha)}r_0^2\gamma^2(10\alpha^2 + 4)] + X^uY^{v-1}[r^{-\alpha}r_0v(-\gamma^2(5 + 4\alpha) - \alpha - 5) + 4r^{(1-\alpha)}v(1 + \alpha)$$
$$+ r^{-(1+\alpha)}r_0^2\gamma^2v(4\alpha + 6)] + X^uY^{v-2}[2r^{-\alpha}r_0\gamma^2v(v - \alpha) + 2r^{1-\alpha}r_0v(-v + \gamma^2 + 1)$$
$$+ 2r^{2-\alpha}v(v + 1)] + X^{u-1}Y^v[r^{-\alpha}r_0u(4\alpha + 5)(\gamma^2 + 1) - 4r^{1-\alpha}(u - \alpha)$$
$$- r^{-(1+\alpha)}r_0^2\gamma^2u(4\alpha - 6)] + X^{u-2}Y^v[2r^{-\alpha}r_0^2\gamma^2u(u - 1) + 2r^{1-\alpha}r_0u(1 - u)$$
$$+ 2r^{2-\alpha}u(u - 1)] + X^{u-1}Y^{v-1}[-4r^{-\alpha}r_0^2\gamma^2uv + 4r^{1-\alpha}r_0uv(\gamma^2 + 1) - 4r^{2-\alpha}uv]. \quad (3.75)$$

Substituting value of shape function $b(r) = r_0 + \gamma^2 r_0(1 - r_0/r)$ in the Ricci scalar given by (3.8) to get

$$R = 2\gamma^2 r_0^2/r^4,$$

and is converted to get

$$r = (2\gamma^2 r_0^2/R)^{1/4}. \tag{3.76}$$

The Ricci scalar given by (3.8) at the throat is given as

$$R_0 = 2\gamma^2/r_0^2,$$

which can be written as

$$r_0 = \gamma\sqrt{2/R_0}.$$

Substituting (3.31),(3.70) and (3.76) in (3.20), we get the specific form $f(R)$ expressed as

$$\begin{aligned} f(R) = {} & \frac{C_1 R}{2} \frac{(R_0 R)^{(\alpha+1)/4}}{\gamma^2 - (R_0/R)^{1/4}(\gamma^2+1) + (R_0/R)^{1/2}} \left[\frac{(R_0/R)^{1/4} - \gamma^2}{R_0^{1/2}}\right]^{(1/2)[(\gamma^2-2\alpha-1)/(\gamma^2-1)]} \\ & \times \left[\frac{(R_0/R)^{1/4} - 1}{{R_0}^{1/2}}\right]^{(1/2)[(\gamma^2(1-2\alpha)-1)/(\gamma^2-1)]} \left[2\gamma^2(\alpha^2+2\alpha+2) + \left(\frac{R_0}{R}\right)^{1/2}(2\alpha+4) \right. \\ & \left. - \left(\frac{R_0}{R}\right)^{1/4}(3\alpha+4)(\gamma^2+1)\right]. \end{aligned} \tag{3.77}$$

3.6 Wormhole Geometries in Nonminimal Curvature-Matter Coupling

In this section, i will review the paper "wormhole geometries supported by a nonminimal curvature-matter coupling [5]". Wormhole geometries in curvature-matter coupled modified gravity are explored, by taking an explicit nonminimal coupling between an arbitrary function of the curvature scalar R. The effective stress-energy tensor is responsible for violation of null energy condition because it contains the coupling between the matter Lagrangian density and the higher order curvature derivatives and matter content. In the presence of a nonminimal R-matter coupling, the general restriction imposed by the NEC violation are presented. Moreover, due to nonlinearity of the equations it becomes difficult to evaluate the exact solution to the gravitational field equations. Thus in this section, several approaches for finding wormhole solutions are figured out and obtained an exact solution by taking into consideration a linear R non-minimal curvature-matter coupling and by taking monotonically decreasing function for the energy density [5].

