# Exponential Convexity of Jensen's Functionals

by

**Iqra Zulfiqar**

A thesis submitted in partial fulfillment of the requirements

for the degree of Master of Science in Mathematics

Supervised by

**Dr. Matloob Anwar**

**Department of Mathematics**

National University of Sciences and Technology

Islamabad, Pakistan

August 2017

FORM TH-4

# National University of Sciences & Technology

## MASTER'S THESIS WORK

We hereby recommend that the dissertation prepared under our supervision by: Ms. Iqra Zulfiqar, Regn No. 00000117080 Titled: Exponential Convexity of Jensen's Functionals be accepted in partial fulfillment of the requirements for the award of **MS** degree.

## Examination Committee Members

1. Name: Dr. Rashid Farooq Signature: ____________
2. Name: Dr. Mujeeb ur Rehman Signature: ____________
3. Name: ____________ Signature: ____________
4. Name: Dr. Tayyab Kamran Signature: ____________

Supervisor's Name: Dr. Matloob Anwar Signature: ____________

Head of Department

28-08-2017
Date

**COUNTERSINGED**

Date: 28/8/17

**Dean/Principal**

# THESIS ACCEPTANCE CERTIFICATE

Certified that final copy of MS thesis written by **Ms. Iqra Zulfiqar**, (Registration No. **00000117080**), of **School of Natural Sciences** has been vetted by undersigned, found complete in all respects as per NUST statutes/regulations, is free of plagiarism, errors, and mistakes and is accepted as partial fulfillment for award of MS/M.Phil degree. It is further certified that necessary amendments as pointed out by GEC members and external examiner of the scholar have also been incorporated in the said thesis.

Signature: ______________

Name of Supervisor Dr. Matloob Anwar

Date: 28-08-2017

Signature (HoD): ______________

Date: 28-08-2017

Signature (Dean/Principal): ______________

Date: 28/8/17

# Dedicated to

*My loving Parents.*

## Abstract

As the title indicates, arguments involving convexity play a crucial role throughout this thesis. For more than a hundred years, convexity has acquired a unique position. Numerous branches of mathematics have extensive application of convex sets properties and functions. Convexity has gained its popularity due to its simplified or easy and intuitive geometric appeal. Convexity has a definite interdisciplinary position. In probability and statistics, the use of convexity to prove Jensen's inequality is standard textbook material.

This thesis is an attempt to bring together refinement of functional form of Jensen's inequality from 1931 to up until recently. This research attempts to develop new operators from the functional form of Jensen's inequality.

Considering significance of Jensen's inequality in multidisciplinary perspective, this thesis intends to analyze this subject comprehensively. For instance, functional form of Jensen's inequality demonstrates few essential features that are satisfied for newly developed operators.

First chapter lays out the basic introduction to the research. Chapter 2 consists of generalization of convex function and presentation of Jensen's inequality for the specified generalized convex function. Further it contains generalization of Čebyšev identity and inequality. Whereas, Chapter 3 presents the affine and functional form of Jensen's inequality. Jensen's inequality appears in many forms as another known form is affine form which has been cited in this thesis. To conclude the thesis, a question is being raised whether the results satisfied through derived operators will be satisfied by affine form of Jensen's inequality or any other form of Jensen's inequality.

# Acknowledgement

My first thanks to the Lord of heavens and earth. To Whom belongs all knowledge and intellect. I thank Him for making this possible. I am thankful to my supervisor Dr. Matloob Anwar for paving this path for me. His utter guidance, kindness and precious time made this whole process look easier and vivid. I owe to thank him for being the sole guide in my research work. I owe a lot to my family and friends for constantly ushering me to concentrate and progress in my research. Their reminders, constant support and continuous encouragement made me get through all frustrating and troublesome phases of each and every research work. Moreover, I thank all my teachers who taught me the skills to research, analyze, review, read and write.

All of the mistakes, negligence, delays, ignorance in this work belongs to me solely and I totally take responsibility for that. I believe that attributing mistakes as experiences is a form of cowardice, hence I open heartedly accept my faults and dont feel guilty. These mistakes will eventually lead me to a new avenue of success someday as they original, and a profound work in good faith.

Iqra Zulfiqar

# Contents

# Chapter 1

# Introduction and Preliminaries

## 1.1 Convexity

Originally convexity belongs to geometry but it is wide spread in other mathematical fields simultaneously like calculus of variation, functional analysis, graph theory, probability theory, complex analysis and many other fields. Moreover, convexity has significant interdisciplinary features and occupies essential position in the field of chemistry, biology and other sciences. However, this aspect of convexity would not be the main focus in this research.

A short historical background of convexity is being given here. Convexity has a history going back to Greek, Egypt and Babylonian times. It is assumed that it is quite younger than numbers but the basic geometric drawings are traced to the initial stages of human civilization. It is difficult to ascertain the first person who first defined convexity. Supposedly "Archimedes" was the first one to define "Convexity". His definitions and postulates remained in the dark for almost two thousand years. Though, the mathematical experts were aware of these. Till $17th$ centaury calculus was at a primitive stage and convexity was not take as a priority.

Convex functions and theories of inequalities have a very close relation. Convexity is a broad subject which also includes theory of convex functions. Convexity is a very powerful property of function. It is known as a natural property of functions. Furthermore, its minimization property makes it unique, novel and beneficial. Due to its minimization characteristic it possess a significant status in optimization theory, calculus of variation and probability theory.

There are number of defining parameters for convex functions. There is no limitation to adopt or choose the suitable definition, according to the context, to evaluate the convexity of a certain given function. A set X is said to be convex set if for any pair of points $x_1$, $x_2 \in X$, line segment joining these points must contain in $X$ that means for all $x_1$, $x_2 \in X$ and for any $\lambda$ such that $0 \leq \lambda \leq 1$ we obtain

$$\lambda x_1 + (1 - \lambda)x_2 \in X.$$

**Definition 1.1.1.** [37] A function $\Psi$ is supposed to be convex on an interval $[c, d]$ if for any pair of points $x_1$, $x_2 \in [a, b]$ and for any $\lambda$ such that $0 \leq \lambda \leq 1$, we have

$$\Psi[\lambda x_1 + (1 - \lambda)x_2] \leq \lambda\Psi(x_1) + (1 - \lambda)\Psi(x_2).$$

A function is strictly convex if the above inequality is strict. If the inequality reversed in above equation, then $\Psi$ is said to be concave function. And it is said to be Affine if

$$\Psi[\lambda x_1 + (1 - \lambda)x_2] = \lambda\Psi(x_1) + (1 - \lambda)\Psi(x_2)$$

holds.

Graphical interpretation of convex function is the line segment for every pair of points must lie on or above the function's graph [41]. It is not always possible to check convexity or concavity by plotting their graphs. So there is another suitable way to check convexity or concavity through second derivative.

**Remark 1.1.1.** [41] If $\Psi$ is twice differentiable on interval $[c, d]$, then a necessary and sufficient condition for function to be convex is that the second derivative is greater than or equal to zero for all $x \in [c, d]$.

## 1.2 Jensen's Inequality

The classical literature of mathematics involves a comprehensive study of the inequalities which is used excessively in mathematics. The critical analysis of inequalities demonstrate the novel features of current mathematics. "Inequalities" by Hardy et all was published in 1934 [12]. This book describes the inequalities in a very efficient and sophisticated manner. These famous mathematicians not only explained and demonstrated this subject with its due moreover they made "inequalities" popular among their peers. "An Introduction to Inequalities" by Beckenbach and Bellman brings forth a well described, brief and comprehensive introduction to inequalities in 1975 [5].

Jensen's inequality is supposed to be the most significant inequality. Its status depends on its vast horizon for modern mathematics and statistics. It is a mechanism to form several classical inequalities. Any improvement, generalization and advancement in Jensen's inequality brings the same to other classical inequalities.

According to the context, Jensen's inequality takes several different forms. In simple words, Jensen's inequality demonstrates that the convex transformation of a mean is less than or equal to the mean applied after convex transformation. Jensen's inequality can be described in other form such as it generalizes the secant line lying above the graph of the convex function and it is the Jensen's inequality for two points.

**Definition 1.2.1.** [11] Lets suppose $\alpha_1, \alpha_2 .....\alpha_n$ are positive numbers such that

$\sum_{i=1}^{n} \alpha_i = 1$ and $g$ is real valued continuous convex function, then

$$g\left[\sum_{i=1}^{n} \alpha_i x_i\right] \leq \sum_{i=1}^{n} \alpha_i g(x_i).$$

If $f$ is concave, then the inequality reverses.

From the perspective of probability theory when a convex function is applied to the expected value of a random variable is always less than or equal to the expected value of the convex function of the random variable, this result is also known as Jensen's inequality.

## 1.3 Exponential Convexity

From now on, $\mathcal{J}$ represents an open interval of $\mathbb{R}$, where $\mathbb{R}$ is set of real numbers.

**Definition 1.3.1.** [7] A function $\psi : \mathcal{J} \longrightarrow \mathbb{R}$ is exponentially convex on $\mathcal{J}$ if it is continuous and

$$\sum_{i,j=1}^{n} \xi_i \xi_j \psi(x_i + x_j) \geq 0,$$

for all $n \in \mathbb{N}$, where $\mathbb{N}$ is set of natural numbers and for every $\xi_i, \xi_j \in \mathbb{R}; i, j = 1, 2, ......., n$ in such a way $x_i + x_j \in \mathcal{J}; \quad 1 \leq i, j \leq n$.

The proposition described below has been taken from [1], which shows an equivalent relation between continuity and exponential convexity.

**Proposition 1.3.1.** [1] *Let $\psi : \mathcal{J} \longrightarrow \mathbb{R}$ the following propositions are equivalent:*

1. *$\psi$ is exponentially convex in $\mathcal{J}$.*

2. *$\psi$ is continuous and*

$$\sum_{i,j=1}^{n} \xi_i \xi_j \psi\left(\frac{x_i + x_j}{2}\right) \geq 0,$$

*for all* $\xi_i, \xi_j \in \mathbb{R}$ *and every* $x_i, x_j \in \mathcal{J}; \quad 1 \le i, j \le n.$

The below mentioned corollary is given in [14], which explains the interesting relation of matrix with the effect of exponential convexity.

**Corollary 1.3.2.** [14] *If* $\psi$ *is exponentially convex function on* $\mathcal{J}$, *then the matrix*

$$\left[\psi(\frac{x_i + x_j}{2})\right]_{i,j=1}^{n} \tag{1.3.1}$$

*is a positive semi-definite matrix. Particularly*

$$det\left[\psi(\frac{x_i + x_j}{2})\right]_{i,j=1}^{n} \ge 0, \tag{1.3.2}$$

*for all* $n \in \mathbb{N}, \quad x_i, x_j \in \mathcal{J}; \quad i, j = 1, 2, ......., n.$

## 1.4 Log-Convex Function

**Definition 1.4.1.** [11] A positive real valued function $g$ is said to be logarithmical convex on interval $[c, d]$ if the composition of logarithmic function with $g$ is a convex function.

Additionally, the corollary which is mentioned below describes the relation between log-convexity and exponential convexity.

**Corollary 1.4.1.** [14] *If* $\psi : \mathcal{J} \longrightarrow (0, \infty)$ *is exponentially convex function, then* $\psi$ *is a log-convex function i.e. for every* $x, y \in \mathcal{J}$ *and every* $\lambda \in [0, 1]$, *we obtain*

$$\psi(\lambda x + (1 - \lambda)y) \le \psi^{\lambda}(x)\psi^{1-\lambda}(y). \tag{1.4.1}$$

# Chapter 2

# Generalization of Convex Function and $\check{C}$eby$\check{s}$ev Identity and Inequality

## 2.1 Generalization of Convex Function

In applied mathematics particularly nonlinear programming and optimization theory, convex functions, convexity and its generalization have a significant position. A number of interdisciplinary applications of this particular property can be observed. In equilibrium and duality theory of Economics convexity has a direct relation. This interesting phenomenon has been studied with several different aspects. But several contemporary issues with respect to notion of convexity make it necessary to further extent its generalization. A deeper look into these modern extension bring forth different dimensions of latest research on convexity. These include an aspect of domain extension to a generalized form or of extending the definition without extending the domain. Some new models in this respect are being mentioned here: pseudo-convex functions [21], quasi-convex functions [2], invex functions [13], preinvex functions

[24], B-vex functions [18], B-preinvex functions [6] and E -convex functions [40].

Here the concept of $\phi$-convex function as generalization of convex functions has been presented. In this chapter generalization of convex function particularly as a $\phi$-convex function has been described. Furthermore a brief introduction of Jensons inequality for the said function will be provided. Subsequently an account of $\phi$-convex function, its generalization as $\phi_d$-convex and $\phi_F$-convex function will be discussed.

### 2.1.1 $\phi$-Convex Function

For a unanimous approach we consider $K$ is an interval in real line $\mathbb{R}$ and $\phi : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is a bifunction excluding specific circumstances.

**Definition 2.1.1.** [9] A function $h : K \longrightarrow \mathbb{R}$ is said to be $\phi$-convex function, if

$$h(\mu x_1 + (1-\mu)x_2) \leq h(x_2) + \mu\phi(h(x_1), h(x_2)), \tag{2.1.1}$$

for every $x_1, x_2 \in K$ and $\mu \in [0, 1]$. Moreover, $\phi$-quasicovex function $h$ satisfies the given condition

$$h(\mu x_1 + (1-\mu)x_2) \leq \max\{h(x_2), h(x_2) + \mu\phi(h(x_1), h(x_2))\}, \tag{2.1.2}$$

for every $x_1, x_2 \in K$ and $\mu \in [0, 1]$. Furthermore, $\phi$-affine $function$ $h$ is achieved if

$$h(\mu x_1 + (1-\mu)x_2) = h(x_2) + \mu\phi(h(x_1), h(x_2)), \tag{2.1.3}$$

for every $x_1, x_2 \in K$ and $\mu \in \mathbb{R}$.

Some essentials to describe function $\phi$ are being given below

**Definition 2.1.2.** A function $\phi$ is considered

**(a)** nonnegatively homogenous if $\phi(\kappa u, \kappa v) = \kappa\phi(u, v)$ for every $u, v \in \mathbb{R}$ and $\kappa \geq 0$;

**(b)** additive if $\phi(u_1, v_1) + \phi(u_2, v_2) = \phi(u_1 + u_2, v_1 + v_2)$ for every $u_1, u_2, v_1, v_2 \in \mathbb{R}$;

**(c)** nonnegatively linear if both condition (a) and (b) are fulfilled;

**(d)** nonnegatively sublinear in first variable if $\phi(\kappa u + v, w) \leq \kappa\phi(u, w) + \phi(v, w)$ for every $u, v, w \in \mathbb{R}$ and $\kappa \geq 0$;

**(e)** nondecreasing in first variable if $u \leq v$ indicates that $\phi(u, w) \leq \phi(v, w)$ for every $u, v, w \in \mathbb{R}$.

The propositions and theorem mentioned below explain the relation between nonnegatively homogenous, additive and nonnegatively linear properties with $\phi$-convex function [9].

