Decomposition of Elasticity Tensor

by

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MASTER'S THESIS WORK

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Abstract

The 4th rank elasticity tensor C_{ijkl} is the constant of proportionality in Hooke's Law. In linear anisotropic elasticity, C_{ijkl} describes the elastic properties of a medium. The decomposition of C_{ijkl} has been studied in 2-dimensions and then in 3-dimensions. There are two ways to decompose the elasticity tensor C_{ijkl} in 3-dimensions. The first one is RS-decomposition which is reducible and the second one is VW-decomposition which is irreducible, under the 3-dimensional general linear group. The irreducible tensors of VW-decomposition of the elasticity tensor is further decomposed under the rotation group.

The properties of VW-decomposition are: uniqueness, irreducible and preservation of the symmetries of the elasticity tensor. It is valid from an algebraic and physical point of view. On the other hand, RS-decomposition is not unique, is reducible and does not preserve the symmetries of the elasticity tensor. It is inferior and fails to have these useful properties from an algebraic and physical point of view. Many physical applications of VW-decomposition are reviewed in the thesis.

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Humera Fatima

Dedicated

to my

parents and husband

Contents

Chapter 1

Introduction

In this chapter, a brief descriptions about basic principles of the theory of linear elasticity, Hooke's law and its importance, objective of the thesis and also a brief introduction about the chapters are given.

Theory of Linear Elasticity

The theory of linear elasticity is a branch of Continuum Mechanics. It is one of the most useful and successful theories in mathematical physics. Elasticity is the characteristic of a solid material to return to its original size and shape after applied forces are removed. It is the ability of the body to resist the distortion in the object because of applied force. Elasticity theory deals with deformable solid bodies. The relationship between stress and strain is stated by Hooke's law. This law is the basic law in the theory of linear elasticity.

Hooke's Law in Physics and Its Importance

Robert Hooke was famous for his law of elasticity which is known as "Hooke's Law". In 1665, he discussed the main concept of the stress, strain and deformation of the elastic objects in a equilibrium state. Hooke's law states that the resorting force of the spring is proportional to the extension or compression of the spring from its equilibrium. Its can also be expressed in the form of a formula as $F = -kx$ where F is a force, k is a spring constant, x is a extension and negative sign indicates that the force is in the opposite direction from the extension. It can also be expressed in terms of stress and strain. Stress is force per unit area within the material , which is caused by an externally applied force. Strain is the relative deformation produced by stress. For relatively small stresses, stress is directly proportional to strain. Hooke first presented his law in the form of a Latin anagram which translates in contemporary language as "extension is directly proportional to force." This law is obeyed by the elastic objects and every spring. It is important in physics since it helps to determine the elasticity of objects. It also helps to calculate elastic potential energy.

The constitutive relation for linear anisotropic elasticity is the generalized Hooke's law which describes the most general linear relationship between stress and strain tensors. The 4th rank elasticity tensor C_{ijkl} emerges from Hooke's law which is the constant of proportionality. Elasticity tensor is also known as stiffness or compliance tensor. Its physical components carry the dimension of force per unit area. It obeys the major and minor (left and right) symmetries. Elasticity tensor, in three dimensions, has 81 components. Due to symmetries of the stress and strain and the strain energy density function, the number of independent components reduces to 21 only. The physical properties of anisotropic elastic materials are described by the tensor such as the elasticity tensor, C_{ijkl} of 4th rank. Moreover, the elastic constants of anisotropic materials are written as 6×6 matrix C_{ijkl} [1]. The components of the elasticity tensor which describes the physical properties of anisotropic elastic materials, depend on the system of coordinate axes and the tensors are usually represented in matrix form.

Objective of the Thesis

In this thesis, we consider the 4th rank elasticity tensor, which results from generalized Hooke's law. Our objectives are

1. To study the decomposition of elasticity tensor under permutation and rotation groups.
2. To study the algebra of the decomposition of the elasticity tensor.
3. To study the decomposition of the elasticity tensor from algebraic and physical point of view.
4. To study the physical applications of the VW-decomposition.

Plan of Work

This thesis is divided into five chapters. The thesis has been organized in the following manner:

In chapter 2, we have reviewed some basics definitions, concepts and results. This chapter contains a brief discussion on tensors, constitutive equation, elasticity tensor and reduction of its components due to the symmetries of the elasticity tensor. Elasticity tensor is also explained in the isotropic and anisotropic material.

In chapter 3, we have reviewed the decomposition of the elasticity tensor in two-dimensional under the rotation group $SO(2)$.

In chapter 4, we have discussed the algebra of the elasticity tensor and its decompositions in 3-dimensional. Also, VW-decomposition is discussed under the permutation and rotation groups. Moreover, the algebraic properties of both decompositions are discussed.

In chapter 5, we discuss some physical applications of the irreducible decomposition of the elasticity tensor.

In chapter 6, we give the summary and conclusions of the thesis.

Chapter 2

Preliminaries

In this chapter, we will recollect some basic definitions, concepts and relevant results that would be used throughout this thesis. We will also be giving some examples to illustrate the concepts. Some important notions and the terminology used are also introduced.

2.1 Basic Concepts in Tensor Analysis

The concept of tensor is a fundamental concept in the elasticity theory. Tensors are also used in many fields such as continuum mechanics, general relativity, differential geometry.

2.1.1 Tensor and Its Linear Transformation

Suppose $\boldsymbol{V}$ be a real vector space and the vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}, \cdots$ are the elements of a vector space $\boldsymbol{V}$.

A Tensor of Order 1

A vector $\mathbf{x}$ is defined to be a tensor of order 1.

A Tensor of Order 2

Let $\mathbf{T} : \boldsymbol{V} \rightarrow \boldsymbol{V}$ be a linear transformation from a vector space $\boldsymbol{V}$ into the same vector space $\boldsymbol{V}$. It is known as a tensor of order 2. It can be written as

$$\mathbf{x} = \mathbf{T}\mathbf{y}, \quad \mathbf{x}, \mathbf{y} \,\epsilon\, \boldsymbol{V}$$

and

$$\mathbf{T}(a\mathbf{x} + b\mathbf{y}) = a\mathbf{T}\mathbf{x} + b\mathbf{T}\mathbf{y}, \quad \forall\ \mathbf{x}, \mathbf{y} \,\epsilon\, \boldsymbol{V},\ \forall\ a, b \,\epsilon\, \mathbb{F}.$$

It satisfies all the axioms of a vector space. Let L_2 denote the vector space of all the tensors of order 2.
The product of two vectors $\mathbf{x}$, $\mathbf{y}$ such that $\mathbf{x} \otimes \mathbf{y} = \mathbf{xy}$, where $\mathbf{x},\ \mathbf{y} \ \epsilon\, \boldsymbol{V}$ and having the values of these two vectors in $\boldsymbol{V}$ such that it is a linear transformation which is known as a tensor product. It can be expressed as

$$(\mathbf{x} \otimes \mathbf{y})\mathbf{z} = \mathbf{xy}(\mathbf{z}) = \mathbf{x}(\mathbf{y.z}),\ \forall\ \mathbf{z}\ \epsilon\ \boldsymbol{V}$$

and

$$\mathbf{xy}(a\mathbf{r} + b\mathbf{s}) = a\mathbf{xy}(\mathbf{r}) + b\mathbf{xy}(\mathbf{s}),\ \forall\ \mathbf{r},\ \mathbf{s}\ \epsilon\ \boldsymbol{V}.$$

Thus the tensor product of two vectors $\mathbf{x}, \mathbf{y}$ is also a tensor of order 2.
Suppose $\{e_m\}(m = 1 \cdots n)$ is an orthonormal basis in $\boldsymbol{V}_n$. Consider $\mathbf{T}$ to be an arbitrary tensor which belongs to L_2. Thus, $\mathbf{T}e_k\ \epsilon\ \boldsymbol{V}_n$ which can be written as

$$\mathbf{T}e_k = T_{mk}e_m\ \forall\ \mathbf{T}\ \epsilon\ L_2$$

which shows that

$$\mathbf{T} = T_{mk}e_me_k,\ \forall\ \mathbf{T}\epsilon\ L_2 \tag{2.1}$$

where $\{e_me_k\}(m, k = 1 \cdots, n)$ represent basis in the vector space L_2 with n^2 dimension. and the components $\mathbf{T}_{mk}$ of $\mathbf{T}$ can be represented as an $m \times m$ matrix

$$\mathbf{T} = (T_{mk}) = \begin{pmatrix} T_{11} & T_{12} & T_{13} & \cdots & T_{1n} \\ T_{21} & T_{22} & T_{23} & \cdots & T_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ T_{n1} & T_{n2} & T_{n3} & \cdots & T_{1n} \end{pmatrix}. \tag{2.2}$$

Suppose $\{e_me_k\}$ $(m, k = 1, \cdots, n)$ and $\{e^{'}_se^{'}_t\}$ $(s, t = 1, \cdots, n)$ are two basis in L_2 correspond the two orthonormal basis $\{e_m\}$ $(m = 1, \cdots, n)$ and $\{e^{'}_m\}$ $(m = 1, \cdots, n)$ in $\boldsymbol{V}_n$. The tensor $\mathbf{T}$ can be written in the form of these two bases as

$$\mathbf{T} = T_{mk}e_me_k = T^{'}_{st}e^{'}_se^{'}_t. \tag{2.3}$$

The component of T_{mk} can be written as

$$\begin{aligned} T_{mk} &= e_m.\mathbf{T}e_k = (q_{mi}e^{'}_i).\mathbf{T}(q_{kj}e^{'}_j), \\ &= q_{mi}q_{kj}Te^{'}_ie^{'}_j, \\ &= q_{mi}q_{kj}T^{'}ij. \end{aligned}$$

This explain the transformation law of the components of a tensor corresponding the change of basis in L_2 which can also be expressed in term of matrix form as

$$\mathbf{T} = \mathbf{QTQ^T}. \tag{2.4}$$

Equation. (2.4) is equivalent to

$$\mathbf{T}^{'} = \mathbf{Q}^{\mathbf{T}}\mathbf{T}\mathbf{Q}.$$

It can be written in component form

$$T^{'}_{st} = q_{ms}q_{kt}T_{mk}. \tag{2.5}$$

Tensors of Order 3

Suppose $\mathbf{T}^{'} : \boldsymbol{V} \rightarrow L_2$ be a linear transformation from the vectors of a vector space $\boldsymbol{V}$ into the vector space of a tensor of order 2.

$$\mathbf{T}^{''} = \boldsymbol{V}(\mathbf{y}) = \boldsymbol{V}\mathbf{y},$$

where $\mathbf{T}^{''}$ denotes a tensor of order 2. Similarly, L_3 denotes the vector space of a tensor of order 3.

2.1.2 Transformation Matrix

Consider $\boldsymbol{V}$, a real vector space. The vectors **x, y, z,...** are elements of the vector space $\boldsymbol{V}$, if they fulfill all the axioms of a vector space $\boldsymbol{V}$. $\boldsymbol{V}_n$ denotes the n-dimensional vector space. Assume two orthonormal bases such as $\{e_k\}$ $(k = 1, 2, \cdots, n)$ and $\{e^{'}_l\}$ $(l = 1, 2, \cdots, n)$ in $\boldsymbol{V}_n$. These basis are related by the following equations

$$e^{'}_l = q_{kl}e_k,$$

and

$$e_k = q^{'}_{lk}e^{'}_l,$$

where the matrix $\mathbf{Q} = [q_{kl}]$ is the transformation matrix and $\mathbf{Q}^{'} = [q^{'}_{lk}]$ is the inverse of $\mathbf{Q} = [q_{kl}]$ matrix. The transformation matrix $\mathbf{Q}$ is given by

$$\mathbf{Q} = \begin{pmatrix} q_{11} & q_{12} & q_{13} & \cdots & q_{1l} \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ q_{k1} & q_{k2} & q_{k3} & \cdots & q_{kl} \end{pmatrix},$$

and the inverse of the matrix $\mathbf{Q}$ is $\mathbf{Q}^{'}$ have the form

$$\mathbf{Q}^{'} = \begin{pmatrix} q^{'}_{11} & q^{'}_{12} & q^{'}_{13} & \cdots & q^{'}_{1l} \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ q^{'}_{k1} & q^{'}_{k2} & q^{'}_{k3} & \cdots & q^{'}_{kl} \end{pmatrix}.$$

Since this matrix represents the change of an orthonormal basis, therefore it is an orthogonal matrix and follows the following properties,

$$\mathbf{Q}\mathbf{Q}^{\mathbf{T}} = \mathbf{I},\ \det\ \mathbf{Q} = \pm 1 \text{ and } \mathbf{Q}^{-1} = \mathbf{Q}^{\mathbf{T}},$$

where $\mathbf{I}$ is the $n \times n$ identity matrix and has the following form

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

2.1.3 Cartesian Tensor

A tensor having components T_{ij}, which transform according to Eq. (2.5) is a Cartesian tensor of order 2. In general components of a Cartesian tensor of order n transform as

$$T^{'}_{j_1 j_2 \cdots j_n} = q_{i_1 j_1} \cdots q_{i_n j_n} T_{i_1 i_2 \cdots i_n}.$$

2.1.4 Symmetric and Antisymmetric Tensors

A tensor $\mathbf{T}_{i_1 i_2 .....i_n}$ is said to be symmetric with respect to i_1 and i_2 if and only if

$$\mathbf{T}_{(i_1 i_2).....i_n} = \mathbf{T}_{i_2 i_1 .....i_n},$$

and generally symmetric tensor is defined as:

$$\mathbf{T}_{(i_1 i_2 .....i_n)} = \frac{1}{n!}[\mathbf{T}_{(\text{sum over all permutations of } i_1 i_2 .....i_n)}].$$

Example: Consider a tensor T_{ijk} of order 3. Then

$$T_{(ijk)} = \frac{1}{6}[T_{ijk} + T_{jki} + T_{kij} + T_{ikj} + T_{jik} + T_{kji}].$$

Interchanging ij, we get

$$T_{(ijk)} = T_{(jik)}.$$

A tensor $\mathbf{T}_{i_1 i_2 .....i_n}$ is said to be antisymmetric with respect to i_1 and i_2 if and only if

$$\mathbf{T}_{[i_1 i_2].....i_n} = -\mathbf{T}_{i_2 i_1 .....i_n}.$$

In general, an antisymmetric tensor is defined as:

$$\mathbf{T}_{[i_1 i_2 .....i_n]} = \frac{1}{n!}[\mathbf{T}_{\text{(sum over all even permuations)}} - \mathbf{T}_{\text{(sum over all odd permutations)}}].$$

Example: Consider a tensor T_{ijk} of order 3. Then

$$T_{[ijk]} = \frac{1}{6}[T_{ijk} + T_{jki} + T_{kij} - T_{kji} - T_{ikj} - T_{jik}].$$

Interchanging ij, we get

$$T_{[ijk]} = -T_{[jik]}.$$

2.1.5 Levi-Civita Tensor

The Levi-Civita tensor [2, 3] also called the permutation tensor [4], antisymmetric tensor, or alternating tensor, is a 3-index mathematical tensor used in particular in tensor calculus. It is named after an Italian mathematician and physicist Tullio Levi-Civita.

In 3 dimensions, the Levi-Civita tensor is defined as follows

$$\epsilon_{ijk} = \begin{cases} +1 & \text{for} \quad (i,j,k) \in (1,2,3), \quad (2,3,1), \quad (3,1,2), \\ -1 & \text{for} \quad (i,j,k) \in (1,3,2), \quad (3,2,1), \quad (2,1,3), \\ 0 & \text{for} \quad i=j \quad \text{or} \quad j=k \quad \text{or} \quad k=i, \end{cases}$$

There are 27 components of Levi-Civita tensor in 3-dimensional space. However, 21 components are zero because the index is repeated. The remaining components are 6, three take the value of +1 if permutation is even and the other three take the value of −1. In index notation, the Levi-Civita tensor i.e. ϵ_{ijk} is very useful when expressing some results in compact form.

The Levi-Civita tensor can be generalized to higher dimensions

$$\epsilon_{ijkl......} = \begin{cases} +1 & \text{if} \quad (i,j,k,l,......) \text{ is an even permutation of } (1,2,3,4,......) \\ -1 & \text{if} \quad (i,j,k,l,......) \text{ is an odd permutation of } (1,2,3,4,......) \\ 0 & \text{if} \quad \text{any two indices are the same.} \end{cases} \}$$

Thus, it is the sign of the permutation in the case of even or odd permutation, zero other-wise.

Properties of Levi-Civita Tensor

The Levi-Civita tensor satisfies the following properties:

$$\begin{aligned} \delta_{ij}\epsilon_{ijk} &= 0, \\ \epsilon_{ijk}\epsilon_{pqk} &= \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}, \\ \epsilon_{ipq}\epsilon_{jpq} &= 2\delta_{ij}, \\ \epsilon_{ijk}\epsilon_{ijk} &= 6, \end{aligned} \tag{2.6}$$

where δ_{ij} is the Kronecker delta [3].

2.1.6 Stress Tensor

Stress tensor is a 2nd rank symmetric tensor which is denoted by σ_{ij}. In the space of 3-dimensions, it has 9 components in which 6 are independent and 3 are dependent. The symmetry of σ_{ij} follows from the assumptions of vanishing torque stresses and vanishing body couples.

2.1.7 Strain Tensor

Strain tensor is also a 2nd rank symmetric tensor which is denoted by ε_{kl}. It has also 9 components in which 6 are independent and the remaining components are dependent. In linear elastic, it is defined in term of the displacement vector i.e.,

$$\varepsilon_{kl} = \frac{1}{2}(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}).$$

2.2 Constitutive Equation

A constitutive equation is the relation between the stress and the strain It is supposed that σ_{ij} is a function of ε_{kl} that is $\sigma_{ij}(\varepsilon_{kl})$ and vice-versa. There is a one-to-one correspondence between stress and strain. Here, the constitutive equations are considered is linear in nature. In the Taylor expansion of the equation, the first order term adequately describe the elastic behaviour of most substances.

$$\sigma_{ij}(\varepsilon_{kl}) = \sigma_{ij}(0) + \frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}}|_{\varepsilon_{kl}=0}\varepsilon_{kl} + \cdots$$

or, since $\sigma_{ij}(0) = 0$, therefore, we have approximately,

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}, \tag{2.7}$$

where

$$C_{ijkl} = \frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}}|_{\varepsilon_{kl}=0}.$$

Equation. (2.7) is called the generalized Hooke's Law. This law of proportionality between stress (σ_{ij}) and strain (ε_{kl}) was first stated in the 17th century by Hooke, for the case of a stretched elastic string.

2.2.1 Elasticity tensor

In the theory of elasticity, the elasticity tensor C_{ijkl} plays a vital role. In generalized Hooke's law, the coefficients C_{ijkl} indicates the most general linear relationship between σ_{ij} and ε_{kl}. These are the components of a fourth rank tensor. Elasticity tensor is also called elastic stiffness tensor.