The action S in modified theories of gravity for nonminimal curvature-matter coupling is expressed as [5]

$$S = \int \left[\frac{1}{2} f_1(R) + [1 + \lambda f_2(R)]\mathcal{L}_m\right]\sqrt{-g}d^4x, \tag{3.78}$$

where $\mathcal{L}_m$ denotes the Lagrangian density of matter, $f_i(R)$ (with $i = 1, 2$) denotes the arbitrary functions of the curvature scalar R and λ is the coupling constant describes strength of the interaction between the matter Lagrangian and $f_2(R)$.
Applying 'δ 'and multiplying both sides by $\delta g^{\mu\nu}$, we get

$$\delta S = \int \left[\frac{1}{2}\frac{\delta(f_1\sqrt{-g})}{\delta g^{\mu\nu}} + \frac{\delta(\mathcal{L}_m\sqrt{-g})}{\delta g^{\mu\nu}} + \lambda\frac{\delta(f_2\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}}\right]\delta g^{\mu\nu}d^4x. \tag{3.79}$$

The variation of action S with respect to the metric $g_{\mu\nu}$ provides the following field equations, So (3.79) can takes the following form

$$\frac{1}{2}\frac{F_1\sqrt{-g}\delta R + f_1\delta\sqrt{-g}}{\delta g^{\mu\nu}} + \frac{\delta(\mathcal{L}_m\sqrt{-g})}{\delta g^{\mu\nu}} + \lambda\frac{F_2\sqrt{-g}\mathcal{L}_m\delta R + f_2\delta(\mathcal{L}_m\sqrt{-g})}{\delta g^{\mu\nu}} = 0, \tag{3.80}$$

where F_i (with $i = 1, 2$) denotes the derivative of $f_i(R)$ (with $i = 1, 2$) with respect to r. Equation (3.80) can be written as

$$F_1\frac{\delta R}{\delta g^{\mu\nu}} + \frac{f_1}{\sqrt{-g}}\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} = -2\lambda F_2\mathcal{L}_m\frac{\delta R}{\delta g^{\mu\nu}} + (1 + \lambda f_2)\frac{\delta(\mathcal{L}_m\sqrt{-g})}{\sqrt{-g}\delta g^{\mu\nu}},$$

or

$$F_1\frac{\delta R}{\delta g^{\mu\nu}} + \frac{f_1}{\sqrt{-g}}\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} = -2\lambda F_2\mathcal{L}_m\frac{\delta R}{\delta g^{\mu\nu}} - 2(1 + \lambda f_2)T^{(m)}_{\mu\nu},$$

where $T^{(m)}_{\mu\nu}$ denotes the matter stress-energy tensor. Using values of $\delta\sqrt{-g}$ and δR from (1.35) and (3.17) respectively in last equation, we get
$F_1(R)R_{\mu\nu} - \frac{1}{2}f_1(R)g_{\mu\nu} - \nabla_\mu\nabla_\nu F_1(R) + g_{\mu\nu}\Box F_1(R)$

$$= -2\lambda F_2(R)\mathcal{L}_m R_{\mu\nu} + 2\lambda(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)F_2(R)\mathcal{L}_m + [1 + \lambda f_2(R)]T^{(m)}_{\mu\nu}. \tag{3.81}$$

Trace Equation

The trace equation can be obtained by contraction of field equation (3.81) which is given by

$g^{\sigma\mu}(F_1R_{\mu\nu} - \frac{1}{2}f_1g_{\mu\nu} - \nabla_\mu\nabla_\nu F_1 + g_{\mu\nu}\Box F_1)$

$$= g^{\sigma\mu}(-2\lambda F_2\mathcal{L}_mR_{\mu\nu} + 2\lambda(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)F_2\mathcal{L}_m + [1+\lambda f_2]T^{(m)}_{\mu\nu}),$$

or

$$F_1R^\sigma{}_\nu - \frac{1}{2}f_1\delta^\sigma{}_\nu - \nabla^\sigma\nabla_\nu F_1 + \delta^\sigma{}_\nu\Box F_1 = -2\lambda F_2\mathcal{L}_mR^\sigma{}_\nu + 2\lambda(\nabla^\sigma\nabla_\nu - \delta^\sigma{}_\nu\Box)F_2\mathcal{L}_m + [1+\lambda f_2]T^\sigma{}_\nu,$$

or

$$F_1R - 2f_1 + 3\Box F_1 = -2\lambda(RF_2\mathcal{L}_m + 3\Box\mathcal{L}_mF_2) + (1+\lambda f_2)T^{(m)}. \tag{3.82}$$

Since,

$$\nabla^\mu G_{\mu\nu} = 0,$$

$$\Rightarrow \qquad \nabla^\mu R_{\mu\nu} = \frac{1}{2}g_{\mu\nu}\nabla^\mu R. \tag{3.83}$$

