Lamb Modes in a Plate or a Cylinder

by

Iqra Batool

A dissertation submitted in partial fulfillment of the requirements

for the degree of Master of Philosophy in Mathematics

Supervised by

Prof. Faiz Ahmad

School of Natural Sciences

National University of Sciences and Technology

Islamabad, Pakistan

© Iqra Batool, 2016

FORM TH-4

National University of Sciences & Technology

MASTER'S THESIS WORK

We hereby recommend that the dissertation prepared under our supervision by: Iqra Batool, Regn No. NUST201463603MSNS78014F Titled: Shape of Lamb modes in Plate or a Cylinder be accepted in partial fulfillment of the requirements for the award of **MS** degree.

Examination Committee Members

1. Name: Dr. Moniba Shams Signature:

2. Name: Dr. Muhammad Ishaq Signature:

3. Name: ________ Signature:

4. Name: Dr. Masood Khan Signature:

Supervisor's Name: Prof. Faiz Ahmed Signature:

Head of Department

08-11-2016
Date

COUNTERSINGED

Date: 08/11/16

Dean/Principal

Abstract

This dissertation deals with the study of Lamb modes in plates and rods. The dispersion relation called Rayleigh-Lamb frequency is derived for symmetric and anti symmetric displacement of a wave propagating in an isotropic and infinite plate of thickness $2h$. The Pochhammer frequency equation, which gives the dispersion relation for a rod of circular cross section is studied. The Lamb modes in plate and rod are graphically represented.

The plateau region shown by Lamb modes in $k-c$ plane is mathematically studied and graphically represented for a plate and a rod. Finally, the anomalous behavior shown by the Lamb modes when the group velocity becomes zero is studied. The symmetric and anti symmetric Lamb modes of plate exhibiting zero group velocity points are graphically shown.

Acknowledgement

My thanks go first to Almighty Allah the most Gracious and Merciful, Who gives power to the weak and helps the helpless.

I would like to express my gratitude to my supervisor, Professor Faiz Ahmad, for directing me troughout this research. I will always be thankful for the learning opportunities he has provided, for his kindness and patience.

I want to thank my parents, who have made enormous efforts to make it possible. Thank you so much for believing in me, for your support, encouragement, endless love and prayers. Special thanks to my uncle, my siblings, my niece and nephew for their prayers and love. I am truly blessed to have you people.

I would also like to thank my class fellows and hostel friends, for always being ready to help me. I wish them all the health and success in life.

Iqra Batool

To My Loving Parents

Contents

Chapter 1

Introduction

A waveguide is a mechanical structure through which waves can propagate. The waves propagating through a waveguide are called guided waves. The most common example of a waveguide is a tube or a pipe. A few examples of guided waves are Rayleigh waves, Love waves, Lamb waves and Stonley waves. The guided waves are named after their investigators.

Guided waves have a variety of applications in non-destructive testing (NDT), seismology and in medicine. A human bone can act as a cylindrical waveguide and can be examined by guided waves. In non-destructive testing technique material is being tested without causing any damage by transmitting ultrasonic waves. In 1961, Worlton [1] gave an experimental confirmation of the use of Lamb waves in NDT. Because of dispersive nature, Lamb modes are very much useful in material characterization and detecting defects. Lamb modes are used to inspect industrial products in aerospace, transportation and pipes.

Lamb waves are the guided waves propagating in plates with free surfaces. These were originally studied by Lamb [2] in 1917. Lamb [2] and Rayleigh [3] gave the frequency relation for waves propagating in an infinite plate. The frequency equation for longitudinal waves in a circular rod of infinite length was given by Pochhammer [4] in 1876 and independently by Chree [5] in 1889.

Some modes in the Lamb modes spectrum exhibit an anomalous behavior at points where the group velocity $\frac{d\omega}{dk}$ vanishes for a finite phase velocity $\frac{\omega}{k}$. The points where the group velocity becomes zero are called zero group velocity points. At these points energy does not propagate and remains trapped under the surface of the plate, causing resonance when a laser source is applied. Zero group velocity points also exhibit the property of separating forward and backward propagating Lamb modes.

Zero group velocity points were first studied by Tolstoy and Usdin [6] in 1957. They observed that group velocity vanishes at a particular point of symmetric mode S_1 for an isotropic plate. Ahmad and Hussain [7] explained analytically the anomalous behavior shown by the S_1 mode of an isotropic material by examining the slope of each mode. Clorennec et al [8] pointed out that zero group velocity Lamb waves can be used to determine the mechanical properties of materials. They have shown that the ZGV resonance that occurs at the minimum frequency of the S_1 and A_2 Lamb modes can be used for measuring the Poisson's ratio and the longitudinal and shear velocities in thin plates os shells.

1.1 Plan of the dissertation:

This thesis concerns with the study of waves propagating through solid plates and rods. The introduction is given in this chapter.

Chapter 2 includes some basic definitions and mathematical preliminaries. The study of governing equations for solid and Halmholtz decomposition of displacement vector into scalar and vector potential is included in this chapter.

Chapter 3 deals with the study of Lamb modes in a plate and a rod in detail. The frequency relation for a plate called, Rayleigh-Lamb dispersion relation, is derived for symmetric and anti symmetric modes by expressing the displacement vector in terms of scalar and vector potentials. The Rayleigh-Lamb frequency relation is graphically represented. Next, the wave propagating in a circular rod is studied. The frequency relation given by Pochhammer is derived and a graph is drawn to describe the shape of longitudinal modes in rods.

Chapter 4 includes the study of the plateau region shown by Lamb modes. We have considered the Rayleigh-Lamb frequency equation and shown that slope of the mode becomes zero at some point of the dispersion curve. Also, we have studied the plateau region in the spectrum of a rod. We have taken Pochhammer frequency relation and shown that slope of the curve must vanish. Plateau region is also graphically represented for a plate and a rod.

Chapter 5 is devoted to the study of zero group velocity points. The Lamb modes spectrum shows an anomalous behavior when frequency$\times$ thickness versus thickness/wavelength is plotted. This is because of the existence of zero group velocity points. This anomalous behavior is graphically shown for symmetric and anti symmetric mode. This chapter also includes the history of some work done on zero group velocity points.

Chapter 2

Preliminaries

In this chapter we will study some basic definitions. Equation of motion for elastic, isotropic and homogeneous solids are studied. Also we have studied the Helmholtz decomposition method in which displacement vector is expressed in terms of scalar and vector potentials.

Elasticity

In physics, elasticity is the ability of a material to resist the applied force and returns to its original shape and size, when applied external force is removed. When force is applied to deform an elastic material, it gives rise to internal stress to resist the deformation and restore it to its original position when force is removed. Rubber and all kinds of springs are examples of elastic material.

The elasticity of a material is characterized by stress−strain relationship. The stress−strain relationship is given by Hooke's Law as

$$\tau_{ij} = C_{ijkl} S_{kl},$$

where τ_{ij} is a stress tensor, S_{kl} is strain tensor and C_{ijkl} is a tensor of rank four called elastic stiffness tensor. Elastic stiffness tensor C_{ijkl} has 81 components but because of symmetries $C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij}$, we are left with 21 independent components.

Isotropic material

A material which is invariant with respect to direction and has same properties in all directions is called an isotropic material. Examples of isotropic material are glass and some metals like aluminium and steel. For an isotropic material, every line (plane) is an axis (plane) of symmetry. So for an isotropic material the elastic stiffness tensor C_{ijkl} should be isotropic. C_{ijkl} can be written as a linear combination of isotropic tensors as

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}),$$

where λ and μ are the Lamé constants. Because of symmetry we have only 2 independent parameters to express the elastic stiffness tensor for isotropic materials. Hooke's Law for isotropic material is written as

$$\tau_{ij} = \lambda S_{kk}\delta_{ij} + 2\mu S_{ij},$$

where τ_{ij}, S_{ij} are respectively the components of the stress tensor and the strain tensor.

2.1 Governing equations for three dimensional solids

For elastic, isotropic and homogeneous solids, the equation of motion for three-dimensional elasticity is

$$\tau_{ij,j} + \rho f_i = \rho\ddot{u}_i, \tag{2.1}$$

The relation between strain and stress, strain and displacement can be described by the Hooke's law and Cauchy relation respectively. The relations are given below:

$$\tau_{ij} = \lambda S_{kk}\delta_{ij} + 2\mu S_{ij}, \tag{2.2}$$

$$S_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \tag{2.3}$$

where u_i are the components of displacement, τ_{ij} are the components of stress tensor, S_{ij} are the deformation tensor components, S_{kk} are the trace of deformation tensor, f_i are the components of volume force, ρ is the density, λ and μ are Lamé coefficients and we have taken the summation convention for $i = 1, 2, 3$.

By using the strain-displacement relation (2.3) in the relation for stress and strain (2.2), we obtain the stress tensor components in terms of displacement vector components as given

$$\tau_{ij} = \lambda(u_{i,i})\delta_{ij} + \mu(u_{i,j} + u_{j,i}). \tag{2.4}$$

Substituting the above relation (2.4) in the equation of motion (2.1) and simplifying, we obtain the equation of motion in terms of displacement as

$$\lambda u_{j,ij} + \mu[u_{i,jj} + u_{j,ij}] + \rho f_i = \rho\ddot{u}_i, \tag{2.5}$$

$$(\lambda + \mu)u_{j,ij} + \mu u_{i,jj} + \rho f_i = \rho\ddot{u}_i. \tag{2.6}$$

In vector form, the equation is as follows:

$$(\lambda + \mu)\boldsymbol{\nabla\nabla.u} + \mu\nabla^2\boldsymbol{u} + \rho\boldsymbol{f} = \rho\ddot{\boldsymbol{u}}. \tag{2.7}$$

In terms of rectangular coordinates, the above equation can be written as

$$\begin{aligned}
(\lambda+\mu)\left(\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_2}{\partial x_1\partial x_2} + \frac{\partial^2 u_3}{\partial x_1\partial x_3}\right) + \mu\nabla^2 u_1 + \rho f_1 &= \rho\ddot{u}_1,\\
(\lambda+\mu)\left(\frac{\partial^2 u_1}{\partial x_1\partial x_2} + \frac{\partial^2 u_2}{\partial x_2^2} + \frac{\partial^2 u_3}{\partial x_2\partial x_3}\right) + \mu\nabla^2 u_2 + \rho f_2 &= \rho\ddot{u}_2,\\
(\lambda+\mu)\left(\frac{\partial^2 u_1}{\partial x_1\partial x_2} + \frac{\partial^2 u_2}{\partial x_2\partial x_3} + \frac{\partial^2 u_3}{\partial x_3^2}\right) + \mu\nabla^2 u_3 + \rho f_3 &= \rho\ddot{u}_3,
\end{aligned} \tag{2.8}$$

where ∇^2 is the Laplace operator and

$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}. \tag{2.9}$$

In the absence of body forces equations of motion become

$$(\lambda + \mu)\boldsymbol{\nabla\nabla.u} + \mu\nabla^2\boldsymbol{u} = \rho\ddot{\boldsymbol{u}}. \tag{2.10}$$

2.2 Displacement potential

The equations of motion (2.10) are coupled. These equations can be uncoupled by Helmholtz decomposition, in which the components of the displacement vectors are expressed in terms of derivatives of scalar and vector potentials as described below

$$\boldsymbol{u} = \boldsymbol{\nabla}\phi + \boldsymbol{\nabla} \times \boldsymbol{\psi}. \tag{2.11}$$

In Cartesian coordinates vector potential $\boldsymbol{\psi}$ is

$$\boldsymbol{\psi} = \psi_x \boldsymbol{e_1} + \psi_y \boldsymbol{e_2} + \psi_z \boldsymbol{e_3}, \tag{2.12}$$

where ψ_x, ψ_y and ψ_z are the components of the vector potential $\boldsymbol{\psi}$.
As given by Achenbach [9] the Helmholtz displacement components are of the form

$$\begin{aligned} u &= \frac{\partial \phi}{\partial x} + \frac{\partial \psi_z}{\partial y} - \frac{\partial \psi_y}{\partial z}, \\ v &= \frac{\partial \phi}{\partial y} + \frac{\partial \psi_x}{\partial z} - \frac{\partial \psi_z}{\partial x}, \\ w &= \frac{\partial \phi}{\partial z} + \frac{\partial \psi_y}{\partial x} - \frac{\partial \psi_x}{\partial y}. \end{aligned} \tag{2.13}$$