**Proposition 2.1.1.** [9] *Suppose a function $h : K \longrightarrow \mathbb{R}$ is $\phi$-convex function as $\phi$ is non negatively homogenous. Then there exist $\kappa \geq 0$ such that $\kappa h : K \longrightarrow \mathbb{R}$ is $\phi$-convex.*

**Proposition 2.1.2.** [9] *Let functions $h, f : K \longrightarrow \mathbb{R}$ are $\phi$-convex functions such that $\phi$ is additive. In that case $h + f : K \longrightarrow \mathbb{R}$ is $\phi$-convex.*

**Theorem 2.1.3.** [9] *Suppose a collection of $\phi$-convex functions $h_j : K \longrightarrow \mathbb{R}$ for $j = 1, 2, ...m$, as $\phi$ is non negatively linear. It follows that for $\kappa_j \geq 0$, $j = 1, 2, ...m$, the function $h = \sum_{j=1}^{m} \kappa_j h_j : K \longrightarrow \mathbb{R}$ is $\phi$-convex.*

It is observed that under specific conditions if sup operation is applied on $\phi$-convex and $\phi$-quasiconvex function, both of these satisfy the closure property.

**Theorem 2.1.4.** [9] *Let $\{h_i : K \longrightarrow \mathbb{R}, i \in I\}$ is a nonempty set of $\phi$-convex($\phi$-quasiconvex) functions which satisfy the given conditions*

**(1)** *there is any $\lambda \in [0, \infty]$ and $\nu \in [-1, \infty]$ which satisfy $\phi(u, v) = \lambda u + \nu v$ for every $u, v \in \mathbb{R}$.*

**(2)** *for every* $u \in K, sup_{i\in I} h_i(u)$ *exists in* $\mathbb{R}$.

*Then the function* $h : K \longrightarrow \mathbb{R}$ *is* $\phi$*-convex (*$\phi$*-quasiconvex) if it satisfies* $h(u) = sup_{i\in I} h_i(u)$ *for every* $u \in K$.

*Proof.* The given results can be achieved for each $u, v \in K$ and $\mu \in [0, 1]$

$$\begin{aligned}
h(\mu u + (1-\mu)v) &= \sup_{i\in I}\{h_i(\mu u + (1-\mu)v)\} \\
&\leq \sup_{i\in I}\{h_i(v) + \mu\phi(h_i(u), h_i(v))\} \\
&= \sup_{i\in I}\{h_i(v) + \mu(\lambda h_i(u) + \nu h_i(v))\} \\
&= \sup_{i\in I}\{(1+\nu\mu)h_i(v) + \lambda\mu h_i(u)\} \\
&\leq (1+\nu\mu)\sup_{i\in I} h_i(v) + \lambda\mu\sup_{i\in I} h_i(u) \\
&= (1+\nu\mu)h(v) + \lambda\mu h(u) \\
&= h(v) + \mu(\lambda h(u) + \nu h(v)) \\
&= h(v) + \mu\phi(h(u), h(v)).
\end{aligned}$$

The same proof is followed if the subjected function is $\phi$-quasiconvex. □

### 2.1.2 Jensen Type Inequality for $\phi$-Convex Function

Jensen's inequality is playing a vital role in applied mathematics. Its close relation with Optimization theory and Calculus of variation qualifies it to be explored and researched for new avenues.

The condition mentioned below will be applied in the proof of Theorem 2.1.5. Whereas Theorem 2.1.5 describes Jensen's inequality with respect to $\phi$-convex function.

Suppose $\phi$-convex function $h : K \longrightarrow \mathbb{R}$, then for $u_1, u_2 \in K$ and $\beta_1 + \beta_2 = 1$, we get

$$h(\beta_1 u_1 + \beta_2 u_2) \leq h(u_2) + \beta_1 \phi(h(u_1), h(u_2)).$$

Moreover, for $m > 2$, then for $u_1, u_2, \dots u_m \in K$, $\sum_{j=1}^{m} \beta_j = 1$ and $\mathrm{T}_j = \sum_{k=1}^{j} \beta_k$, we get

$$\begin{aligned} h(\sum_{j=1}^{m} \beta_j u_j) &= h((\mathrm{T}_{m-1} \sum_{j=1}^{m-1} \frac{\beta_j}{\mathrm{T}_{m-1}} u_j) + \beta_m u_m) \\ &\leq h(u_m) + \mathrm{T}_{m-1} \phi(h(\sum_{j=1}^{m-1} \frac{\beta_j}{\mathrm{T}_{m-1}} u_j), h(u_m)). \end{aligned}$$

Theorem 2.1.5 is very significant for this research. As it describes the essential position of Jensen's inequality for $\phi$-convex function.

**Theorem 2.1.5.** [9] *Suppose $h : K \longrightarrow \mathbb{R}$ is a $\phi$-convex function and $\phi$ is nondecreasing nonnegatively sublinear in first variable. If $\mathrm{T}_j = \sum_{k=1}^{j} \beta_k$ for $j = 1, \cdots, m$ as $\mathrm{T}_m = 1$, then we get*

$$h(\sum_{j=1}^{m} \beta_j u_j) \leq h(u_m) + \sum_{j=1}^{m-1} \mathrm{T}_j \phi_h(u_j, u_{j+1}, \cdots, u_m),$$

*where $\phi_h(u_j, u_{j+1}, \cdots, u_m) = \phi(\phi_h(u_j, u_{j+1}, \cdots, u_{m-1}), h(u_m))$ and $\phi_h(u) = h(u)$ for every $u \in K$.*

*Proof.* The below mention proof is obtained because $\phi$ act as nondecreasing nonnegatively sublinear on first variable

$$
\begin{aligned}
h(\sum_{j=1}^{m}\beta_j u_j) &\leq h(u_m) + \mathrm{T}_{m-1}\phi(h(\sum_{j=1}^{m-1}\frac{\beta_j}{\mathrm{T}_{m-1}}u_j), h(u_m)) \\
&= h(u_m) + \mathrm{T}_{m-1}\phi(h(\frac{\mathrm{T}_{m-2}}{\mathrm{T}_{m-1}}\sum_{j=1}^{m-2}\frac{\beta_j}{\mathrm{T}_{m-2}}u_j + \frac{\beta_{m-1}}{\mathrm{T}_{m-1}}u_{m-1}), h(u_m)) \\
&\leq h(u_m) + \mathrm{T}_{m-1}\phi(h(u_{m-1}) + \frac{\mathrm{T}_{m-2}}{\mathrm{T}_{m-1}}\phi(h(\sum_{j=1}^{m-2}\frac{\beta_j}{\mathrm{T}_{m-2}}u_j), h(u_{m-1})), h(u_m)) \\
&\leq h(u_m) + \mathrm{T}_{m-1}\phi(h(u_{m-1}), h(u_m)) + \mathrm{T}_{m-2}\phi(\phi(h(\sum_{j=1}^{m-2}\frac{\beta_j}{\mathrm{T}_{m-2}}u_j), h(u_{m-1})), h(u_m)) \\
&\leq \dots \\
&\leq h(u_m) + \mathrm{T}_{m-1}\phi(h(u_{m-1}), h(u_m)) + \mathrm{T}_{m-2}\phi(\phi(h(u_{m-2}), h(u_{m-1})), h(u_m)) \\
&+ \dots + \mathrm{T}_1\phi(\phi(\dots\phi(\phi(h(u_1), h(u_2)), h(u_3))\dots)), h(u_{m-1})), h(u_m)) \\
&= h(u_m) + \mathrm{T}_{m-1}\phi_h(u_{m-1}, u_m) + \mathrm{T}_{m-2}\phi_h(u_{m-2}, u_{m-1}, u_m) + \dots \\
&+ \mathrm{T}_1\phi_h(u_1, u_2, \dots, u_{m-1}, u_m) \\
&= h(u_m) + \sum_{j=1}^{m-1}\mathrm{T}_j\phi_h(u_j, u_{j+1}, \dots, u_m).
\end{aligned}
$$

The proof is complete. □

**Example 2.1.1.** [9] Let $h(u) = u^2$ and $\phi(u, v) = u(1 + 2v)$ for every $u, v \in \mathbb{R}^+ = [0, \infty)$. Here function $\phi$ is nondecreasing nonnegatively sublinear in first variable and $h$ is $\phi$-convex so $(\beta_1 u_1 + \beta_2 u_2)^2 \leq u_2^2 + \beta_1 u_1^2(1 + 2u_2^2)$, for $u_1, u_2 \in \mathbb{R}^+$ and $\beta_1 + \beta_2 \geq 0$ as $\beta_1 + \beta_2 = 1$. Furthermore, for $u_1, u_2, \dots u_m \in \mathbb{R}^+$ and $\beta_1, \beta_2, \dots \beta_m$ with $\sum_{j=1}^{m}\beta_j = 1$ then by applying Theorem 2.1.5, we get

$$
(\sum_{j=1}^{m}\beta_j u_j)^2 \leq u_m^2 + \sum_{j=1}^{m-1}\mathrm{T}_j[u_j^2(1 + 2u_{j+1}^2)(1 + 2u_{j+2}^2)\dots(1 + 2u_m^2)].
$$

### 2.1.3 $\phi_d$-Convex and $\phi_F$-Convex Function

Here onwards $\phi_d$-convex function and $\phi_F$-convex function has been described as $\phi$-convex function's generalized form and produces some results.

**Definition 2.1.3.** [9] Suppose $\mathbb{R}^+$ is a collection of nonnegative real numbers and $d : \mathbb{R}\times\mathbb{R}\times[0,1] \longrightarrow \mathbb{R}^+$ is a function which simultaneously satisfies that $\mu d(u,v,\mu) \in [0,1]$ for every $u, v \in \mathbb{R}$ and $\mu \in [0,1]$. A function $h : K \longrightarrow \mathbb{R}$ is said to be $\phi_d$-convex function if

$$h(\mu u + (1-\mu)v) \leq h(v) + \mu d(u,v,\mu)\phi(h(u),h(v))$$

for every $u, v \in \mathbb{R}$ and $\mu \in [0,1]$.

It is presented by Theorem 2.1.6 that the family of $\phi_d$-convex function and $\phi$-quasiconvex function are equivalent.

**Theorem 2.1.6.** [9] *Let the function be $h : K \longrightarrow \mathbb{R}$. The below mention assumptions are equivalent:*

**(a)** *There exist function $d$ such that $h$ is $\phi_d$-convex function.*

**(b)** *$h$ is $\phi$-quasiconvex function.*

*Proof.* $(a) \longrightarrow (b)$ For every $u, v \in K$ and $\mu \in [0,1]$,

$$h(\mu u+(1-\mu)v) \leq h(v)+\mu d(u,v,\mu)\phi(h(u),h(v)) \leq \max\{h(v), h(v)+\phi(h(u),h(v))\}.$$

$(b) \longrightarrow (a)$ For every $u, v \in K$ and $\mu \in [0,1]$, let us define

$$d(u,v,\mu) = \begin{cases} \frac{1}{\mu}, & if \mu \in [0,1] \text{ and } h(v) \leq h(v) + \phi(h(u),h(v)); \\ \\ 0, & \mu = 0 \text{ or } h(v) > h(v) + \phi(h(u),h(v)). \end{cases}$$

It is observed that $\mu d(u,v,\mu) \in [0,1]$. The earlier defined function provide the given below relation

$$\begin{aligned} h(\mu u + (1-\mu)v) &\leq max\{h(v), h(v) + \phi(h(u), h(v))\} \\ &= \mu d(u,v,\mu)(h(v) + \phi(h(u), h(v))) + (1 - \mu d(u,v,\mu))h(v) \\ &= h(v) + \mu d(u,v,\mu)\phi(h(u), h(v)). \end{aligned}$$

The proof is complete. □

**Definition 2.1.4.** Let $B$ be a subset of real space $\mathbb{R}$ which is $F$-convex if and only if there is a function $F : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies that $\mu F(u) + (1-\mu)F(v) \in B$, for every $u, v \in B$ and $0 \leq \mu \leq 1$.

**Definition 2.1.5.** Let $B \subseteq \mathbb{R}$ is $F$-convex. Then $F(B) \subseteq B$.

**Definition 2.1.6.** Consider $B$ be a $F$-convex. A function $h : B \longrightarrow \mathbb{R}$ is supposed to be $\phi_F$-convex if

$$h(\mu F(u) + (1-\mu)F(v)) \leq h(F(v)) + \mu\phi(h(F(u), h(F(v)))).$$

It simultaneously be called $\phi_F$-quasiconvex if

$$h(\mu F(u) + (1-\mu)F(v)) \leq \max\{h(F(v)), h(F(v)) + \phi(h(F(u)), h(F(v)))\}.$$

The restriction of a $\phi_F$-convex ($\phi_F$-quasiconvex) function is a $\phi$-convex ($\phi$-quasiconvex) function. Theorem 2.1.7 depicts this result. It is shown in Theorem 2.1.8 that $\phi_F$-convex function is equivalent to $\phi$-convex ($\phi$-quasiconvex) function if it is restricted on an appropriate domain, for detail see [9].

**Theorem 2.1.7.** [9] *Let $B \subseteq \mathbb{R}$ be a $F$-convex set such that $C \subseteq F(B)$ be a nonempty convex set. If $h : B \longrightarrow \mathbb{R}$ is a $\phi_F$-convex ($\phi_F$-quasiconvex), then it's restriction $\bar{h} : C \longrightarrow \mathbb{R}$ can be presented as*

$$\bar{h}(u') = h(u') \quad for\ every\ u' \in C,$$

*is a $\phi$-convex ($\phi$-quasiconvex) on $C$.*

*Proof.* For $u', v' \in C$ there exist $u, v \in B$ which satisfy that $u' = F(u)$ and $v' = F(v)$. As $C$ is convex set, so for every $\mu \in [0,1]$, can be expressed as $\mu u' + (1-\mu)v' \in C$. Hence

$$\begin{aligned}\bar{h}(\mu u' + (1-\mu)v') &= h(\mu F(u) + (1-\mu)F(v)) \\ &\leq h(F(v)) + \mu\phi(h(F(u)), h(F(v))) \\ &= \bar{h}(v') + \mu\phi(\bar{h}(u'), \bar{h}(v')).\end{aligned}$$

This completes the proof. □

**Theorem 2.1.8.** [9] *Let $B \subseteq \mathbb{R}$ be a $F$-convex set such that $C \subseteq F(B)$ be a nonempty convex set. The function $h : B \longrightarrow \mathbb{R}$ is said to be $\phi_F$-convex ($\phi_F$-quasiconvex) iff it's restriction $\bar{h} : F(B) \longrightarrow \mathbb{R}$ is represented as*

$$\bar{h}(u') = h(u') \quad \textit{for every } u' \in F(B),$$

*is $\phi$-convex ($\phi$-quasiconvex) on $F(B)$.*

*Proof.* Necessary condition here is same as in Theorem 2.1.7. For sufficient condition, suppose $u, v \in B$. Thus $F(u), F(v) \in F(B)$. Since $\bar{h}$ is $\phi$-convex on $F(B)$ so the result is

$$\bar{h}(\mu F(u) + (1-\mu)F(v)) \leq \bar{h}(F(v)) + \mu\phi(\bar{h}(F(u)), \bar{h}(F(v))).$$

As we know $\bar{h}$ is the restriction of $h$ on $F(B)$ so

$$h(\mu F(u) + (1-\mu)F(v)) \leq h(F(v)) + \mu\phi(h(F(u)), h(F(v))).$$

This completes the proof. □

## 2.2 Generalization of Čebyšev Identity and Inequality

According to A. M. Ostrowski [25], monotonic functions $h'$ and $r'$ satisfy continuity on $[c, d]$ and function $l$ satisfies the positivity of integration, generalized Čebyšev inequality for monotonic functions $h'$ and $r'$ is given below

$$\mathcal{T}(h,r,l) = h'(\xi)r'(\eta)\mathcal{T}(u-c,u-c,l), \tag{2.2.1}$$

where

$$\mathcal{T}(h,r,l) = \int_c^d l(u)du \int_c^d l(u)h(u)r(u)du - \int_c^d l(u)h(u)du \int_c^d l(u)r(u)du. \tag{2.2.2}$$

Some other generalizations are also available in [30]. Some more generalization by J. Pecaric are given in [31] through following functional

$$\mathcal{C}(h,l) = \int_c^d \int_c^d l(u,v)h(u,u)dvdu - \int_c^d \int_c^d l(u,v)h(u,v)dvdu. \tag{2.2.3}$$

Here functions $l$ and $h$ are integrable over $J^2 = [c,d] \times [c,d]$.