Also from (3.81), we have

$$(\Box\nabla_\nu - \nabla_\nu\Box)F_i = R_{\mu\nu}\nabla^\mu F_i, \qquad \textit{where} \qquad i = 1, 2. \tag{3.84}$$

Taking covariant derivative of (3.81), we get the following expression

$F_1\nabla^\mu R_{\mu\nu} + R_{\mu\nu}\nabla^\mu F_1 - \frac{1}{2}\nabla^\mu f_1g_{\mu\nu} - \nabla^\mu\nabla_\mu\nabla_\nu F_1 + g_{\mu\nu}\nabla^\mu\Box F_1$

$$= -2\lambda\nabla^\mu F_2\mathcal{L}_mR_{\mu\nu} - 2\lambda F_2\mathcal{L}_m\nabla^\mu R_{\mu\nu} + 2\lambda\nabla^\mu\nabla_\mu\nabla_\nu F_2\mathcal{L}_m - 2\lambda g_{\mu\nu}\nabla^\mu\Box F_2\mathcal{L}_m + (1+\lambda f_2)\nabla^\mu T^{(m)}_{\mu\nu} + \lambda\nabla^\mu f_2T^{(m)}_{\mu\nu}.$$

Using (3.83) and (3.84) in the above equation provides the following form

$$\nabla^\mu T^{(m)}_{\mu\nu} = \frac{\lambda F_2}{1+\lambda f_2}\left[g_{\mu\nu}\mathcal{L}_m - T^{(m)}_{\mu\nu}\right]\nabla^\mu R, \tag{3.85}$$

which follows that the coupling between the higher order curvature derivative and the matter terms describes an exchange between energy and momentum.

3.6.1 Specific case: $f_1(R) = R$

For simplicity, consider a special case $f_1(R) = R$, for which (3.81) can be written as $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} \equiv G_{\mu\nu}$

$$= [1 + \lambda f_2(R)]T^{(m)}_{\mu\nu} - 2\lambda[F_2(R)\mathcal{L}_m R_{\mu\nu} - (\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)F_2(R)\mathcal{L}_m]. \tag{3.86}$$

Equation (3.86) may be expressed as effective gravitational field equation, which can be written as following manner

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = T^{\text{eff}}_{\mu\nu},$$

where

$$T^{\text{eff}}_{\mu\nu} = [1 + \lambda f_2(R)]T^{(m)}_{\mu\nu} - 2\lambda[F_2(R)\mathcal{L}_m R_{\mu\nu} - (\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)F_2(R)\mathcal{L}_m]. \tag{3.87}$$

3.6.2 Specific Case: $f_2(R) = R$ and $\mathcal{L}_m = -\rho(r)$

Consider the specific case of $f_2(R) = R$ and choose the Lagrangian density of the form $\mathcal{L}_m = -\rho(r)$ then field equation (3.86) reduces to the following form

$$G_{\mu\nu} = [1 + \lambda R]T^{(m)}_{\mu\nu} + 2\lambda[\rho R_{\mu\nu} - (\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)\rho], \tag{3.88}$$

where $\Box\rho = g^{\mu\nu}\nabla_\mu\nabla_\nu\rho(r)$ is calculated by using metric given by (3.3). The non-vanishing components are given by

$$\Box\rho(r) = g^{tt}\nabla_t\nabla_t\rho(r) + g^{rr}\nabla_r\nabla_r\rho(r) + g^{\theta\theta}\nabla_\theta\nabla_\theta\rho(r) + g^{\phi\phi}\nabla_\phi\nabla_\phi\rho(r),$$

$$\begin{aligned} &= [-e^{-2\Phi}(\rho(r)_{,tt} - \Gamma^r{}_{tt}\rho(r)_{,r})] + [(1 - b/r)(\rho(r)_{,rr} - \Gamma^r{}_{rr}\rho(r)_{,r})] \\ &+ [(1/r^2)(\rho(r)_{,\theta\theta} - \Gamma^r{}_{\theta\theta}\rho(r)_{,r})] + [(1/r^2\sin^2\theta)(F(r)_{,\phi\phi} - \Gamma^r{}_{\phi\phi}\rho(r)_{,r})]. \end{aligned} \tag{3.89}$$

Substituting the values from (2.5) in (3.89), we get

$$\begin{aligned} \Box\rho &= \left(1 - \frac{b}{r}\right)\left(\rho'' - \frac{b'r - b}{2r^2(1 - b/r)}\rho'\right) + \left(1 - \frac{b}{r}\right)\frac{\rho'}{r} + \left(1 - \frac{b}{r}\right)\frac{\rho'}{r}, \\ &= \left(1 - \frac{b}{r}\right)\left(\rho'' - \frac{b'r - b}{2r^2(1 - \frac{b}{r})}\rho' + \frac{2\rho'}{r}\right), \end{aligned} \tag{3.90}$$

where prime describes radial derivative.