Now by taking into account that $\boldsymbol{\nabla . \nabla}\phi = \nabla^2\phi$ and $\boldsymbol{\nabla . \nabla} \times \boldsymbol{\psi} = 0$, and using the equation (2.11) into the equation (2.10), we have

$$\begin{aligned} (\lambda+\mu)\boldsymbol{\nabla}\nabla^2\phi + \mu\nabla^2\boldsymbol{\nabla}\phi + \mu\nabla^2\boldsymbol{\nabla} \times \boldsymbol{\psi} &= \rho[\boldsymbol{\nabla}\ddot{\phi} + \boldsymbol{\nabla} \times \ddot{\boldsymbol{\psi}}], \\ (\lambda+2\mu)\boldsymbol{\nabla}\nabla^2\phi - \rho\boldsymbol{\nabla}\ddot{\phi} + \mu\nabla^2\boldsymbol{\nabla} \times \boldsymbol{\psi} - \rho\boldsymbol{\nabla} \times \ddot{\boldsymbol{\psi}} &= 0, \\ \boldsymbol{\nabla}[(\lambda+2\mu)\nabla^2\phi - \rho\ddot{\phi}] + \boldsymbol{\nabla} \times [\mu\nabla^2\boldsymbol{\psi} - \rho\ddot{\boldsymbol{\psi}}] &= 0. \end{aligned} \tag{2.14}$$

Equation (2.11) satisfies the equation of motion (2.10), if the following uncoupled equations are satisfied.

$$\nabla^2\phi = \frac{1}{c_L^2}\ddot{\phi}^2, \tag{2.15}$$

$$\nabla^2\boldsymbol{\psi} = \frac{1}{c_T^2}\ddot{\boldsymbol{\psi}}. \tag{2.16}$$

Here c_L and c_T are speeds of longitudinal and transverse wave, respectively. Where $c_L^2 = \frac{\lambda+2\mu}{\rho}$ and $c_T^2 = \frac{\mu}{\rho}$.
Equations for vector potential $\boldsymbol{\psi}$ in component form can be written as

$$\begin{aligned} \nabla^2\psi_x =& \frac{1}{c_T^2}\ddot{\psi}_x, \\ \nabla^2\psi_y =& \frac{1}{c_T^2}\ddot{\psi}_y, \\ \nabla^2\psi_z =& \frac{1}{c_T^2}\ddot{\psi}_z. \end{aligned} \tag{2.17}$$

The relation between displacement components and potentials in cylindrical coordinates are given by Achenbach [9] as

$$\begin{aligned} u &= \frac{\partial \phi}{\partial r} + \frac{1}{r}\frac{\partial \psi_z}{\partial \theta} - \frac{\partial \psi_\theta}{\partial z}, \\ v &= \frac{1}{r}\frac{\partial \phi}{\partial \theta} + \frac{\partial \psi_r}{\partial z} - \frac{\partial \psi_z}{\partial r}, \\ w &= \frac{\partial \phi}{\partial z} + \frac{1}{r}\frac{\partial (\psi_\theta r)}{\partial r} - \frac{1}{r}\frac{\partial \psi_r}{\partial \theta}. \end{aligned} \tag{2.18}$$

The scalar potential in cylindrical coordinates is the same as in rectangular coordinates, where the components of vector potential in cylindrical coordinates satisfy the following equations

$$\begin{aligned} \nabla^2 \psi_r - \frac{\psi_r}{r^2} - \frac{2}{r^2}\frac{\partial \psi_\theta}{\partial \theta} &= \frac{1}{c_T^2}\ddot{\psi}_r, \\ \nabla^2 \psi_\theta - \frac{\psi_\theta}{r^2} + \frac{2}{r^2}\frac{\partial \psi_r}{\partial \theta} &= \frac{1}{c_T^2}\ddot{\psi}_\theta, \\ \nabla^2 \psi_z &= \frac{1}{c_T^2}\ddot{\psi}_z, \end{aligned} \tag{2.19}$$

where ψ_r, ψ_θ and ψ_z are the components of the vector potential $\boldsymbol{\psi}$ in cylindrical coordinates and ∇^2 is defined as

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}. \tag{2.20}$$

Chapter 3

Shape of Lamb Modes

Lamb waves are the guided waves propagating in the traction free solid plates. Waves in plates were first studied by Lamb. Lamb waves have infinite number of modes for symmetric and anti symmetric displacement. Lamb modes provide relation between speed of wave and frequency or wave number.

This chapter includes the study of Lamb modes in a plate and a rod. A time harmonic wave passing through a plate in plane strain is considered and dispersion relation for symmetric and anti symmetric modes is calculated. Frequency relation for a rod of infinite length is also derived. Lamb mode dispersion curves are graphically represented.

3.1 Lamb modes in a plate

Consider a time-harmonic wave propagating in a plate of thickness $2h$. For motion in plane strain in x_1x_2-plane, we have

$$u_1 = u_1(x_1, x_2, t), u_2 = u_2(x_1, x_2, t), u_3 = 0 \qquad \text{and} \qquad \frac{\partial}{\partial x_3}() = 0.$$

Then equations of motion (2.10), under the plane strain condition will take the form

$$\begin{aligned} \frac{\partial \tau_{11}}{\partial x_1} + \frac{\partial \tau_{12}}{\partial x_2} &= \rho \ddot{u}_1, \\ \frac{\partial \tau_{12}}{\partial x_1} + \frac{\partial \tau_{22}}{\partial x_2} &= \rho \ddot{u}_2. \end{aligned} \tag{3.1}$$

And Hooke's law becomes

$$\begin{aligned}
\tau_{11} &= \lambda\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right) + 2\mu\frac{\partial u_1}{\partial x_1}, \\
\tau_{12} &= \mu\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right), \\
\tau_{22} &= \lambda\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right) + 2\mu\frac{\partial u_2}{\partial x_2}.
\end{aligned} \tag{3.2}$$

Using equation (3.2) in equation (3.1), we have

$$\lambda\left(\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_2}{\partial x_1 \partial x_2}\right) + 2\mu\frac{\partial^2 u_1}{\partial x_1^2} + \mu\left(\frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_2}{\partial x_2 \partial x_1}\right) = \rho\ddot{u}_1, \tag{3.3}$$

$$\mu\left(\frac{\partial^2 u_1}{\partial x_1 \partial x_2} + \frac{\partial^2 u_2}{\partial x_1^2}\right) + \lambda\left(\frac{\partial^2 u_1}{\partial x_1 \partial x_2} + \frac{\partial^2 u_2}{\partial x_2^2}\right) + 2\mu\frac{\partial^2 u_2}{\partial x_2^2} = \rho\ddot{u}_2. \tag{3.4}$$

Rearranging

$$\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\lambda+\mu}{\lambda+2\mu}\frac{\partial^2 u_2}{\partial x_1 \partial x_2} + \frac{\mu}{\lambda+2\mu}\frac{\partial^2 u_1}{\partial x_2^2} = \frac{1}{c_L}\ddot{u}_1, \tag{3.5}$$

$$\frac{\partial^2 u_2}{\partial x_2^2} + \frac{\lambda+\mu}{\lambda+2\mu}\frac{\partial^2 u_1}{\partial x_1 \partial x_2} + \frac{\mu}{\lambda+2\mu}\frac{\partial^2 u_2}{\partial x_1^2} = \frac{1}{c_L}\ddot{u}_2. \tag{3.6}$$

The Helmholtz decomposition displacement under the plane strain condition is

$$u_1 = \frac{\partial \phi}{\partial x_1} + \frac{\partial \psi_3}{\partial x_2}, \tag{3.7}$$

$$u_2 = \frac{\partial \phi}{\partial x_2} - \frac{\partial \psi_3}{\partial x_1}. \tag{3.8}$$

Hence the stress relations become

$$\tau_{11} = \lambda\left(\frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_2^2}\right) + 2\mu\left(\frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \psi_3}{\partial x_1 \partial x_2}\right), \tag{3.9}$$

$$\tau_{12} = \mu\left(2\frac{\partial^2 \phi}{\partial x_1 \partial x_2} + \frac{\partial^2 \psi_3}{\partial x_2^2} - \frac{\partial^2 \psi_3}{\partial x_1^2}\right), \tag{3.10}$$

$$\tau_{22} = \lambda\left(\frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_2^2}\right) + 2\mu\left(\frac{\partial^2 \phi}{\partial x_2^2} - \frac{\partial^2 \psi_3}{\partial x_1 \partial x_2}\right), \tag{3.11}$$

where ϕ and ψ_3 satisfy the following wave equations

$$\frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_2^2} = \frac{1}{c_L^2}\ddot{\phi}, \tag{3.12}$$

$$\frac{\partial^2 \psi_3}{\partial x_1^2} + \frac{\partial^2 \psi_3}{\partial x_2^2} = \frac{1}{c_T^2}\ddot{\psi}_3. \tag{3.13}$$

Now, consider the solution of wave equation of the form

$$\phi = \Phi(x_2) \exp i(kx_1 - \omega t), \tag{3.14}$$

$$\psi_3 = \Psi(x_2) \exp i(kx_1 - \omega t), \tag{3.15}$$

where ω and k are the frequency and wave number respectively.
Substituting equations (3.14) and (3.15) in equation (3.12) and (3.13), we get

$$\frac{d^2\Phi}{dx_2^2} + p^2\Phi = 0, \tag{3.16}$$

$$\frac{d^2\Psi}{dx_2^2} + q^2\Phi = 0, \tag{3.17}$$

where $p^2 = \frac{\omega^2}{c_L^2} - k^2$ and $q^2 = \frac{\omega^2}{c_T^2} - k^2$. The solutions of the above equations are

$$\Phi(x_2) = A\sin(px_2) + B\cos(px_2), \tag{3.18}$$

$$\Psi(x_2) = C\sin(qx_2) + D\cos(qx_2). \tag{3.19}$$

Using above relations in equations (3.7)-(3.11), (3.14) and (3.15), and omitting the exponential term in the sequel because it appears in all the expressions and does not play any role in determining the relation for frequency.

$$\begin{aligned}
\phi &= A\sin(px_2) + B\cos(px_2),\\
\psi_3 &= C\sin(qx_2) + D\cos(qx_2),\\
u_1 &= ik[A\sin(px_2) + B\cos(px_2)] + q[C\cos(qx_2) - D\sin(qx_2)],\\
u_2 &= pA\cos(px_2) - pB\sin(px_2) - ik[C\sin(qx_2) + D\cos(qx_2)],\\
\tau_{11} &= -\lambda[(k^2+p^2)A\sin(px_2) + (k^2+p^2)B\cos(px_2)] + 2\mu[-k^2(A\sin(px_2) + B\cos(px_2))\\
&\quad - ikq(C\cos(qx_2) + D\sin(qx_2))],\\
\tau_{12} &= \mu[2ikp(A\cos(px_2) - B\sin(px_2)) + (k^2-q^2)C\sin(qx_2) + (k^2-q^2)D\cos(qx_2)],\\
\tau_{22} &= -\lambda[(k^2+p^2)A\sin(px_2) + (k^2+p^2)B\cos(px_2)] - 2\mu[p^2A\sin(px_2) + p^2B\cos(px_2)\\
&\quad + ikqC\cos(qx_2) - ikqD\sin(qx_2)].
\end{aligned}$$

Now we split the wave propagating in the plate into symmetric and anti symmetric modes to find the frequency relations. The displacement in x_1-direction is symmetric (anti symmetric) with respect to $x_2 = 0$ if u_1 contains cosines (sines). And if u_2 contains sines (cosines) then displacement in the x_2-direction will be symmetric (anti

symmetric).
The equations for symmetric modes are given below:

$$\Phi = B\cos(px_2), \tag{3.20}$$
$$\Psi = C\sin(qx_2), \tag{3.21}$$
$$u_1 = ikB\cos(px_2) + qC\cos(qx_2), \tag{3.22}$$
$$u_2 = -pB\sin(px_2) - ikC\sin(qx_2), \tag{3.23}$$
$$\tau_{21} = \mu[-2ikpB\sin(px_2) + (k^2 - q^2)C\sin(qx^2)], \tag{3.24}$$
$$\tau_{22} = -\lambda(k^2 + p^2)B\cos(px_2) - 2\mu[p^2B\cos(px_2) + ikqC\cos(qx_2)]. \tag{3.25}$$

The equations for anti symmetric modes are as follows:

$$\Phi = A\sin(px_2), \tag{3.26}$$
$$\Psi = D\cos(qx_2), \tag{3.27}$$
$$u_1 = ikA\sin(px_2) - qD\sin(qx_2), \tag{3.28}$$
$$u_2 = pA\cos(px_2) - ikD\cos(qx_2), \tag{3.29}$$
$$\tau_{21} = \mu[2ikpA\cos(px_2) + (k^2 - q^2)D\cos(qx_2)], \tag{3.30}$$
$$\tau_{22} = -\lambda(k^2 + p^2)A\sin(px^2) - 2\mu[p^2A\sin(px_2) - ikqD\sin(qx_2)]. \tag{3.31}$$

The frequency equation can be obtained by using traction free boundary conditions. The traction free boundary conditions for a plate of thickness $2h$ are

$$\tau_{21} = \tau_{22} = 0 \qquad \text{at} \qquad x_2 = \pm h. \tag{3.32}$$

First, considering the case for symmetric modes and applying the above boundary conditions, we get the two homogeneous equations for constants B and C.

