The Spectral Method

by

Ifra Mumtaz

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Supervised by

Prof. Faiz Ahmad

School of Natural Sciences

National University of Sciences and Technology

Islamabad, Pakistan

Dedicated to

My Loving Parents

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Ifra Mumtaz

Abstract

The spectral method has attracted much attention in recent research of numerical computing for solving differential equations. The purpose of this thesis is to study the basic theory of this method. We study the properties of sets of orthogonal polynomials. If the points of interpolation are chosen at zeros of a certain polynomial then the approximation of a function by the first n members of an orthogonal set becomes optimum in the least square sense.

The spectral method for the solution of nonlinear boundary value problems are described in detail and are applied to a few examples. The solutions are compared and the results are plotted graphically. We also find the approximate solution of Blasius boundary value problem.

Plan of Thesis

The present dissertation is arranged as follows:
In Chapter 1, we describe the brief introduction of the polynomial approximation and the basic concept of the spectral method.
In Chapter 2, we describe the properties of orthogonal polynomials, which are important in applications. In this we discuss existence of orthogonal polynomials, recurrence relation, zeros of orthogonal polynomials and Favard's Theorem.
In Chapter 3, we describe the least square approximation. The problem of least square ties in with the early history of orthogonal polynomials. We also discuss the discrete least square approximation and orthogonal polynomial least square approximation.
In Chapter 4, we find the approximate solutions of the boundary value problems by using the spectral method. We also find the approximate solution of Blasius boundary value problem and compare it with numerical solution.

Contents

Chapter 1

Introduction

Weierstrass' Theorem states that a function continuous on a closed and bounded interval can be approximated by a polynomial.

Theorem: Let $I \subset \mathbb{R}$ be a closed and bounded interval and f a contiuous function on I. Then for every $\epsilon > 0$, there exist a polynomial P such that

$$|f(x) - P(x)| < \epsilon$$

for all x in I.

For the proof see [1] . It is easy to see that it is impossible to approximate a discontinuous function by a polynomial.

Example: Let

$$f(x) = \begin{cases} 0, & 0 \leq x < \frac{1}{2} \\ 1, & \frac{1}{2} \leq x \leq 1. \end{cases}$$

Let $\epsilon = 0.1$ and suppose that a polynomial P exists so that

$$|f(x) - P(x)| < 0.1, \qquad 0 \leq x \leq 1.$$

This would mean

$$f(x) - 0.1 < P(x) < f(x) + 0.1, \qquad 0 \leq x \leq 1.$$

Hence

$$-0.1 < P(\frac{1}{2}-) < 0.1, \quad \text{for } \ 0 \leq x < \frac{1}{2}$$

and

$$0.9 < P(\frac{1}{2}+) < 1.1, \quad \text{for } \ \frac{1}{2} \leq x \leq 1.$$

$P(x)$ being a polynomial must be continuous at $x = \frac{1}{2}$, and

$$P(\frac{1}{2}) = P(\frac{1}{2}-) = P(\frac{1}{2}+).$$

And the above inequalities would imply

$$0.9 < P(\frac{1}{2}) < 0.1,$$

which is absurd. This example hence shows that, it is impossible to approximate a discontinuous function by a polynomial.

Weierstrass' Theorem guarantees the existence of a polynomial but it is not so easy to find it. However if a function is $(n+1)-$ times differentiable at a point x_0 and

$$|f^{n+1}(x)| \leq M,$$

for all x in $|x - x_0| < a$, then

$$\left| f(x) - \sum_{i=0}^{n} \frac{f^i\,(x_0)}{i!}(x-x_0)^i \right| \leq \frac{M}{(n+1)!}\,|x-x_0|^{n+1}\,.$$

The above result requires differentiability upto order $n+1$ and boundedness of the derivative which may not be easy to meet.

Let $\{\varphi_n(x)\}_{n=0}^{\infty}$ be a sequence of functions orthogonal on $[a, b]$ with respect to a weight function $w(x)$ i.e.

$$< \varphi_n, \varphi_m > \ = \int_a^b \varphi_n(x)\varphi_m(x)w(x)dx = 0, \qquad n \neq m. \tag{1.1}$$

Let f be a piecewise continuous function on $[a, b]$.

Let

$$c_n = < f, \varphi_n >, \qquad n = 0, 1, ...$$

The real number c_n is called the n -th Fourier coefficient of f with respect to $\{\varphi_n\}_{n=0}^{\infty}$. If we let

$$g_n(x) = \sum_{k=0}^{n} a_k \varphi_k,$$

$g_n(x)$ is the best approximation to $f(x)$ if and only if a_k are chosen so that

$$a_k = c_k.$$

The best approximation is meant in the sense that

$$\|f(x) - g(x)\|,$$

is minimized by the above choice.

If $\{\varphi_n\}_{n=0}^{\infty}$ is chosen to be a sequence of orthogonal polynomials, then $g_n(x)$ provides a polynomial approximation to f on $[a, b]$.

The infinite series

$$\sum_{k=0}^{\infty} a_k \varphi_k,$$

is called the Fourier series for f. It can be shown that for every simple set of orthogonal polynomials $\{\varphi_n\}_{n=0}^{\infty}$ the Fourier series converges to f i.e

$$\lim_{n \to \infty} \left\| f - \sum_{k=0}^{n} a_k \varphi_k \right\| = 0.$$

In this thesis, we shall discuss polynomial approximation to a function , but using instead of the inner product (1.1), we use the discrete orthogonality condition discussed as in chapter 3. Here the coefficients $a_k : k = 0, 1, ..., n$ in

$$g_n(x) = \sum_{k=0}^{n} a_k \varphi_k,$$

are found by solving the $(n + 1) \times (n + 1)$ system of equations formed by evaluating the above equation at the $n + 1$ zeros of the polynomial $\varphi_{n+1}(x)$. This method is known in the literature as the spectral method. Spectral methods were developed in a long series of papers by Steven Orszag starting in 1969. Spectral methods are a class of teachniques used in applied mathematics. This is a powerful method for the solution of ordinary and partial differential equations. In this method we try to approximate the functions (solution of o.d.e's , p.d.e's, etc) by mean of a truncated series of orthogonal functions (polynomials). We shall apply this method to solve a nonlinear boundary value problem which is important in the analysis of fluid flow in a boundary layer.

Chapter 2

Properties of Orthogonal Polynomials

In this chapter we shall discuss some basic properties of orthogonal polynomials, which are important in applications.

Two real-valued functions f and g are said to be orthogonal on $[a,b]$, with respect to a positive weight function $w(x)$ on (a,b), if the inner product

$$< f,g > \ = \int_a^b w(x)f(x)g(x)dx = 0.$$

When $w(x) = 1$, then f and g are said to be simply orthogonal.

A sequence of polynomials $\{\varphi_n\}_{n=0}^{\infty}$ is called an orthogonal set of polynomials, if

$$< \varphi_m, \varphi_n > \ = 0 \qquad \text{for} \quad m \neq n.$$

Furthermore, if $< \varphi_n, \varphi_n > = 1, \ n = 0, 1, ...$ then the set of polynomials is called an orthonormal set.

A sequence of polynomials $\{\varphi_n\}_{n=0}^{\infty}$, is called a simple set of polynomials, if φ_n is of degree n.

Throughout this chapter, we consider simple sets of orthogonal polynomial.

Theorem 1.[2]

Let $\{\varphi_n\}_{n=0}^{\infty}$ be a simple set of polynomials and let Q_k be an arbitrary polynomial of degree k. Then Q_k is a linear combination of polynomials φ_0, $\varphi_1, \ldots..,$ φ_k.

$$Q_k(x) = A_k x^k + A_{k-1} x^{k-1} + ... + A_0 x^0.$$

Proof:

We shall prove this theorem with the help of mathematical induction.

For $m = 0$, we have:

$$\begin{aligned} Q_0 &= A_0 x^0 \qquad \text{and} \qquad \varphi_0 = a_0 x^0, \\ a_0 Q_0 &= a_0 A_0 x^0, \qquad (2.1) \\ A_0 \varphi_0 &= A_0 a_0 x^0. \qquad (2.2) \end{aligned}$$

Subtracting Eq. (2.1) and Eq. (2.2) we get

$$\begin{aligned} a_0 Q_0 - A_0 \varphi_0 &= a_0 A_0 x^0 - A_0 a_0 x^0, \\ &= 0. \\ a_0 Q_0 &= A_0 \varphi_0, \\ Q_0 &= C \varphi_0. \end{aligned}$$

in which $C = \frac{A_0}{a_0}$. Thus the theorem is true for $m = 0$.

Suppose that theorem is true for $m \leq k$, where k is any non-negative integer.