Stress Tensor and Symmetry of Elasticity Tensor

Since σ_{ij} is a symmetric tensor i.e. $\sigma_{ij} = \sigma_{ji}$ due to this fact the elasticity tensor obeys a symmetry i.e.,

$$C_{ijkl} = C_{jikl}. \tag{2.8}$$

The symmetries of σ_{ij} for the elasticity tensor has Eq. (2.8) can be written as

$$C_{ijkl} - C_{jikl} = 0,$$
$$C_{[ij]kl} = 0.$$

The above symmetry is called left minor symmetry of the elasticity tensor.

Strain Tensor and Symmetry of Elasticity Tensor

Since ε_{kl} is also a symmetric tensor i.e. $\varepsilon_{kl} = \varepsilon_{lk}$ due to this fact the elasticity tensor obeys a symmetry i.e.,

$$C_{ijkl} = C_{ijlk}. \tag{2.9}$$

The symmetries ϵ_{kl} for the elasticity tensor has Eq. (2.9) can also be written as

$$C_{ijkl} - C_{ijlk} = 0,$$
$$C_{ij[kl]} = 0.$$

This symmetry is called the right minor symmetry of the elasticity tensor.

Strain Energy Function and Symmetry of Elasticity Tensor

The strain energy function of a material which is deformed is defined as

$$\Omega = \frac{1}{2}\sigma_{ij}\epsilon_{ij}. \tag{2.10}$$

When we substitute of generalized Hooke's law in Eq. (2.10), the given expression is in the form

$$\Omega = \frac{1}{2}C_{ijkl}\epsilon_{ij}\epsilon_{kl}. \tag{2.11}$$

The existence of strain energy function defines an elastic continuum. This implies that C_{ijkl} is symmetric under permutations of pairs of subscripts ij and kl. This can be derived as follows.

Differentiating both sides of the above equation with respect to ε_{ij}, ε_{kl}, respectively, we get

$$\frac{\partial^2\Omega}{\partial\varepsilon_{ij}\partial\varepsilon_{kl}} = C_{ijkl},$$

where the indices $i, j, k, l = 1, 2, 3$.

If Ω has continuous first and second order derivatives, then we can write [5]

$$\frac{\partial^2\Omega}{\partial\varepsilon_{ij}\partial\varepsilon_{kl}} = \frac{\partial^2\Omega}{\partial\varepsilon_{kl}\partial\varepsilon_{ij}}$$

which shows that

$$C_{ijkl} = C_{klij}. \tag{2.12}$$

Eq. (2.12) can be written as

$$C_{ijkl} - C_{klij} = 0.$$

The above symmetry is called the major symmetry of the elasticity tensor.

Reduction of Components of Elasticity Tensor

In 3 dimensional space, the total number of independent components of the 4th rank elasticity tensor are $3^4 = 81$. The reduction of independent components from 81 to 56 is due to the symmetry of stress tensor ($C_{ijkl} = C_{jikl}$) and the further reduction in the components of the tensor C_{ijkl} from 56 to 36 is due to the the symmetry of strain tensor ($C_{ijkl} = C_{ijlk}$). Moreover, the reduction of 36 components to 21 independent components is due the the strain energy function.

Elasticity Tensor in Voigt's Notation

Voigt's notation is very useful to express the 81 independent elastic components i.e. C_{ijkl} into 6×6 symmetric matrix of elasticity tensor i.e. C_{IJ}.

C_{ijkl} is the standard "longhand" notation of the elasticity tensor. However, C_{IJ} is the standard "shorthand" notation [6, 7]. A pair of indices ij is replaced by I and kl is replaced by J respectively.

$$\begin{array}{lll} (11) \longleftrightarrow (1) & (22) \longleftrightarrow (2) & (33) \longleftrightarrow (3) \\ (23) = (32) \longleftrightarrow (4) & (13) = (31) \longleftrightarrow (5) & (12) = (21) \longleftrightarrow (6) \end{array} \tag{2.13}$$

Hence, the elasticity tensor matrix C_{IJ}, after using the above relation in Eq. (2.13), takes the form as follows:

$$\begin{pmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1131} & C_{1112} \\ * & C_{2222} & C_{2233} & C_{2223} & C_{2231} & C_{2212} \\ * & * & C_{3333} & C_{3323} & C_{3331} & C_{3312} \\ * & * & * & C_{2323} & C_{2331} & C_{2312} \\ * & * & * & * & C_{3131} & C_{3112} \\ * & * & * & * & * & C_{1212} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ * & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ * & * & C_{33} & C_{34} & C_{35} & C_{36} \\ * & * & * & C_{44} & C_{45} & C_{46} \\ * & * & * & * & C_{55} & C_{56} \\ * & * & * & * & * & C_{66} \end{pmatrix} \tag{2.14}$$

where the remaining non-zero components are independent of each other. In both matrices, those entries which are dependent due to the symmetry of the tensor components denoted by the $*$. Voigt's notation is only applicable as the left minor and right minor symmetries are valid and due to the major symmetry.

2.3 Isotropic and Anisotropic

"Isotropic" and "Anisotropic" are associated words which are antonyms. In terms of structure, the word isotropic means "equal direction". Anisotropic is derived from it by adding the Greek prefix "an" which opposes the meaning of its base word.

1. Anisotropic: a physical property which have a different value, measured in different directions.

2. Isotropic: a physical property which have a same value, measured in different directions.

Anisotropic Material
Anisotropic materials are those materials in which the components of the elasticity tensor depend on the coordinates.

Isotropic Material
Isotropic materials are those materials in which the elastic components of the elasticity tensor do not depend upon the coordinates.

2.3.1 Elastic Tensor For an Isotropic Material

A tensor is called isotropic if it has same components in every co-ordinate system. In this system, properties of the crystal do not depend on the direction. In 3 dimensions, there are only 3 independent isotropic tensors of rank 4th i.e., $\delta_{ij}\delta_{kl}$, $\delta_{ik}\delta_{jl}$ and $\delta_{il}\delta_{jk}$. If C_{ijkl} is to be isotropic it must be the linear combination of these 3 tensors i.e.,

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu_1\delta_{ik}\delta_{jl} + \mu_2\delta_{il}\delta_{jk}, \tag{2.15}$$

interchanging $i \leftrightarrow j$ then

$$C_{ijkl} = \lambda\delta_{ji}\delta_{kl} + \mu_1\delta_{jk}\delta_{il} + \mu_2\delta_{jl}\delta_{ik}, \tag{2.16}$$

in the isotropic material, elasticity tensor can be defined as

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{2.17}$$

The above equation is the constitutive law for an isotropic, linear elastic and homogeneous material where $\mu = \mu_1 = \mu_2$ and λ and μ are the Lame's constants [8] and μ is also known as shear modulus. In shear modulus, σ_{12}, the component of stress tensor, does not vanish but all other components of the stress tensor vanish. It is defined as

$$\mu = \frac{1}{2}\frac{\sigma_{12}}{\varepsilon_{12}}.$$

Hooke's Law For Isotropic Material
Using Eq. (2.17) into generalized Hooke's law equation then we have

$$\begin{aligned}
\sigma_{ij} &= \{\lambda\delta_{ij}\delta_{kl} + \mu(\delta_{il}\delta_{jk} + \delta_{ik}\delta_{lj})\}\varepsilon_{kl} \\
&= \lambda\delta_{ij}\delta_{kl}\varepsilon_{kl} + \mu(\delta_{il}\delta_{jk}\varepsilon_{kl} + \delta_{ik}\delta_{lj}\varepsilon_{kl}) \\
&= \lambda\delta_{ij}\varepsilon_{kk} + \mu(\varepsilon_{ij} + \varepsilon_{ij}) \\
&= \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij}.
\end{aligned}$$

Matrix For Isotropic Material
The elasticity tensor can be express in matrix form for isotropic material as

$$C_{IJ} = \begin{pmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1131} & C_{1112} \\ * & C_{2222} & C_{2233} & C_{2223} & C_{2231} & C_{2212} \\ * & * & C_{3333} & C_{3323} & C_{3331} & C_{3312} \\ * & * & * & C_{2323} & C_{2331} & C_{2312} \\ * & * & * & * & C_{3131} & C_{3112} \\ * & * & * & * & * & C_{1212} \end{pmatrix} = \begin{pmatrix} 2\mu+\lambda & \lambda & \lambda & 0 & 0 & 0 \\ * & 2\mu+\lambda & \lambda & 0 & 0 & 0 \\ * & * & 2\mu+\lambda & 0 & 0 & 0 \\ * & * & * & \mu & 0 & 0 \\ * & * & * & * & \mu & 0 \\ * & * & * & * & * & \mu \end{pmatrix}.$$

There are 12 non-zero components of which two are independent.

2.3.2 Cubic Crystal

In cubic crystals, they have at least three dyad axes (A_2) and four triad axes (A_3). Taking the dyad axes (A_2) with the coordinate axes, a rotation about the triad axis (A_3) through an angle $\frac{2\pi}{3}$ gives a cyclic permutation of the axes. The constants of the elasticity tensor C_{ijkl} should be unchanged under the cyclic permutation of the indices such as $(123) \rightarrow (231) \rightarrow (312)$. In Voigt notaion this implies that:

$$C_{IJ} = \begin{pmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ * & C_{11} & C_{12} & 0 & 0 & 0 \\ * & * & C_{11} & 0 & 0 & 0 \\ * & * & * & C_{44} & 0 & 0 \\ * & * & * & * & C_{44} & 0 \\ * & * & * & * & * & C_{44} \end{pmatrix} \tag{2.18}$$

where for the cubic crystal, there are 9 non zero components but with 3 independent components i.e. C_{11}, C_{12} and C_{44}.

Chapter 3

Plane Elasticity Tensors

In this chapter, we study the decomposition of the plane elasticity tensor in two dimensions under $SO(2)$ which is the rotation group. We review the decomposition by Vianello and Forte [9].

In 2 dimensions the elasticity tensor C_{ijkl} is called plane elasticity tensor. In plane elasticity, the matrix of C_{ijkl} is

$$C = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{12} & C_{22} & C_{23} \\ C_{13} & C_{23} & C_{33} \end{pmatrix},$$

where C be the matrix of components of the plane elasticity tensor and is symmetric. $\mathbb{E}$ is the vector space of the plane elasticity tensors. The dimension of the plane elasticity tensor is 6.

In 2 dimensions, the rotation matrix is

$$Q = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

3.1 Decomposition of Plane Elasticity Tensors

Let $\boldsymbol{V}$ be a two-dimensional Euclidean vector space. Tensors defined on $\boldsymbol{V}$ are called plane tensors. Let $\{e_i\}$ $(i = 1, 2)$ be an orthogonal basis for $\boldsymbol{V}$, then $\{e_i \otimes e_j\}$ $(i, j = 1, 2)$ is a basis for tensors of rank 2 and $\{e_i \otimes e_j \otimes e_k \otimes e_l\}$ $(i, j, k, l = 1, 2)$ is a basis for the elasticity tensor of rank 4.

A traceless tensor $\mathbb{T}$ of rank 2 is such that $\mathbb{T}_{ii} = 0$. A traceless tensor $\mathbb{T}$ of rank 4 is such that $\mathbb{T}_{iikl} = 0$, $(k, l = 1, 2)$. A totally symmetric tensor is such that its components are invariant with respect to any permutation of its indices. A traceless symmetric tensor is called harmonic. Let $O(2)$ be the set of orthogonal tensors Q and $SO(2)$ its subgroup of rotations with determinant equal to $+1$. We shall consider irreducible decomposition of the elasticity tensor C with the symmetries $C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij}$ as a sum of scalars λ, μ and tensors L, M where L is a harmonic tensor of rank 2 and M is harmonic tensor of rank 4. This decomposition is invariant under $O(2)$ because if

$$\begin{aligned} L^{'} &= (Q * L)_{mn} = q_{mi}q_{nj}L_{ij}, \\ M^{'} &= (Q * M)_{mnst} = q_{mi}q_{nj}q_{sk}q_{tl}M_{ijkl}, \end{aligned}$$

and then $L^{'}$ and $M^{'}$ are harmonic if L, M are harmonic. This decomposition is in the form [9, 10]

$$C_{ijkl} = M_{ijkl} + \frac{1}{6}[\delta_{ij}L_{kl} + L_{ij}\delta_{kl} + \delta_{ik}L_{lj} + L_{ik}\delta_{lj} + \delta_{il}L_{jk}] + \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{lj} + \delta_{il}\delta_{jk}), \tag{3.1}$$

where

$$\lambda = \frac{1}{4}(\frac{3}{2}C_{ppqq} - C_{pqpq}), \quad \mu = \frac{1}{4}(C_{pqpq} - \frac{1}{2}C_{ppqq}), \quad L_{ik} = \frac{1}{12}(2C_{ipkp} - C_{pqpq}\delta_{ik}),$$

$$\begin{aligned} M_{ijkl} = C_{ijkl} &- \frac{1}{6}(\delta_{ij}C_{kplp} + \delta_{kl}C_{ipjp} + \delta_{ik}C_{lpjp} + \delta_{lj}C_{ipkp} + \delta_{il}C_{jpkp} + \delta_{jk}C_{iplp}) \\ &+ \frac{1}{12}[C_{pqpq}(5\delta_{ij}\delta_{kl} - \delta_{ik}\delta_{lj} - \delta_{il}\delta_{jk})] - \frac{1}{8}[C_{ppqq}(3\delta_{ij}\delta_{kl} - \delta_{ik}\delta_{lj} - \delta_{il}\delta_{jk})]. \end{aligned}$$

This decomposition is irreducible because the space of harmonic tensors of rank 2 or 4 does not contain a subspace invariant under $O(2)$.

First we consider the decomposition of a symmetric tensor T of rank 2 into a scalar and a harmonic tensor of rank 2. We write T as a 2×2 matrix,

$$T_{ij} = \begin{pmatrix} T_{11} & T_{12} \\ T_{12} & T_{22} \end{pmatrix}.$$

Its decomposition will be of the form

$$\begin{pmatrix} T_{11} & T_{12} \\ T_{12} & T_{22} \end{pmatrix} = \alpha \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \beta & \gamma \\ \gamma & -\beta \end{pmatrix}.$$

Thus,

$$\begin{aligned} \alpha + \beta &= T_{11}, \\ \alpha - \beta &= T_{22}, \\ \gamma &= T_{12}. \end{aligned}$$

Hence,

$$\alpha = \frac{T_{11} + T_{22}}{2},$$
$$\beta = \frac{T_{11} - T_{22}}{2},$$
$$\gamma = T_{12}.$$

The tensor $\mathbb{T}$ is expressed as

$$\mathbb{T} = \alpha\delta_{ij} + \beta(e_1 \otimes e_1 - e_2 \otimes e_2) + \gamma(e_1 \otimes e_2 + e_2 \otimes e_1),$$

or

$$\mathbb{T} = \alpha \mathrm{I} + \sqrt{2}\beta E_1 + \sqrt{2}\gamma E_2,$$

where

$$E_1 = \frac{1}{\sqrt{2}}(e_1 \otimes e_1 - e_2 \otimes e_2),$$
$$E_2 = \frac{1}{\sqrt{2}}(e_1 \otimes e_2 + e_2 \otimes e_1)$$

Under $SO(2)$, E_1, E_2 transform as

$$\begin{aligned}
E_1' &= Q(\theta) * E_1 \\
&= Q(\theta) * \frac{1}{\sqrt{2}}(e_1 \otimes e_1 - e_2 \otimes e_2) \\
&= \frac{1}{\sqrt{2}}\{Q(\theta)e_1 \otimes Q(\theta)e_1 - Q(\theta)e_2 \otimes Q(\theta)e_2\} \\
&= \frac{1}{\sqrt{2}}\{(\cos\theta e_1 + \sin\theta e_2) \otimes (\cos\theta e_1 + \sin\theta e_2) - (-\sin\theta e_1 + \cos\theta e_2) \otimes (-\sin\theta e_1 + \cos\theta e_2)\} \\
&= \frac{1}{\sqrt{2}}\{\cos 2\theta(e_1 \otimes e_1) - \cos 2\theta(e_2 \otimes e_2) + 2\cos\theta\sin\theta(e_1 \otimes e_2) + 2\cos\theta\sin\theta(e_2 \otimes e_1)\} \\
&= \cos 2\theta \frac{1}{\sqrt{2}}(e_1 \otimes e_1 - e_2 \otimes e_2) + \sin 2\theta \frac{1}{\sqrt{2}}(e_1 \otimes e_2 + e_2 \otimes e_1) \\
&= \cos 2\theta E_1 + \sin 2\theta E_2,
\end{aligned}$$

$$
\begin{aligned}
E_2' &= Q(\theta) * E_2 \\
&= Q(\theta) * \frac{1}{\sqrt{2}}(e_1 \otimes e_2 + e_2 \otimes e_1) \\
&= \frac{1}{\sqrt{2}}\{Q(\theta)e_1 \otimes Q(\theta)e_2 + Q(\theta)e_2 \otimes Q(\theta)e_1\} \\
&= \frac{1}{\sqrt{2}}\{(\cos\theta e_1 + \sin\theta e_2) \otimes (-\sin\theta e_1 + \cos\theta e_2) + (-\sin\theta e_1 + \cos\theta e_2) \otimes (\cos\theta e_1 + \sin\theta e_2)\} \\
&= \frac{1}{\sqrt{2}}\{-\sin 2\theta(e_1 \otimes e_1) + \cos 2\theta(e_1 \otimes e_2) + \cos 2\theta(e_2 \otimes e_1) + \sin 2\theta(e_2 \otimes e_2)\} \\
&= -\sin 2\theta \frac{1}{\sqrt{2}}(e_1 \otimes e_1 - e_2 \otimes e_2) + \cos 2\theta \frac{1}{\sqrt{2}}(e_1 \otimes e_2 + e_2 \otimes e_1) \\
&= -\sin 2\theta E_1 + \cos 2\theta E_2,
\end{aligned}
$$

which is a rotation of the vector (E_1, E_2) through 2θ. Thus invariants of $\mathbb{T}$ are

$$\alpha = \frac{T_{11} + T_{22}}{2},$$

and

$$(\sqrt{2}\beta)^2 + (\sqrt{2}\alpha)^2 = \frac{(T_{11} - T_{22})^2}{2} + 2T_{12}^2.$$

Chapter 4

Algebra of the Decomposition of the Elasticity Tensor

The set of all general elasticity tensors especially all 4th rank tensors with the major and minor symmetries forms a vector space. The vector space of the elasticity tensors is denoted by $\boldsymbol{C}$. The dimension of the vector space $\boldsymbol{C}$ is 21.

In the literature two types of decompositions are found

1. Reducible Decomposition

2. Irreducible Decomposition.

Reducible Decomposition

If it is possible to express a vector space $C = C_1 \oplus C_2$ where C_1 and C_2 are invariant subspaces under a group then C is said to be decomposed into $C_1 \oplus C_2$. This decomposition is reducible, not unique and does not preserve the major and minor symmetries of the elasticity tensor. It is also known as RS-decomposition.

Irreducible Decomposition

If it is not possible to further decompose C_1 or C_2 then the above decomposition is said to be irreducible otherwise it is called reducible. This decomposition is irreducible, unique and preserves the major and minor symmetries of the elasticity tensor. It is also called VW-decomposition.

