Hermite-Hadamard Inequality for Preinvex Functions on Time Scales

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Dedication

This thesis is dedicated to my respectable Supervisor, Teachers, Parents and Siblings for their endless devotion, support and encouragement.

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Glory be to Almighty Allah. Our Creator, Sustainer, Fashioner, the Most Merciful. All praise is for Him only. I am highly grateful to Almighty Allah for showering His countless blessings upon me and giving me the ability and strength to complete this thesis successfully and blessing me more than I deserve. Also Darood upon Muhammad (PBUH) the One who is Mercy to the whole universe.

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Irfanullah

Abstract

In the second half of the 20th century a number of generalizations of convex functions have been made. The basic purpose of these generalizations was to weaken convexity conditions as much as possible. Among these generalizations one of the basic interest is invex functions which were initially studied by Hanson and named by Craven. Hanson noticed that the convexity conditions in Kuhn-Tucker conditions for mathematical programming problems can be weakened further. The invexity requires the differentiability conditions and those non differentiable are called Preinvex functions introduces by Weir and Jeyakumar also generalize convex functions. Like convex functions the characterization of these functions in terms of invexity of epigraph is possible.

Any closed subset of $\mathbb{R}$ is called time scale. The theory of time scales goes back to German mathematician Stephen Hilger. He introduced time scales in his PhD thesis. The main theme of time scales calculus is to unify integral and differential calculus with that of finite differences and provides a formal courtesy to study the differences between discrete and continuous analysis.

Dinu in 2008 investigated convex functions and some related inequalities like Jensen and Hermite-Hadamard on time scales, latter Abe-i-kpeng in 2016 studied Quasi-convex functions on time scales, however, a vast class of Preinvex functions on time scales has not been examined up until now. This proposal tries to incorporate these capacities on time scales and presents Jensen and Hermite-Hadamard inequalities for this class.

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Chapter 1

Introduction

In this chapter we present an introduction and background of the convex functions and time scales.

Any closed subset of $\mathbb{R}$ is called time scale. The theory of time scales goes back to German mathematician Stephen Hilger. Who introduced time scales in his PhD thesis [24]. The main theme of time scales calculus is to unify integral and differential calculus with that of finite differences and provides a formal courtesy to study the differences between discrete and continuous analysis. The utilizations of time scales analysis are very generous and has gotten a great deal of consideration lately. The far-reaching ones among others incorporates the dynamic equations, which contains both differential and difference equations, which are of curious interest in biology, mathematical modeling and engineering. Other applications are in the fields of economics, networks, physics, optimization which have come lately [26].

Many results concerning continuous analysis are carried over discrete analysis quite comfortably but some seems to be fully clashing, the study of time scales helps us to understand why such discrepancies occur between these two cases. For further study of time scales it is referred to study [25, 11].

The idea of convexity is straightforward and characteristic, and can be followed back to Archimedes regarding his well known estimation of π. This thought has immediate and roundabout effects in our regular day to day existence through its various applications in industry, business, medicines, art etc. The theory of convex functions is a part of general theory of convexity since a convex function is one whose epigraph is a convex set.

It is an essential theory which contacts pretty much every part of mathematics, likely out of the blue we experience with this theory in graphical analysis in which we learn the second derivative test in recognizing convexity of a graph. We likewise meet this theory in tracing maxima and minima of a function of several variable. We can also observe convexity in Mathematical programming, Optimization theory and engineering etc. A great research work in this field has done by J.L.W.V Jensen [27, 28]. Also in 20th century enormous research was done by Hardy, Littlewood and Póyla [23] on publishing first book in inequalities.

In the second half of the 20th century a number of generalizations of convex functions have been made in mathematics and also in professional disciplines such as engineering and economics. These generalizations were usually made from a particular problem, the basic purpose of these generalizations was to weaken convexity conditions as much as possible. Among these generalizations one of basic interest is invex functions which were initially studied by Hanson [22] and named by Craven [13], Hanson noticed that the convexity conditions in Kuhn-Tucker conditions for mathematical programming problems can be weakened further. Some properties of invex functions were studied by Ben-Israel and Mond [8]. The invexity requires the differentiability conditions. In [40, 32, 46], the class of Preinvex functions, not necessarily differentiable, has been introduced. This calss contains convex functions as subclass. Like convex functions the characterization of these functions in terms of invexity of epigraph is possible. Some properties and inequalities like Jensen and Hermite-Hadamrd inequalities were studied by Weir and Mond [43] , Noor [36], Yang and Li [46] and Mohan and Neogy [32].

Dinu in 2008 [16, 17] investigated convex functions and some related inequalities like Jensen and Hermite-Hadamard on time scales, latter Abe-i-kpeng in 2016 [1] studied quasi-convex functions on time scales, however, a vast class of preinvex functions on time scales has not been examined up until now, this proposal tries to incorporate these capacities on time scales and presents Jensen and Hermite-Hadamard inequalities for this class.

Chapter 2 covers few basic concepts related to the field of study. It includes the study related to concepts of convex sets, convex functions their generalizations. An introduction to time scales is also discussed here. Chapter 3 is devoted to the study of Invex functions and their relation with generalizations of convex functions. Also the detail review of invex sets and preinvex functions is incorporated here. In Chapter 4 different inequalities have been studied for convex functions and preinvex functions. Also therein time scales versions of some inequalities are present. In Chapter 5 we introduce the notion of invex set and preinvex functions on time scales and also we have studied Jensen and Hermite-Hadamard inequalities for these functions. Chapter 6 incorporates the conclusion.

Chapter 2

Preliminaries

In this chapter, some prerequisite ideas and concepts are discussed that reader should familiar with. It mainly includes some preliminary definitions of convex functions and their generalizations. Also the relation between these generalizations has been shown here.

2.1 Convex Functions

Definition 2.1.1. A set $\mathfrak{X} \subseteq \mathbb{R}^n$ is said to be convex if, for every pair of points $u_1, u_2 \in \mathfrak{X}$, the segment with u_1 and u_2 as end points lies entirely inside $\mathfrak{X}$, otherwise called not convex. Geometrically we can represent convex and non convex sets by figures 2.1.1 and 2.1.2 respectively.

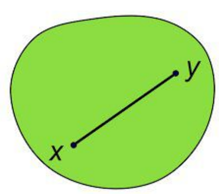

Figure 2.1.1: Convex set

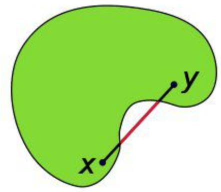

Figure 2.1.2: Not Convex set

Formally a convex set can be defined in following way:

Definition 2.1.2. A set $\mathfrak{X} \subseteq \mathbb{R}^n$ is convex if $u_1, u_2 \in \mathfrak{X}$ we have,

$$\alpha u_1 + (1-\alpha)u_2 \in \mathfrak{X} \text{ for all } \alpha \in [0,1] \tag{2.1.1}$$

Example 2.1.3. Some examples of convex sets are given below.

- Empty set and singleton sets are convex traditionally.
- $\mathbb{R}^n$ is convex.
- The line through x_o and in the direction of $u : \{y \in \mathbb{R}^n \text{ such that } y = x_o + tu, t \in \mathbb{R}\}$.

We want to characterize convex set in terms of convex combination. For this we need to define convex combination.
A point $u = \alpha u_1 + (1-\alpha)u_2$ is called convex combination of u_1 and u_2. The set of all convex combinations of u_1 and u_2 is called as convex hull written as:

$$Conv\{u_1, u_2\} = \{\alpha u_1 + \beta u_2 : \alpha + \beta = 1\}.$$

with

$$\alpha = \frac{u - u_1}{u_2 - u_1} \text{ and } \beta = \frac{u_2 - u}{u_2 - u_1} \quad \text{for} \quad u \in [u_1, u_2] \quad \text{and} \quad u_1 \neq u_2.$$

Definition 2.1.4. A convex combinations of finitely many points $u_i \in \mathbb{R}$ with $i =$

$1, 2, 3, ..., k$ is a point u of the form,

$$u = u_1\lambda_1 + u_2\lambda_2 + \ldots + u_k\lambda_k, \quad with \quad \lambda_1 + \lambda_2 + \ldots + \lambda_k = 1,\ \lambda_i \geq 0. \tag{2.1.2}$$

We are also interested in convex function, therefore, definition is given as:

Definition 2.1.5. Let $\mathfrak{X} \subseteq \mathbb{R}^n$ be such that $\mathfrak{X}$ is convex. A function $\Phi : \mathfrak{X} \to \mathbb{R}$ is called to be convex if for all $s_1, s_2 \in \mathfrak{X}$ and $\lambda \in [0, 1]$ we get

$$\Phi(\lambda s_1 + (1-\lambda)s_2) \leq \lambda\Phi(s_1) + (1-\lambda)\Phi(s_2). \tag{2.1.3}$$

Φ is called strictly convex if the inequality is strict for $\lambda \in (0, 1)$ and $s_1 \neq s_2$. For $\lambda = \frac{1}{2}$

$$\Phi(\frac{s_1+s_2}{2}) \leq \frac{\Phi(s_1)+\Phi(s_2)}{2} \quad \forall s_1, s_2 \in \mathfrak{X} \tag{2.1.4}$$

which is called Jensen convex function. A function Φ is concave if $-\Phi$ is convex and is strictly concave if $-\Phi$ is strictly convex.

A convex function can be defined geometrically as follows:
A function Φ is convex if the chord connecting any pair of points in its graph rests on or above its points. Φ is known as strictly convex if the chord lies above its graph. A concave function can be defined in the similar words but in opposite direction and can be seen in Figure 2.1.3.

Example. $\Phi(x) = \alpha x^2 + \beta x + \gamma$ is convex if $\alpha > 0$ and concave if $\alpha < 0$.

Epigraph of a function Φ is defined as,

$$\text{epi}\Phi = \{(x, \beta) \in \mathfrak{X} \times \mathbb{R} : \Phi(x) \leq \beta\}. \tag{2.1.5}$$

epiΦ is convex set in $\mathbb{R}^{n+1}$ if and only if Φ is convex function [45].

In a similar way, hypograph of Φ is defined as,

$$\text{hypo}\Phi = \{(x, \beta) \in \mathfrak{X} \times \mathbb{R} : \Phi(x) \geq \beta\}. \tag{2.1.6}$$

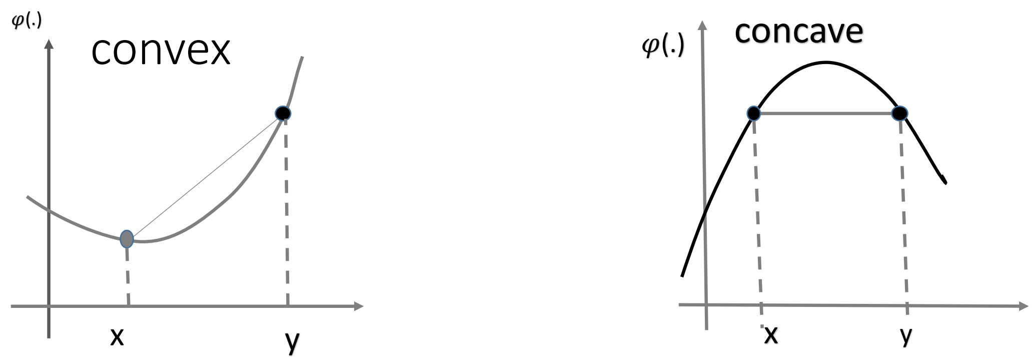

Figure 2.1.3: Convex and concave function

If hypograph of Φ is convex then Φ is concave, See in [29].

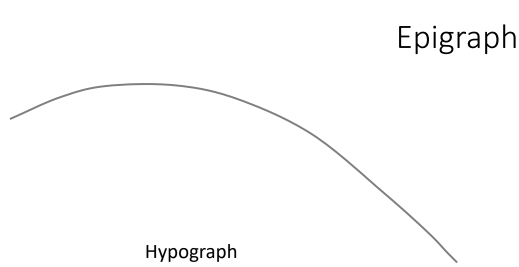

Figure 2.1.4: Epigraph and Hypograph

If a function Φ is differentiable on $\mathfrak{X}^o \subseteq \mathbb{R}^n$(interior of convex set $\mathfrak{X}$) then Φ is convex on $\mathfrak{X}^o$ if and only if,

$$\Phi(s_1) - \Phi(s_2) \geq (s_1 - s_2)^T \nabla\Phi(s_2) \text{ for all } s_1, s_2 \in \mathfrak{X}, \tag{2.1.7}$$

where $\nabla\Phi(s_2)$ calculates gradient of Φ at $s_2 \in \mathfrak{X}^o$.

Some basic characteristics of convex functions follow from [33].

Theorem 2.1.6. *If $\Phi : \mathfrak{X} \to \mathbb{R}$ be a convex function on $\mathfrak{X} \subseteq \mathbb{R}^n$(where $\mathfrak{X}$ is convex), then,*

1) *Every local minimum of Φ is global minimum.*

2) *The set $\mathbf{C} = \{u : u$ is minimum of $\Phi\}$ is convex set.*

3) *If Φ is differentiable on $\mathfrak{X}^o$(Interior of $\mathfrak{X}$), then every stationary point s_2 is global minimizer, that is, $\nabla\Phi(s_2) = 0 \implies \Phi(s_2) \leq \Phi(s_1)\ \forall s_1 \in \mathfrak{X}^o$.*

Theorem 2.1.7. *[12] A necessary condition for a convex function Φ defined on convex set $\mathfrak{X} \subseteq \mathbb{R}^n$ is that, the lower level set $L_\Phi(\delta) = \{u \in \mathfrak{X} : \Phi(u) \leq \delta\}$ is convex for every $\delta \in \mathbb{R}$.*

Note. The condition in Theorem 2.1.7 is not sufficient generally. For example

$$\Phi(x) = \log x, \quad x \in \mathbb{R}$$

has lower level sets convex but is not convex.

2.2 Quasi-Convex and Pseudo-Convex Functions.

One way to generalize the convexity of a function is to relax convexity conditions, and consider category of functions wherefore the convexity of lower level sets is sufficient. These types of functions are called quasi-convex functions. Clearly we can see that this class strictly contains the class of convex functions.