$$-\mu 2ikp\sin(ph)B + \mu[(k^2 - q^2)\sin(qh)]C = 0, \tag{3.33}$$
$$-(\lambda(k^2 + p^2) + 2\mu p^2)\cos(ph)B - 2\mu ikq\cos(qh)C = 0. \tag{3.34}$$

Now the determinant of the coefficient must vanish because equations are homogeneous, which yields:

$$4i^2\mu^2k^2pq\sin(ph)\cos(qh) + \mu(k^2 - q^2)[\lambda(k^2 + p^2) + 2\mu p^2]\sin(qh)\cos(ph) = 0, \tag{3.35}$$

$$
\begin{aligned}
4^2\mu^2k^2pq\tan(ph) &= \mu(k^2-q^2)[\lambda(k^2+p^2)+2\mu p^2]\tan(qh),\\
\frac{\tan(qh)}{\tan(ph)} &= \frac{4^2\mu^2k^2pq}{\mu(k^2-q^2)[\lambda(k^2)+(\lambda+2\mu)p^2]},\\
&= \frac{4c_T^2k^2pq}{(k^2-q^2)[(c_L^2-2c_T^2)k^2+c_L^2p^2]},\\
&= \frac{4c_T^2k^2pq}{(k^2-q^2)[(c_L^2-2c_T^2)k^2+c_L^2(k^2\frac{c^2}{c_L^2}-k^2)]},\\
&= \frac{4c_T^2k^2pq}{(k^2-q^2)(c^2-2c_T^2)k^2},\\
&= \frac{4k^2pq}{(k^2-q^2)(\frac{c^2}{c_T^2}-2)k^2},\\
\frac{\tan(qh)}{\tan(ph)} &= \frac{4k^2pq}{(k^2-q^2)(q^2+k^2-2k^2)},
\end{aligned}
$$

The above equation can be simplified as

$$
\frac{\tan(qh)}{\tan(ph)} = -\frac{4k^2pq}{(q^2-k^2)^2}. \tag{3.36}
$$

Now, using traction free boundary condition for anti symmetric mode we get:

$$
\mu 2ikp\cos(ph)A+\mu[(k^2-q^2)\cos(qh)]D=0, \tag{3.37}
$$

$$
[-\lambda(k^2+p^2)\sin(ph)-2\mu p^2\sin(ph)]A+2\mu ikq\sin(qh)D=0. \tag{3.38}
$$

Equating the determinant of coefficients of above system of equations to zero

$$
4\mu^2i^2k^2pq\tan(qh) = -\mu(k^2-q^2)[\lambda k^2+(\lambda+2\mu)p^2]\tan(ph), \tag{3.39}
$$

$$
\begin{aligned}
4\mu k^2pq\tan(qh) &= (k^2-q^2)[\lambda k^2+(\lambda+2\mu)p^2]\tan(ph),\\
\frac{\tan(ph)}{\tan(qh)} &= \frac{4\mu k^2pq}{(k^2-q^2)[\lambda k^2+(\lambda+2\mu)p^2]},\\
&= \frac{4c_T^2k^2pq}{(k^2-q^2)[(c_L^2-2c_T^2)k^2+p^2c_L^2]},\\
&= \frac{4c_T^2k^2pq}{(k^2-q^2)[(c_L^2-2c_T^2)k^2+(\frac{k^2c^2}{c_L^2}-k^2)c_L^2]},\\
&= \frac{4c_T^2k^2pq}{(k^2-q^2)(c^2-2c_T^2)k^2},\\
\frac{\tan(ph)}{\tan(qh)} &= \frac{4k^2pq}{(k^2-q^2)(\frac{c^2k^2}{c_T^2}-2k^2)},
\end{aligned}
$$

Hence

$$\frac{\tan(ph)}{\tan(qh)} = -\frac{4k^2pq}{(q^2-k^2)^2}. \tag{3.40}$$

Equations (3.36) and (3.40) represent the Rayleigh-Lamb frequency equations for symmetric and anti symmetric modes respectively.
The Rayleigh-Lamb frequency equation gives relation between the frequency and the wave number and yields infinite number of branches for symmetric and anti symmetric modes

The following figures show the dispersion curves for symmetric and anti symmetric Lamb modes in an aluminum plate.

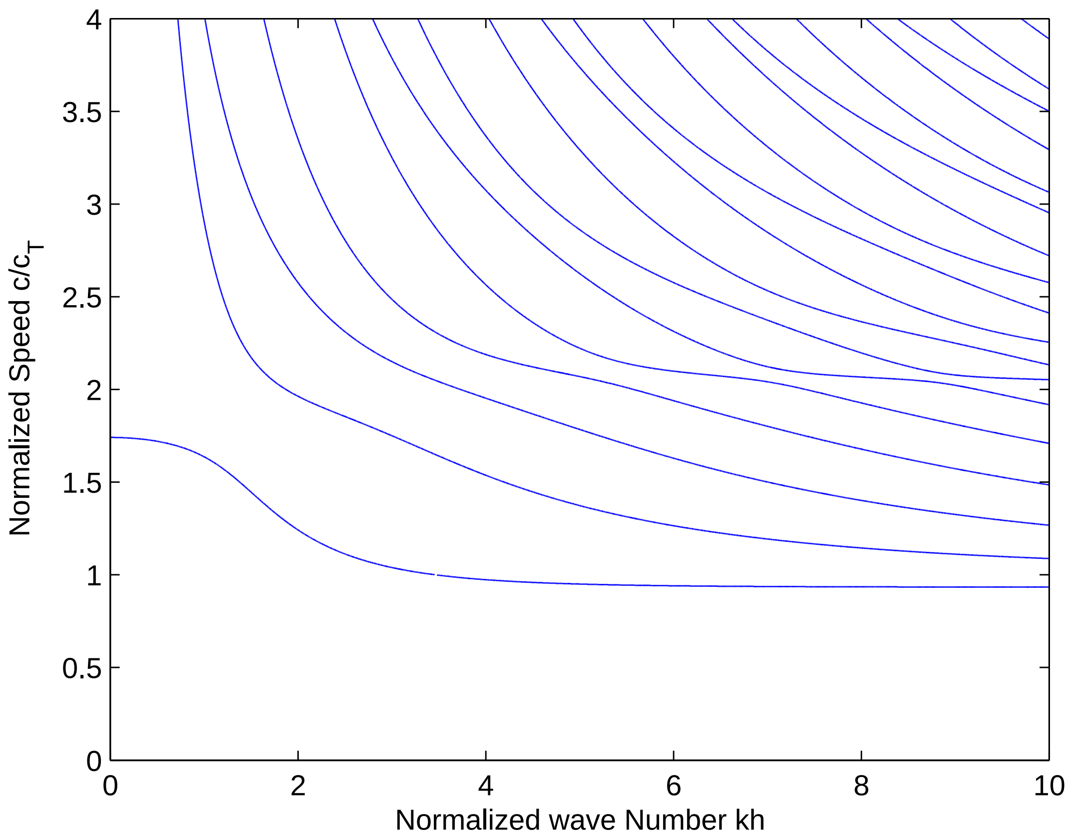

Figure 3.1: Shape of symmetric Lamb modes for aluminum plate.

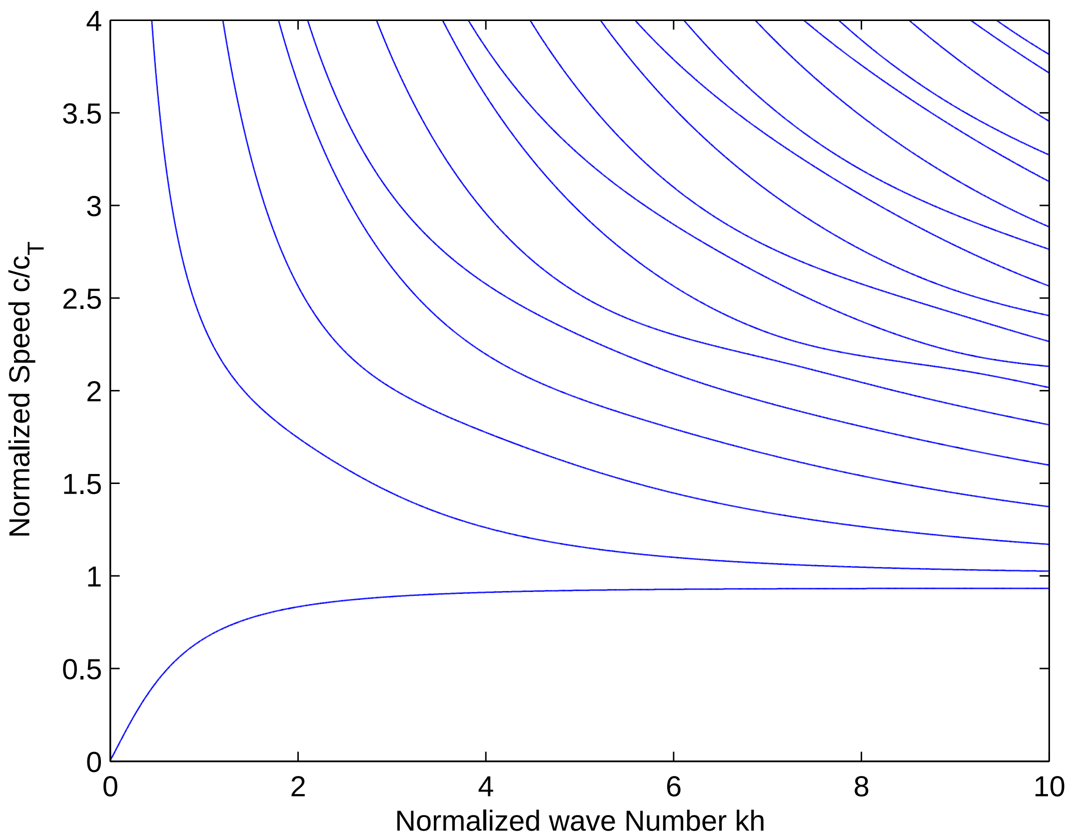

Figure 3.2: Shape of anti symmetric Lamb modes in an aluminum plate.

The figures showing the symmetric and anti symmetric Lamb modes in a copper plate are given below

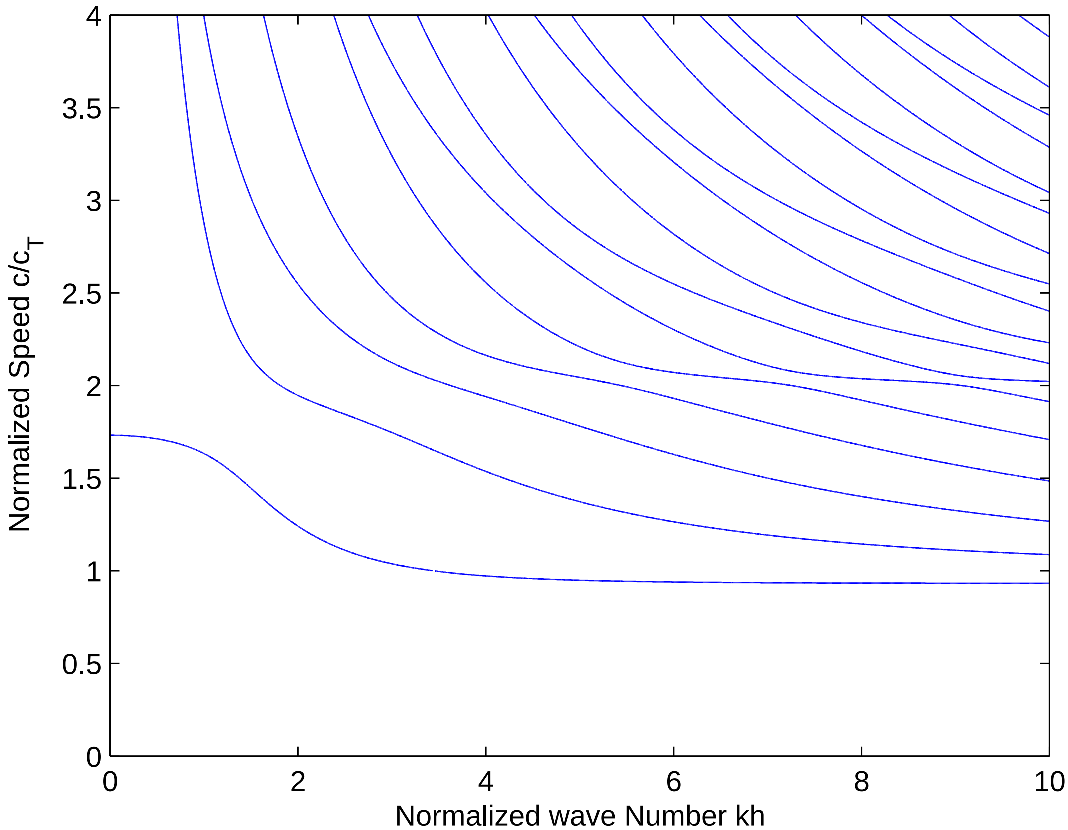

Figure 3.3: Shape of symmetric Lamb modes for copper plate.