Gravitational Field Equations

Using values of the Ricci tensor (3.8), the Einstein tensor (3.9), $\Box\rho$ from (3.90) and stress-energy tensor (3.27) in (3.88), then the gravitational field equations can be expressed by the following relationships

$$2\lambda r\rho''(b-r)+\lambda\rho'(rb'+3b-4r)+\rho(r^2+2\lambda b')-b'=0, \tag{3.91}$$

$$4\lambda r\rho'(b-r)+2\lambda\rho(b-b'r)-rp_r(r^2+2\lambda b')-b=0, \tag{3.92}$$

$$4\lambda r^2\rho''(b-r)+2\lambda r\rho'(rb'+b-2r)-2\lambda\rho(rb'+b)-2rp_t(r^2+2\lambda b')+b-rb'=0. \tag{3.93}$$

Form of Shape Function

The form of general solution of (3.91) for $b(r)$ is expressed as following form

$$b(r)=\left[\int\frac{re^{[h(r)]}(-r\rho+2\lambda r\rho''+4\lambda\rho')}{\lambda(r\rho'+2\rho)-1}dr+C\right]e^{[-h(r)]}, \tag{3.94}$$

where C denotes the integration constant and $h(r)$ is defined as

$$h(r)=\lambda\int\frac{3\rho'+2r\rho''}{\lambda(r\rho'+2\rho)-1}dr.$$

Radial and Transverse Pressures

From (3.92) and (3.93), the radial and lateral pressures are given by

$$p_r(r)=\frac{4\lambda r\rho'(b-r)+2\lambda\rho(b-b'r)-b}{r(r^2+2\lambda b')}, \tag{3.95}$$

$$p_t(r)=\frac{4\lambda r^2\rho''(b-r)+2\lambda r\rho'(rb'+b-2r)-2\lambda\rho(rb'+b)+b-rb'}{2r(r^2+2\lambda b')}. \tag{3.96}$$

3.6.3 Specific Energy Density: $\rho(r)=\rho_0(r_0/r)^\alpha$

Consider the monotonically decreasing function of energy density which is given as

$$\rho(r)=\rho_0\left(\frac{r_0}{r}\right)^\alpha, \tag{3.97}$$

with energy density at the throat $\rho_0>0$ and $\alpha>0$. Using value of $\rho(r)$ from (3.97) into (3.94), provides the shape function $b(r)$ which is expressed as

$$b(r) = \left(\frac{r\rho_0}{\alpha-3}\right)\left(\frac{r_0}{r}\right)^{\alpha}\Bigg(C + 2\alpha\lambda(\alpha-3)hypergeom\Bigg(\left[\frac{\alpha-1}{\alpha}, -\frac{1+\alpha}{\alpha-2}\right], \left[\frac{2\alpha-1}{\alpha}\right],$$
$$-\rho_0\lambda\left(\frac{r_0}{r}\right)^{\alpha}(\alpha-2)\Bigg) - r^2 hypergeom\Bigg(\left[-\frac{\alpha+1}{\alpha-2}, \frac{\alpha-3}{\alpha}\right], \left[\frac{2\alpha-3}{\alpha}\right],$$
$$-\rho_0\lambda(\frac{r_0}{r})^{\alpha}(\alpha-2)\Bigg)\Bigg)\left(\rho_0\lambda\left(\frac{r_0}{r}\right)^{\alpha}(\alpha-2)+1\right)^{(1-2\alpha)/(\alpha-2)}, \quad (3.98)$$