Definition 2.2.1. A function $\Phi : \mathfrak{X} \to \mathbb{R}$ (where $\mathfrak{X}$ is convex) is called quasi-convex on $\mathfrak{X}$ if its lower level sets,

$$L_{\Phi}(\delta) = \{u \in \mathfrak{X} : \Phi(u) \leq \delta\} \tag{2.2.1}$$

are convex for every $\delta \in \mathbb{R}$. Φ is quasi-concave given that $-\Phi$ is quasi-convex, that is, its upper level set,

$$U_{\Phi}(\delta) = \{u \in \mathfrak{X} : \Phi(u) \geq \delta\} \tag{2.2.2}$$

are convex for every $\delta \in \mathbb{R}$.

We can prove that a function $\Phi : \mathfrak{X} \to \mathbb{R}$ is quasi-convex on $\mathfrak{X}$ if and only if $\forall\ u_1, u_2 \in \mathfrak{X}$ and $\alpha \in [0, 1]$

$$\Phi(\alpha u_1 + (1-\alpha)u_2) \leq max\{\Phi(u_1), \Phi(u_2)\}. \tag{2.2.3}$$

If there is strictness in the above inequality then Φ is said strictly quasi-convex function.

We have picked up the following example from [12].

Example. Define $\Phi(x) = \begin{cases} \frac{|u|}{u}, & u \neq 0; \\ 0, & u = 0. \end{cases}$

We find that Φ is quasi convex but it is not convex.

Quasi-convex functions on contradiction to convex functions can have local minimum points that are not absolute and that the stationary points of a (differentiable) quasi-convex functions are not necessarily global minimum points.

We have the following theorems from [12].

Theorem 2.2.2. *Let $K \subseteq \mathbb{R}^n$ is convex set, and $\Phi : K \to \mathbb{R}$,*

i) Φ is convex implies that Φ is quasi-convex.

ii) If Φ is quasi-convex on K, then the set $\{u : u \text{ is global minimum point of } \Phi\}$ is convex.

The following theorem (Theorem 1.4 [33]) characterizes the quasi-convexity of differentiable functions.

Theorem 2.2.3. *Let $\Phi : K \to \mathbb{R}$ (where K is convex) be differentiable function. The necessary and sufficient condition for Φ to be quasi-convex on K is*

$$u_1, u_2 \in K, \, \Phi(u_1) \leq \Phi(u_2) \implies (u_1 - u_2)^T \nabla \Phi(u_1) \leq 0. \tag{2.2.4}$$

holds.

In [39] pseudo-convex functions are also introduced and defined as;

Definition 2.2.4. A function $\Phi : \mathfrak{X} \to \mathbb{R}$ ($\mathfrak{X}$ is open) is said to be pseudo-convex if for all $u_1, u_2 \in \mathfrak{X}$ and $\alpha \in (0, 1)$, then,

$$\Phi(u_1) < \Phi(u_2) \implies \Phi(\alpha u_1 + (1 - \alpha) u_2) \leq \Phi(u_1) + (\alpha - 1) k(u_1, u_2) \tag{2.2.5}$$

where $k(u_1, u_2) > 0$.

Among the major properties of convex functions of critical point being a global minimum point does not hold for quasi-convex functions, therefore, a wider class of differentiable functions which contains the class of differentiable convex functions was introduced by Mangasarian in [31].

Definition 2.2.5. A differentiable function $\Phi : \mathfrak{X} \to \mathbb{R}$ ($\mathfrak{X} \subseteq \mathbb{R}^n$ is open) is called pseudo-convex if for every $u_1, u_2 \in \mathfrak{X}$

$$(u_1 - u_2)^T \nabla \Phi(u_2) \geq 0 \implies \Phi(u_1) \geq \Phi(u_2), \tag{2.2.6}$$

or, equivalently

$$\Phi(u_1) < \Phi(u_2) \implies (u_1 - u_2)^T \nabla \Phi(u_2) < 0. \tag{2.2.7}$$

For this class the stationary point $y'(\nabla\Phi(y')=0)$ is for sure global minimizer.

Theorem 2.2.6. *[12] Let $\Phi : \mathfrak{C} \to \mathbb{R}$ (where $\mathfrak{C} \subseteq \mathbb{R}^n$ is open convex) be differentiable function.*

1) If c is critical point of Φ and Φ is pseudoconvex, then c is global minimum.

2) In case Φ is pseudoconvex, afterwards Φ is quasi-convex.

3) If $\nabla\Phi(u) \neq 0$, for all $u \in \mathfrak{C}$, once Φ is pseudoconvex implies Φ is quasi-convex and conversely.

2.3 Time Scales Theory

In this section, we discuss focal ideas and meanings of the time scale analysis started by Hilger in 1988 under the supervision of Bernd Aulbach. All through this part the likenesses and contrasts in considering the time scales as in the $\mathbb{R}$ and $\mathbb{Z}$ setup are commented. Consideration is given to the ideas, for example, continuity, rd-continuity, differentiability which are relevant in the analysis of hybrid continuous and discrete systems. Basic concepts of convex analysis are briefly discussed.

A time scales $\mathbb{T}$ is non void subset of $\mathbb{R}$ which is closed (with subspace topology induced from $\mathbb{R}$). Therefore, the real numbers $\mathbb{R}$, integers $\mathbb{Z}$, natural numbers $\mathbb{N}$, any closed interval $[a, b]$, are time scales. Further, the set $[1, 3] \cup [5, 6] \cup \mathbb{N}_0$($\mathbb{N}_0$ is set of non-negative integers), the Cantor set are also some examples. In contrary $\mathbb{Q}$ (the set of rationals) is not a time scale.

Operations on time scales [11] :

The forward jump and backward jump operators speak to the nearest point in the time scales on the right and left of a given point x individually.

Formally we define as below.

For $x \in \mathbb{T}$, the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(x) = min\{p \in \mathbb{T} : p > x\}. \tag{2.3.1}$$

and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ by

$$\rho(x) = max\{p \in \mathbb{T} : p < x\}. \tag{2.3.2}$$

We set the convention as, $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$.

Classification of points. Let $\mathbb{T}$ be a time scale, a point $x \in \mathbb{T}$ is classified as below,

i) x is termed as right scattered, if $\sigma(x) > x$.

ii) x is cited to be left scattered, if $\rho(x) < x$.

iii) x is isolated, if $\rho(x) < x < \sigma(x)$.

iv) x is right dense, if $x < max\mathbb{T}$ and $\sigma(x) = x$.

v) If $x > min\mathbb{T}$ and $\rho(x) = x$, then, x is left dense.

vi) If x is both left and right dense, then x is said to be dense.

The Figure 2.3.1 is taken from [11] which classifies the points of time scales.

The functions $\mu, \nu : \mathbb{T} \to [0, \infty)$ defined by

$$\mu(x) = \sigma(x) - x \quad \text{and} \quad \nu(x) = x - \rho(x)$$

are called the forward and backward graininess functions respectively.

Example 2.3.1. For $\mathbb{T} = \mathbb{R}$. We have $\sigma(x) = x$ and $\rho(x) = x$, that is, every $x \in \mathbb{T}$ is dense and $\mu(x) = 0$ and for $\mathbb{T} = \mathbb{Z}$, $\sigma(x) = x+1$ $\rho(t) = x-1$ with $\mu(x) = x+1-x = 1$. We see that every $x \in \mathbb{T}$ is dense and $x \in \mathbb{Z}$ is scattered.

The Table 2.3.1 gives forward and backward jumps for different time scales.

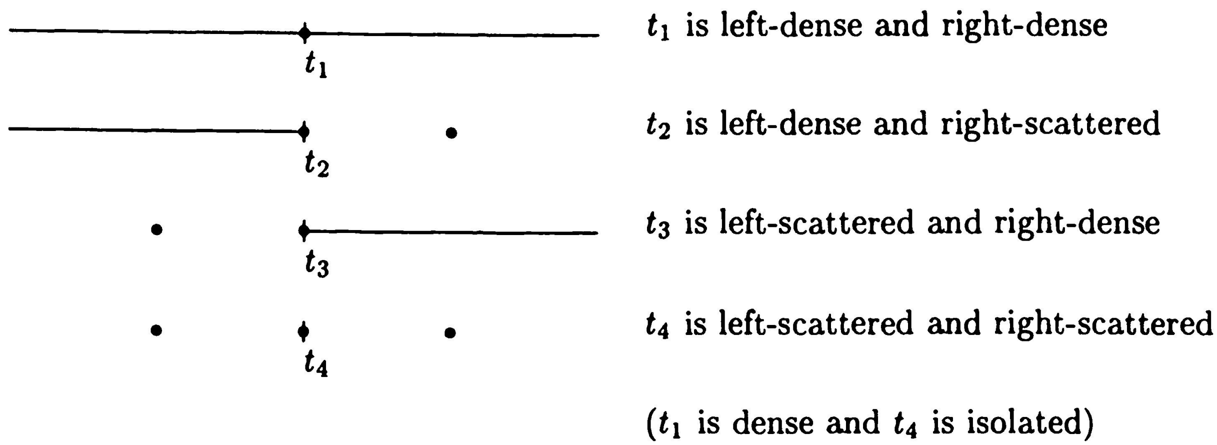

Figure 2.3.1: Classification of points.

$\mathbb{T}$	$\sigma(x)$	$\rho(x)$
$\mathbb{R}$	$x,\ \forall x$	$x,\ \forall x$
$\mathbb{Z}$	$x+1,\ \forall x$	$x-1,\ \forall x$
$h\mathbb{Z},\ h>0$	$x+h,\ \forall x$	$x-h,\ \forall x$
$h\mathbb{Z},\ h<0$	$x-h\,\forall x$	$x+h\,\forall x$
$\mathbb{N}^k,\ k\in\mathbb{N}$	$\left(1+\sqrt[k]{x}\right)^k$	$\left(\sqrt[k]{x}-1\right)\ for\, x\neq 1$ and $1\ for\, x=1$
$q^{\mathbb{Z}}\cup\{0\},\ q>1$	$qx\,\forall x$	$\frac{x}{q}\ for\, x\neq 1$ and $1\ for\, x=1$
$p^{\mathbb{N}_0}\cup\{0\},\ p\in(0,1)$	$\frac{x}{p}\ for\, x\neq 1\ and\, 1\ for\, x=1$	$px\ for\, x\neq 0$ and $0\ for\, x=0$

Table 2.3.1: Jump operators.

The next table shows forward and backward graininess for time scales in above table.

$\mathbb{T}$	$\mu(x)$	$\nu(x)$
$\mathbb{R}$	0	0
$\mathbb{Z}$	1	-1
$h\mathbb{Z},\ h > 0$	h	h
$h\mathbb{Z},\ h < 0$	$-h$	$-h$
$\mathbb{N}^k,\ k \in \mathbb{N}$	$\left(1+\sqrt[k]{x}\right)^k - 1$	$x - \left(\sqrt[k]{x}-1\right) for\, x \neq 1,$ $and\, x - 1\, for\, x = 1$
$q^{\mathbb{Z}} \cup \{0\},\ q > 1$	$x(q-1)$	$x(1-\frac{1}{q})\, for\, x \neq 1$ $x - 1\, for\, x = 1$
$p^{\mathbb{N}_0} \cup \{0\},\ p \in (0,1)$	$\frac{1-x}{p}\, for\, x \neq 1$ and 0 for $x = 1$.	$x(1-p)\, for\, x \neq 0$ $x\, for\, x = 0$

Table 2.3.2: Graininess functions

Let $I = [x, y] \subseteq \mathbb{R}$ with $x, y \in \mathbb{T}$, then the time scale interval is defined as,

$$I_{\mathbb{T}} = [x, y]_{\mathbb{T}} = [x, y] \cap \mathbb{T} = \{u \in \mathbb{T} : x \leq u \leq y\} \tag{2.3.3}$$

We define the sets:

$$\mathbb{T}^k = \begin{cases} \mathbb{T} - (\rho(\max \mathbb{T}), \max \mathbb{T}], & if \;\; \max \mathbb{T} < \infty; \\ \mathbb{T}, & otherwise. \end{cases} \tag{2.3.4}$$

$$\mathbb{T}_k = \begin{cases} \mathbb{T} - [\min \mathbb{T}, \sigma(\min \mathbb{T})), & if \min \mathbb{T} > -\infty; \\ \mathbb{T}, & otherwise. \end{cases} \tag{2.3.5}$$

Example 2.3.2. Let $\mathbb{T} = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$. Then $\sup \mathbb{T} = 1$ and $\rho(1) = \frac{1}{2}$. Therefore $\mathbb{T}^k = \mathbb{T} - \left(\frac{1}{2}, 1\right] = \left\{\frac{1}{n} : n \in \mathbb{N}\backslash\{1\}\right\} \cup \{0\}$.

If $[r, s]_{\mathbb{T}}$ be a time scale interval then,

$$[r, s]^k_{\mathbb{T}} = \begin{cases} [r, s], & \text{when } s \text{ is left-dense in } \mathbb{T}; \\ [r, s), & \text{when } s \text{ is left-scattered in} \mathbb{T}. \end{cases}$$

2.3.1 Time Scales Calculus

In this section we discuss delta and nabla calculus on time scales, we can also find diamond-alpha calculus in various books on time scales for instance we refer interested readers to [11].

Definition 2.3.3. The number $\Phi^{\Delta}(x_0)$ (if exists) having the property that for every $\varepsilon > 0$, there is a neighborhood $O_{\mathbb{T}}$ of x_0 so that

$$\left|\Phi(\sigma(x_0)) - \Phi(s) - \Phi^{\Delta}(x_0)[\sigma(x_0) - s]\right| \leq \varepsilon|\sigma(x_0) - s| \quad for \quad all \quad s \in O_{\mathbb{T}}. \tag{2.3.6}$$

is called the delta\Hilger derivative of $\Phi : \mathbb{T} \to \mathbb{R}$ on the point x_0.
The number $\Phi^{\nabla}(x_0)$ (if it survives) having the property that for every $\varepsilon > 0$, there is a neighborhood $U_{\mathbb{T}}$ of x_0 so that

$$\left|\Phi(\rho(x_0)) - \Phi(s) - \Phi^{\nabla}(x_0)[\rho(x_0) - s]\right| \leq \varepsilon|\rho(x_0) - s| \quad for \quad all \quad s \in U_{\mathbb{T}}. \tag{2.3.7}$$

is called the nabla derivative of $\Phi : \mathbb{T} \to \mathbb{R}$ on the point x_0.

