Approximate Closed-form Solution of System of Partial Differential Equations

by

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MASTER'S THESIS WORK

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Dedicated to

my beloved parents and siblings
for their best support

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With humble and sincerest words, I am immeasurably indebted to **Almighty Omnipotent**, who endowed upon me the potential and ability to contribute a drop of material in the existing ocean of knowledge. I invoke peace for **Holy Prophet Hazrat Muhammad (PBUH)** who is forever a torch of guidance and courage for the entire human race.

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Abstract

Most of the natural phenomena are completely described by a system of nonlinear PDEs, because, often a single PDE fails to explain all the properties. In fluid mechanics, boundary layer problems are those in which conservation of more than one quantities is necessary i.e. mass, momentum, and energy of the flowing fluid are studied simultaneously. The exact solution to such system of PDEs is difficult to find. Therefore, we go for reducing system of nonlinear PDEs to a system of ODEs via similarity transformations (invariant transformations) which hopefully can be solved for exact solutions. Unfortunately, the reduced system of ODEs is again a difficult task to solve for exact solution and we rely over its numerical solution. But it is difficult to get a solution of system of PDEs using the numerical solution of reduced system of ODEs. A technique in [1] is then useful for determining approximate closed-form solution of the original PDE. In this thesis, we extend the technique described in [1] for a system of PDEs and implement it to a boundary layer problem. Residues of the system of ODEs and PDEs are calculated with respect to the approximate closed-form solutions. The results strongly support the approximate closed-form solution presented in the thesis.

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Chapter 1

Introduction

Mathematical models interpret our thoughts for a number of phenomena occurring in the world. We acquire a thorough knowledge of these phenomena by translating them into a concise language i.e. Mathematics. Models are formulated by well-defined rules. Solving the model which come up with real world problem means to find a solution of mathematical equations present in it. Many natural phenomena are governed by relations (equations) involving rates at which things happen (derivatives). Equations which contain derivatives are differential equations. In vast fields of science and engineering, the natural phenomena are modeled by non-linear partial differential equations (PDEs) because most of the processes are non-linear in nature. For example, shock waves occur in explosions, traffic flow, glacier waves, airplanes breaking the sound barrier, and so on which are modeled by non-linear hyperbolic PDEs [2]. The key defining property of a partial differential equation is that there are more than one independent variables $x, y, \ldots$ and there is a dependent variable, u, that is an unknown function of these variables $u(x, y, \ldots)$, where a PDE is an identity that relates the independent variables, the dependent variable u, and the partial derivatives of u [2].

Definition 1.1. *[3] Let* $u : \Omega \to \mathbb{R}$, $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \Omega \subseteq \mathbb{R}(n \geq 2)$, *we usually write* $u_{x_i} = \frac{\partial u}{\partial x_i}$, $u_{x_i x_j} = \frac{\partial^2 u}{\partial x_i x_j}$, $u_{x_i x_j x_k} = \frac{\partial^3 u}{\partial x_i x_j x_k}$ *and so on. A vector of the form* $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$*, where each component* α_i *is a non negative integer, is called a multi-index of order* $\mid \alpha \mid= (\alpha_1 + \alpha_2 + \ldots + \alpha_n)$*. Given a multi index* α*, define*

$$D^\alpha u(\boldsymbol{x}) := \frac{\partial^\alpha u(\boldsymbol{x})}{\partial x_1^{\alpha_1} \ldots x_n^{\alpha_n}} = \partial_{\alpha_1}^{x_1} \ldots \partial_{\alpha_n}^{x_n}.$$

Fix an integer $k \geq 1$ *such that* $D^k u(\boldsymbol{x}) := \{D^\alpha u(\boldsymbol{x}) :\mid \alpha \mid= k\}$*. An expression of the form:*

$$F(D^k u(\boldsymbol{x}), D^{k-1} u(\boldsymbol{x}), \ldots, Du(\boldsymbol{x}), u(\boldsymbol{x}), \boldsymbol{x}) = 0, \tag{1.1}$$

is called a k^{th} *order partial differential equation, where* $F : \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times \ldots \times \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}$ *is given and* $u : \Omega \to \mathbb{R}$ *is the unknown.*

The solution of a PDE refers to all possible functions $u(\mathbf{x})$ which are in explicit form. If it is difficult to find a solution explicitly then we deduce the existence and other properties of its solution. Solution of PDE also satisfies the additional boundary conditions on some part of the boundary of the domain. A single PDE can scarcely describe a natural phenomena, therefore, a system of coupled PDEs is needed for complete description. For example, let us say that we want to compute the distribution of heat with a microwave oven then we must first compute the electrical wave E that generates the heat which is given by the Helmholtz equation

$$\Delta E + \omega^2 E = 0,$$

where ω is the frequency of the wave and secondly, we must solve the heat equation

$$\Delta T =| E |^2,$$

for the temperature, T, within the microwave oven since T depends on E hence this is a coupled problem with two partial differential equations [4]. In a system of partial differential equations several unknown dependent variables are involved.

Definition 1.2. *[3] An expression of the form:*

$$\boldsymbol{F}(D^k \boldsymbol{u}(\boldsymbol{x}), D^{k-1}\boldsymbol{u}(\boldsymbol{x}), \ldots, D\boldsymbol{u}(\boldsymbol{x}), \boldsymbol{u}(\boldsymbol{x}), \boldsymbol{x}) = 0, \tag{1.2}$$

is called a k^{th} order system of partial differential equations, where $\boldsymbol{F} : \mathbb{R}^{mn^k} \times \mathbb{R}^{mn^{k-1}} \times \ldots \times \mathbb{R}^{mn} \times \mathbb{R}^m \times \Omega \to \mathbb{R}^m$ is given $\boldsymbol{u} : \Omega \to \mathbb{R}^m$ is the unknown.

In this definition, we have m number of scalar PDEs with m unknowns i.e. $u_1, u_2, \ldots, u_m$. In general, solution of every PDE cannot be obtained using some specific method. So every problem has to be dealt separately and if non of the techniques yield a solution then there is always an existence theory which shows the existence of solution and vice versa. Various partial differential equations that are proved important in various fields of science and engineering are studied in computational mathematics. Solution of such equations help us to understand the prominent features of many naturally occurring phenomena. The term “exact” generalizes a solution that describes the entire physics and mathematics of a problem. Thus an analytical solution of a given PDE is often referred to as its exact solution. The solution also satisfies the boundary and initial conditions, if any. In computational mathematics our goal is to find solution of various types of PDEs. A well posed problem is accompanied with all features to solve its PDEs.

Definition 1.3. *[3] A problem is said to be well-posed if it satisfies the following:*

1. *the problem actually has a solution,*
2. *the solution is unique,*
3. *the solution continuously depends on the variables and parameters of the original PDE.*

If any of the above is not satisfied then the problem is said to be ill-posed. Thus by solving PDEs we mean to find an explicit formula, if possible, which satisfies above criteria. Solution of a PDE of order k should be infinitely differentiable more specifically at least k times differentiable. Now let us take example of scalar conservation law, which is given by a PDE as follows:

$$u_t + uu_x = 0, \qquad\qquad u(x,0) = f(x).$$

Mathematical modeling of various one dimensional phenomena in fluid dynamics e.g. gas dynamics, traffic flow, etc, and particularly formation as well as propagation of shock waves is governed by above PDE. Since this equation models the physical phenomena so there is a solution but a shock wave has discontinuous solution. Therefore, the study of conservation laws lead us to the solutions which are not continuous. In general, the conservation laws are not described by classical solution defined in definition (1.3). The well-posed conservative problems are then defined by weak solutions.

Definition 1.4. *If we are able to write the partial differential equation of order k such that no derivative of solution appears, this is the weak formulation and its solution is weak solution which may not be differentiable k times.*

Some of the difficulties entailed while solving PDEs include high order, non-linearity and large number of independent variables. Finding an explicit formula of solution for a PDE is often difficult. So if analytical methods fail to provide a solution (either strong or weak form), we use some numerical method to observe the behavior of solution. A numerical method results in the form of numbers. However, our goal is to find the solution in the form of some function where the variables and parameters of the original problem are involved.

Remark 1.1. *A numerical method is an explicit scheme to yield the solution of PDEs involved in some problem. The study and applications of numerical method is Numerical Analysis.*

In the field of computational mathematics, non-linear PDEs whose analytical solution (closed-form) is difficult to find is investigated through various numerical methods. The method of transforming the nonlinear PDEs to ordinary differential equations (ODEs) is important. This method is constructive in the analysis of PDEs in many physical problems. A procedure was developed in [1] that uses the numerical solution of the reduced ODE to find an approximate closed-form solution of the original PDE. System of PDEs is harder to solve as compared to a single PDE. In this thesis, we apply the procedure developed in [1] to a system of non-linear PDEs. The convergence of the residuals by the approximate closed-form solution is also checked to observe their accuracy.

The system of equations in the incompressible boundary layer flow problem is a non-linear PDE system composed of the continuity, the momentum, and the energy equations which shows the conservation of mass, momentum, and energy, respectively. In this system, effect of the velocity field $(u(x, y), v(x, y))$ and the temperature distribution $T(x, y)$ is to be analyzed with respect to the flow directions x and y of the fluid. In order to solve such system of PDEs, one often uses similarity transformations, that reduces the system of PDEs to a system of ODEs. However, exact solution of the reduced system of ODEs is difficult to be obtained in most of the cases. Therefore, one looks for the numerical solution of the reduced system. However, difficulty arises in using the numerical solution in similarity transformations to get the solution of the system of PDEs. To deal with such situation we approximate the numerical solution of the reduced system of ODEs by closed-form function, that can easily be used in the similarity transformations to get an approximate closed-form solution of the original system of PDEs.

In Chapter 2, a review of a single PDE with the example of non-linear diffusion equation is discussed. Approximate closed-form solution for a PDE is calculated and compared with its numerical solution. Comparison shows that the difference between values of approximate closed-form and numerical solution is insignificant. In Chapter 3, a system of PDEs which govern the phenomena of incompressible boundary layer flow in two dimensions is considered. The approximate closed-form solution of the system is calculated and the convergence of the residuals of this solution is checked and relative errors are presented.

1.1 Basic Definitions

Fluid mechanics concerns with the behavior of fluids which deforms continuously under the action of shear stress.

Viscosity

Dynamic viscosity μ is a measure of the resistance between a layer of a fluid. Fluids such as paint, honey, engine oil, etc have much higher viscosities than water [5]. Also the *density* ρ of a fluid is its mass per unit volume which highly varies in gases and increases nearly proportionally to the pressure level whereas density in fluids is nearly constant. The ratio of dynamic viscosity to density is called *kinematic viscosity* ν, which is written as

$$\nu = \frac{\mu}{\rho},$$

SI units of ν is $m^2 s^{-1}$.

Newton's Law of Viscosity

The relationship between shear stress τ and strain rate is given by

$$\tau = \mu \frac{du}{dy},$$

which is dimensionally consistent, therefore, μ has dimensions of stress-time i.e. $(\frac{M}{LT})$ [6]. The fluids observing this relation are **Newtonian fluids**.

Specific Heat of a Fluid

Specific heat also termed as heat capacity is the amount of heat required to raise a temperature of a unit mass of the fluid by one degree which can be done at constant volume or at constant pressure. It is denoted by c_p. The SI unit is $Jkg^{-1}K^{-1}$ [7].

Thermal Diffusivity

Thermal diffusivity measures the change in temperature produced in unit volume of the material by the amount of heat that flows in unit time through a unit area. We have

$$\alpha = \frac{\lambda}{\rho\ c_p},$$

where λ is the thermal conductivity of the material and SI units of α are m^2s^{-1}.

Boundary Layer Theory

It is a thin layer near a surface which is also called the boundary layer. The boundary layer divides the flow past a surface into two regions. A region near the surface where the viscosity's effect are important while the region outside of boundary layer where the effect of viscosity is negligible.

Laminar Flow

In laminar flow the layer of fluids are well defined and consistent. *Reynolds number* is the ratio of inertial forces to viscous forces. Mathematically

$$Re = \frac{\rho U L}{\mu},$$

where ρ is the density, U is the characteristic velocity, L is the characteristic length and μ is the dynamic viscosity. If the Reynolds number is small then the flow is laminar otherwise its turbulent.

Steady Flow

A flow in which properties like velocity, pressure, temperature, etc do not depend on time. Mathematically

$$\frac{\partial P}{\partial t} = 0,$$

where P is any fluid property.

Incompressible Fluid

A fluid in which density ρ of the fluid is constant known as incompressible fluid. Mathematically, we write

$$\frac{d\rho}{dt} = 0.$$

Prandtl Number

The Prandtl Number is a dimensionless number approximating the ratio of momentum diffusivity (kinematic viscosity) ν to thermal diffusivity α and can be expressed as:

$$Pr = \frac{\nu}{\alpha}.$$

Prandtl number of gases is about 1. In this case both momentum and heat flux dissipate in a fluid medium approximately at the same rate. The momentum and thermal boundary layers will be identical for $Pr = 1$.

Eckert Number

The Eckert number, Ec, is a dimensionless quantity useful in determining the relative importance in a heat transfer situation of the kinetic energy of a flow. It is the ratio of the kinetic energy to the enthalpy

$$Ec = \frac{U^2}{c_p \Delta T},$$

where U is the flow velocity of the fluid, c_p is the specific heat at constant pressure, and $\Delta T = T_w - T_\infty$ is the temperature difference between the surface and the free stream [8].

1.2 Governing Equations of Fluid Flow

The motion of a fluid can be described by kinematic and dynamic conservation laws for mass, momentum, and energy, these will be formulated in terms of independent spatial variables $\vec{x} = (x, y, z)$ and temporal variable t. The dependent variables are denoted as:

1. velocity vector $\vec{u} = (u, v, w)$,
2. density ρ,
3. pressure p,
4. temperature T.