Let Q_{k+1}be any polynomial of degree $k + 1$,

$$Q_{k+1}(x) = A_{k+1} x^{k+1} + A_k x^k + ...A_0,$$

And let

$$\varphi_{k+1}(x) = a_{k+1} x^{k+1} + a_k x^k + ...a_0.$$

And if we write

$$a_{k+1} Q_{k+1}(x) = a_{k+1} A_{k+1} x^{k+1} + a_{k+1} A_k x^k + ... + a_{k+1} A_0, \qquad (2.3)$$

$$A_{k+1}\varphi_{k+1}(x) = A_{k+1}a_{k+1}x^{k+1} + A_{k+1}a_k x^k + ... + A_{k+1}a_0. \tag{2.4}$$

subtracting Eq. (2.3) and Eq. (2.4), and choose $C_{k+1} = \frac{A_{k+1}}{a_{k+1}}$,

$$\begin{aligned}
a_{k+1}Q_{k+1}(x) - A_{k+1}\varphi_{k+1}(x) &= a_{k+1}A_{k+1}x^{k+1} + a_{k+1}A_k x^k + ... + a_{k+1}A_0 \\
&\quad -a_{k+1}A_{k+1}x^{k+1} - a_k A_{k+1}x^k - .... - a_0 A_{k+1}, \\
Q_{k+1}(x) - C_{k+1}\varphi_{k+1}(x) &= \text{Polynomial of degree } \leq k. \\
&= C_k\varphi_k + ...C_1\varphi_1 + C_0\varphi_0, \\
Q_{k+1}(x) &= \sum_{i=0}^{k+1} C_i\varphi_i.
\end{aligned}$$

Thus if the theorem is true for $m \leq k$ it is true for $m = k+1$. Since it is true for $m = 0$, it is true for every non-negative integer m.

Next we discuss a necessary and sufficient condition for orthogonality.

Theorem 2.

A simple set of polynomials $\{\varphi_n\}_{n=0}^{\infty}$ is an orthogonal set w.r.t. weight function, $w(x)$ on the interval (a,b) *iff*

$$< \varphi_n, I^m > \; = \int_a^b w(x)\varphi_n(x)x^m dx = 0. \tag{2.5}$$

where

$$I(x) = x, \qquad I^m(x) = x^m.$$

for all x and m a nonnegative integer less than n.

Proof:

Suppose $\{\varphi_n\}_{n=0}^{\infty}$ is orthogonal then the condition (2.5) is satisfied. Let n and m be any positive integer s.t. $0 \leq m < n$. By Theorem 1, there exist constants C_0, C_1,......, C_m such that.

$$I^m = C_0\varphi_0 + C_1\varphi_1 + ... + C_m\varphi_m,$$

then

$$\begin{aligned}< \quad \varphi_n, I^m > \quad &= C_0 < \varphi_n, \varphi_0 > + C_1 < \varphi_n, \varphi_1 > + ... + C_m < \varphi_n, \varphi_m >, \\ < \quad \varphi_n, I^m > \quad &= 0.\end{aligned}$$

Conversely if

$$\int_a^b \varphi_n(x) w(x) x^m dx = 0, \qquad m < n.$$

Let φ_n and φ_m be two polynomial $n > m$

$$\begin{aligned}\varphi_m &= a_m x^m + a_{m-1} x^{m-1} + ... + a_0 x^0, \\ \varphi_m &= a_m I^m + a_{m-1} I^{m-1} + ... + a_0 I^0,\end{aligned}$$

then

$$\begin{aligned}< \quad \varphi_n, \varphi_m > \quad &= a_m < \varphi_n, I^m > + a_{m-1} < \varphi_n, I^{m-1} > + ... + a_0 < \varphi_n, I^0 >, \\ < \quad \varphi_n, \varphi_m > \quad &= 0.\end{aligned}$$

2.1 Existence

We shall prove the existence of orthogonal polynomials with the help of Gram - Schmidt process.

Gram - Schmidt Process

The Gram - Schmidt process is used to construct a sequence of orthogonal polynomials with respect to an inner product.

If we have a linearly independent set of continuous functions $\{f_i\}_{i=1}^n$, then the Gram - Schmidt process generates an orthogonal set of functions $\{g_i\}_{i=1}^n$, in the following manner

$$\begin{aligned}g_1 &= f_1, \\ g_2 &= f_2 - \frac{< g_1, f_2 >}{< g_1, g_1 >} g_1,\end{aligned}$$

and

$$< g_1, g_2 > \quad = 0,$$

so, first two members are mutually orthogonal. And if

$$g_n = f_n - \frac{< g_1, f_n >}{< g_1, g_1 >} g_1 - \frac{< g_2, f_n >}{< g_2, g_2 >} g_2 - ... - \frac{< g_{n-1}, f_n >}{< g_{n-1}, g_{n-1} >} g_{n-1.}$$

then by the help of mathematical induction we can prove that all of its members are mutually orthogonal. [3].

Example: We can generate a sequence of orthogonal polynomials w.r.t, an inner product

$$< f, g > \quad = \quad \int_a^b w(x) f(x) g(x) dx.$$

Here we take, $w(x) = x, (a, b) = (0, 1)$ and linearly independent sequence $\{1, x, x^2, ...\}$.

Take $g_1 = f_1 = 1$, then we have

$$\begin{aligned} g_2(x) &= f_2 - \frac{< g_1, f_2 >}{< g_1, g_1 >} g_1, \\ &= x - \frac{< 1, x >}{< 1, 1 >} 1, \\ &= x - \frac{2}{3}, \end{aligned}$$

since

$$< 1, x > \quad = \quad \int_0^1 x^2 dx = \frac{1}{3} \quad \text{and} \quad < 1, 1 > \quad = \quad \int_0^1 x dx = \frac{1}{2}.$$

Further

$$\begin{aligned} g_3(x) &= f_3 - \frac{< g_1, f_3 >}{< g_1, g_1 >} g_1 - \frac{< g_2, f_3 >}{< g_2, g_2 >} g_2, \\ &= x^2 - \frac{< 1, x^2 >}{< 1, 1 >} 1 - \frac{< x - \frac{2}{3}, x^2 >}{< x - \frac{2}{3}, x - \frac{2}{3} >} (x - \frac{2}{3}), \\ &= x^2 - \frac{6}{5} x + \frac{3}{10}, \end{aligned}$$

since

$$< 1, x^2 >= \int_0^1 x^3 dx = \frac{1}{4}\ ,\ < x-\frac{2}{3}, x^2 >= \int_0^1 x^3 (x-\frac{2}{3}) dx = \frac{1}{30} \text{ and } < x-\frac{2}{3}, x-\frac{2}{3} >= \int_0^1 x(x-\frac{2}{3})^2 = \frac{1}{36}.$$

Here $g_1(x) = 1$, $g_2(x) = x - \frac{2}{3}$ and $g_3(x) = x^2 - \frac{6}{5}x + \frac{3}{10}$, are the first three monic polynomials on the interval $(0, 1)$.

By repeating this process we obtain any member of orthogonal polynomials of the set.

We also note that each pair of these polynomials are orthogonal. As

$$\begin{aligned} < \ 1, x - \frac{2}{3} > \ &= \int_0^1 x(x - \frac{2}{3}) dx = 0, \\ < \ 1, x^2 - \frac{6}{5}x + \frac{3}{10} > \ &= \int_0^1 x(x^2 - \frac{6}{5}x + \frac{3}{10}) dx = 0, \\ < \ x - \frac{2}{3}, x^2 - \frac{6}{5}x + \frac{3}{10} > \ &= \int_0^1 x(x - \frac{2}{3})(x^2 - \frac{6}{5}x + \frac{3}{10}) dx = 0. \end{aligned}$$

Theorem 3.

Let $\{\varphi_n\}_{n=0}^{\infty}$ be a set of orthogonal polynomials and let Q_m be an arbitrary polynomial of degree m, then

$$Q_m = C_0\varphi_0 + C_1\varphi_1 + ... + C_m\varphi_m,$$

where

$$C_k = \frac{< Q_m, \varphi_k >}{\| \varphi_k \|^2}, \qquad k = 1,\ 2, ...,\ m. \tag{2.6}$$

Proof:

From Theorem 1, there exist constants C_i such that

$$Q_m = \sum_{i=0}^{m} C_i \varphi_i. \tag{2.7}$$

Multiply both sides of Eq. (2.7) by $w(x)$ and $\varphi_k(x)$, where k is an arbitrary integer $(0 \leq k \leq m)$ and integrate from a to b

$$\int_a^b Q_m \varphi_k(x) w(x) dx = C_0 \int_a^b \varphi_0(x)\varphi_k(x)w(x)dx + C_1 \int_a^b \varphi_1(x)\varphi_k(x)w(x)dx + ... + C_m \int_a^b \varphi_m(x)\varphi_k(x)w(x)dx.$$

since $(\varphi_i, \varphi_k) = 0$ when $i \neq k$, $(i = 0, 1, ..., m)$ we have

$$\begin{aligned} & < Q_m, \varphi_k >= C_k < \varphi_k, \varphi_k >, \\ & = C_k \parallel \varphi_k \parallel^2, \\ C_k & = \frac{< Q_m, \varphi_k >}{\parallel \varphi_k \parallel^2}. \end{aligned}$$

where $k = 0, 1, ..., m$ and $\parallel \varphi_k \parallel \neq 0$.