We say that Φ is delta differentiable on $\mathbb{T}^k$ if $\Phi^{\Delta}(x)$ exists for all $x \in \mathbb{T}^k$ and nabla diffrentiable accordingly.

For $\mathbb{T} = \mathbb{R}$ the delta and nabla derivatives are classical derivative.

Theorem 2.3.4. *If delta (nabla) derivative of a function Φ exists, then it is unique.*

The following theorem can be seen in [11].

Theorem 2.3.5. *A function Φ is delta differentiable at point x_0 with*

$$\Phi^{\Delta}(x_0) = \frac{\Phi(\sigma(x_0)) - \Phi(x_0)}{\sigma(x_0) - x_0}, \tag{2.3.8}$$

if Φ is continuous at right-scattered point x_0.

If x_0 is right dense, then Φ is differentiable at x_0 if and only if the limit

$$\lim_{s\to x_0}\frac{\Phi(x_0)-\Phi(s)}{x_0-s} \tag{2.3.9}$$

exists as a finite number. In this case,

$$\Phi^{\Delta}(x_0)=\lim_{s\to x_0}\frac{\Phi(x_0)-\Phi(s)}{x_0-s}. \tag{2.3.10}$$

A simple useful formula for delta diffrentiable Φ is

$$\Phi(\sigma(x))=\Phi(x)+\mu(x)\Phi^{\Delta}(x). \tag{2.3.11}$$

Similarly, for nabla derivative

$$\Phi^{\nabla}(x_0)=\frac{\Phi(x_0)-\Phi(\rho(x_0))}{x_0-\rho((x_0))}, \tag{2.3.12}$$

for continuous Φ at left-scattered point x_0 and

$$\Phi^{\nabla}(x_0)=\lim_{s\to x_0}\frac{\Phi(x_0)-\Phi(s)}{x_0-s}, \tag{2.3.13}$$

for left-dense point x_0. Here the useful formula becomes

$$\Phi(\rho(x))=\Phi(x)-\nu(x)\Phi^{\nabla}(x). \tag{2.3.14}$$

A time scale $\mathbb{T}$ is said to be regular if

- $\sigma(\rho(x))=x$, for all$x\in\mathbb{T}$.
- $\rho(\sigma(x))=x$, for all$x\in\mathbb{T}$.

We move toward the product and quotient rule of delta derivative. All these results also hold for nabla derivative.

Theorem 2.3.6. *Suppose $\Phi, \phi : \mathbb{T} \to \mathbb{T}$ are delta differentiable on $\mathbb{T}^k$. Then*

$$(\Phi\phi)^\Delta = \Phi^\Delta \phi + \Phi^\sigma \phi^\Delta = \Phi\phi^\Delta + \Phi^\Delta \phi^\sigma.$$

$$\left(\frac{\Phi}{\phi}\right)^\Delta = \frac{\Phi^\Delta \phi - \Phi\phi^\Delta}{\phi\phi^\sigma}.$$

By using product rule we can have,

$$(\Phi^n)^\Delta = \Phi^\Delta \sum_{p=0}^{n} \Phi^p (\Phi^\sigma)^{n-1-p}.$$

Let $\Phi(x) = (x-a)^m$ for $a \in \mathbb{R}$ and $m \in \mathbb{R}$. then $\Phi^\Delta(x) = \sum_{k=0}^{m-1}(x-a)^k(\sigma(x)-a)^{m-1-k}$ and $(\frac{1}{\Phi})^\Delta(x) = -\sum_{k=0}^{m-1} \frac{1}{(x-a)^{m-k}} \frac{1}{(\sigma(x)-a)^{k+1}}$.

$\mathbb{T}$	$\sigma(x)$	$\mu(x)$	Φ^Δ
$\mathbb{R}$	$x,\ \forall x$	0	$\Phi'(x)$
$\mathbb{T}$	$x+1,\ \forall x$	1	$\Delta\Phi(x)$
$h\delta Z,\ \delta > 0$	$x+\delta,\ \forall x$	δ	$\frac{\Phi(x+\delta)-\Phi(x)}{\delta}$
$p^{\mathbb{Z}} \cup \{0\},\ p > 1$	$px\,\forall x$	$x(p-1)$	$\frac{\Phi(px)-\Phi(x)}{x(p-1)}$
$h\delta Z,\ \delta < 0$	$x-\delta\,\forall x$	$-\delta$	$\frac{\Phi(x-\delta)-\Phi(x)}{-\delta}$
$\mathbb{N}^c,\ c \in \mathbb{N}$	$\left(1+\sqrt[c]{x}\right)^c$	$\left(1+\sqrt[c]{x}\right)^c - 1$	$\frac{\Phi((1+\sqrt[c]{x})^c)-\Phi(x)}{(1+\sqrt[c]{x})^c-1}$
$\mathbb{N}_0^2$	$\left(\sqrt{x}+1\right)^2$	$1+\sqrt{2}$	$\frac{\Phi((1+\sqrt{x})^2)-\Phi(x)}{1+\sqrt{x}}$

Table 2.3.3: Delta derivatives

Definition 2.3.7. The function $\Phi : \mathbb{T} \to \mathbb{R}$ is called regulated if its right-limit exists finitely at right dense points while left limit exists and is finite at left-dense points.

Definition 2.3.8. A function $\Phi : \mathbb{T} \longrightarrow \mathbb{R}$ is said to be rd-continuous if it is continuous at right dense points in $\mathbb{T}$ and at left dense points in $\mathbb{T}$ its left limit exists (finite). The set of all rd-continuous functions is denoted by $C_{rd} = C_{rd}(\mathbb{T}) =$

$C_{rd}(\mathbb{T}, \mathbb{R})$.

Definition 2.3.9. A function $\Phi : \mathbb{T} \longrightarrow \mathbb{R}$ is said to be ld-continuous if it is continuous at left dense points in $\mathbb{T}$ and at right dense points in $\mathbb{T}$ its right limit exists (finite). The set of all rd-continuous functions is denoted by $C_{ld} = C_{ld}(\mathbb{T}) = C_{ld}(\mathbb{T}, \mathbb{R})$.

We only shall discuss delta integrals on time scales with keeping in mind that all these results hold for nabla integrals also.

Definition 2.3.10. Let $\Phi : \mathbb{T} \to \mathbb{R}$ be a continuous function, it is called pre differentiable with (region of differentiation) D if,

- $D \subset \mathbb{T}^k$,
- $\mathbb{T}^k \backslash D$ is countable and has no right scattered member of $\mathbb{T}$,
- and Φ is differentiable at each $t \in D$.

For delta differentiable function Φ and $a, b \in \mathbb{T}$, we define the Cauchy integral by

$$\int_a^b \Phi^{\Delta}(t)\Delta t = \Phi(a) - \Phi(b). \tag{2.3.15}$$

Theorem 2.3.11. *(Existence of anti derivatives). To every rd-continuous function there is an anti derivative, in particular if $t_0 \in \mathbb{T}$, then F taken by*

$$F(x) := \int_{x_0}^{x} \Phi(s)\Delta s \quad for\, all \quad t \in \mathbb{T}.$$

is anti derivative of Φ.

Theorem 2.3.12. *If $a, b, c \in \mathbb{T}, \alpha \in \mathbb{R}$ and $\Phi, \phi \in C_{rd}$, then*

1. $\int_a^b (\Phi(x) + \phi(x))\Delta x = \int_a^b \Phi(x)\Delta x + \int_a^b \phi(x)\Delta x$,
2. $\int_a^b \alpha\Phi(x)\Delta x = \alpha \int_a^b \Phi(x)\Delta x$,
3. $\int_a^b \Phi(x)\Delta x = -\int_b^a \Phi(x)\Delta x$,

4. $\int_a^a \Phi(x)\Delta x = 0,$

5. $\int_a^b \Phi(x)\Delta x = \int_a^c \Phi(x)\Delta x + \int_c^b \Phi(x)\Delta x,\ a < c < b.$

6. $\left| \int_a^b \Phi(x)\Delta x \right| \le \int_a^b |\Phi(x)|\Delta x.$

7. If $\Phi(x) \ge 0\ \forall\ x$, then $\int_a^b \Phi(x)\Delta x \ge 0.$

Theorem 2.3.13. *If $\Phi \in C_{rd}$ and $x \in \mathbb{T}^k$, then*

$$\int_x^{\sigma(x)} \Phi(s)\Delta s = \mu(x)\Phi(x).$$

Integration by parts is given by

$$\int_a^b \Phi(x)\phi^{\Delta}(x)\Delta x = \Phi(x)\phi(x)|_a^b - \int_a^b \Phi^{\Delta}(x)\phi^{\sigma}(x)\Delta x. \tag{2.3.16}$$

We can also read infinite integrals as,

$$\int_a^{\infty} \Phi(x)\Delta x = \lim_{b\to\infty} \int_a^b \Phi(x)\Delta x. \tag{2.3.17}$$

If $\mathbb{T} = \mathbb{R},$ then

$$\int_a^b \Phi(x)\Delta x = \int_a^b \Phi(x)dx.$$

If $\mathbb{T}$ =$\mathbb{Z}$, then

$$\int_a^b \Phi(x)\Delta x = \sum_{x=a}^{b-a} \Phi(x).$$

Generally integration on discrete time scales can be seen by

$$\int_a^b \Phi(x)\Delta x = \sum_{x=\in(a,b)} \Phi(x)\mu(x). \tag{2.3.18}$$

Now we focus our attention toward exponential and logarithmic functions on time scales.

2.4 Exponential, Logarithmic and Convex Functions on Time Scales.

In this section we discuss the exponential function, natural logarithm and convex functions over time scales.

Let $\gamma : \mathbb{T} \to \mathbb{R}$ be such that for all $x \in \mathbb{T}$, $1 + \mu(x)\gamma(x) \neq 0$ then, γ is called regressive. We define $\mathfrak{R}$ to be the set of all regressive and rd-continuous functions. The set $\mathfrak{R}^+ = \{\gamma \in \mathfrak{R} : 1 + \mu(x)\gamma(x) > 0, \text{ for all } x \in \mathbb{T}\}$ denotes all positive regressive and rd-continuous functions. It can be easily seen that $\mathfrak{R}$ forms an abelian group under the operation $\oplus$ defined by $\gamma \oplus \delta := \gamma + \delta + \mu\gamma\delta$. If $\gamma \in \mathfrak{R}$, then the exponential function can be defined by

$$e_\gamma(x,u) = exp\Big(\int_u^x \zeta_{\mu(r)}(\gamma(r))\Delta r\Big) \text{ , for } x \in \mathbb{T}, u \in \mathbb{T}^k, \tag{2.4.1}$$

where $\zeta_\mu(w)$ is cylindrical transformation, which is defined by

$$\zeta_\mu(w) = \begin{cases} \frac{log(1+\mu w)}{\mu}, & \mu \neq 0; \\ z, & \mu = 0. \end{cases} \tag{2.4.2}$$

The additive inverse in this group is given by

$$\ominus\gamma := -\frac{\gamma}{1+\mu\gamma}. \tag{2.4.3}$$

Here $\ominus$ defines the subtraction on the set of regressive functions defined by

$$\gamma \ominus \delta := \gamma \oplus (\ominus\delta). \tag{2.4.4}$$

The following properties can be seen in [11].

Theorem 2.4.1. *If $\gamma, \delta \in \mathfrak{R}$ and $t_o \in \mathbb{T}$, then*

1 $e_\gamma(x,x) = 1$ and $e_o(x,y) = 1$;

2 $e_\gamma(\sigma(x),y) = (1+\mu(x)\gamma(x))e_\gamma(x,y)$;

3 $\frac{1}{e_\gamma(x,y)} = e_{\ominus\gamma}(x,y) = e_\gamma(y,x)$;

4 $\frac{e_\gamma(x,y)}{e_\delta(x,y)} = e_{\gamma\ominus\delta}(x,y)$;

5 $e_\gamma(x,y)e_\delta(x,y) = e_{\gamma\oplus\delta}(x,y)$;

6 if $\gamma \in \mathfrak{R}^+$, then $e_\gamma(x,x_o) > 0$ for all $x \in \mathbb{T}$;

7 $e_\gamma^\Delta(x,x_o) = \gamma(x)e_\gamma(x,x_o)$;

8 $\left(\frac{1}{e_\gamma(.,y)}\right)^\Delta = -\frac{\gamma(x)}{e_\gamma^\sigma(.,y)}$.

The following table taken from [9] represents exponential function in different time scales.

$\mathbb{T}$	$e_\alpha(x,x_o)$	x_o	$\gamma(x)$	$e_\gamma(x,x_o)$
$\mathbb{T}$	$e^{\alpha(x-x_o)}$	0	1	e^x
$\mathbb{Z}$	$(1+\alpha)^{x-x_o}$	0	1	2^x
$p\mathbb{Z}$	$(1+\alpha p)^{(x-x_o)/p}$	0	1	$(1+p)^{x/p}$
$\frac{1}{n}\mathbb{Z}$	$(1+\frac{\alpha}{n})^{n(x-x_o)}$	0	1	$\left[\left(1+\frac{1}{n}\right)^n\right]^x$
$q^{\mathbf{N}_o}$	$\cdots$	1	$\frac{1-x}{(q-1)x^2}$	$\sqrt{x}e^{-\frac{ln^2(x)}{2ln(2)}}$
$2^{\mathbf{N}_o}$	$\cdots$	1	$\frac{1-x}{x^2}$	$\sqrt{x}e^{-\frac{ln^2(x)}{ln(4)}}$
$\mathbf{N}_o^2$	$\cdots$	0	1	$2^{\sqrt{x}}(\sqrt{x})!$
$\{\sum_{k=1}^n \frac{1}{k} :\ n \in \mathbf{N}\}$	$\binom{n+\alpha-xo}{n-x_o}$, $x = \sum_{k=1}^n \frac{1}{k}$	0	1	$n+1$, $x = \sum_{k=1}^n \frac{1}{k}$

Table 2.4.1: Exponential functions

The following result is referred as substitution rule and can be found in [11].