1.2.1 Conservation of mass

Let us take an arbitrary small volume element dV within a flowing fluid and the velocity of the fluid is $\vec{u}$ having density ρ along an arbitrary fixed closed surface element dS as shown in the following figure (1.1). We will start by the definition of mass m within the control volume V, thus across dS the local volume rate of flow is,

$$(\hat{n} \cdot \vec{u})dS,$$

where $\hat{n}$ is a unit normal vector directed outwardly as shown in the figure (1.1). If the flow is in outward direction then the above dot product is taken as positive and negative if the flow is inward. The local mass rate of flow is

$$(\hat{n} \cdot \rho\vec{u})dS.$$

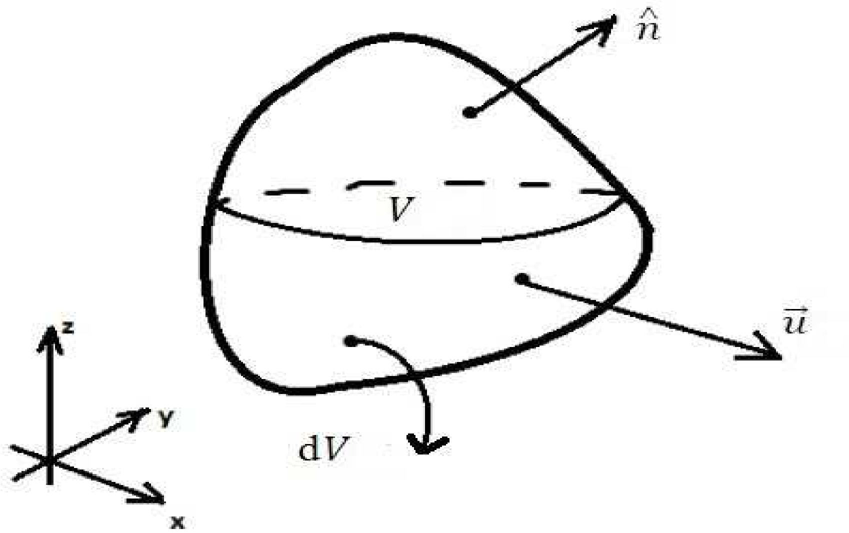

Figure 1.1: Schematic view of control volume [9].

According to the law of conservation of mass, the total mass of fluid within V will increase only because of net influx of fluid across the bounding surface S. Mathematically,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho dV = -\int_S (\hat{n}\cdot\rho\vec{u})dS. \tag{1.3}$$

Definition 1.5. *[6] Gauss divergence theorem says that if V is a closed region in space enclosed by a surface S, then*

$$\int_V (\nabla\cdot\vec{u})dV = \int_S (\hat{n}\cdot\vec{u})dS.$$

Thus we have from equation (1.3)

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho dV &= -\int_V (\nabla \cdot \rho\vec{u})dV,\\
\int_V \frac{\partial \rho}{\partial t} dV &= -\int_V (\nabla \cdot \rho\vec{u})dV,\\
\int_V \Big(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\vec{u})\Big)dV &= 0,
\end{aligned}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\vec{u}) = 0, \tag{1.4}$$

which is the equation of continuity for a compressible fluid. For incompressible fluids the above equation (1.4) becomes

$$\nabla \cdot (\rho\vec{u}) = 0 \qquad \text{or} \qquad \nabla \cdot \vec{u} = 0.$$

1.2.2 Conservation of Momentum

The Navier-Stokes equation describes the conservation of momentum within the control volume V. If volume flow rate $(\hat{n} \cdot \vec{u})dS$ is multiplied with momentum per unit volume $\rho\vec{u}$ (i.e.),

$$(\hat{n} \cdot \vec{u})\rho\vec{u}dS = \hat{n} \cdot (\rho\vec{u}\vec{u})dS,$$

which is the rate at which momentum is carried out across the surface element dS where $\rho\vec{u}\vec{u}$ is the momentum flux. In addition to the momentum transport by the flow, there will also be the momentum transferred by means of the molecular motions and interactions within the fluid which will be denoted by π. Thus, according to the law of conservation of momentum, the total momentum within V will increase due to the net influx (internal forces) and the external force of gravity [6]. Mathematically

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\vec{u}dV = -\int_S (\hat{n}\cdot\rho\vec{u}\vec{u})\,dS - \int_S (\hat{n}\cdot\pi)dS + \int_V \rho\vec{g}dV,$$

by using Gauss divergence theorem

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\vec{u}dV &= -\int_V (\nabla\cdot\rho\vec{u}\vec{u})dV - \int_V \nabla\cdot\pi dV + \int_V \rho\vec{g}dV,\\ \int_V \left(\frac{\partial}{\partial t}(\rho\vec{u}) + \nabla\cdot(\rho\vec{u}\vec{u}) + (\nabla\cdot\pi) - \rho\vec{g}\right)dV &= 0,\\ \frac{\partial}{\partial t}(\rho\vec{u}) + \nabla\cdot(\rho\vec{u}\vec{u}) + (\nabla\cdot\pi) - \rho\vec{g} &= 0. \qquad (1.5)\end{aligned}$$

Using the identity

$$\nabla\cdot(\rho\vec{u}\vec{u}) = \rho(\vec{u}\cdot\nabla)\vec{u} + \vec{u}(\nabla\cdot(\rho\vec{u}))$$

and simplifying using continuity equation we have

$$\begin{aligned}\rho\left(\frac{\partial\vec{u}}{\partial t} + (\vec{u}\cdot\nabla)\vec{u}\right) &= -\nabla\cdot\pi + \rho\vec{g},\\ \rho(\frac{\mathrm{d}\vec{u}}{\mathrm{d}t}) &= -\nabla\cdot\pi + \rho\vec{g}, \qquad (1.6)\end{aligned}$$

The surface forces occurred due to the stresses on the sides of the control volume. They include forces due to pressure on the sides by the molecules and also the viscous or frictional forces in a moving fluid. If the fluid is at rest then the pressure is the only surface force. Thus the sum of the viscous stresses τ_{ij} which arises from the motion of the velocity gradients plus the hydrostatic pressure is the surface force π_{ij} (π_{ij} is the stress in j direction on a face normal to i-axis). These stresses are written as

$$\pi_{ij} = -\Big(p + \frac{2}{3}\mu\nabla\cdot\vec{u}\Big)\delta_{ij} + \mu\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big), \qquad (1.7)$$

where δ_{ij} is the Kronecker delta

$$\delta_{ij} = \begin{cases} 1, & if\ i\ =\ j; \\ 0, & if\ i\ \neq\ j, \end{cases}$$

and p is the hydrostatic pressure [10]. For an incompressible flow $\nabla \cdot \vec{u} = 0$ and the simplified Navier-Stokes equation for an incompressible fluid is given by

$$\rho(\frac{\mathrm{d}\vec{u}}{\mathrm{d}t}) = -\nabla p + \mu \nabla^2 \vec{u} + \rho \vec{g}.$$

1.2.3 Conservation of Energy

Let us look at various forms of heat flow in the control volume V. If local volume rate of flow across dS i.e. $(\hat{n} \cdot \vec{u})dS$ is multiplied by $\frac{1}{2}\rho\vec{u}^2$, we get the rate of convective flows of kinetic energy across dS

$$(\hat{n} \cdot \frac{1}{2}\rho\vec{u}^2\vec{u})dS.$$

If $\hat{U}$ is the internal energy per unit mass then the rate of convective flows of internal energy across dS is given by

$$(\hat{n} \cdot \rho\hat{U}\vec{u})dS.$$

In addition to the rate of convective flows of kinetic and internal energies across dS, there may be a transfer of energy occurred due to the molecular motions which is a mode of transport linked with the heat conduction. Let $\vec{q}$ be the corresponding heat flux then heat flow across dS by the heat conduction will be

$$(\hat{n} \cdot \vec{q})dS.$$

The rate of doing work by the fluid outside V on the fluid inside V will be

$$-(\hat{n}\cdot\pi)\cdot\vec{u}dS.$$

According to the first law of thermodynamics, which states that
"Rate of increase of the sum of kinetic and internal energies equal the rate of energy addition both by the bulk flow and the conduction plus the rate at which the fluid outside V is doing work on the fluid inside V and the energy addition due to gravitational force on the fluid".
Mathematically

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V\left(\frac{1}{2}\rho\vec{u}^2+\rho\hat{U}\right)dV=-\int_S\hat{n}\cdot(\frac{1}{2}\rho\vec{u}^2+\rho\hat{U})\vec{u}dS-\int_S(\hat{n}\cdot\vec{q})dS-\int_S((\hat{n}\cdot\pi)\cdot\vec{u})dS+\int_V(\rho\vec{g}\cdot\vec{u})dV,$$

by using the Gauss divergence theorem

$$\int_V\frac{\partial}{\partial t}\left(\frac{1}{2}\rho\vec{u}^2+\rho\hat{U}\right)dV=\int_V\left(-\nabla\cdot\left(\frac{1}{2}\rho\vec{u}^2+\rho\hat{U}\right)\vec{u}-\nabla\cdot\vec{q}-\nabla\cdot(\pi\cdot\vec{u})+\rho\vec{g}\cdot\vec{u}\right)dV$$

or

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho\vec{u}^2+\rho\hat{U}\right)=-\nabla\cdot\left(\frac{1}{2}\rho\vec{u}^2+\rho\hat{U}\right)\vec{u}-\nabla\cdot\vec{q}-\nabla\cdot(\pi\cdot\vec{u})+\rho\vec{g}\cdot\vec{u}. \quad (1.8)$$

Now from momentum equation (1.5)

$$\frac{\partial}{\partial t}(\rho\vec{u})=-\nabla\cdot(\rho\vec{u}\vec{u})-(\nabla\cdot\pi)+\rho\vec{g},$$

now taking dot product with $\vec{u}$

$$\frac{\partial}{\partial t}(\frac{1}{2}\rho\vec{u}^2)=-\nabla\cdot(\frac{1}{2}\rho\vec{u}^2\vec{u})-(\nabla\cdot\pi)\cdot\vec{u}+\rho\vec{g}\cdot\vec{u}, \quad (1.9)$$

Substituting equation (1.9) in equation (1.8), we get

$$\frac{\partial}{\partial t}(\rho\hat{U})=-\nabla\cdot(\rho\hat{U})\vec{u}-\nabla\cdot\vec{q}+(\nabla\cdot\pi)\cdot\vec{u}-(\nabla\cdot(\pi\cdot\vec{u})). \quad (1.10)$$

Now we can write

$$\frac{\partial}{\partial t}(\rho\hat{U}) = \hat{U}\frac{\partial \rho}{\partial t} + \rho\frac{\partial \hat{U}}{\partial t}, \tag{1.11}$$

using continuity equation in the first term of the equation (1.11), we get

$$\frac{\partial}{\partial t}(\rho\hat{U}) = \nabla \cdot (\rho\vec{u})\hat{U} + \rho\frac{\partial \hat{U}}{\partial t},$$

and using the following identity,

$$\nabla \cdot (\rho\hat{U})\vec{u} = (\rho\hat{U})(\nabla \cdot \vec{u}) + \vec{u} \cdot (\nabla(\rho\hat{U})),$$

this leads us to the following form of equation (1.10),

$$\rho\frac{\mathrm{d}\hat{U}}{\mathrm{d}t} = -\nabla \cdot \vec{q} - \nabla \cdot (\pi \cdot \vec{u}) + (\nabla \cdot \pi) \cdot \vec{u}, \tag{1.12}$$

in which the stress tensor π_{ij} is defined as,

$$\begin{aligned}
\pi_{ij} &= -p\delta_{ij} + \tau_{ij}, \\
\tau_{ij} &= \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \lambda\frac{\partial u_k}{\partial x_k}\delta_{ij},
\end{aligned}$$

and heat flux $\vec{q}$ is given by Fourier's law of heat conduction,

$$\vec{q} = -k\nabla T,$$

where k is the thermal conductivity which indicates the rate at which heat energy is transferred through a medium by conduction process. The most familiar form of the energy balance equation is as follows,

$$\rho\frac{\mathrm{d}T}{\mathrm{d}t} = -\frac{\mathrm{d}p}{\mathrm{d}t} + \nabla \cdot (k\nabla T) + \tau_{ij}\frac{\partial u_i}{\partial x_j},$$

in which last term refers to the viscous dissipiation.

Chapter 2

Approximate Closed-form Solution of a PDE

The history of diffusion goes back to several centuries as many natural and technical process are governed by diffusion, for example, molecules of the perfume diffuses into air, sugar molecules in the crystals diffuses into water slowly, conduction process, flat soda, etc. In 1785, Jan Ingenhousz (Netherlands) wrote about random motion of coal dust particles in alcohol also in 1827, Scottish botanist Robert Brown saw tiny particles, smaller than pollen, wiggling around in water [11]. The natural phenomena of diffusive transportation is of great interest and the method of finding approximate closed-form solution for such a problem is disscused in [1] and [12]. This chapter gives a review of their work in which the flow of gas through a semi-infinite porous medium at uniform pressure $P_0 > 0$ at $t = 0$ is governed by the following non-linear PDE, [12]

$$\frac{\partial}{\partial x}\left(u\frac{\partial u}{\partial x}\right) = c\frac{\partial u}{\partial t}, \tag{2.1}$$

for which we have the following conditions:

$$u(x,0) = P_0, \qquad 0 \le x < \infty, \tag{2.2}$$

$$u(0,t) = P_1(< P_0), \qquad 0 < t < \infty, \tag{2.3}$$

$$u(\infty,t) = P_0, \qquad 0 < t < \infty. \tag{2.4}$$

The flow of gas through porous medium is viscous and as the gas flows the pressure P_0 is reduced to $P_1 > 0$. The lower pressure is then maintained. Let $P_0 = 1$ then we have

$$u(x,0) = 1, \qquad 0 \le x < \infty, \tag{2.5}$$

$$u(0,t) = P_1(< 1), \qquad 0 < t < \infty, \tag{2.6}$$

$$u(\infty,t) = 1, \qquad 0 < t < \infty. \tag{2.7}$$

2.1 Reduction of PDE to ODE

The PDE in equation (2.1) is transformed to an ODE using following similarity transformations [13]

$$z = \frac{x}{2}\sqrt{\frac{c}{t}}, \tag{2.8}$$

$$w = \frac{(1-u^2)}{\alpha}, \qquad \text{where} \quad \alpha = 1 - P_1^2. \tag{2.9}$$