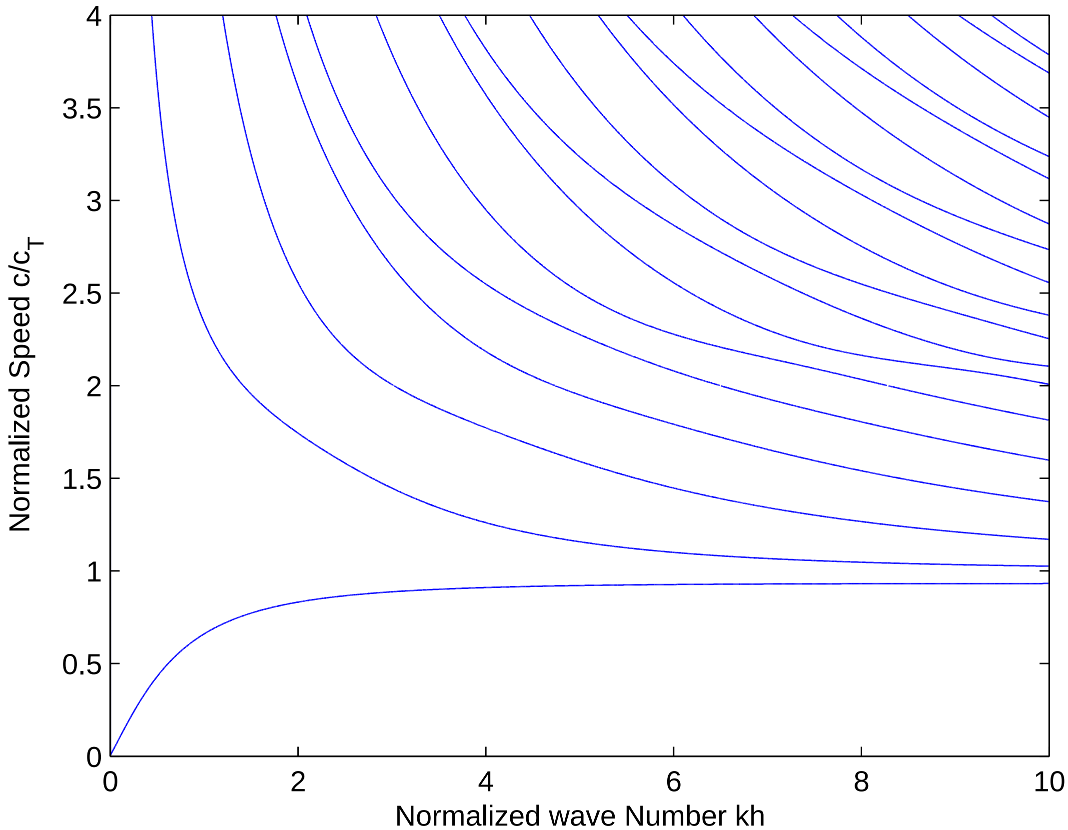

Figure 3.4: Shape of anti symmetric Lamb modes for copper plate.

The dispersion curves for symmetric and anti symmetric Lamb modes in a steel plate are given in the following figures.

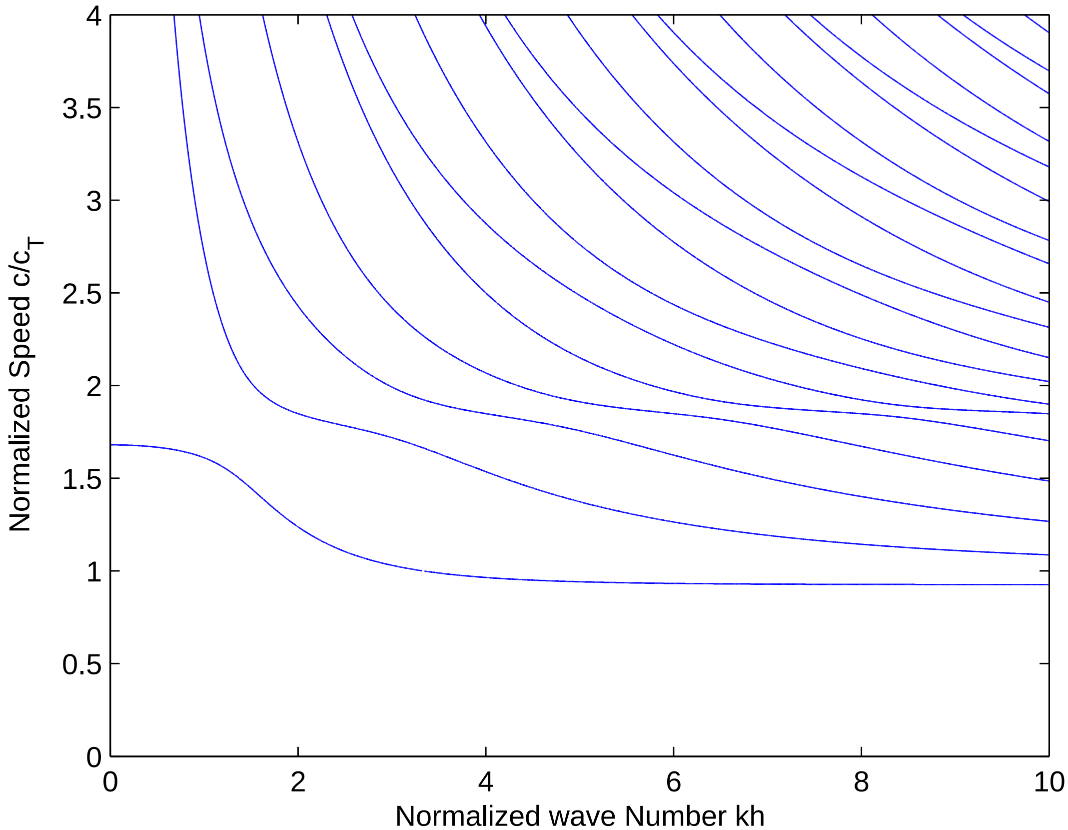

Figure 3.5: Dispersion curves for symmetric Lamb modes in steel plate.

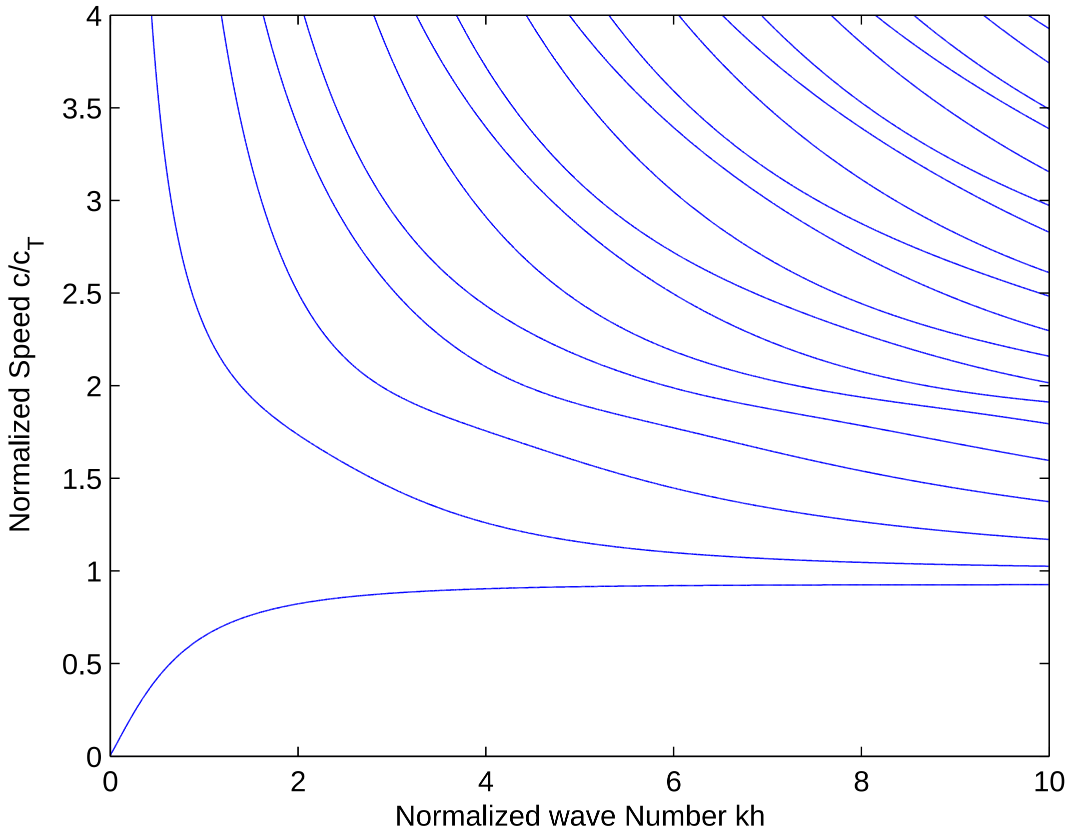

Figure 3.6: Shape of anti symmetric Lamb modes for steel plate.

3.2 A simple formula for Lamb modes in a plate:

Ahmad [10] gave a simple formula to draw the dispersion curves for plates when phase velocity is between c_T and c_L. To find the formula define

$$\omega = ck \qquad \text{and} \qquad u = hk.$$

Here the symbol u is defined as $u = hk$ and should not be confused with the displacement vector $\boldsymbol{u}$.
Also let $c_T < c < c_L$. Then

$$\begin{aligned} ph =& hk\sqrt{\frac{c^2}{c_L^2} - 1} = u\sqrt{\frac{c^2}{c_L^2} - 1}, \\ qh =& hk\sqrt{\frac{c^2}{c_T^2} - 1} = u\sqrt{\frac{c^2}{c_T^2} - 1} \end{aligned}$$

and Rayleigh-Lamb equation for symmetric modes (3.36) become

$$\begin{aligned} \frac{\tan\left(u\sqrt{\frac{c^2}{c_T^2} - 1}\right)}{\tan\left(u\sqrt{\frac{c^2}{c_L^2} - 1}\right)} =& \frac{-4u^4\sqrt{\frac{c_2}{c_L^2} - 1}\sqrt{\frac{c_2}{c_T^2} - 1}}{\left(u^2\left(\frac{c^2}{c_T^2} - 1\right) - u^2\right)^2}, \\ \frac{\tan u\sqrt{\frac{c^2}{c_T^2} - 1}}{i\tanh u\sqrt{1 - \frac{c^2}{c_L^2}}} =& - \frac{4i\sqrt{1 - \frac{c^2}{c_L^2}}\sqrt{\frac{c^2}{c_T^2} - 1}}{\left[1 - \left(\frac{c^2}{c_T^2} - 1\right)\right]^2}, \\ \frac{\tan u\sqrt{\frac{c^2}{c_T^2} - 1}}{\tanh\left(u\sqrt{1 - \frac{c^2}{c_L^2}}\right)} =& \frac{u\sqrt{1 - \frac{c^2}{c_L^2}}\sqrt{\frac{c^2}{c_T^2} - 1}}{\left[1 - \left(\frac{c^2}{c_T^2} - 1\right)\right]^2}. \end{aligned}$$

Above equation can be written as

$$\frac{\tan(u\alpha)}{\tanh(u\beta)} = \gamma, \tag{3.41}$$

where

$$\begin{aligned} \alpha =& \sqrt{\frac{c^2}{c_T^2} - 1}, \\ \beta =& \sqrt{1 - \frac{c^2}{c_L^2}} \end{aligned}$$

and

$$\gamma = \frac{4\alpha\beta}{(1-\alpha^2)^2}.$$

The function $\tanh x$ is an increasing function and bounded above by 1. For any $u\beta > 0$, $1-\epsilon < \tanh(u\beta) < 1$, and as $u\beta$ becomes larger ϵ becomes smaller. For

$$\beta > 3, \tag{3.42}$$

we have $\epsilon < 0.005$ and we can use the approximation

$$\tanh(u\beta) \approx 1. \tag{3.43}$$

The above approximation is valid in most parts of the region $c_T < c < c_L$ because $u\beta > 3$ for the modes $S_i \geq 2$. Now equation (3.41) can be written as

$$\tan(u\alpha) = \gamma, \tag{3.44}$$

which has the solution

$$u_n = \frac{\tan^{-1}(\gamma) + n\pi}{\alpha}, n = 1, 2, 3.... \tag{3.45}$$

In the region where equation (3.43) is valid, accurate dispersion curves for plates can be drawn by using equation (3.45) which is an explicit solution of the Rayleigh-Lamb equation.