**Proposition 2.2.1.** [31] *Suppose $l : J^2 \longrightarrow \mathbb{R}$ is an integrable function in such a way*

$$U(u,u) = \bar{U}(u,u) \quad for\ every\ u \in [c,d] \tag{2.2.4}$$

*and either*

$$U(u,v) \geq 0, \quad c \leq v \leq u \leq d; \quad \bar{U}(u,v) \geq 0, \quad c \leq u \leq v \leq d$$

*or the reverse of previously mentioned inequality is also valid. where*

$$U(u,v) = \int_u^d \int_c^v l(w,z)dzdw$$

*and*

$$\bar{U}(u,v) = \int_c^u \int_v^d l(w,z)dzdw$$

*If $h : J^2 \longrightarrow \mathbb{R}$ has continuous partial derivatives $h_{(1,0)} = \frac{\partial}{\partial u}h(u,v), h_{(0,1)} = \frac{\partial}{\partial v}h(u,v)$, and $h_{(1,1)} = \frac{\partial^2}{\partial u \partial v}h(u,v)$. Then there exist $\xi, \eta \in [c,d]$ in such a way*

$$\mathcal{C}(h,l) = h_{(1,1)}(\xi,\eta)\mathcal{C}((u-c)(v-c),l).$$

Suppose we have two integrable functions $l : J^2 \longrightarrow \mathbb{R}$ and $o : J \longrightarrow \mathbb{R}$. Hence the notations given below are used for our convenience in theorems discussed now onwards

$$\bar{L}^{(m,n)}(u,v) = \int_u^d \int_v^d l(w,z) \frac{(w-u)^m}{m!} \frac{(w-v)^n}{n!} dz dw, \tag{2.2.5}$$

$$L^{(m,n)}(u,v) = \int_u^d \int_v^d l(w,z) \frac{(w-u)^m}{m!} \frac{(z-v)^n}{n!} dz dw, \tag{2.2.6}$$

$$O^{(m,n)}(u) = \int_w^d o(w) \frac{(w-u)^m}{m!} \frac{(w-c)^n}{n!} dw, \tag{2.2.7}$$

$$\begin{aligned} S(u,v) = & \int_{max\{u,v\}}^d \int_c^d l(w,z) \frac{(w-u)^Q}{Q!} \frac{(w-v)^P}{P!} dz dw \\ & - \int_u^d \int_v^d l(w,z) \frac{(w-u)^Q}{Q!} \frac{(z-v)^P}{P!} dz dw, \end{aligned} \tag{2.2.8}$$

$$\begin{aligned} \bar{S}(u,v) = & \int_{max\{u,v\}}^d o(w) \frac{(w-u)^Q}{Q!} \frac{(w-v)^P}{P!} dw \\ & - \int_u^d \int_v^d l(w,z) \frac{(w-u)^Q}{Q!} \frac{(z-v)^P}{P!} dz dw, \end{aligned} \tag{2.2.9}$$

$$E_0(u,v) = \frac{(u-c)^{Q+1}(v-c)^{P+1}}{(Q+1)!(P+1)!}. \tag{2.2.10}$$

Here a definition from [38] will be used. Suppose $B = [c,d] \times [e,f]$ represents a rectangle in $\mathbb{R}^2$ and $S(B)$ represents the system of rectangles $[u_1,u_2] \times [v_1,v_2]$ contained in $B$.

The function $E_x : S(B) \longrightarrow \mathbb{R}$ is categorized as function of rectangles associated with $x$. Where $E_x([u_1,u_2] \times [v_1,v_2]) = x(u_1,v_1) - x(u_2,v_1) - x(u_1,v_2) + x(u_2,v_2)$ for $[u_1,u_2] \times [v_1,v_2] \in S(B)$ and $x$ is function from $B$ to $\mathbb{R}$.

**Definition 2.2.1.** Under below given conditions, a function $x : B \longrightarrow \mathbb{R}$ is supposed to be absolute continuous on $B$ as carathéodory,

**(a)** The function $E_x$ connected with $x$ is absolutely continuous which means for each $\epsilon > 0$, there exist $\delta > 0$ where $Q_1, Q_2, \ldots Q_k \in S(B)$ are mutually non-overlapping rectangles while $\sum_{j=1}^{k} |Q_j| \leq \delta$, and $|.|$ represents area of rectangle, then $\sum_{j=1}^{k} E_x(Q_j) \leq \epsilon$.

**(b)** The functions $x(c,.) : [c,d] \longrightarrow \mathbb{R}$ and $x(.,e) : [e,f] \longrightarrow \mathbb{R}$ satisfy the absolute continuity.

Suppose 3 functions $h, l : J^2 \longrightarrow \mathbb{R}$ and $o : J \longrightarrow \mathbb{R}$ where $l$ and $o$ are integrable and $h_{(Q,P)}$ exists. Also it is absolute continuous as caratheodory. Then $\bar{\mathcal{C}}(h,l)$ provided here

$$\begin{aligned}
\bar{\mathcal{C}}(h,l) = \mathcal{C}(h,l) &- \sum_{m=0}^{Q} \sum_{n=0}^{P} h_{(m,n)}(c,c)[\bar{L}^{(m,n)}(c,c) - \bar{L}^{(m,n)}(c,c)] \\
&- \sum_{n=0}^{P} \int_c^d h_{(Q+1,n)}(u,c)[\bar{L}^{(Q,n)}(u,c) - L^{(Q,n)}(u,c)]du \qquad (2.2.11) \\
&- \sum_{m=0}^{Q} \int_c^d h_{(m,P+1)}(c,v)[\bar{L}^{(m,P)}(c,v) - L^{(m,P)}(c,v)]dv,
\end{aligned}$$

where $\mathcal{C}(h,l)$ is described in (2.2.3).

**Lemma 2.2.2.** [14] *Suppose two functions $l, h : J^2 \longrightarrow \mathbb{R}$ where $l$ is supposed to be integrable and $h_{(Q+1,P)}$ and $h_{(Q,P+1)}$ exists. Also both functions satisfy the absolute continuity. Then we get*

$$\int_c^d \int_c^d l(u,v)h(u,v)dvdu = \sum_{m=0}^{Q}\sum_{n=0}^{P} h_{(m,n)}(c,c)L^{(m,n)}(c,c)$$
$$+ \sum_{n=0}^{P}\int_c^d h_{(Q+1,n)}(u,c)L^{(Q,n)}(u,c)du$$
$$+ \sum_{m=0}^{Q}\int_c^d h_{(m,P+1)}(c,v)L^{(m,P)}(c,v)dv \tag{2.2.12}$$
$$+ \int_c^d \int_c^d h_{(Q+1,P+1)}(u,v)L^{(Q,P)}(u,v)dvdu,$$

*where* $L^{(m,n)}$ *is described in* (2.2.5).

*Proof.* Suppose $G(v) = h(u,v)$, that is, we assume a function $h(u,v)$ as a function of variable $v$. Then function $G$ is given below

$$\begin{aligned} h(u,v) = G(v) &= \sum_{n=0}^{P} G^{(n)}(c)\frac{(v-c)^n}{n!} + \int_c^v G^{P+1}(z)\frac{(v-z)^P}{P!}dz \\ &= \sum_{n=0}^{P} h_{(0,n)}(u,c)\frac{(v-c)^n}{n!} + \int_c^v h_{(0,P+1)}(u,z)\frac{(v-z)^P}{P!}dz. \end{aligned} \tag{2.2.13}$$

Here in the above mentioned expression we used $G^{(n)}(c) = h_{(0,n)}(u,c)$ and $G^{(P+1)}(z) = h_{(0,P+1)}(u,z)$. Now if we take product of above expression with $L(u,v)$ and then apply integration through variable $v$ over $[c,d]$. The result will be

$$\begin{aligned} \int_c^d L(u,v)h(u,v)dv &= \sum_{n=0}^{P} h_{(0,n)}(u,c)\int_c^d L(u,v)\frac{(v-c)^n}{n!}dv \\ &+ \int_c^d (\int_c^v L(u,v)h_{(0,P+1)}(u,z)\frac{(v-z)^P}{P!}dz)dv. \end{aligned} \tag{2.2.14}$$

Suppose the representation of functions as $u \longmapsto h_{(0,j)}(u,c)$ and $u \longmapsto h_{(0,P+1)}(u,z)$

then by applying Taylor expansions:

$$h_{(0,n)}(u,c)=\sum_{m=0}^{Q}h_{(m,n)}(c,c)\frac{(u-c)^m}{m!}+\int_c^u h_{(Q+1,n)}(w,c)\frac{(u-w)^Q}{Q!}dw,$$
$$h_{(0,P+1)}(u,z)=\sum_{m=0}^{Q}h_{(m,P+1)}(c,z)\frac{(u-c)^m}{m!}+\int_c^u h_{(Q+1,P+1)}(w,z)\frac{(u-w)^Q}{Q!}dw.$$

Substituting above mentioned expression in (2.2.14), we obtain

$$\begin{aligned}\int_c^d L(u,v)h(u,v)dv &= \sum_{n=0}^{P}\left(\sum_{m=0}^{Q}h_{(m,n)}(c,c)\frac{(u-c)^m}{m!}+\int_c^u h_{(Q+1,n)}(w,c)\frac{(u-w)^Q}{Q!}dw\right)\\
&\int_c^d L(u,v)\frac{(v-c)^n}{n!}dv+\int_c^d\left(\left(\int_c^v L(u,v)\left(\sum_{m=0}^{Q}h_{(m,P+1)}(c,z)\frac{(u-c)^m}{m!}\right.\right.\right.\\
&\left.\left.\left.+\int_c^u h_{(Q+1,P+1)}(w,z)\frac{(u-w)^Q}{Q!}dw\right)\frac{(v-z)^P}{P!}dz\right)dv\\
&=\sum_{n=0}^{P}\left(\sum_{m=0}^{Q}h_{(m,n)}(c,c)\frac{(u-c)^m}{m!}\right)\int_c^d L(u,v)\frac{(v-c)^n}{n!}dv\\
&+\sum_{n=0}^{P}\left(\int_c^u h_{(Q+1,n)}(w,c)\frac{(u-w)^Q}{Q!}dw\right)\int_c^d L(u,v)\frac{(v-c)^n}{n!}dv\\
&+\int_c^d\int_c^v L(u,v)\left(\sum_{m=0}^{Q}h_{(m,P+1)}(c,z)\frac{(u-c)^m}{m!}\right)\frac{(v-z)^P}{P!}dzdv\\
&+\int_c^d\int_c^v\left(\int_c^u L(u,v)h_{(Q+1,P+1)}(w,z)\frac{(u-w)^Q}{Q!}dw\right)\frac{(v-z)^P}{P!}dzdv.\end{aligned}$$

Now if we apply integration to the above expression over $[c,d]$ by variable $u$ then we obtain

$$\begin{aligned}
&\int_c^d \int_c^d L(u,v)h(u,v)dvdu \\
&= \int_c^d \left[\sum_{n=0}^{P}\left(\sum_{m=0}^{Q} h_{(m,n)}(c,c)\frac{(u-c)^m}{m!}\right)\int_c^d L(u,v)\frac{(v-c)^n}{n!}dv\right]du \\
&+ \int_c^d \left[\sum_{n=0}^{P}\left(\int_c^u h_{(Q+1,n)}(w,c)\frac{(u-w)^Q}{Q!}dw\right)\int_c^d L(u,v)\frac{(v-c)^n}{n!}dv\right]du \\
&+ \int_c^d \left[\int_c^d \int_c^v L(u,v)\left(\sum_{m=0}^{Q} h_{(m,P+1)}(c,z)\frac{(u-c)^m}{m!}\right)\frac{(v-z)^P}{P!}dzdv\right]du \\
&+ \int_c^d \left[\int_c^d \int_c^v \left(\int_c^u L(u,v)h_{(Q+1,P+1)}(w,z)\frac{(u-w)^Q}{Q!}dw\right)\frac{(v-z)^P}{P!}dzdv\right]du.
\end{aligned}$$

By changing order of summation in the first summand, and applying integral linearity, we obtain

$$\sum_{m=0}^{Q}\sum_{n=0}^{P}\int_c^d \int_c^d L(u,v)h_{(m,n)}(c,c)\frac{(u-c)^m}{m!}\frac{(v-c)^n}{n!}dvdu.$$

Subsequently, the second summand will be expressed as below

$$\begin{aligned}
&\int_c^d \left[\sum_{n=0}^{P}\left(\int_c^u h_{(Q+1,n)}(w,c)\frac{(u-w)^Q}{Q!}dw\right)\int_c^d L(u,v)\frac{(v-c)^n}{n!}dv\right]du \\
&= \int_c^d \left[\sum_{n=0}^{P}\left(\int_c^u \int_c^d L(u,v)\frac{(v-c)^n}{n!}h_{(Q+1,n)}(w,c)\frac{(u-w)^Q}{Q!}dvdw\right)\right]du \\
&= \sum_{n=0}^{P}\int_c^d \int_c^u \int_c^d L(u,v)h_{(Q+1,n)}(w,c)\frac{(u-w)^Q}{Q!}\frac{(v-c)^n}{n!}dvdwdu \\
&= \sum_{n=0}^{P}\int_c^d \int_w^d \int_c^d L(u,v)h_{(Q+1,n)}(w,c)\frac{(u-w)^Q}{Q!}\frac{(v-c)^n}{n!}dvdudw.
\end{aligned}$$

In previous equation Fubini theorem is used for variable $w$ and $u$. Let's clarify that variable $u$ is altered from $c$ to $d$ while $w$ is altered from $c$ to $u$. We get variable $w$

altered from $c$ to $d$ and variable $u$ into $w$ to $d$, when integration order is changed.