Equation (2.9) can be re-written as

$$u = \sqrt{1-\alpha w}. \tag{2.10}$$

Now differentiating equations (2.8) and (2.10) w.r.t 'x', we have

$$\frac{\partial z}{\partial x} = \frac{1}{2}\sqrt{\frac{c}{t}}, \tag{2.11}$$

$$\frac{\partial u}{\partial x} = -\frac{\alpha}{4\sqrt{1-\alpha w}}\frac{c}{t}\frac{\partial w}{\partial z}, \tag{2.12}$$

and again differentiating equations (2.8) and (2.10) w.r.t 't', we have

$$\frac{\partial z}{\partial t} = -\frac{x}{4t}\sqrt{\frac{c}{t}}, \tag{2.13}$$

$$\frac{\partial u}{\partial t} = -\frac{\alpha x}{8t\sqrt{1-\alpha w}}\frac{c}{t}\frac{\partial w}{\partial z}. \tag{2.14}$$

and

$$u\frac{\partial u}{\partial x} = -\frac{\alpha}{4}\sqrt{\frac{c}{t}}\frac{\partial w}{\partial z}. \tag{2.15}$$

Differentiating equation (2.15) w.r.t 'x'

$$\frac{\partial}{\partial x}\left(u\frac{\partial u}{\partial x}\right) = -\frac{\alpha c}{8t}\frac{\partial^2 w}{\partial z^2}. \tag{2.16}$$

Substituting derivatives in equation (2.1), we get

$$w'' + \frac{2z}{\sqrt{1-\alpha w}}w' = 0, \tag{2.17}$$

and the corresponding boundary conditions transform as follows:

$$w = 1 \quad \text{when} \quad z = 0, \tag{2.18}$$

$$w = 0 \quad \text{when} \quad z \to \infty. \tag{2.19}$$

2.2 Numerical Solution of the Reduced ODE

The exact solution of the non-linear reduced ODE is difficult to find so we find a numerical solution using MATLAB built-in solver *bvp4c*. We need to convert equation (2.17) into a system of first order ODEs and corresponding boundary conditions. The numerical solution of equation (2.17) is shown in figure (2.1):

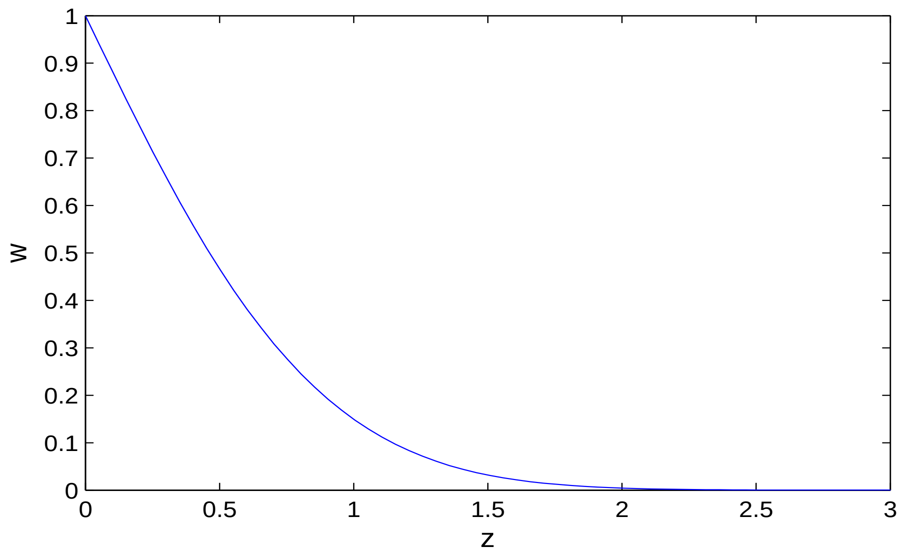

Figure 2.1: Numerical solution of the reduced ODE.

2.3 Approximate Closed-form Solution of the Reduced ODE

A lower solution of equation (2.17) is considered as a good initial guess in terms of error function "erf" i.e.

$$\text{initial approximation} = w_{lower} = 1 - \text{erf}\left(\frac{z}{\sqrt{P_1}}\right).$$

An initial approximation can be improved when the lower solution is considered as follows:

$$1 - \operatorname{erf}(kz), \tag{2.20}$$

with $k \approx k_0 = \dfrac{1}{\sqrt{P_1}}$. As there is a decrement in value of k from k_0, a solution close enough to numerical solution is attained. For some appropriate value of n and values $\varepsilon > 0,\ \delta_i > 0$, we have a sequence of values

$$k = k_i = k_0 - \delta_i, (i = 1, 2, \ldots, n).$$

Equation (2.20) generates a sequence of curves w_{k_i} which uniformly approaches the numerical solution. Thus we say that

$$w_{approx.} = w_{k_i},$$

which lies in some $\varepsilon - band$ around the graph of numerical solution w_{num}.

2.3.1 Initial Approximate Closed-form Solution for $P_1 = 0.9$

For $P_1 = 0.9$ lower solution of the reduced ODE becomes

$$w_{lower} = 1 - \operatorname{erf}(\frac{z}{\sqrt{0.9}}),$$

or

$$w_{lower} \approx 1 - \operatorname{erf}(1.054092z). \tag{2.21}$$

The graph of w_{num} and w_{lower} is shown in figure (2.2).

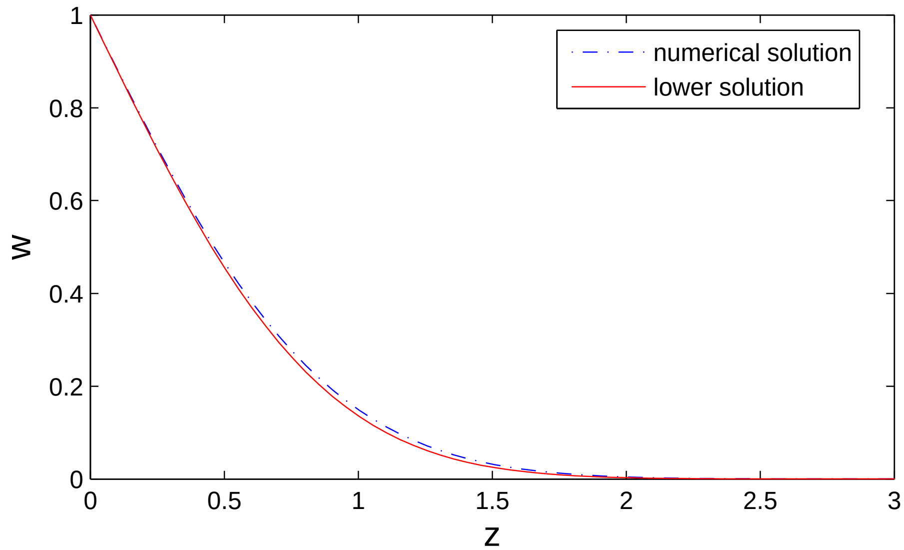

Figure 2.2: Comparison between w_{num} and w_{lower} with $P_1 = 0.9$.

2.3.2 Refined Approximate Closed-form Solution for $P_1 = 0.9$

Now decreasing the value of $h_0 = 1.054092$ to $h_n = 1.0111$, we get the approximate closed-form solution for the reduced ODE with $P_1 = 0.9$ as below

$$w_{approx.} = 1 - \text{erf}(1.0111z). \tag{2.22}$$

The numerical w_{num}, initial approximate closed-form w_{lower} and refined approximate closed-form $w_{approx.}$ solutions are shown in figure (2.3).

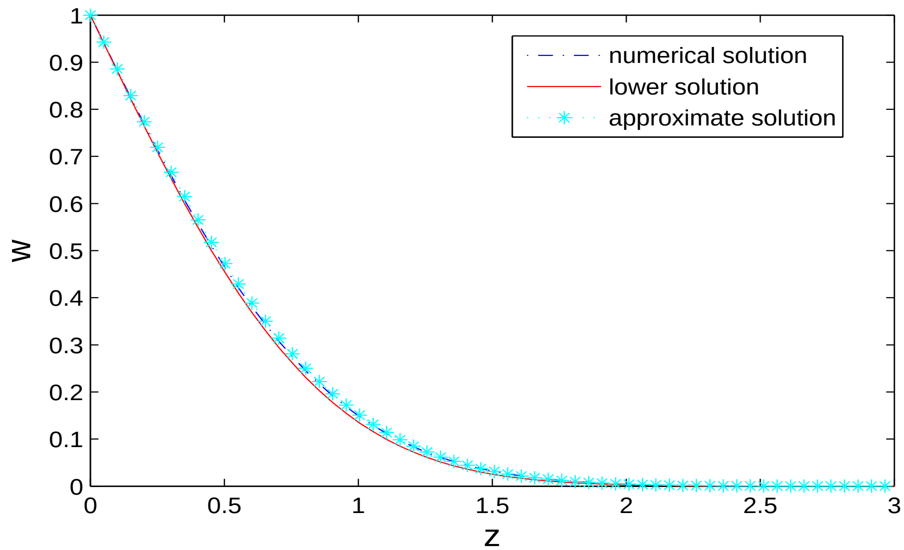

Figure 2.3: Graph of w_{num}, w_{lower} and $w_{approx.}$ with $P_1 = 0.9$.

2.4 Comparison between Numerical and Approximate Closed-form Solution of Reduced ODE

The graph of differences in values at each point between two solutions is represented as:

$$\text{diff}_{nl}(z) = w_{num}(z) - w_{lower}(z),$$
$$\text{diff}_{na}(z) = w_{num}(z) - w_{approx.}(z),$$

which is shown in figure (2.4). The maximum absolute differences are

$$max \mid w_{num}(z) - w_{lower}(z) \mid = 0.0143,$$

and

$$max \mid w_{num}(z) - w_{approx.}(z) \mid = 0.0081.$$

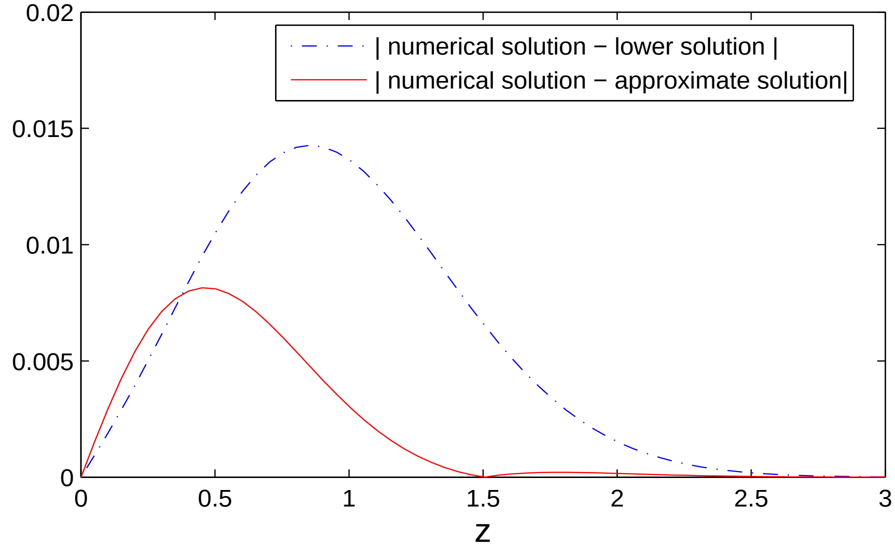

Figure 2.4: Graph of $\mid w_{num} - w_{lower} \mid$ and $\mid w_{num} - w_{approx.} \mid$ with $P_1 = 0.9$.

2.5 Approximate Closed-form Solution for Diffusion Problem

An approximate closed-form solution for diffusion problem can be obtained by applying the inverse similarity transformations to the approximate closed-form solution of the reduced ODE. Substituting the approximate closed-form solution from equation (2.22) in transformations given by equations (2.8) and

(2.9), we get the approximate closed-form solution of the original PDE as

$$u(x,t) = \sqrt{0.81 + 0.19\mathrm{erf}(1.0111\frac{x}{2}\sqrt{\frac{c}{t}})}, \tag{2.23}$$

which satisfies the boundary conditions in equations (2.5), (2.6) and (2.7) as follows

$$\begin{aligned} \lim_{t\to 0} u(x,t) &= \lim_{t\to 0} \sqrt{0.81 + 0.19\mathrm{erf}(1.0111\frac{x}{2}\sqrt{\frac{c}{t}})}, \\ &= \sqrt{0.81 + 0.19\mathrm{erf}(\infty)}, \\ &= 1. \end{aligned} \tag{2.24}$$

$$\begin{aligned} \lim_{x\to 0} u(x,t) &= \lim_{x\to 0} \sqrt{0.81 + 0.19\mathrm{erf}(1.0111\frac{x}{2}\sqrt{\frac{c}{t}})}, \\ &= \sqrt{0.81 + 0.19\mathrm{erf}(0)}, \\ &= 0.9. \end{aligned} \tag{2.25}$$

$$\begin{aligned} \lim_{x\to \infty} u(x,t) &= \lim_{x\to \infty} \sqrt{0.81 + 0.19\mathrm{erf}(1.0111\frac{x}{2}\sqrt{\frac{c}{t}})}, \\ &= \sqrt{0.81 + 0.19\mathrm{erf}(\infty)}, \\ &= 1. \end{aligned} \tag{2.26}$$

The surface plot for approximate closed-form solution in equation (2.23) of PDE in equation (2.1) is shown in figure (2.5).

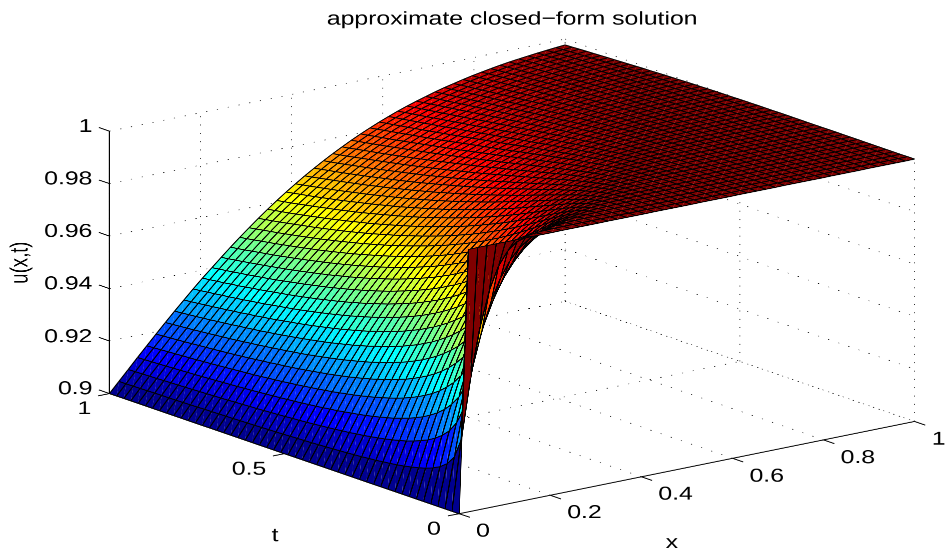

Figure 2.5: Approximate closed-form solution for diffusion problem.