For validity of the approximation, let $\tanh(u\beta) = (1-\epsilon)$ then equation (3.41) becomes

$$\tan(u\alpha) = \gamma(1-\epsilon),$$

or

$$u_n = \frac{\tan^{-1}(\gamma(1-\epsilon)) + n\pi}{\alpha}$$

and the error du is

$$du = \frac{\gamma\epsilon}{\alpha(1+\gamma^2)} = \frac{4\beta\epsilon}{(1-\alpha^2)^2(1+\gamma^2)}.$$

Since ϵ is very small, so approximation (3.43) gives good results for the range $c_T < c < c_L$. Ahmad pointed out that the difference between the exact and approximated curves is less than 1% for S_1 mode and less than 4 parts in 5000 for S_2 mode. For higher modes the difference is invisible and curves overlap. The following figure shows the symmetric Lamb modes for isotropic aluminium plate. It is found that approximated curves slightly differ for S_1 and S_2 modes only. For higher modes the exact and approximated curves are same.

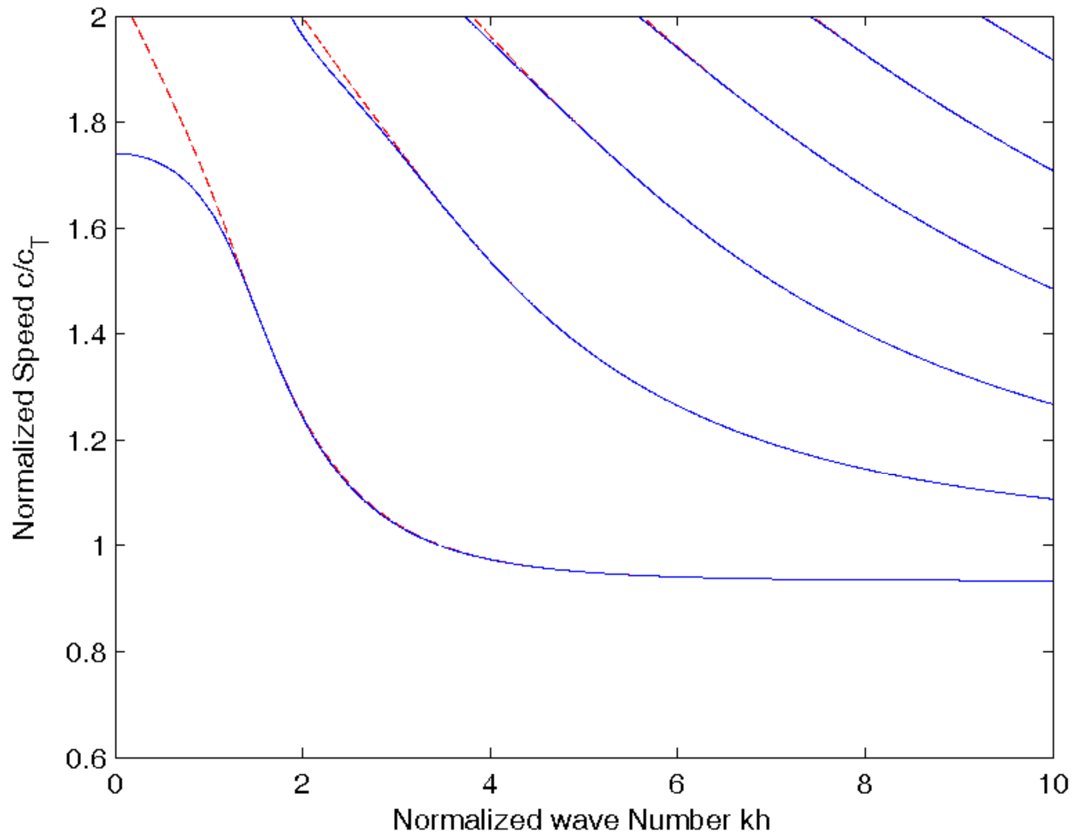

Figure 3.7: Shape of symmetric Lamb modes for exact and approximate formula for an aluminum plate. Solid lines show Lamb modes for exact formula and dotted lines show approximated Lamb modes.

3.3 Lamb modes in a cylinder

Consider a homogeneous, isotropic and linearly elastic circular rod of infinite length and radius a. The displacement equations of motion for cylinder can be written as

$$\nabla^2 u - \frac{u}{r^2} - \frac{2}{r^2}\frac{\partial v}{\partial \theta} + \frac{1}{1-2\nu}\frac{\partial \Delta}{\partial r} = \frac{1}{c_T^2}\frac{\partial^2 u}{\partial t^2}, \tag{3.46}$$

$$\nabla^2 v - \frac{v}{r^2} + \frac{2}{r^2}\frac{\partial u}{\partial \theta} + \frac{1}{1-2\nu}\frac{1}{r}\frac{\partial \Delta}{\partial \theta} = \frac{1}{c_T^2}\frac{\partial^2 v}{\partial t^2}, \tag{3.47}$$

$$\nabla^2 w + \frac{1}{1-2\nu}\frac{\partial \Delta}{\partial z} = \frac{1}{c_T^2}\frac{\partial^2 w}{\partial t^2}, \tag{3.48}$$

where Laplacian (∇^2) and dilatation (Δ) are defined as

$$\Delta = \frac{\partial u}{\partial r} + \frac{1}{r}\left(\frac{\partial v}{\partial \theta} + u\right) + \frac{\partial w}{\partial z}, \tag{3.49}$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}. \tag{3.50}$$

The displacement vector in cylindrical coordinates is

$$u = \frac{\partial \phi}{\partial r} + \frac{1}{r}\frac{\partial \psi_z}{\partial \theta} - \frac{\partial \psi_\theta}{\partial z}, \tag{3.51}$$

$$v = \frac{1}{r}\frac{\partial \phi}{\partial \theta} + \frac{\partial \psi_r}{\partial z} - \frac{\partial \psi_z}{\partial r}, \tag{3.52}$$

$$w = \frac{\partial \phi}{\partial z} + \frac{1}{r}\frac{\partial (\psi_\theta r)}{\partial r} - \frac{1}{r}\frac{\partial \psi_r}{\partial \theta}, \tag{3.53}$$

where scalar potential ϕ satisfies the equation

$$\nabla^2 \phi = \frac{1}{c_L^2}\frac{\partial^2 \phi}{\partial t^2}. \tag{3.54}$$

And the components of vector potential $\boldsymbol{\psi}$ satisfy the following equations

$$\nabla^2 \psi_r - \frac{\psi_r}{r^2} - \frac{2}{r^2}\frac{\partial \psi_\theta}{\partial \theta} = \frac{1}{c_T^2}\frac{\partial^2 \psi_r}{\partial t^2}, \tag{3.55}$$

$$\nabla^2 \psi_\theta - \frac{\psi_\theta}{r^2} + \frac{2}{r^2}\frac{\partial \psi_r}{\partial \theta} = \frac{1}{c_T^2}\frac{\partial^2 \psi_\theta}{\partial t^2}, \tag{3.56}$$

$$\nabla^2 \psi_z = \frac{1}{c_T^2}\frac{\partial^2 \psi_z}{\partial t^2}. \tag{3.57}$$

The components of vector potential must satisfy an additional condition $\boldsymbol{\nabla .\psi} = 0$. The relevant stress-strain relations are

$$\tau_{rr} = \lambda\Delta + 2\mu\frac{\partial u}{\partial r}, \tag{3.58}$$

$$\tau_{r\theta} = \mu\left[\frac{1}{r}\left(\frac{\partial u}{\partial \theta} - v\right) + \frac{\partial v}{\partial r}\right], \tag{3.59}$$

$$\tau_{rz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right). \tag{3.60}$$

By considering traction free condition at $r = a$, we have $\tau_{rr} = \tau_{r\theta} = \tau_{rz} = 0$.
Now consider the scalar potential ϕ. A wave propagating in the z-direction is of the form

$$\phi = \Phi(r)\Theta(\theta)\exp[i(kz - \omega t)] \tag{3.61}$$

Replacing equation (3.61) in equation (3.54) and applying method of separation of variables.

$$\frac{d^2\Phi}{dr^2}\Theta(\theta) + \frac{1}{r}\frac{d\Theta}{dr}\Theta(\theta) + \frac{1}{r^2}\frac{d^2\Theta}{d\theta^2}\Phi(r) - k^2\Phi(r)\Theta(\theta) = -\frac{\omega^2}{c_L^2}\Phi(r)\Theta(\theta). \tag{3.62}$$

Let

$$\frac{1}{\Phi(r)}\left(r^2\frac{d^2\Phi}{dr^2} + r\frac{d\Phi}{dr}\right) + r^2\left(\frac{\omega^2}{c_L^2} - k^2\right) = -\frac{1}{\Theta(\theta)}\frac{d^2\Theta}{d\theta^2} = \lambda, \tag{3.63}$$

$$\frac{d^2\Theta}{d\theta^2} + \lambda\Theta = 0, \tag{3.64}$$

$$\frac{1}{\Phi(r)}\left(r^2\frac{d^2\Phi}{dr^2} + r\frac{d\Phi}{dr}\right) + r^2\left(\frac{\omega^2}{c_L^2} - k^2\right) = \lambda. \tag{3.65}$$

Equation (3.64) has the solution

$$\Theta = A\cos\sqrt{\lambda}\theta + B\sin\sqrt{\lambda}\theta, \tag{3.66}$$

where $\sqrt{\lambda} = n$ is an integer.
Equation (3.65) becomes

$$\frac{1}{\Phi(r)}\left(r^2\frac{d^2\Phi}{dr^2} + r\frac{d\Phi}{dr}\right) + r^2\left(\frac{\omega^2}{c_L^2} - k^2\right) = n^2, \tag{3.67}$$

$$r^2\frac{d^2\Phi}{dr^2} + r\frac{d\Phi}{dr} + (r^2p^2 - n^2)\Phi(r) = 0, \tag{3.68}$$

which is a Bessel equation and has the Bessel function of first kind of order n, $J_n(pr)$ as a bounded solution. Using solutions of equations (3.64) and (3.65) in equation (3.61), we have the expression for scalar potential as

$$\phi = [A_1 \cos(n\theta) + A_2 \sin(n\theta)]\, J_n(pr) \exp[i(kz - \omega t)], \tag{3.69}$$

where $p = \sqrt{\frac{\omega^2}{c_L^2} - k^2}$.
The solution for the wave equation governing ψ_z can be obtained by same method and written as

$$\psi_z = [B_1 \cos(n\theta) + B_2 \sin(n\theta)] J_n(qr) \exp[i(kz - \omega t)], \tag{3.70}$$

where $q = \sqrt{\frac{\omega^2}{c_T^2} - k^2}$.
Now, coupled equations (3.55) and (3.56) indicate that sin-dependence on θ in ψ_r is in accordance with the cosin-dependence on θ in ψ_θ and vice versa. Consider

$$\psi_r = \Psi_r(r) \sin(n\theta) \exp[i(kz - \omega t)], \tag{3.71}$$
$$\psi_\theta = \Psi_\theta(r) \cos(n\theta) \exp[i(kz - \omega t)]. \tag{3.72}$$

Using above relations in equations (3.55) and (3.56), we get

$$\frac{d^2\Psi_r}{dr^2} + \frac{1}{r}\frac{d\Psi_r}{dr} + \frac{1}{r^2}(-n^2\Psi_r + 2n\Psi\theta - \Psi_r) - k^2\Psi_r + \frac{\omega^2}{c_T^2}\Psi_r = 0, \tag{3.73}$$
$$\frac{d^2\Psi_\theta}{dr^2} + \frac{1}{r}\frac{d\Psi_\theta}{dr} + \frac{1}{r^2}(-n^2\Psi_\theta + 2n\Psi r - \Psi_\theta) - k^2\Psi_\theta + \frac{\omega^2}{c_T^2}\Psi_\theta = 0. \tag{3.74}$$

Now by adding and subtracting the above two equations we have the solution for $\Psi_r + \Psi_\theta$ and $\Psi_r - \Psi_\theta$ as

$$\Psi_r + \Psi_\theta = 2C_1 J_{n+1}(qr), \tag{3.75}$$
$$\Psi_r - \Psi_\theta = 2C_2 J_{n+1}(qr). \tag{3.76}$$

From above equations we get the expressions for Ψ_r and Ψ_θ as

$$\Psi_r = C_1 J_{n-1}(qr) + C_2 J_{n+1}(qr), \tag{3.77}$$
$$\Psi_\theta = C_1 J_{n-1}(qr) - C_2 J_{n+1}(qr). \tag{3.78}$$

Thus we have determined the scalar and vector potentials in terms of four arbitrary constants but the displacement vector is specified by three constants and we have only three boundary conditions. To eliminate one constant, we required an additional condition $\Psi_r = -\Psi_\theta$, which implies $C_1 = 0$, and we have following set of potentials,

$$\phi = A_1 J_n(pr)\cos(n\theta)\exp[i(kz - \omega t)], \tag{3.79}$$

$$\psi_z = B_1 J_n(qr)\sin(n\theta)\exp[i(kz - \omega t)], \tag{3.80}$$

$$\psi_r = C_2 J_{n+1}(qr)\sin(n\theta)\exp[i(kz - \omega t)], \tag{3.81}$$

$$\psi_\theta = -C_2 J_{n+1}(qr)\cos(n\theta)\exp[i(kz - \omega t)]. \tag{3.82}$$

Using the above relations in stress equations and applying traction free boundary condition, we get three homogeneous equations and the frequency equation can be obtained by putting the determinant of coefficients equal to zero.