4.1 Reducible Decomposition of Elasticity Tensor

The elasticity tensor can be decomposed into two tensorial parts as mentioned by Cowin [11], Campanella and Tonton [12], Podio-Guidugli [13], Weiner [14], and Hauss*ü*hl [15]. This decomposition is given by

$$C_{ijkl} = R_{ijkl} + S_{ijkl}. \tag{4.1}$$

The first reducible part R_{ijkl} may be obtained by the symmetrization of the elasticity tensor,

$$R_{ijkl} := C_{i(jk)l},$$

the second reducible part of the elasticity tensor may be obtained by the anti-symmetrization given by

$$S_{ijkl} := C_{i[jk]l}.$$

4.1.1 Tensors R_{ijkl} and S_{ijkl}

The tensors R_{ijkl} and S_{ijkl} both fulfill the major symmetry.

Proposition 1: The major symmetry holds for both tensors R_{ijkl} and S_{ijkl}.

Proof: By definition,

$$\begin{aligned}
R_{ijkl} - R_{klij} &= C_{i(jk)l} - C_{k(li)j} \\
&= \frac{1}{2}(C_{ijkl} + C_{ikjl}) - \frac{1}{2}(C_{klij} + C_{kilj}) \\
&= \frac{1}{2}[C_{ijkl} + C_{ikjl} - C_{klij} - C_{kilj}] \\
&= \frac{1}{2}[C_{ijkl} + C_{kijl} - C_{ijkl} - C_{kilj}] \\
&= 0.
\end{aligned}$$

Similarly,

$$\begin{aligned}
S_{ijkl} - S_{klij} &= C_{i[jk]l} - C_{k[li]j} \\
&= \frac{1}{2}(C_{ijkl} - C_{ikjl}) - \frac{1}{2}(C_{klij} - C_{kilj}) \\
&= \frac{1}{2}[C_{ijkl} - C_{ikjl} - C_{klij} + C_{kilj}] \\
&= \frac{1}{2}[C_{ijkl} - C_{kijl} - C_{ijkl} + C_{kilj}] \\
&= 0.
\end{aligned}$$

Hence, proved that the major symmetry holds for R_{ijkl} and S_{ijkl}. However, the tensors R_{ijkl} and S_{ijkl} do not fulfill the minor symmetries, as proved below.

Proposition 2: The minor symmetries do not hold for the tensors R_{ijkl} and S_{ijkl}.

Proof: By definition,

$$\begin{aligned}
R_{ijkl} &= \frac{1}{2}(C_{ijkl} + C_{ikjl}) \\
R_{[ij]kl} &= \frac{1}{2}(C_{[ij]kl} + C_{[i|k|j]l}) \\
&= \frac{1}{2}[\frac{1}{2}(C_{ijkl} - C_{jikl}) + \frac{1}{2}(C_{ikjl} - C_{jkil})] \\
&= \frac{1}{4}(C_{ijkl} - C_{jikl} + C_{ikjl} - C_{jkil}) \\
&= \frac{1}{4}(C_{kijl} - C_{kjil}) \\
&= \frac{1}{4}(2C_{k[ij]l}) \\
&= \frac{1}{2}C_{k[ij]l} \\
&= \frac{1}{2}S_{kijl}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
S_{ijkl} &= \frac{1}{2}(C_{ijkl} - C_{ikjl}) \\
S_{[ij]kl} &= \frac{1}{2}(C_{[ij]kl} - C_{[i|k|j]l}) \\
&= \frac{1}{2}[\frac{1}{2}(C_{ijkl} - C_{jikl}) - \frac{1}{2}(C_{ikjl} - C_{jkil})] \\
&= \frac{1}{4}(C_{ijkl} - C_{jikl} - C_{ikjl} + C_{jkil}) \\
&= \frac{1}{4}(C_{kijl} - C_{kjil}) \\
&= \frac{1}{4}(2C_{k[ij]l}) \\
&= -\frac{1}{2}C_{k[ij]l} \\
&= -\frac{1}{2}S_{kijl}.
\end{aligned}$$

Hence the above proposition is proved. Also $R_{[ij]kl} = -S_{[ij]kl}$ and these expressions are not equal to zero ($R_{[ij]kl} = -S_{[ij]kl} \neq 0$). It follows from Proposition 1 that

$R_{ij[kl]} = R_{[kl]ij}$ and $S_{ij[kl]} = S_{[kl]ij}$. Now, these expressions are shown to be equal. By definition,

$$\begin{aligned} R_{ijkl} &= C_{i(jk)l} = \frac{1}{2}(C_{ijkl} + C_{ikjl}) \\ R_{ij[kl]} &= \frac{1}{2}(C_{ij[kl]} + C_{i[k|j|l]}) \\ &= \frac{1}{2}[\frac{1}{2}(C_{ijkl} - C_{ijlk}) + \frac{1}{2}(C_{ikjl} - C_{iljk})] \\ &= \frac{1}{4}(C_{ijkl} - C_{ijlk} + C_{ikjll} - C_{iljk}) \\ &= \frac{1}{4}(C_{ijkl} - C_{klij} + C_{ikjl} - C_{iljk}) \\ &= \frac{1}{4}(C_{ikjl} - C_{iljk}) \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} R_{klij} &= C_{k(li)j} = \frac{1}{2}(C_{klij} + C_{kilj}) \\ R_{[kl]ij} &= \frac{1}{2}(C_{[kl]ij} + C_{[k|i|l]j}) \\ &= \frac{1}{2}[\frac{1}{2}(C_{klij} - C_{lkij}) + \frac{1}{2}(C_{kilj} - C_{likj})] \\ &= \frac{1}{4}(C_{klij} - C_{lkij} + C_{kilj} - C_{likj}) \\ &= \frac{1}{4}(C_{klij} - C_{lkij} + C_{kilj} - C_{likj}), \\ &= \frac{1}{4}(C_{ikjl} - C_{iljk}). \end{aligned} \tag{4.3}$$

Thus, from Eqs. (4.2) and (4.3) we have $R_{ij[kl]} = R_{[kl]ij}$. Similarly, one can also prove that $S_{ij[kl]} = S_{[kl]ij}$. Since the minor symmetries for both tensors (R_{ijkl} and S_{ijkl}) do not hold therefore these tensors do not belong to the vector space of elasticity tensors $\boldsymbol{C}$ and they cannot be expressed in Voigt's notation.

4.1.2 Vector Space $\boldsymbol{R}$

The set of tensors $\{R_{ijkl}; i, j, k, l = 1, 2, 3, 4\}$ form a vector space denoted by $\boldsymbol{R}$. The dimension of the vector space $\boldsymbol{R}$ is 21.

$$\begin{aligned} R_{ijkl} &= \frac{1}{2}(C_{ijkl} + C_{ikjl}) \\ R_{1111} &= \frac{1}{2}(C_{1111} + C_{1111}) \\ &= C_{1111}. \end{aligned}$$

The tensor R_{ijkl} cannot be written in Voigt's notation because it does not hold the minor symmetries. The independent components of tensor R_{ijkl} have been calculated below.

$$\begin{aligned}
&R_{1111} = C_{11}, \quad R_{1113} = C_{15}, \quad R_{1112} = C_{16}, \quad R_{2222} = C_{22}, \\
&R_{2221} = C_{26}, \quad R_{3333} = C_{33}, \quad R_{3332} = C_{34}, \quad R_{3331} = C_{35}, \\
&R_{2331} = C_{45}, \quad R_{3221} = C_{46}, \quad R_{3113} = C_{55}, \quad R_{3112} = C_{56}, \\
&R_{1122} = \frac{1}{2}(C_{12} + C_{66}), \quad R_{1133} = \frac{1}{2}(C_{13} + C_{55}), \quad R_{2223} = C_{24}, \\
&R_{1123} = \frac{1}{2}(C_{14} + C_{56}), \quad R_{2233} = \frac{1}{2}(C_{23} + C_{44}), \quad R_{2332} = C_{44}, \\
&R_{2231} = \frac{1}{2}(C_{25} + C_{46}), \quad R_{1233} = \frac{1}{2}(C_{36} + C_{45}), \quad R_{1221} = C_{66}.
\end{aligned} \tag{4.4}$$

These components are linearly independent.

4.1.3 Vector Space $\boldsymbol{S}$

The set of tensors $\{S_{ijkl}; i, j, k, l = 1, 2, 3\}$ forms a vector space denoted by $\boldsymbol{S}$. The dimension of the vector space $\boldsymbol{S}$ is 6.

$$S_{ijkl} = \frac{1}{2}(C_{ijkl} - C_{ikjl})$$
$$S_{1122} = \frac{1}{2}(C_{1122} - C_{1212}).$$

The independent components of the antisymmetric 4th rank tensor S_{ijkl} can be written explicitly in Voigt's notation as

$$S_{1122} = \frac{1}{2}(C_{12} - C_{66}), \quad S_{1133} = \frac{1}{2}(C_{13} - C_{55}), \quad S_{1123} = \frac{1}{2}(C_{14} - C_{56}),$$

$$S_{2233} = \frac{1}{2}(C_{23} - C_{44}), \quad S_{2231} = \frac{1}{2}(C_{25} - C_{46}), \quad S_{1233} = \frac{1}{2}(C_{36} - C_{45}).$$

Since in Voigt's notation all components of elasticity tensor (C_{IJ} with $I \leq J$) are assumed to be linearly independent. Therefore, these components are also linearly independent.

4.1.4 Algebraic Properties of the Tensors R_{ijkl} and S_{ijkl}

We can observe some of the basic features of the tensors R_{ijkl} and S_{ijkl}.

Inconsistency

There are many ways in which elasticity tensor can be expressed in terms of both R_{ijkl} and S_{ijkl} tensors. Generally, the components of elasticity tensor, say C_{1223} can be defined as:

$$C_{1223} = R_{1223} + S_{1223},$$

where, we have used the definition of both R_{1223} and S_{1223} tensors i.e.,

$$\begin{aligned} R_{1223} &= C_{1(22)3} = \frac{1}{2}(C_{1223} + C_{1223}) = C_{1223}. \\ S_{1223} &= C_{1[22]3} = \frac{1}{2}(C_{1223} - C_{1223}) = 0. \end{aligned} \tag{4.5}$$

Further,

$$C_{1223} \overset{(4.5)}{=} R_{1223} + \underbrace{S_{1223}}_{0} \overset{maj}{=} R_{2312} = R_{2132} \overset{(4.4)}{=} C_{46}. \tag{4.6}$$

We can be written C_{1223} in another way as:

$$C_{1223} = C_{2123} = R_{2123} + S_{2123} = \frac{1}{2}(C_{46} + C_{25}) + \frac{1}{2}(C_{46} - C_{25}) = C_{46},$$

where the components of R_{2123} and S_{2123} are given by

$$\begin{aligned} R_{2123} &= \frac{1}{2}(C_{2123} + C_{2213}) = \frac{1}{2}(C_{64} + C_{25}) = \frac{1}{2}(C_{46} + C_{25}), \\ S_{2123} &= \frac{1}{2}(C_{2123} - C_{2213}) = \frac{1}{2}(C_{64} - C_{25}) = \frac{1}{2}(C_{46} - C_{25}). \end{aligned}$$

The result in Eq. (4.6) is recovered, but it has been achieved with the help of non-vanishing component of the vector space $\boldsymbol{S}$.

Reducibility

In general, the tensor R_{ijkl} is not a completely symmetric tensor and it allows more finer decomposition. Therefore,

$$R_{ijkl} = R_{(ijkl)} + D_{ijkl}. \tag{4.7}$$

Consequently, from the Eq. (4.7) the elasticity tensor can be further decomposed into three tensorial parts i.e.,

$$C_{ijkl} = R_{(ijkl)} + S_{ijkl} + D_{ijkl}.$$

Vector Spaces

The vector spaces of $\boldsymbol{R}$ and $\boldsymbol{S}$ are "partial" vector spaces. The dimensions of $\boldsymbol{C}$, $\boldsymbol{R}$ and $\boldsymbol{S}$ are 21, 21 and 6 respectively. These two vector spaces are not the subspaces of the vector space $\boldsymbol{C}$.
Hence, in this way, the RS-decomposition is not possible as there exists a problem from an algebraic point of view. Now, we take another decomposition of the elasticity tensor which is compatible from an algebraic point of view.

4.2 Irreducible Decomposition of Elasticity Tensor

We shall show that the unique irreducible decomposition of C_{ijkl} under the linear group $GL(3,R)$ is

$$C_{ijkl} = V_{ijkl} + W_{ijkl}, \tag{4.8}$$

where V_{ijkl} and W_{ijkl} are the 4th rank tensors. The tensor V_{ijkl} is the first irreducible part of the irreducible decomposition of the elasticity tensor and is defined as

$$V_{ijkl} := C_{(ijkl)},$$

where the tensor V_{ijkl} is obtained by complete symmetrization of the indices of the elasticity tensor. The term $C_{(ijkl)}$ is defined as:

$$\begin{aligned} C_{(ijkl)} = \frac{1}{4!}(&C_{ijkl} + C_{iklj} + C_{iljk} + C_{ilkj} + C_{ikjl} + C_{ijlk} + C_{jkli} + C_{jlik} \\ &+C_{jikl} + C_{jilk} + C_{jlki} + C_{jkil} + C_{klij} + C_{kijl} + C_{kjli} + C_{kjil} \\ &+C_{kilj} + C_{klji} + C_{lijk} + C_{ljki} + C_{lkij} + C_{lkji} + C_{ljik} + C_{likj}). \end{aligned}$$

The more compact form of first irreducible part is:

$$V_{ijkl} = \frac{1}{3}(C_{ijkl} + C_{iklj} + C_{iljk}). \tag{4.9}$$

The second irreducible part of the irreducible decomposition of the elasticity tensor is W_{ijkl} and is given by

$$W_{ijkl} = C_{ijkl} - V_{ijkl}, \tag{4.10}$$

or

$$W_{ijkl} := \frac{1}{3}(2C_{ijkl} - C_{ilkj} - C_{iklj}). \tag{4.11}$$

If we apply total symmetrization on the indices of both sides of Eq. (4.10) then we get the new result i.e.,

$$W_{(ijkl)} = C_{(ijkl)} - V_{(ijkl)} = 0.$$

Proposition 3: The tensor W satisfies the additional symmetry

$$W_{i(jkl)} = 0 \qquad \text{or} \qquad W_{ijkl} + W_{iklj} + W_{iljk} = 0.$$

Proof:

$$W_{i(jkl)} = \frac{1}{3!}(W_{ijkl} + W_{iklj} + W_{iljk} + W_{ijkl} + W_{ilkj} + W_{ikjl}) \tag{4.12}$$

By definition,

$$\begin{aligned}
W_{ijkl} &= \frac{1}{3}(2C_{ijkl} - C_{ilkj} - C_{iklj}), \quad & W_{iklj} &= \frac{1}{3}(2C_{iklj} - C_{ijlk} - C_{iljk}),\\
W_{iljk} &= \frac{1}{3}(2C_{iljk} - C_{ikjl} - C_{ijkl}), \quad & W_{ijlk} &= \frac{1}{3}(2C_{ijlk} - C_{iklj} - C_{ilkj}),\\
W_{ilkj} &= \frac{1}{3}(2C_{ilkj} - C_{ijkl} - C_{ikjl}), \quad & W_{ikjl} &= \frac{1}{3}(2C_{ikjl} - C_{iljk} - C_{ijlk}),
\end{aligned}$$

using these terms in Eq. (4.12) and then we have

$$\begin{aligned}
W_{i(jkl)} = \frac{1}{18}&(2C_{ijkl} - C_{ilkj} - C_{iklj} + 2C_{iklj} - C_{ijlk} - C_{iljk} + 2C_{iljk} - C_{ikjl} - C_{ijkl}\\
&+ 2C_{ijlk} - C_{iklj} - C_{ilkj} + 2C_{ilkj} - C_{ijkl} - C_{ikjl} + 2C_{ikjl} - C_{iljk} - C_{ijlk})\\
&= 0.
\end{aligned}$$

Hence the proposition 3 is proved.

4.2.1 Tensors V_{ijkl} and W_{ijkl}

The major symmetry holds for the tensors V_{ijkl} and W_{ijkl}.

Proposition 4: The partial tensors V_{ijkl} and W_{ijkl} possess the major symmetry

$$V_{ijkl} - V_{klij} = 0, \qquad W_{ijkl} - W_{klij} = 0.$$

Proof :

$$\begin{aligned}
V_{ijkl} - V_{klij} &= \frac{1}{3}(C_{ijkl} + C_{iklj} + C_{iljk}) - \frac{1}{3}(C_{klij} + C_{kijl} + C_{kjli})\\
&= \frac{1}{3}(C_{ijkl} + C_{iklj} + C_{iljk} - C_{klij} - C_{kijl} - C_{kjli}) = 0.
\end{aligned}$$

$$W_{ijkl} - W_{klij} = \frac{1}{3}(2C_{ijkl} - C_{ilkj} - C_{iklj}) - \frac{1}{3}(2C_{klij} - C_{kjil} - C_{kijl})$$
$$= \frac{1}{3}(2C_{ijkl} - C_{ilkj} - C_{iklj} - 2C_{klij} + C_{kjil} + C_{kijl}) = 0.$$

Hence prosposition 4 is proved.

The partial tensors V_{ijkl} and W_{ijkl} hold the minor symmetries.

Proposition 5 : The minor symmetries hold for the partial tensors V_{ijkl} and W_{ijkl}

$$V_{[ij]kl} = V_{ij[kl]} = 0, \qquad W_{[ij]kl} = W_{ij[kl]} = 0.$$

Proof : By definition,

$$\begin{aligned} V_{[ij]kl} &= \frac{1}{3}(C_{[ij]kl} + C_{[i|kl|j]} + C_{i|l|j]k}) \\ &= \frac{1}{3}[\frac{1}{2}(C_{ijkl} - C_{jikl}) + \frac{1}{2}(C_{iklj} - C_{jkli}) + \frac{1}{2}(C_{iljk} - C_{jlik})] \\ &= \frac{1}{6}(C_{ijkl} - C_{jikl} + C_{iklj} - C_{jkli} + C_{iljk} - C_{jlik}) = 0. \end{aligned}$$

$$\begin{aligned} W_{[ij]kl} &= \frac{1}{3}(2C_{[ij]kl} - C_{[i|lk|j]} - C_{[i|kl|j]}) \\ &= \frac{1}{3}[\frac{1}{2}2(C_{ijkl} - C_{jikl}) - \frac{1}{2}(C_{ilkj} - C_{jlki}) - \frac{1}{2}(C_{iklj} - C_{jkli})] \\ &= \frac{1}{6}(2C_{ijkl} - 2C_{jikl} + C_{ilkj} - C_{jlki} + C_{iklj} - C_{jkli}) = 0. \end{aligned}$$

Similarly, we can prove the right minor symmetry for V_{ijkl} and W_{ijkl} tensors. Hence proposition 5 is proved.