2.6 Numerical Solution for Diffusion Problem

The numerical solution of diffusion problem is obtained using MATLAB built-in function *pdepe*( ) in PDE Toolbox and is shown in figure (2.6).

2.7 Comparison between Numerical and Closed-form Solution for Diffusion Problem

A comparison between results obtained from approximate closed-form and numerical solutions of original PDE for diffusion problem where $P_1 = 0.9$ shows a maximum difference of 0.0023.

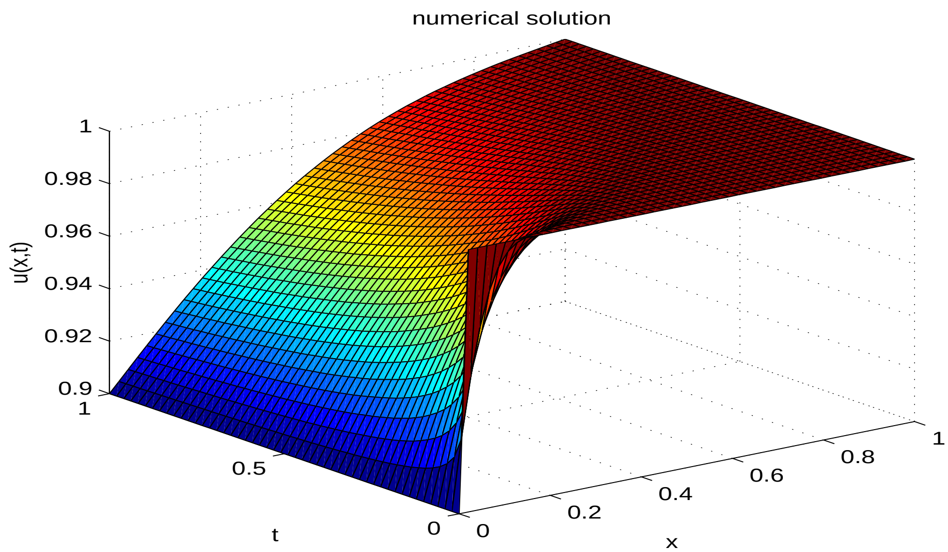

Figure 2.6: Numerical solution for diffusion problem.

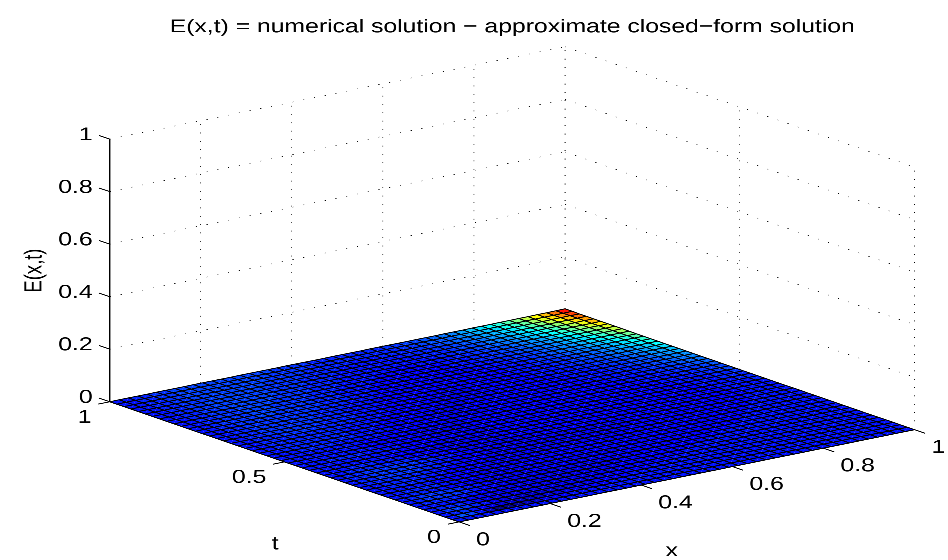

Figure 2.7: Comparison between numerical and closed-form solution for diffusion problem.

Chapter 3

Approximate Closed-form Solution of System of PDEs

A great deal of interest has been generated in the area of two-dimensional boundary layer flow over a continuous moving solid surface in the recent years, due to its numerous and wide-ranging applications, in various fields like aerodynamics extrusion of polymer sheets such as a polymer fiber which is extruded continuously from a die with an understood assumption that the fiber is inextensible, etc [14]. The manufacturing processes such as glass blowing, continuous casting, and spinning of fibers involve interaction of stretched sheet with the ambient fluid, both thermally and mechanically. In many situations one encounters the boundary layer flow over the non-linear stretching surfaces. For example, in a melt-spinning process, the extrudate is stretched into a filament while it is drawn from the die and finally, this surface solidifies while it passes through an effectively controlled cooling system in

order to acquire the top quality property of the final product [14]. The pioneering work on the steady boundary layer flow due to a continuously moving surface was done by Sakiadis [15]. Prandtl showed for the first time in 1904, usually the viscosity of a fluid only plays a role in a thin layer [16]. A thin layer of viscous fluid that is close to the surface of a wall which is in contact with a moving stream has the thickness say L and the flow velocity varies from zero at the wall to the free stream velocity u_{∞}. In this region the flow sticks to the wall because of its viscosity. Such a layer is termed as a momentum boundary layer. Also the temperature of the wall T_w is different from that of the free stream temperature T_{∞}, variation of temperature in this small region forms a thermal boundary layer. In this thesis, we have considered a fluid over a semi-infinite flat plate situated at $y = 0$. Let the ambient fluid is at rest i.e. $u_{\infty} = 0$ and the flat plate is set to move impulsively at some time with constant velocity u_w. At the same time the temperature of the flat plate T_w is suddenly raised from the surrounding fluid temperature T_{∞} thus, both momentum and thermal boundary layers are developed.
When the surface is moved impulsively in an ambient fluid, the inviscid flow is developed almost immediately but the viscous flow within the boundary layer develops slowly and it becomes a fully developed flow after sometime. In the light of above assumptions the system of PDEs which governs this

phenomena is as follows:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{3.1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \upsilon\frac{\partial^2 u}{\partial y^2}, \tag{3.2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} + \frac{\upsilon}{c_p}\left(\frac{\partial u}{\partial y}\right)^2, \tag{3.3}$$

where α is the thermal diffusivity and c_p is the specific heat at a constant pressure of the fluid. The boundary conditions on velocity and temperature are

$$\begin{aligned} &u(x,0) = u_w, && u(x,\infty) = 0, \\ &v(x,0) = 0, && \\ &T(x,0) = T_w, && T(x,\infty) = T_\infty. \end{aligned} \tag{3.4}$$

3.1 Reduction to a Non-linear System of ODEs

Consider the following similarity transformations as given in [17],

$$\eta = y\sqrt{\frac{u_w}{x\nu}}, \tag{3.5}$$

$$\Psi(x,y) = \sqrt{\nu u_w x}\ f(\eta), \tag{3.6}$$

$$\theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty}, \tag{3.7}$$

the velocities of the fluid in x and y directions take the form

$$\begin{aligned} &u(x,y) = \Psi_y, && v(x,y) = -\Psi_x, \\ &u(x,y) = u_w f'(\eta), && v(x,y) = \frac{1}{2}\sqrt{\frac{\nu u_w}{x}}(\eta f'(\eta) - f(\eta)). \end{aligned} \tag{3.8}$$

From equations (3.8), we have

$$\frac{\partial u}{\partial x} = -\frac{y u_w \sqrt{u_w}}{2x\sqrt{x\nu}} f'', \tag{3.9}$$

$$\frac{\partial u}{\partial y} = \frac{u_w \sqrt{u_w}}{\sqrt{x\nu}} f'', \tag{3.10}$$

$$\frac{\partial v}{\partial y} = \frac{y u_w \sqrt{u_w}}{2x\sqrt{x\nu}} f''. \tag{3.11}$$

Differentiating equation (3.10) w.r.t 'y', we get

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{u_w \sqrt{u_w}}{\sqrt{x\nu}} f''\right), \tag{3.12}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{u_w^2}{x\nu} f'''. \tag{3.13}$$

Using equations (3.9) and (3.11), the PDE (3.1) vanishes. The non-linear PDE (3.2) reduces to the following non-linear ODE by using equations (3.9), (3.10) and (3.13) with the transformations u and v given by equations (3.8)

$$f''' + ff'' = 0. \tag{3.14}$$

From equation (3.7), we have

$$\frac{\partial T}{\partial x} = \frac{\theta'}{2x}\eta(T_\infty - T_w), \tag{3.15}$$

$$\frac{\partial T}{\partial y} = -\frac{\theta'}{y}\eta(T_\infty - T_w), \tag{3.16}$$

$$\frac{\partial^2 T}{\partial y^2} = -\frac{u_w}{x\nu}\theta''(T_\infty - T_w). \tag{3.17}$$

Using equation (3.10) alongwith the transformations given by equations (3.8), the non-linear PDE (3.3) takes the following form

$$\theta'' + Pr\ f\ \theta' + Pr\ Ec\ f''^2 = 0, \tag{3.18}$$

where

$$Pr = \frac{\nu}{\alpha}, \qquad\qquad Ec = \frac{U^2}{c_p(T_w - T_\infty)}.$$

The corresponding boundary conditions are transformed as

$$\begin{aligned} &f(0) = 0, \qquad && f'(0) = 1, \qquad && f'(\infty) = 0, \\ &\theta(0) = 1, \qquad && \theta(\infty) = 0. && \end{aligned} \tag{3.19}$$

This completes the reduction of the initial boundary value problem of system of PDEs given by equations (3.1), (3.2), (3.3) and (3.4) to a boundary value problem of system of ODEs given by equations (3.14), (3.18) and (3.19).

3.2 Numerical Solution of the Reduced Non-linear System of ODEs

It is difficult to solve the reduced non-linear system of ODEs analytically, so a numerical solution using *bvp*5*c* is obtained using MATLAB, and is shown in figure (3.1) and (3.2).

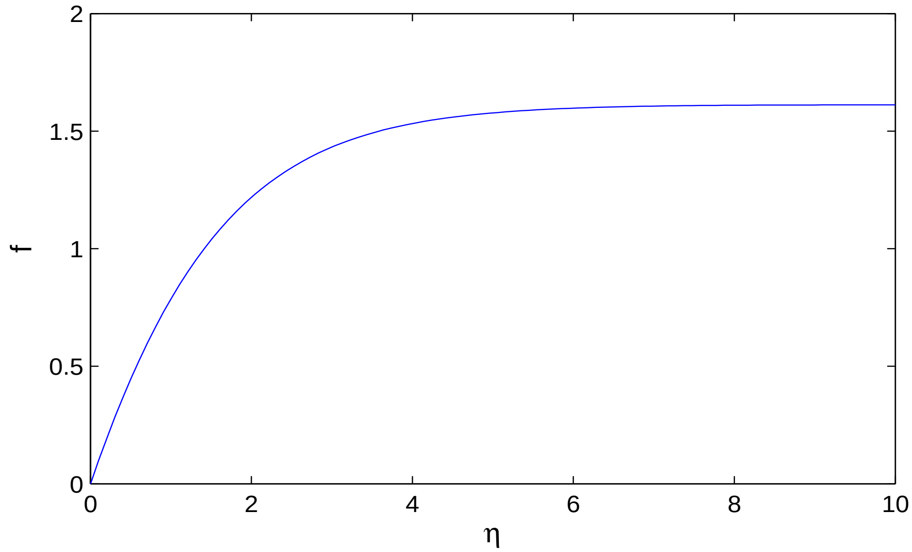

Figure 3.1: Numerical solution of the function f.

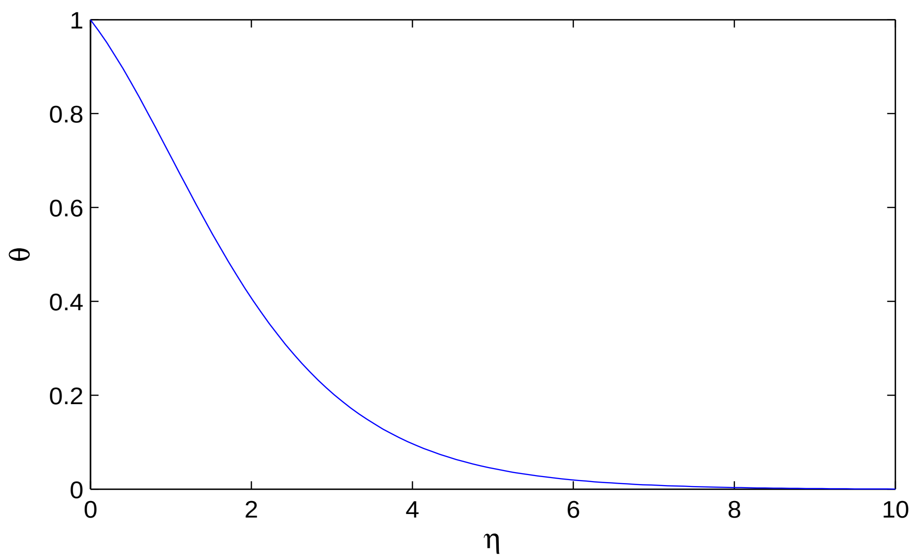

Figure 3.2: Numerical solution of the function θ.

3.3 Approximate Closed-form Solution of the Reduced System of ODEs

Numerical solution of the reduced system of ODEs cannot describe the system from which it is deduced i.e. the system of PDEs. Hence we need to determine the approximate closed-form solution from the above numerical solution which leads us to the approximate closed-form solution of the system of PDEs.