2.2 Three Term Recurrence Relation

Theorem 4.

The polynomials of an orthogonal set satisfy a recurrence relation of the form

$$x\varphi_n = A_n \varphi_{n+1}(x) + B_n \varphi_n(x) + C_n \varphi_{n-1}(x) \qquad n \geq 1,$$

where A_n, B_n and C_n are constants that may depends on n.

Proof:

Since $x\varphi_n$ is a polynomial of degree $n+1$. From Theorem 3, we have

$$x\varphi_n = \sum_{k=1}^{n+1} a_{n,k} \varphi_k(x),$$

where

$$a_{n,k} = \frac{< x\varphi_n,\ \varphi_k >}{\parallel \varphi_k \parallel^2} = \frac{< \varphi_n,\ x\varphi_k >}{\parallel \varphi_k \parallel^2},$$

$$a_{n,k} = 0 \ \text{ for } \ k+1 < n \ \text{ or } \ k < n-1,$$

$$\begin{aligned} x\varphi_n & = \sum_{k=1}^{n+1} a_{n,k} \varphi_k(x), \\ & = a_{n,1}\varphi_1(x) + a_{n,2}\varphi_2(x) + ... + a_{n,n-1}\varphi_{n-1}(x) + a_{n,n}\varphi_n(x) + a_{n,n+1}\varphi_{n+1}(x), \end{aligned}$$

as

$$a_{n,k} = 0 \quad \text{for} \quad k < n-1,$$

so

$$x\varphi_n = a_{n,n-1}\varphi_{n-1}(x) + a_{n,n}\varphi_n(x) + a_{n,n+1}\varphi_{n+1}(x).$$

Setting $A_n = a_{n,n+1}$, $B_n = a_{n,n}$ and $C_n = a_{n,n-1}$, the recurrence relation becomes

$$x\varphi_n = A_n\varphi_{n+1}(x) + C_n\varphi_{n-1}(x) + B_n\varphi_n(x). \tag{2.8}$$

Theorem 5.

If $\parallel \varphi_n \parallel$ is a constant independent of n then $C_n = A_{n-1}$ in the recurrence relation

$$x\varphi_n = A_n\varphi_{n+1}(x) + B_n\varphi_n(x) + A_{n-1}\varphi_{n-1}(x).$$

Proof:

Assume $\parallel \varphi_n \parallel$ is constant, which we can take to be unity, in the Eq. (2.6), *then*

$$\begin{aligned} a_{n,k} &= \frac{< x\varphi_n, \varphi_k >}{\parallel \varphi_n \parallel^2} \\ &= < x\varphi_n, \varphi_k > \\ &= \int_a^b x\varphi_n(x)\varphi_k(x)w(x)dx \\ &= \int_a^b \varphi_n(x)(x\varphi_k(x))w(x)dx \\ &= < \varphi_n,\ x\varphi_k > \\ &= < x\varphi_k,\ \varphi_n >, \\ &= a_{k,n}. \end{aligned}$$

As $a_{n,k} = a_{k,n}$

And we know from Theorem 4, that

$$\begin{aligned} A_n &= a_{n+1,n} \\ A_{n-1} &= a_{n,n-1} \\ A_{n-1} &= a_{n-1,n} \\ A_{n-1} &= C_n. \end{aligned}$$

2.3 Christoffel-Darboux Formula

A sequence of orthonormal polynomials $\{\varphi_n\}_{n=0}^{\infty}$ *satisfies the Christoffel-Darboux identities*

$$\sum_{i=0}^{n} \varphi_i(x)\varphi_i(y) = \frac{a_n}{a_{n+1}} \frac{\varphi_{n+1}(x)\,\varphi_n(y) - \varphi_{n+1}(y)\,\varphi_n(x)}{x-y}, \; x \neq y \tag{2.9}$$

and

$$\sum_{i=0}^{n} [\varphi_i(x)]^2 = \frac{a_n}{a_{n+1}} [\varphi_{n+1}^{'}(x)\varphi_n(x) - \varphi_n^{'}(x)\varphi_{n+1}(x)]. \tag{2.10}$$

where a_n *is the leading coefficient of polynomial* $\varphi_n(x)$.

Proof:

We derive these results from three term recurrence relation

$$x\varphi_n(x) = A_n\varphi_{n+1}(x) + B_n\varphi_n(x) + C_n\varphi_{n-1}(x).$$

Here we take $C_n = A_{n-1}$, and arrange the equation as

$$x\varphi_n(x) = A_n\varphi_{n+1}(x) + B_n\varphi_n(x) + A_{n-1}\varphi_{n-1}(x), \tag{2.11}$$

and

$$y\varphi_n(y) = A_n\varphi_{n+1}(y) + B_n\varphi_n(y) + A_{n-1}\varphi_{n-1}(y). \tag{2.12}$$

Multiply Eq. (2.11) through by $\varphi_n(y)$ and Eq. (2.12) by $\varphi_n(x)$, subtract the results, we obtain

$$\begin{aligned}(x-y)\varphi_n(x)\varphi_n(y) &= A_n[\varphi_{n+1}(x)\varphi_n(y) - \varphi_n(x)\varphi_{n+1}(y)] \\ &\quad + A_{n-1}[\varphi_{n-1}(x)\varphi_n(y) - \varphi_n(x)\varphi_{n-1}(y)].\end{aligned}$$

Change the index to i, in the above equation and sum over i=0 to n. We get

$$(x-y)\sum_{i=0}^{n} \varphi_i(x)\varphi_i(y) = A_n[\varphi_{n+1}(x)\varphi_n(y) - \varphi_n(x)\varphi_{n+1}(y)]. \tag{2.13}$$

Now comparing the coefficients of x^{n+1} in the recurrence relation (2.8), we get the value of A_n

as

$$A_n = \frac{a_n}{a_{n+1}}.$$

Substitute the value of A_n in the above Eq. (2.13)

$$\sum_{i=0}^{n} \varphi_i(x)\varphi_i(y) = \frac{a_n}{a_{n+1}} \frac{[\varphi_{n+1}(x)\varphi_n(y) - \varphi_n(x)\varphi_{n+1}(y)]}{x - y}. \tag{2.14}$$

Which proves (2.9)

Now add and subtract $\varphi_{n+1}(x)\varphi_n(x)$ to the numerator of the right hand side of the Eq. (2.14), and letting $y \to x$, and apply the L'Hospital rule, we get

$$\sum_{i=0}^{n} [\varphi_i(x)]^2 = \frac{a_n}{a_{n+1}} [\varphi'_{n+1}(x)\varphi_n(x) - \varphi'_n(x)\varphi_{n+1}(x)].$$

2.4 Zeros of Orthogonal Polynomials

Theorem 6.

The nth degree polynomial φ_n of an orthogonal set has n real distinct zeros, all of which lie in the interval (a, b).

Proof:

Since φ_0 is a nonzero constant polynomial, and

$$< \varphi_0,\ \varphi_n > \ = \int_a^b \varphi_0 \varphi_n(x) w(x) dx$$

$$0 = \varphi_0 \int_a^b \varphi_n(x) w(x) dx,$$

$$\Rightarrow \qquad \int_a^b \varphi_n(x) w(x) dx = 0.$$

Since $w(x) > 0$ on (a, b), $\varphi_n(x)$ must changes sign at least one point in the interval. Since $\varphi_n(x)$ is a polynomial of degree n, it can change sign at most n times.

Suppose it changes sign k-times at $x_1,\ x_2, ..., x_k$ where $k < n$. Define

$$\psi_k(x) = (x - x_1)(x - x_2)(x - x_3)....(x - x_k).$$

Now

$$\int_a^b \psi_k(x)\varphi_n(x)w(x)dx = 0 \quad \text{if} \quad k < n.$$

$\varphi_n(x)$ changes sign at x_1, $\psi_k(x)$ also changes sign at x_1 but product doesn't changes sign. Similarly for x_2, $x_3, ..., x_k$, and

$$\int_a^b \varphi_n(x)\psi_k(x)w(x)dx \neq 0.$$

This is a contradiction. So we conclude that $k = n$. Thus φ_n changes sign at n distinct points in (a, b) and has n real distinct zeros in this interval.