3.3.1 Frequency spectrum for longitudinal waves

The axially symmetric longitudinal waves are characterized by radial and axial displacements. Putting $n = 0$ in equations (3.79) – (3.82), we get

$$\phi = AJ_0(pr)\exp[i(kz - \omega t)], \tag{3.83}$$

$$\psi_\theta = CJ_1(qr)\exp[i(kz - \omega t)]. \tag{3.84}$$

The relations for radial and axial displacement are

$$u = \frac{\partial \phi}{\partial r} + \frac{1}{r}\frac{\partial \psi_z}{\partial \theta} - \frac{\partial \psi_\theta}{\partial z}, \tag{3.85}$$

$$w = \frac{\partial \phi}{\partial z} + \frac{1}{r}\frac{\partial(\psi_\theta r)}{\partial r} - \frac{1}{r}\frac{\partial \psi_r}{\partial \theta}. \tag{3.86}$$

Using above relations in relations for τ_r and τ_{rz} and applying traction free boundary condition. Then relation for τ_r at $r = a$ becomes

$$
\begin{aligned}
(\lambda+2\mu)\left[-p^2 J_1'(pa)A - ikqJ_1'(qa)C\right] + \frac{\lambda}{a}\left[-pJ_1(pa)A - ikJ_1(qa)C\right]& \\
+\lambda\left[-k^2 J_0(pa)A + ikqJ_0(qa)C\right] = 0,& \\
\left[-p^2(\lambda+2\mu)\left(J_0(pa) - \frac{J_1(pa)}{pa}\right) - \lambda\left(\frac{p}{a}J_1(pa) + k^2 J_0(pa)\right)\right] A& \\
+\left[-ikq(\lambda+2\mu)\left(J_0(qa) - \frac{J_1(qa)}{qa}\right) + ik\lambda\left(\frac{-1}{a}J_1(qa) + qJ_0(qa)\right)\right] C = 0,& \\
\left[-p^2(\lambda+2\mu)J_0(pa) + \frac{2\mu}{a}pJ_1(pa) - \lambda k^2 J_0(pa)\right] A& \\
+2\mu\left[-ikqJ_0(qa) + \frac{ik}{a}J_1(qa)\right] C = 0,& \\
\left[\frac{2\mu}{a}pJ_1(pa) + \rho(-p^2c_L^2 - k^2c_L^2 + 2c_T^2k^2)J_0(pa)\right] A& \\
+2\mu\left[-ikqJ_0(qa) + \frac{ik}{a}J_1(qa)\right] C = 0,& \\
\left[\frac{2\mu}{a}pJ_1(pa) - \mu(q^2-k^2)J_0(pa)\right] A + 2\mu\left[-ikqJ_0(qa) + \frac{ik}{a}J_1(qa)\right] C = 0,&
\end{aligned}
$$

$$
\left[\frac{p}{a}J_1(pa) - \frac{1}{2}(q^2-k^2)J_0(pa)\right] A + \left[-ikqJ_0(qa) + \frac{ik}{a}J_1(qa)\right] C = 0. \tag{3.87}
$$

Also relation for τ_{rz} vanishes at $r = a$.

$$
-ikpJ_1(pa)A + k^2 J_1(qa)C - ikpJ_1(pa)A - q^2 J_1(qa)C = 0, \tag{3.88}
$$

$$
\left[-2ikpJ_1(pa)\right] A + \left[(k^2-q^2)J_1(qa)\right] C = 0. \tag{3.89}
$$

Now the determinant of the coefficient of equations (3.87) and (3.89) must vanish, which yields,

$$
\begin{aligned}
\frac{1}{2}(q^2-k^2)^2 J_0(pa)J_1(qa) - \frac{p}{a}(q^2-k^2)J_1(pa)J_1(qa) -& \\
\left(-2pqk^2 J_0(qa)J_1(pa) + \frac{2p}{a}k^2 J_1(pa)J_1(qa)\right) = 0,&
\end{aligned} \tag{3.90}
$$

$$
(q^2-k^2)^2 J_0(pa)J_1(qa) - \frac{2p}{a}(q^2+k^2)J_1(pa)J_1(qa) + 4pqk^2 J_0(qa)J_1(pa) = 0, \tag{3.91}
$$

which is the Pochhammer frequency equation.

The following figure shows frequency as a function of wave number for $k = 1.8811$.

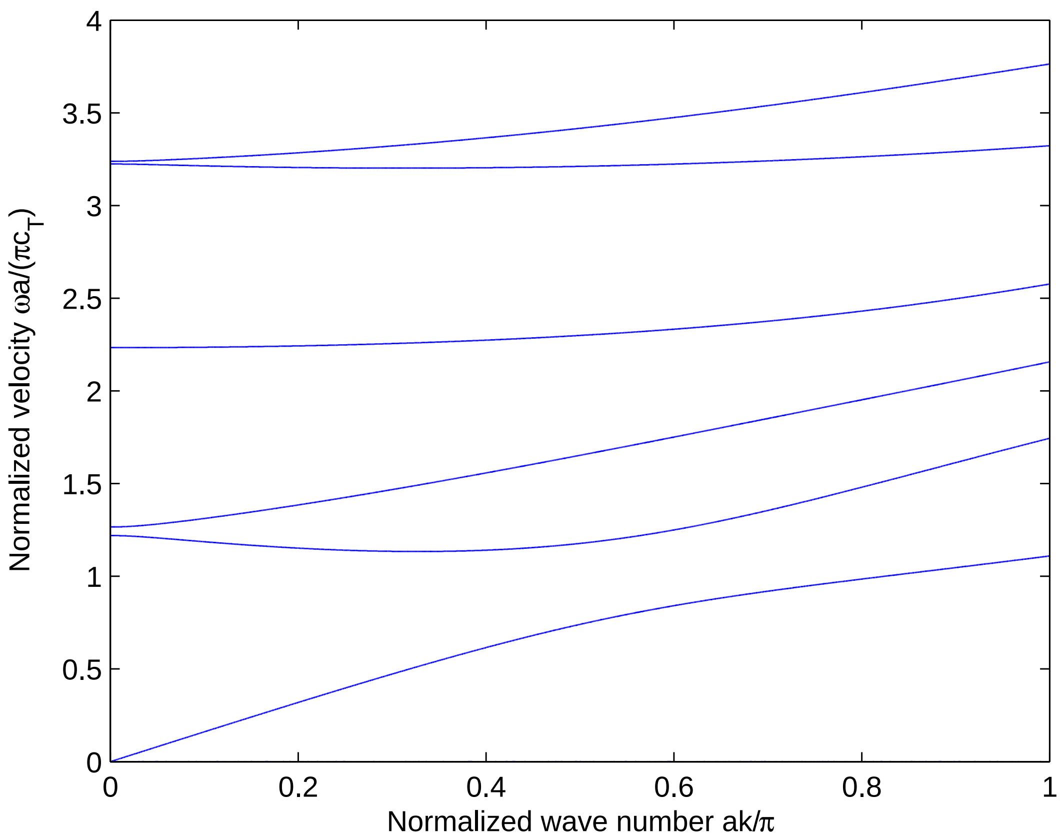

Figure 3.8: Shape of longitudinal modes in cylinder.

Chapter 4

Plateau Region

This chapter deals with the study of the plateau or flat region appearing in the spectrum of Lamb modes for plates and cylinders. When normalized phase speed is plotted in terms of normalized wave number then graph shows a region where slope of the curves become nearly zero is called plateau region. Mathematical reasoning for plateau region in a plate and a cylinder is described and existence of plateau region is graphically represented in this chapter.

4.1 Plateau region for a plate

Consider an isotropic plate of thickness $2h$. Let

$$p = \sqrt{\frac{\omega^2}{c_L^2} - k^2} \qquad \text{and} \qquad q = \sqrt{\frac{\omega^2}{c_{T^2}} - k^2},$$

where ω, k ,c_L and c_T denote frequency, wave number, speed of longitudinal mode and speed of transverse mode respectively. Dispersion relation for symmetric modes of plate is given by Achenbach [9] as

$$\frac{\tan(qh)}{\tan(ph)} = -\frac{4k^2pq}{(q^2-k^2)^2}, \tag{4.1}$$

$$(q^2-k^2)^2\tan(qh) = -4k^2pq\tan(ph), \tag{4.2}$$

$$(q^2-k^2)^2\sin(qh)\cos(ph) + 4k^2pq\sin(ph)\cos(qh) = 0, \tag{4.3}$$

$$(q^2h^2-k^2h^2)^2\sin(qh)\cos(ph) + 4(hk)^2(ph)(qh)\sin(ph)\cos(qh) = 0. \tag{4.4}$$

Let $\omega = ck$, $u = hk$, $p_1 = \sqrt{\frac{c^2}{c_L} - 1}$ and $q_1 = \sqrt{\frac{c^2}{{c_T}^2} - 1}$ then we have $ph = up_1$ and $qh = uq_1$. So equation (4.4) becomes

$$(u^2{q_1}^2 - u^2)^2 \sin(uq_1)\cos(up_1) + 4u^2(up_1)(uq_1)\sin(up_1)\cos(uq_1) = 0, \tag{4.5}$$
$$(q_1^2 - 1)^2 \sin(uq_1)\cos(up_1) + 4p_1q_1\sin(up_1)\cos(uq_1) = 0. \tag{4.6}$$

When $c = c_L$, $p_1 = 0$ the above equation becomes:

$$({q_1}^2 - 1)^2 \sin(uq_1) = 0, \tag{4.7}$$
$$u_n q_1 = \arcsin(0), \tag{4.8}$$
$$u_n q_1 = n\pi. \tag{4.9}$$

Denote the left side of equation (4.6) by $f(u, y)$. The slope of the mode at any point (u, c) on the curve is given by the formula

$$\frac{dc}{du} = -\frac{\frac{\partial f}{\partial u}}{\frac{\partial f}{\partial c}}, \tag{4.10}$$
$$\frac{\partial f}{\partial u} = h(u, c) + g(u, c), \tag{4.11}$$

where

$$h(u, c) = q_1({q_1}^2 - 1)^2 \cos(uq_1)\cos(up_1), \tag{4.12}$$

and

$$g(u, c) = 4p_1q_1[p_1\cos(up_1)\cos(uq_1) - q_1\sin(up_1)\sin(uq_1)] - p_1({q_1}^2 - 1)^2 \sin(up_1)\sin(uq_1). \tag{4.13}$$

When $c \approx c_L$, as a consequence $p_1 \approx 0$ then $g(u, c)$ will be small. Also $h(u, c)$ will vanish because $\cos(uq_1)$ must vanish between two consecutive zeroes of $\sin(uq_1)$. So the slop of a mode at any point (u, c) on the curve will be zero.
The following figure shows the plateau region shown by Lamb mode spectrum of a plate.

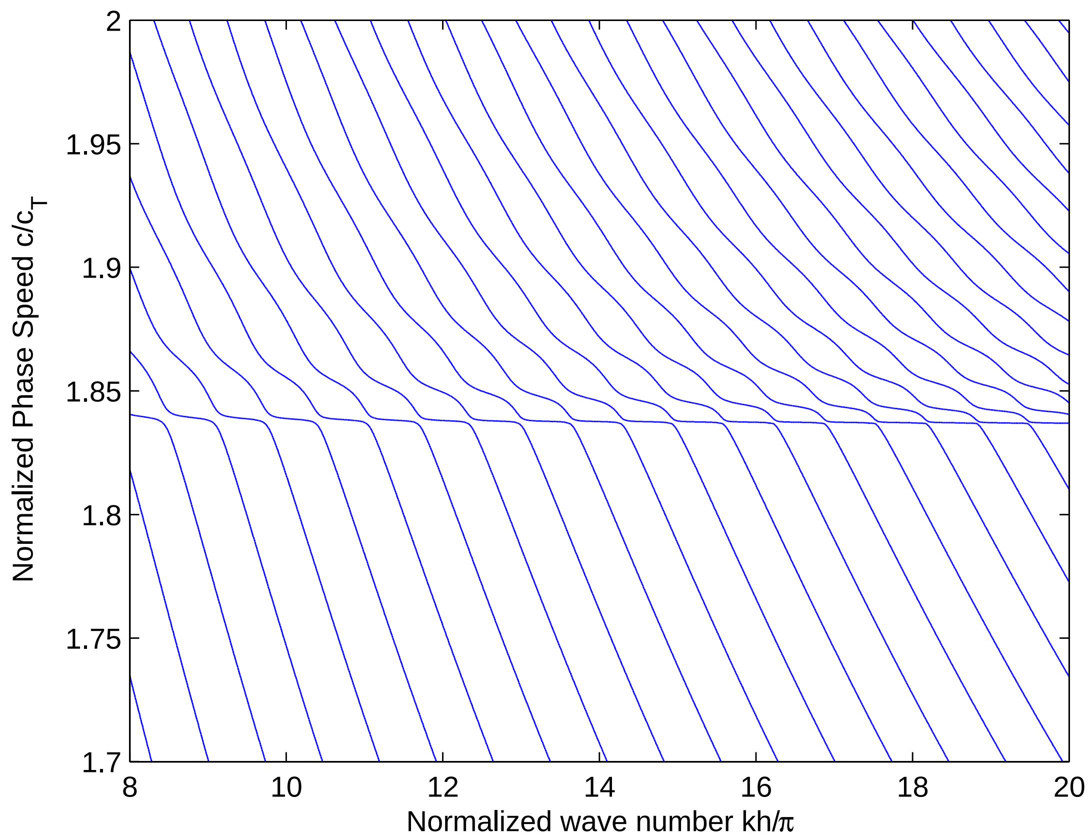

Figure 4.1: Plateau region in a steel plate.