3.3.1 Approximate Closed-form Solution for the Function f

The graph of the numerical solution of the function f, in figure (3.1) is similar to the graph of function "$-e^{-\eta}$" therefore, we assume an initial approximation for the function f as

$$f_{initial}(\eta) = \alpha - \beta e^{-\gamma\eta}, \tag{3.20}$$

where α, β and γ are constants to be determined. Notice that $f'(\infty) = 0$ is satisfied by $f_{initial}$ if γ is positive. Using the condition $f(0) = 0$ in equation (3.20), we have $\beta = \alpha$, and $f'(0) = 1$ implies $\gamma = \frac{1}{\alpha}$. So $f_{initial}$ takes the form

$$f_{initial}(\eta) = \alpha(1 - e^{-\frac{\eta}{\alpha}}), \tag{3.21}$$

which satisfies the boundary conditions. Using different data points on the graph of f, we get an appropriate value of the unknown constant $\alpha = 1.6116$, to have

$$f_{initial}(\eta) = 1.6116(1 - e^{-\frac{\eta}{1.6116}}). \tag{3.22}$$

Figure (3.3) shows $f_{initial}(\eta)$ alongwith f_{num} and the difference between the numerical and its initial approximate closed-form solutions is shown in figure (3.4) which is similar to the graph of function $\eta e^{-\eta}$. In the table (3.1), we can observe that the maximum difference between the values is 0.0743.

Table 3.1: Table showing the difference between the initial approximate solution and the numerical solution for the function f.

η	f_{num}	$f_{initial}$	$f_{num} - f_{initial}$
0.0000	0.0000	0.0000	0.0000
0.5263	0.4655	0.4490	0.0165
1.0526	0.8165	0.7729	0.0436
1.5789	1.0706	1.0066	0.0640
2.1053	1.2489	1.1752	0.0737
2.6316	1.3711	1.2968	0.0743
3.1579	1.4535	1.3845	0.0691
3.6842	1.5085	1.4478	0.0608
4.2105	1.5449	1.4934	0.0515
4.7368	1.5688	1.5263	0.0425

η	f_{num}	$f_{initial}$	$f_{num} - f_{initial}$
5.2632	1.5845	1.5501	0.0344
5.7895	1.5946	1.5672	0.0274
6.3158	1.6012	1.5796	0.0216
6.8421	1.6054	1.5885	0.0169
7.3684	1.6081	1.5949	0.0131
7.8947	1.6097	1.5996	0.0101
8.4211	1.6107	1.6029	0.0078
8.9474	1.6112	1.6053	0.0059
9.4737	1.6115	1.6071	0.0044
10.000	1.6116	1.6083	0.0032

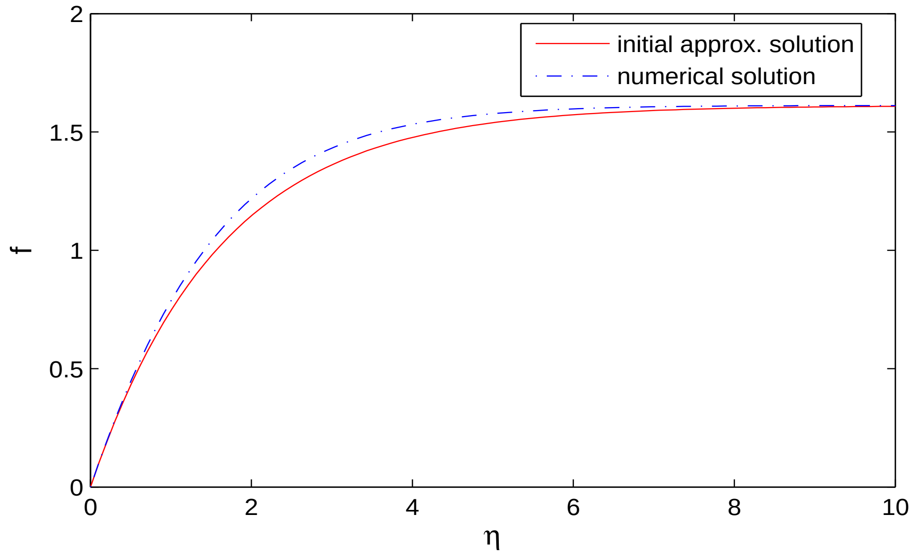

Figure 3.3: Graph of $f_{initial}$.

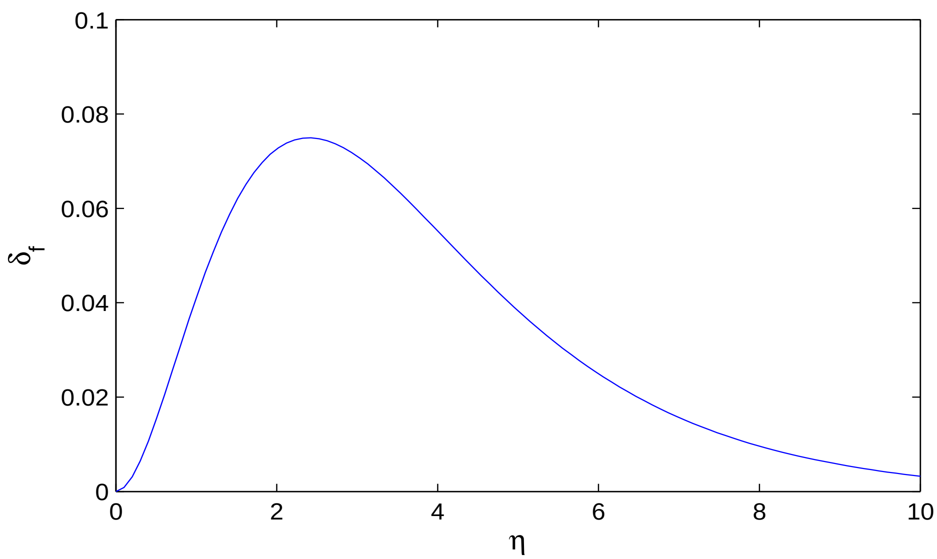

Figure 3.4: Graph of $f_{num} - f_{initial}$.

In correspondence to figure (3.4), let the following expression approximates the graph of the difference between the numerical and initial approximate solutions

$$\delta_f(\eta) = \alpha\ \eta^{\beta}\ e^{-\gamma\eta}. \tag{3.23}$$

Let us take three points $(\eta_1, \delta_1), (\eta_2, \delta_2)$ and (η_3, δ_3) to determine the expressions for α, β and γ, then the equation (3.23) gives

$$\begin{aligned} \delta_1 &= \alpha\ \eta_1^{\beta}\ e^{-\gamma\eta_1}, & (3.24)\\ \delta_2 &= \alpha\ \eta_2^{\beta}\ e^{-\gamma\eta_2}, & (3.25)\\ \delta_3 &= \alpha\ \eta_3^{\beta}\ e^{-\gamma\eta_3}. & (3.26) \end{aligned}$$

Dividing equation (3.24) by equation (3.25) and taking 'ln' on both sides, we have

$$\begin{aligned} \ln\left(\frac{\delta_1}{\delta_2}\right) &= \ln\left(\frac{\eta_1^{\beta}\ e^{-\gamma\eta_1}}{\eta_2^{\beta}\ e^{-\gamma\eta_2}}\right), \\ &= \beta \ln\left(\frac{\eta_1}{\eta_2}\right) + \gamma(\eta_2 - \eta_1). \end{aligned} \tag{3.27}$$

Now dividing equation (3.24) by equation (3.26) and taking 'ln' on both sides, to get

$$\begin{aligned} \ln\left(\frac{\delta_1}{\delta_3}\right) &= \ln\left(\frac{\eta_1^{\beta}\ e^{-\gamma\eta_1}}{\eta_3^{\beta}\ e^{-\gamma\eta_3}}\right), \\ &= \beta \ln\left(\frac{\eta_1}{\eta_3}\right) + \gamma(\eta_3 - \eta_1). \end{aligned} \tag{3.28}$$

Multiply equation (3.27) by $(\eta_3 - \eta_1)$ and equation (3.28) by $(\eta_2 - \eta_1)$, we get

$$\begin{aligned} (\eta_3 - \eta_1)\ln\left(\frac{\delta_1}{\delta_2}\right) &= \beta(\eta_3 - \eta_1)\ln\left(\frac{\eta_1}{\eta_2}\right) + \gamma(\eta_3 - \eta_1)(\eta_2 - \eta_1), & (3.29)\\ (\eta_2 - \eta_1)\ln\left(\frac{\delta_1}{\delta_3}\right) &= \beta(\eta_2 - \eta_1)\ln\left(\frac{\eta_1}{\eta_3}\right) + \gamma(\eta_2 - \eta_1)(\eta_3 - \eta_1). & (3.30) \end{aligned}$$

Subtract equation (3.30) from equation (3.29), to have

$$(\eta_3-\eta_1)\ln\left(\frac{\delta_1}{\delta_2}\right)-(\eta_2-\eta_1)\ln\left(\frac{\delta_1}{\delta_3}\right)=\beta(\eta_3-\eta_1)\ln\left(\frac{\eta_1}{\eta_2}\right)-\beta(\eta_2-\eta_1)\ln\left(\frac{\eta_1}{\eta_3}\right),$$

which gives us the expression for β as

$$\beta=\frac{(\eta_3-\eta_1)\ln\left(\frac{\delta_1}{\delta_2}\right)-(\eta_2-\eta_1)\ln\left(\frac{\delta_1}{\delta_3}\right)}{(\eta_3-\eta_1)\ln\left(\frac{\eta_1}{\eta_2}\right)-(\eta_2-\eta_1)\ln\left(\frac{\eta_1}{\eta_3}\right)}. \tag{3.31}$$

Using equation (3.31) in equation (3.27) and simplifying, we have

$$\gamma=\frac{\ln\left(\frac{\delta_1}{\delta_2}\right)-\beta\ln\left(\frac{\eta_1}{\eta_2}\right)}{\eta_2-\eta_1}, \tag{3.32}$$

and finally equation (3.24) with equations (3.31) and (3.32) gives the expression for α as

$$\alpha=\frac{\delta_1}{\eta_1^{\beta}\ e^{-\gamma\eta_1}}. \tag{3.33}$$

Now choosing some appropriate points from the table (3.1), we get the following approximation to the graph in figure (3.4),

$$\delta_f(\eta)=0.06516394126\eta^{3.100000001}e^{-1.089970273\eta}, \tag{3.34}$$

which is shown in figure (3.5). Adding this difference approximation given by equation (3.34) in equation (3.22), we get the approximate closed-form solution for the function 'f' as

$$f_{approx}(\eta)=1.6116(1-e^{-\frac{\eta}{1.6116}})+0.06516394126\eta^{3.100000001}e^{-1.089970273\eta}, \tag{3.35}$$

which satisfies the associated boundary conditions as well. Clearly,

$f_{approx}(\eta = 0) = 0$. Differentiating equation (3.35), we get

$$\begin{aligned} f'_{approx}(\eta) &= e^{-0.6205013651\eta} + 0.2020082180\eta^{2.100000001}e^{-1.089970273\eta} \\ &\quad - 0.07102675884\eta^{3.100000001}e^{-1.089970273\eta}, \\ f'_{approx}(0) &= 1, \end{aligned}$$

and as $\eta \to \infty$ then we have

$$\begin{aligned} \lim_{\eta\to\infty} f'_{approx}(\eta) = \lim_{\eta\to\infty} \Big(& e^{-0.6205013651\eta} + 0.2020082180\eta^{2.100000001}e^{-1.089970273\eta} \\ & - 0.07102675884\eta^{3.100000001}e^{-1.089970273\eta}\Big). \end{aligned}$$

The first term in the above equation clearly goes to 0 when $\eta \to \infty$ and for the rest of the terms we use L'Hospitals rule, writing remaining terms as follows:

$$\begin{aligned} & e^{-1.089970273\eta}\Big(0.2020082180\eta^{2.100000001} - 0.07102675884\eta^{3.100000001}\Big) \\ & \qquad = \frac{0.2020082180\eta^{2.100000001} - 0.07102675884\eta^{3.100000001}}{e^{1.089970273\eta}}, \end{aligned}$$

and differentiating upto four times we get

$$\begin{aligned} \lim_{\eta\to\infty} \frac{0.2020082180\eta^{2.100000001} - 0.07102675884\eta^{3.100000001}}{e^{1.089970273\eta}} &= \lim_{\eta\to\infty} \frac{\frac{-0.04199750895}{\eta^{1.899999999}} - \frac{0.05086226261}{\eta^{0.899999999}}}{1.411427627e^{1.089970273\eta}}, \\ &= 0. \end{aligned}$$

The difference between the numerical and the approximate closed-form solutions for the function 'f' is shown in table (3.2) where

$$max \mid f_{num} - f_{approx} \mid = 0.0194.$$

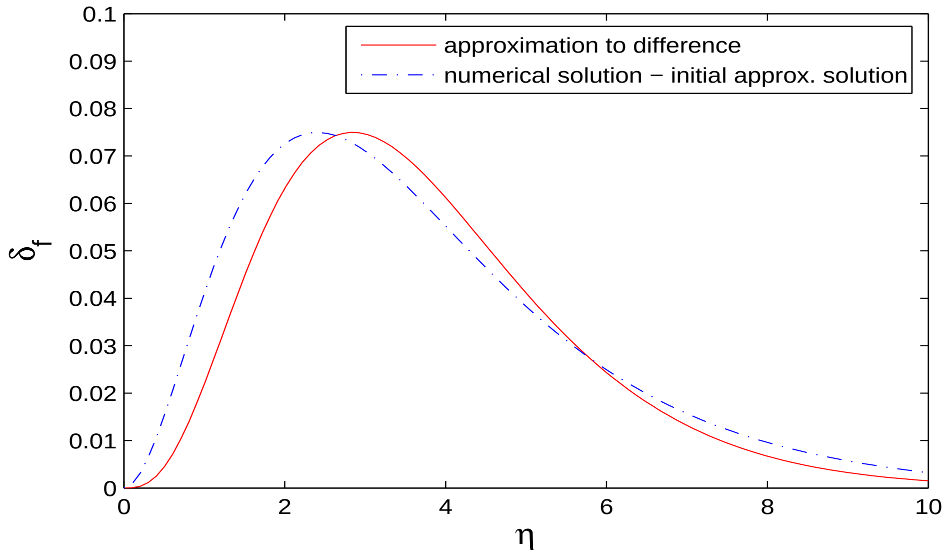

Figure 3.5: Approximation for the difference in graph of the function f.