Interlacing Property

If $\{\varphi_n(x)\}_{n=0}^{\infty}$ *is the sequence of orthogonal polynomials, then the zeros of* $\varphi_n(x)$ and $\varphi_{n+1}(x)$ *separate each other.*

Proof:

We prove this property by the help of Christoffel-Darboux formula. Since for all $\varphi_n(x)$, the leading coefficient a_n can be taken as positive, then Eq. (2.10) gives

$$\varphi_{n+1}^{'}(x)\varphi_n(x) - \varphi_n^{'}(x)\varphi_{n+1}(x) > 0 , \qquad -\infty < x < \infty. \tag{2.15}$$

Let x_k and x_{k+1} be the consecutive zeros of $\varphi_n(x)$.Then

$$\varphi_n^{'}(x_k)\varphi_n'(x_{k+1}) < 0. \tag{2.16}$$

Substituting $x = x_k$ in Eq. (2.15) we get

$$-\varphi_n^{'}(x_k)\varphi_{n+1}(x_k) > 0,$$

and similarly

$$-\varphi_n^{'}(x_{k+1})\varphi_{n+1}(x_{k+1}) > 0.$$

Multiply the above two inequalities together, we have

$$\varphi_n^{'}(x_k)\varphi_n^{'}(x_{k+1})\varphi_{n+1}(x_k)\varphi_{n+1}(x_{k+1}) > 0,$$

from Eq. (2.16) we have

$$\varphi_{n+1}(x_k)\,\varphi_{n+1}(x_{k+1}) < 0,$$

so $\varphi_{n+1}(x)$ has a zero among each pair of adjacent zeros of $\varphi_n(x)$.

Let $x_{n,n}$ denote the greatest zero of $\varphi_n(x)$. We note that $\varphi_n(x) \to \infty$ as $x \to \infty$, we must have $\varphi_n'(x_{n,n}) > 0$, and Eq. (2.15) becomes

$$\varphi_{n+1}(x_{n,n}) < 0.$$

But $\varphi_{n+1}(x) \to \infty$ as $x \to \infty$, so $\varphi_{n+1}(x)$ must have a zero to the right of $x_{n,n}$. Similarly, if $x_{1,n}$ is the smallest zero of $\varphi_n(x)$, then $\varphi_{n+1}(x)$ must have a zero to the left of $x_{1,n}$. Thus

$$x_{1,n+1} < x_{1,n} < x_{2,n+1} < ... < x_{n,n+1} < x_{n,n} < x_{n+1,n+1}.$$

Lemma:

If $\{\varphi_n\}$ be the set of orthogonal polynomials that corresponds to the positive weight function $w(x)$ on the finite interval (a,b). Let w be of the form

$$w(x) = (x-a)^{\alpha}(b-x)^{\beta} \qquad \textit{where} \quad \alpha > -1,\ \beta > -1,$$

then

$$\int_a^b \frac{d}{dx}\left[(x-a)(x-b)\,\varphi_n'(x)\,w(x)\right] Q(x)\,dx = 0,$$

for every polynomial Q of degree less than n.

Proof:

$$\begin{aligned}
L.H.S &= \int_a^b \frac{d}{dx}\left[(x-a)(x-b)\varphi_n^{'}(x)\,w(x)\right]Q(x)\,dx,\\
&= (x-a)(x-b)\varphi_n^{'}(x)\,w(x)Q(x)\mid_a^b - \int_a^b (x-a)(x-b)\varphi_n^{'}(x)\,w(x)Q^{'}(x)\,dx,\\
&= -\int_a^b \varphi_n^{'}(x)(x-a)(x-b)\,w(x)Q^{'}(x)\,dx,\\
&= -\varphi_n(x)(x-a)(x-b)\,w(x)Q^{'}(x)\mid_a^b + \int_a^b \varphi_n(x)\frac{d}{dx}\left[(x-a)(x-b)\,w(x)Q^{'}(x)\right]dx,\\
&= \int_a^b \varphi_n(x)\frac{d}{dx}\left[(x-a)(x-b)\,w(x)Q^{'}(x)\right]dx,
\end{aligned}$$

$$= \int_a^b \varphi_n(x)\frac{d}{dx}\left[(x-a)(x-b)(x-a)^{\alpha}(b-x)^{\beta}Q^{'}(x)\right]dx. \tag{2.17}$$

Now

$$\begin{aligned}
&\frac{d}{dx}\left[-(x-a)^{\alpha+1}(b-x)^{\beta+1}Q^{'}(x)\right],\\
=\ & -(x-a)^{\alpha}(\alpha+1)(b-x)^{\beta+1}Q^{'}(x) - (x-a)^{\alpha+1}(b-x)^{\beta}(\beta+1)Q^{'}(x) - (x-a)^{\alpha+1}(b-x)^{\beta+1}Q^{''}(x),\\
=\ & (\alpha+1)(x-b)w(x)Q^{'}(x) - (x-a)(\beta+1)w(x)Q^{'}(x) + (x-a)(x-b)w(x)Q^{''}(x),\\
=\ & w(x)\left[(\alpha+1)(x-b)Q^{'}(x) - (x-a)(\beta+1)Q^{'}(x) + (x-a)(x-b)Q^{''}(x)\right].
\end{aligned}$$

Now Eq. (2.17) implies that

$$\begin{aligned}
&\int_a^b \varphi_n(x)[w(x)\{(\alpha+1)(x-b)Q^{'}(x) - (x-a)(\beta+1)Q^{'}(x) + (x-a)(x-b)Q^{''}(x)\}]dx,\\
=\ & \int_a^b \varphi_n(x)w(x)(\alpha+1)(x-b)Q^{'}(x)dx - \int_a^b \varphi_n(x)w(x)(x-a)(\beta+1)Q^{'}(x)dx\\
& + \int_a^b \varphi_n(x)w(x)(x-a)(x-b)Q^{''}(x)\,dx.
\end{aligned}$$

As Q is a polyniomial of degree less than n, $Q^{'}$ is a polynomial of degree less than $n-1$, and $Q^{''}$ is a polynomial of degree less than $n-2$. So

$$\int_a^b \varphi_n(x)w(x)[\{(\alpha+1)(x-b)Q^{'}(x) - (x-a)(\beta+1)Q^{'}(x) + (x-a)(x-b)Q^{''}(x)\}]dx = 0.$$

2.5 Differential Equation

Theorem 7.

Let $\{\varphi_n\}$ be the set of orthogonal polynomials that corresponds to the positive weight function $w(x)$ on the finite interval $(a,\ b)$. Let w be of the form

$$w(x) = (x-a)^{\alpha}\,(b-x)^{\beta} \qquad \text{where} \quad \alpha > -1,\ \beta > -1,$$

then φ_n satisfies the second order differential equation

$$(x-a)(x-b)\varphi_n'' + [(2+\alpha+\beta)x - a(1+\beta) - b(1+\alpha)]\varphi_n' = [n^2 + (\alpha+\beta+1)n]\varphi_n.$$

Proof:

We prove this result by the help of above lemma

$$\int_a^b \frac{d}{dx}\left[(x-a)\,(x-b)\,\varphi_n'\,(x)\,w\,(x)\right] Q\,(x)\,dx = 0,$$

for every polynomial $Q\,(x)$ of degree less than n.

Now

$$\begin{aligned}
& \frac{d}{dx}\left[(x-a)\,(x-b)\,\varphi_n'\,(x)\,w\,(x)\right], \\
= \; & \frac{d}{dx}\left[-\,(x-a)^{\alpha+1}\,(b-x)^{\beta+1}\,\varphi_n'\,(x)\right], \\
= \; & w\,(x)\left[-\,(\alpha+1)\,(b-x)\,\varphi_n' + (x-a)\,(\beta+1)\,\varphi_n' - (x-a)\,(x-b)\,\varphi_n''\right].
\end{aligned}$$

Therefore

$$\int_a^b w\,(x)\left[(x-a)\,(x-b)\,\varphi_n'' + \{(x-a)\,(\beta+1) + (\alpha+1)\,(x-b)\}\,\varphi_n'\right] Q\,(x)\,dx = 0,$$

for every polynomial $Q\,(x)$ of degree less than n.

The expression in brackets must be a constant multiple of φ_n. Thus there is a constant A_n such that

$$\frac{d}{dx}\left[(x-a)\,(x-b)\,\varphi_n'\,(x)\right] = A_n\varphi_n. \tag{2.18}$$

If we write

$$\varphi_n(x) = a_n x^n + a_{n-1} x^{n-1} + ...a_0,$$

and compare the coefficient of x^n on both sides of the above equation, we find that

$$A_n = \left[n^2 + (\alpha + \beta + 1)\, n\right],$$

and hence putting the value of A_n in Eq. (2.18) we get

$$(x-a)(x-b)\,\varphi_n'' + \left[(2+\alpha+\beta)\,x - a(1+\beta) - b(1+\alpha)\right]\varphi_n' = \left[n^2 + (\alpha+\beta+1)\,n\right]\varphi_n.$$

If the weight function is $w(x) = ce^{-\alpha x}(x-a)^{\beta}$ on (a, ∞), then φ_n satisfies the differential equation

$$(x-a)\,\varphi_n'' + (a\alpha + \beta + 1 - \alpha x)\,\varphi_n' = -n\alpha\varphi_n(x).$$