4.2.2 Vector Spaces of $\boldsymbol{V}$ and $\boldsymbol{W}$

The vector space of elasticity tensor C_{ijkl} is denoted by $\boldsymbol{C}$ and has dimensions 21. The vector space of the partial tensors V_{ijkl} and W_{ijkl}, satisfy all the properties of C. These irreducible tensors are subspaces of the vector space C. The irreducible decomposition of the elasticity tensor means the reduction of C into the direct sum of its subspace V for the tensor V_{ijkl} and subspace W for the tensor W_{ijkl} is given by

$$C = V \oplus W. \tag{4.13}$$

The intersection between the vector spaces V and W is empty. It is a unique decomposition of the corresponding tensors. The sum of the dimensions of the two subspaces V and W is equal to 21. The tensors V_{ijkl} and W_{ijkl} are the two irreducible parts of the elasticity tensor. These tensors preserve their symmetries and also satisfy the minor and major symmetries of the elasticity tensor.

Vector Space of $\boldsymbol{V}$

The dimension of the vector space $\boldsymbol{V}$ is 15. Consider

$$\begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ * & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ * & * & C_{33} & C_{34} & C_{35} & C_{36} \\ * & * & * & C_{44} & * & * \\ * & * & * & C_{54} & C_{55} & * \\ * & * & * & C_{64} & C_{65} & C_{66} \end{pmatrix}.$$

If the tensor is totally symmetric

$$\begin{aligned} C_{12} &= C_{1122} = C_{1212} = C_{66}, \\ C_{13} &= C_{1133} = C_{1313} = C_{55}, \\ C_{14} &= C_{1123} = C_{1213} = C_{65}, \\ C_{23} &= C_{2233} = C_{2323} = C_{44}, \\ C_{25} &= C_{2213} = C_{2123} = C_{64}, \\ C_{36} &= C_{3312} = C_{3132} = C_{54}, \end{aligned}$$

In this manner only 15 independent components are left. Hence the dimension of $\boldsymbol{V}$ is 15.

Vector Space of $\boldsymbol{W}$

The dimension of the vector space $\boldsymbol{W}$ is 6. Eqs. (4.13) shows that the vector space of the tensors W_{ijkl} has dimension 6.

V_{ijkl} and W_{ijkl} Tensors in Voigt's Notation

The irreducible decomposition can be written in Voigt's notation because this decomposition holds the minor symmetries. In Voigt's notation the Eq. (4.8) is given by

$$C_{IJ} = V_{IJ} + W_{IJ} \quad \text{with} \quad C_{[IJ]} = V_{[IJ]} = W_{[IJ]} = 0. \tag{4.14}$$

$V_{[IJ]}$ and $W_{[IJ]}$ are the 6×6 matrix. They have 15 and 6 independent components respectively. By using the definition of the tensor V_{ijkl} and W_{ijkl} we will calculate the components of the tensors V_{ijkl} and W_{ijkl} in Voigt's notation as mentioned by Voigt [6]. First, by using the definition of the tensor V_{ijkl} i.e.,

$$V_{1111} = \frac{1}{3}(C_{1111} + C_{1111} + C_{1111}), \tag{4.15}$$

where $i, j, k, l = 1$. We write Eq. (4.15) in Voigt's notation i.e.,

$$V_{11} = \frac{1}{3}(C_{11} + C_{11} + C_{11}) = C_{11},$$

Similarly,

$$\begin{aligned} V_{22} &= C_{22}, \quad V_{33} = C_{33}, \quad V_{15} = C_{15}, \quad V_{16} = C_{16}, \\ V_{26} &= C_{26}, \quad V_{24} = C_{24}, \quad V_{34} = C_{34}, \quad V_{35} = C_{35}, \end{aligned}$$

$$\begin{aligned} V_{12} &= \frac{1}{3}(C_{12} + 2C_{66}), \quad V_{13} = \frac{1}{3}(C_{13} + 2C_{55}), \quad V_{14} = \frac{1}{3}(C_{14} + 2C_{56}), \\ V_{23} &= \frac{1}{3}(C_{23} + 2C_{44}), \quad V_{25} = \frac{1}{3}(C_{25} + 2C_{46}), \quad V_{36} = \frac{1}{3}(C_{36} + 2C_{45}). \end{aligned} \tag{4.16}$$

Now, we have use the definition of the tensor W_{ijkl} then put $i, j, k, l = 1$ i.e.,

$$\begin{aligned} W_{1111} &= \frac{1}{3}(2C_{1111} - C_{1111} - C_{1111}), \\ W_{11} &= 0. \end{aligned}$$

The first component of the tensor W_{ijkl} is equal to zero in Voigt's notation.
Similarly,

$$\begin{aligned} W_{12} &= \frac{2}{3}(C_{12} - C_{66}), \quad W_{13} = \frac{2}{3}(C_{13} - C_{55}), \quad W_{14} = \frac{2}{3}(C_{14} - C_{56}), \\ W_{23} &= \frac{2}{3}(C_{23} - C_{44}), \quad W_{25} = \frac{2}{3}(C_{25} - C_{46}), \quad W_{36} = \frac{2}{3}(C_{36} - C_{45}). \end{aligned} \tag{4.17}$$

These 6 components are linearly independent. Explicitly, we can presented the decomposition of Eq. (4.14) as follows:

$$\begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ * & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ * & * & C_{33} & C_{34} & C_{35} & C_{36} \\ * & * & * & C_{44} & C_{45} & C_{46} \\ * & * & * & * & C_{55} & C_{56} \\ * & * & * & * & * & C_{66} \end{pmatrix} = \begin{pmatrix} V_{11} & V_{12} & V_{13} & V_{14} & V_{15} & V_{16} \\ * & V_{22} & V_{23} & V_{24} & V_{25} & V_{26} \\ * & * & V_{33} & V_{34} & V_{35} & V_{36} \\ * & * & * & V_{44} & V_{45} & V_{46} \\ * & * & * & * & V_{55} & V_{56} \\ * & * & * & * & * & V_{66} \end{pmatrix} \tag{4.18}$$

$$+ \begin{pmatrix} 0 & W_{12} & W_{13} & W_{14} & 0 & 0 \\ * & 0 & W_{23} & 0 & W_{25} & 0 \\ * & * & 0 & 0 & 0 & W_{36} \\ * & * & * & W_{44} & W_{45} & W_{46} \\ * & * & * & * & W_{55} & W_{56} \\ * & * & * & * & * & W_{66} \end{pmatrix}.$$

Where $*$ denotes the dependent components of the tensors V_{ijkl} and W_{ijkl} e.g. $V_{12} = V_{21}$. These components $V_{44} = V_{23}$, $V_{45} = V_{36}$, $V_{46} = V_{25}, V_{55} = V_{13}$, $V_{56} = V_{14}$, $V_{66} = V_{12}$, $W_{44} = -\frac{1}{2}W_{23}$, $W_{45} = -\frac{1}{2}W_{36}$, $W_{46} = -\frac{1}{2}W_{25}$, $W_{55} = -\frac{1}{2}W_{13}$, $W_{56} = -\frac{1}{2}W_{14}$, $W_{66} = -\frac{1}{2}W_{12}$ are independent. Now, we have to use the values of these components in Eq. (4.18).

$$\begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ * & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ * & * & C_{33} & C_{34} & C_{35} & C_{36} \\ * & * & * & C_{44} & C_{45} & C_{46} \\ * & * & * & * & C_{55} & C_{56} \\ * & * & * & * & * & C_{66} \end{pmatrix} = \begin{pmatrix} \mathbf{V_{11}} & \mathbf{V_{12}} & \mathbf{V_{13}} & \mathbf{V_{14}} & \mathbf{V_{15}} & \mathbf{V_{16}} \\ * & \mathbf{V_{22}} & \mathbf{V_{23}} & \mathbf{V_{24}} & \mathbf{V_{25}} & \mathbf{V_{26}} \\ * & * & \mathbf{V_{33}} & \mathbf{V_{34}} & \mathbf{V_{35}} & \mathbf{V_{36}} \\ * & * & * & V_{23} & V_{36} & V_{25} \\ * & * & * & * & V_{13} & V_{14} \\ * & * & * & * & * & V_{12} \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & \mathbf{W_{12}} & \mathbf{W_{13}} & \mathbf{W_{14}} & 0 & 0 \\ * & 0 & \mathbf{W_{23}} & 0 & \mathbf{W_{25}} & 0 \\ * & * & 0 & 0 & 0 & \mathbf{W_{36}} \\ * & * & * & -\frac{1}{2}W_{23} & -\frac{1}{2}W_{36} & -\frac{1}{2}W_{25} \\ * & * & * & * & -\frac{1}{2}W_{13} & -\frac{1}{2}W_{14} \\ * & * & * & * & * & -\frac{1}{2}W_{12} \end{pmatrix}.$$

Here, the independent components of the tensors V_{ijkl} and W_{ijkl} are represented by boldface and these three matrices are symmetric.

4.3 Comparing the Irreducible and Reducible Decompositions with each other

VW and RS are two different decompositions of the elasticity tensor. Now, we want to compare these two decompositions.

$$\underbrace{C_{ijkl}}_{21} = \underbrace{R_{ijkl}}_{21} + \underbrace{S_{ijkl}}_{6} = \underbrace{V_{ijkl}}_{15} + \underbrace{W_{ijkl}}_{6}. \tag{4.19}$$

The dimensions of the corresponding above mentioned vector spaces have shown clearly. We explain that the two tensors W_{ijkl} and S_{ijkl} can be formulate in terms of each other because these tensors have the same dimensions. S_{ijkl} is the part of reducible decomposition which is antisymmetric with respect to the indices j and k.

Proposition 6: The reducible part S_{ijkl} can be expressed in terms of the tensor W_{ijkl} which is irreducible elasticity tensor as given by

$$S_{ijkl} = W_{i[jk]l}. \tag{4.20}$$

the inverse of the above expression can also reads,

$$W_{ijkl} = \frac{4}{3} S_{ij(kl)}.$$

Proof: By definition,

$$\begin{aligned} W_{ijkl} &= \frac{1}{3}(2C_{ijkl} - C_{ilkj} - C_{iklj}) \\ W_{i[jk]l} &= \frac{1}{3}(2C_{i[jk]l} - C_{il[kj]} - C_{i[k|l|j]}) \\ &= \frac{1}{3}[C_{ijkl} - C_{ikjl} - \frac{1}{2}(C_{iklj} - C_{ijlk})] \\ &= \frac{1}{6}(2C_{ijkl} - 2C_{ikjl} - C_{iklj} + C_{ijlk}) \\ &= \frac{1}{2}(C_{ijkl} - C_{ikjl}) \\ &= S_{ijkl}. \end{aligned}$$

Hence the first part of this proposition is proved. Since

$$\begin{aligned} S_{ijkl} &= W_{i[jk]l} \\ S_{ijkl} &= \frac{1}{2}(W_{ijkl} - W_{ikjl}) \\ &= \frac{1}{2}(W_{ijkl} - W_{iklj}) \end{aligned}$$

$$S_{ij(kl)} = \frac{1}{2}[W_{ij(kl)} - W_{i(kl)j}]. \tag{4.21}$$

Here, W fulfills the right minor symmetric such that

$$W_{ij(kl)} = \frac{1}{2}(W_{ijkl} + W_{ijlk}) = \frac{1}{2}(2W_{ijkl}) = W_{ijkl}.$$

Using the value of $W_{ij(kl)}$ in Eq. (4.21) then we get,

$$\begin{aligned} S_{ij(kl)} &= \frac{1}{2}[W_{ijkl} - W_{i(kl)j}] \\ &= \frac{1}{2}[W_{ijkl} - \frac{1}{2}(W_{iklj} + W_{ilkj})] \\ &= \frac{1}{4}(2W_{ijkl} - W_{iklj} - W_{ilkj}) \end{aligned} \tag{4.22}$$

From proposition 3, we have $W_{iklj} = -W_{ijkl} - W_{iljk}$, using this value in Eq. (4.22).

$$\begin{aligned} S_{ij(kl)} &= \frac{1}{4}[2W_{ijkl} - (-W_{ijkl} - W_{iljk}) - W_{ilkj}] \\ &= \frac{1}{4}(3W_{ijkl}) = \frac{3}{4}W_{ijkl}. \end{aligned}$$

In other word, we can also write

$$W_{ijkl} = \frac{4}{3}S_{ij(kl)}.$$

Hence the second part is also proved.

Proposition 7: The reducible tensor R_{ijkl} can also be defined in terms of the irreducible tensors of the elasticity tensor

$$R_{ijkl} = V_{ijkl} + W_{i(jk)l}.$$

Proof: The tensor R_{ijkl} and S_{ijkl}, can be defined in terms of the irreducible parts. Now, using Eq. (4.20) in Eq. (4.19) then we have

$$C_{ijkl} = R_{ijkl} + W_{i[jk]l} = V_{ijkl} + W_{ijkl} \tag{4.23}$$

again solve Eq. (4.23) with respect to the tensor R_{ijkl},

$$\begin{aligned} R_{ijkl} + W_{i[jk]l} &= V_{ijkl} + W_{ijkl} \\ R_{ijkl} &= V_{ijkl} + W_{ijkl} - W_{i[jk]l} \\ &= V_{ijkl} + W_{ijkl} - \frac{1}{2}(W_{ijkl} - W_{ikjl}) \\ &= V_{ijkl} - \frac{1}{2}(2W_{ijkl} - W_{ijkl} + W_{ikjl}) \\ &= V_{ijkl} - \frac{1}{2}(W_{ijkl} + W_{ikjl}). \end{aligned}$$

Hence the proposition 7 have proved.

4.3.1 Irreducible Tensor W_{ijkl} and Symmetric 2nd Rank Tensor

Haussühl introduced, Δ_{mn}, a symmetric 2nd rank tensor [16] associated with W_{ijkl}

$$\Delta_{mn} = \frac{1}{4}\varepsilon_{mil}\varepsilon_{njk}W_{ijkl}, \tag{4.24}$$

where $\varepsilon_{ijk} = 0, \pm 1$ is a Levi-Civita tensor. It is necessary to represent the tensor W_{ijkl} as a symmetric 2nd rank tensor in 3D space. Applying the operator $(\frac{1}{2}\varepsilon_{mij})$, we can easily map a index pair ij which is antisymmetric to the corresponding vector index m. Since we know that the tensor W_{ijkl} has four indices, so we can apply the operator ε twice.

Proposition 8: The irreducible tensor W_{ijkl} of the elasticity tensor can be represented as a symmetric second rank tensor is given in Eq. (4.24) with the inverse

$$S_{ijkl} = \epsilon_{ikm}\epsilon_{jln}\Delta_{mn} \quad \text{or} \quad W_{ijkl} = \epsilon_{im(k}\epsilon_{l)jn}\Delta_{mn}, \tag{4.25}$$

Proof: First, using the definition of Δ_{mn} and apply anti-symmetrization over the indices m and n.

$$\begin{aligned}
\Delta_{[mn]} &= \frac{1}{4}\varepsilon_{[m|ik|}\varepsilon_{n]jl}W_{ijkl}, \\
&= \frac{1}{4}[\frac{1}{2}(\varepsilon_{mik}\varepsilon_{njl} - \varepsilon_{nik}\varepsilon_{mjl})W_{ijkl}], \\
&= \frac{1}{8}(\varepsilon_{mik}\varepsilon_{njl} - \varepsilon_{nik}\varepsilon_{mjl})W_{ijkl}, \\
&= \frac{1}{8}\varepsilon_{mik}\varepsilon_{njl}W_{ijkl} - \frac{1}{8}\varepsilon_{nik}\varepsilon_{mjl}W_{ijkl},
\end{aligned}$$

Now, interchanging the indices $i \longleftrightarrow j$ and $k \longleftrightarrow l$ then

$$\begin{aligned}
\Delta_{[mn]} &= \frac{1}{8}\varepsilon_{mik}\varepsilon_{njl}W_{ijkl} - \frac{1}{8}\varepsilon_{njl}\varepsilon_{mik}W_{jilk}, \\
&= \frac{1}{8}\varepsilon_{mik}\varepsilon_{njl}(W_{ijkl} - W_{jilk}), \\
&= 0.
\end{aligned}$$

So,

$$\begin{aligned}
\Delta_{[mn]} &= \frac{1}{2}(\Delta_{mn} - \Delta_{nm}), \\
\Delta_{mn} &= \Delta_{nm}.
\end{aligned}$$

We take the second rank symmetric tensor

$$\Delta_{mn} = \frac{1}{4}\varepsilon_{muv}\varepsilon_{nwx}W_{uwxv} \tag{4.26}$$

$$\begin{aligned}
\epsilon_{ikm}\epsilon_{jln}\Delta_{mn} =& \frac{1}{4}\epsilon_{ikm}\epsilon_{jln}\epsilon_{muv}\epsilon_{nwx}W_{uwxv} \\
=& \frac{1}{4}[(\epsilon_{mik}\epsilon_{muv})(\epsilon_{njl}\epsilon_{nwx})]W_{uwxv} \\
=& \frac{1}{4}[(\delta_{iu}\delta_{kv} - \delta_{iv}\delta_{ku})(\delta_{jw}\delta_{lx} - \delta_{jx}\delta_{lw})]W_{uwxv} \\
=& \frac{1}{4}(W_{ijlk} - W_{iljk} - W_{kjli} + W_{klji}) \\
=& \frac{1}{2}(W_{ijlk} - W_{iljk}) = W_{i[jl]k} \\
=& S_{ijlk}.
\end{aligned}$$

Hence we have proved the first part of this proposition.

$$W_{ijkl} = \epsilon_{im(k}\epsilon_{l)jn}\Delta_{mn} \tag{4.27}$$

using Eq. (4.26) in above equation then we have

$$\begin{aligned}
W_{ijkl} =& \frac{1}{2}(\epsilon_{imk}\epsilon_{ljn} + \epsilon_{iml}\epsilon_{kjn})\Delta_{mn} \\
=& \frac{1}{2}(\epsilon_{imk}\epsilon_{ljn} + \epsilon_{iml}\epsilon_{kjn})(\frac{1}{4}\epsilon_{muv}\epsilon_{nwx}W_{uwxv}) \\
=& \frac{1}{8}(\epsilon_{imk}\epsilon_{ljn}\epsilon_{muv}\epsilon_{nwx} + \epsilon_{iml}\epsilon_{kjn}\epsilon_{muv}\epsilon_{nwx})W_{uwxv} \\
=& \frac{1}{8}(\delta_{iv}\delta_{ku}\delta_{lw}\delta_{jx} - \delta_{iv}\delta_{ku}\delta_{lx}\delta_{jw} - \delta_{iu}\delta_{kv}\delta_{lw}\delta_{jx} + \delta_{iu}\delta_{kv}\delta_{lx}\delta_{jw} + \delta_{iv}\delta_{lu}\delta_{kw}\delta_{jx} - \delta_{iv}\delta_{lu}\delta_{kx}\delta_{jw} \\
& - \delta_{iu}\delta_{lv}\delta_{kw}\delta_{jx} + \delta_{iu}\delta_{lv}\delta_{kx}\delta_{jw})W_{uwxv} \\
=& \frac{1}{8}(W_{klji} - W_{kjli} - W_{iljk} + W_{ijlk} + W_{lkji} - W_{ljki} - W_{ikjl} + W_{ijkl})
\end{aligned}$$

by using the symmetries of the tensor W_{ijkl} then we have

$$W_{ijkl} = \epsilon_{im(k}\epsilon_{l)jn}\Delta_{mn}.$$

Hence the second of this proposition have proved.