The spacetime geometry is asymptotically flat that is $b(r)/r \to 0$ for $r \to \infty$ if $\alpha \geq 3$. In particular, imposing $\alpha = 3$, then (3.94) provides the following solution for $b(r)$

$$b(r) = \Bigg[\frac{6}{7}\rho_0^5\lambda^5\frac{r_0^{15}}{r^{14}} - \frac{1}{12}\rho_0^5\lambda^4\frac{r_0^{15}}{r^{12}} + \frac{48}{11}\rho_0^4\lambda^4\frac{r_0^{12}}{r^{11}} - \frac{4}{9}\rho_0^4\lambda^3\frac{r_0^{12}}{r^9} + 9\rho_0^3\lambda^3\frac{r_0^9}{r^8}$$
$$-\rho_0^3\lambda^2\frac{r_0^9}{r^6} + \frac{48}{5}\rho_0^2\lambda^2\frac{r_0^6}{r^4} - \frac{4}{3}\rho_0^2\lambda\frac{r_0^6}{r^3} + 6\rho_0\lambda\frac{r_0^3}{r^2} + \rho_0 r_0^3\ln(r) + C\Bigg]\left(1+\lambda\rho_0\left[\frac{r_0}{r}\right]^3\right)^{-5}. \quad (3.99)$$

Consider the condition $b(r_0) = r_0$, through which constant of intergration is deduced and given by

$$C = r_0\Bigg[(1+\lambda\rho_0)^5 - \Bigg(\frac{6}{7}\rho_0^5\lambda^5 - \frac{1}{12}\rho_0^5\lambda^4 r_0^2 + \frac{48}{11}\rho_0^4\lambda^4 - \frac{4}{9}\rho_0^4\lambda^3 r_0^2 + 9\rho_0^3\lambda^3 - \rho_0^3\lambda^2 r_0^2 + \frac{48}{5}\rho_0^2\lambda^2$$
$$-\frac{4}{3}\rho_0^2\lambda r_0^2 + 6\rho_0\lambda + \rho_0 r_0^2\ln(r_0)\Bigg)\Bigg]. \quad (3.100)$$

In Figs. (3.7) and (3.8), the graphs are drawn for the values $r_0 = 1$ and $\rho_0 = 0.75$. The shape function (3.99) is plotted in Fig.(3.7). The fundamental conditions for wormhole i.e. $b/r \to 0$ as $r \to \infty$ and $b(r) < r$ are followed. At throat null energy condition is violated for the normal matter threading, as shown in Fig.(3.8).

3.6.4 General Relativistic Case: $\lambda = 0$

By taking into account $\lambda = 0$ and the energy density ρ given in (3.97), the field equation (3.91) provides the following form of shape function

$$b(r) = r_0 + \rho_0 r_0^3 \ln\left(\frac{r_0}{r}\right). \quad (3.101)$$

Differentiating $b(r)$ with respect to r, we have

$$b'(r) = \frac{\rho_0 r_0^3}{r}.$$

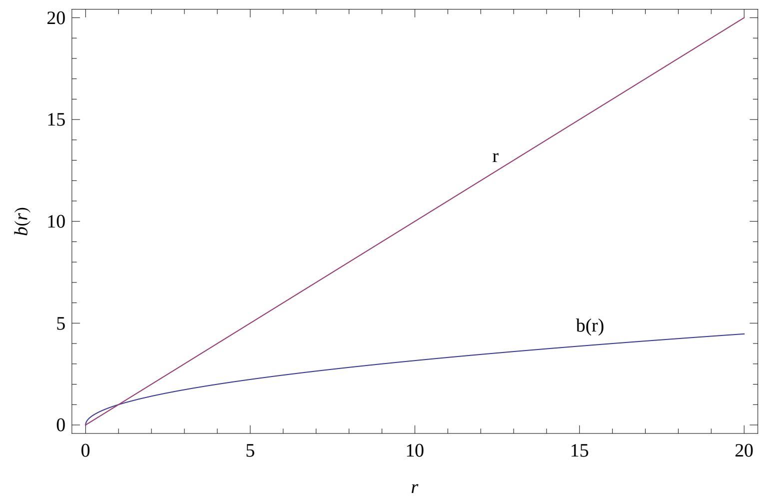

Figure 3.7: The shape function $b(r)$ is plotted for the values $\lambda = 0.1$, $r_0 = 1$ and $\rho_0 = 0.75$. The conditions, $b/r \to 0$, as $r \to \infty$, and $b(r) < r$ are obeyed.