Theorem 2.4.2. *Let $v : \mathbb{T} \to \mathbb{R}$ be strictly increasing and $\tilde{\mathbb{T}} := v(\mathbb{T})$ be a time scale. If $\Phi : \mathbb{T} \to \mathbb{R}$ be an rd-continuous function and v is differentiable with rd-continous derivative, then for $a, b \in \mathbb{T}$*

$$\int_a^b \Phi(x)v^{\Delta}(x)\Delta x = \int_{v(a)}^{v(b)} (\Phi \circ v^{-1})(y)\tilde{\Delta} y, \tag{2.4.5}$$

or,

$$\int_a^b \Phi(x)v^{\nabla}(x)\nabla x = \int_{v(a)}^{v(b)} (\Phi \circ v^{-1})(y)\tilde{\nabla} y. \tag{2.4.6}$$

Substitution rule for decreasing fnction $v : \mathbb{T} \to \mathbb{R}$ is given and proved in [4]. The result stated as follows:

Theorem 2.4.3. *Let $v : \mathbb{T} \to \mathbb{R}$ be strictly decreasing and $\tilde{\mathbb{T}} := v(\mathbb{T})$ be a time scale. If $\Phi : \mathbb{T} \to \mathbb{R}$ be a continuous function and v is differentiable wit rd-continuous derivative, then for $a, b \in \mathbb{T}$*

$$\int_a^b \Phi(x)(-v^{\Delta})(x)\Delta x = \int_{v(a)}^{v(b)} (\Phi \circ v^{-1})(y)\tilde{\nabla} y, \tag{2.4.7}$$

or,

$$\int_a^b \Phi(x)(-v^{\nabla})(x)\nabla x = \int_{v(a)}^{v(b)} (\Phi \circ v^{-1})(y)\tilde{\Delta} y. \tag{2.4.8}$$

We refer the reader to consult [11] fo more and detailed calculus of time scales.

Moving toward logarithmic function on time scales we must refer to an open question asked by Martin Bohner in [10].

$$Define\, a\, "nice"\, logarithmic\, function\, on\, time\, scales.$$

To answer this open problem Dorota Mozyrska and Delfm F.M. Torres in [34] discussed natural logarithm and its properties.

Definition 2.4.4. Let $\mathbb{T}$ be a time scale with $1 \in \mathbb{T}$, there is atleast one $x \in \mathbb{T}$ such that $0 < x \neq 1$. The natural logarithm is defined as

$$L_{\mathbb{T}} = \int_1^x \frac{1}{s} \Delta s, \qquad x \in \mathbb{T} \cap (0, +\infty). \tag{2.4.9}$$

The properties and examples of defined natural logarithm can be seen in [34].

Now we pay our attention to convex functions and some related inequalities on time scales. Convex functions on time scales are discussed in [34, 16]

Definition 2.4.5. [16] A function $\Phi : \mathbb{T} \to \mathbb{R}$ is called convex on $I_{\mathbb{T}}$, if for $t, s \in I_{\mathbb{T}} = I \cap \mathbb{T}$ (with I be an interval in $\mathbb{R}$) and $\alpha \in [0,1]$ such that $\alpha t + (1-\alpha)s \in I_{\mathbb{T}}$ we have,

$$\Phi(\alpha t + (1-\alpha)s) \leq \alpha \Phi(t) + (1-\alpha)\Phi(s). \tag{2.4.10}$$

Φ is called strictly convex if the inequality is strict and $\alpha \in (0,1)$. Φ is concave if $-\Phi$ is convex.

In [16] (Remark 3.2) Dinu showed that (2.4.10) is equivalent to definition of convexity given by Mozyrska and Torres in [34], that is

A continuous function $\Phi : \mathbb{T} \to \mathbb{R}$ is said to be convex on $I_{\mathbb{T}}$ (where $I_{\mathbb{T}}$ is time scale interval) if for all $t_1, t_2 \in I_{\mathbb{T}}$

$$(t_2 - t)\Phi(t_1) + (t_1 - t_2)\Phi(t) + (t - t_1)\Phi(t_2) \geq 0, \quad t \in I_{\mathbb{T}}. \tag{2.4.11}$$

Φ is called concave if the inequality reverses. The following theorem can be seen in [16].

Theorem 2.4.6. *A differentiable function $\Phi : I_{\mathbb{T}} \to \mathbb{R}$ is convex (concave) on $I_{\mathbb{T}}$, if Φ^{Δ} is non decreasing (non increasing) on $I_{\mathbb{T}}^k$.*

More properties and examples of convex functions on time scales can be studied in [34, 16].

Chapter 3

Invex and Preinvex Functions

3.1 Invex Functions

Hanson in [22] pointed out that the sufficiency of Krush-kuhn-Tucker conditions and weak duality could be obtained by substituting the linear term $(u_1 - u_2)$, appearing in convexity for differentiable convex functions, by an arbitrary vector valued functions, usually denoted by $\varpi(u_1, u_2)$ and also called "kernel" and Craven in [13] named these functions invex functions abbreviation for "Invariant Convex".

Definition 3.1.1. A diffrentiable function $f : \mathfrak{C} \to \mathbb{R}$ (with $\mathfrak{C} \subseteq \mathbb{R}^n$ be an open set) is called invex if there exists a vector function $\varpi : \mathfrak{C} \times \mathfrak{C} \to \mathbb{R}^n$ such that,

$$f(u_1) - f(u_2) \geq \varpi(u_1, u_2)^T \nabla f(u_2), \qquad \forall u_1, u_2 \in K. \tag{3.1.1}$$

It can be observed that the specific class of (differentiable) convex functions is obtained from this group by making $\varpi(u_1, u_2) = u_1 - u_2$.

In [49], functions with stationary points as global minimizers are observed and their applications in mathematical programming. In [14] Craven and Glover proved that the class of invex functions is equivalent to the class of functions whose stationary points are global minima. The following theorem was first stated Crave and Glover.

Theorem 3.1.2. *Th necessary and sufficient condition for a function f to be invex is that every stationary point is global minimum.*

The proof of this theorem can be directly seen in [14, 33].

Corollary 3.1.3. *A function f with no stationary points is invex.*

Definition 3.1.4. If for a function $f : \mathbb{R}^n \to \mathbb{R}$

$$\varpi(u, v)^T \nabla f(v) \geq 0 \implies f(u) - f(v) \geq 0. \tag{3.1.2}$$

holds, then it is called pseudoinvex, whereas, f is quasiinvex with respect to ϖ if,

$$f(u) - f(v) \leq 0 \implies \varpi(u, v)^T \nabla f(v) \leq 0, \tag{3.1.3}$$

holds.

3.1.1 Invexity with other Generalizations of Convexity

Here the relationship of invexity with other generalizations of convexity is discussed. The counterpart of convexity and generalized convexity is, concavity and generalized concavity receptively, same for invexity is incavity.

Firstly, note that:

1. A function is differentiable and convex then, it is invex by taking ($\varpi(x, y) = x - y$) but not converse. Here we take, for example, $\Phi(x) = \log(x)$, $x \in \mathbb{R}$, which is invex, because of no stationary points but it is not convex.

2. A differentiable pseudo-convex function on $\mathbb{R}^n$ is also invex, but not in opposite direction if $n > 1$. For $n = 1$ the two classes coincide. (Remark 2.3 [41]).

3. There exist invex functions which are not quasi-convex and there are also functions which are quasi-convex but not invex. In other words, the intersection of the classes of invex functions and quasi-convex functions is non empty (Remark 2.4 [41]).

4. In light of theorem of stationary points the groups of pseudoinvex and invex functions is one and only one.

This relationship can be understand by diagram given in [8].

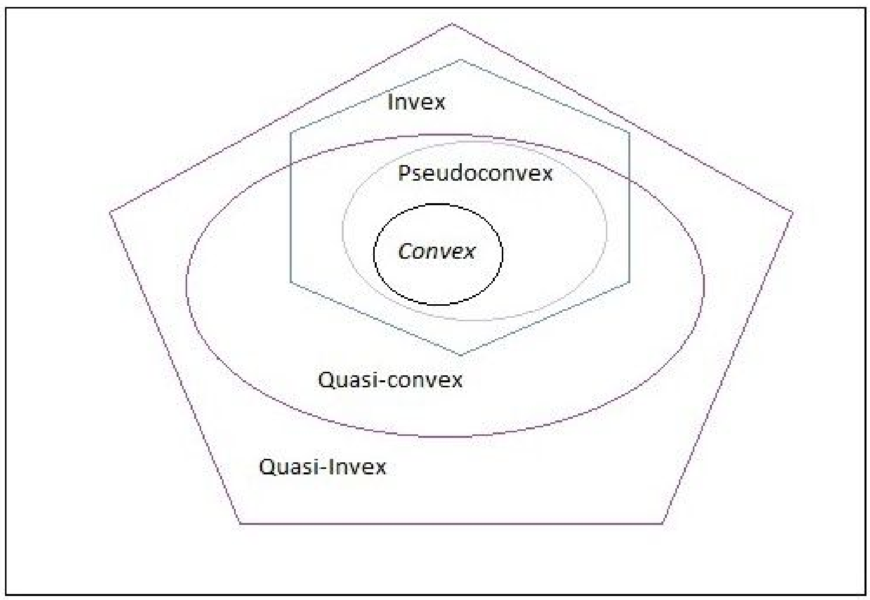

Figure 3.1.1: Invexity and generalized convexity

3.2 Preinvex Functions

Since invexity requires the diffrentiability conditions, in [8, 44] a class of preinvex functions, not necessarily differentiable was introduced.

Definition 3.2.1. (Mohan and Neogy [32]) A subset $\mathbf{S} \subseteq \mathbb{R}^n$ is called invex with to

$\varpi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ if

$$v + \alpha\varpi(u, v) \in \mathbf{S} \;\; u, v \in \mathbf{S},\; \alpha \in [0, 1]. \tag{3.2.1}$$

From definition we can conclude that $\mathbf{S}$ has a path originating from u and is limited to $\mathbf{S}$. It would not be necessary that v should be one of the end points of this path. However, if we require that u should be end point then we define $\varpi(u, v) = u - v$, which reduces the definition to convexity.

We pick up the following example of invex set from [32].

Example 3.2.2. $S = [-7, -2] \cup [2, 10]$ is invex set in $\mathbb{R}$ respecting $\varpi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by,

$$\varpi(u, v) = \begin{cases} u - v, & uv \geq 0 : \\ -7 - v, & u \geq 0,\ v \leq 0 : \\ 2 - v, & u \leq 0,\ v \geq 0. \end{cases} \tag{3.2.2}$$

More examples of invex sets can be found in [43].

The following proposition from [32] enables us to construct invex sets in $\mathbb{R}^n$, starting from invex set in $\mathbb{R}$. However, the general problem of recognizing the classes of invex sets in $\mathbb{R}^n$ which is useful in optimization theory remains open.

Proposition 3.2.3. *Suppose $K_1 \subseteq \mathbb{R}$, $K_2 \subseteq \mathbb{R}$ such that they are invex with respect to ϖ_1 and ϖ_2 respectively. Then $K_1 \times K_2 \subseteq \mathbb{R}^2$ is invex respecting ϖ defined by*

$$\varpi\begin{pmatrix} u_1, & v_1 \\ u_2, & v_2 \end{pmatrix} = \begin{pmatrix} \varpi_1(u_1, v_1) \\ \varpi_2(u_2, v_2) \end{pmatrix}. \tag{3.2.3}$$

Example 3.2.4. Suppose we are given with two invex sets $K_1 = [-6, -3] \cup [1, 6]$ and

$K_2 = [-7, -1] \cup [2, 13]$ respecting ϖ_1 and ϖ_2 respectively defined by

$$\varpi_1 = \begin{cases} u_1 - v_1, & u_1 v_1 \geq 0; \\ -6 - v_1, & u_1 \geq 0,\ v_1 \leq 0; \\ 1 - v_1, & u_1 \leq 0,\ v_1 \geq 0. \end{cases} \text{ and } \varpi_2 = \begin{cases} u_2 - v_2, & u_2 v_2 \geq 0; \\ -7 - v_2, & u_2 \geq 0,\ v_2 \leq 0; \\ 2 - v_2, & u_2 \leq 0,\ v_2 \geq 0. \end{cases}$$

then we can easily prove that $K_1 \times K_2$ is invex with respect to $\varpi = \begin{pmatrix} \varpi_1 \\ \varpi_2 \end{pmatrix}$.

The following definition is adopted from [43].

Definition 3.2.5. Let Φ be a function defined in invex set K respecting ϖ. Φ is said to be preinvex with respect to ϖ if,

$$\Phi(v + \lambda\varpi(u, v)) \leq \lambda\Phi(u) + (1 - \lambda)\Phi(v), \qquad \forall u, v \in K,\ \forall\lambda \in [0, 1]. \tag{3.2.4}$$

We can see that, the class of convex functions is strictly contained in preinvex functions by taking $\varpi(u, v) = u - v$.

Example 3.2.6. [40]. Let $\Phi : \mathbb{R} \to \mathbb{R}$ be given as $\Phi(u) = -|u|$, then Φ is preinvex respecting

$$\varpi(u, v) = \begin{cases} v - u, & uv \geq 0; \\ u - v, & uv < 0. \end{cases}$$

Like pseudo-convex and quasi-convex functions we also have prepseudoinvex and prequasiinvex functions. Let us look at them one by one with relation with preinvexity.

Definition 3.2.7. Let K be an invex set respecting some ϖ. A function Φ is said to be prepseudoinvex respecting ϖ if,

$$\Phi(u) < \Phi(v) \implies \Phi(v + \alpha\varpi(u, v)) \leq \Phi(v) + \alpha(\alpha - 1)b(u, v) \text{ with } \alpha \in (0, 1), \tag{3.2.5}$$

where b is positive function.

The following result can be seen in [41].

Theorem 3.2.8. *A function Φ is preinvex with respect to ϖ on some invex set K only if Φ is prepseudoinvex respecting the same ϖ.*

Definition 3.2.9. Let K be an invex set with respect to ϖ. A function Φ is said to be prequasiinvex on K if we have,

$$\Phi(v + \alpha\varpi(u, v)) \leq max\{\Phi(u), \Phi(v)\}, \tag{3.2.6}$$

for all $u, v \in K$ and $\alpha \in (0, 1)$.