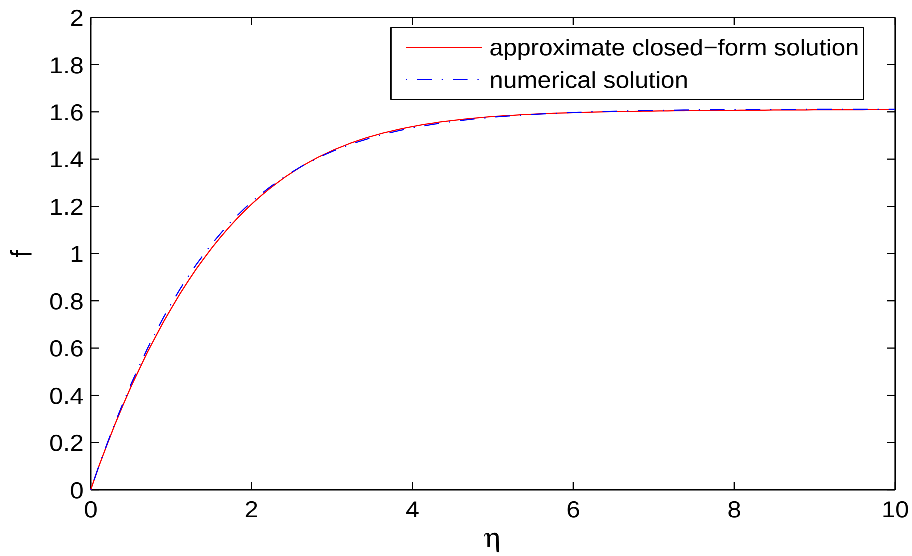

Figure 3.6: Approximate closed-form solution of the function f.

Table 3.2: Table showing the difference between the approximate and the numerical solutions.

η	f_{num}	f_{approx}	$\mid f_{num} - f_{approx} \mid$	η	f_{num}	f_{approx}	$\mid f_{num} - f_{approx} \mid$
0.0000	0.0000	0.0000	0.0000	5.2632	1.5845	1.5863	0.0018
0.5263	0.4655	0.4540	0.0115	5.7895	1.5946	1.5946	0.0000
1.0526	0.8165	0.7972	0.0194	6.3158	1.6012	1.5998	0.0014
1.5789	1.0706	1.0546	0.0160	6.8421	1.6054	1.6031	0.0023
2.1053	1.2489	1.2412	0.0077	7.3684	1.6081	1.6053	0.0028
2.6316	1.3711	1.3711	0.0000	7.8947	1.6097	1.6068	0.0029
3.1579	1.4535	1.4581	0.0046	8.4211	1.6107	1.6079	0.0028
3.6842	1.5085	1.5147	0.0062	8.9474	1.6112	1.6087	0.0025
4.2105	1.5449	1.5505	0.0056	9.4737	1.6115	1.6094	0.0021
4.7368	1.5688	1.5727	0.0038	10.000	1.6116	1.6099	0.0017

3.3.2 Approximate Closed-form Solution for the Function θ

Now the graph of numerical solution for the function 'θ' in figure (3.1) is similar to the graph of function $e^{-\eta}$. So let us assume the initial approximate closed-form solution for the function 'θ' as follows

$$\theta_{initial}(\eta) = \alpha e^{-\beta\eta}, \tag{3.36}$$

which is when subjected to the following boundary conditions,

$$\theta(0) = 1, \qquad \theta(\infty) = 0, \tag{3.37}$$

becomes

$$\theta_{initial}(\eta) = e^{-\beta\eta}, \tag{3.38}$$

where $\beta > 0$ is arbitrary which satisfies the boundary conditions. Now using different data points on the graph of θ, we get an appropriate value of the unknown constant $\beta = 0.6910$ and equation (3.38) becomes

$$\theta_{initial}(\eta) = e^{-0.6910\eta}, \tag{3.39}$$

which is shown in the following figure (3.7). The difference between the numerical solution of 'θ' and its initial approximate closed-form solution is shown in figure (3.8).

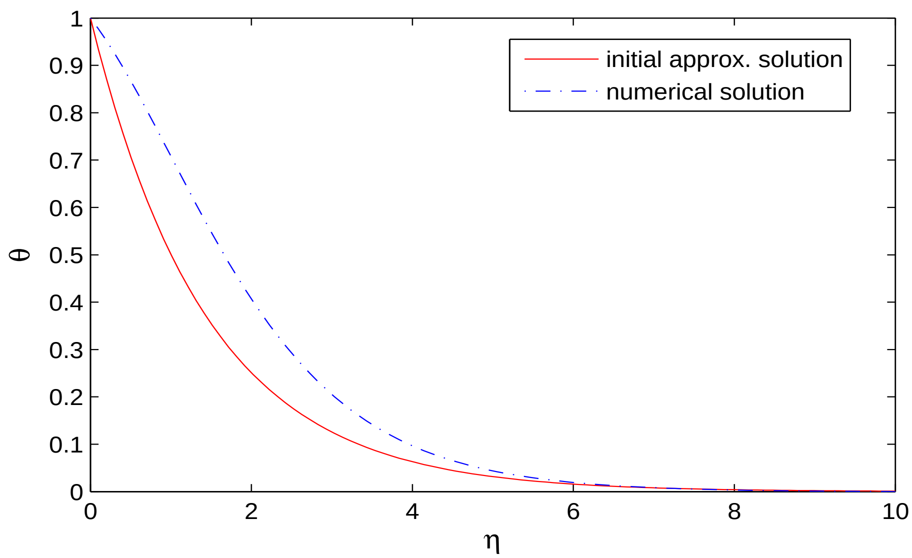

Figure 3.7: Initial approximation for the function θ.

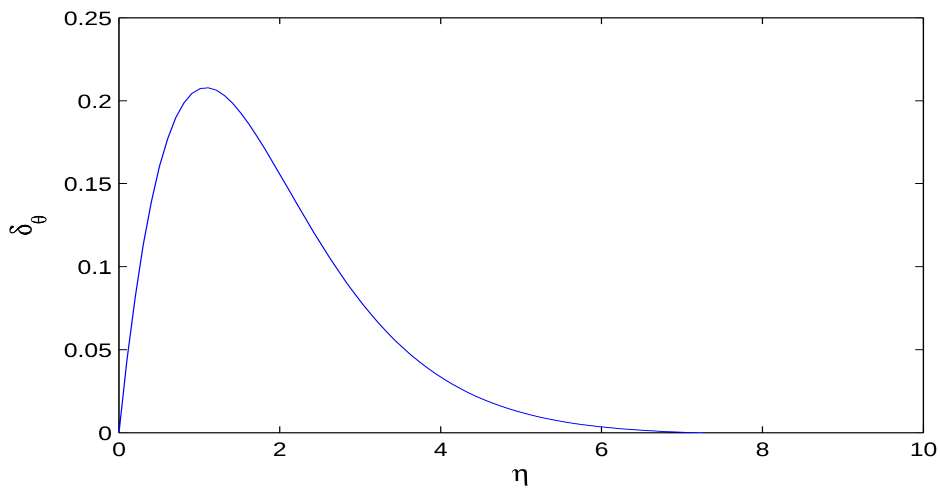

Figure 3.8: Graph of $\theta_{num} - \theta_{initial}$.

Table 3.3: Table showing the difference between the numerical solution and the initial approximate solution.

η	θ_{num}	$\theta_{initial}$	$\theta_{num} - \theta_{initial}$
0.0000	1.0000	1.0000	0.0000
0.5263	0.8595	0.6951	0.1644
1.0526	0.6910	0.4832	0.2078
1.5789	0.5244	0.3359	0.1885
2.1053	0.3804	0.2335	0.1469
2.6316	0.2668	0.1623	0.1046
3.1579	0.1828	0.1128	0.0700
3.6842	0.1232	0.0784	0.0447
4.2105	0.0820	0.0545	0.0275
4.7368	0.0541	0.0379	0.0162

η	θ_{num}	$\theta_{initial}$	$\theta_{num} - \theta_{initial}$
5.2632	0.0354	0.0263	0.0091
5.7895	0.0230	0.0183	0.0047
6.3158	0.0148	0.0127	0.0021
6.8421	0.0095	0.0088	0.0006
7.3684	0.0059	0.0061	0.0002
7.8947	0.0036	0.0043	0.0007
8.4211	0.0021	0.0030	0.0009
8.9474	0.0011	0.0021	0.0010
9.4737	0.0004	0.0014	0.0010
10.000	0.0000	0.0010	0.0010

The graph of figure (3.8) is similar to the graph in figure (3.4). Following the similar procedure as adopted in subsection (3.3.1), we get the following closed-form approximation of the graph in figure (3.8)

$$\delta_\theta(\eta) = 0.9660913990\eta^{2.100000001}e^{-1.562329683\eta}, \tag{3.40}$$

which approximates the difference $\theta_{num} - \theta_{initial}$ as shown in figure (3.9). Adding equation (3.40) in equation (3.39), we have

$$\theta_{approx.}(\eta) = e^{-0.6910\eta} + 0.9660913990\eta^{2.100000001}e^{-1.562329683\eta}, \tag{3.41}$$

which satisfies the boundary conditions $\theta(0) = 1$ and $\lim_{\eta\to\infty} \theta(\eta) = 0$.
$\theta_{approx}(\eta)$ is shown in figure (3.10).

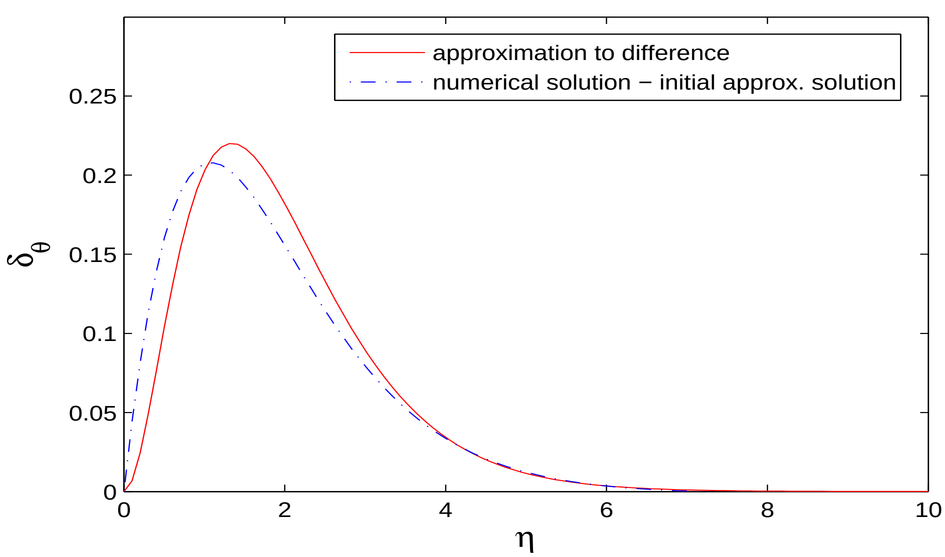

Figure 3.9: Approximation to the difference $\theta_{num} - \theta_{initial}$.

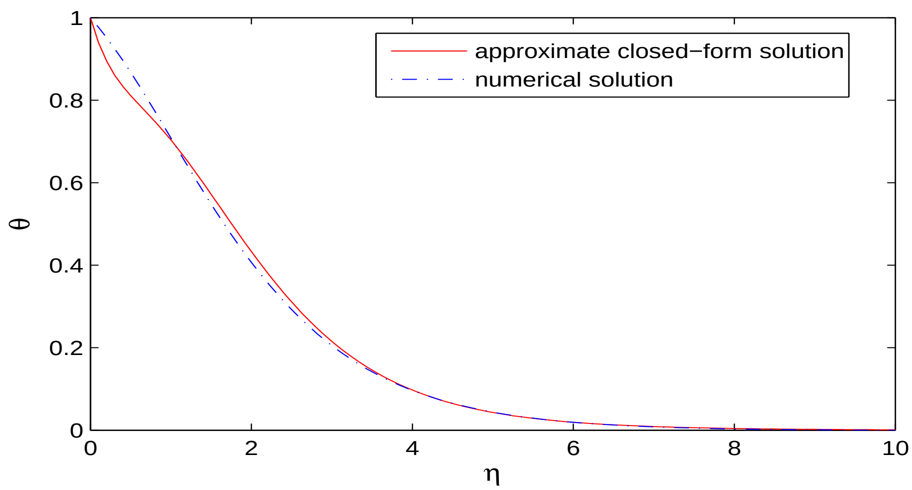

Figure 3.10: Approximate closed-form solution of the function θ.

Table 3.4: Table showing the difference between the approximate closed-form and the numerical solutions of function 'θ'.

η	θ_{num}	θ_{approx}	$\mid \theta_{num} - \theta_{approx} \mid$
0.0000	1.0000	1.0000	0.0000
0.5263	0.8595	0.8054	0.0541
1.0526	0.6910	0.6909	0.0001
1.5789	0.5244	0.5498	0.0254
2.1053	0.3804	0.4055	0.0251
2.6316	0.2668	0.2830	0.0162
3.1579	0.1828	0.1906	0.0078
3.6842	0.1232	0.1257	0.0025
4.2105	0.0820	0.0820	0.0000
4.7368	0.0541	0.0534	0.0007

η	θ_{num}	θ_{approx}	$\mid \theta_{num} - \theta_{approx} \mid$
5.2632	0.0354	0.0348	0.0006
5.7895	0.0230	0.0229	0.0002
6.3158	0.0148	0.0151	0.0003
6.8421	0.0095	0.0101	0.0006
7.3684	0.0059	0.0068	0.0009
7.8947	0.0036	0.0046	0.0010
8.4211	0.0021	0.0031	0.0010
8.9474	0.0011	0.0021	0.0010
9.4737	0.0004	0.0015	0.0011
10.000	0.0000	0.0010	0.0010

The difference between the numerical and the approximate closed-form solution for the function 'θ' is shown in the table (3.4), where $max \mid \theta_{num.} - \theta_{approx.} \mid = 0.0541$.

3.3.3 Residual Analysis of the Approximate Closed-form Solution of the Reduced System of ODEs

The residuals of the reduced system of ODEs given by equations (3.14) and (3.18), with respect to their approximate closed-form solutions given by equations (3.35) and (3.41), respectively are shown in figure (3.11). Tables (3.5) and (3.6) give values of the residuals for equations (3.14) and (3.18), respectively, with $Pr = Ec = 1$ for different values of η.