4.4 Irreducible Decomposition Under the Rotation Group SO(3)

The special orthogonal group, denoted by $SO(3)$, is the subgroup of orthogonal matrices with determinant +1. Under this group, the elasticity tensor of 21 independent components is irreducibly decomposed into the sum of five independent pieces [17]

$$C_{ijkl} = \sum_{a=1}^{5} {}^{(a)}C_{ijkl} = ({}^{(1)}V_{ijkl} +^{(2)} V_{ijkl} +^{(3)} V_{ijkl}) + ({}^{(1)}W_{ijkl} +^{(2)} W_{ijkl}), \qquad (4.28)$$

where these pieces are invariant and unique under the action of rotation group $SO(3)$. The vector space of the elasticity tensor into five subspaces can be written as

$$C = ({}^{(1)}C \oplus^{(2)} C \oplus^{(3)} C) \oplus ({}^{(4)}C \oplus^{(5)} C).$$

Since V_{ijkl} is a totally symmetric 4th rank tensor. Now, we construct the unique totally symmetric 2nd rank tensor and scalar with the help of contraction of V_{ijkl} with the Kronecker delta.

$$\begin{aligned} V_{ij} &:= \delta_{kl} V_{ijkl} = V_{ijkk}, \\ &= \delta_{kl} \frac{1}{3}(C_{ijkl} + C_{iklj} + C_{iljk}), \\ &= \frac{1}{3}(C_{ijkk} + C_{ikkj} + C_{ikjk}), \\ &= \frac{1}{3}(C_{ijkk} + 2C_{ikkj}), \end{aligned}$$

and the contraction of the Kronecker delta with the totally symmetric 2nd rank tensor is

$$\begin{aligned} V &:= \delta_{ij} V_{ij} = V_{iikk}, \\ &= \frac{1}{3}(C_{iikk} + 2C_{ikki}), \end{aligned}$$

The traceless part of the totally symmetric tensor V_{ij} is defined as

$$T_{ij} := V_{ij} - \frac{1}{3} V \delta_{ij}, \qquad \text{with} \qquad \delta_{ij} T_{ij} = 0,$$

where T_{ij} is a traceless tensor because its trace is equal to zero $T_{ii} = V_{ii} - \frac{1}{3} V \delta_{ii} = 0$. The sub-tensor ${}^{(1)}V_{ijkl}$ of the irreducible tensor V_{ijkl} is defined as

$${}^{(1)}V_{ijkl} := \eta V \delta_{(ij} \delta_{kl)},$$

$$
\begin{aligned}
{}^{(1)}V_{ijkl} =& \eta V[\frac{1}{24}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{lj} + \delta_{il}\delta_{jk} + \delta_{il}\delta_{kj} + \delta_{ik}\delta_{jl} + \delta_{ij}\delta_{lk} + \delta_{jk}\delta_{li} + \delta_{jl}\delta_{ik} \\
&+ \delta_{ji}\delta_{kl} + \delta_{ji}\delta_{lk} + \delta_{jl}\delta_{ki} + \delta_{jk}\delta_{il} + \delta_{kl}\delta_{ij} + \delta_{ki}\delta_{jl} + \delta_{kj}\delta_{li} + \delta_{kj}\delta_{il} \\
&+ \delta_{ki}\delta_{lj} + \delta_{kl}\delta_{ji} + \delta_{li}\delta_{jk} + \delta_{lj}\delta_{ki} + \delta_{lk}\delta_{ij} + \delta_{lk}\delta_{ji} + \delta_{lj}\delta_{ik} + \delta_{li}\delta_{kj})],
\end{aligned}
$$

$$
{}^{(1)}V_{ijkl} = \eta V[\frac{8}{24}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{lj} + \delta_{il}\delta_{jk})]. \tag{4.29}
$$

The second sub-tensor ${}^{(2)}V_{ijkl}$ of the tensor V_{ijkl} is defined as

$$
{}^{(2)}V_{ijkl} := \xi T_{(ij}\delta_{kl)},
$$

$$
{}^{(2)}V_{ijkl} = \xi\frac{4}{24}(T_{ij}\delta_{kl} + T_{ik}\delta_{jl} + T_{il}\delta_{jk} + T_{jk}\delta_{il} + T_{jl}\delta_{ik} + T_{kl}\delta_{ij}). \tag{4.30}
$$

The third sub-tensor ${}^{(3)}V_{ijkl}$ is defined as

$$
{}^{(3)}V_{ijkl} := R_{ijkl},
$$

where the tensor $R_{ijkl} := V_{ijkl} -^{(1)} V_{ijkl} -^{(2)} V_{ijkl}$ is the remainder which is totally traceless. The trace of the tensor R_{ijkl} is also equal to zero. Now we calculate the values of η and ξ

$$
\begin{aligned}
R_{iikl} &= V_{iikl} -^{(1)} V_{iikl} -^{(2)} V_{iikl} \\
0 &= V_{kl} - \frac{40}{24}\eta V\delta_{kl} - \frac{28}{24}\xi T_{kl} \\
&= V_{kl} - \frac{40}{24}\eta V\delta_{kl} - \frac{28}{24}\xi(V_{kl} - \frac{1}{3}V\delta_{kl}) \\
&= V_{kl} - \frac{5}{3}\eta V\delta_{kl} - \frac{7}{6}\xi V_{kl} + \frac{7}{18}\xi V\delta_{kl}
\end{aligned} \tag{4.31}
$$

from Eq. (4.31) comparing the coefficients of the V_{kl} and $V\delta_{kl}$ then we have

$$
\eta = \frac{1}{5}, \qquad \xi = \frac{6}{7}.
$$

Using the values of η and ξ in Eqs. (4.29) and (4.30) then we have

$$
{}^{(1)}V_{ijkl} = \frac{1}{15}V(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{lj} + \delta_{il}\delta_{jk}),
$$

and

$$
{}^{(2)}V_{ijkl} = \frac{1}{7}(T_{ij}\delta_{kl} + T_{ik}\delta_{jl} + T_{il}\delta_{jk} + T_{jk}\delta_{il} + T_{jl}\delta_{ik} + T_{kl}\delta_{ij}).
$$

The vector spaces of these sub-tensors are denoted by ${}^{(1)}V$, ${}^{(2)}V$ and ${}^{(3)}V$. These sub-spaces are mutually orthogonal such that ${}^{(a)}V_{ijkl}{}^{(b)}V_{ijkl} = 0$ for $a, b = 1, 2, 3$ but $a \neq b$. If we take $a = 1, b = 2$ then we can show that

$$\begin{aligned}
{}^{(1)}V_{ijkl}{}^{(2)}V_{ijkl} = \frac{1}{105}V[&\delta_{ij}\delta_{kl}T_{ij}\delta_{kl} + \delta_{ij}\delta_{kl}T_{ik}\delta_{jl} + \delta_{ij}\delta_{kl}T_{il}\delta_{jk} + \delta_{ij}\delta_{kl}T_{jk}\delta_{il} + \delta_{ij}\delta_{kl}T_{jl}\delta_{ik} + \delta_{ij}\delta_{kl}T_{kl}\delta_{ij} \\
&+ \delta_{ik}\delta_{jl}T_{ij}\delta_{kl} + \delta_{ik}\delta_{jl}T_{ik}\delta_{jl} + \delta_{ik}\delta_{jl}T_{il}\delta_{jk} + \delta_{ik}\delta_{jl}T_{jk}\delta_{il} + \delta_{ik}\delta_{jl}T_{jl}\delta_{ik} + \delta_{ik}\delta_{jl}T_{kl}\delta_{ij} \\
&+ \delta_{il}\delta_{jk}T_{ij}\delta_{kl} + \delta_{il}\delta_{jk}T_{ik}\delta_{jl} + \delta_{il}\delta_{jk}T_{il}\delta_{jk} + \delta_{il}\delta_{jk}T_{jk}\delta_{il} + \delta_{il}\delta_{jk}T_{jl}\delta_{ik} + \delta_{il}\delta_{jk}T_{kl}\delta_{ij}] \\
= \frac{1}{105}&V(T_{ii}\delta_{kk} + T_{ii} + T_{ii} + T_{jj} + T_{jj} + T_{kk}\delta_{ii} + T_{ii} + T_{kk}\delta_{ii} + T_{ii} \\
&+ T_{ii}T_{jj}\delta_{ii} + T_{jj} + T_{ii} + T_{ii} + T_{ii}\delta_{jj} + T_{jj}\delta_{ii} + T_{kk} + T_{jj}) = 0.
\end{aligned}$$

The dimension of the vector spaces ${}^{(1)}V$, ${}^{(2)}V$ and ${}^{(3)}V$ are

$$\begin{aligned}
V &= {}^{(1)}V \oplus {}^{(2)}V \oplus {}^{(3)}V, \\
15 &= 1 + 5 + 9.
\end{aligned}$$

We know that the symmetric 2nd rank tensor has 6 dimensions but totally symmetric 2nd rank tensor has 5 $(6-1=5)$ because of the relation $T_{ii} = T_{11}+T_{22}+T_{33} = 0$. Since symmetric 4th rank tensor has 21 dimensions but totally symmetric 4th rank tensor has 9 dimensions. The reduction in dimensions is due to the relations $R_{ijkl} = R_{ikjl}$ and the tensor R_{iikl}. This relation $R_{ijkl} = R_{ikjl}$ reduces the dimensions to 15 $(21 - 6 = 15)$ while R_{iikl} further reduces to 9 $(15 - 6 = 9)$.

We turn now to the second irreducible 4th rank tensor with 6 independent components. It can be represented as a symmetric 2nd rank tensor Δ_{mn} is given in Eq. (4.25). In order to decompose the second irreducible tensor W_{ijkl}, it is more appropriate to use its representation by the tensor density as

$$W := \delta_{mn}\Delta_{mn}. \tag{4.32}$$

Using the definition of Δ_{mn} in Eq. (4.32) and we have

$$\begin{aligned}
W &= \frac{1}{3}\delta_{mn}\epsilon_{mil}\epsilon_{njk}W_{ijkl} \\
&= \frac{1}{3}(\delta_{ij}\delta_{kl} - \delta_{ik}\delta_{jl})W_{ijkl} \\
&= \frac{1}{3}(W_{iikk} - W_{ikik}) \\
&= \frac{1}{3}(C_{iikk} - C_{ikik}).
\end{aligned}$$

Δ_{ij} can be decomposed into the scalar and traceless parts:

$$\Delta_{ij} = Q_{ij} + \frac{1}{3}W\delta_{ij} \tag{4.33}$$

where the symmetric and traceless tensor Q_{ij} is

$$Q_{ij} := \Delta_{ij} - \frac{1}{3}W\delta_{ij}.$$

The decomposition of second irreducible tensor W_{ijkl} under the rotation group is

$$W_{ijkl} =^{(1)} W_{ijkl} +^{(2)} W_{ijkl}.$$

By using Eq. (4.33) in Eq. (4.25), we get the values of the sub-tensors ${}^{(1)}W_{ijkl}$ and ${}^{(2)}W_{ijkl}$

$$\begin{aligned}
W_{ijkl} &= \frac{1}{2}(\epsilon_{imk}\epsilon_{ljn} + \epsilon_{iml}\epsilon_{kjn})\Delta_{mn} \\
&= \frac{1}{2}(\epsilon_{imk}\epsilon_{ljn} + \epsilon_{iml}\epsilon_{kjn})(Q_{mn} + \frac{1}{3}W\delta_{mn}) \\
&= \frac{1}{2}(\epsilon_{imk}\epsilon_{ljn} + \epsilon_{iml}\epsilon_{kjn})Q_{mn} + \frac{1}{6}W(\epsilon_{imk}\epsilon_{ljn} + \epsilon_{iml}\epsilon_{kjn})\delta_{mn} \\
&= \frac{1}{6}W(2\delta_{ij}\delta_{kl} - \delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}) + \frac{1}{2}(\delta_{ik}Q_{jl} + \delta_{jk}Q_{il} + \delta_{il}Q_{jk} + \delta_{jl}Q_{ik} - 2\delta_{ij}Q_{kl} - 2\delta_{kl}Q_{ij})
\end{aligned}$$

where the scalar part is

$${}^{(1)}W_{ijkl} := \frac{1}{6}W(2\delta_{ij}\delta_{kl} - \delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}),$$

and the remainder part is

$${}^{(2)}W_{ijkl} := \frac{1}{2}(\delta_{ik}Q_{jl} + \delta_{jk}Q_{il} + \delta_{il}Q_{jk} + \delta_{jl}Q_{ik} - 2\delta_{ij}Q_{kl} - 2\delta_{kl}Q_{ij}).$$

The vector space of these sub-tensors are denoted by ${}^{(1)}W$ and ${}^{(2)}W$ and the corresponding dimensions of these vectorspaces are

$$\begin{aligned}
W &= {}^{(1)}W \oplus {}^{(2)}W, \\
6 &= 1 + 5.
\end{aligned}$$

These subspaces are mutually orthogonal to eachother,

$${}^{(1)}W_{ijkl}{}^{(2)}W_{ijkl} = 0.$$

Theorem : The elasticity tensor C_{ijkl} is decomposed into 5 pieces under the action of the group $SO(3)$ [17]

$$C_{ijkl} = ({}^{(1)}V_{ijkl} +^{(2)} V_{ijkl} +^{(3)} V_{ijkl}) + ({}^{(1)}W_{ijkl} +^{(2)} W_{ijkl}).$$

This decomposition corresponds to the direct sum decomposition of the vector space of the elasticity tensor into five subspaces

$$C = ({}^{(1)}C \oplus^{(2)} C \oplus^{(3)} C) \oplus ({}^{(4)}C \oplus^{(5)} C),$$

with the dimensions

$$21 = (1 + 5 + 9) + (1 + 5).$$

The irreducible pieces are orthogonal to one another: for $a \neq b$

$${}^{(a)}C_{ijkl}{}^{(b)}C_{ijkl} = 0.$$

The Euclidean squares, $C^2 = C_{ijkl}C_{ijkl}$ and ${}^{(a)}C_{ijkl} =^{(a)} C_{ijkl}{}^{(a)}C_{ijkl}$ with $a = 1 \cdots 5$, fulfill the "Pythagorean theorem:"

$$C = ({}^{(1)}C^2 \oplus^{(2)} C^2 \oplus^{(3)} C^3) \oplus ({}^{(4)}C^2 \oplus^{(5)} C^2).$$

It should be noticed that the reducible decomposition RS cannot be understood directly by the elasticity tensor because these tensors do not fulfill the symmetries of the elasticity tensor.

Chapter 5

Applications of the Irreducible Decomposition

In the previous chapter, we have discussed the RS-decomposition and VW-decomposition of the elasticity tensor. The first decomposition is reducible and the second decomposition is irreducible. In this chapter, physical applications of the VW-decomposition discussed by Itin [18] are reviewed. The irreducible parts V_{ijkl} and W_{ijkl} can be used in all physical applications because the reducible parts R_{ijkl} and S_{ijkl} cannot be interpreted directly as elasticity tensors.

5.1 Cauchy Relations and Cauchy Factor

Definition of Cauchy Relations

The Cauchy relations are defined so that the second irreducible tensor of the VW-decomposition of the elasticity tensor W_{ijkl} vanishes ($W_{ijkl} = 0$). It is noticed that from proposition 8 the tensor S_{ijkl} and a symmetric 2nd rank tensor Δ_{mn} also vanishes and from Eq. (4.8) we conclude that elasticity tensor C_{ijkl} is totally symmetric tensor ($C_{ijkl} = C_{(ijkl)}$). Moreover, according to the definition of the tensor S_{ijkl}, Eq. (4.1) can be written as

$$C_{ijkl} = C_{ikjl}, \tag{5.1}$$

where Eq. (5.1) are called Cauchy relations. An alternative form of Eq. (5.1) is

$$C_{ijkl} - C_{ikjl} = 0.$$

During the early days of modern linear elasticity theory, Cauchy formulated molecular models for elastic bodies based on 15 independent elastic constants. According to Eq. (5.1), there are 6 non-zero components of Cauchy relations that holds. In addition to the relations due to the minor and major symmetries, Cauchy relations give the following relations [6, 7]:

$$C_{1122} = C_{1212}, \qquad C_{1133} = C_{3131}, \qquad C_{2233} = C_{2323},$$
$$C_{1123} = C_{1213}, \qquad C_{2231} = C_{2321}, \qquad C_{3312} = C_{3132}.$$

In Voigt notation, these non-zero components can be written as:

$$C_{12} = C_{66}, \qquad C_{13} = C_{55}, \qquad C_{23} = C_{44}$$
$$C_{14} = C_{56}, \qquad C_{25} = C_{46}, \qquad C_{36} = C_{45}.$$

A lattice-theoretical approach shows that the Cauchy relations are valid provided the following conditions hold [19]:

- The central forces are the forces of interaction between atoms or molecules of a crystal.
- Every atom and molecule is a center of symmetry.
- The interaction forces between the building blocks of a crystal can be well approximated by a harmonic potential [20, 21].

Cauchy versus non-Cauchy Parts in Elasticity

In Cauchy relations, the tensor W_{ijkl} is equal to zero. As a consequence, the totally symmetric tensor V_{ijkl} is called the "Cauchy part" of the elasticity tensor C_{ijkl} which is also known as "main part" while "non-Cauchy part" of the elasticity is called W_{ijkl}. It is also known as "deviation part". When we take S_{ijkl} tensor as a deviation part in the RS-decomposition, it means that R_{ijkl} is the co-partner of the tensor S_{ijkl} and has 21 independent components. It is also noticed that R_{ijkl} itself becomes an elasticity tensor C_{ijkl}. Moreover, it is assumed that the tensor S_{ijkl} is equal to zero then the tensor R_{ijkl} is restricted to 15 independent components. The RS-decomposition creates problem for the identification of the deviation part. The identification of the deviation part can be solved by using the VW-decomposition which is an irreducible.