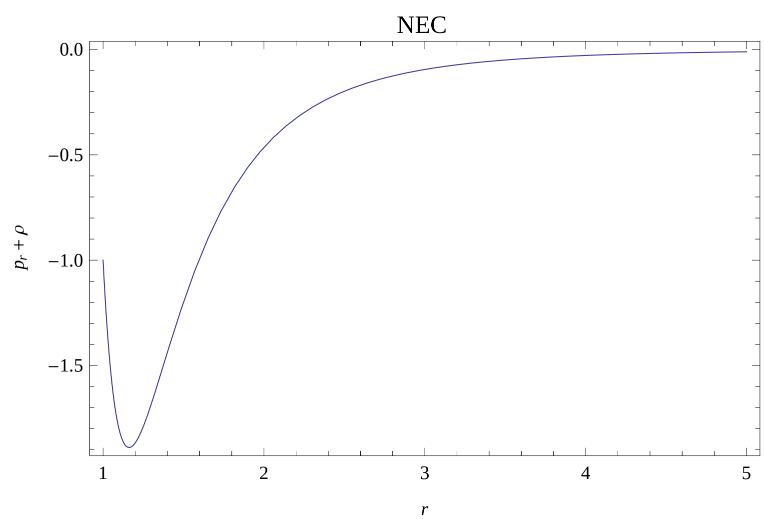

Figure 3.8: The null energy condition NEC profile for the values $\lambda = 0.1$, $r_0 = 1$ and $\rho_0 = 0.75$.

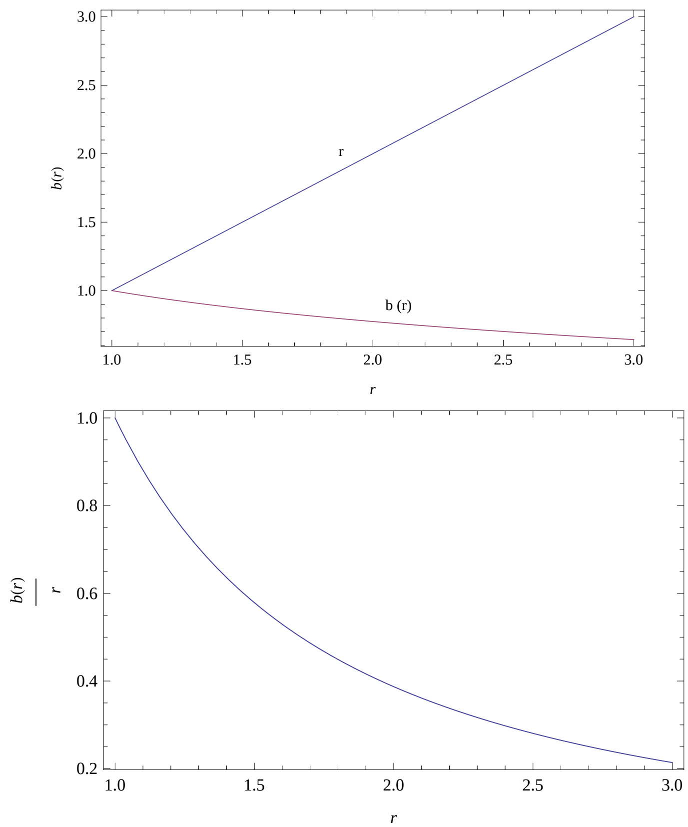

Figure 3.9: The wormhole conditions, $b(r) < r$ and $b/r \to 0$, as $r \to \infty$, are obeyed.

At the throat, the flaring out condition for wormhole, $b'(r_0) < 1$ is imposed the restriction $\rho_0 r_0^2 < 1$. The radial pressure $p_r(r)$ and transverse pressure $p_t(r)$ are given by

$$p_r(r) = -\frac{r_0}{r^3}\left[1 + \rho_0 r_0^2 \ln\left(\frac{r}{r_0}\right)\right], \tag{3.102}$$

$$p_t(r) = \frac{r_0}{r^3}\left[1 + \rho_0 r_0^2\left(\ln\left(\frac{r}{r_0}\right) - 1\right)\right], \tag{3.103}$$

respectively.

The graphs are drawn for the values $r_0 = 1$ and $\rho_0 = 0.75$ in Figs. (3.9) and (3.10). The conditions for the wormhole, $b(r) < r$ and $b/r \to 0$ as $r \to \infty$ are satisfied in Fig.(3.9). From graph in Fig.(3.10) by considering the condition at the throat i.e.