It can be seen in ([6], p.60) that a prequasiinvex function on K possesses the minimum property on every segment $[v, v + \varpi(u, v)]$ for all $u, v \in K$.

Like quasiconvex functions prequsiinvex functions can also be characterized by it's lower level sets. We have the next propositions from[41].

Proposition 3.2.10. *A function $\Phi : K \to \mathbb{R}$ (where K is invex respecting ϖ) is prequasiinvex respecting ϖ only if it's lower sets are invex respecting ϖ.*

Proposition 3.2.11. *Let $\Phi : K \to \mathbb{R}$ (where K is invex respecting $\varpi(u, v) \neq 0$ whenever $u \neq v$) is prequasiinvex respecting ϖ. Then every local minimum of Φ is absolute minimum and the set of all these points in invex respecting ϖ.*

The following theorem is proved in [8].

Theorem 3.2.12. *If Φ is differentiable and is preinvex with respect to some ϖ then, Φ is invex with respect to the same ϖ.*

The converse of the above theorem is not true in general. It seems that preinvexity is stronger condition, as shown by Pini [41] by the following example.

Example 3.2.13. The function $\Phi(u) = u^2$ is invex respecting ϖ defined by

$$\varpi = \begin{cases} \frac{u^2 - v^2}{2v}, & v \neq 0; \\ 0, & v = 0. \end{cases}$$

but not preinvex.

In [32], Mohan and Neogy imposed a condition called "Condition C" on ϖ with which a differentiable function which is invex on K, respecting ϖ, is also preinvex on K.

Let $\varpi : \mathbb{R}^{\mathbf{n}} \times \mathbb{R}^{\mathbf{n}} \to \mathbb{R}^{\mathbf{n}}$; we say the function ϖ satisfies condition C if for any $u, v \in \mathbb{K}$

$$\varpi(v, v + \lambda\varpi(u, v)) = -\lambda\varpi(u, v), \tag{3.2.7}$$

and

$$\varpi(u, v + \lambda\varpi(u, v)) = (1 - \lambda)\varpi(u, v), \tag{3.2.8}$$

for all $\lambda \in [0, 1]$.

An important consequence of Condition C is,

$$\varpi(v + \alpha_1\varpi(u, v), v + \alpha_2\varpi(u, v)) = (\alpha_1 - \alpha_2)\varpi(u, v)\, \alpha_1, \alpha_2 \in [0, 1]. \tag{3.2.9}$$

The following results are taken from [32].

Theorem 3.2.14. *Let K be an invex set respecting ϖ, with ϖ satisfying Condition C and O be an open set such that $K \subseteq O$. Let $\Phi : O \to \mathbb{R}$ be differentiable function, then we have the following,*

- *Φ is invex on K respecting ϖ only if Φ is preinvex on K respecting ϖ.*
- *Φ is quasiinvex on K respecting ϖ then Φ is prequasiinvex on K respecting ϖ.*

The following results show us the relation of prepseudoinvex functions with quasiinvex and prequasiinvex functions.

Theorem 3.2.15. *Let $K \subseteq \mathbb{R}^n$ be an invex set with respect to some ϖ and $\Phi : \mathbb{R}^n \to \mathbb{R}$ be differentiable and prepseudoinvex on K, then Φ is quasiinvex on K.*

Theorem 3.2.16. *If $\Phi : \mathbb{R}^n \to \mathbb{R}$ be prepseudoinvex then Φ is prequasiinvex respecting the same ϖ.*

Chapter 4

Inequalities

This chapter presents discrete and integral versions of some inequalities for convex functions and same inequalities for preinvex functions and after that we discuss unified version of inequalities for convex functions on time scales.

4.1 Inequalities for Convex Functions

Let us first start with Jensen's inequality taken from [30].

Theorem 4.1.1. *Let $\varPhi : I = [x, y] \to \mathbb{R}$ is convex function, suppose $u_k \in I$, $k = 1, 2, 3, ..., n$ and $\alpha_k \in \mathbf{R}^+, k \in \{1, 2, .., n\}$ then the following holds*

$$\varPhi\Big(\frac{\sum_{k=1}^{n} \alpha_k u_k}{\sum_{k=1}^{n} \alpha_k}\Big) \leq \frac{\sum_{k=1}^{n} \alpha_k \varPhi(u_k)}{\sum_{k=1}^{n} \alpha_k}. \tag{4.1.1}$$

If $\sum_{k=1}^{n} \alpha_k = 1$ then

$$\varPhi\Big(\sum_{i=1}^{n} \alpha_k u_k\Big) \leq \sum_{k=1}^{n} \alpha_k \varPhi(u_k). \tag{4.1.2}$$

In literature this is called Jensen's Inequality. It was proved by J.L.W.V Jensen in [27, 28].

He applied the well-known inductive approach used via Cauchy (1821) within the proof respecting the arithmetic-geometric mean inequality. However, inequality appears, under specific assumptions, plenty earlier. Jensen himself pointed out in the appendix in accordance with his paper that O. Hölder proved inequality in 1889, supposing to that amount Φ is a doubly differentiable function on I such as $\Phi''(u) \geq 0$ on that interval. This assumption is of the case about double differentiable functions equivalent together with the consideration that Φ is convex.The above inequality was once proved, after Hölder, the usage of the identical assumptions by R. Henderson in 1895. However, namely a long way again as much 1875 a specific cas of the above inequality, the case when $\alpha_k = 1$, $\forall k = 1, 2, .., n$ was proved by J. Grolous by using an utility of the centroid method. This is, so far so we may want to find, the first inequality for convex functions to show up within the mathematical literature. J. Grolous also introduced the assumption that $\Phi''(u) \geq 0$, but that may be viewed from the textual content itself as it is enough according to expect as Φ is a convex function, in the geometric sense (see [37]).

It is also known that the assumption $\alpha_k > 0$ can be relaxed at the expense of restricting u_k, $k = 1, 2, 3, ..., n$ more severely [42]. Namely, if $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ is a real n-tuple such that for every $j \in \{1, 2, 3, ..., n\}$,

$$0 < \sum_{k=1}^{j} \alpha_k \leq \sum_{k=1}^{n} \alpha_k, \tag{4.1.3}$$

then for any monotonic n-tuple $u = (u_1, u_2, ..., u_n) \in I^n$ (increasing or decreasing) we get $\bar{u} = \sum_{k=1}^{n} \alpha_k x_k \in I$, and for any function Φ convex on I still (4.1.1) holds. Under such assumptions this inequality is called the Jensen–Steffensen inequality for convex functions and (4.1.3) are called Steffensen's conditions due to J. F. Steffensen.

The Jensen inequality for integrals takes the form as follows, see [27].

Theorem 4.1.2. *Let* $\Phi \in C(I, \mathbb{R})$ *is convex on* $I \subseteq \mathbb{R}$ *and* $g \in C([x, y], \mathbb{R})$*, with*

$x, y \in \mathbb{R}$ and $x < y$, then,

$$\Phi\Big(\frac{\int_x^y g(u)du}{y-x}\Big) \leq \frac{\int_x^y \Phi(g(u))du}{y-x}. \tag{4.1.4}$$

Now we present an inequality that is sharply related in accordance with Jensen inequality. It is Hermite-Hadamard inequality(also called the first fundamental inequality) for convex function. It was once first established by Hermite in 1881. But, it was additionally proved by Hadamard of 1893 whoever was no longer aware of Hermit's work. Therefore, in general, this inequality is called Hermite-Hadamard inequality and offers to us an estimate for the integral arithmetic mean:

$$\Phi\Big(\frac{x_1+x_2}{2}\Big) \leq \frac{1}{x_2-x_1}\int_{x_1}^{x_2} \Phi(u)du \leq \frac{\Phi(x_1)+\Phi(x_2)}{2}. \tag{4.1.5}$$

with $x_1, x_2 \in \mathbb{R}$ with $x_1 < x_2$ and Φ is convex function on $[x_1, x_2]$.

Hadamard's inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found, see [19, 20, 18, 21].

Theorem 4.1.3. *A function Φ is convex on convex set $K = [x_1, x_2]$ if and only if it satisfies the Hermite-Hadamard Inequality (4.1.5).*

Now we move toward two useful inequalities with enormous usage: Hölder's and Minkowski's Inequality. We shall also present some of their variants. For this purpose we firstly introduce Young's Inequality [47, 48].

In 1912, Young presented the following inequality called Young's inequality

$$xy \leq \int_0^x \Phi(u)du + \int_0^y (\Phi^{-1})(v)dv, \tag{4.1.6}$$

for any real valued continuous function, $\Phi : [0, \infty) \to [0, \infty)$ satisfying $\Phi(0) = 0$ with Φ is strictly increasing on $[0, \infty)$ and $x, y \in [0, \infty)$. The equality holds if and only if $y = f(x)$. The classical Young's inequality is a useful consequence and can be obtained by setting $\Phi(u) = u^{p-1}$with $\frac{1}{p} + \frac{1}{q} = 1$ and is given as ,

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q}. \tag{4.1.7}$$

Equality holds if and only if $x^p = y^q$.
The Hölder's inequality can be found in [15], which is states as follow:

Theorem 4.1.4. *Let $u_1, u_2, ..., u_n$ and $v_1, v_2, ..., v_n$ be any positive real numbers and $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then,*

$$\sum_{k=1}^{n} u_k v_k \leq \Big(\sum_{k=1}^{n} u_k^p\Big)^{\frac{1}{p}} \Big(\sum_{i=1}^{n} v_k^q\Big)^{q}. \tag{4.1.8}$$

Equality occurs if and only if

$$\frac{u_1^p}{v_1^q} = \frac{u_2^p}{v_2^q} = ... = \frac{u_n^p}{v_n^q}.$$

The inequality holds in reverse if $p < 1 \, and \, p \neq 0$.
In integrals Hölder's inequality takes the form

$$\int_x^y |\Phi(u)\phi(u)| du \leq \Big[\int_x^y |\Phi(u)|^p du\Big]^{\frac{1}{p}} \Big[\int_x^y |\phi(u)|^q du\Big]^{\frac{1}{q}}, \tag{4.1.9}$$

where $x, y \in \mathbb{R}$ and $\Phi, \phi \in C([x, y], \mathbb{R})$.

In the above theorem if we let $p = q = 2$, then the inequality is called Cauchy's inequality. The Minkowski's inequality can be seen in [15], which states that,

Theorem 4.1.5. *Let $u_1, u_2, u_3, ..., u_n$and $v_1, v_2, v_3, ..., v_n$ be positive real numbers and $p > 1$. Then*

$$\Big(\sum_{k=1}^{n} (u_k + v_k)^p\Big)^{\frac{1}{p}} \leq \Big(\sum_{k=1}^{n} u_k^p\Big)^{\frac{1}{p}} + \Big(\sum_{k=1}^{n} v_k^p\Big)^{\frac{1}{p}}. \tag{4.1.10}$$

Equality holds if and only if

$$\frac{u_1}{v_1} = \frac{u_2}{v_2} = ... = \frac{u_n}{v_n}.$$

The integral version of Minkowski's inequality can be seen in [23], which is given as,

$$\Big(\int_x^y |\Phi(u) + \phi(u)|^p du\Big)^{\frac{1}{p}} \leq \Big(\int_x^y |\Phi(u)|^p du\Big)^{\frac{1}{p}} + \Big(\int_x^y |\phi(u)| du\Big)^{\frac{1}{p}}. \tag{4.1.11}$$

4.2 Inequalities for Preinvex Functions

Now we want to see some inequalities of convex functions for preinvex functions. We shall discuss Jensen and Hermite-Hadamard inequalities with some refinements for preinvex functions but before that we give a theorem under validity of Condition C which shows us where preinvexity coincides with convexity. The results in this section are adopted from [40, 36].

Theorem 4.2.1. *Let Φ be a preinvex function on invex set K with respect to ϖ satisfying condition C. Then the function Φ is convex on the segment $[u, u + \varpi(v, u)]$ for every pair of points $u, v \in K$.*

Corollary 4.2.2. *Let $0 \leq \alpha_1, \alpha_2, \alpha_3, ...\alpha_n \leq 1$ be such that*

$$\sum_{k=1}^{n} \alpha_k = 1.$$

Let

$$\{t_k\}_{k=1}^n \subset [0, 1],$$

and

$$t = \sum_{k=1}^{n} \alpha_k t_k.$$

Then under the hypothesis of the theorem(4.1.1) Φ satisfies the inequalities,

$$\Phi(u+t\varpi(v,u)) \leq \sum_{k=1}^{n} \alpha_k \Phi(u+t_k\varpi(v,u)) \leq (1-t)\Phi(u) + t\Phi(u+\varpi(v,u)), \quad (4.2.1)$$

and

$$\Phi\Big(u+\sum_{k=1}^{n}\alpha_k(1-t_k)\varpi(v,u)\Big) \leq t\Phi(u)+(1-t)\Phi(u+\Phi(v,u))$$

$$\leq \Phi(u)+\Phi(u+\varpi(v,u)) - \sum_{k=1}^{n}\alpha_k\Phi(u+t_k\varpi(v,u)) \quad (4.2.2)$$

for every pair of points $u, v \in K$.

The first inequality in (4.2.1) provides Jensen inequality for preinvex functions, we can write it as

$$\Phi\Big(\sum_{k=1}^{n}\alpha_k(u+t_k\varpi(v,u)) \leq \sum_{k=1}^{n}\alpha_k\Phi(u+t_k\varpi(v,u)).$$

Theorem 4.2.3. *Let $S \subseteq \mathbb{R}$ be invex set regarding some ϖ that satisfies condition C. Assume $\Phi : S \to \mathbb{R}$ be a preinvex function on S with regards to the same ϖ. Then the following holds*

$$\Phi\Big(u_1+\frac{\varpi(u_2,u_1)}{2}\Big) \leq \frac{1}{||\varpi(u_2,u_1)||}\int_{u_1}^{u_1+\varpi(u_2,u_1)}\Phi(u)du \leq \frac{\Phi(u_1)+\Phi(u_1+\varpi(u_2,u_1))}{2}, \quad (4.2.3)$$

for every pair $u_1u_2 \in S$ with $\varpi(u_2,u_1) \neq 0$.