The above figure shows that when $c = c_L$, then the slope of the dispersion curves become nearly equal to zero showing the plateau region for a steel plate.

4.2 Plateau region for a cylinder

Dispersion relation for longitudinal modes of a homogeneous and isotropic infinite circular rod of radius a is given by Achenbach[9] is given as,

$$\frac{2p}{a}(q^2 + k^2)J_1(pa)J_1(qa) - (q^2 - k^2)^2 J_0(pa)J_1(qa) - 4k^2 pqJ_1(pa)J_0(qa) = 0, \quad (4.14)$$

where $p = \sqrt{\frac{\omega^2}{c_L^2} - k^2}$ and $q = \sqrt{\frac{\omega^2}{c_{T^2}} - k^2}$.
Here ω, k, c_L, and c_T denote frequency, wave number, speed of longitudinal modes

and speed of transverse modes respectively.
Now multiplying equation (4.14) by a^4 on both sides. we get:

$$\begin{aligned} 2ap(a^2q^2+a^2k^2)J_1(pa)J_1(qa)-(a^2q^2-a^2k^2)^2J_0(pa)J_1(qa) \\ -4a^2k^2(ap)(aq)J_1(pa)J_0(qa)=0. \end{aligned} \tag{4.15}$$

Let $\omega = ck$, $u = hk$, $p_1 = \sqrt{\frac{c^2}{c_L}-1}$ and $q_1 = \sqrt{\frac{c^2}{c_T{}^2}-1}$ then we have $ph = up_1$ and $qh = uq_1$. So equation (4.15) becomes

$$\begin{aligned} 2up_1(u^2q_1^2+u^2)J_1(up_1)J_1(uq_1)-(u^2q_1^2-u^2)^2J_0(up_1)J_1(uq_1) \\ -4u^2(up_1)(uq_1)J_1(up_1)J_0(uq_1)=0, \\ 2u^3p_1\left(q_1^2+1\right)J_1(up_1)J_1(uq_1)-u^4(q_1^2-1)^2J_0(up_1)J_1(uq_1) \\ -4u^4p_1q_1J_1(up_1)J_0(uq_1)=0 \end{aligned}$$

and

$$2p_1\left(q_1^2+1\right)J_1(up_1)J_1(uq_1)-u\left(2-\frac{c^2}{c_T^2}\right)^2J_0(up_1)J_1(uq_1)-4up_1q_1J_1(up_1)J_0(uq_1)=0. \tag{4.16}$$

When $c = c_L$ then $p = 0$ and the modes cross the horizontal line $c = c_L$ such that

$$u_n\sqrt{\frac{c_L{}^2}{c_T{}^2}-1} = \qquad \text{n-th zero of } J_1(x) \qquad . \tag{4.17}$$

Let the left side of equation (4.16) is $f(u, y)$, then the slope of the curve at any point (u, c) is given as

$$\frac{dc}{du} = -\frac{\frac{\partial f}{\partial u}}{\frac{\partial f}{\partial c}}, \tag{4.18}$$

$$\frac{\partial f}{\partial u} = g(u,c)+h(u,c), \tag{4.19}$$

where

$$h(u,c) = -(2-\frac{c^2}{c_T^2})^2J_0(up_1)[(uq_1)J_1'(uq_1)+J_1(uq_1)], \tag{4.20}$$

and

$$g(u,c) = 2p_1(q_1^2+1)\frac{\partial}{\partial u}\left(J_1(up_1)J_1(uq_1)\right)-\left(2-\frac{c^2}{c_T^2}\right)^2uJ_0'(up_1)J_1(uq_1) \tag{4.21}$$

$$-4p_1q_1\frac{\partial}{\partial u}\left(J_1(up_1)J_0(uq_1)\right), \tag{4.22}$$

where ($'$) indicates the derivative with respect to argument. Now, $g(u, c)$ is small because $p \approx 0$ at $c = c_L$, and $h(u.c)$ will also be small because of the following:

Lemma 4.2.1. *The function $xJ_n'(x) + J_n(x)$ vanishes at least once between two consecutive zeros of $J_n(x)$.*

Proof. Let x_k and x_{k+1} are two consecutive zeroes of $J_n(x)$. Since $J_n(x)$ vanishes at these two points, so $J_n'(x_i) \neq 0$ for $i = k, k+1$. This is because vanishing of both $J_n(x)$ and its derivative at any point will imply $J_n(x) \equiv 0$, which is false. Hence without lost of generality, we can assume $J_n(x) > 0$, $x_k < x < x_{k+1}$, which implies $J_n'(x_k) > 0$ and $J_n'(x_{k+1}) < 0$. Now define

$$f(x) = xJ_n'(x_k) + J_n(x).$$

Since $f(x_k) > 0$ and $f(x_{k+1}) < 0$, so lemma is proved. $\square$

The following figure shows the plateau region in cylinder.

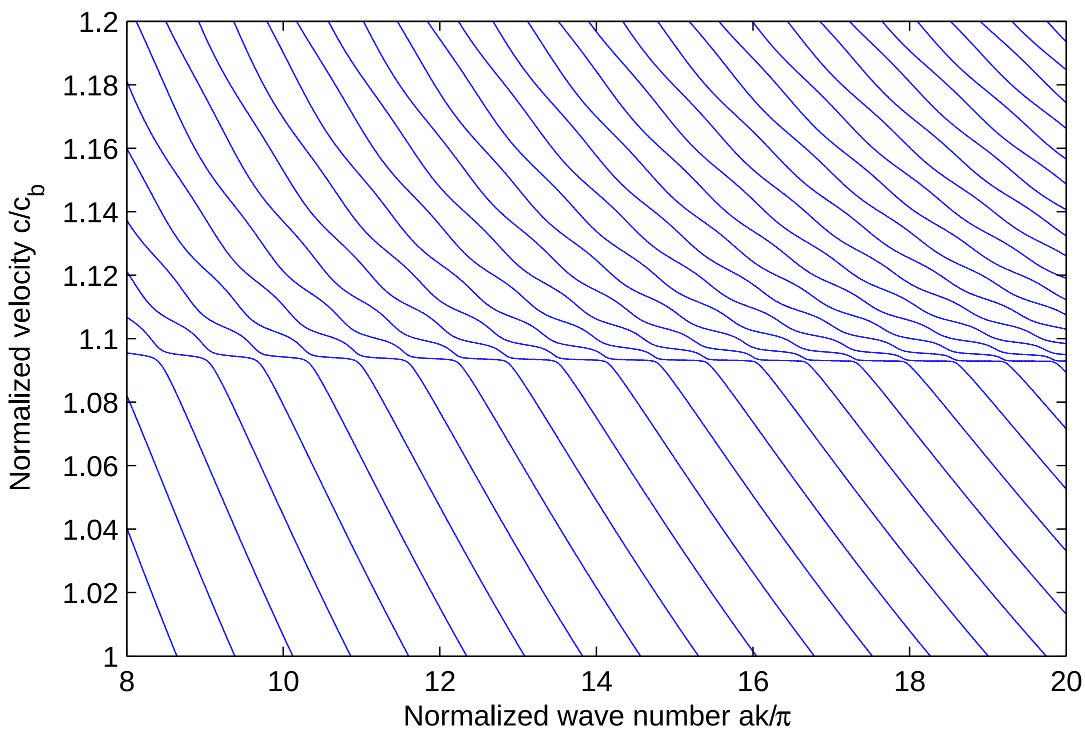

Figure 4.2: Plateau region in a cobalt cylinder.

Chapter 5

Zero Group Velocity

Lamb modes are the dispersion curves showing a relation between phase velocity and frequency or wave number. Lamb modes show an anomalous behavior where group velocity becomes zero for finite wavenumber, when frequency×thickness versus thickness/wavelength is plotted. This anomalous behavior is studied in this chapter.

Group velocity at any point of a dispersion curve can be obtained by measuring the slope of the curve at that point. A Lamb mode has a zero group velocity point if it undergoes a change in sign of the slope. The point at which slope of the curve becomes zero is called zero group velocity point. Tolstoy and Usdin [6] pointed out that the group velocity vanishes at a particular point of dispersion curve for S_1 Lamb mode. Other Lamb modes can also have zero group velocity points. Existence of zero group velocity points depend on the mechanical properties of the plate.

Zero group velocity points are of great physical importance because at these points energy does not propagate in the plane of the plate and remains trapped under the source, which provides strong and easily detectable resonance frequency of plate. The sharp resonance effect was first observed with the symmetric Lamb mode of first order S_1. S_1-ZGV can be excited by a laser pulse. Mechanical vibrations involving the S_1 Lamb mode at the ZGV point are longitudinal. And transverse vibrations can be observed for antisymmetric Lamb mode A_2 by using a laser source. Zero group velocity points also exhibit the property of separating forward and backward propagating Lamb waves. Backward propagation of waves occur in the region where the slope of the dispersion curve is zero that is group velocity and phase velocity have opposite signs.

ZGV modes have many applications like characterization of plates and cylindrical structures, measurement of thickness of plate, determination of poisson's ratio of materials etc. Measurement of both transverse and longitudinal speed is an important feature of ZGV resonance. Although ZGV frequencies can be used to determine the material properties, but their use in material characterization is not straight forward. This is because we can not easily access their location in $k-\omega$ space. Lamb waves dispersion curves can be normalized by thickness of plate and wave velocity, leaving the dimensionless Poisson's ratio as the only parameter governing shape of spectrum.

5.1 Zero group velocity modes of a plate

Zero group velocity for symmetric mode:

Consider an isotropic and infinite plate of thickness $2h$. The Rayleigh-Lamb dispersion relation for symmetric modes is given by Achenbach [9] as:

$$\frac{\tan(qh)}{\tan(ph)} = -\frac{4pqk^2}{(q^2-k^2)^2},$$

$$f_1 = (q^2-k^2)^2 \sin(qh)\cos(ph) + 4pqk^2 \sin(ph)\cos(qh) = 0, \tag{5.1}$$

where

$$p^2 = \frac{\omega^2}{c_T^2} - k^2,$$
$$q^2 = \frac{\omega^2}{c_L^2} - k^2.$$

In the above relation ω and k are respectively the frequency and wave number of the wave. c_L and c_T denote the speed of the longitudinal and transvers waves respectively.

The propagation of Lamb modes can be represented by the set of dispersion curves. To draw the dispersion curves showing relation between wave number k

and frequency ω for symmetric Lamb modes of an infinite isotropic plate, we choose $x = \frac{kh}{\pi}$ and $y = \frac{\omega h}{\pi}$. The dispersion relation (5.1) becomes

$$(h^2q^2 - x^2\pi^2)^2 \sin(qh)\cos(ph) + 4(ph)(qh)(x^2\pi^2)\sin(ph)\cos(qh) = 0, \tag{5.2}$$

where

$$ph = \pi\sqrt{\frac{y^2}{c_L^2} - x^2},$$
$$qh = \pi\sqrt{\frac{y^2}{c_T^2} - x^2}.$$

The dispersion curves are shown in the following figure.