Now the $3rd$ summand will be expressed:

$$\int_c^d \left[\int_c^d \int_c^v L(u,v) \left(\sum_{m=0}^{Q} h_{(m,P+1)}(c,z)\frac{(u-c)^m}{m!}\right) \frac{(v-z)^P}{P!} dzdv\right] du$$
$$= \sum_{m=0}^{Q} \int_c^d \int_c^d \int_c^v L(u,v) h_{(m,P+1)}(c,z)\frac{(u-c)^m}{m!} \frac{(v-z)^P}{P!} dzdvdu$$
$$= \sum_{m=0}^{Q} \int_c^d \int_c^d \int_z^d L(u,v) h_{(m,P+1)}(c,z)\frac{(u-c)^m}{m!} \frac{(v-z)^P}{P!} dvdzdu$$
$$= \sum_{m=0}^{Q} \int_c^d \int_c^d \int_z^d L(u,v) h_{(m,P+1)}(c,z)\frac{(u-c)^m}{m!} \frac{(v-z)^P}{P!} dvdudz.$$

Here in the above expression Fubini theorem is applied for replacing $z$ and $v$ and then $z$ and $u$.

Now the $4th$ summand is expressed:

$$\int_c^d \left[\int_c^d \int_c^v \left(\int_c^u L(u,v) h_{(Q+1,P+1)}(w,z)\frac{(u-w)^Q}{Q!} dw\right) \frac{(v-z)^P}{P!} dzdv\right] du$$
$$= \int_c^d \int_c^d \int_c^v \int_c^u L(u,v) h_{(Q+1,P+1)}(w,z)\frac{(u-w)^Q}{Q!} \frac{(v-z)^P}{P!} dwdzdvdu$$
$$= \int_c^d \int_c^d \int_w^d \int_z^d L(u,v) h_{(Q+1,P+1)}(w,z)\frac{(u-w)^Q}{Q!} \frac{(v-z)^P}{P!} dvdudzdw.$$

In the above expression Fubini theorem is applied multiple times.

Consequently using all these results, we obtain:

$$\int_c^d \int_c^d L(u,v)h(u,v)dvdu$$
$$= \sum_{m=0}^{Q}\sum_{n=0}^{P}\int_c^d \int_c^d L(u,v)h_{(m,n)}(c,c)\frac{(u-c)^m}{m!}\frac{(v-c)^n}{n!}dvdu$$
$$+ \sum_{n=0}^{P}\int_c^d \int_w^d \int_c^d L(u,v)h_{(Q+1,n)}(w,c)\frac{(u-w)^Q}{Q!}\frac{(v-c)^n}{n!}dvdudw$$
$$+ \sum_{m=0}^{Q}\int_c^d \int_c^d \int_z^d L(u,v)h_{(m,P+1)}(c,z)\frac{(u-c)^m}{m!}\frac{(v-z)^P}{P!}dvdudz$$
$$+ \int_c^d \int_c^d \int_w^d \int_z^d L(u,v)h_{(Q+1,P+1)}(w,z)\frac{(u-w)^Q}{Q!}\frac{(v-z)^P}{P!}dvdudzdw.$$

If variables (of right side) in above expression substituted as $u \longleftrightarrow w$ and $v \longleftrightarrow z$, we obtain

$$\int_c^d \int_c^d L(u,v)h(u,v)dvdu$$
$$= \sum_{m=0}^{Q}\sum_{n=0}^{P}\int_c^d \int_c^d L(w,z)h_{(m,n)}(c,c)\frac{(w-c)^m}{m!}\frac{(z-c)^n}{n!}dzdw$$
$$+ \sum_{n=0}^{P}\int_c^d \int_u^d \int_c^d L(w,z)h_{(Q+1,n)}(u,c)\frac{(w-u)^Q}{Q!}\frac{(z-c)^n}{n!}dzdwdu$$
$$+ \sum_{m=0}^{Q}\int_c^d \int_c^d \int_v^d L(w,z)h_{(m,P+1)}(c,v)\frac{(w-c)^m}{m!}\frac{(z-v)^P}{P!}dzdwdv$$
$$+ \int_c^d \int_c^d \int_u^d \int_v^d L(w,z)h_{(Q+1,P+1)}(u,v)\frac{(w-u)^Q}{Q!}\frac{(z-v)^P}{P!}dzdwdvdu.$$

By using equation (2.2.6), we obtain required result. □

**Theorem 2.2.3.** [16] *Suppose two function $l, h : J^2 \longrightarrow \mathbb{R}$ where $l$ is integrable and*

*$h_{(Q+1,P)}$ and $h_{(Q,P+1)}$ exist and satisfy absolute continuity. Then we obtain*

$$\begin{aligned}\mathcal{C}(h,l) &= \int_c^d\int_c^d l(u,v)h(u,u)dvdu - \int_c^d\int_c^d l(u,v)h(u,v)dvdu. \\ &= \sum_{m=0}^{Q}\sum_{n=0}^{P} h_{(m,n)}(c,c)\left[\bar{L}^{(m,n)}(c,c) - L^{(m,n)}(c,c)\right] \\ &+ \sum_{n=0}^{P}\int_c^d h_{(Q+1,n)}(u,c)\left[\bar{L}^{(Q,n)}(u,c) - L^{(Q,n)}(u,c)\right]du \\ &+ \sum_{m=0}^{Q}\int_c^d h_{(m,P+1)}(c,v)\left[\bar{L}^{(m,P)}(c,v) - L^{(m,P)}(c,v)\right]dv \\ &+ \int_c^d\int_c^d h_{(Q+1,P+1)}(u,v)S(u,v)dvdu.\end{aligned}$$

*Here $\bar{L}^{(m,n)}, L^{(m,n)}$ and $S(u,v)$ are described in* (2.2.5), (2.2.6) *and* (2.2.8) *respectively.*

*Proof.* To prove the above mentioned identity we have to obtain this expression first $\int_c^d\int_c^d l(u,v)h(u,u)dvdu$. First of all we use Taylor expansion of two variables on $h(u,u)$ and then multiply the resultant with $l(u,v)$. Integrate the obtained result over $[c,d]\times[c,d]$ by variables $u$ and $v$ to obtain

$$\begin{aligned}&\int_c^d\int_c^d l(u,v)h(u,u)dvdu \\ &= \int_c^d\left[\sum_{n=0}^{P}\left(\sum_{m=0}^{Q} h_{(m,n)}(c,c)\frac{(u-c)^m}{m!}\right)\int_c^d l(u,v)\frac{(u-c)^n}{n!}dv\right]du \\ &+ \int_c^d\left[\sum_{n=0}^{P}\left(\int_c^u h_{(Q+1,n)}(w,c)\frac{(u-w)^Q}{Q!}dw\right)\int_c^d l(u,v)\frac{(u-c)^n}{n!}dv\right]du \\ &+ \int_c^d\left[\int_c^d\int_c^u l(u,v)\left(\sum_{m=0}^{Q} h_{(m,P+1)}(c,z)\frac{(u-c)^m}{m!}\right)\frac{(u-z)^p}{p!}dzdv\right]du \\ &+ \int_c^d\left[\int_c^d\int_c^u\left(\int_c^u l(u,v)h_{(Q+1,P+1)}(w,z)\frac{(u-w)^Q}{Q!}dw\right)\frac{(u-z)^P}{P!}dzdv\right]du.\end{aligned}$$

Change the order of summation in the earliest first summand and apply linearity of integral then we obtain

$$\sum_{m=0}^{Q}\sum_{n=0}^{P}\int_c^d\int_c^d l(u,v)h_{(m,n)}(c,c)\frac{(u-c)^m}{m!}\frac{(u-c)^n}{n!}dvdu.$$

On second summand if Fubini's theorem is applied then the obtained result is as follows

$$\int_c^d\left[\sum_{n=0}^{P}\left(\sum_c^u h_{(Q+1,n)}(w,c)\frac{(u-w)^Q}{Q!}\right)\int_c^d l(u,v)\frac{(u-c)^n}{n!}dv\right]du$$
$$=\int_c^d\left[\sum_{n=0}^{P}\left(\int_c^u\int_c^d l(u,v)\frac{(u-c)^n}{n!}h_{(Q+1,n)}(w,c)\frac{(u-w)^Q}{Q!}dvdw\right)\right]du$$
$$=\sum_{n=0}^{P}\int_c^d\int_c^u\int_c^d l(u,v)h_{(Q+1,n)}(w,c)\frac{(u-w)^Q}{Q!}\frac{(u-c)^n}{n!}dvdwdu$$
$$=\sum_{n=0}^{P}\int_c^d\int_w^d\int_c^d l(u,v)h_{(Q+1,n)}(w,c)\frac{(u-w)^Q}{Q!}\frac{(u-c)^n}{n!}dvdudw.$$

Similarly, the third summand will transform into

$$\int_c^d\left[\int_c^d\int_c^u l(u,v)\left(\sum_{m=0}^{Q}h_{(m,P+1)}(c,z)\frac{(u-c)^m}{m!}\right)\frac{(u-z)^p}{p!}dzdv\right]du$$
$$=\sum_{m=0}^{Q}\int_c^d\int_c^d\int_c^u l(u,v)h_{(m,P+1)}(c,z)\frac{(u-c)^m}{m!}\frac{(u-z)^p}{p!}dzdvdu$$
$$=\sum_{m=0}^{Q}\int_c^d\int_c^d\int_z^d l(u,v)h_{(m,P+1)}(c,z)\frac{(u-c)^m}{m!}\frac{(u-z)^p}{p!}dvdudz.$$

Finally, the fourth summand can be expressed as

$$\int_c^d\left[\int_c^d\int_c^u\left(\int_c^u l(u,v)h_{(Q+1,P+1)}(w,z)\frac{(u-w)^Q}{Q!}dw\right)\frac{(u-z)^P}{P!}dzdv\right]du$$
$$=\int_c^d\int_c^d\int_c^u\int_c^u l(u,v)h_{(Q+1,P+1)}(w,z)\frac{(u-w)^Q}{Q!}\frac{(u-z)^P}{P!}dwdzdvdu$$
$$=\int_c^d\int_c^d\int_{max\{w,z\}}^d\int_c^d l(u,v)h_{(Q+1,P+1)}(w,z)\frac{(u-w)^Q}{Q!}\frac{(u-z)^P}{P!}dvdudzdw.$$

By adding the above mentioned expressions we obtain

$$\int_c^d \int_c^d l(u,v)h(u,u)dvdu$$
$$= \sum_{m=0}^{Q}\sum_{n=0}^{P}\int_c^d \int_c^d l(u,v)h_{(m,n)}(c,c)\frac{(u-c)^m}{m!}\frac{(u-c)^n}{n!}dvdu$$
$$+ \sum_{n=0}^{P}\int_c^d \sum_{w}^{d}\int_c^d l(u,v)h_{(Q+1,n)}(w,c)\frac{(u-w)^Q}{Q!}\frac{(u-c)^n}{n!}dvdudw$$
$$+ \sum_{m=0}^{Q}\int_c^d \int_c^d \int_z^d l(u,v)h_{(m,P+1)}(c,z)\frac{(u-c)^m}{m!}\frac{(u-z)^p}{p!}dvdudz$$
$$+ \int_c^d \int_c^d \int_{max\{w,z\}}^d \int_c^d l(u,v)h_{(Q+1,P+1)}(w,z)\frac{(u-w)^Q}{Q!}\frac{(u-z)^P}{P!}dvdudzdw.$$

If on right side variables are replaced $u \longleftrightarrow w, v \longleftrightarrow z$. So this relation is obtained

$$\int_c^d \int_c^d l(u,v)h(u,u)dvdu$$
$$= \sum_{m=0}^{Q}\sum_{n=0}^{P}\int_c^d \int_c^d l(w,z)h_{(m,n)}(c,c)\frac{(w-c)^m}{m!}\frac{(w-c)^n}{n!}dzdw$$
$$+ \sum_{n=0}^{P}\int_c^d \int_u^d \int_c^d l(w,z)h_{(Q+1,n)}(u,c)\frac{(w-u)^Q}{Q!}\frac{(w-c)^n}{n!}dzdwdu$$
$$+ \sum_{m=0}^{Q}\int_c^d \int_c^d \int_v^d l(w,z)h_{(m,P+1)}(c,v)\frac{(w-c)^m}{m!}\frac{(w-v)^p}{p!}dzdwdv$$
$$+ \int_c^d \int_c^d \int_{max\{u,v\}}^d \int_c^d l(w,z)h_{(Q+1,P+1)}(u,v)\frac{(w-u)^Q}{Q!}\frac{(w-v)^P}{P!}dzdwdvdu.$$

The earliest described notions provide:

$$
\begin{aligned}
&\int_c^d \int_c^d l(u,v)h(u,u)dvdu \\
&= \sum_{m=0}^{Q}\sum_{n=0}^{P} h_{(m,n)}(c,c)\bar{L}^{(m,n)}(c,c) + \sum_{n=0}^{P}\int_c^d h_{(Q+1,n)}(u,c)\bar{L}^{(Q,n)}(u,c)du \\
&+ \sum_{m=0}^{Q}\int_c^d h_{(m,P+1)}(c,v)dv + \int_c^d\int_c^d h_{(Q+1,P+1)}(u,v)\int_{max\{u,v\}}^d\int_c^d \\
&l(w,z)\frac{(w-u)^Q}{Q!}\frac{(w-v)^P}{P!}dzdwdvdu.
\end{aligned}
$$

Where $\bar{L}^{(m,n)}$ is described in (2.2.5). Substituting the said relation $\int_c^d\int_c^d l(u,v)h(u,u)dvdu$ and Lemma 2.2.2 in

$$
\mathcal{C}(h,l) = \int_c^d\int_c^d l(u,v)h(u,u)dvdu - \int_c^d\int_c^d l(u,v)h(u,v)dvdu.
$$

The desired identity is obtained. □

**Theorem 2.2.4.** [16] *Suppose $l, h : J^2 \longrightarrow \mathbb{R}$ are two functions where $l$ is integrable and $h$ is $(Q+1, P+1)$-convex. Then*

$$
\bar{\mathcal{C}}(h,l) \geq 0
$$

*If*

$$
S(u,v) \geq 0 \qquad \forall u,v \in [c,d],
$$

*where $\bar{\mathcal{C}}(h,l)$ and $S(u,v)$ are described in* (2.2.11) *and* (2.2.8) *respectively.*

*Proof.* $h$ can be approximated uniformly on $J^2$ by polynomials which have non-negative partial derivatives of order $(Q+1, P+1)$, if $h$ is $(Q+1, P+1)$-convex function. Clearly , Bernstein polynomials

$$
\mathcal{B}^{q,p}(u,v) = \sum_{m=0}^{q}\sum_{n=0}^{p}\binom{q}{m}\binom{p}{n} h(c+m\mathrm{h}, c+n\mathrm{k})(u-c)^m(d-u)^{q-m}(v-c)^n(d-v)^{p-n},
$$

where h $= \frac{(d-c)}{q}$ and k $= \frac{(d-c)}{p}$, converge uniformly to $h$ on $J^2$ as $q \longrightarrow \infty, p \longrightarrow \infty$. Also $h$ is continuous. Moreover, the formula given below may be proved by induction

$$\begin{aligned}
&\mathcal{B}^{q,p}_{(Q+1,P+1)}(u,v)\\
&= (Q+1)!(P+1)! \binom{q}{Q+1} \binom{p}{P+1} \sum_{m=0}^{q-Q-1} \sum_{n=0}^{p-P-1} \binom{q-Q-1}{m} \binom{p-P-1}{n}\\
&\times (\triangle^{Q+1,P+1}_{\mathrm{h,k}} h(c+m\mathrm{h}, c+n\mathrm{k}))(u-c)^m(d-u)^{q-Q-1-m}(v-c)^n(d-v)^{p-P-1-n}.
\end{aligned}$$

Because $h$ is $(Q+1,P+1)$-convex, $\triangle^{(Q+1,P+1)}_{\mathrm{h,k}} \geq 0$ for $\mathrm{h,k} > 0$, so $\mathcal{B}^{q,p}_{(Q+1,P+1)} \geq 0$. As $S(u,v)$ satisfies continuity and $\mathcal{B}^{q,p}_{(Q+1,P+1)} \geq 0$ on $J^2$ so by (2.2.11) it results as

$$\begin{aligned}
&\bar{\mathcal{C}}(\mathcal{B}^{q,p}, l)\\
&= \int_c^d \int_c^d \mathcal{B}^{q,p}_{(Q+1,P+1)}(u,v) \bigg[ \int_{max\{w,z\}}^d \int_c^d l(w,z) \frac{(u-w)^Q}{Q!} \frac{(u-z)^P}{P!} dwdz\\
&- \int_u^d \int_v^d l(w,z) \frac{(w-u)^Q}{Q!} \frac{(z-v)^P}{P!} dzdw \bigg] dvdu \geq 0.
\end{aligned}$$

It may be presented as

$$\bar{\mathcal{C}}(\mathcal{B}^{q,p}, l) = \int_c^d \int_c^d \mathcal{B}^{q,p}_{(Q+1,P+1)}(u,v) S(u,v) dvdu.$$

Considering $q, p \longrightarrow \infty$ by suitable sequence, the uniform convergence of $\mathcal{B}^{q,p}_{(Q+1,P+1)}$ to $h_{(Q+1,P+1)}$ fulfils the requirement. □

# Chapter 3

# Affine and Functional Form of Jensen's Inequality

There exist numerous inequalities in maths literature. They are of tremendous importance due to their extensive use in mathematics. Among all other inequalities jensen's inequality is not only the most important inequality which was introduced by the Danish mathematician John Jensen in 1906, but also it has been observed that many other famous and significant inequalities are formed by modifying this inequality. In this chapter, we described the affine and functional form of Jensen's inequality.