Also, if the weight function is $w(x) = ce^{-\alpha x + \beta x}$ on $(-\infty, \infty)$, then φ_n satisfies the differential equation

$$\varphi_n^{''} + (\beta - 2\alpha x)\varphi_n^{'} = -2\alpha n\varphi_n(x).$$

Definition: Let $\{\mu_n\}_{n=0}^{\infty}$ be a sequence of complex numbers and let $\mathcal{L}$ be a complex valued function defined on the vector space of all polynomials by

$$\begin{aligned} \mathcal{L}\left[x^n\right] &= \mu_n, \qquad n = 0, 1, 2, ... \\ \mathcal{L}\left[\beta_1\pi_1(x) + \beta_2\pi_2(x)\right] &= \beta_1\mathcal{L}[\pi_1(x)] + \beta_2\mathcal{L}[\pi_2(x)], \end{aligned}$$

for all complex numbers β_j and all polynomial $\pi_j(x)$ $(j = 1, 2)$. Then $\mathcal{L}$ is called the moment functional determined by the moment sequence $\{\mu_n\}$. The number μ_n is called the moment of order n. [4]

2.6 Favard's Theorem.

Let $\{a_n\}_{n=0}^{\infty}$ *and* $\{b_n\}_{n=0}^{\infty}$, *be two arbitrary sequences of complex numbers, and let* $\{P_n\}_{n=0}^{\infty}$, *be a sequence of polynomials defined by the relation*

$$P_n(x) = (x - a_n) P_{n-1}(x) - b_n P_{n-2}(x), \quad n = 1, 2, 3..., \tag{2.19}$$

where $P_{-1}(x) = 0$ *and* $P_0(x) = 1$. *Then, there exists a moment functional* $\mathcal{L}$ *such that*

$$\mathcal{L}[1] = b_1, \qquad \mathcal{L}[P_n\ P_m] = 0 \text{ if } n \neq m.$$

Proof: [5].

To prove this theorem, we will define the functional $\mathcal{L}$ by induction on P_n, the linear space of polynomials with degree n. We set

$$\mathcal{L}[1] = \mu_0 = b_1, \qquad \mathcal{L}[P_n] = 0, \quad n = 1, 2, 3, \ldots . \tag{2.20}$$

So, by using the three term recurrence relation, we find all the moments in the following way:

For $n = 1$, the Eq. (2.19) becomes

$$\begin{aligned} 0 &= \mathcal{L}[P_1] = \mathcal{L}[x - a_1] \\ &= \mathcal{L}[x] - a_1 \mathcal{L}[1] \\ &= \mu_1 - a_1 b_1 \qquad \text{where } \mathcal{L}[x^n] = \mu_n, \quad n = 0, 1, 2, \ldots \\ &\Longrightarrow \mu_1 = a_1 b_1. \end{aligned}$$

For $n = 2$

$$\begin{aligned} 0 &= \mathcal{L}[P_2] = \mathcal{L}[(x - a_2)P_1 - b_2 P_0] \\ &= \mathcal{L}[(x - a_2)(x - a_1) - b_2 P_0] \\ &= \mu_2 - (a_1 + a_2)\mu_1 + (a_1 a_2 - b_2) b_1 \\ &\Longrightarrow \mu_2 = (a_1 + a_2)\mu_1 - (a_1 a_2 - b_2) b_1. \end{aligned}$$

Continuing this process, we can find μ_{n+1} by using three term recurrence relation.

By replacing n by $n+1$ in Eq. (2.19), we get

$$xP_n(x) = P_{n+1}(x) + a_{n+1}P_n(x) + b_{n+1}P_{n-1}(x), \quad n \geq 1, \tag{2.21}$$

$$\mathcal{L}[xP_n(x)] = \mathcal{L}[P_{n+1}(x)] + a_{n+1}\mathcal{L}[P_n(x)] + b_{n+1}\mathcal{L}[P_{n-1}(x)],$$

By using Eq. (2.20), we obtain

$$\mathcal{L}\left[xP_n(x)\right] = 0, \quad n \geq 2.$$

Multiplying both sides of Eq. (2.21) by x and using the last result, we then find

$$\mathcal{L}\left[x^2P_n(x)\right] = 0, \quad n \geq 3.$$

Continuing in this way, we conclude

$$\mathcal{L}\left[x^kP_n(x)\right] = 0, \quad 0 \leq k < n.$$

It follows that for $m \neq n$, $\mathcal{L}\left[P_m(x)P_n(x)\right] = 0$.

So, therefore Favard's Theorem states that corresponding to any three term recurrence relation a set of orthogonal polynomials exist, according to Riesz's Theorem [6] , which states that

"Every bounded linear functional on a Hilbert space can be represented in term of inner product."

Finally, we have

$$\begin{aligned}
\mathcal{L}\left[x^nP_n\right] &= \mathcal{L}\left[x^{n-1}(P_{n+1} + a_{n+1}P_n + b_{n+1}P_{n-1})\right], \\
&= \mathcal{L}[x^{n-1}P_{n+1}] + a_{n+1}\mathcal{L}[x^{n-1}P_n] + b_{n+1}\mathcal{L}[x^{n-1}P_{n-1}], \\
&= b_{n+1}\mathcal{L}[x^{n-1}P_{n-1}].
\end{aligned}$$

And we also find that for $n = 1$

$$\begin{aligned}\mathcal{L}\left[xP_1\right] &= b_2\mathcal{L}[x^0P_0], \\ &= b_2\mathcal{L}[1], \\ &= b_2b_1,\end{aligned}$$

for $n = 2$

$$\begin{aligned}\mathcal{L}\left[x^2P_2\right] &= b_3\mathcal{L}[xP_1], \\ &= b_3b_2b_1.\end{aligned}$$

and so in general

$$\mathcal{L}\left[x^nP_n\right] = b_{n+1}b_n...b_1.$$

Chapter 3

Orthogonal Polynomials and Least-Square Approximation

Some materiel of this chapter is based on [7].

Polynomials must include in approximation theory, and are of principal importance. Approximation theory is concerned with fitting function to a given set of data and finding the best function in a certain class that can be used to represent the set of data.

Here we consider the least square approximation.

In the least square approximation, a function $f(x)$ is defined on some interval $[a, b]$, we approximate it by a polynomial, such that the error is minimized.

Let $\{\varphi_n(x)\}_{n=0}^{\infty}$ be a set of functions defined on an interval $[a, b]$, and let $w(x)$ be a positve weight function on (a, b).

Suppose the following k-sum

$$p_k(x) = a_k\varphi_k + ...a_1\varphi_1 + a_0\varphi_0,$$

approximates an arbitrary function $f(x)$ on $[a, b]$. Define the error

$$e(x) = [f(x) - p_k(x)].$$

Here we first discuss the discrete least square approximation.

3.1 Discrete Least Square Approximation

Choose a discrete set of nodes $x_i, i = 0, 1, ..., k$ with $a < x_0 < x_1 .... < x_k < b$, so that the sum

$$S = \sum_{i=0}^{k} w(x_i) e^2(x_i), \tag{3.1}$$

is minimum.

Here

$$p_k(x) = \sum_{m=0}^{k} a_m \varphi_m(x).$$

Now from Eq. (3.1) we have

$$\begin{aligned} S &= \sum_{i=0}^{k} w(x_i) f^2(x_i) - 2\sum_{i=0}^{k} w(x_i) f(x_i) \sum_{m=0}^{k} a_m \varphi_m(x_i) + \sum_{i=0}^{k} w(x_i) \left(\sum_{m=0}^{k} a_m \varphi_m(x_i) \right)^2, \\ S &= \sum_{i=0}^{k} w(x_i) f^2(x_i) - 2\sum_{i=0}^{k}\sum_{m=0}^{k} w(x_i) f(x_i) a_m \varphi_m(x_i) + \sum_{m=0}^{k}\sum_{n=0}^{k} a_m a_n \left(\sum_{i=0}^{k} w(x_i) \varphi_m(x_i) \varphi_n(x_i) \right). \end{aligned}$$

To find the minimum , we first solve the system of $k+1$ equations in $a_0, a_1, ..., a_k$, we must have

$$\begin{aligned} \frac{\partial S}{\partial a_m} &= 0, m = 0, 1..., k. \\ 0 &= 0 - 2\sum_{i=0}^{k} w(x_i) f(x_i) \varphi_m(x_i) + 2\sum_{n=0}^{k} a_n \sum_{i=0}^{k} w(x_i) \varphi_m(x_i) \varphi_n(x_i), \end{aligned}$$

$$\sum_{i=0}^{k} w(x_i) f(x_i) \varphi_m(x_i) = \sum_{n=0}^{k} a_n \sum_{i=0}^{k} w(x_i) \varphi_m(x_i) \varphi_n(x_i). \tag{3.2}$$

Matrix of the above system is becomes diagonal if we require the functions $\varphi_m(x)$ to satisfy the following discrete orthogonality condition.

$$\sum_{i=0}^{k} w(x_i) \varphi_m(x_i) \varphi_n(x_i) = 0, \text{if } m \neq n.$$

Take $m = n$ in Eq. (3.2), and we get the coefficient a_m as

$$a_m = \frac{\sum_{i=0}^{k} w(x_i) f(x_i) \varphi_m(x_i)}{\sum_{i=0}^{k} w(x_i) \varphi_m^2(x_i)}, m = 0, 1....k.$$