Definition of Cauchy Factor

We can define the Cauchy factor, a dimensionless quantity as

$$0 \leq F_{Cauchy} = \sqrt{\frac{V_{ijkl}V_{ijkl}}{C_{ijkl}C_{ijkl}}} \leq 1. \tag{5.2}$$

In a tetragonal material, the matrices V_{ijkl} and W_{ijkl} are

$$V_{ijkl} = \begin{pmatrix} C_{11} & \frac{1}{3}(C_{12}+2C_{66}) & \frac{1}{3}(C_{13}+2C_{55}) & 0 & 0 & C_{16} \\ * & C_{11} & \frac{1}{3}(C_{13}+2C_{55}) & 0 & 0 & -C_{16} \\ * & * & C_{33} & 0 & 0 & 0 \\ * & * & * & \frac{1}{3}(C_{13}+2C_{55}) & 0 & 0 \\ * & * & * & * & \frac{1}{3}(C_{13}+2C_{55}) & 0 \\ * & * & * & * & * & \frac{1}{3}(C_{12}+2C_{66}) \end{pmatrix},$$

and

$$W_{ijkl} = \begin{pmatrix} 0 & \frac{2}{3}(C_{12}-C_{66}) & \frac{2}{3}(C_{13}-C_{55}) & 0 & 0 & 0 \\ * & 0 & \frac{2}{3}(C_{13}-C_{55}) & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 \\ * & * & * & \frac{1}{3}(C_{44}-C_{23}) & 0 & 0 \\ * & * & * & * & \frac{1}{3}(C_{44}-C_{23}) & 0 \\ * & * & * & * & * & \frac{1}{3}(C_{66}-C_{12}) \end{pmatrix}.$$

By taking the square of both tensors V_{ijkl} and W_{ijkl}, we get the following form

$$V_{ijkl}V_{ijkl} = 2(C_{11})^2 + 6\{\frac{1}{3}(C_{12}+2C_{66})\}^2 + 12\{\frac{1}{3}(C_{13}+2C_{55})\}^2 + (C_{33})^2 + 8(C_{16})^2,$$

and

$$W_{ijkl}W_{ijkl} = 2\{\frac{2}{3}(C_{12}-C_{66})\}^2 + 4\{\frac{2}{3}(C_{13}-C_{55})\}^2 + 8\{\frac{1}{3}(C_{44}-C_{23})\}^2 + 4\{\frac{1}{3}(C_{66}-C_{12})\}^2.$$

Consider indium which is a tetragonal material whose $C_{11} = 4.53\times10^{10}Nm^{-2}, C_{12} = 4.0\times10^{10}Nm^{-2}, C_{13} = 4.15\times10^{10}Nm^{-2}, C_{33} = 4.51\times10^{10}Nm^{-2}, C_{44} = 0.65\times10^{10}Nm^{-2}, C_{66} = 1.21\times10^{10}Nm^{-2}$ and $C_{16} = 0\times10^{10}Nm^{-2}$. Using these values in above equations we get

$$V_{ijkl}V_{ijkl} = V^2 = 128.46,$$
$$W_{ijkl}W_{ijkl} = W^2 = 43.05,$$
$$C_{ijkl}C_{ijkl} = C^2 = 128.46 + 43.05 = 171.51.$$

By substituting the above values in Eq. (5.2), the Cauchy factor is

$$F_{Cauchy} = 0.9.$$

In a hexagonal material, the matrices V_{ijkl} and W_{ijkl} are

$$V_{ijkl} = \begin{pmatrix} C_{11} & \frac{1}{3}(C_{11}) & \frac{1}{3}(C_{13}+2C_{55}) & 0 & 0 & 0 \\ * & C_{11} & \frac{1}{3}(C_{13}+2C_{55}) & 0 & 0 & 0 \\ * & * & C_{33} & 0 & 0 & 0 \\ * & * & * & \frac{1}{3}(C_{13}+2C_{55}) & 0 & 0 \\ * & * & * & * & \frac{1}{3}(C_{13}+2C_{55}) & 0 \\ * & * & * & * & * & \frac{1}{3}C_{11} \end{pmatrix}$$

and

$$W_{ijkl} = \begin{pmatrix} 0 & \frac{2}{3}(C_{12}-C_{66}) & \frac{2}{3}(C_{13}-C_{55}) & 0 & 0 & 0 \\ * & 0 & \frac{2}{3}(C_{13}-C_{55}) & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 \\ * & * & * & \frac{1}{3}(C_{44}-C_{23}) & 0 & 0 \\ * & * & * & * & \frac{1}{3}(C_{44}-C_{23}) & 0 \\ * & * & * & * & * & \frac{3C_{11}-C_{12}}{2} \end{pmatrix}$$

Similarly, we also get

$$V_{ijkl}V_{ijkl} = 2(C_{11})^2 + 6(\frac{1}{3}C_{11})^2 + 12\{\frac{1}{3}(C_{13}+2C_{55})\}^2 + (C_{33})^2,$$

and

$$W_{ijkl}W_{ijkl} = 2\{\frac{2}{3}(C_{12}-C_{66})\}^2+4\{\frac{2}{3}(C_{13}-C_{55})\}^2+8\{\frac{1}{3}(C_{44}-C_{23})\}^2+4\{\frac{3C_{11}-C_{12}}{2}\}^2,$$

where $C_{66} = \frac{1}{2}(3C_{11}-C_{12})$. Consider Beryllium which is a hexagonal material whose $C_{11} = 29.23 \times 10^{10} Nm^{-2}, C_{12} = 2.67 \times 10^{10} Nm^{-2}, C_{13} = 1.4 \times 10^{10} Nm^{-2}, C_{33} = 33.64 \times 10^{10} Nm^{-2}$ and $C_{44} = 16.25 \times 10^{10} Nm^{-2}$. Using these values in above equations we get

$$\begin{aligned} V_{ijkl}V_{ijkl} &= V^2 = 4942.32, \\ W_{ijkl}W_{ijkl} &= W^2 = 9227.33, \\ C_{ijkl}C_{ijkl} &= C^{2=}14169.65. \end{aligned}$$

By substituting the above values in Eq. (5.2), the Cauchy factor is

$$F_{Cauchy} = 0.6.$$

5.2 Strain Energy Function

The strain energy function is expressed by Eq. (2.10). When the generalized Hooke law is used in Eq. (2.10) results into Eq. (2.12). Because of the irreducible decomposition Eq. (2.12) can be written as:

$$\begin{aligned} \Omega &= \frac{1}{2}(V_{ijkl}+W_{ijkl})\varepsilon_{ij}\varepsilon_{kl}, \\ &= \frac{1}{2}V_{ijkl}\varepsilon_{ij}\varepsilon_{kl} + \frac{1}{2}W_{ijkl}\varepsilon_{ij}\varepsilon_{kl}. \end{aligned}$$

The above expression of the strain energy function can be separated into two parts, one is a Cauchy and other is a non-Cauchy part which makes good sense in physics.

$$\Omega_{(C)} = \frac{1}{2}V_{ijkl}\varepsilon_{ij}\varepsilon_{kl}, \qquad \Omega_{(nC)} = \frac{1}{2}W_{ijkl}\varepsilon_{ij}\varepsilon_{kl}.$$

Where $\Omega_{(C)}$ and $\Omega_{(nC)}$ are Cauchy and non-Cauchy parts respectively.
Therefore the tensors V_{ijkl} and W_{ijkl} with respect to their dimensions and symmetries, are C_{ijkl} themselves. Since the strain tensor ε_{ij} can be expressed in terms of the displacement gradients according to

$$\varepsilon_{(ij)} = \frac{\partial u_j}{\partial x_i}. \tag{5.3}$$

Using Eq. (5.3) in Eq. (2.12) then we get

$$\Omega = \frac{1}{2}C_{ijkl}\frac{\partial u_j}{\partial x_i}\frac{\partial u_l}{\partial x_k}. \tag{5.4}$$

5.3 Null Lagrangian in Linear Elasticity

Euler Lagrange Equation

The Euler Lagrange equation is a second order partial differential equation whose solutions are the functions for which a given functional is stationary. A differentiable functional at its local maxima and minima. The Euler Lagrange equation is given as

$$L(t, q, \dot{q}) = \frac{\partial L}{\partial q} - \frac{d}{\partial t}(\frac{\partial L}{\partial \dot{q}}), \tag{5.5}$$

where t is time, q is coordinate point and $\dot{q}$ is derivative of q with respect to time t.

Null Lagrangian

A null Lagrangian is one whose Euler Lagrange equation vanish identically [22].

Existence of Null Lagrangian in Linear Elasticity

In the theory of linear elasticity, a null Lagrangian defined as that part of the strain energy functional, which does not play any role in the equilibrium equation.

The RS-decomposition is used in Eq. (5.4) then the energy density of a material which is deformed is formulated explicitly as

$$\Omega = \frac{1}{2}(R_{ijkl} + S_{ijkl})\frac{\partial u_j}{\partial x_i}\frac{\partial u_l}{\partial x_k} = \frac{1}{2}R_{ijkl}\frac{\partial u_j}{\partial x_i}\frac{\partial u_l}{\partial x_k} + \frac{1}{2}S_{ijkl}\frac{\partial u_j}{\partial x_i}\frac{\partial u_l}{\partial x_k}, \tag{5.6}$$

since C_{ijkl} holds the minor symmetries. The left minor symmetry of the elasticity tensor C_{ijkl} is used in the above equation and again we rewrite the Eq. (5.6) as

$$\Omega = \frac{1}{2}C_{ijkl}\frac{\partial u_i}{\partial x_j}\frac{\partial u_l}{\partial x_k} = \frac{1}{2}R_{ijkl}\frac{\partial u_i}{\partial x_j}\frac{\partial u_l}{\partial x_k} + \frac{1}{2}S_{ijkl}\frac{\partial u_i}{\partial x_j}\frac{\partial u_l}{\partial x_k}. \tag{5.7}$$

Obviously, the sum of two tensors R_{ijkl} and S_{ijkl} are not changed while every single term of Eq. (5.6) did change because the left minor symmetry does not hold for the tensors R_{ijkl} and S_{ijkl}. Since the last term of Eq. (5.7) may be written as

$$\frac{1}{2}S_{ijkl}\frac{\partial}{\partial x_j}(u_i\frac{\partial u_l}{\partial x_k}) = \frac{1}{2}S_{ijkl}\frac{\partial u_i}{\partial x_j}\frac{\partial u_l}{\partial x_k} + \frac{1}{2}S_{ijkl}u_i\frac{\partial}{\partial x_j}(\frac{\partial u_l}{\partial x_k})$$

$$\frac{1}{2}S_{ijkl}\frac{\partial u_i}{\partial x_j}\frac{\partial u_l}{\partial x_k} = \frac{1}{2}S_{ijkl}\frac{\partial}{\partial x_j}(u_i\frac{\partial u_l}{\partial x_k}) - \frac{1}{2}S_{ijkl}u_i\frac{\partial}{\partial x_j}(\frac{\partial u_l}{\partial x_k}) \tag{5.8}$$

using the Eq. (5.8) in the last term of Eq. (5.7) as

$$\Omega = \frac{1}{2}R_{ijkl}\frac{\partial u_i}{\partial x_j}\frac{\partial u_l}{\partial x_k} + \frac{1}{2}S_{ijkl}\frac{\partial}{\partial x_j}(u_i\frac{\partial u_l}{\partial x_k}) - \frac{1}{2}S_{ijkl}u_i\frac{\partial}{\partial x_j}(\frac{\partial u_l}{\partial x_k}). \tag{5.9}$$

Since $S_{i(jk)l} = 0$ and $\frac{\partial}{\partial x_{[j}}\frac{\partial}{\partial x_{k]}} = 0$, the last term of the above equation vanishes and the remaining terms are

$$\Omega = \frac{1}{2}R_{ijkl}\frac{\partial u_i}{\partial x_j}\frac{\partial u_l}{\partial x_k} + \frac{1}{2}S_{ijkl}\frac{\partial}{\partial x_j}(u_i\frac{\partial u_l}{\partial x_k}), \tag{5.10}$$

where in Eq. (5.10) the second term is total derivative term [23]. It is also known as S-term. Thus, the first term, R-term, is involved in the variational principle to determine the equations of motion. This term is also involved in the equilibrium equation for solving the problem of null Lagrangian for the theory of linear elasticity. This result was described by Lancia *et al* [23]. By using propositions 6 and 7 in Eq. (5.10) then we have

$$\Omega = \frac{1}{2}(V_{ijkl} + W_{i(jk)l})\frac{\partial u_i}{\partial x_j}\frac{\partial u_l}{\partial x_k} + \frac{1}{2}W_{i[jk]l}\frac{\partial}{\partial x_j}(u_i\frac{\partial u_l}{\partial x_k})$$

$$\Omega = \frac{1}{2}[V_{ijkl} + \frac{1}{2}(W_{ijkl} + W_{ikjl})]\frac{\partial u_i}{\partial x_j}\frac{\partial u_l}{\partial x_k} + \frac{1}{2}W_{i[jk]l}\frac{\partial}{\partial x_j}(u_i\frac{\partial u_l}{\partial x_k}). \tag{5.11}$$

Subsequently, using proposition 3 in Eq (5.11).

$$\Omega = \frac{1}{2}[V_{ijkl} - \frac{1}{2}W_{iljk}]\frac{\partial u_i}{\partial x_j}\frac{\partial u_l}{\partial x_k} + \frac{1}{2}W_{i[jk]l}\frac{\partial}{\partial x_j}(u_i\frac{\partial u_l}{\partial x_k}). \tag{5.12}$$

Therefore, in Eq. (5.12) the tensor W_{ijkl} is included in the total derivative term. If Cauchy relations hold then this tensor vanishes. However, in the first part of strain energy functional the tensors V_{ijkl} and W_{iljk} appear together.

The problem is to identify the null Lagrangian part of the elasticity Lagrangian and then identify the set of elastic constants which contribute to the equilibrium equation. Eq. (2.12) can be turned into RS-decomposition as shown in Eq. (5.10) hence it acts as a Lagrangian functional. Also it generates the equilibrium equation along with the variation relative to displacement field.

Our considerations, the equilibrium conditions for the Lagrangian given in Eq. (5.12). The variation of this Lagrangian up to a total derivative term reads as

$$\begin{aligned}\delta\Omega &= \frac{1}{2}[V_{ijkl} - \frac{1}{2}W_{iljk}]\delta(\frac{\partial u_i}{\partial x_j}\frac{\partial u_l}{\partial x_k}) \\ &= \frac{1}{2}[V_{ijkl} - \frac{1}{2}W_{iljk}][(\frac{\partial}{\partial x_j}(\delta u_i))\frac{\partial u_l}{\partial x_k} + \frac{\partial u_i}{\partial x_j}(\frac{\partial}{\partial x_k}(\delta u_l))].\end{aligned} \tag{5.13}$$

Since, both the tensors V_{ijkl} and W_{iljk} hold the major and minor symmetries. The last term of Eq. (5.13) can be added:

$$\begin{aligned}\delta\Omega &= \frac{1}{2}[V_{ijkl} - \frac{1}{2}W_{iljk}][(\frac{\partial}{\partial x_j}(\delta u_i))\frac{\partial u_l}{\partial x_k} + \frac{\partial u_l}{\partial x_k}(\frac{\partial}{\partial x_j}(\delta u_i))] \\ &= (V_{ijkl} - \frac{1}{2}W_{iljk})[(\frac{\partial}{\partial x_j}(\delta u_i))\frac{\partial u_l}{\partial x_k}].\end{aligned} \tag{5.14}$$

The above equation can also be written as

$$\begin{aligned}\delta\Omega &= [V_{ijkl} - \frac{1}{2}W_{iljk}][\frac{\partial}{\partial x_j}(\delta u_i\frac{\partial u_l}{\partial x_k}) - \delta u_i\frac{\partial}{\partial x_j}(\frac{\partial u_l}{\partial x_k})] \\ &= (V_{ijkl} - \frac{1}{2}W_{ijkl})\frac{\partial}{\partial x_j}(\delta u_i\frac{\partial u_l}{\partial x_k}) - (V_{ijkl} - \frac{1}{2}W_{iljk})\delta u_i\frac{\partial}{\partial x_j}(\frac{\partial u_l}{\partial x_k})\end{aligned}$$

and the equilibrium condition is taken as:

$$(V_{ijkl} - \frac{1}{2}W_{iljk})\delta u_i\frac{\partial}{\partial x_j}(\frac{\partial u_l}{\partial x_k}) = 0. \tag{5.15}$$

By using the propositions 3 and 7 in Eq. (5.15) and knowing that $\delta u_i \neq 0$, the above equation can be written as:

$$\begin{aligned}&(V_{ijkl} + W_{i(jk)l})\frac{\partial}{\partial x_j}(\frac{\partial u_l}{\partial x_k}) = 0 \\ &R_{ijkl}\frac{\partial}{\partial x_j}(\frac{\partial u_l}{\partial x_k}) = 0.\end{aligned}$$

Since $R_{ijkl} = C_{i(jk)l}$ and $\frac{\partial}{\partial x_{[j}}\frac{\partial}{\partial x_{k]}} = 0$, the standard equilibrium equation becomes

$$C_{i(jk)l}\frac{\partial}{\partial x_j}(\frac{\partial u_l}{\partial x_k}) = 0,$$

or

$$C_{ijkl}\frac{\partial}{\partial x_j}(\frac{\partial u_l}{\partial x_k}) = 0. \tag{5.16}$$

The total derivative term is the part of strain energy functional. It does not play any role in the equilibrium equation Eq. (5.16) in which the set of elastic constants contribute but does not remove any subset of the elastic constants. Thus, null Lagrangian does not exist for an arbitrary material.

5.4 Acoustic Wave Propagation

The equation of motion comes from the fundamental 2nd law of dynamics

$$F = ma, \tag{5.17}$$

where F is the forces, m is the mass and a is the acceleration. Eq. (5.17) can also be written as

$$\begin{aligned}
&\sum F = ma,\\
&\text{body forces} + \text{surface forces} = m\frac{\partial^2 u_i}{\partial t^2},\\
&\rho b_i + \frac{\partial \sigma_{ij}}{\partial x_j} = \rho\frac{\partial^2 u_i}{\partial t^2},
\end{aligned}$$

where m is the mass density, b_i is the body force density. When we ignore the body forces then

$$\frac{\partial \sigma_{ij}}{\partial x_j} = \rho\frac{\partial^2 u_i}{\partial t^2}, \tag{5.18}$$

and using the generalized Hooke's law in Eq. (5.18) then we have

$$C_{ijkl}\frac{\partial^2 u_l}{\partial x_j \partial x_k} = \rho\frac{\partial^2 u_i}{\partial t^2}.$$

This is called equation of motion. In the theory of linear elasticity, the wave propagation in anisotropic media is describe as:

$$C_{ijkl}\frac{\partial^2 u_l}{\partial x_j \partial x_k} - \rho\delta_{il}\frac{\partial^2 u_l}{\partial t^2} = 0, \tag{5.19}$$

where u_l is the displacement co-vector that depends on time as well as space coordinates. The elasticity tensor, C_{ijkl}, mass density, ρ, and Kronecker delta, δ_{ij}, are coefficients (constants). In reference [24] a solution is assumed i.e.

$$u_l = U_l e^{i(\xi n_i x_i - \omega t)}, \tag{5.20}$$

A system of three homogeneous algebraic equations is obtained

$$(C_{ijkl}\xi^2 n_j n_k - \rho\delta_{il}\omega^2)U_l = 0, \tag{5.21}$$

which has a non-trivial solution if and only if the characteristic equation holds

$$\det(C_{ijkl}\xi^2 n_j n_k - \rho\delta_{il}\omega^2) = 0. \tag{5.22}$$

The algebraic Eq. (5.22) takes the form

$$(\Gamma_{il} - v^2\delta_{il})U_l = 0, \tag{5.23}$$

where

$$\Gamma_{il} = \frac{1}{\rho}C_{ijkl}n_j n_k, \tag{5.24}$$

$$v = \frac{\omega}{\xi},$$

and the characteristic equation becomes

$$\det(\Gamma_{il} - v^2\delta_{il}) = 0. \tag{5.25}$$

The Christoffel tensor becomes symmetric with respect to the major and minor symmetries of the elasticity tensor as below

$$\begin{aligned}
\Gamma_{li} &= \frac{1}{\rho}C_{ljki}n_j n_k, \\
&= \frac{1}{\rho}C_{kilj}n_j n_k, \\
&= \frac{1}{\rho}C_{ikjl}n_j n_k, \\
&= \frac{1}{\rho}C_{ijkl}n_j n_k, \\
&= \Gamma_{il}.
\end{aligned}$$

Therefore, its eigenvalues and eigenvectors are real and orthogonal respectively.