An overview of scientific historic background of Jensen's inequality is provided here. Due to their unique features Jensen and other inequalities were given significant attention throughout last centaury. Further information can be availed from these references [32], [8], [22], [33] and [27]. [29] Deals with numerous convex functions and their inequalities, [36] provides applicability of convex analysis.

## 3.1 Affine and Functional Form of Jensen's Inequality for Continues Convex Function

This section compromises of significant results associated with Jensen's inequality i.e. functional form of Jenssen's inequality as well as extension of Jensen's inequality to affine combination for continuous convex function.

**Definition 3.1.1.** Suppose $\mathbb{Y}$ is a real vector space. If pairs of $u, v \in \mathbb{Y}$ points with coefficient $\mu, \nu \in \mathbb{R}$ are combined, the result is binomial combination

$$\mu u + \nu v \tag{3.1.1}$$

by fulfilling mentioned below these two conditions

$$\mu, \nu \geq 0 \qquad \mu + \nu = 1 \tag{3.1.2}$$

then the above expression in (3.1.1) is said to be convex. Its graphical interpretation is the line segment enclosed by $u$ and $v$. A convex set $\mathcal{J} \subseteq \mathbb{Y}$ must contain all binomial convex combination of its points. If the given below inequality

$$h(\mu u + \nu v) \leq \mu h(u) + \nu h(v) \tag{3.1.3}$$

satisfied for all binomial convex combination $\mu u + \nu v$ of the set $\mathcal{J}$, then the function $h : \mathcal{J} \longrightarrow \mathbb{R}$ is said to be convex.

The expression in (3.1.1) is said to be affine if $\mu + \nu = 1$, and its geometric interpretation is line segment passing through $u$ and $v$. An affine set must contain all binomial affine combination of its points. If the equality in expression (3.1.3) holds for every binomial affine combination $\mu u + \nu v$ of the set $\mathcal{J}$, then the function $h : \mathcal{J} \longrightarrow \mathbb{R}$ is said to be affine.

According to mathematical induction, binomial combination may b substituted with finite combinations. Subsequently expression (3.1.3) converted into the well known Jensen's inequality, see [19].

**Definition 3.1.2.** [26] Suppose $\mathcal{X}$ is a non-empty set in such a way $\mathbb{Y}$ is a subspace of the vector space containing all real functions on the domain $\mathcal{X}$. Lets consider $\mathbb{Y}$ consists of the unit function which is expressed as $I(y) = 1$ for all $y \in \mathcal{X}$. Suppose $\mathcal{J} \subseteq \mathbb{R}$ is an interval in such a way $\mathbb{Y}_{\mathcal{J}} \subseteq \mathbb{Y}$ is subset consisting of all functions whose image lie in $\mathcal{J}$. Let $\mu f + \nu g$ be a convex combination of function $f, g \in \mathbb{Y}_{\mathcal{J}}$, then the number convex combination $\mu f(y) + \nu g(y)$ is in $\mathcal{J}$ for all $y \in \mathcal{X}$, where it shows the function set $\mathbb{Y}_{\mathcal{J}}$ is convex.

**Definition 3.1.3.** [26] Let $R : \mathbb{Y} \longrightarrow \mathbb{R}$ be a positive linear functional for all non-negative function $f \in \mathbb{Y}$ i.e. $R(f) \geq 0$ is supposed to be unital(normalized) if $R(I) = 1$ holds. The closed interval of real numbers with the image of the function $f \in \mathbb{Y}$ contains the number $R(f)$ for every unital positive linear functional $R$.

Functional form of Jensen's inequality is ratify for convex function of one variable by Jensen in 1931[20]. The same may be written in another form as:

**Theorem 3.1.1.** [20] *Suppose $\mathcal{J} \subseteq \mathbb{R}$ is a closed interval, and suppose function $f \in \mathbb{Y}_{\mathcal{J}}$. Suppose $h : \mathcal{J} \longrightarrow \mathbb{R}$ is a continuous convex function in such a way $h(f) \in \mathbb{Y}$. Then for every unital positive linear functional $R : \mathbb{Y} \longrightarrow \mathbb{R}$ fulfills the inclusion*

$$R(f) \in \mathcal{J} \tag{3.1.4}$$

*and the inequality*

$$h(R(f)) \leq R(h(f)). \tag{3.1.5}$$

*The reverse inequality in* (3.1.5) *is obtain, in case of $h$ is concave. Moreover, equality in expression* (3.1.5) *is achieved if $h$ is affine.*

There is restriction on interval $\mathcal{J}$, that it is closed. In other case it is observed that $R(f) \notin \mathcal{J}$, see [23]. In [34], it is observed that inequality mentioned in (3.1.5) is not achieved if $h$ is not continuous function. So $h$ must be continuous function.

To bring simplicity, basic formula regarding convex functions are being mentioned on interval of real numbers. The below mentioned affine combination is unique for every real number $y$ in case if $u$ and $v$ are different real numbers such that $u < v$,

$$y = \frac{v-y}{v-u}u + \frac{y-u}{v-u}v. \tag{3.1.6}$$

The above mentioned binomial combination is said to be convex iff the real number $y$ contain in interval $[u, v]$. Suppose $[u, v]$ belongs to an interval $\mathcal{J} \subseteq \mathbb{R}$, and suppose $h^{line}_{\{u,v\}} : \mathbb{R} \longrightarrow \mathbb{R}$ is a function whose geometric interpretation is line that crosses from $(u, h(u))$ and $(v, h(v))$ for function $h$'s graph. Using the affinity of $h^{line}_{\{u,v\}}$, we get

$$h^{line}_{\{u,v\}}(y) = \frac{v-y}{v-u}h(u) + \frac{y-u}{v-u}h(v) \quad for \;\; y \in \mathbb{R}, \tag{3.1.7}$$

and using the convexity of $h$, it is obtained that

$$h(y) \leq h^{line}_{\{u,v\}}(y) \qquad if \;\; y \in [u, v] \tag{3.1.8}$$

and

$$h(y) \geq h^{line}_{\{u,v\}}(y) \qquad if \;\; y \in \mathcal{J}/(u, v). \tag{3.1.9}$$

**Lemma 3.1.2.** [28] *Suppose $\mu, \nu, \omega \in [0, 1]$ are the coefficients in such a way $\mu + \nu - \omega = 1$. Suppose $u, v, w \in \mathbb{R}$ are the points in such a way $w \in conv\{u, v\}$. Then the affine combination $\mu u + \nu v - \omega w$ belongs to $conv\{u, v\}$, and for each convex function $h : conv\{u, v\} \longrightarrow \mathbb{R}$ fulfills the inequality*

$$h(\mu u + \nu v - \omega w) \leq \mu h(u) + \nu h(v) - \omega h(w). \tag{3.1.10}$$

*Proof.* Involving the convex combination $w = \kappa u + \lambda v$, it indicates that

$$\mu u + \nu v - \omega w = [\mu(1-\kappa) + \kappa(1-\nu)]u + [\nu(1-\lambda) + \lambda(1-\mu)]v. \tag{3.1.11}$$

In the above mention equality (3.1.11) the coefficients written in square parenthesis are nonnegative and their sum is equal to 1. Hence the right side of the above equation (3.1.11) represents the convex combination of points $u$ and $v$. Thus, the combination $\mu u + \nu v - \omega w$ contains in $conv\{u, v\}$. Suppose $u = v$ then the inequality mentioned in (3.1.10) becomes the trivial form $h(u) \leq h(v)$.

Let $u \neq v$, then using the convexity of $h$ and applying affinity of $h^{line}_{\{u,v\}}$, we obtain

$$h(\mu u + \nu v - \omega w) \leq h^{line}_{\{u,v\}}(\mu u + \nu v - \omega w) \tag{3.1.12}$$

$$= \mu h(u) + \nu h(v) - \omega h^{line}_{\{u,v\}}(w) \tag{3.1.13}$$

$$\leq \mu h(u) + \nu h(v) - \omega h(w) \tag{3.1.14}$$

completing the proof. □

**Theorem 3.1.3.** [28] *Suppose* $\mu_m, \nu_n, \omega_k \geq 0$ *are the coefficients in such a way their sum* $\mu = \sum_{m=1}^{N} \mu_m, \nu = \sum_{n=1}^{M} \nu_n, \omega = \sum_{k=1}^{L} \omega_k$ *fulfil* $\mu + \nu - \omega = 1$, *where* $\mu, \nu \in (0, 1]$. *Suppose* $u_m, v_n, w_k \in \mathbb{R}$ *are the points in such a way* $w_k \in Conv\{u, v\}$, *where* $u$ *and* $v$ *are defined as*

$$u = \frac{1}{\mu}\sum_{m=1}^{N} \mu_m u_m, \qquad v = \frac{1}{\nu}\sum_{n=1}^{M} \nu_n v_n. \tag{3.1.15}$$

*Then the affine combination mentioned below*

$$\sum_{m=1}^{N} \mu_m u_m + \sum_{n=1}^{M} \nu_n v_n - \sum_{k=1}^{L} \omega_k w_k \tag{3.1.16}$$

*contained in* $Conv\{u, v\}$, *and every convex function* $h : Conv\{u, v\} \longrightarrow \mathbb{R}$ *fulfils the inequality which is given below*

$$h(\sum_{m=1}^{N} \mu_m u_m + \sum_{n=1}^{M} \nu_n v_n - \sum_{k=1}^{L} \omega_k w_k) \leq \sum_{m=1}^{N} \mu_m h(u_m) + \sum_{n=1}^{N} \nu_n h(v_n) - \sum_{k=1}^{L} \omega_k h(w_k). \tag{3.1.17}$$

*Proof.* As $\mu = 1 - \nu + \omega$, we get $\mu \geq \omega$ and subsequently $\nu \geq \omega$. Let $\omega = 0$, then the affine combination expressed in equation (3.1.16) becomes the convex combination $\mu u + \nu v$ contain in $conv\{u, v\}$, and the inequality expressed in equation (3.1.17) becomes the famous Jensen's inequality. Let $\omega > 0$, then including points $u, v$ and

$$w = \frac{1}{\omega}\sum_{k=1}^{L} \omega_k w_k \tag{3.1.18}$$

in expression (3.1.16), we obtain the combination $\mu u + \nu v - \omega w$ which contain in $conv\{u, v\}$ through Lemma 3.1.2. For the case $u = v$, the inequality mentioned in equation (3.1.17) is trivially true. Thus, it is supposed that $u \neq v$ and apply the function $h^{line}_{\{u,v\}}$. Through the affinity of $h^{line}_{\{u,v\}}$ to the convex combination in equation (3.1.18), and considering the inequalities $h^{line}_{\{u,v\}}(w_k) \geq h(w_k)$, we obtain

$$\begin{aligned} h^{line}_{\{u,v\}}(w) &= \frac{1}{\omega}\sum_{k=1}^{L} \omega_k h^{line}_{\{u,v\}}(w_k) \\ &\geq \frac{1}{\omega}\sum_{k=1}^{L} \omega_k h(w_k). \end{aligned} \tag{3.1.19}$$

Applying the inequality in equation (3.1.12), through Jensen's inequality to $h(u)$ and $h(v)$, and in the end strictly applying the inequality in equation (3.1.19) respecting minus, we obtain

$$\begin{aligned} h\Big(\sum_{m=1}^{N} \mu_m u_m + \sum_{n=1}^{M} \nu_n v_n - \sum_{k=1}^{L} \omega_k w_k\Big) &= h(\mu u + \nu v - \omega w) \\ &\leq \mu h(u) + \nu h(v) - \omega h^{line}_{\{u,v\}}(w) \\ &\leq \sum_{m=1}^{N} \mu_m h(u_m) + \sum_{n=1}^{M} \nu_n h(v_n) - \sum_{k=1}^{L} \omega_k h(w_k), \end{aligned}$$

finishing the proof. □

For convex function and a pair of normalized(unital) positive linear functional an interesting result is observed as given below

**Theorem 3.1.4.** [26] *Suppose $\mathcal{J} \subset \mathbb{R}$ is a closed interval such that $[a, b] \subset \mathcal{J}$. Suppose function $f \in \mathbb{Y}_{[a,b]}$ and function $g \in \mathbb{Y}_{\mathcal{J}\setminus(a,b)}$. Suppose function $h : \mathcal{J} \longrightarrow \mathbb{R}$ is a convex function in such a way $h(f), h(g) \in \mathbb{Y}$. Now if two positive linear functional $R, T : \mathbb{Y} \longrightarrow \mathbb{R}$ fulfil*

$$R(f) = T(g), \tag{3.1.20}$$

*then*

$$R(h(f)) \leq T(h(g)). \tag{3.1.21}$$

*Proof.* To demonstrate the inequality in equation (3.1.21) we consider two cases, one is $u < v$ and the other is $u = v$. Let $u < v$ then it demonstrate that $h(f) \leq h^{line}_{\{u,v\}}(f)$ from equation (3.1.8), and equation (3.1.9) implies $h^{line}_{\{u,v\}}(g) \leq h(g)$. Through given inequalities and application of affinity of $h^{line}_{\{u,v\}}$, we obtain

$$\begin{aligned} R(h(f)) &\leq R(h^{line}_{\{u,v\}}(f)) = h^{line}_{\{u,v\}}(L(g)) \\ &= h^{line}_{\{u,v\}}(T(g)) = T(h^{line}_{\{u,v\}}(g)) \\ &\leq T(h(g)), \end{aligned} \tag{3.1.22}$$

the desired result is achieved for this case.