As

$$S = \sum_{i=0}^{k} w(x_i) f^2(x_i) - 2\sum_{i=0}^{k}\sum_{m=0}^{k} w(x_i) f(x_i) a_m \varphi_m(x_i) + \sum_{m=0}^{k}\sum_{n=0}^{k} a_m a_n \left(\sum_{i=0}^{k} w(x_i) \varphi_m(x_i) \varphi_n(x_i)\right).$$

Take $n = m$ in third term

$$S = \sum_{i=0}^{k} w(x_i) f^2(x_i) - 2\sum_{i=0}^{k}\sum_{m=0}^{k} w(x_i) f(x_i) a_m \varphi_m(x_i) + \sum_{i=0}^{k}\sum_{m=0}^{k} w(x_i) a_m^2 \varphi_m^2(x_i). \quad (3.3)$$

As

$$f(x) \approx \sum_{m=0}^{k} a_m \varphi_m(x_i). \quad (3.4)$$

Take inner product with $\sum_{i=0}^{k} w(x_i) \varphi_j(x_i)$,

$$\sum_{i=0}^{k} w(x_i) \varphi_j(x_i) f(x) \approx \sum_{i=0}^{k} w(x_i) \varphi_j(x_i) \sum_{m=0}^{k} a_m \varphi_m(x_i).$$

Take $j = m$

$$\sum_{i=0}^{k} w(x_i) \varphi_m(x_i) f(x) \approx \sum_{i=0}^{k}\sum_{m=0}^{k} w(x_i) a_m \varphi_m^2(x_i).$$

Eq. (3.3) implies that

$$
\begin{aligned}
S_{\min} &= \sum_{i=0}^{k} w(x_i)\, f^2(x_i) - 2\sum_{m=0}^{k} a_m \left(\sum_{i=0}^{k} w(x_i)\, f(x_i)\, \varphi_m(x_i)\right) + \sum_{i=0}^{k}\sum_{m=0}^{k} w(x_i)\, a_m^2 \varphi_m^2(x_i)\,. \\
S_{\min} &= \sum_{i=0}^{k} w(x_i)\, f^2(x_i) - 2\sum_{m=0}^{k} a_m \left(\sum_{i=0}^{k}\sum_{m=0}^{k} w(x_i)\, a_m \varphi_m^2(x_i)\right) + \sum_{i=0}^{k}\sum_{m=0}^{k} w(x_i)\, a_m^2 \varphi_m^2(x_i)\,. \\
S_{\min} &= \sum_{i=0}^{k} w(x_i)\, f^2(x_i) - 2\sum_{m=0}^{k}\sum_{i=0}^{k} w(x_i)\, a_m^2 \varphi_m^2(x_i) + \sum_{i=0}^{k}\sum_{m=0}^{k} w(x_i)\, a_m^2 \varphi_m^2(x_i)\,.
\end{aligned}
$$

And the minimum of S becomes

$$
S_{\min} = \sum_{i=0}^{k} w(x_i) \left\{ f^2(x_i) - \sum_{m=0}^{k} a_m^2 \varphi_m^2(x_i) \right\}. \tag{3.5}
$$

3.2 Orthogonal Polynomials Least Square Approximation

Let $\{\varphi_i(x)\}_{i=0}^{\infty}$ be a simple set of polynomials orthogonal with respect to the weight function $w(x)$ on $[a, b]$. It is easily shown that members of the set satisfy a three term recurrence relation of the form

$$
x\varphi_n(x) = A_n\varphi_{n+1}(x) + B_n\varphi_n(x) + C_n\varphi_{n-1}(x). \tag{3.6}
$$

Without loss of generality, we can normalise the polynomials that $\|\varphi_n(x)\| = 1$, for every n. Then $A_{n-1} = C_n$ and the above recurrence relation can be written in the form

$$
x\varphi_n(x) = A_n\varphi_{n+1}(x) + B_n\varphi_n(x) + A_{n-1}\varphi_{n-1}(x).
$$

[2] . Assume $\varphi_{-1}(x) \equiv 0$ and write first four equations of the above set.

$$
\begin{aligned}
B_0\varphi_0(x) \quad + A_0\varphi_1(x) \quad + 0 \quad + 0 &= x\varphi_0(x), \\
A_0\varphi_0(x) \quad + B_1\varphi_1(x) \quad + A_1\varphi_2(x) + 0 &= x\varphi_1(x), \\
0 \quad + A_1\varphi_1(x) + B_2\varphi_2(x) + A_2\varphi_3(x) &= x\varphi_2(x), \\
0 \quad + 0 \quad + A_2\varphi_2(x) \quad + B_3\varphi_3(x) &= x\varphi_3(x) - A_3\varphi_4(x).
\end{aligned}
$$

The above system is equivalent to

$$\begin{bmatrix} B_0 & A_0 & 0 & 0 \\ A_0 & B_1 & A_1 & 0 \\ 0 & A_1 & B_2 & A_2 \\ 0 & 0 & A_2 & B_3 \end{bmatrix} \begin{bmatrix} \varphi_0(x) \\ \varphi_1(x) \\ \varphi_2(x) \\ \varphi_3(x) \end{bmatrix} = x \begin{bmatrix} \varphi_0(x) \\ \varphi_1(x) \\ \varphi_2(x) \\ \varphi_3(x) \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ A_3\varphi_4(x) \end{bmatrix}. \tag{3.7}$$

Note that the matrix of the system is symmetric. It is well-known that the polynomial $\varphi_n(x)$ has exactly n real distinct zeros in (a,b). Let x_i, $i = 0,1,2,3$ be the zeros of $\varphi_4(x)$. It is clear that the matrix in (3.7) has eigenvalues x_i with the eigenvectors $[\varphi_0(x_i), \varphi_1(x_i), \varphi_2(x_i), \varphi_3(x_i)]^T$. In general case, the $k \times k$ tridiagonal matrix has eigenvalues x_l where $\varphi_{k+1}(x_l) = 0$, $l = 0,1,...,k$. Also eigenvectors are mutually orthogonal. Define

$$c(x_r) = \sqrt{\sum_{i=0}^{k} \varphi_i^2(x_r)},$$

as the norm of the $r - th$ eigenvector. Therefore the matrix A of order $k+1$ formed by the normalised eigenvectors,

$$\begin{bmatrix} \frac{\varphi_0(x_0)}{c(x_0)} & \frac{\varphi_0(x_1)}{c(x_1)} & .. & .. & \frac{\varphi_0(x_k)}{c(x_k)} \\ \frac{\varphi_1(x_0)}{c(x_0)} & \frac{\varphi_1(x_1)}{c(x_1)} & .. & .. & \frac{\varphi_1(x_k)}{c(x_k)} \\ ... & ... & .. & .. & .. \\ ... & ... & .. & .. & .. \\ \frac{\varphi_k(x_0)}{c(x_0)} & \frac{\varphi_k(x_1)}{c(x_1)} & .. & .. & \frac{\varphi_k(x_k)}{c(x_k)} \end{bmatrix}$$

has the property

$$AA^T = A^T A = I,$$

one of whose consequences is the desired discrete orthogonality relation:

$$\sum_{i=0}^{k} w(x_i)\varphi_l(x_i)\varphi_m(x_i) = 0, \ if \ l \neq m.$$

In the above $w(x) = \frac{1}{c^2(x)}$.

Chapter 4

Application to a Non-Linear Boundary Value Problem

4.1 Introduction

Consider the boundary value problem

$$y'' = f(x, y, y'), \quad y(a) = y_1, \quad y(b) = y_2. \tag{4.1}$$

In this chapter we shall find an approximate solution of the boundary value problem by using spectral method. This method employs a set of orthogonal polynomials, $\{\varphi_n\}_{n=0}^{\infty}$, to represents the unknown function. The unknown function is approximated by a sum of first $k+1$ member of the set. Let

$$y_k(x) = a_k\varphi_k(x) + a_{k-1}\varphi_{k-1}(x) + ...a_0\varphi_0(x).$$

Put in the given equation and define the residual

$$Rl_k(x) = y_k^{''} - f(x, y_k, y_k'). \tag{4.2}$$

This residual is evaluated at suitably chosen $k-1$ collocation points to give $k-1$ equations which together with two boundary conditions yield a set of $k+1$ equations in as many unknowns $a_0, a_1, ..., a_k$. A solution of this set is substituted in $y_k(x)$ provides an approximate solution to the boundary value problem.

In this method we use the Legendre polynomial in the solution.