New we look at a refinement of this inequality and with that we shall close this section.

Theorem 4.2.4. *Let $S \subseteq \mathbb{R}$ be invex set regarding some ϖ that satisfies condition C. Assume $\Phi : S \to \mathbb{R}$ be a preinvex function on S with regards to the same ϖ.*

Then the following holds

$$\Phi\Big(u + \frac{\varpi(v,u)}{2}\Big) \le t\Phi\Big(u + \frac{t}{2}\varpi(v,u)\Big) + (1-t)\Phi\Big(u + \frac{1+t}{2}\varpi(v,u)\Big)$$

$$\le \frac{1}{||\varpi(v,u)||}\int_u^{u+\varpi(v,u)} \Phi(u)du \tag{4.2.4}$$

$$\le \frac{t\Phi(a) + (1-t)\Phi(u+\varpi(v,u)) + \Phi(u + t\varpi(v,u))}{2}$$

$$\le \frac{\Phi(u) + \Phi(u+\varpi(v,u))}{2}$$

for every pair $u, v \in S$ *with* $\varpi(v,u) \neq 0$, *and* $t \in [0,1]$.

4.3 Inequalities on Time Scales

The purpose of this section is to present extension of inequalities on time scales that are discussed in previous section.

The following are unified versions of Young's inequality and are adopted from [45, 3]

Theorem 4.3.1. *Let* Φ *be rd-continuous on* $[0,c]_{\mathbb{T}} = [0,c] \cap \mathbb{T}$ *for* $c > 0$, *strictly increasing, with* $\Phi(0) = 0$. *Then for* $x \in [0,c]_{\mathbb{T}}$ *and* $y \in [0, \Phi(c)]$ *the inequality*

$$xy \le \int_0^x \Phi^\sigma(u)\Delta u + \int_0^y (\Phi^{-1})^\sigma(v)\Delta v. \tag{4.3.1}$$

For $\mathbb{T} = \mathbb{R}$ this yields classical young's inequality.

and for $\mathbb{T} = \mathbb{Z}$ and $\Phi(u) = u$, this theorem says that

$$xy \le \sum_{u=0}^{x-1}(u+1) + \sum_{v=0}^{y-1}(v+1) = \frac{1}{2}x(x+1) + \frac{1}{2}y(y+1). \tag{4.3.2}$$

Theorem 4.3.2. *Let* $\mathbb{T}$ *be any time scales (unbounded above) with* $0 \in \mathbb{T}$. *Further, suppose that* $\Phi : [0,\infty)_{\mathbb{T}} \to \mathbb{R}$ *is a real-valued function satisfying*

- $\Phi(0) = 0$;
- Φ *is continuous on* $[0, \infty)_{\mathbb{T}}$*, rd-continuous at* 0;
- Φ *is strictly increasing on* $[0, \infty)_{\mathbb{T}}$ *such that* $\Phi(\mathbb{T})$ *is also a time scales.*

Then for any $x \in [0, \infty)_{\mathbb{T}}$ *and* $y \in [0, \infty)_{\Phi(\mathbb{T})}$*, we have*

$$2xy \leq \int_0^x [\Phi(u) + \Phi^{\sigma}(u)]\Delta u + \int_0^y [\Phi^{-1}(v) + \Phi^{-1}(\sigma(v))]\Delta v \tag{4.3.3}$$

with equality if and only if $y = \Phi(x)$.

Many more versions of Young's inequality can be found in literature. Now let us look at Hölder's inequality which can be found in [2].

Theorem 4.3.3. *Let* $x, y \in \mathbb{T}$*. For rd-continuous* $\Phi, \phi : [x, y] \to \mathbb{R}$ *we have*

$$\int_x^y |\Phi(u)\phi(u)|\Delta u \leq \Big[\int_x^y |\Phi(u)|^p \Delta u\Big]^{\frac{1}{p}} \Big[\int_x^y |\phi(u)|^q \Delta u\Big]^{\frac{1}{q}}. \tag{4.3.4}$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

The proof of this is similar as that of classical Hölder's Inequality.

For $p = q = 2$ this inequality yields Cauchy inequality for time scales.

Next we present unified version of Minkowski's Inequality which can be seen in [2] and it's proof is similar to that of Minkowski inequality in classical version.

Theorem 4.3.4. *Let* $x, y \in \mathbb{T}$*. For rd-continuous functions* $\Phi, \phi : [x, y] \to \mathbb{R}$ *we have*

$$\Big(\int_x^y |\Phi(u) + \phi(u)|^p \Delta u\Big)^{\frac{1}{p}} \leq \Big(\int_x^y |\Phi(u)|^p \Delta u\Big)^{\frac{1}{p}} + \Big(\int_x^y |\phi(u)|\Delta u\Big)^{\frac{1}{p}}.$$

There seems no big difference between these inequalities on time scales and that in classical inequalities.

Let us move toward other inequalities for convex function to see that there is difference or not and after that we are closing this chapter.

Firstly we see Jensen Inequality from [2].

Theorem 4.3.5. *Let* $\phi \in C_{rd}([x,y],(r,s))$ *where* $x,y \in \mathbb{T}$ *and* $r,s \in \mathbb{R}$. *If* $\Phi \in C((r,s),\mathbb{R})$ *is convex then,*

$$\Phi\left(\frac{\int_x^y \phi(u)\Delta u}{y-x}\right) \leq \frac{\int_x^y \Phi(\phi(u))\Delta u}{y-x}. \tag{4.3.5}$$

Hermite-Hadamard inequality for time scales is discussed in [5] for $\mathbb{Z}$ and for arbitrary time scale it can be seen in [4].

Theorem 4.3.6. *Suppose* $\Phi : \mathbb{Z} \to \mathbb{R}$ *is a convex function on* $[a,b]_{\mathbb{Z}}$ *with* $a,b \in \mathbb{Z}$, $a < b$, *and* $a+b$ *is even number. Then*

$$\Phi\Big(\frac{a+b}{2}\Big) \leq \frac{1}{2(b-a)}\Big[\int_a^b \Phi(x)\Delta x + \int_a^b \Phi(x)\nabla x\Big] \leq \frac{\Phi(a)+\Phi(b)}{2} \tag{4.3.6}$$

holds.

Theorem 4.3.7. *Suppose* $\Phi : \mathbb{T} \to \mathbb{R}$ *be a convex function on* $[a,b]_{\mathbb{T}}$. *Then*

$$\begin{aligned}\Phi\Big(m_[a,b]\Big) &\leq \frac{1}{b-a}\Big[\int_{[a,b]_{\mathbb{T}}} k(x)\Phi(x)\Delta x - \int_{[a,b]_{\mathbb{T}}} k(x)\phi^{\nabla}(x)\Phi(x)\nabla x\Big] \\ &\leq m_{[0,1]}\Phi(a) + (1-m_{[0,1]})\Phi(b),\end{aligned}$$

where $\phi : [a,b]_{\mathbb{T}} \to [a,b]_{\mathbb{T}}$ *and* $k : [a,b]_{\mathbb{T}} \to \mathbb{R}^+$ *defined by*

$$\phi(\lambda a + (1-\lambda)b) = (1-\lambda)a + \lambda b,$$

with $\lambda \in [0,1]$, *and*

$$k(x) = \begin{cases} \frac{\phi(x)-m}{\phi(x)-x}, & x \neq m_[a,b]; \\ \frac{1}{2}, & x = m_[a,b]. \end{cases}$$

where $m_{[a,b]}$ *is midpoint of* $[a,b]_{\mathbb{T}}$ *and* $m_{[0,1]}$ *is midpoint of* $[0,1]$.

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Chapter 5

Preinvex Functions on Time Scales

The theory of time scales developed by Hilger was actually the theory of unification of discrete and continuous analysis. Many people worked in this theory and developed the concepts of calculus and analysis in this theory. Like other people C. Dinu in 2008 introduced the concept of convex functions on time scales. After C. Dinu, Abe-i-kpeng in 2016 put quasi-convex functions in the context of time scales. However, a vast class of preinvex functions has not been examined up until now on time scales. So, the basic thing in this chapter is that we discuss preinvex functions in time scales and also discuss Hermite-Hadamard inequality for these functions.

5.1 Preinvex Functions

Definition 5.1.1. Let $\mathbb{R}$ be the set of reals and $\mathbb{T}$ be a time scale. A subset $K_{\mathbb{T}} \subseteq \mathbb{T}$ is $\mathbb{T}-$*invex* or simply *invex* respecting $\eta_{\mathbb{T}}$ if and only if there exists an invex set $K \subseteq \mathbb{R}$ respecting η such that, $K_{\mathbb{T}} = K \cap \mathbb{T}$ where $\eta_{\mathbb{T}}$ is restriction of η to time scale $\mathbb{T}$ that is $\eta_{\mathbb{T}} = \eta$ for all $(x, y) \in \mathbb{T} \times \mathbb{T}$.

Formally the definition becomes as under.

Definition 5.1.2. Let $\mathbb{T}$ be a time scales , a subset $K_{\mathbb{T}} \subseteq \mathbb{T}$ is called an invex set with respect to $\eta_{\mathbb{T}} : K_{\mathbb{T}} \times K_{\mathbb{T}} \to \mathbb{R}$ if $\forall a, b \in K_{\mathbb{T}}$ and $\lambda \in [0,1]$ the generated segment $[a, a + \lambda\eta_{\mathbb{T}}(b,a)]$ is contained in $K_{\mathbb{T}}$.

Example 5.1.3. Let $\mathbb{T} = \mathbb{R}$, then $K_{\mathbb{T}} = [-3,-1] \cup [0,5]$ is invex respecting

$$\eta(a,b) = \begin{cases} a-b, & ab \geq 0; \\ -3-y, & a \geq 0, b \leq 0; \\ -y, & a \leq 0, b \geq 0; \end{cases}$$

Definition 5.1.4. Let $K_{\mathbb{T}} \subseteq \mathbb{T}$ be an invex set respecting $\eta_{\mathbb{T}}$. A function $\Phi : K_{\mathbb{T}} \longrightarrow \mathbb{R}$ is said to be preinvex respecting $\eta_{\mathbb{T}}$, if

$$\Phi(a + \lambda\eta_{\mathbb{T}}(b,a)) \leq \lambda\Phi(b) + (1-\lambda)\Phi(a), \tag{5.1.1}$$

holds for all $a, b \in K_{\mathbb{T}}$ *and* $\lambda \in [0,1]$.

Theorem 5.1.5. *A function $f : K_{\mathbb{T}} \to \mathbb{R}$ is preinvex with respect to $\eta_{\mathbb{T}}$ if and only if the epigraph of f is invex set in $\mathbb{T} \times \mathbb{R}$.*

Proof. Let $epif = \Big\{(x,y) \in \mathbb{T} \times \mathbb{R} : f(x) \leq y\Big\}$ is an invex set respecting $\acute{\eta}\Big((x_1,y_1),(x_2,y_2)\Big) = \Big(\eta_{\mathbb{T}}(x_2,x_1), y_2 - y_1\Big)$, then for (x_1,y_1), $(x_2,y_2) \in epif$ and $\lambda \in [0,1]$

$$\begin{aligned} (x_1,y_1) + \lambda\acute{\eta}\Big((x_1,y_1),(x_2,y_2)\Big) &= (x_1,y_1) + \lambda\Big(\eta_{\mathbb{T}}(x_2,x_1), y_2 - y_1\Big) \\ &= \Big(x_1 + \lambda\eta_{\mathbb{T}}(x_2,x_1), \lambda y_2 + (1-\lambda)y_1\Big) \in epif. \end{aligned}$$

Then by invexity of $epif$, we have

$$f(x_1 + \lambda\eta_{\mathbb{T}}(x_2,x_1)) \leq \lambda y_2 + (1-\lambda)y_1.$$

We also have

$$\lambda f(x_2) + (1-\lambda) f(x_1) \leq \lambda y_2 + (1-\lambda) y_1.$$

From these two inequalities we deduce that,

$$f(x_1 + \lambda \eta_{\mathbb{T}}(x_2, x_1)) \leq \lambda f(x_2) + (1-\lambda) f(x_1).$$

This prove preinvexity of f.

Conversely, we prove invexity of $epif$ respecting $\acute{\eta}$, when f is given to be preinvex. For this, since (x_1, y_1), $(x_2, y_2) \in epif$ and $\lambda \in [0,1]$ we have,

$$(x_1, y_1) + \lambda \acute{\eta}\Big((x_1, y_1), (x_2, y_2)\Big) = (x_1, y_1) + \lambda\Big(\eta_{\mathbb{T}}(x_2, x_1), y_2 - y_1\Big)$$

$$= \Big(x_1 + \lambda \eta_{\mathbb{T}}(x_2, x_1), \lambda y_2 + (1-\lambda) y_1\Big).$$

By preinvexity of f,

$$\begin{aligned} f(x_1 + \lambda \eta_{\mathbb{T}}(x_2, x_1)) &\leq \lambda f(x_2) + (1-\lambda) f(x_1) \\ &\leq \lambda y_2 + (1-\lambda) y_1. \end{aligned} \tag{5.1.2}$$

Hence we are done with the proof. □

Theorem 5.1.6. *Let $K_{\mathbb{T}} = K \cap \mathbb{T}$ be an invex set respecting $\eta_{\mathbb{T}}$ in $\mathbb{T}$ where K is invex with respect to some η in $\mathbb{R}$. A function $f : K_{\mathbb{T}} \to \mathbb{R}$ is preinvex respecting $\eta_{\mathbb{T}}$ if and only if there exists a preinvex function $\tilde{f} : \mathbb{R} \to \mathbb{R}$ respecting some η and such that $\tilde{f}(t) = f(t)$ for all $t \in K_{\mathbb{T}}$.*

Proof. Sufficiency follows from preinvexity of $\tilde{f}$.

To see necessary part let us assume that f is invex set with respect to $\eta_{\mathbb{T}}$on $K_{\mathbb{T}}$.