Consider the case $u = v$. Now if $u$ is the interior point of $\mathcal{J}$, then the series of inequalities in equation (3.1.22) holds with any supporting line function $h^{line}_{\{u\}}$ at $u$, because $h(u) = h(f) = h^{line}_{\{u\}}(f)$ and $h^{line}_{\{u\}}(g) \leq h(g)$ [26]. Now if $u$ is the boundary point of $\mathcal{J}$, then applying continuity of $h$ for every $\epsilon \geq 0$ we can get the interior point $w$ of $\mathcal{J}$ such that $h(u) \leq h(w) + \epsilon$, which means $h(f) \leq h^{line}_{\{w\}}(f) + \epsilon$. Using the function $h^{line}_{\{w\}}$ to equation (3.1.22), we obtain

$$R(h(f)) \leq T(h(g)) + \epsilon, \tag{3.1.23}$$

let $\epsilon$ approaches to zero, we get the desired inequality. □

Generalized form of Theorem 3.1.1 is above mentioned Theorem 3.1.4, as it indicates the given below corollary

**Corollary 3.1.5.** [26] *Suppose $\mathcal{J} \subset \mathbb{R}$ is a closed interval and function $f \in \mathbb{Y}_{\mathcal{J}}$. Suppose continuous convex function $h : \mathcal{J} \longrightarrow \mathbb{R}$ in such a way $h(f) \in \mathbb{Y}$. Where a normalized (unital) positive linear functional $R : \mathbb{Y} \longrightarrow \mathbb{R}$ fulfils the result* (3.1.20) $\Longrightarrow$ (3.1.21) *of above Theorem* 3.1.4 *for $R = T$, then*

$$h(R(f)) \leq R(h(f)). \tag{3.1.24}$$

*Proof.* Considering function $f_0 = R(f)I$, and applying the condition

$$R(f_0) = R(f) \tag{3.1.25}$$

with functions $f_0$ and $f$ fulfilling the requirements of Theorem 3.1.4, we can get

$$R(h(f_0)) \leq R(h(f)). \tag{3.1.26}$$

As $h(f_0) = h(R(f_0))I$ constant function value of $h(R(f))$, the left side of the inequality (3.1.26) is equal to $h(R(f))$ which gives the desired result. □

The extension of Theorem 3.1.4 through many unital functional is as mentioned below

**Corollary 3.1.6.** [26] *Suppose* $[a_1, b_1] \subseteq \ldots \subseteq [a_{n-1}, b_{n-1}] \subseteq \mathcal{J}$. *Suppose function* $f_1 \in \mathbb{Y}_{[a_1,b_1]}$ *and function* $f_k \in \mathbb{Y}_{[a_k,b_k]\setminus(a_{k-1},b_{k-1})}$ *for* $k = 2, 3, \ldots, n-1$, *and function* $f_n \in \mathbb{Y}_{\mathcal{J}\setminus(a_{n-1},b_{n-1})}$. *Suppose a continuous convex function* $h : \mathcal{J} \longrightarrow \mathbb{R}$ *in such a way* $h(f_m) \in \mathbb{Y}$. *If an n-tuple of unital positive linear functionals* $R_m : \mathbb{Y} \longrightarrow \mathbb{R}$ *fulfils*

$$R_m(f_m) = R_{m+1}(f_{m+1}) \quad for \quad m = 1, \ldots n-1, \tag{3.1.27}$$

*then*

$$R_m(h(f_m)) \leq R_{m+1}(h(f_{m+1})) \quad for \quad m = 1, \ldots n-1. \tag{3.1.28}$$

For further generalization of Theorem 3.1.1 and Theorem 3.1.4, if the normalized(unital) functional $R$ is substituted with $N$ functionals $R_m$ fulfilling $\sum_{m=1}^{N} R_m(I) = 1$.

If above mentioned functional collection and Theorem 3.1.1 are combined in discrete form of jensen's inequality, the mentioned below results are achieved

**Corollary 3.1.7.** [26] *Suppose* $\mathcal{J} \subset \mathbb{R}$ *is a closed interval and function* $f_1, \ldots f_N \in \mathbb{Y}_{\mathcal{J}}$. *Suppose* $h : \mathcal{J} \longrightarrow \mathbb{R}$ *is a continuous convex function in such a way* $h(f_m) \in \mathbb{Y}$.

*Then every n-tuple of positive linear functional* $R_m : \mathbb{Y} \longrightarrow \mathbb{R}$ *with* $\sum_{m=1}^{N} R_m(1) = 1$ *fulfils the inclusion*

$$\sum_{m=1}^{N} R_m(f_m) \in \mathcal{J} \tag{3.1.29}$$

*and the inequality*

$$g(\sum_{m=1}^{N} R_m(f_m)) \leq \sum_{m=1}^{N} R_m(g(f_m)). \tag{3.1.30}$$

*Proof.* Without loss of generality, we can presume for every number $\lambda_m = R_m(I) > 0$, and consider unital positive linear functionals $S_m = (\frac{1}{\lambda_m})R_m$. Then every $S_m(f_m) \in \mathcal{J}$ by the inclusion in expression (3.1.4), hence proved that the convex combination

$$\sum_{m=1}^{N} R_m(f_m) = \sum_{m=1}^{N} \lambda_m S_m(f_m) \tag{3.1.31}$$

contains in the interval $\mathcal{J}$. By application of Jensen's inequality in its discrete form to expression (3.1.31), also applying the inequalities $h(S_m(f_m)) \leq S_m(h(f_m))$ obtained through equation (3.1.5), we get the desired results. □

The extension of Theorem 3.1.4 provides two collections of positive linear functional. Every collection should strictly fulfill the sum condition considering the unit function.

**Theorem 3.1.8.** [26] *Suppose* $\mathcal{J} \subset \mathbb{R}$ *is a closed interval and* $[a, b] \in \mathcal{J}$. *Suppose function* $f_1, \ldots f_N \in \mathbb{Y}_{[a,b]}$ *and* $g_1, g_2, \ldots, g_M \in \mathbb{Y}_{\mathcal{J}\setminus(a,b)}$. *Suppose* $h : \mathcal{J} \longrightarrow \mathbb{R}$ *is a continuous convex function in such a way* $h(f_m), h(g_n) \in \mathbb{Y}$. *If a couple of n-tuple of positive linear functional* $R_m, T_n : \mathbb{Y} \longrightarrow \mathbb{R}$ *with* $\sum_{m=1}^{N} R_m(1) = \sum_{n=1}^{M} T_n(1) = 1$ *fulfil*

$$\sum_{m=1}^{N} R_m(f_m) = \sum_{n=1}^{M} T_n(g_n), \tag{3.1.32}$$

*then the inequality given below is valid*

$$\sum_{m=1}^{N} R_m(h(f_m)) \leq \sum_{n=1}^{M} T_n(h(g_n)). \tag{3.1.33}$$

*Proof.* The sum $\sum_{m=1}^{N} R_m(fm)$ is contain in $[u, v]$ by equation (3.1.31). The proof is similar to Theorem 3.1.4. In the case $u < v$, the equation (3.1.22) is used considering the equality

$$\sum_{m=1}^{N} R_m(h_{\{u,v\}}^{line}(f_m)) = h_{\{u,v\}}^{line}(\sum_{m=1}^{N} R_m(f_m)) \tag{3.1.34}$$

for $N - tuples\ \ f_m$ and $R_m$, and the similar expression is also obtained for $m$-tuples $g_n$ and $T_n$. □

## 3.2 Affine and Functional Form of Jensen's Inequality for 3-Convex Function

I. A. Baloch, J. Pečarić, M. Pralijak [4] gave the well defined new class of function which is mentioned below

**Definition 3.2.1.** Suppose $b \in J^0$, where $J$ be any interval(open, closed or semi open in either direction) in $\mathbb{R}$ and $J^0$ is its interior. A function $h : J \longrightarrow \mathbb{R}$ is supposed to b 3-convex function at point $b$ (respectively 3-concave function at point $b$)if there exist a constant $\alpha$ in such a way that the function $H(y) = h(y) - \frac{\alpha}{2}y^2$ is concave (respectively convex) on $J \cap (-\infty, b]$ and convex (respectively concave) on $J \cap [b, \infty)$. A function $h$ is 3-concave function at point $b$ if $-h$ is 3-convex function at point $b$.

A function is said to be 3-convex function on an interval iff it is 3-convex function at every point of the interval. Here $\mathcal{K}_1^b(J)$ and $\mathcal{K}_2^b(J)$ represents the class of all 3-convex function at point $b$ and the class of 3-concave function at point $b$ respectively.

**Theorem 3.2.1.** [4] *Suppose* $\mu_m, \nu_n, \omega_k \geq 0$ *and* $\xi_m, \eta_n, \zeta_k \geq 0$ *are the coefficients in such a way their sum* $\mu = \sum_{m=1}^{N} \mu_m, \nu = \sum_{n=1}^{M} \nu_n, \omega = \sum_{k=1}^{L} \omega_k$ *fulfil* $\mu + \nu - \omega = 1$, *where* $\mu, \nu \in (0, 1]$ *and* $\xi = \sum_{m=1}^{N} \xi_m, \eta = \sum_{n=1}^{M} \eta_n, \zeta = \sum_{k=1}^{L} \zeta_k$ *fulfil* $\xi + \eta - \zeta = 1$ *and* $\xi, \eta \in (0, 1]$ . *Suppose* $u_m, v_n, w_k \in [a, b]$ *are the points in such a way* $w_k \in$

*$Conv\{u,v\}$ and $p_m, q_n, r_k \in [b,c]$ are the points in such a way $r_k \in Conv\{p,q\}$, where $u, v, p$ and $q$ are defined as*

$$u = \frac{1}{\mu}\sum_{m=1}^{N}\mu_m u_m,\ \ v = \frac{1}{\nu}\sum_{n=1}^{M}\nu_n v_n,\ \ p = \frac{1}{\mu}\sum_{m=1}^{N}\xi_m p_m,\ \ q = \frac{1}{\nu}\sum_{n=1}^{M}\eta_n q_n.$$

*Now, if*

$$\sum_{m=1}^{N}\mu_m(u_m)^2 + \sum_{n=1}^{M}\nu_n(v_n)^2 - \sum_{k=1}^{L}\omega_k(w_k)^2 - \left(\sum_{m=1}^{N}\mu_m u_m + \sum_{n=1}^{M}\nu_n v_n - \sum_{k=1}^{L}\omega_k w_k\right)^2$$
$$= \sum_{m=1}^{N}\xi_m(p_m)^2 + \sum_{n=1}^{M}\eta_n(q_n)^2 - \sum_{k=1}^{L}\zeta_k(r_k)^2 - \left(\sum_{m=1}^{N}\xi_m p_m + \sum_{n=1}^{M}\eta_n q_n - \sum_{k=1}^{L}\zeta_k r_k\right)^2 \tag{3.2.1}$$

*and also there exists $b \in J^{\circ}(J = [a,c])$ such that*

$$max\{max_m\{u_m\}, max_n\{v_n\}, max_k\{w_k\}\} \le b \le min\{min_m\{p_m\}, min_n\{q_n\}, min_k\{r_k\}\}.$$

*Then for all $h \in \mathcal{K}_1^b(J)$, the below mentioned inequality is valid*

$$\sum_{m=1}^{N}\mu_m h(u_m) + \sum_{n=1}^{M}\nu_n h(v_n) - \sum_{k=1}^{L}\omega_k h(w_k) - h\left(\sum_{m=1}^{N}\mu_m u_m + \sum_{n=1}^{M}\nu_n v_n - \sum_{k=1}^{L}\omega_k w_k\right)$$
$$= \sum_{m=1}^{N}\xi_m h(p_m) + \sum_{n=1}^{M}\eta_n h(q_n) - \sum_{k=1}^{L}\zeta_k h(r_k) - h\left(\sum_{m=1}^{N}\xi_m p_m + \sum_{n=1}^{M}\eta_n q_n - \sum_{k=1}^{L}\zeta_k r_k\right). \tag{3.2.2}$$

*Proof.* As we know $h \in \mathcal{K}_1^b(J)$, so there exist a constant $\alpha$ in such a way $H(y) = h(y) - \frac{\alpha}{2}y^2$ is concave on $J \cap (-\infty, b]$ and $u_m, v_n, w_k \in [a,b]$ are the points in such a way $w_k \in Conv\{u, v\}$, hence by applying inequality (3.1.17) we obtain

$$0 \geq \sum_{m=1}^{N} \mu_m H(u_m) + \sum_{n=1}^{M} \nu_n H(v_n) - \sum_{k=1}^{L} \omega_k H(w_k) - H\left(\sum_{m=1}^{N} \mu_m u_m + \sum_{n=1}^{M} \nu_n v_n - \sum_{k=1}^{L} \omega_k w_k\right)$$
$$= \sum_{m=1}^{N} \mu_m h(u_m) + \sum_{n=1}^{M} \nu_n h(v_n) - \sum_{k=1}^{L} \omega_k h(w_k) - h\left(\sum_{m=1}^{N} \mu_m u_m + \sum_{n=1}^{M} \nu_n v_n - \sum_{k=1}^{L} \omega_k w_k\right)$$
$$- \frac{\alpha}{2}\left\{\sum_{m=1}^{N} \mu_m (u_m)^2 + \sum_{n=1}^{M} \nu_n (v_n)^2 - \sum_{k=1}^{L} \omega_k (w_k)^2 - \left(\sum_{m=1}^{N} \mu_m u_m + \sum_{n=1}^{M} \nu_n v_n - \sum_{k=1}^{L} \omega_k w_k\right)^2\right\}$$

Furthermore, as $h \in \mathcal{K}_1^b(J)$ is convex on $J \cap [b, \infty)$, thus for $p_m, q_n, r_k \in [b, c]$ are points in such a way $r_k \in Conv\{p, q\}$, then by applying inequality (3.1.17), we get

$$0 \geq \sum_{m=1}^{N} \mu_m H(p_m) + \sum_{n=1}^{M} \nu_n H(q_n) - \sum_{k=1}^{L} \omega_k H(r_k) - H\left(\sum_{m=1}^{N} \mu_m p_m + \sum_{n=1}^{M} \nu_n q_n - \sum_{k=1}^{L} \omega_k r_k\right)$$
$$= \sum_{m=1}^{N} \mu_m h(p_m) + \sum_{n=1}^{M} \nu_n h(q_n) - \sum_{k=1}^{L} \omega_k h(r_k) - h\left(\sum_{m=1}^{N} \mu_m p_m + \sum_{n=1}^{M} \nu_n q_n - \sum_{k=1}^{L} \omega_k r_k\right)$$
$$- \frac{\alpha}{2}\left\{\sum_{m=1}^{N} \mu_m (p_m)^2 + \sum_{n=1}^{M} \nu_n (q_n)^2 - \sum_{k=1}^{L} \omega_k (r_k)^2 - \left(\sum_{m=1}^{N} \mu_m p_m + \sum_{n=1}^{M} \nu_n q_n - \sum_{k=1}^{L} \omega_k r_k\right)^2\right\}$$