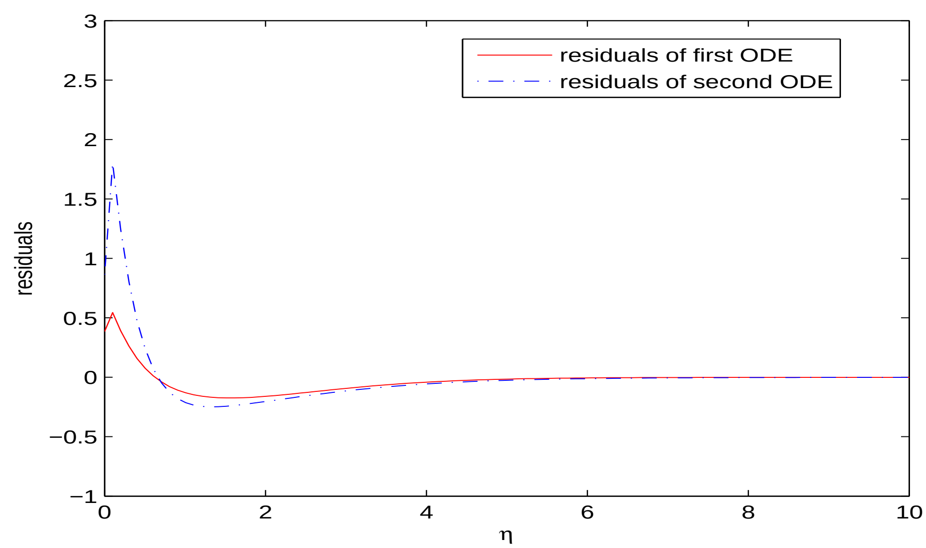

Figure 3.11: Residuals for the system of ODEs.

Table 3.5: Table showing the residuals for $f'''(\eta) + f(\eta)f''(\eta) = 0$.

η	Residuals	η	Residuals
0.0000	0.3850219441	5.2632	-0.0118161682
0.5263	0.0612030390	5.7895	-0.0068314244
1.0526	-0.1384880536	6.3158	-0.0040005845
1.5789	-0.1743087983	6.8421	-0.0024351401
2.1053	-0.1550527289	7.3684	-0.0015789942
2.6316	-0.1192493839	7.8947	-0.0011044087
3.1579	-0.0835325433	8.4211	-0.0008280371
3.6842	-0.0546321943	8.9474	-0.0006523037
4.2105	-0.0339198418	9.4737	-0.0005278954
4.7368	-0.0202683429	10.000	-0.0004312413

Table 3.6: Table showing the residuals for $\theta'' + Prf\theta' + PrEcf''^2 = 0$.

η	Residuals	η	Residuals
0.0000	0.8625029441	5.2632	-0.0207606455
0.5263	0.1999315611	5.7895	-0.0137479758
1.0526	-0.2242285067	6.3158	-0.0091662274
1.5789	-0.2399013184	6.8421	-0.0061603431
2.1053	-0.1941108639	7.3684	-0.0041721124
2.6316	-0.1451904223	7.8947	-0.0028443897
3.1579	-0.1033586304	8.4211	-0.0019495748
3.6842	-0.0709328181	8.9474	-0.0013417278
4.2105	-0.0475262938	9.4737	-0.0009261877
4.7368	-0.0314568530	10.000	-0.0006407319

3.4 Approximate Closed-form Solution of the System of PDEs

Substituting the expressions of f from equation (3.35) and θ from equation (3.41) in the transformations given by equations (3.5), (3.7) and (3.8), we get the approximate closed-form solution for the system of PDEs as

$$u(x,y) = u_w\left(e^{-\frac{y}{1.6116}\sqrt{\frac{u_w}{x\nu}}} + e^{-1.089970273y\sqrt{\frac{u_w}{x\nu}}}\right.$$
$$\left.\left(0.2020082180\left(y\sqrt{\frac{u_w}{x\nu}}\right)^{2.100000001} - 0.07102675884\left(y\sqrt{\frac{u_w}{x\nu}}\right)^{3.100000001}\right)\right), \tag{3.42}$$

$$v(x,y) = 0.5\sqrt{\frac{u_w\nu}{x}}\left(y\sqrt{\frac{u_w}{x\nu}}e^{-\frac{y}{1.6116}\sqrt{\frac{u_w}{x\nu}}} + e^{-1.089970273y\sqrt{\frac{u_w}{x\nu}}}\left(0.1368442767\left(y\sqrt{\frac{u_w}{x\nu}}\right)^{3.100000001}\right.\right.$$
$$\left.\left. - 0.07102675884\left(y\sqrt{\frac{u_w}{x\nu}}\right)^{4.100000001}\right) - 1.6116\left(1 - e^{-\frac{y}{1.6116}\sqrt{\frac{u_w}{x\nu}}}\right)\right), \tag{3.43}$$

$$T(x,y) = \left(T_w - T_\infty\right)\left(e^{-0.691y\sqrt{\frac{u_w}{x\nu}}} + 0.9660913990\left(y\sqrt{\frac{u_w}{x\nu}}\right)^{2.100000001} e^{-1.562329683y\sqrt{\frac{u_w}{x\nu}}}\right) + T_\infty. \tag{3.44}$$

u, v and T are shown in the figures (3.12), (3.13) and (3.14) respectively.

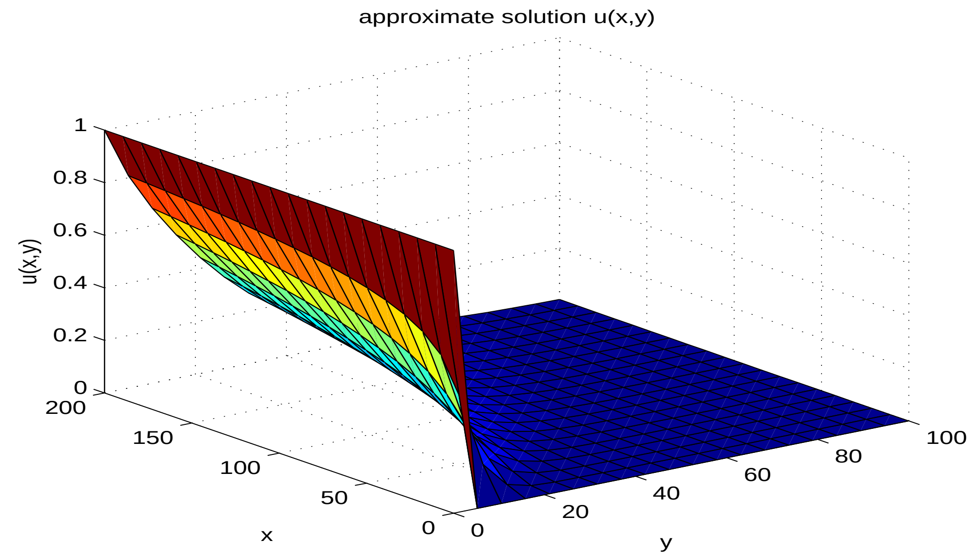

Figure 3.12: Approximate closed-form solution of $u(x, y)$.

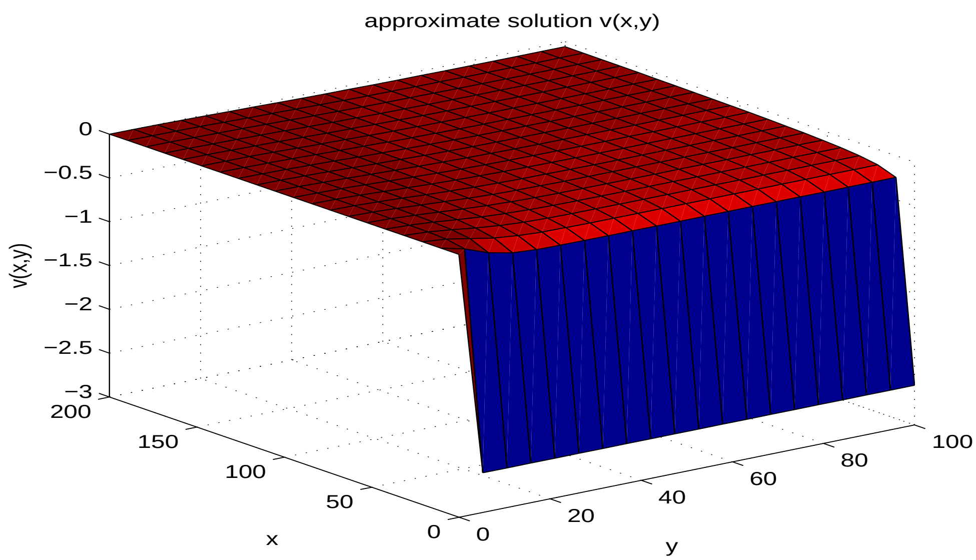

Figure 3.13: Approximate closed-form solution of $v(x, y)$.

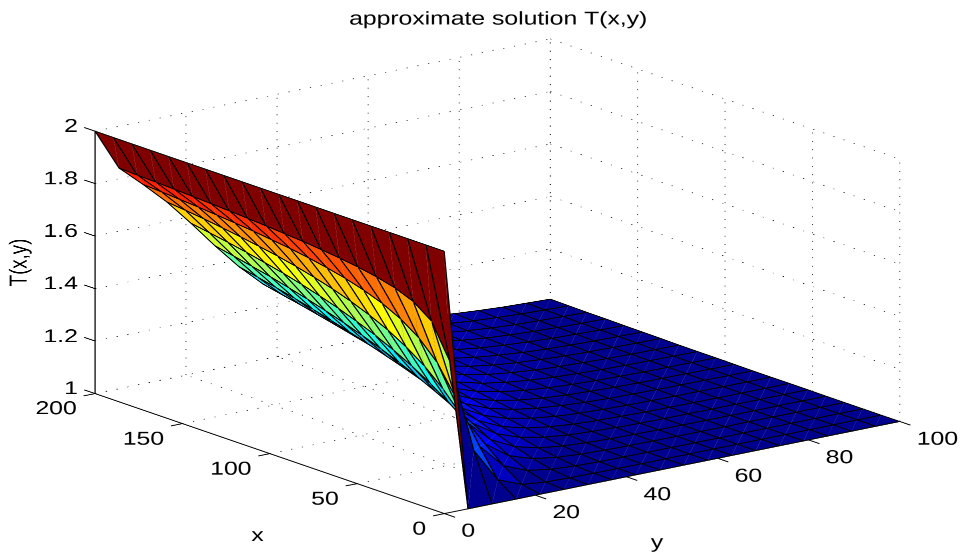

Figure 3.14: Approximate closed-form solution of $T(x, y)$.

To check that u, v and T satisfy the boundary conditions given by equations (3.4), notice that as $y \to 0$ with $x > 0$ then $u(x, 0) = u_w$ holds and when $(x, y) \to (0, 0)$, we have

$$\lim_{(x,y)\to(0,0)} u(x,y) = \lim_{(x,y)\to(0,0)} \left(u_w \left(e^{-\frac{y}{1.6116}\sqrt{\frac{u_w}{x\nu}}} + e^{-1.089970273y\sqrt{\frac{u_w}{x\nu}}} \left(0.2020082180 \left(y\sqrt{\frac{u_w}{x\nu}} \right)^{2.100000001} - 0.07102675884 \left(y\sqrt{\frac{u_w}{x\nu}} \right)^{3.100000001} \right) \right) \right).$$

To evaluate the above limit, we choose the path $y = mx$, that yields 4

$$\lim_{x\to 0} u(x,y) = \lim_{x\to 0} \left(u_w \left(e^{-\frac{mx}{1.6116}\sqrt{\frac{u_w}{x\nu}}} + e^{-1.089970273mx\sqrt{\frac{u_w}{x\nu}}} \left(0.2020082180 \left(mx\sqrt{\frac{u_w}{x\nu}} \right)^{2.100000001} - 0.07102675884 \left(mx\sqrt{\frac{u_w}{x\nu}} \right)^{3.100000001} \right) \right) \right),$$

or

$$\lim_{x\to 0} u(x,y) = \lim_{x\to 0}\left(u_w\left(e^{-\frac{m\sqrt{x}}{1.6116}\sqrt{\frac{u_w}{\nu}}} + e^{-1.089970273m\sqrt{x}\sqrt{\frac{u_w}{\nu}}}\right.\right.$$
$$\left.\left.\left(0.2020082180\left(m\sqrt{x}\sqrt{\frac{u_w}{\nu}}\right)^{2.100000001} - 0.07102675884\left(m\sqrt{x}\sqrt{\frac{u_w}{\nu}}\right)^{3.100000001}\right)\right)\right),$$
$$= u_w.$$

Also as $x \to 0$ when $y > 0$, the first term in equation (3.42) goes to zero

$$\lim_{x\to 0} u_w\left(e^{-\frac{y}{1.6116}\sqrt{\frac{u_w}{x\nu}}}\right) = 0,$$

and for the other terms, treat y as a constant and using L'Hospitals rule repeatedly, we have

$$0.2020082180 u_w y^{2.100000001}\left(\sqrt{\frac{u_w}{\nu}}\right)^{2.100000001} \lim_{x\to 0}\frac{\left(\frac{1}{\sqrt{x}}\right)^{2.100000001}}{e^{1.089970273y\sqrt{\frac{u_w}{x\nu}}}},$$
$$= 0.03603603973\nu\sqrt{\frac{\nu}{u_w}}\frac{1}{y^{0.899999999}}\left(\sqrt{\frac{u_w}{\nu}}\right)^{2.100000001} \lim_{x\to 0}\frac{x^{0.45}}{e^{1.089970273y\sqrt{\frac{u_w}{x\nu}}}} = 0.$$

Similarly third term goes to 0 and finally we have

$$\lim_{x\to 0} u(x,y) = 0.$$

Thus, we have verified the left boundary condition given by equation (3.4) for u. The right boundary condition given by equation (3.4) for u also holds as first term in equation (3.42) clearly goes to zero as

$$\lim_{y\to\infty} u_w\left(e^{-\frac{y}{1.6116}\sqrt{\frac{u_w}{x\nu}}}\right) = 0,$$

and for the rest of the terms we form the following expression using L'Hospitals rule,

$$\lim_{y\to\infty} A\frac{y^B}{e^{Cy}} = A\lim_{y\to\infty}\left(\prod_{n=0}^{i}(B-n)\right)\frac{y^{(B-i)}}{C^i e^{Cy}},$$

until $(B - i) < 0$ where A, B and C are constants, and the limit goes to zero. Equation (3.43) clearly goes to 0 when $y = 0$ and finally following the similar procedure as for $u(x, y)$, the left as well as right boundary conditions for $T(x, y)$ can be verified.