Decomposition of the Christoffel Tensor

Under the action of the group $GL(3, R)$, a symmetric tensor, by itself cannot be decomposed directly. But the VW-decomposition of the C_{ijkl} generates the corresponding decomposition of the Γ_{il}. By using the VW-decomposition in Eq. (5.24)

$$\begin{aligned}\Gamma_{il} &= (V_{ijkl} + W_{ijkl})n_j n_k,\\ \Gamma_{il} &= V_{ijkl}n_j n_k + W_{ijkl}n_j n_k,\\ \Gamma_{il} &= V_{il} + W_{il},\end{aligned}$$

where $V_{il} := V_{ijkl}n_j n_k = V_{li}$ and $W_{il} := W_{ijkl}n_j n_k = W_{li}$. These two tensors which are symmetric correspond to the Cauchy and non-Cauchy parts of the C_{ijkl}. Using the values of Cauchy (V_{ijkl}) and non-Cauchy (W_{ijkl}) parts

$$\begin{aligned}V_{il} = V_{ijkl}n_j n_k &= \frac{1}{3\rho}(C_{ijkl} + C_{iklj} + C_{iljk})n_j n_k,\\ W_{il} = W_{ijkl}n_j n_k &= \frac{1}{3\rho}(2C_{ijkl} - C_{iklj} - C_{ilkj})n_j n_k.\end{aligned}$$

Here, the two tensors V_{il} and W_{il} are called Cauchy Christoffel and non-Cauchy Christoffel tensors respectively.

Proposition 9: For every elasticity tensor, C_{ijkl}, and wave co-vector, n_i,

$$W_{il}n_l = 0.$$

Proof: Since W_{il} is non-Cauchy Christoffel tensor

$$\begin{aligned}W_{il}n_l &= W_{ijkl}n_j n_k n_l,\\ &= W_{i(jkl)}n_j n_k n_l,\\ &= 0.\end{aligned}$$

Proposition 10: The determinant of the non-Cauchy Christoffel tensor, W_{il}, is

$$\det\,(W_{ij}) = 0.$$

Using the decomposition of Christoffel tensor in Eq. (5.23) and also can be written as

$$(V_{il} + W_{il} - v^2\delta_{il})U_l = 0,$$

where the characteristic equation is

$$\det\,(V_{il} + W_{il} - v^2\delta_{il}) = 0. \tag{5.26}$$

From the proposition 10, the Eq. (5.26) takes the form

$$\det\ (W_{il} - v^2\delta_{il}) = 0,$$

where V_{il} is equal to zero and det $(W_{il}) = 0$ so at least one of its eigen value is equal to zero. It is noticed that Christoffel tensor is real and also symmetric, thus all its eigenvalues are real and the related eigenvectors are orthogonal. We need to satisfy the condition of positive definiteness of the matrix Γ_{ij} to get three real positive eigenvalues such that

i. all eigenvalues are distinct ($v_1^2 > v_2^2 > v_3^2$),

ii. two eigenvalues are equal ($v_1^2 > v_2^2 = v_3^2$ or $v_1^2 = v_2^2 > v_3^2$),

iii. or three eigenvalues are equal ($v_1^2 = v_2^2 = v_3^2$).

5.5 Polarization of Acoustic Waves

Acoustic wave propagation in an elastic medium is an eigenvector problem given by Eq. (5.23) in which the phase velocity v^2 is the eigenvalues and in general three distinct real positive solutions correspond to three independent waves i.e. ${}^{(1)}U_l$, ${}^{(2)}U_l$ and ${}^{(3)}U_l$ and are called acoustic polarizations [24]. On the basis of polarization vector waves can be classified as:

- Longitudinal Wave
- Transverse Wave

There are three pure polarization of acoustic waves for the isotropic material. One is "longitudinal wave" that is also called compression wave and the other two are "transverse waves" which are also called shear waves.
The polarization is directed along the propagation vector called longitudinal wave is given by

$$\overrightarrow{U} \times \overrightarrow{n} = 0,$$

and the polarization is normal to the propagation vector called transverse wave is given by

$$\overrightarrow{U}.\overrightarrow{n} = 0.$$

In general, the three pure polarization waves do not exist for anisotropic materials.

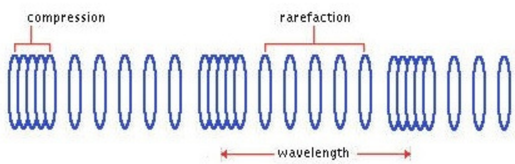

Figure 1: Longitudinal Wave

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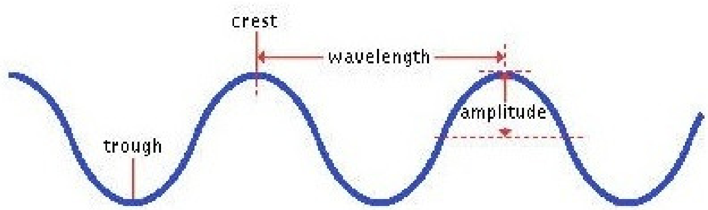

Figure 2: Transverse Wave

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Now, we introduce a vector and scalar in term of Christoffel tensor and direction vector such as

$$V_i := \Gamma_{ij} n_j, \qquad\qquad V := \Gamma_{ij} n_i n_j.$$

Since V_i and V depend only on the Cauchy part of the elasticity tensor given by proposition 9

$$V_i = V_{ij} n_j, \qquad\qquad V = V_{ij} n_i n_j.$$

Proposition 11: Suppose the vector n_i denotes an allowed direction for the propagation of a longitudinal wave then the velocity v_L of the longitudinal wave in the direction of the vector n_i is calculated by the Cauchy part of the elasticity tensor such as

$$v_L = \sqrt{V}.$$

Proof: Consider $u_j = \alpha n_j$ for the longitudinal wave. According to Eq. (5.23) becomes

$$\begin{aligned}
&(\Gamma_{ij} - v^2\delta_{ij})u_j = 0,\\
&(\Gamma_{ij} - v^2\delta_{ij})\alpha n_j = 0,\\
&\alpha\Gamma_{ij}n_j - v^2\delta_{ij}\alpha n_j = 0,\\
&v^2\delta_{ij}n_j = \Gamma_{ij}n_j,\\
&v^2\delta_{ij}n_i n_j = \Gamma_{ij}n_i n_j,\\
&v^2 n_j n_j = \Gamma_{ij}n_i n_j,\\
&v^2 = V,\\
&v = \sqrt{V},
\end{aligned}$$

where $n_j n_j = 1$ and $\Gamma_{ij}n_i n_j = V$.

Proposition 12: The three purely polarized waves such as one longitudinal and the two transverse waves can propagate through a medium with a given elasticity tensor in the direction $\overrightarrow{n}$ if and only if

$$V_i = V n_i.$$

Proof: Since Γ_{ij} is a matrix which is symmetric, the eigenvalues and the eigenvectors are real and orthogonal, respectively. One of the eigenvectors of pure polarizations points in the direction of $\overrightarrow{n}$ if and only if pure polarizations are three. Suppose

$$\begin{aligned}
&V_i = \Gamma_{ij}n_j,\\
&V_i n_i = \Gamma_{ij}n_i n_j,\\
&V_i n_i = V,\\
&V_i n_i n_i = V n_i,\\
&V_i = v_L^2 n_i.
\end{aligned}$$

Hence, the directions of the purely polarized waves depend on the Cauchy part of the elasticity tensor.

5.6 Examples

5.6.1 Isotropic Media

The elasticity tensor is defined by Eq. (2.17). The first irreducible part V_{ijkl} of the elasticity tensor for the isotropic bodies can be expressed as

$$\begin{aligned}
V_{ijkl} = C_{(ijkl)} &= \lambda(\delta_{(ij}\delta_{kl)}) + \mu(\delta_{(ik}\delta_{lj)} + \delta_{(il}\delta_{jk)}),\\
&= (\lambda + 2\mu)\delta_{(ij}\delta_{kl)},
\end{aligned} \tag{5.27}$$

where $\delta_{(ij}\delta_{kl)} = \delta_{(ik}\delta_{lj)} = \delta_{(il}\delta_{jk)}$. we can also write Eq.(5.27) as

$$V_{ijkl} = \frac{(\lambda + 2\mu)}{3}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{lj} + \delta_{il}\delta_{jk}).$$

The second irreducible part W_{ijkl} of the elasticity tensor for the isotropic bodies is given by

$$\begin{aligned} W_{ijkl} &= \lambda(\delta_{ij}\delta_{kl}) + \mu(\delta_{ik}\delta_{lj} + \delta_{il}\delta_{jk}) + \frac{(\lambda + 2\mu)}{3}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{lj} + \delta_{il}\delta_{jk}), \\ &= \lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{lj} + \mu\delta_{il}\delta_{jk} - \frac{(\lambda + 2\mu)}{3}\delta_{ij}\delta_{kl} - \frac{(\lambda + 2\mu)}{3}\delta_{ik}\delta_{lj} - \frac{(\lambda + 2\mu)}{3}\delta_{il}\delta_{jk}, \end{aligned}$$

$$= \frac{1}{3}(3\lambda\delta_{ij}\delta_{kl} + 3\mu\delta_{ik}\delta_{lj} + 3\mu\delta_{il}\delta_{jk} - \lambda\delta_{ij}\delta_{kl} - \lambda\delta_{ik}\delta_{lj} - \lambda\delta_{il}\delta_{jk} - 2\mu\delta_{ij}\delta_{kl} - 2\mu\delta_{ik}\delta_{lj} - 2\mu\delta_{il}\delta_{jk}),$$

$$\begin{aligned} &= \frac{1}{3}[2\lambda\delta_{ij}\delta_{kl} - 2\mu\delta_{ij}\delta_{kl} - (\lambda - \mu)\delta_{ik}\delta_{lj} - (\lambda - \mu)\delta_{il}\delta_{jk}], \\ &= \frac{1}{3}[2(\lambda - \mu)\delta_{ij}\delta_{kl} - (\lambda - \mu)\delta_{ik}\delta_{lj} - (\lambda - \mu)\delta_{il}\delta_{jk}], \end{aligned}$$

$$W_{ijkl} = \frac{\lambda - \mu}{3}(2\delta_{ij}\delta_{kl} - \delta_{ik}\delta_{lj} - \delta_{il}\delta_{jk}). \tag{5.28}$$

Putting the values of the Cauchy and non-Cauchy parts in Eq. (4.8) then we get

$$C_{ijkl} = \frac{(\lambda + 2\mu)}{3}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{lj} + \delta_{il}\delta_{jk}) + \frac{(\lambda - \mu)}{3}(2\delta_{ij}\delta_{kl} - \delta_{ik}\delta_{lj} - \delta_{il}\delta_{jk}), \tag{5.29}$$

consider

$$\alpha := \frac{(\lambda + 2\mu)}{3}, \qquad \beta := \frac{(\lambda - \mu)}{3}.$$

Putting the values of α and β in Eq. (5.29) then we obtain

$$\begin{aligned} C_{ijkl} &= \alpha(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{lj} + \delta_{il}\delta_{jk}) + \beta(2\delta_{ij}\delta_{kl} - \delta_{ik}\delta_{lj} - \delta_{il}\delta_{jk}), \\ &= \alpha(\delta_{ij}\delta_{kl} + 2\delta_{i(k}\delta_{l)j}) + \beta(2\delta_{ij}\delta_{kl} - 2\delta_{i(k}\delta_{l)j}), \\ &= \alpha\delta_{ij}\delta_{kl} + 2\alpha\delta_{i(k}\delta_{l)j} + 2\beta\delta_{ij}\delta_{kl} - 2\beta\delta_{i(k}\delta_{l)j}, \\ &= (\alpha + 2\beta)\delta_{ij}\delta_{kl} + 2(\alpha - \beta)\delta_{i(k}\delta_{l)j}. \end{aligned}$$

The RS-decomposition of the elasticity tensor for the isotropic material can also be expressed as

$$\begin{aligned} R_{ijkl} = C_{i(jk)l} &= \lambda\delta_{i(j}\delta_{k)l} + \mu(\delta_{i(k}\delta_{j)} + \delta_{il}\delta_{jk}), \\ &= (\lambda + \mu)\delta_{i(j}\delta_{k)l} + \mu\delta_{il}\delta_{jk}, \end{aligned}$$

where $\delta_{i(j}\delta_{k)l}$ and $\delta_{(ik)} = \delta_{ik}$. Now for the tensor S_{ijkl} as

$$\begin{aligned} S_{ijkl} = C_{i[jk]l} &= \lambda\delta_{i[j}\delta_{k]l} + \mu(\delta_{i[k}\delta_{|l|j]} + \delta_{il}\delta_{[jk]}), \\ &= \lambda\delta_{i[j}\delta_{k]l} + \mu(\delta_{i[k}\delta_{|l|j}), \\ &= (\lambda - \mu)\delta_{i[j}\delta_{k]l}, \end{aligned}$$

where $\delta_{[jk]} = 0$ and $\delta_{i[k}\delta_{|l|j]} = -\delta_{i[j}\delta_{k]l}$. Putting the values of the tensors R_{ijkl} and S_{ijkl} in Eq. (4.1) and then we have

$$\begin{aligned} C_{ijkl} &= (\lambda + \mu)\delta_{i(j}\delta_{k)l} + \mu\delta_{il}\delta_{jk} + (\lambda - \mu)\delta_{i[j}\delta_{k]l}, \\ &= (\lambda + \mu)[\frac{1}{2}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl})] + \mu\delta_{il}\delta_{jk} + (\lambda - \mu)[\frac{1}{2}(\delta_{ij}\delta_{kl} - \delta_{ik}\delta_{jl})], \\ &= \frac{(\lambda + \mu)}{2}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl}) + \mu\delta_{il}\delta_{jk} + \frac{(\lambda - \mu)}{2}(\delta_{ij}\delta_{kl} - \delta_{ik}\delta_{jl}). \end{aligned}$$

Consider

$$\alpha^{'} = \frac{(\lambda + \mu)}{2}, \qquad \beta^{'} = \mu, \qquad \gamma^{'} = \frac{(\lambda - \mu)}{2}.$$

Using these values in the above expression then we have

$$C_{ijkl} = \alpha^{'}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl}) + \beta^{'}\delta_{il}\delta_{jk} + \gamma^{'}(\delta_{ij}\delta_{kl} - \delta_{ik}\delta_{jl}).$$

We know that in the Cauchy relations , the second irreducible part of the elasticity tensor and the tensor S_{ijkl} are equal to zero. In isotropic media, the Cauchy relations are defined as

$$\lambda = \mu. \tag{5.30}$$

Putting Eq. (5.30) in VW-decomposition then we have

$$\begin{aligned} C_{ijkl} &= (\frac{\mu + 2\mu}{3})[\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{lj} + \delta_{il}\delta_{jk}] + (\frac{\mu - \mu}{3})[2\delta_{ij}\delta_{kl} - \delta_{ik}\delta_{lj} - \delta_{il}\delta_{jk}], \\ &= \mu(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{lj} + \delta_{il}\delta_{jk}). \end{aligned}$$

and again putting the same condition in RS-decomposition then we obtain

$$\begin{aligned} C_{ijkl} &= (\frac{\mu + \mu}{2})[\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl}] + \mu\delta_{il}\delta_{jk} + \frac{\mu - \mu}{2}[\delta_{ij}\delta_{kl} - \delta_{ik}\delta_{jl}], \\ &= \mu(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \end{aligned}$$

With respect to the Eq. (5.30), the elasticity tensor is equal for RS- and VW-decompositions. Therefore, when Cauchy relations hold then the tensors R_{ijkl} and V_{ijkl} are same for the isotropic media.