We define $\tilde{f} : K \to \mathbb{R}$ by

$$\tilde{f}(x) = \begin{cases} f(x), & \text{if } x \in K_{\mathbb{T}}; \\ f(y) + \frac{f(t)-f(y)}{\eta_{\mathbb{T}}(y,t)}(x-y), & x \in (y,t) \text{ with } x \in K \backslash K_{\mathbb{T}} \text{ and } t, y \in K_{\mathbb{T}}; \\ f(y), & x \in (t,y) \text{ with } x \in K \backslash K_{\mathbb{T}} \text{ and } y \in K_{\mathbb{T}}; \\ f(t), & x \in (t,y) \text{ with } x \in K \backslash K_{\mathbb{T}} \text{ and } t \in K_{\mathbb{T}}. \end{cases}$$

The proof of $\tilde{f}$ is preinvex is divided into three cases:

Case 1: $x, y \in K_{\mathbb{T}}$ then we are done.

Case 2: $x \in K_{\mathbb{T}}$ and $y \in K \backslash K_{\mathbb{T}}$ then we argument as follows, We have either $x + \lambda\eta(y,x) \in K_{\mathbb{T}}$ or in $K \backslash K_{\mathbb{T}}$. In first subcase we are done. But, if $x + \lambda\eta(y,x) \in K \backslash K_{\mathbb{T}}$ then we can find $t \in K_{\mathbb{T}}$ such that $t > y$ and $x + \lambda\eta(y,x) \in (x,t)$, and so

$$\tilde{f}(x + \lambda\eta(y,x)) = f(x) + \frac{f(t) - f(x)}{\eta_{\mathbb{T}}(x,t)}(x + \lambda\eta(y,x) - x)$$

By using the fact that $\eta(y,x) \leq \eta_{\mathbb{T}}(x,t)$, we obtain

$$\tilde{f}(x + \lambda\eta(y,x)) \leq f(x) + \frac{f(t) - f(x)}{\eta_{\mathbb{T}}(x,t)}\lambda\eta_{\mathbb{T}}(x,t)$$

$$\tilde{f}(x + \lambda\eta(y,x)) \leq (1-\lambda)f(x) + \lambda f(t)$$

And

$$\tilde{f}(x + \lambda\eta(y,x)) \leq (1-\lambda)f(x) + \lambda f(y).$$

Case 3: $x, y \in K \backslash K_{\mathbb{T}}$ then preinvexity of $\tilde{f}$ follows from previous two cases.

□

Remark 5.1.7. In the above result if $\eta_{\mathbb{T}}(a,b) = b-a$ we get Theorem (3.4) of [16].

We can define Condition C for time scales as follows.

Definition 5.1.8. Let $K_{\mathbb{T}} \subseteq \mathbb{T}$ be an invex set respecting a function $\eta_{\mathbb{T}}$. We say that $\eta_{\mathbb{T}}$ satisfies condition C if the following equalities holds

$$\eta_{\mathbb{T}}(x, x + t\eta_{\mathbb{T}}(y,x)) = -t\eta_{\mathbb{T}}(y,x). \tag{5.1.3}$$

$$\eta_{\mathbb{T}}(y, x + t\eta_{\mathbb{T}}(y,x)) = (1-t)\eta_{\mathbb{T}}(y,x). \tag{5.1.4}$$

Theorem 5.1.9. *Let $a, b \in \mathbb{T}$ be a pair of points, let $[a,b] = I \cap \mathbb{T}$ be an invex set respecting a mapping $\eta_{\mathbb{T}}$. Then $\eta_{\mathbb{T}}(y,x)$ is collinear with $b-a$ for every pair of points in $[a,b]$.*

Proof. Let $x, y \in [a,b]$, $x = a + t_1(b-a)$, $t_1 \in [0,1]$. Since

$$x + t\eta_{\mathbb{T}}(y,x) \in [a,b]$$

then,

$$x + t\eta_{\mathbb{T}}(y,x) = a + t_2(b-a),$$

$$a + t_1(b-a) + t\eta_{\mathbb{T}}(y,x) = a + t_2(b-a),$$

$$\eta_{\mathbb{T}}(y,x) = \frac{t_2 - t_1}{t}(b-a).$$

This proves the collinearity . □

Corollary 5.1.10. *Let $K_{\mathbb{T}} \subseteq \mathbb{T}$ be an invex set with respect to a mapping $\eta_{\mathbb{T}}$. Let $a, b \in K_{\mathbb{T}}$ be a pair of points such that the generated time scale interval $[a, a+\eta_{\mathbb{T}}(b,a)]$ is invex respecting $\eta_{\mathbb{T}}$. Then $\eta_{\mathbb{T}}(y,x)$ is collinear with $\eta_{\mathbb{T}}(b,a)$ for every pair of points x and y.*

Proof. Let x , $y \in [a, a + \eta_{\mathbb{T}}(b,a)]$ then $x = a + t_1\eta_{\mathbb{T}}(b,a)$, where $t_1 \in [0,1]$ and since

$[a, a + \eta_{\mathbb{T}}(b, a)]$ is invex respecting $\eta_{\mathbb{T}}(y, x)$, then

$$x + t\eta_{\mathbb{T}}(y, x) \in [a, a + \eta_{\mathbb{T}}(b, a)] \text{ with } t \in [0, 1].$$

Therefore,

$$x + t\eta_{\mathbb{T}}(y, x) = a + t_2\eta_{\mathbb{T}}(b, a).$$

Inserting value of x and simplifying results in,

$$\eta_{\mathbb{T}}(b, a) = \frac{t}{t_2 - t_1}\eta_{\mathbb{T}}(y, x).$$

This shows that the two are collinear. □

Theorem 5.1.11. *Let $K_{\mathbb{T}} \subseteq \mathbb{T}$ be an invex set respecting $\eta_{\mathbb{T}}$ that satisfies condition C, and let $f : K_{\mathbb{T}} \longrightarrow \mathbb{R}$ be a preinvex function with respect to $\eta_{\mathbb{T}}$. Then the function f is convex on the generated time scale interval $[a, a + \eta_{\mathbb{T}}(b, a)]$ for every pair of points $a, b \in K_{\mathbb{T}}$.*

Proof. Let $x, y \in [a, a + \eta_{\mathbb{T}}(b, a)]$ then $x = a + t_1\eta_{\mathbb{T}}(b, a)$ and $y = a + t_2\eta_{\mathbb{T}}(b, a)$ we have,

$$x + \lambda\eta_{\mathbb{T}}(y, x) = a + t_1\eta_{\mathbb{T}}(b, a) + \lambda\eta_{\mathbb{T}}(a + t_1\eta_{\mathbb{T}}(b, a), a + t_2\eta_{\mathbb{T}}(b, a))$$

$$= a + \eta_{\mathbb{T}}(b, a) + \lambda(t_2 - t_1)\eta_{\mathbb{T}}(b, a)$$

$$= a + \eta_{\mathbb{T}}(b, a) - \lambda[(t_1 - t_2)\eta_{\mathbb{T}}(b, a) + a - a]$$

$$= (1 - \lambda)(a + t_1\eta_{\mathbb{T}}(b, a)) + \lambda(a + t_2\eta_{\mathbb{T}}(b, a))$$

$$= (1 - \lambda)x + \lambda y.$$

Now consider,

$$f((1 - \lambda)x + \lambda y) = f(x + \lambda\eta_{\mathbb{T}}(y, x)) \leq (1 - \lambda)f(x) + \lambda f(y).$$

which proves convexity of f. □

5.2 Hermite-Hadamard Inequality

In this section we discuss time scales version of Hermite-Hadamard inequality for preinvex functions and compare that with that given for reals.

First we present some notations adopted from [4].

Let $\mathbb{T}$ be a time scale with all points as right scattered and $a, b \in \mathbb{T}$ then by $[a, b]_{\mathbb{T}}$ we mean time scale interval, let us define $\mathbb{T}_{[a,b]} = \{\frac{b-t}{b-a}; t \in [a, b]_{\mathbb{T}}\}$. Clearly it can be seen that $\mathbb{T}_{[a,b]} \subset [0, 1]$ and $\mathbb{T}_{[a,b]} = [0, 1]$ if $\mathbb{T}$ is continuous, also we can see there is bijection between $[a, b]_{\mathbb{T}}$ and $\mathbb{T}_{[a,b]}$.

We first establish the inequality for $\mathbb{Z}$ and then for general time scales.

Theorem 5.2.1. *Suppose that $\Phi : K_{\mathbb{Z}} \to \mathbb{R}$ be a preinvex function respecting $\varpi_{\mathbb{Z}}$ which satisfy condition C and Φ is rd-continuous with $K_{\mathbb{Z}}$ an invex set respecting the same $\varpi_{\mathbb{Z}}$, and let $a, b \in K_{\mathbb{Z}}$ be such that $I = [a, a + \varpi_{\mathbb{Z}}(b, a)]$ has an odd number of points then, prove that*

$$\Phi\Big(a + \frac{\varpi_{\mathbb{Z}}(b, a)}{2}\Big) \leq \frac{1}{2\varpi_{\mathbb{Z}}(b, a)}\Big[\int_a^{a+\varpi_{\mathbb{Z}}(b,a)} \Phi(x)\Delta x + \int_a^{a+\varpi_{\mathbb{Z}}(b,a)} \Phi(x)\nabla x\Big] \tag{5.2.1}$$

$$\leq \frac{\Phi(a) + \Phi(a + \varpi_{\mathbb{Z}}(b, a))}{2}$$

holds.

Proof. Let

$$I_{[0,1]} = \mathbb{Z}_{[a,a+\varpi_{\mathbb{Z}}(b,a)]}$$

then, for fix $\lambda \in I_{[0,1]}$ define

$$x = a + \lambda\varpi_{\mathbb{Z}}(b, a), \qquad y = a + (1 - \lambda)\varpi_{\mathbb{Z}}(b, a).$$

then, it can be easily seen that $x, y \in I$ and we can write

$$a + \frac{\varpi_{\mathbb{Z}}(b, a)}{2} = \frac{x + y}{2} = \frac{a + \lambda\varpi_{\mathbb{Z}}(b, a) + a + (1 - \lambda)\varpi_{\mathbb{Z}}(b, a)}{2}$$

and by Theorem (5.1.11) $\varPhi$ is convex on I. We have,

$$\varPhi\Big(a+\frac{\varpi_{\mathbb{Z}}(b,a)}{2}\Big)\leq\frac{1}{2}\Big(\varPhi(a+\lambda\varpi_{\mathbb{Z}}(b,a))+\varPhi(a+(1-\lambda)\varpi_{\mathbb{Z}}(b,a))\Big).$$

Integrating both sides over $I_{[0,1]}$ we have,

$$\int_{I_{[0,1]}}\varPhi\Big(a+\frac{\varpi_{\mathbb{Z}}(b,a)}{2}\Big)\Delta\lambda\leq\frac{1}{2}\Big[\int_{I_{[0,1]}}\varPhi(a+\lambda\varpi_{\mathbb{Z}}(b,a))\Delta\lambda+\int_{I_{[0,1]}}\varPhi(a+(1-\lambda)\varpi_{\mathbb{Z}}(b,a))\Delta\lambda\Big].$$

Now let us look closely at both integrals on right side.

First we conside

$$\int_{I_{[0,1]}}\varPhi(a+\lambda\varpi_{\mathbb{Z}}(b,a))\Delta\lambda.$$

Let us define $v:I\to I_{[0,1]}$by $v(x)=\frac{x-a}{\varpi_{\mathbb{Z}}(b,a)}=\lambda$, v is increasing clearly and we have $x=v^{-1}(\lambda)$, by using substitution rule (2.4.5) we have

$$\int_{I_{[0,1]}}\varPhi(a+\lambda\varpi_{\mathbb{Z}}(b,a)\Delta\lambda=\int_{v(a)}^{v(a+\varpi_{\mathbb{Z}}(b,a))}(\varPhi\circ v^{-1})(\lambda)\Delta\lambda$$

$$=\int_{a}^{a+\varpi_{\mathbb{Z}}(b,a)}\varPhi(x)v^{\Delta}(x)=\frac{1}{\varpi_{\mathbb{Z}}(b,a)}\int_{a}^{a+\varpi_{\mathbb{Z}}(b,a)}\varPhi(x)\Delta x,$$

where $v^{\Delta}(x)=\frac{1}{\varpi_{\mathbb{Z}}(b,a)}$.

Next let us observe the second integral

$$\int_{I_{[0,1]}}\varPhi(a+(1-\lambda)\varpi_{\mathbb{Z}}(b,a))\Delta\lambda.$$

Define,

$$u(x)=\frac{a+\varpi_{\mathbb{Z}}(b,a)-x}{\varpi_{\mathbb{Z}}(b,a)}=1-\lambda,$$

it can be seen that $v^{-1}(1-\lambda)=u^{-1}(\lambda)$ is decreasing and we have by substitution rule (2.4.7), it can be obtain that,

$$\int_{I_{[0,1]}} \Phi(a+(1-\lambda)\varpi_{\mathbb{Z}}(b,a))\Delta\lambda = -\int_{u(a)}^{u(a+\varpi_{\mathbb{Z}}(b,a))} (\Phi \circ u^{-1})(\lambda)\Delta\lambda$$

$$= -\int_{a}^{a+\varpi_{\mathbb{Z}}(b,a)} \Phi(x)u^{\nabla}(x)\nabla x = \frac{1}{\varpi_{\mathbb{Z}}(b,a)} \int_{a}^{a+\varpi_{\mathbb{Z}}(b,a)} \Phi(x)\nabla x,$$

where

$$u^{\nabla}(x) = -\frac{1}{\varpi_{\mathbb{Z}}(b,a)}.$$

This proves left side of (5.2.1), that is,

$$\Phi\Big(a+\frac{\varpi_{\mathbb{Z}}(b,a)}{2}\Big) \leq \frac{1}{2\varpi_{\mathbb{Z}}(b,a)}\Big[\int_{a}^{a+\varpi_{\mathbb{Z}}(b,a)} \Phi(x)\Delta x + \int_{a}^{a+\varpi_{\mathbb{Z}}(b,a)} \Phi(x)\nabla x\Big]. \tag{5.2.2}$$

Now we move to the proof of right side,

Since Φ is convex on I, we have,

$$\Phi(a+\lambda\varpi_{\mathbb{Z}}(b,a)) = \Phi((1-\lambda)a+\lambda(a+\varpi_{\mathbb{Z}}(b,a)))$$

$$\leq (1-\lambda)\Phi(a)+\lambda\Phi(a+\varpi_{\mathbb{Z}}(b,a))$$

that is

$$\Phi(a+\lambda\varpi_{\mathbb{Z}}(b,a)) \leq (1-\lambda)\Phi a+\lambda\Phi(a+\varpi_{\mathbb{Z}}(b,a)).$$

and

$$\Phi(a+(1-\lambda)\varpi_{\mathbb{Z}}(b,a)) \leq \lambda\Phi a+(1-\lambda)\Phi(a+\varpi_{\mathbb{Z}}(b,a)).$$

adding these two and integrating over $I_{[0,1]}$, we get

$$\frac{1}{\varpi_{\mathbb{Z}}(b,a)}\Big[\int_{a}^{a+\varpi_{\mathbb{Z}}(b,a)} \Phi(x)\Delta x + \int_{a}^{a+\varpi_{\mathbb{Z}}(b,a)} \Phi(x)\nabla x\Big] \leq \Phi(a)+\Phi(a+\varpi_{\mathbb{Z}}(b,a)). \tag{5.2.3}$$

Combining (5.2.2) and (5.2.3) we have the required result.