The set of Legendre polynomial $P_n(x)$ is simply orthogonal on $[-1,1]$. They are defined by the recurrence relation

$$\begin{aligned} P_0 &= 1, \\ P_1(x) &= x, \\ nP_n(x) &= (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x), \qquad n \geq 2. \end{aligned}$$

We are concentrating to find the solution on $[0,1]$ instead of the interval $[-1,1]$. For this purpose we transform the interval by defining $x = 2x_1 - 1$, and the shifted Legendre polynomials are defined by the relation

$$\begin{aligned} P_0 &= 1, \\ P_1(x) &= 2x-1, \\ nP_n(x) &= (2x-1)(2n-1)P_{n-1}(x) - (n-1)P_{n-2}(x), \qquad n \geq 2. \end{aligned}$$

4.2 Applications

In this section we shall find an approximate solution of the Blasius problem. For a discussion of this problem see Boyd [8] and Ahmad [9].

The two dimensional steady-state laminar viscous flow over a semi-infinite plate modeled by the nonlinear differential equation

$$\begin{aligned} f^{'''}(\eta) + \beta_0 f^{''}(\eta) f(\eta) &= 0, \quad \eta \in [0,\infty) \qquad \text{with boundary conditions} \qquad (4.3) \\ f(0) &= f^{'}(0) = 0, \qquad f^{'}(\infty) = 1, \end{aligned}$$

where η and $f(\eta)$ are respectively the dimensionless coordinate and the dimensionless stream functions defined in such a manner that the set of two partial differential Navier-Stokes equations reduces to the single ordinary differential equation (4.3). It is relatively easy to find a series or a numerical solution of the above problem and physical parameters of interest such as the shear-stress distribution along the surface, the drag on the surface and the boundary layer thickness are easily evaluated.

The main hurdle in the solution of the above problem, called the Blasius problem, is the absence of the second derivative $f^{''}(0)$. Once this derivative has been correctly evaluated an analytical solution of the boundary value problem may be readily found. Blasius found the following power series solution of the problem with $\beta_0 = \frac{1}{2}$.

$$f(\eta) = \sum_{k=0}^{\infty} \left(-\frac{1}{2}\right)^k \frac{A_k \sigma^{k+1}}{(3k+2)!} \eta^{3k+2}, \tag{4.4}$$

where $A_0 = A_1$ and $A_k = \sum_{r=0}^{k-1} \binom{3k-1}{3r} A_r A_{k-r-1}$, $k \geq 2, \sigma$ represents the unknown $f''(0)$. Solved (4.3), (with $\beta_0 = \frac{1}{2}$) numerically and found

$\sigma = 0.33206$.

Several authors have devised numerical algorithms to find good approximations to $f^{''}(0)$. Asaithambi solved (4.3) with $\beta_0 = 1$ and found $f^{''}(0)$ denoted by α to be $\alpha = 0.469600$.

Fang et al. [10] have shown that the substitution

$$f(\eta) = \frac{1}{\sqrt{\beta_0}} F\left(\sqrt{\beta_0}\eta\right)$$

transform Eq. (4.3) into

$$F^{'''} + FF^{''} = 0.$$

Therefore it is sufficient to consider the Blasius problem with $\beta_0 = 1$. Henceforth we shall treat the problem with $\beta_0 = 1$. Liao applied his homotopy analysis method to the Blasius problem and obtained the solution to a high level of accuracy. J.H. He has used an iterative perturbation technique to find an approximate analytic solution of the Blasius problem. [11]. Abbasbandy has used a modified version of the Adomian decomposition method to find a numerical soution while Cortell has studied the dependence of the solution on the parameter β_0.

Crocco proposed a further transformation of the Blasius problem in the $1940's$, and independently Wang [12] made use of the same transformations which helps to approach the Blasius problem from a new perspective. They used an ingenious idea to transform the Blasius problem into a simpler problem governed by a second order differential equation. They used the transformation

$$x = f^{'}(\eta), \; y = f^{''}(\eta), \tag{4.5}$$

to transform (4.3) to

$$\begin{aligned} \frac{d^2y}{dx^2} + \frac{x}{y} &= 0, \quad x \in [0,1) \qquad \text{with the boundary conditions} \\ y(0) &= f^{''}(0), \quad y^{'}(0) = 0, \quad \lim_{x \to 1} y(x) = 0. \end{aligned} \tag{4.6}$$

Wang used the Adomian decompostion method to solve (4.6), and found

$$y(x) = \alpha - \frac{x^3}{6\alpha} - \frac{x^6}{180\alpha^3} - \frac{x^9}{2160\alpha^5} - \frac{x^{12}}{19008\alpha^7} \cdots \tag{4.7}$$

To find α the equation $y(1) = 0$ is solved for α. He solved this equation retaining six terms of the series (4.7) and found $\alpha = 0.453539$. Hashim improved this value to $\alpha = 0.466799$. Recently Ahmad [9] improved this value to $\alpha = 0.469606$.

As an example of the application of the Spectral method to the boundary value problems, we shall find an analytical expression for the solution of the problem (4.6).

$$yy'' + x = 0, \qquad y'(0) = 0, \quad y(1) = 0.$$

Let

$$f(x) = y_{20}(x) = \sum_{i=0}^{20} a_i P_i(x). \tag{4.8}$$

The boundary condition $y'_{20}(0) = 0$ leads to

$$2a_1 - 3a_2 + 6a_3 - 10a_4 + 15a_5 - 21a_6 + 28a_7 - 36a_8 + 45a_9 - 55a_{10} + 66a_{11} -$$

$$78a_{12} + 91a_{13} - 105a_{14} + 120a_{15} - 136a_{16} + 153a_{17} - 171a_{18} + 190a_{19} - 210a_{20} = 0,$$

and boundary condition $y_{20}(1) = 0$ leads to

$$a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + a_{10} +$$

$$a_{11} + a_{12} + a_{13} + a_{14} + a_{15} + a_{16} + a_{17} + a_{18} + a_{19} + a_{20} = 0.$$

Let x_i denote the i-th zero of 21st Legendre functions i.e $P_{21}(x)$. These are

$$\begin{aligned}
x_1 &= 0.00312391,\ x_2 = 0.0163866,\ x_3 = 0.0399503,\\
x_4 &= 0.0733183,\ x_5 = 0.11578,\ x_6 = 0.166431,\ x_7 = 0.224191,\\
x_8 &= 0.287829,\ x_9 = 0.355989,\ x_{10} = 0.427219,\ x_{11} = 0.5,\\
x_{12} &= 0.572779,\ x_{13} = 0.644012,\ x_{14} = 0.712125,\ x_{15} = 0.77593,\\
x_{16} &= 0.833427,\ x_{17} = 0.886945,\ x_{18} = 0.927271,\ x_{19} = 0.959121,\\
x_{20} &= 0.986469,\ x_{21} = 0.997427.
\end{aligned}$$

Ninteen more equations are obtained by setting the residual to zero at the point $x_2, ..., x_{20}$. When this system of equations is solved, we find the following values for the coefficients a_0, $a_1, ..., a_{20}$.

$$\begin{aligned}
a_0 &= 0.36805,\ a_1 = -0.187637,\ a_2 = -0.117454,\ a_3 = -0.0378429,\\
a_4 &= -0.0111564,\ a_5 = -0.00573766,\ a_6 = -0.00313192,\ a_7 = -0.00186456,\\
a_8 &= -0.00116731,\ a_9 = -0.000750723,\ a_{10} = -0.000489022,\ a_{11} = -0.00031856,\\
a_{12} &= -0.000205192,\ a_{13} = -0.000129189,\ a_{14} = -0.0000784632,\ a_{15} = -0.0000452055,\\
a_{16} &= -0.0000241291,\ a_{17} = -0.0000115009,\ a_{18} = -4.59043 \times 10^{-6},\ a_{19} = -1.34687 \times 10^{-6},\\
a_{20} &= -2.10276 \times 10^{-7}.
\end{aligned}$$

Inserting the above values in Eq. (4.8) and simplifying we get the following expressions as our approximate solution to non-linear boundary value problem

$$\begin{aligned} f(x) &= 0.468677 + 5.55112 \times 10^{-17}x - 0.0000104644x^2 - 0.35401x^3 - \\ & 0.0902364x^4 + 2.65177x^5 - 47.4162x^6 + 560.855x^7 - 4653.09x^8 + \\ & 28062.2x^9 - 126146.x^{10} + 429705.x^{11} - 1.12011 \times 10^6 x^{12} + \\ & 2.24271 \times 10^6 x^{13} - 3.44112 \times 10^6 x^{14} + 4.00875 \times 10^6 x^{15} - 3.48076 \times 10^6 x^{16} + \\ & 2.1807 \times 10^6 x^{17} - 930927.x^{18} + 242252.x^{19} - 28985.8x^{20}. \end{aligned} \tag{4.9}$$

Table 1, contains the approximate solution and , numerical solution produced by the bvp4c program of MATLAB, and their relative error for $i = 5$.

Table 1.

x	Approximate solution	Numerical solution	Relative Error
0	0.448547	0.4696	0.0448318
0.1	0.448179	0.4693	0.0450053
0.2	0.44556	0.4668	0.0455013
0.3	0.43846	0.4600	0.0468261
0.4	0.424537	0.4467	0.049615
0.5	0.401127	0.4244	0.0548322
0.6	0.36504	0.3902	0.0644798
0.7	0.312349	0.3406	0.0829448
0.8	0.238185	0.2700	0.1178333
0.9	0.136528	0.1689	0.1916637
1	1.11022×10^{-16}	0	0

Table 2, contains the approximate solution and , numerical solution produced by the bvp4c program of MATLAB, and their relative error for $i = 10$.