$b'_0 < 1$, one may easily confirm that the positive parameter for nonminimal coupling lies in the range $0 < \lambda < 0.16$. The Eq. (3.104) is depicted in Fig. (3.10), which exposed the possibility to minimize the violation of null energy condition for increasing values of λ.

3.6.5 Minimizing NEC Violation

Normal matter minimizes the violation of null energy condition at the throat by increasing in the value of parameter λ. The null energy condition of the matter threading wormhole at throat is expressed as

$$(\rho + p_r) \mid_{r_0} = \frac{\rho_0^2 \lambda (2\lambda + r_0^2) + \rho_0 (\lambda + r_0^2) - 1}{3\rho_0 \lambda (2\lambda + r_0^2) + r_0^2}. \tag{3.104}$$

By taking into consideration the values $\rho_0 = 0.75$, $r_0 = 1$ and $b'_0 < 1$, one may verify that the nonminimal coupling parameter λ lies in the range $0 < \lambda < 1/6$. From this, wormhole geometries supported by a nonminimal curvature-matter coupling is constructed. Although finding exact solutions of the wormhole at the throat satisfying the energy conditions is a tough task. It is only possible by minimizing the NEC violation at the throat of normal matter.

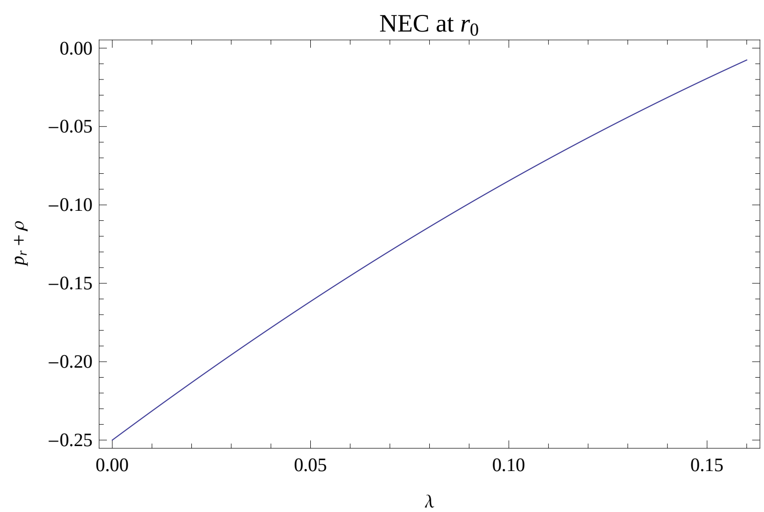

Figure 3.10: The general relativistic deviation for $\lambda = 0.1$, $r_0 = 1$ and $\rho_0 = 0.75$, considered at the throat.

Chapter 4

CONCLUSION

In general relativity, a fundamental fact of static traversable wormholes is the NEC violation. In spite of this fact, it was shown that NEC and the WEC can be avoided for time-dependent wormhole solutions, for some specific interval of time at the throat. This fact can be solved by taking into consideration the modified EFEs, one may impose the stress-energy tensor threading the wormhole satisfies the null energy condition.

In chapter 3, the possibility that $f(R)$ modified theories of gravity can support the wormholes are explored. In this analysis, to make calculations simple and easy, the redshift function is considered as a constant. Field equations are solved by taking into account the specific equation of state and by specify the shape function $b(r)$. From gravitational field equations the function $F(r)$ is figure out. The curvature scalar $R(r)$ is deduced from its definition and by using wormhole metric. The exact solutions of function $f(R)$ are found from trace equation. Energy conditions are also checked for specific solutions.

Furthermore, wormhole geometries in curvature-matter coupled modified gravity are investigated. This is done by considering an explicit nonminimal coupling between an arbitrary function of the scalar curvature R, and the Lagrangian density of matter. The nonlinearity of the equations create difficulty in finding the exact solutions to the field equations. Thus numerous approaches for deducing wormhole solutions are outlined and obtained an exact solution by taking into account a linear R nonminimal curvature-matter coupling and by considering function for the energy density which is explicit monotonically decreasing. Exact wormhole solutions are found where the normal matter minimizes the violation of NEC at the throat.

Likewise, one may also work out numerous results for the forms of $f(R)$ gravity by selecting different equation of states and by the choice of several shape functions.

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