Now we discuss the described iequality for general discrete time scales and show that the continous case follows from here. □

Theorem 5.2.2. *Suppose that $\Phi : K_{\mathbb{T}} \to \mathbb{R}$ be a preinvex function respecting $\varpi_{\mathbb{T}}$ which satisfy condition C and Φ is rd-continuous with $K_{\mathbb{T}}$an invex set respecting the same $\varpi_{\mathbb{T}}$, and let $a, b \in K_{\mathbb{T}}$ be such that $I = [a, a + \varpi_{\mathbb{T}}(b, a)]$ has an odd number of points and there exists functions $\phi : I \to I$ and $k : I \to \mathbb{R}^+$ defined by*

$$\phi(a + \lambda\varpi_{\mathbb{T}}(b, a)) = a + (1 - \lambda)\varpi_{\mathbb{T}}(b, a)$$

with $\lambda \in \mathbb{T}_{[a,a+\varpi_{\mathbb{T}}(b,a)]}$, and

$$k(x) = \begin{cases} \frac{\phi(x)-m}{\phi(x)-x}, & x \neq m; \\ \frac{1}{2}, & x = m, \end{cases}$$

where m is midpoint of I. Then

$$\Phi\Big(m\Big) \leq \frac{1}{\varpi_{\mathbb{T}}(b, a)}\Big[\int_a^{a+\varpi_T(b,a)} k(x)\Phi(x)\Delta x - \int_a^{a+\varpi_{\mathbb{T}}(b,a)} k(x)\phi^{\nabla}(x)\Phi(x)\nabla x\Big] \quad (5.2.4)$$

$$\leq m_{[0,1]}\Phi(a) + (1 - m_{[0,1]})\Phi(a + \varpi_{\mathbb{T}}(b, a)).$$

holds, where $m_{[0,1]}$ is midpoint of $\mathbb{T}_{[a,a+\varpi_{\mathbb{T}}(b,a)]}$.

Proof. Let

$$I_{[0,1]} = \mathbb{T}_{[a,a+\varpi_{\mathbb{T}}(b,a)]}$$

then, for fix $\lambda \in I_{[0,1]}$ define

$$x = a + \lambda\varpi_{\mathbb{T}}(b, a), \quad y = a + (1 - \lambda)\varpi_{\mathbb{T}}(b, a).$$

then, it can be easily seen that $x, y \in I$ and we have $m_{[x,y]} = m$ where $m_{[x,y]}$ is midpoint of the segment $[x, y]$ and m is midpoint of I, we have for $x \neq m$

$$m_{[x,y]} = \frac{y - m_{[x,y]}}{y - x} x + \frac{m_{[x,y]} - x}{y - x} y.$$

$$m = \frac{y - m}{y - x} x + \frac{m - x}{y - x} y,$$

with $m = m_{[x,y]}$.

By using convexity on the segment I we have

$$\Phi(m) \leq \frac{y - m}{y - x} \Phi(x) + \frac{m - x}{y - x} \Phi(y),$$

$$\Phi\Big(m\Big) \leq k(x)\Phi(x) + k(y)\Phi(y).$$

Integrating both sides over $I_{[0,1]}$ we have

$$\int_{I_{[0,1]}} \Phi(m)\Delta\lambda \leq \int_{I_{[0,1]}} k(x)\Phi(x)\Delta\lambda + \int_{I_{[0,1]}} k(y)\Phi(y)\Delta\lambda,$$

Now let us look closely at both integrals on right side.

First we consider

$$\int_{I_{[0,1]}} k(x)\Phi(x)\Delta\lambda.$$

Let us define

$$v : I \to I_{[0,1]}$$

by

$$v(x) = \frac{x - a}{\varpi_{\mathbb{T}}(b, a)} = \lambda,$$

v is increasing clearly and we have $x = v^{-1}(\lambda)$. By setting $G := k \cdot \Phi$. Then we have $k(x)\Phi(x) = G(v^{-1}(\lambda))$. And using substitution rule (2.4.5) we have

$$\int_{I_{[0,1]}} G(v^{-1}(\lambda))\Delta\lambda = \int_{v(a)}^{v(a+\varpi_{\mathbb{T}}(b,a))} (G \circ v^{-1})(\lambda)\Delta\lambda$$

$$= \int_a^{a+\varpi_{\mathbb{T}}(b,a)} G(x)v^{\Delta}(x) = \frac{1}{\varpi_{\mathbb{T}}(b,a)} \int_a^{a+\varpi_{\mathbb{T}}(b,a)} G(x)\Delta x$$

$$= \frac{1}{\varpi_{\mathbb{T}}(b,a)} \int_a^{a+\varpi_{\mathbb{T}}(b,a)} k(x)\Phi(x)\Delta x.$$

where $v^{\Delta}(x) = \frac{1}{\varpi_{\mathbb{T}}(b,a)}$.

Next let us observe the second integral

$$\int_{I_{[0,1]}} k(x)\Phi(x)\Delta\lambda.$$

Define $u(x) = \frac{a+\varpi_{\mathbb{Z}}(b,a)-x}{\varpi_{\mathbb{Z}}(b,a)} = 1-\lambda$,it can be seen that $v^{-1}(1-\lambda) = u^{-1}(\lambda)$ is decreasing and we have by substitution rule 2.4.7, it can be obtained that,

$$\int_{I_{[0,1]}} G(v^{-1}(\lambda))\Delta\lambda = -\int_{u(a)}^{u(a+\varpi_{\mathbb{T}}(b,a))} (G \circ u^{-1})(\lambda)\Delta\lambda$$

$$= -\int_a^{a+\varpi_{\mathbb{T}}(b,a)} G(x)u^{\nabla}(x)\nabla x = \frac{-1}{\varpi_{\mathbb{T}}(b,a)} \int_a^{a+\varpi_{\mathbb{T}}(b,a)} k(x)\phi^{\nabla}(x)\Phi(x)\nabla x,$$

where $u^{\nabla}(x) = \frac{u(x)-u(\rho(x))}{x-\rho(x)}$

$= \frac{v(y)-v(\rho(y))}{x-\rho(x)}$

$= \frac{\frac{y-a}{\varpi_{\mathbb{T}}(b,a)} - \frac{\rho(y)-a}{\varpi_{\mathbb{T}}(b,a)}}{x-\rho(x)}$

$= \frac{1}{\varpi_{\mathbb{T}}(b,a)} \frac{y-\rho(y)}{x-\rho(x)}$

$= \frac{1}{\varpi_{\mathbb{T}}(b,a)} \frac{\phi(x)-\phi(\rho(x))}{x-\rho(x)}$

$= \frac{\phi^{\nabla}(x)}{\varpi_{\mathbb{T}}(b,a)}$.

This proves left side of (5.2.4) , that is,

$$\Phi(m) \leq \frac{1}{\varpi_{\mathbb{T}}(b,a)} \Big[\int_a^{a+\varpi_{\mathbb{Z}}(b,a)} k(x)\Phi(x)\Delta x - \int_a^{a+\varpi_{\mathbb{Z}}(b,a)} k(x)\phi^{\nabla}(x)\Phi(x)\nabla x \Big]. \quad (5.2.5)$$

Now we move to the proof of right side,

Since Φ is convex on I, we have,

$$\Phi(x) = \Phi((1-\lambda)a + \lambda(a + \varpi_{\mathbb{T}}(b,a)))$$

$$\leq (1-\lambda)\Phi(a) + \lambda\Phi(a + \varpi_{\mathbb{T}}(b,a))$$

that is

$$\Phi(x) \leq u(x)\Phi(a) + v(x)\Phi(a + \varpi_{\mathbb{T}}(b,a))$$

$$\implies k(x)\Phi(x) \leq k(x)u(x)\Phi(a) + k(x)v(x)\Phi(a + \varpi_{\mathbb{T}}(b,a)).$$

and

$$k(y)\Phi(y) \leq k(y)u(y)\Phi(a) + k(y)v(y)\Phi(a + \varpi_{\mathbb{T}}(b,a)).$$

Adding these two inequalities we have,

$$k(x)\Phi(x)+k(y)\Phi(y) \leq [k(x)u(x)+k(y)u(y)]\Phi(a)+[k(x)v(x)+k(y)v(y)]\Phi(a+\varpi_{\mathbb{T}}(b,a)), \tag{5.2.6}$$

by simple calculation

$$u(x)k(x) + u(y)k(y) = \frac{a + \varpi_{\mathbb{T}}(b,a) - x}{\varpi_{\mathbb{T}}(b,a)} \cdot \frac{y-m}{y-x} + \frac{a + \varpi_{\mathbb{T}}(b,a) - y}{\varpi_{\mathbb{T}}(b,a)} \cdot \frac{x-m}{x-y}$$

we have

$$= \frac{a + \varpi_{\mathbb{T}}(b,a) - m}{\varpi_{\mathbb{T}}(b,a)} = u(m) = m_{[0,1]},$$

and

$$v(x)k(x) + v(y)k(y) = \frac{m-a}{\varpi_{\mathbb{T}}(b,a)} = v(m) = 1 - m_{[0,1]}.$$

Integrating both sides of (5.2.6) over $I_{[0,1]}$, we get

$$\frac{1}{\varpi_{\mathbb{T}}(b,a)}\Big[\int_a^{a+\varpi_{\mathbb{T}}(b,a)} k(x)\Phi(x)\Delta x - \int_a^{a+\varpi_{\mathbb{T}}(b,a)} k(x)\phi^{\nabla}(x)\Phi(x)\nabla x\Big]$$

$$\leq m_{[0,1]}\Phi(a) + (1-m_{[0,1]})\Phi(a+\varpi_{\mathbb{T}}(b,a)). \tag{5.2.7}$$

□

Corollary 5.2.3. *For $\mathbb{T}=\mathbb{Z}$. Prove that the above inequality becomes the inequality given in (5.2.3).*

Proof. Here $\phi(x) = 2a + \varpi_{\mathbb{Z}}(b,a) - x$ and $k(x) = \frac{1}{2}$ with $\phi^{\nabla}(x) = -1$. Inserting this data we have the desired inequality. □

Corollary 5.2.4. *Let $\mathbb{T}=\mathbb{R}$ then prove that (5.2.7) becomes Hermite-Hadamard inequality for preinvex functions given by*

$$\Phi\Big(a+\frac{\varpi(b,a)}{2}\Big) \leq \frac{1}{\varpi(b,a)}\int_a^{a+\varpi(b,a)} \Phi(x)dx \leq \frac{\Phi(a)+\Phi(a+\varpi(b,a))}{2}.$$

Proof. Since $\Phi : K \to \mathbb{R}$ is actually an extension of $\theta : K_{\mathbb{Z}} \to \mathbb{R}$, and we know that θ satisfies (5.2.5), so as Φ, and also we have,

$$\lim_{\mu(t)\to 0} \sum_{t\in[a,a+\varpi(b,a))} \mu(t)\Phi(t) = \int_a^{a+\varpi(b,a)} \Phi(x)dx,$$

and

$$\lim_{\mu(t)\to 0} \sum_{t\in(a,a+\varpi(b,a)]} \mu(t)\Phi(t) = \int_a^{a+\varpi(b,a)} \Phi(x)dx.$$

Hence it follows the result. □

Theorem 5.2.5. *(Jensen Inequality). Let $\mathbb{T}$be time scales and $K_{\mathbb{T}}$ be an invex set respecting $\varpi_{\mathbb{T}}$. Also, let $a,b \in K_{\mathbb{T}}$ we have $[a, a+\varpi_{\mathbb{T}}(b,a)] \subseteq K_{\mathbb{T}}$. Let*

$$g : [a, a+\varpi_{\mathbb{T}}(b,a)] \to K$$

be and rd-continuous function. Then prove that every preinvex function

$$\Phi : [a, a + \varpi(b, a)] \to \mathbb{R}$$

satisfy the following inequality

$$\Phi\Big(\frac{\int_a^{a+\varpi_{\mathbb{T}}(b,a)} g(x)\Delta x}{\varpi_{\mathbb{T}}(b, a)}\Big) \leq \frac{\int_a^{a+\varpi_{\mathbb{T}}(b,a)} \Phi(g(x))\Delta x}{\varpi_{\mathbb{T}}(b, a)}.$$

Proof. By using convexity of Φ on $[a, a + \varpi(b, a)]$ and Jensen inequality for convex functions we arrive at the desired result. □

Chapter 6

Conclusion

This chapter finishes up the thesis by expressing and summarizing the inferences and findings. The knowledge assists the reader to understand the essence of the study and parting ways for future undertakings identified with this territory of research.

We have given Jensen and Hermite-Hadamard inequalities in this settings of time scale which were previously proved for real numbers. We have observed that our findings are consistent with mentioned inequalities in the case preinvex functions on reals and for convex functions on time scales. In future, we can carry out refinements of Hermite-Hadamard inequality for preinvex functions on time scales. Also, we can define an analogue of invex functions, named as delta invex functions and study applications of these functions in optimization, engineering and computer science.

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