Table 2.

x	Approximate solution	Numerical solution	Relative Error
0	0.46518	0.4696	0.0094123
0.1	0.464822	0.4693	0.0095419
0.2	0.46231	0.4668	0.0096187
0.3	0.455466	0.4600	0.0098565
0.4	0.442018	0.4467	0.0104884
0.5	0.419488	0.4244	0.011574
0.6	0.384974	0.3902	0.0133931
0.7	0.334737	0.3406	0.0172137
0.8	0.263285	0.2700	0.0248704
0.9	0.160235	0.1689	0.0513025
1	5.66214×10^{-15}	0	0

Table 3, contains the approximate solution and , numerical solution produced by the bvp4c program of MATLAB, and their relative error for $i = 15$.

Table 3.

x	Approximate solution	Numerical solution	Relative Error
0	0.472512	0.4696	0.00620102
0.1	0.471108	0.4693	0.00385255
0.2	0.467508	0.4668	0.00151671
0.3	0.459618	0.4600	0.0008304
0.4	0.445208	0.4467	0.00334
0.5	0.421746	0.4244	0.0062535
0.6	0.386404	0.3902	0.0097283
0.7	0.335352	0.3406	0.0154081
0.8	0.263141	0.2700	0.0254037
0.9	0.159442	0.1689	0.0559976
1	-8.52709×10^{-11}	0	0

Table 4, contains the approximate solution and , numerical solution produced by the bvp4c program of MATLAB, and their relative error for $i = 20$.

		Table 4.	
x	Eq. (4.9)	Numerical solution	Relative Error
0	0.468677	0.4696	0.0019655
0.1	0.468322	0.4693	0.0020841
0.2	0.465829	0.4668	0.0020801
0.3	0.459036	0.4600	0.0020957
0.4	0.445692	0.4467	0.0022565
0.5	0.42334	0.4244	0.0024976
0.6	0.389115	0.3902	0.0027806
0.7	0.339342	0.3406	0.0036935
0.8	0.268633	0.2700	0.0050631
0.9	0.167088	0.1689	0.0107282
1	-6.14091×10^{-9}	0	0

The agreement between these solution is good.

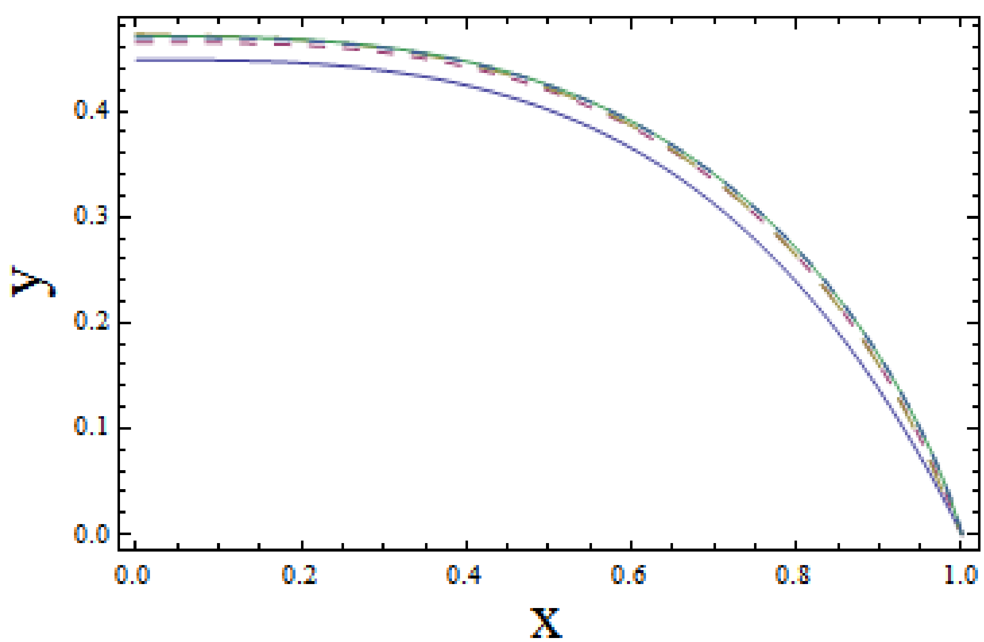

Fig 1 : The approximate solutions for $i = 5, 10, 15, 20$ are compared with the numerical solution.

Example 2:

$$y'' + 2yy' = 0, \qquad y(0) = 1, \quad y(1) = \frac{1}{2}.$$

Let

$$f(x) = y_{10}(x) = \sum_{i=0}^{10} a_i P_i(x). \tag{4.10}$$

The boundary condition $y_{10}(0) = 1$ leads to

$$a_0 - a_1 + a_2 - a_3 + a_4 - a_5 + a_6 - a_7 + a_8 - a_9 + a_{10} = 1,$$

and boundary condition $y_{10}(1) = \frac{1}{2}$ leads to

$$a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + a_{10} = \frac{1}{2}.$$

Let x_i denote the $i-th$ zero of $11th$ Legender functions i.e $P_{11}(x)$. These are

$$\begin{aligned} x_1 &= 0.0108857,\ x_2 = 0.0564687,\ x_3 = 0.134924, \\ x_4 &= 0.240452,\ x_5 = 0.365228,\ x_6 = 0.5,\ x_7 = 0.634772, \\ x_8 &= 0.759548,\ x_9 = 0.865076,\ x_{10} = 0.943534, \\ x_{11} &= 0.989114. \end{aligned}$$

Nine more equations are obtained by setting the residual to zero at the point $x_2, ..., x_{10}$. When this system of equations solved we find the following values for the coefficients a_0, $a_1, ..., a_{10}$.

$$\begin{aligned} a_0 &= 0.693145,\ a_1 = -0.238327,\ a_2 = 0.0545675, \\ a_3 &= -0.0112394,\ a_4 = 0.00220532,\ a_5 = -0.000419716, \\ a_6 &= 0.0000792958,\ a_7 = -0.0000140458,\ a_8 = 3.02692 \times 10^{-6}, \\ a_9 &= -2.55567 \times 10^{-7},\ a_{10} = 1.86686 \times 10^{-7}. \end{aligned}$$

Inserting the above values in Eq. (4.10) and simplifying we get the following expressions as our approximate solution to non-linear boundary value problem

$$\begin{aligned} f(x) &= 0.761673 - 0.523355x + 1.00001x^2 - 0.999883x^3 + 0.997954x^4 \\ &\quad -0.983085x^5 + 0.917895x^6 - 0.744965x^7 + 0.462478x^8 - 0.184883x^9 \\ &\quad +0.0344914x^{10} - 0.238327(-1 + 2x). \end{aligned} \tag{4.11}$$

Now the exact solution of the problem is

$$y = \frac{1}{1+x}. \tag{4.12}$$

In Table 2, we compare, at some randomly selected points, the approximate solution, Eq. (4.11), with the exact solution, Eq. (4.12).

Table 2.

x	Eq. (4.11)	Eq. (4.12)	Relative Error
0	1	1	0
0.1	0.90909	0.909091	0.0000011
0.2	0.833332	0.833333	0.0000012
0.3	0.769229	0.769231	0.0000026
0.4	0.714283	0.714286	0.0000042
0.5	0.666664	0.666667	0.0000045
0.6	0.624997	0.625	0.0000048
0.7	0.588232	0.588235	0.0000051
0.8	0.555552	0.555556	0.0000072
0.9	0.526311	0.526316	0.0000095
1	0.5	0.5	0

In Fig 2, the approximate solution (solid line) represented by Eq. (4.11) is compared with the exact solution (dotted line).

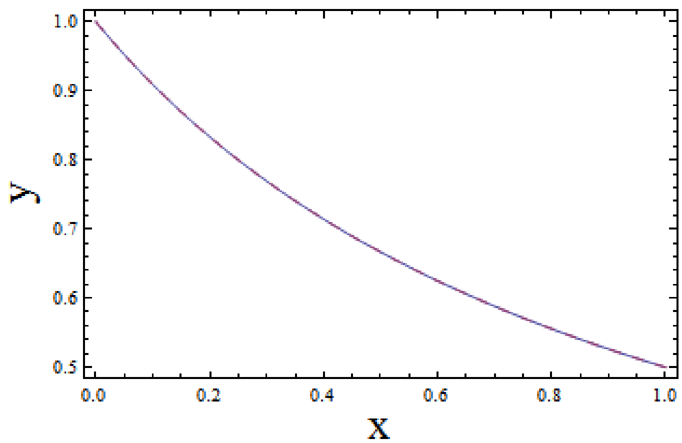

Figure 2:

The two curves appear to be identical on the scale.

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