Through previous relation, the following is observed

$$\sum_{m=1}^{N} \mu_m h(u_m) + \sum_{n=1}^{M} \nu_n h(v_n) - \sum_{k=1}^{L} \omega_k h(w_k) - h\left(\sum_{m=1}^{N} \mu_m u_m + \sum_{n=1}^{M} \nu_n v_n - \sum_{k=1}^{L} \omega_k w_k\right)$$
$$- \frac{\alpha}{2}\left\{\sum_{m=1}^{N} \mu_m (u_m)^2 + \sum_{n=1}^{M} \nu_n (v_n)^2 - \sum_{k=1}^{L} \omega_k (w_k)^2 - \left(\sum_{m=1}^{N} \mu_m u_m + \sum_{n=1}^{M} \nu_n v_n - \sum_{k=1}^{L} \omega_k w_k\right)^2\right\}$$
$$\leq 0 \leq$$
$$= \sum_{m=1}^{N} \mu_m h(u_m) + \sum_{n=1}^{M} \nu_n h(v_n) - \sum_{k=1}^{L} \omega_k h(w_k) - h\left(\sum_{m=1}^{N} \mu_m u_m + \sum_{n=1}^{M} \nu_n v_n - \sum_{k=1}^{L} \omega_k w_k\right)$$
$$- \frac{\alpha}{2}\left\{\sum_{m=1}^{N} \mu_m (u_m)^2 + \sum_{n=1}^{M} \nu_n (v_n)^2 - \sum_{k=1}^{L} \omega_k (w_k)^2 - \left(\sum_{m=1}^{N} \mu_m u_m + \sum_{n=1}^{M} \nu_n v_n - \sum_{k=1}^{L} \omega_k w_k\right)^2\right\}$$

Hence

$$\sum_{m=1}^{N}\mu_m h(u_m)+\sum_{n=1}^{M}\nu_n h(v_n)-\sum_{k=1}^{L}\omega_k h(w_k)-h\left(\sum_{m=1}^{N}\mu_m u_m+\sum_{n=1}^{M}\nu_n v_n-\sum_{k=1}^{L}\omega_k w_k\right)$$
$$-\frac{\alpha}{2}\left\{\sum_{m=1}^{N}\mu_m (u_m)^2+\sum_{n=1}^{M}\nu_n (v_n)^2-\sum_{k=1}^{L}\omega_k (w_k)^2-\left(\sum_{m=1}^{N}\mu_m u_m+\sum_{n=1}^{M}\nu_n v_n-\sum_{k=1}^{L}\omega_k w_k\right)^2\right\}$$
$$\leq \sum_{m=1}^{N}\mu_m h(u_m)+\sum_{n=1}^{M}\nu_n h(v_n)-\sum_{k=1}^{L}\omega_k h(w_k)-h\left(\sum_{m=1}^{N}\mu_m u_m+\sum_{n=1}^{M}\nu_n v_n-\sum_{k=1}^{L}\omega_k w_k\right)$$
$$-\frac{\alpha}{2}\left\{\sum_{m=1}^{N}\mu_m (u_m)^2+\sum_{n=1}^{M}\nu_n (v_n)^2-\sum_{k=1}^{L}\omega_k (w_k)^2-\left(\sum_{m=1}^{N}\mu_m u_m+\sum_{n=1}^{M}\nu_n v_n-\sum_{k=1}^{L}\omega_k w_k\right)^2\right\}$$

by applying (3.2.1), we obtain desired result.

□

**Theorem 3.2.2.** [4] *Suppose $\mathcal{J}\subset\mathbb{R}$ is a closed interval such that $[a,b]\subset\mathcal{J}$ and function $g_k\in\mathbb{Y}_{\mathcal{J}\setminus(a,b)}$ for $k=1,2$. Suppose a continuous convex function $h\in\mathcal{K}_1^b(\mathcal{J})$ in such a way $h(f_k),h(g_k)\in\mathbb{Y}$. Now if two positive linear functionals $R,T:\mathbb{Y}\longrightarrow\mathbb{R}$ fulfils*

$$R(f_k)=T(g_k)\quad \textit{and}\ T(g_1^2)-R(f_1^2)=T(g_2^2)-R(f_2^2),\quad k=1,2, \tag{3.2.3}$$

*then the inequality*

$$T(h(g_1))-R(h(f_1))\leq T(h(g_2))-R(h(f_2)) \tag{3.2.4}$$

*is valid.*

*Proof.* Because $h\in\mathcal{K}_1^b(\mathcal{J})$, there exist a constant $\alpha$ in such a way $H(y)=h(y)-\frac{\alpha}{2}y^2$ is concave on $\mathcal{J}\cap(-\infty,b]$, hence by reversing equation (3.1.21) for $H$ on $\mathcal{J}\cap(-\infty,b]$, we obtain

$$\begin{aligned}0 &\geq T(H(g_1))-R(H(f_1))\\ &= T(h(g_1))-R(h(f_1))-\frac{\alpha}{2}(T(g_1^2)-R(f_1^2))\end{aligned}$$

Furthermore, as $H(y) = h(y) - \frac{\alpha}{2}y^2$ is convex on $\mathcal{J} \cap (b, \infty]$, hence by (3.1.21) for $H$ on $\mathcal{J} \cap (-\infty, b]$, we obtain

$$
\begin{aligned}
0 &\geq T(H(g_2)) - R(H(f_2)) \\
&= T(h(g_2)) - R(h(f_2)) - \frac{\alpha}{2}(T(g_2^2) - R(f_2^2))
\end{aligned}
$$

So, the following expression is obtained

$$
\begin{gathered}
T(h(g_1)) - R(h(f_1)) - \frac{\alpha}{2}(T(g_1^2) - R(f_1^2)) \\
\leq 0 \leq \\
T(h(g_2)) - R(h(f_2)) - \frac{\alpha}{2}(T(g_2^2) - R(f_2^2)).
\end{gathered}
$$

Hence

$$
\begin{aligned}
&T(h(g_1)) - R(h(f_1)) - \frac{\alpha}{2}(T(g_1^2) - R(f_1^2)) \\
&\qquad \leq T(h(g_2)) - R(h(f_2)) - \frac{\alpha}{2}(T(g_2^2) - R(f_2^2)),
\end{aligned}
$$

Hence, from (3.2.3), we obtain the desired result. □

**Remark 3.2.1.** Theorem 3.2.2 provides

$$T(h(g_1)) - R(h(f_1)) \leq \frac{\alpha}{2}(T(g_1^2) - R(f_1^2)) \tag{3.2.5}$$

and

$$T(h(g_2)) - R(h(f_2)) \geq \frac{\alpha}{2}(T(g_2^2) - R(f_2^2)). \tag{3.2.6}$$

According to equation (3.2.3), we obtain refinement of (3.2.4) as below

$$
\begin{aligned}
&T(h(g_1)) - R(h(f_1)) \leq \\
&\qquad \frac{\alpha}{2}(T(g_1^2) - R(f_1^2))(= \frac{\alpha}{2}(T(g_2^2) - R(f_2^2))) \\
&\qquad \leq T(h(g_2)) - R(h(f_2))
\end{aligned}
$$

# Chapter 4

# Functionals Generated by Functional Form of Jensen's Inequality for Exponentially and Log-convex Functions

As an observable fact, the implication of functionals is evident in the mathematical literature. Consequently, in this section some new functionals have been derived. Let the family of functions is defined as:

$$h^t(f) = \begin{cases} \frac{f^t}{t(t-1)(t-2)...(t-n)}, & t \notin \{0, 1, 2, \cdots, n\}, \\ \\ \frac{f^t \log(f)}{(-1)^n t!(n-t)!}, & t \in \{0, 1, 2, \cdots, n\}. \end{cases} \tag{4.0.1}$$

Clearly $h^t_{n+1} = f^{t-n-1} = e^{(t-n-1)\log(f)} \geq 0$ for $f \in \mathbb{R}$, so $h^t(f)$ is $n{+}1{-}convex function$ and $f \longmapsto h^t_{n+1}(f)$ is exponentially convex function on $\mathbb{R}$. Furthermore, according to Corollary 1.4.1, this function $h^t_{n+1}(f)$ is log-convex function.

## 4.1 Exponential Convexity of Jensen's Functional

The functional being used below $\Lambda_t$ has been derived through Corollary 3.1.5.

**Theorem 4.1.1.** *Let*

$$\Lambda_t = R(h^t(f)) - h^t(R(f)) \tag{4.1.1}$$

*be a linear functional and $h^t$ is a well defined function explained earlier in* (4.0.1) *then it satisfies the following statements.*

1. $t \longmapsto \Lambda_t$ *is continuous on* $\mathbb{R}$.
2. $t \longmapsto \Lambda_t$ *is exponentially convex function.*
3. *If* $t \longmapsto \Lambda_t$ *is positive function on* $\mathbb{R}$ *then* $t \longmapsto \Lambda_t$ *is log-convex function on* $\mathbb{R}$.
4. *For every* $k \in \mathbb{N}$ *and* $t_1, t_2, ....t_k \in \mathbb{R}$ *the matrix*

$$[\Lambda_{\frac{t_i+t_j}{2}}]_{i,j=1}^k$$

*be a positive semi-definite. Particulary*

$$det[\Lambda_{\frac{t_i+t_j}{2}}]_{i,j=1}^k \geq 0.$$

5. *If* $t \longmapsto \Lambda_t$ *is differentiable on* $\mathbb{R}$ *then for each* $\mathrm{s}, \mathrm{t}, \mathrm{u}, \mathrm{v} \in \mathbb{R}$ *while* $\mathrm{s} \leq \mathrm{u}$ *and* $\mathrm{t} \leq \mathrm{v}$, *we obtain*

$$\mathfrak{m}_{\mathrm{s},\mathrm{t}}(f) \leq \mathfrak{m}_{\mathrm{u},\mathrm{v}}(f) \tag{4.1.2}$$

*where*

$$\mathfrak{m}_{\mathrm{s},\mathrm{t}}(f) = \begin{cases} \left(\frac{\Lambda_\mathrm{s}}{\Lambda_\mathrm{t}}\right)^{\frac{1}{\mathrm{s}-\mathrm{t}}}, & \mathrm{s} \neq \mathrm{t}, \\ \\ exp\left(\frac{\frac{d}{d\mathrm{s}}\Lambda_\mathrm{s}}{\Lambda_\mathrm{s}}\right), & \mathrm{s} = \mathrm{t}. \end{cases} \tag{4.1.3}$$

*Proof.* $(i)$ If we use L. Hopital rule and apply limit, we get

$$\lim_{t\to 0}\Lambda_t = \lim_{t\to 0}(R(h^t(f)) - h^t(R(f))) = \frac{R\log(f)}{(-1)^n n!} - \frac{\log(R(f))}{(-1)^n n!} = \Lambda_0$$

By using similar technique we can obtain

$$\lim_{t\to 0}\Lambda_t = \Lambda_k \qquad\qquad k = 1, 2, 3\dots n.$$

(ii). Suppose the function is defined as

$$w(f) = \sum_{i,j=1}^{K} \alpha_i\alpha_j h^{\frac{t_i+t_j}{2}}(f),$$

where $t_i \in \mathbb{R},\ \alpha_i \in \mathbb{R},\ i = 1, 2, \dots k.$

As the function $t \longmapsto h^t_{n+1}$ is exponentially convex, we have

$$w_{n+1}(f) = \sum_{i,j=1}^{K} \alpha_i\alpha_j h_{n+1}^{\frac{t_i+t_j}{2}}(f) = \sum_{i,j=1}^{K} \alpha_i\alpha_j e^{t_{ij}-n-\log(f)} \geq 0, \quad where \;\; t_{ij} = \frac{t_i+t_j}{2}$$

which indicates that $w$ is $n+1-convex function$ on $\mathbb{R}$ along with it $\Lambda(w) \geq 0$. Therefore

$$\sum_{i,j=1}^{K} \alpha_i\alpha_j \Lambda(h^{\frac{t_i+t_j}{2}}) \geq 0.$$

It is proved that the function $t \longmapsto \Lambda_t$ is an exponentially convex function on $\mathbb{R}$.

$(iii)$. It is deduced from $(ii)$.

$(iv)$. This is deduction of Corollary 1.3.2.

$(v)$. The inequality described below is extracted by the convex function's definition $\phi$ [29]

$$\frac{\phi(\mathrm{s}) - \phi(\mathrm{t})}{\mathrm{s}-\mathrm{t}} \leq \frac{\phi(\mathrm{u}) - \phi(\mathrm{v})}{\mathrm{u}-\mathrm{v}},$$

for all $\mathrm{s}, \mathrm{t}, \mathrm{u}, \mathrm{v} \in J \subset \mathbb{R}$ such that $\mathrm{s} \leq \mathrm{u}, \mathrm{t} \leq \mathrm{v}, \mathrm{s} \neq \mathrm{t}, \mathrm{u} \neq \mathrm{v}$.
Since by $(iii)$, $\Lambda_{\mathrm{t}}$ is log-convex function, hence we substitute $\phi(f) = \log(\Lambda_{\mathrm{t}})$ in above equation, we have

$$\frac{\log \Lambda_{\mathrm{s}} - \log \Lambda_{\mathrm{t}}}{\mathrm{s} - \mathrm{t}} \leq \frac{\log \Lambda_{\mathrm{u}} - \log \Lambda_{\mathrm{v}}}{\mathrm{u} - \mathrm{v}},$$

for $\mathrm{s} \leq \mathrm{u}, \mathrm{t} \leq \mathrm{v}, \mathrm{s} \neq \mathrm{t}, \mathrm{u} \neq \mathrm{v}$, that is analogues to equation (4.3.3). The illustration for $\mathrm{s} = \mathrm{t}$ and/or $\mathrm{u} = \mathrm{v}$ are certainly deduced through above inequality while considering corresponding limits. □

## 4.2 Exponential Convexity of Difference of Unital Positive Linear Functional

The functional described below is deduced from Theorem 3.1.4.

**Theorem 4.2.1.** *Let*

$$\Lambda_t = T(h^t(g)) - R(h^t(f)) \tag{4.2.1}$$

*be a linear functional and $h^t$ is a well defined function explained earlier in* (4.0.1) *then it satisfies the following statements.*

1. $t \longmapsto \Lambda_t$ *is continuous on* $\mathbb{R}$.
2. $t \longmapsto \Lambda_t$ *is exponentially convex function.*
3. *If* $t \longmapsto \Lambda_t$ *is positive function on* $\mathbb{R}$ *then* $t \longmapsto \Lambda_t$ *is log-convex function on* $\mathbb{R}$.