3.4.1 Residual Analysis of Approximate Closed-form Solution of System of PDEs

Using the approximate closed-form solution in equations (3.42), (3.43) and (3.44), we find the residuals of the original system of PDEs given by equations (3.1) - (3.3). The residual surfaces for equations (3.1), (3.2) and (3.3) are shown in figures (3.15), (3.16) and (3.17) respectively.

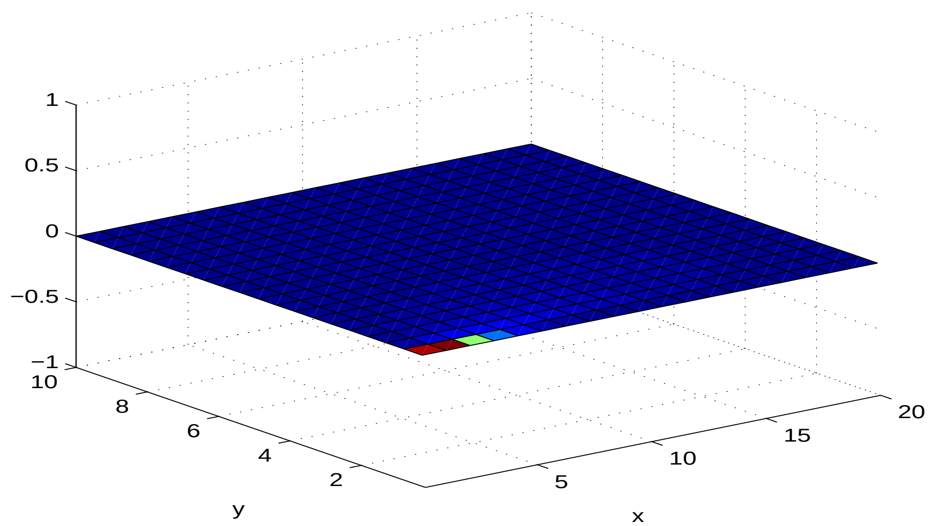

Figure 3.15: Residuals for the continuity equation.

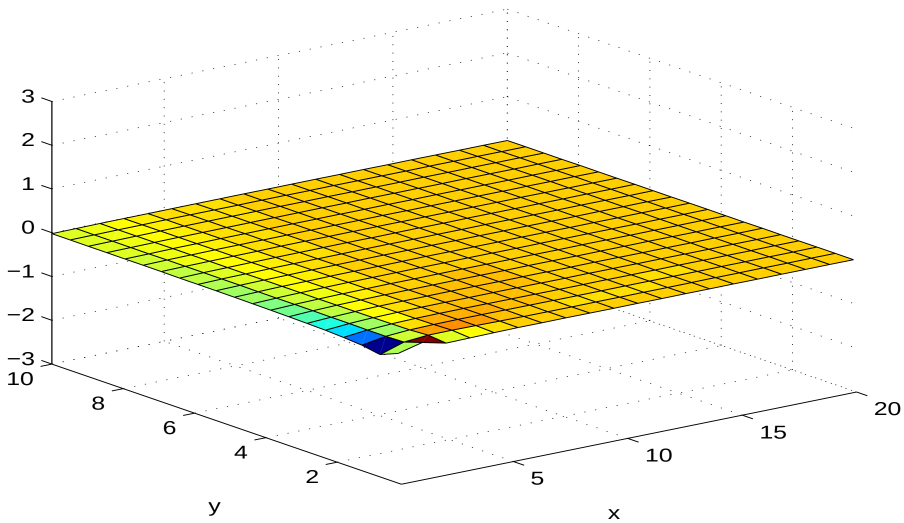

Figure 3.16: Residuals for the momentum equation.

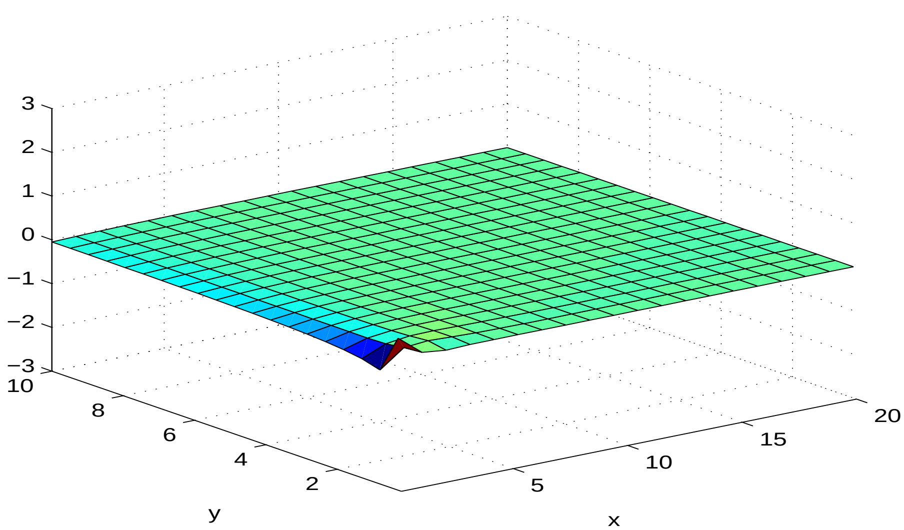

Figure 3.17: Residuals for the energy equation.

Tables (3.7) , (3.8) and (3.9) show the residuals of the continuity, momentum, and energy equations, respectively, for some values of x and y.

Table 3.7: Table showing residuals for the continuity equation; all values are multiples of 10^{-10}.

x \ y	0.2900	1.3121	2.3342	3.8674	5.4005	6.9337	7.9558	8.4668	9.4889	10.0000
0.1000	1.922	1.039	0.191	0.015	0.001	0.000	0.000	0.000	0.000	0.000
1.1474	0.058	0.194	0.194	0.117	0.057	0.027	0.016	0.013	0.008	0.006
2.1947	0.022	0.086	0.108	0.091	0.060	0.036	0.025	0.021	0.015	0.012
3.2421	0.012	0.051	0.070	0.069	0.054	0.037	0.028	0.024	0.018	0.016
4.2895	0.008	0.034	0.050	0.055	0.046	0.035	0.028	0.025	0.019	0.017
5.3368	0.006	0.025	0.038	0.044	0.040	0.032	0.027	0.025	0.020	0.018
6.3842	0.004	0.019	0.030	0.037	0.035	0.030	0.026	0.024	0.020	0.018
7.4316	0.004	0.016	0.025	0.031	0.031	0.027	0.024	0.022	0.019	0.018
8.4789	0.003	0.013	0.021	0.027	0.028	0.025	0.023	0.021	0.018	0.017
9.5263	0.002	0.011	0.018	0.024	0.025	0.023	0.021	0.020	0.018	0.017
10.5737	0.002	0.009	0.015	0.021	0.022	0.021	0.020	0.019	0.017	0.016
11.6211	0.002	0.008	0.013	0.019	0.020	0.020	0.019	0.018	0.016	0.015
12.6684	0.002	0.007	0.012	0.017	0.019	0.018	0.017	0.017	0.015	0.015
13.7158	0.001	0.006	0.011	0.015	0.017	0.017	0.016	0.016	0.015	0.014
14.7632	0.001	0.006	0.010	0.014	0.016	0.016	0.015	0.015	0.014	0.014
15.8105	0.001	0.005	0.009	0.013	0.015	0.015	0.015	0.014	0.013	0.013
16.8579	0.001	0.005	0.008	0.012	0.014	0.015	0.014	0.014	0.013	0.012
17.9053	0.001	0.004	0.007	0.011	0.013	0.014	0.013	0.013	0.012	0.012
18.9526	0.001	0.004	0.007	0.010	0.012	0.013	0.012	0.012	0.012	0.011
20.0000	0.001	0.004	0.006	0.009	0.011	0.012	0.012	0.012	0.011	0.011

Table 3.8: Table showing residuals for the momentum equation; all values are multiples of 10^{-1}.

x \ y	0.2900	1.3121	2.3342	3.8674	5.4005	6.9337	7.9558	8.4668	9.4889	10.000
0.1000	-0.353	-0.222	-0.020	0.003	0.000	0.000	0.000	0.000	0.000	0.000
1.1474	-2.021	0.177	0.146	-0.006	-0.019	-0.006	-0.002	-0.000	0.001	0.001
2.1947	-1.312	-0.035	0.115	0.042	0.020	-0.011	-0.008	-0.007	-0.004	-0.002
3.2421	-0.977	-0.111	0.070	0.053	0.037	-0.005	-0.007	-0.007	-0.006	-0.005
4.2895	-0.780	-0.140	0.036	0.052	0.042	0.002	-0.004	-0.005	-0.006	-0.005
5.3368	-0.651	-0.152	0.013	0.046	0.041	0.007	0.000	-0.002	-0.004	-0.004
6.3842	-0.559	-0.155	-0.004	0.040	0.038	0.011	0.004	0.001	-0.002	-0.003
7.4316	-0.490	-0.154	-0.016	0.033	0.034	0.014	0.007	0.004	-0.000	-0.001
8.4789	-0.436	-0.151	-0.025	0.028	0.030	0.015	0.009	0.006	0.002	0.000
9.5263	-0.394	-0.147	-0.031	0.022	0.026	0.016	0.010	0.008	0.003	0.002
10.5737	-0.359	-0.143	-0.036	0.018	0.022	0.017	0.011	0.009	0.005	0.003
11.6211	-0.329	-0.139	-0.040	0.014	0.019	0.017	0.012	0.010	0.006	0.004
12.6684	-0.305	-0.134	-0.043	0.010	0.016	0.017	0.012	0.010	0.007	0.005
13.7158	-0.283	-0.130	-0.045	0.007	0.013	0.016	0.013	0.011	0.007	0.006
14.7632	-0.265	-0.126	-0.046	0.004	0.011	0.016	0.013	0.011	0.008	0.006
15.8105	-0.249	-0.122	-0.047	0.002	0.009	0.015	0.013	0.011	0.008	0.007
16.8579	-0.234	-0.118	-0.048	-0.000	0.007	0.015	0.013	0.011	0.008	0.007
17.9053	-0.222	-0.115	-0.048	-0.002	0.005	0.014	0.012	0.011	0.009	0.007
18.9526	-0.210	-0.111	-0.049	-0.004	0.003	0.013	0.012	0.011	0.009	0.008
20.0000	-0.200	-0.108	-0.049	-0.005	0.002	0.013	0.012	0.011	0.009	0.008

Table 3.9: Table showing residuals for the energy equation; all values are multiples of 10^{-1}.

x \ y	0.2900	1.3121	2.3342	3.8674	5.4005	6.9337	7.9558	8.4668	9.4889	10.000
0.1000	4.765	-0.211	0.005	0.001	0.000	0.000	0.000	0.000	0.000	0.000
1.1474	-3.951	0.523	0.114	-0.023	-0.008	-0.001	0.000	-0.001	0.000	0.000
2.1947	-2.786	0.195	0.195	0.012	-0.012	-0.006	-0.003	-0.002	-0.000	-0.000
3.2421	-2.143	0.006	0.178	0.044	-0.004	-0.008	-0.005	-0.004	-0.002	-0.001
4.2895	-1.744	-0.097	0.142	0.061	0.006	-0.006	-0.006	-0.005	-0.003	-0.003
5.3368	-1.473	-0.156	0.107	0.069	0.016	-0.002	-0.005	-0.005	-0.004	-0.003
6.3842	-1.276	-0.191	0.076	0.071	0.023	0.001	-0.003	-0.004	-0.004	-0.004
7.4316	-1.126	-0.211	0.051	0.070	0.028	0.005	-0.001	-0.002	-0.004	-0.004
8.4789	-1.009	-0.222	0.029	0.067	0.032	0.008	0.001	-0.001	-0.003	-0.003
9.5263	-0.914	-0.228	0.012	0.062	0.034	0.011	0.003	0.001	-0.002	-0.002
10.573	-0.835	-0.231	-0.002	0.057	0.036	0.014	0.005	0.003	-0.001	-0.002
11.621	-0.769	-0.231	-0.014	0.053	0.036	0.016	0.007	0.004	0.000	-0.001
12.668	-0.713	-0.230	-0.024	0.048	0.037	0.018	0.009	0.006	0.001	0.000
13.715	-0.665	-0.227	-0.032	0.043	0.036	0.019	0.010	0.007	0.003	0.001
14.763	-0.623	-0.224	-0.039	0.038	0.036	0.020	0.012	0.008	0.004	0.002
15.810	-0.586	-0.221	-0.044	0.034	0.035	0.021	0.013	0.009	0.004	0.003
16.857	-0.553	-0.217	-0.049	0.030	0.034	0.021	0.014	0.010	0.005	0.004
17.905	-0.524	-0.214	-0.053	0.026	0.033	0.022	0.014	0.011	0.006	0.004
18.952	-0.497	-0.210	-0.057	0.023	0.031	0.022	0.015	0.012	0.007	0.005
20.000	-0.474	-0.206	-0.060	0.020	0.030	0.022	0.015	0.013	0.008	0.006

Chapter 4

Conclusions

In this thesis, we have solved a system of PDEs for a particular boundary layer flow problem following some simple steps. Firstly, the exact solution of such system is hard to find so we have applied the similarity transformations under which system of PDEs is reduced to a system of ODEs. Also exact solution of reduced system of the ODEs is hard to find, so we found the numerical solution using *bvp*5*c* in MATLAB. Our next step was to approximate the numerical solution by some closed-form functions. To check the error involved between the numerical and the approximate closed-form solution of the reduced system of ODEs, we found the residuals by approximate closed-form solution for this system, that supports validity of our approximate closed-form solution for the reduced system of ODEs. On the next stage we again considered the similarity transformations to write approximate closed-form solution of the original system of PDEs. To check the error involved in our approximate closed-form solution of the system of PDEs, we work out residuals, with respect to the approximate closed-form solution, of the system of PDEs, that again supports the approximate closed-form solution of the system of PDEs. It will be interesting to determine numerical solution of the original system of PDEs for the boundary layer problem so that one can compare the results with the approximate closed-form solution.

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