Cauchy Factor for Isotropic Media

The matrices of the tensors V_{ijkl} and W_{ijkl} for isotropic media are

$$V_{ijkl} = \begin{pmatrix} C_{11} & \frac{1}{3}C_{11} & \frac{1}{3}C_{11} & 0 & 0 & 0 \\ * & C_{11} & \frac{1}{3}C_{11} & 0 & 0 & 0 \\ * & * & C_{11} & 0 & 0 & 0 \\ * & * & * & \frac{1}{3}C_{11} & 0 & 0 \\ * & * & * & * & \frac{1}{3}C_{11} & 0 \\ * & * & * & * & * & \frac{1}{3}C_{11} \end{pmatrix}. \tag{5.31}$$

$$W_{ijkl} = \begin{pmatrix} 0 & \frac{1}{3}(3C_{12}-C_{11}) & \frac{1}{3}(3C_{12}-C_{11}) & 0 & 0 & 0 \\ * & 0 & \frac{1}{3}(3C_{12}-C_{11}) & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 \\ * & * & * & -\frac{1}{6}(3C_{12}-C_{11}) & 0 & 0 \\ * & * & * & * & -\frac{1}{6}(3C_{12}-C_{11}) & 0 \\ * & * & * & * & * & -\frac{1}{6}(3C_{12}-C_{11}) \end{pmatrix}. \tag{5.32}$$

By taking the square of both tensors V_{ijkl} and W_{ijkl}, we get the following form

$$\begin{aligned} V_{ijkl}V_{ijkl} &= V^2 = 3(C_{11})^2 + 18\{\frac{1}{3}(C_{11})\}^2, \\ W_{ijkl}W_{ijkl} &= W^2 = 6\{\frac{1}{3}(3C_{12}-C_{11})\}^2 + 12\{-\frac{1}{6}(3C_{12}-C_{11})\}^2. \end{aligned} \tag{5.33}$$

Consider aluminium which is an isotropic material whose $C_{11} = 10.73 \times 10^{10} Nm^{-2}, C_{12} = 6.08 \times 10^{10} Nm^{-2}, C_{44} = 2.83 \times 10^{10} Nm^{-2}$. Using these values in Eq. (5.33) we get

$$\begin{aligned} V_{ijkl}V_{ijkl} &= V^2 = 581.05, \\ W_{ijkl}W_{ijkl} &= W^2 = 56.40, \\ C_{ijkl}C_{ijkl} &= C^2 = 637.45. \end{aligned}$$

The Cauchy factor is

$$F_{Cauchy} = 0.9.$$

Isotropic is a special case in which the main difference between these two decompositions become obvious. The VW-decomposition determines the existence of two parameters which are linearly independent of the isotropic medium while the RS-decomposition also determines the existence of three parameters which are linearly dependent. Now, calculate the characteristic velocities of the acoustic waves. We first find the Christoffel matrices from the tensors V_{il} and W_{il}

$$\begin{aligned}V_{il} = V_{ijkl}n_j n_k &= \frac{(\lambda+2\mu)}{3}[\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{lj}+\delta_{il}\delta_{jk}]n_j n_k,\\ &= \alpha(\delta_{ij}\delta_{kl}n_j n_k+\delta_{ik}\delta_{lj}n_j n_k+\delta_{il}\delta_{jk}n_j n_k),\\ &= (n_i n_l + n_i n_l + \delta_{il}) = \alpha(\delta_{il}+2n_i n_l),\end{aligned}$$

by replacing $l = j$ in the above equation then we have

$$V_{ij} = \alpha(\delta_{ij}+2n_i n_j), \tag{5.34}$$

$$\begin{aligned}W_{il} = W_{ijkl}n_j n_k &= \frac{\lambda-\mu}{3}[2\delta_{ij}\delta_{kl}-\delta_{ik}\delta_{lj}-\delta_{il}\delta_{jk}]n_j n_k,\\ &= \beta(2\delta_{ij}n_j n_k-\delta_{ik}\delta_{lj}n_j n_k-\delta_{il}\delta_{jk}n_j n_k),\\ &= \beta(2n_i n_l - n_i n_l - \delta_{il}) = -\beta(\delta_{il}-n_i n_l),\end{aligned}$$

and if we replace j with l then

$$W_{ij} = -\beta(\delta_{ij}-n_i n_j). \tag{5.35}$$

Eqs. (5.34) and (5.35) are Christoffel matrices

$$\begin{aligned}V_{ij}n_j &= \alpha(\delta_{ij}+2n_i n_j)n_j,\\ &= \alpha(\delta_{ij}n_j)+2\alpha n_i(n_j n_j),\\ &= \alpha n_i + 2\alpha n_i,\end{aligned}$$

$$V_i = 3\alpha n_i, \tag{5.36}$$

where V_i is a vector. Consider,

$$\begin{aligned}&V_{ij}n_j = 3\alpha n_i,\\ &V_{ij}n_i n_j = 3\alpha n_i n_i,\\ &V_j n_j = 3\alpha,\\ &V = 3\alpha,\end{aligned} \tag{5.37}$$

where V is a scalar. We take the equation $\Gamma_{il} = V_{il} + W_{il}$ and using the values of V_{il} and W_{il} in this equation then we obtain

$$\begin{aligned}\Gamma_{il} &= \alpha(\delta_{il}+2n_i n_l)-\beta(\delta_{il}-n_i n_l),\\ &= \alpha\delta_{il}+2\alpha n_i n_l-\beta\delta_{il}+\beta n_i n_l,\\ &= (\alpha-\beta)\delta_{il}+(2\alpha+\beta)n_i n_l.\end{aligned} \tag{5.38}$$

Using Eq. (5.38) in Eq. (5.25) then we get

$$\det[v^2\delta_{il} - (\alpha - \beta)\delta_{il} + (2\alpha + \beta)n_i n_l] = 0,$$

in terms of the parameters such as α and β, the characteristic equation for the acoustic waves take the form

$$\det[(v^2 - \alpha + \beta)\delta_{ij} - (2\alpha + \beta)n_i n_j] = 0. \tag{5.39}$$

The velocity of longitudinal wave given below

$$v_1^2 = V = 3\alpha = (\lambda - 2\mu), \tag{5.40}$$

is a first solution of the above equation. Using the value of longitudinal wave velocity in Eq. (5.39) then $\det(\delta_{ij} - n_i n_j) = 0$ which means that v_1^2 is an eigen value of the Eq. (5.39). Putting the value of $v^2 = (\alpha - \beta)$ in Eq. (5.39) then we have $\det(n_i n_j) = 0$. The velocity of transverse waves are $v_2^2 = v_3^2 = (\alpha - \beta) = \mu$.

5.6.2 Anisotropic Media

Proposition 13: The most general type of an anisotropic medium that allows propagation of purely polarized waves in an arbitrary direction has an elasticity tensor of the form [18]

$$C_{ijkl} = \begin{pmatrix} \alpha & \frac{\alpha}{3}+2\rho_1 & \frac{\alpha}{3}+2\rho_2 & 2\rho_3 & 0 & 0 \\ * & \alpha & \frac{\alpha}{3}+2\rho_4 & 0 & 2\rho_5 & 0 \\ * & * & \alpha & 0 & 0 & 2\rho_6 \\ * & * & * & \frac{\alpha}{3}-\rho_4 & -\rho_6 & -\rho_5 \\ * & * & * & * & \frac{\alpha}{3}-\rho_2 & -\rho_3 \\ * & * & * & * & * & \frac{\alpha}{3}-\rho_1 \end{pmatrix},$$

where $\rho_1, ...\rho_6$ are arbitrary parameters. In this medium, the longitudinal waves velocity is $v_L = \sqrt{3\alpha} = \sqrt{\lambda - 2\mu}$.

Cubic crystal

In cubic crystal, there are three elastic constants which are linearly independent of each other. In a properly chosen coordinate system, they can be put into the Voigt matrix given by Eq. (2.18) [24].

$$\begin{pmatrix} C_{1111} & C_{1112} & C_{1133} & C_{1123} & C_{1131} & C_{1112} \\ * & C_{2222} & C_{2233} & C_{2223} & C_{2231} & C_{2212} \\ * & * & C_{3333} & C_{3323} & C_{3331} & C_{3312} \\ * & * & * & C_{2323} & C_{2331} & C_{2312} \\ * & * & * & * & C_{3131} & C_{3112} \\ * & * & * & * & * & C_{1212} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ * & C_{11} & C_{12} & 0 & 0 & 0 \\ * & * & C_{11} & 0 & 0 & 0 \\ * & * & * & C_{66} & 0 & 0 \\ * & * & * & * & C_{66} & 0 \\ * & * & * & * & * & C_{66} \end{pmatrix}.$$

By using the cubic crystal matrix in Eq. (4.18) then we have

$$\begin{pmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ * & C_{11} & C_{12} & 0 & 0 & 0 \\ * & * & C_{11} & 0 & 0 & 0 \\ * & * & * & C_{66} & 0 & 0 \\ * & * & * & * & C_{66} & 0 \\ * & * & * & * & * & C_{66} \end{pmatrix} = \begin{pmatrix} V_{11} & V_{12} & V_{12} & 0 & 0 & 0 \\ * & V_{11} & V_{12} & 0 & 0 & 0 \\ * & * & V_{11} & 0 & 0 & 0 \\ * & * & * & V_{66} & 0 & 0 \\ * & * & * & * & V_{66} & 0 \\ * & * & * & * & * & V_{66} \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & W_{12} & W_{12} & 0 & 0 & 0 \\ * & 0 & W_{12} & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 \\ * & * & * & W_{66} & 0 & 0 \\ * & * & * & * & W_{66} & 0 \\ * & * & * & * & * & W_{66} \end{pmatrix}.$$

By using the Eqs. (4.16) and (4.17) in the above matrices then we get Cauchy and non-Cauchy parts in the matrices form. The Cauchy part is

$$V_{ijkl} = \begin{pmatrix} C_{11} & \frac{1}{3}(C_{12}+2C_{66}) & \frac{1}{3}(C_{12}+2C_{66}) & 0 & 0 & 0 \\ * & C_{11} & \frac{1}{3}(C_{12}+2C_{66}) & 0 & 0 & 0 \\ * & * & C_{11} & 0 & 0 & 0 \\ * & * & * & \frac{1}{3}(C_{12}+2C_{66}) & 0 & 0 \\ * & * & * & * & \frac{1}{3}(C_{12}+2C_{66}) & 0 \\ * & * & * & * & * & \frac{1}{3}(C_{12}+2C_{66}) \end{pmatrix},$$

$$V_{ijkl} = \begin{pmatrix} \alpha^{'} & \beta^{'} & \beta^{'} & 0 & 0 & 0 \\ * & \alpha^{'} & \beta^{'} & 0 & 0 & 0 \\ * & * & \alpha^{'} & 0 & 0 & 0 \\ * & * & * & \beta^{'} & 0 & 0 \\ * & * & * & * & \beta^{'} & 0 \\ * & * & * & * & * & \beta^{'} \end{pmatrix},$$

where $C_{11} = \alpha^{'}$ and $\frac{1}{3}(C_{12} + 2C_{66}) = \beta^{'}$. The non-Cauchy part is

$$W_{ijkl} = \begin{pmatrix} 0 & \frac{2}{3}(C_{12}-C_{66}) & \frac{2}{3}(C_{12}-C_{66}) & 0 & 0 & 0 \\ * & 0 & \frac{2}{3}(C_{12}-C_{66}) & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 \\ * & * & * & \frac{1}{3}(C_{66}-C_{12}) & 0 & 0 \\ * & * & * & * & \frac{1}{3}(C_{66}-C_{12}) & 0 \\ * & * & * & * & * & \frac{1}{3}(C_{66}-C_{12}) \end{pmatrix}.$$

$$W_{ijkl} = \begin{pmatrix} 0 & 2\gamma^{'} & 2\gamma^{'} & 0 & 0 & 0 \\ * & 0 & \gamma^{'} & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 \\ * & * & * & -\gamma^{'} & 0 & 0 \\ * & * & * & * & -\gamma^{'} & 0 \\ * & * & * & * & * & -\gamma^{'} \end{pmatrix},$$

where $W_{12} = C_{12} = \frac{2}{3}(C_{12} - C_{66}) = \gamma^{'}$ and $W_{66} = \frac{1}{3}(C_{66} - C_{12}) = -\gamma^{'}$. Therefore, $\alpha^{'}$, $\beta^{'}$ and $\gamma^{'}$ are new elastic constants. The elasticity tensor C_{ijkl} is expressed in terms of these constants. The Cauchy part of the Christoffel tensor can be written in the form

$$V_{il} = \begin{pmatrix} \alpha^{'} n_1^2 + \beta^{'}(n_2^2 + n_3^2) & 2\beta^{'} n_1 n_2 & 2\beta^{'} n_1 n_3 \\ * & \alpha^{'} n_2^2 + \beta^{'}(n_1^2 + n_3^2) & 2\beta^{'} n_2 n_3 \\ * & * & \alpha^{'} n_3^2 + \beta^{'}(n_1^2 + n_2^2) \end{pmatrix}.$$

The vector V_i of the Cauchy Christoffel tensor is

$$V_i = \begin{pmatrix} \alpha^{'} n_1^3 + 3\beta^{'} n_1 n_2^2 + 3\beta^{'} n_1 n_3^2 \\ \\ \alpha^{'} n_2^3 + 3\beta^{'} n_1^2 n_2 + 3\beta^{'} n_2 n_3^2 \\ \\ \alpha^{'} n_3^3 + 3\beta^{'} n_1^2 n_3 + 3\beta^{'} n_2^2 n_3 \end{pmatrix}.$$

and the scalar V of the Cauchy Christoffel tensor takes the form

$$V = (\alpha^{'} - 3\beta^{'})(n_1^4 + n_2^4 + n_3^4) + 3\beta^{'},$$

respectively. While the non-Cauchy part of the Christoffel tensor takes the form

$$W_{il} = \begin{pmatrix} -\gamma^{'}(n_2^2 + n_3^2) & -\gamma^{'} n_1 n_2 & -\gamma^{'} n_1 n_3 \\ * & -\gamma^{'}(n_1^2 + n_3^2) & -\gamma^{'} n_2 n_3 \\ * & * & -\gamma^{'}(n_1^2 + n_2^2) \end{pmatrix}.$$

The corresponding vector $W_{ij}n_j = W_i$ and scalar $W_{ij}n_in_j = W$ are equal to zero. Now, we calculate the longitudinal velocity for cubic crystal.

1. Edges: In edges, $\overrightarrow{n} = (1, 0, 0)$. The unit vector is $\hat{n} = (1, 0, 0)$. The longitudinal velocity for edges is

$$\begin{aligned} V &= (\alpha^{'} - 3\beta^{'})(n_1^4 + n_2^4 + n_3^4) + 3\beta^{'}, \\ &= \alpha^{'} - 3\beta^{'} + 3\beta^{'} = \alpha^{'}, \\ v_L &= \sqrt{\alpha^{'}} = \sqrt{C_{11}}. \end{aligned}$$

2. Face diagonals: In face diagonals, $\overrightarrow{n} = (1, 1, 0)$. The unit vector is $\hat{n} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$. The longitudinal velocity for face diagonals is

$$
\begin{aligned}
V &= (\alpha^{'} - 3\beta^{'})(n_1^4 + n_2^4 + n_3^4) + 3\beta^{'}, \\
&= (\alpha^{'} - 3\beta^{'})[(\frac{1}{\sqrt{2}})^4 + (\frac{1}{\sqrt{2}})^4 + 0] + 3\beta^{'}, \\
&= \frac{\alpha^{'} - 3\beta^{'}}{2} + 3\beta^{'}, \\
&= \frac{\alpha^{'} + 3\beta^{'}}{2}, \\
v_L &= \sqrt{\frac{\alpha' + 3\beta'}{2}}, \\
&= \sqrt{\frac{(C_{11} + C_{12} + 2C_{66})}{2}}.
\end{aligned}
$$

3. Space diagonals: In space diagonals, $\overrightarrow{n} = (1, 1, 1)$. The unit vector is $\hat{n} = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$. The longitudinal velocity for space diagonals is

$$
\begin{aligned}
V &= (\alpha^{'} - 3\beta^{'})(n_1^4 + n_2^4 + n_3^4) + 3\beta^{'}, \\
&= (\alpha^{'} - 3\beta^{'})[(\frac{1}{\sqrt{3}})^4 + (\frac{1}{\sqrt{3}})^4 + (\frac{1}{\sqrt{3}})^4] + 3\beta^{'}, \\
&= \frac{\alpha^{'} - 3\beta^{'}}{3} + 3\beta^{'}, \\
&= \frac{\alpha^{'} + 6\beta^{'}}{3}, \\
v_L &= \sqrt{\frac{\alpha' + 6\beta'}{3}}, \\
&= \sqrt{\frac{(C_{11} + 2C_{12} + 4C_{66})}{3}}.
\end{aligned}
$$

There are extensive materials with same S and arbitrary W-tensor which have exactly the same directions and velocities of the longitudinal waves. This can be seen in the light of proposition 14 and 15. The elasticity tensor C_{ijkl} can be expressed for such materials as:
From proposition 13, $C_{11} = \alpha$, using the value of C_{11} in $\alpha^{'}$ then we get

$$\alpha^{'} = \alpha.$$

$C_{12} = \frac{\alpha}{3} + 2\rho_1$ and $C_{66} = \frac{\alpha}{3} - \rho_1$. By using the value of these components in $\beta^{'}$ and $\gamma^{'}$ then we have

$$\begin{aligned}\beta^{'} &= \frac{1}{3}\{\frac{\alpha}{3} + 2\rho_1 + 2(\frac{\alpha}{3} - \rho_1)\}\\ &= \frac{1}{3}\{\frac{\alpha}{3} + 2\rho_1 + \frac{2\alpha}{3} - 2\rho_1\}\\ &= \frac{\alpha}{3}.\end{aligned}$$

$$\begin{aligned}\gamma^{'} &= \frac{1}{3}\{C_{12} - C_{66}\}\\ &= \frac{1}{3}\{\frac{\alpha}{3} + 2\rho_1 - \frac{\alpha}{3} + \rho_1\}\\ &= \rho_1.\end{aligned}$$

The new matrix of the elasticity tensor is

$$C_{ijkl} = \begin{pmatrix} \alpha^{'} & \beta^{'} + 2\rho_1 & \beta^{'} + 2\rho_2 & 2\rho_3 & 0 & 0 \\ * & \alpha^{'} & \beta^{'} + 2\rho_4 & 0 & 2\rho_5 & 0 \\ * & * & \alpha^{'} & 0 & 0 & 2\rho_6 \\ * & * & * & \beta^{'} - \rho_4 & -\rho_6 & -\rho_5 \\ * & * & * & * & \beta^{'} - \rho_2 & -\rho_3 \\ * & * & * & * & * & \beta^{'} - \rho_1 \end{pmatrix}.$$

Chapter 6

Summary and Conclusion

In this thesis, we have studied the decompositions of elasticity tensor under 2 and 3 dimensions and some problems relating to its theory. In this chapter, we summarize our thesis. In the theory of linear anisotropic elasticity, the properties of the elastic medium are described by the 4th rank elasticity tensor C_{ijkl}. There are two ways to decompose the elasticity tensor C_{ijkl} under the general linear group. The first reducible decomposition of elasticity tensor is RS-decomposition which is frequently used in the literature. The elasticity tensor decomposed into a partially symmetric tensor R_{ijkl} and partially antisymmetric tensor S_{ijkl}. The vector spaces of $\boldsymbol{R}$ and $\boldsymbol{C}$ are same such as 21 dimensions and the vector space of $\boldsymbol{S}$ is 6 dimensions. The tensors R_{ijkl} and S_{ijkl} hold the major symmetry of the C_{ijkl} but do not obey the minor symmetries of the elasticity tensor. Moreover, due to the lack of the minor symmetries, they do not represent elasticity tensor C_{ijkl}. The partial symmetric tensor R_{ijkl} can further be decomposed. Consequently, this decomposition does not correspond to a direct sum decomposition of the vector space defined by C.

The second irreducible decomposition is VW-decomposition. In this decomposition, the elasticity tensor C_{ijkl} is decomposed into the completely symmetric part V plus the remainder W. Under the 3-dimensional general linear group, it is irreducible and unique. In VW-decomposition, the first irreducible part, denoted by V_{ijkl}, consists of 15 independent components and the other irreducible part, denoted by W_{ijkl}, has 6 independent components. The irreducible decomposition of the tensor C_{ijkl} yields the decomposition of the corresponding tensor space C into a direct sum of two subspaces such as $V \subset C$ and $W \subset C$. The VW-decomposition is more superior than RS-decomposition because it is irreducible, unique and preserves the minor and major symmetries of the elasticity tensor. We have concluded that the VW-decomposition is more suitable decomposition than RS-decomposition. It is valid from algebraic and physical point of view.

In the framework of the VW-decomposition (irreducible decomposition) of the elasticity tensor, we have studied the physical applications of its decomposition. The first physical application of VW-decomposition is Cauchy relations. Cauchy relations hold if and only if the second irreducible part of this decomposition is equal to zero. In Cauchy relations, there are two types of elasticity: one is Cauchy type i.e. V and other is non-Cauchy type i.e. W which measures the deviation from V (Cauchy part). The second application is strain energy density function. It is split into two parts (Cauchy and non-Cauchy), makes good sense in physics. The other applications, for the acoustic wave propagation define the Cauchy and non-Cauchy parts of the Christoffel tensor (Γ_{il}). The interesting results are obtained for the Christoffel tensor (Γ_{il}) which mentioned in propositions 9 and 10. Also, examine the polarizations of elastic wave. The Cauchy part of the Christoffel tensor determine the propagation of longitudinal wave (see proposition 11). We have presented (see proposition 13) a complete new class of anisotropic materials which allow pure polarizations to propagate in arbitrary directions, similarly as in isotropic material.

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