4. *For every* $k \in \mathbb{N}$ *and* $t_1, t_2, ....t_k \in \mathbb{R}$ *the matrix*

$$[\Lambda_{\frac{t_i+t_j}{2}}]_{i,j=1}^{k}$$

*be a positive semi-definite. Particulary*

$$det[\Lambda_{\frac{t_i+t_j}{2}}]_{i,j=1}^{k} \geq 0.$$

5. *If* $t \longmapsto \Lambda_t$ *is differentiable on* $\mathbb{R}$ *then for each* $\mathrm{s}, \mathrm{t}, \mathrm{u}, \mathrm{v} \in \mathbb{R}$ *while* $\mathrm{s} \leq \mathrm{u}$ *and* $\mathrm{t} \leq \mathrm{v}$*, we obtain*

$$\mathfrak{m}_{\mathrm{s},\mathrm{t}}(f) \leq \mathfrak{m}_{\mathrm{u},\mathrm{v}}(f) \tag{4.2.2}$$

*where*

$$\mathfrak{m}_{\mathrm{s},\mathrm{t}}(f) = \begin{cases} (\frac{\Lambda_{\mathrm{s}}}{\Lambda_{\mathrm{t}}})^{\frac{1}{\mathrm{s}-\mathrm{t}}}, & \mathrm{s} \neq \mathrm{t}, \\ \\ exp(\frac{\frac{d}{d\mathrm{s}}\Lambda_{\mathrm{s}}}{\Lambda_{\mathrm{s}}}), & \mathrm{s} = \mathrm{t}. \end{cases} \tag{4.2.3}$$

*Proof.* Parallel to the proof of Theorem 4.1.1. □

The functional described below is deduced from The Corollary 3.1.7.

**Theorem 4.2.2.** *Let*

$$\Lambda_t = \sum_{i=1}^{n} R_i(h^t(f_i)) - h^t(\sum_{i=1}^{n} R_i(f_i)) \tag{4.2.4}$$

*be a linear functional and* $h^t$ *is a well defined function explained earlier in* (4.0.1) *then it satisfies the following statements.*

1. $t \longmapsto \Lambda_t$ *is continuous on* $\mathbb{R}$.

2. $t \longmapsto \Lambda_t$ *is exponentially convex function.*

3. *If* $t \longmapsto \Lambda_t$ *is positive function on* $\mathbb{R}$ *then* $t \longmapsto \Lambda_t$ *is log-convex function on* $\mathbb{R}$.

4. *For every* $k \in \mathbb{N}$ *and* $t_1, t_2, ....t_k \in \mathbb{R}$ *the matrix*

$$[\Lambda_{\frac{t_i+t_j}{2}}]_{i,j=1}^{k}$$

*be a positive semi-definite. Particulary*

$$det[\Lambda_{\frac{t_i+t_j}{2}}]_{i,j=1}^{k} \geq 0.$$

5. *If* $t \longmapsto \Lambda_t$ *is differentiable on* $\mathbb{R}$ *then for each* $\mathrm{s}, \mathrm{t}, \mathrm{u}, \mathrm{v} \in \mathbb{R}$ *while* $\mathrm{s} \leq \mathrm{u}$ *and* $\mathrm{t} \leq \mathrm{v}$*, we obtain*

$$\mathfrak{m}_{\mathrm{s,t}}(f) \leq \mathfrak{m}_{\mathrm{u,v}}(f) \tag{4.2.5}$$

*where*

$$\mathfrak{m}_{\mathrm{s,t}}(f) = \begin{cases} (\frac{\Lambda_{\mathrm{s}}}{\Lambda_{\mathrm{t}}})^{\frac{1}{\mathrm{s-t}}}, & \mathrm{s} \neq \mathrm{t}, \\ \\ exp(\frac{\frac{d}{d\mathrm{s}}\Lambda_{\mathrm{s}}}{\Lambda_{\mathrm{s}}}), & \mathrm{s} = \mathrm{t}. \end{cases} \tag{4.2.6}$$

*Proof.* similar to the proof of Theorem 4.1.1. □

The functional described below is deduced from Theorem 3.1.6.

**Theorem 4.2.3.** *Let*

$$\Lambda_t = R_{i+1}(h^t(f_{i+1})) - R_i(h^t(f_i)) \tag{4.2.7}$$

*be a linear functional and* $h^t$ *is a well defined function explained earlier in* (4.0.1) *then it satisfies the following statements.*

1. $t \longmapsto \Lambda_t$ *is continuous on* $\mathbb{R}$.

2. *$t \longmapsto \Lambda_t$ is exponentially convex function.*

3. *If $t \longmapsto \Lambda_t$ is positive function on $\mathbb{R}$ then $t \longmapsto \Lambda_t$ is log-convex function on $\mathbb{R}$.*

4. *For every $k \in \mathbb{N}$ and $t_1, t_2, ....t_k \in \mathbb{R}$ the matrix*

$$[\Lambda_{\frac{t_i+t_j}{2}}]_{i,j=1}^{k}$$

*be a positive semi-definite. Particulary*

$$det[\Lambda_{\frac{t_i+t_j}{2}}]_{i,j=1}^{k} \geq 0.$$

5. *If $t \longmapsto \Lambda_t$ is differentiable on $\mathbb{R}$ then for each $\mathrm{s}, \mathrm{t}, \mathrm{u}, \mathrm{v} \in \mathbb{R}$ while $\mathrm{s} \leq \mathrm{u}$ and $\mathrm{t} \leq \mathrm{v}$, we obtain*

$$\mathfrak{m}_{\mathrm{s},\mathrm{t}}(f) \leq \mathfrak{m}_{\mathrm{u},\mathrm{v}}(f) \tag{4.2.8}$$

*where*

$$\mathfrak{m}_{\mathrm{s},\mathrm{t}}(f) = \begin{cases} (\frac{\Lambda_{\mathrm{s}}}{\Lambda_{\mathrm{t}}})^{\frac{1}{\mathrm{s}-\mathrm{t}}}, & \mathrm{s} \neq \mathrm{t}, \\ \\ exp(\frac{\frac{d}{d\mathrm{s}}\Lambda_{\mathrm{s}}}{\Lambda_{\mathrm{s}}}), & \mathrm{s} = \mathrm{t}. \end{cases} \tag{4.2.9}$$

*Proof.* Parallel to the proof of Theorem 4.1.1. □

## 4.3 Exponential Convexity of Jensen's Functional of Many Unital Functional

The functional described below is deduced from Theorem 3.1.8.

**Theorem 4.3.1.** *Let*

$$\Lambda_t = \sum_{j=1}^{m} T_j(h^t(g_j)) - \sum_{i=1}^{n} R_i(h^t(f_i)) \tag{4.3.1}$$

*be a linear functional and* $h^t$ *is a well defined function explained earlier in* (4.0.1) *then it satisfies the following statements.*

1. $t \longmapsto \Lambda_t$ *is continuous on* $\mathbb{R}$.

2. $t \longmapsto \Lambda_t$ *is exponentially convex function.*

3. *If* $t \longmapsto \Lambda_t$ *is positive function on* $\mathbb{R}$ *then* $t \longmapsto \Lambda_t$ *is log-convex function on* $\mathbb{R}$.

4. *For every* $k \in \mathbb{N}$ *and* $t_1, t_2, ....t_k \in \mathbb{R}$ *the matrix*

$$[\Lambda_{\frac{t_i+t_j}{2}}]_{i,j=1}^{k}$$

*be a positive semi-definite. Particulary*

$$det[\Lambda_{\frac{t_i+t_j}{2}}]_{i,j=1}^{k} \geq 0.$$

5. *If* $t \longmapsto \Lambda_t$ *is differentiable on* $\mathbb{R}$ *then for each* $\mathrm{s}, \mathrm{t}, \mathrm{u}, \mathrm{v} \in \mathbb{R}$ *while* $\mathrm{s} \leq \mathrm{u}$ *and* $\mathrm{t} \leq \mathrm{v}$*, we obtain*

$$\mathfrak{m}_{\mathrm{s,t}}(f) \leq \mathfrak{m}_{\mathrm{u,v}}(f) \tag{4.3.2}$$

*where*

$$\mathfrak{m}_{\mathrm{s,t}}(f) = \begin{cases} (\frac{\Lambda_{\mathrm{s}}}{\Lambda_{\mathrm{t}}})^{\frac{1}{\mathrm{s-t}}}, & \mathrm{s} \neq \mathrm{t}, \\ \\ exp(\frac{\frac{d}{d\mathrm{s}}\Lambda_{\mathrm{s}}}{\Lambda_{\mathrm{s}}}), & \mathrm{s} = \mathrm{t}. \end{cases} \tag{4.3.3}$$

*Proof.* Parallel to the proof of Theorem 4.1.1. □

## 4.4 Conclusion

In this thesis, we have discussed the generalization of Convex function as well as Jensen's type inequality for generalized Convex function. Along with it the generalization of $\check{C}$eby$\check{s}$ev identity and inequality are also analyzed. Moreover, refinement of Jensen's inequality for affine and functional form has been presented, through the functional form of Jensen's inequality we have derived some new functionals that satisfied exponential and log convexity. It has been observed that Jensen's inequality appears in many form as one of them is affine form which has been thoroughly analyzed in the subjected thesis.

This research has observed several potential areas of research for Jensens Inequality. For instance, an open end question can be raised that whether we can obtain operators from the affine form of Jensen's inequality such that they satisfy the same results as the operators generated through the functional form of Jensen's inequality?

# Bibliography


[1] M. Anwar, J. Jaksetic, J. Pečariċ and Atiq ur Rehman, *Exponential convexity, positive semi-definite matrices and fundamental inequalities,* Journal of mathematical inequalities, 4, **2**(2010), 171-189.

[2] K. J. Arrow and A. D. Enthoven, *Quasiconvex Programming*, Econometrica, **29**(1961), 779-800.

[3] I. A. Baloch And S. S. Dragomir, *Affine and functional form of jensen's inequality for 3-convex function at a point*, Arxive, ID1601.06123, (2015).

[4] I. A. Baloch, J. Pečarić, M. Praljak, *Generalization of Levinson's inequality*, J. Math. Inequal. **9**(2015), 571-586.

[5] E. F. Beckenbach, R. Bellman, *An Introduction to Inequalities*, Random House Inc. (1975).

[6] C. R. Bector and c. singh, *B-vex functions*, J. Optimiz. Theory. App. **71**(1991), 237-254.

[7] S. N. Bernstien, *Sur les fonction absolument monotones,* Acta Math, **52**(1929), 1-66.

[8] M. Bjelica, *Refinement and converse of Brunk-Olkin inequality*, J. Math. Anal. Appl. **272**(1998), 462-467.

[9] M. Eshaghi Gordji, M. Rostamian Delavar and M. De La Sen, *On $\phi$-Convex functions*, J. Math. Inequal. **10**(2016), 173-183.

[10] C. Fulga and V. Preda, *Nonlinear programming with E-preinvex and local E-preinvex functions*, Eur. J. Oper. Res. **192**(2009), 737-743.

[11] I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series, and Products*, San Diego, CA: Academic Press, (2000).

[12] G. Hardy, J. E. Littlewood, G. Polya, *Inequalities*, Cambridge University Press, (1999).

[13] M. A. Hanson, *On sufficiency of the Kuhn-Tucker conditions*, J. Optimiz. Theory. App. **80**(1981), 545-550.

[14] A. R. Khan, Josip Pečariĉ and Sanja Varosanec, *popviciu characterization of positivity of sums and integrals for convex functions of higher order,* Journal of mathematical inequalities, $7, \mathbf{2}$(2013), 195-212.

[15] A. R. Khan, Josip Pečariĉ and Sanja Varosanec, *Positivity of Sums and Integrals for Convex Functions of Higher Order of $n$ Variables*, J. Math. Inequal. **19**(2016), 221-247.

[16] A. R. Khan, Josip Pečariĉ and Marjan Praliak, *Generalized Čebyšev and Ky Fan Identities and Inequalities*, J. Math. Inequal. **10**(2016), 185-204.

[17] S. T. Lay, *Convex Sets and Their Applications*, New York Wiley, (1979).

[18] X. F. Li, J. L. Dong and Q. H. Liu, *Lipschitz B-vex functions and nonsmooth programming*, J. Optimiz. Theory. App. **3**(1997), 557-573.

[19] J. L. W. V. Jensen, *Om konvekse Funktioner og Uligheder Mellem Middelvrdier*, Nyt Tidsskrift for Matematik. B. **16**(1905), 49-68.

[20] B. Jessen, *Bemrkninger om konvekse Funktioner og Uligheder imellem Middelvrdier. I.*, Matematisk Tidsskrift. B. **2**(1931), 17-28.

[21] O. L. Mangasarian, *Pseudo-Convex Functions*, Siam. J. Control. **3**(1965), 281-290.

[22] A. Mcd. Mercer, *A variant of Jensens inequality*, Journal of Inequalities in Pure and Applied Mathematics, **4**(2003), *Article*73.

[23] J. Mićić, Z. Pavić, J. Pečarić, *The inequalities for quasiarithmetic means*, Abstr. Appl. Anal. **2012**(2012), *Article ID* 203145.

[24] S. R. Mohan and S. K. Neogy, *On invex sets and preinvex functions*, J. Math. Anal. Appl. **189**(1995), 901-908.

[25] A. M. Ostrowski, *On the integral inequality*, Aequationes Math. **4**(1970), 358-373.

[26] Z. Pavić, *Generalization of the functional form of jensen's inequality*, Journal of advanced inequalities and application, **33**(2014).

[27] Z. Pavić, *Generalizations of Jensen-Mercers Inequality*, Journal of Pure and Applied Microbiology, **11**(2014), 19-36.

[28] Z. Pavić, *Certain Inequalities for Convex functions*, J. Math. Inequal. **9**(2015), 1349-1364.

[29] J. Pečarić, F. Proschan and Y. L. Tong, *Convex functions, Partial Orderings and Statistical Applications,* Academic Press, New York, (1992).

[30] J. Pečarič, *On the Ostrowskis Generalization of Čebyševs Inequality*, J. Math. Anal. Appl. **102**(1984), 479-487.

[31] J. Pečarič, *Some Further Remarks on the Ostrowskis Generalization of Čebyševs Inequality*, J. Math. Anal. Appl. **123**(1987), 18-33.

[32] J. E. Pečarič, *A simple proof of the Jensen-Steffensen inequality*, Amer. Math. Monthly **91**(1984), 195-196.

[33] J. Pečarič, J. Perušič AND K.SMOLJAK, Mercer and Wu-Srivastava generalisations of Steffensens inequality, Appl. Math. Comput. **219**(2013), 10548-10558.

[34] I. Rasa, *A note on Jensens inequality, in Itinerant Seminar on Functional Equations*, pproximation and Convexity, Universitatea Babes-Bolyai, Cluj-Napoca, Romania, (1988), 275-280.

[35] A. W. Robert and D. E. Varbeg, *Convex functions*, Academic Press, (1973).

[36] R. T. Rockafellar, *Convex Analysis*, Princeton University Press, New Jersey, USA, (1972).

[37] W. Rudin, *Principles of Mathematical Analysis*, New York McGraw-Hill, (1976).

[38] J. Šremr, *Absolutely Continuous Functions of Two Variables in the Sense of Carathéodory*, Electron. J. Differential Equations, **154**(2010), 1-11.

[39] Y. R. Syau and E. S. Lee, *Some properties of E-convex functions*, Appl. Math. Lett. **18**(2005), 1074-1080.

[40] E. A. Youness, *E-convex sets, E-convex functions and E-convex programming*, J. Optimiz. Theory. App. **102**(1999), 439-450.

[41] R. Webster, *Convexity*, Oxford University Press, (1995).