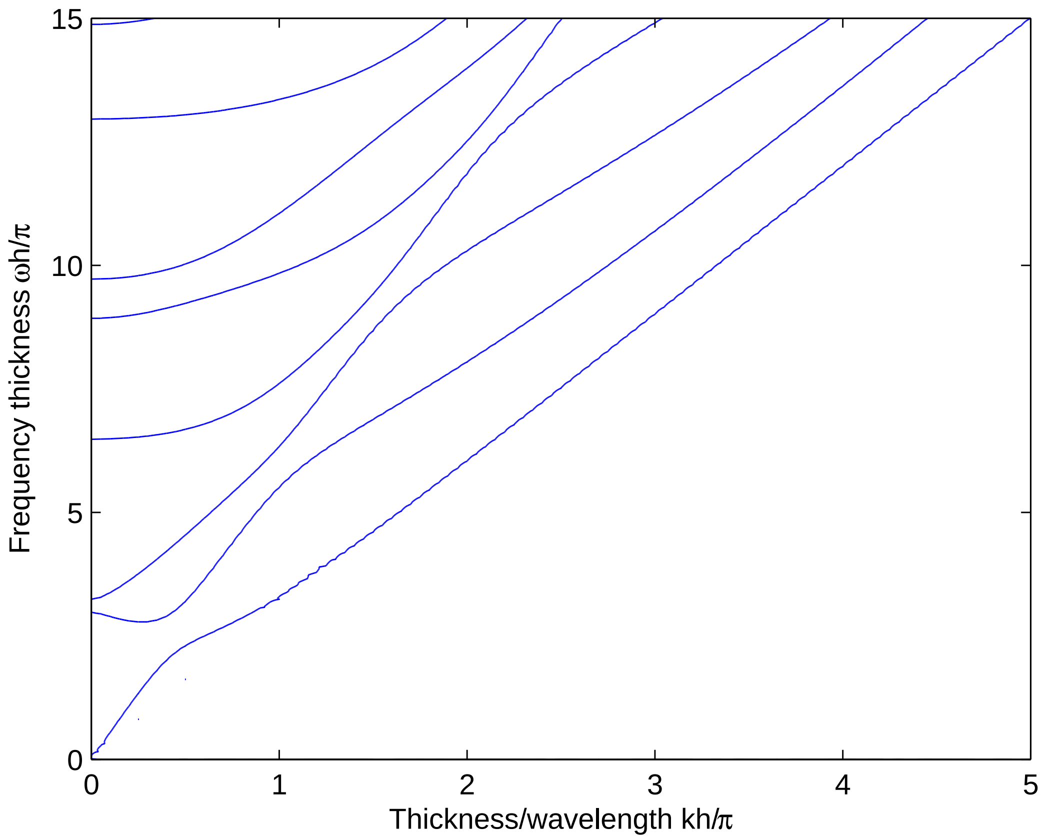

Figure 5.1: Symmetric Lamb mode dispersion curves for a plate of thickness $2h$ with longitudinal velocity $c_L = 5.95km/s$ and transverse velocity $c_T = 3.24km/s$.

In the above figure we have plotted the frequency×thickness ($\frac{\omega h}{\pi}$) versus thickness/ wavelength ($\frac{kh}{\pi}$), which shows the dispersion curves for the symmetric modes propagating in a plate. The most striking feature of the above figure is that S_1 mode exhibits an anomalous behavior at point where the group velocity $\frac{d\omega}{dk}$ is zero while the phase velocity $\frac{\omega}{k}$ remains finite.

Zero group velocity for antisymmetric mode:

The dispersion relation for antisymmetric Lamb mode is

$$\frac{\tan(qh)}{\tan(ph)} = -\frac{(q^2-k^2)^2}{4k^2pq},$$

$$f_2 = (q^2-k^2)^2 \sin(ph)\cos(qh) + 4k^2pq\cos(ph)\sin(qh) = 0, \tag{5.3}$$

where p and q are the same as defined for symmetric mode. We have taken $x = \frac{kh}{\pi}$ and $y = \frac{\omega h}{\pi}$ and drawn the graph for antisymmetric Lamb modes as shown in the following figure.

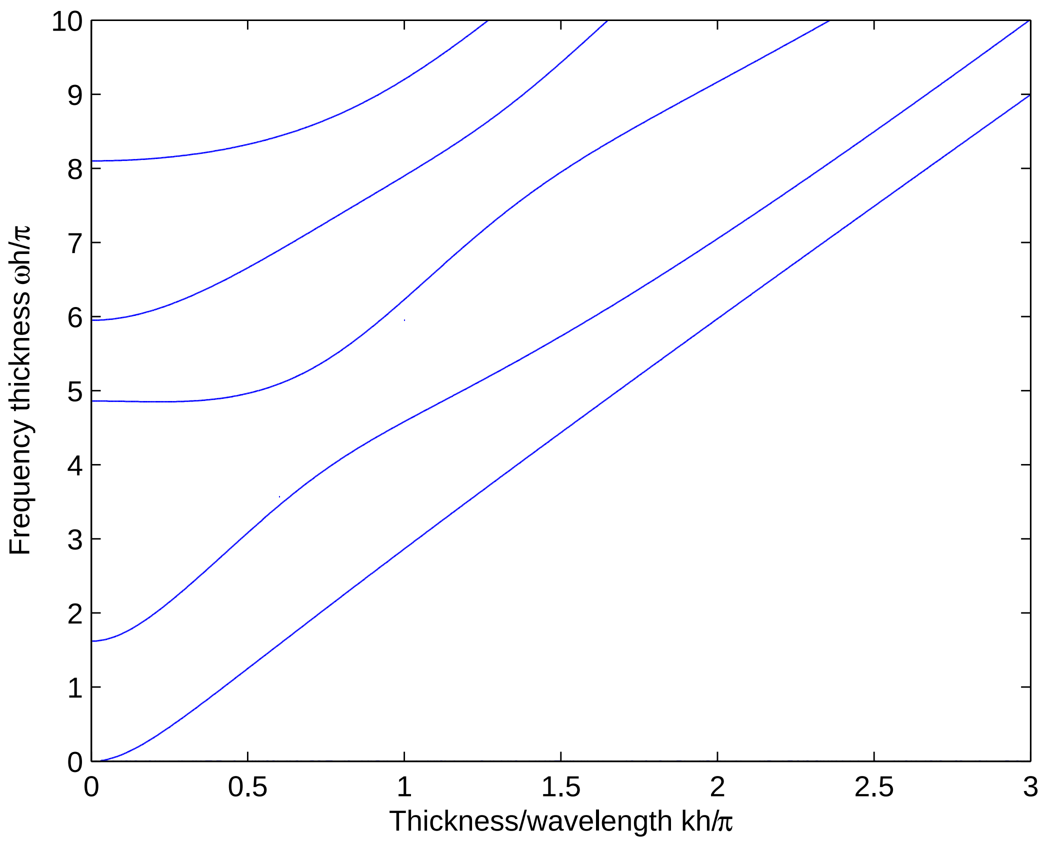

Figure 5.2: Antisymmetric Lamb mode dispersion curves for a plate of thickness $2h$ with longitudinal velocity $c_L = 5.95km/s$ and transverse velocity $c_T = 3.24km/s$.

The above figure shows a relation between frequency and wave number for antisymmetric Lamb mode of a plate of thickness $2h$ with $k = \frac{c_L}{c_T} = 1.8364$. We can see that antisymmetric mode A_2 exhibit a zero group velocity point for a finite wave number.

Clorennec et al [8] demonstrated that the accurate value of poisson's ratio ν in the plates or shells can be determined by measuring two zero group velocity resonance frequencies simultaneously. And if thickness of the plate is known than we can also find longitudinal velocity c_L and transverse velocity c_T. They measured the Poisson's ratio by taking the ratio of the S_1S_2 and A_2A_3 ZGV resonance.

Grünsteidl et al [11] described a method for inverse characterization of a plat using ZGV Lamb modes. They used the experimentally measured ZGV frequencies of S_1 and A_2 modes for thin aluminium and tungsten plates to characterize these materials. Their approach can be summarized as follows. Define parameters $kh = u$ and $fh = \omega$. Also let the locations of the S_1 and A_2 resonance in the Lamb mode spectrum be (u_1, ω_1) and (u_2, ω_2) respectively. Then equation (5.1) can be written as

$$f_1(u_1, \omega_1, c_T, c_L) = 0. \tag{5.4}$$

At zero group velocity point $\frac{\partial \omega_1}{\partial u_1} = 0$ which requires

$$\frac{\partial f_1}{\partial u_1} = 0. \tag{5.5}$$

And the equations corresponding to the A_2 resonance are

$$f_2(u_2, \omega_2, c_T, c_L) = 0, \tag{5.6}$$

$$\frac{\partial f_2}{\partial u_2} = 0. \tag{5.7}$$

The set of equations (5.4)-(5.7) contains six parameters $u_1, u_2, \omega_1, \omega_2, c_T, c_L$. By substituting experimentally measured values of resonance frequencies ω_1 and ω_2, we are left with the set of four equations containing four parameters which can be numerically solved. The solution given by Grünsteidl et al is given in the following table.

Material	ω_1 MHzmm	ω_2 MHzmm	c_T m/sec	c_L m/sec	u_1	u_2	Poisson ratio
Tungsten	2.248	3.997	2678	4785	1.725	1.667	0.272
Aluminium	2.750	4.587	3058	6021	1.625	0	0.326

Table 5.1: Solution of Grünsteidl et al.

The solution with vanishing u_1 or u_2 corresponds to thickness resonance and solution with non zero u is a zero group velocity resonance. But the problem is that the solution given by Grünsteidl et al is not unique. For a given pair of resonance frequencies ω_1 and ω_2, the following cases can arise.

1. Both are thickness resonances.
2. One is a thickness and the other is a ZGV resonance.
3. Both are ZGV resonances.

Each case will produce a distinct set of material properties. Since there is no prior knowledge of these parameters so the calculations will lead to a set of possible answers only one of which will represent the required solution. The possible solutions for the Aluminium plate with the frequencies $\omega_1 = 2.750$ MHzmm and $\omega_2 = 4.587$ MHzmm is given in the following table.

	u_1	u_2	c_L m/sec	c_T m/sec	Poisson ratio
1	1.625	0	6021	3058	0.326
2	0	1.649	5500	3072.5	0.273
3	0.958	3.7136	7761	2763	0.427
4	0	0	5500	3058	0.276

Table 5.2: Multiple solutions

A similar table can be constructed for the tungsten plate. In order to characterize the material one would need to know whether the mode is thickness or zero group velocity mode.

Chapter 6

Conclusion

We have studied the two well-known dispersion relations i.e. the Rayleigh-Lamb relation for a plate and the Pochhammer relation for a cylinder. Also a MATLAB program is developed to draw the dispersion curves associated with these relations.

We have discussed an approximate formula to draw the dispersion curves for a plate in the interval $c_T < c < c_L$. Curves produced by the approximate formula agree very well with those of exact relation when phase velocity is between c_T and c_L.

We have also examined the behavior of Lamb modes in a plate and a cylinder when $c \approx c_L$. And found that the Lamb modes spectrum shows a region where slope of the curves become nearly zero giving the appearance of plateau region. The existence of plateau region is theoretically explained.

We have studied the anomalous behavior shown by the Lamb modes spectrum when frequency$\times$ thickness versus thickness/wavelength is plotted. This is because of the existence of zero group velocity points. Such points are of great physical importance.

Bibliography

[1] D. Worlton, "Experimental Confirmation of Lamb Waves at Megacycle Frequencies", Journal of Applied Physisc, **32**, $967 - 971$, (1961).

[2] H. Lamb, "On waves in an elastic plate", Proceedings of the Royal Society of London Series, **93**, $114 - 128$, (1917).

[3] J. Rayleigh, *The Theory of Sound*, Dover, New York, (1945).

[4] L. Pochhammer, "Ueber die Fortpflanzungsgeschwindigkeiten kleiner Schwingungen in einem unbegrenzten isotropen Kreiscylinder", Journal fr die reine und angewandte Mathematik, **81**, $324 - 336$, (1876).

[5] C. Chree, "The equations of an isotropic elastic solid in polar and cylindrical coordinates, their solutions and applications", Trans. Camb. Phil. Soc. Math. Phys. Sci., **14**, $250 - 369$, (1889).

[6] I. Tolstoy and E. Usdin, "Wave propagating in elastic plates: low and high mode dispersion," J.Acoust. Soc. Am. **29**, 37 – 42, (1957).

[7] F. Ahmad and T. Hussain, "Anomalous Dispersion of the S_1 Lamb Mode", Advances in Acoustics and Vibration, **2013**, 903934, (2013).

[8] D. Clorennec, Claire Prada, and Daniel Royer, "Local and noncontact measurements of bulk acoustic wave velocities in thin isotropic plates and shells using zero group velocity", Journal of Applied Physisc, **101**, 034908, (2007).

[9] J. D. Achenbach, *Waves Propagation in Elastic Solids*, North Holland, New York, (1980).

[10] F.Ahmad, "A simple formula for the Lamb modes in a plate", Acoustical society of America, **111**, (2002).

[11] C. Grünsteidl, T. W. Murry, T. Berer, and I. A. Veres, "Inverse characterization of plates using zero group velocity Lamb modes", Ultrasonics, **65**, 1 − 4, (2016).