COLLIDING IMPULSIVE PLANE GRAVITATIONAL WAVES

By

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Dedicated to

My Mother

and

My Grandfather

Acknowledgement

After completing the writing of the whole thesis, it's surprising that I find writing this part so difficult. But yes it is, for the journey has been long and almost every person around me, known-unknown, at each step appears to have left some impact on me and thus, contributed to this work. Hence, my acknowledgements would attempt best to cover those whom I directly can point out as contributors to this work in their own unique unmatched ways.

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Abstract

In this thesis the main area of work concerns the study of colliding impulsive plane gravitational waves. These are the exact solutions of vacuum Einstein's field equations describing the mutual scattering of impulsive plane gravitational waves. Khan and Penrose obtained an exact solution for colliding plane impulsive gravitational waves with the remarkable feature that the spacetime before the collision is flat and after the collision is not only curved, but develops a future curvature singularity. We probe the curvature by considering the momentum imparted to test particles by the colliding gravitational waves so as to try to understand how the singularity develops.

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Chapter 1

Introduction

The theory of General Relativity (GR) was formulated by Albert Einstein in 1915 and published in 1916. In the beginning of the twentieth century, the paradigms in physics were broken and lots of innovative ideas arose in the field of physics and Einstein's contributions were prominent among those innovations. He gave a completely different interpretation of gravity, energy, matter and even space and time. General Relativity is the field theory of gravitation described in terms of matter and geometry. The relation between matter and geometry is described by a system of partial differential equations called Einstein's field equations,

$$R_{ab} - \frac{1}{2}g_{ab}R + \Lambda g_{ab} = \kappa T_{ab}, \quad (a, b = 0, 1, 2, 3) \tag{1.1}$$

where R_{ab}, R, κ and Λ are respectively the Ricci tensor, the Ricci scalar, the coupling constant and the cosmological constant. The cosmological constant is negligible in all non-cosmological situations. And g_{ab} is the metric tensor, T_{ab} is the stress-energy tensor. The left side of Eq.(1.1) represents the geometry of the spacetime while the right side demonstrates the matter content of the spacetime.

In GR the metric tensor is the field and the nonlinear interaction of the metric tensor and its inverse in the Lagrangian makes the field equations nonlinear. Consequently the whole theory becomes nonlinear. The nonlinearity of GR and its mathematical difficulties made it more complex. However, later on some developments in the theory made it easier and complexities were diminished by using the symmetry methods and finding the exact solutions of Einstein field equations.

General Relativity predicts, the existence of curvature disturbances produced by acceleration of matter on a flat and empty space. These ripples in the curvature of spacetime are called gravitational waves and they travel at the speed of light. In Newtonian mechanics gravity is a force which acts between two bodies. Einstein presented gravity as the curvature of spacetime determined by the distribution of matter-energy. Another audacious idea of Einstein's theory is that gravity is the dynamic field while in Newton's theory it is static. Newton's theory is specified by a scalar potential and change in the potential produced by a change in mass density is instantaneous. Hence potential doest not satisfy a wave equation therefore Newton's theory does not allow for gravitational waves.

Mathematically, linearization of Einstein's field equations naturally leads to the existence of gravitational waves. The exact gravitational waves solutions are non-static (time varying) solutions of vacuum field equations, for them the stress-energy tensor will be zero. The first exact solution of cylindrical gravitational wave was presented by Einstein and Rosen in 1937. Later, Bondi and Robinson discovered the exact plane wave solution in 1957. These solutions will be discussed in the third chapter in detail.

Gravitational waves carry energy away from the source and interact with matter very weakly. As these waves move away from the source, the amplitude decreases. That is why, their direct detection has been difficult in the past.

On Sept 14, 2015 for the first time the gravitational waves signal was detected simultaneously by twin Laser Interferometer Gravitational-Waves Observatory (LIGO) [1]. The observed signal matches with the waveform of the merger of binary black holes orbiting around each other subsequently results in a single massive black hole.

There had already been strong indirect evidence for the existence of gravitational waves in the past. In 1974, Hulse and Taylor discovered a pulsar with an unseen companion named as, PSR B1913+16 [2]. The binary pulsar system should lose energy in the form of gravitational radiation emission which results in an orbital decay of the system. The orbital period of this binary system decreases at $\dot{T}_b = 2.4 \times 10^{-12}$ seconds per second [3]. This is precisely the value predicted by GR [3]. This correspondence can be seen in Figure 1.1. Hulse and Taylor were awarded the Nobel prize in physics for this pioneering work in 1993 [4].

In the first chapter of the thesis, we will discuss basic concepts of differential geometry briefly which are a prerequisite to understand GR. In the second chapter, we analyse GR and the weak field approximation which leads us to gravitational waves. In the third chapter, the exact solutions of gravitational waves are discussed with special emphasis on colliding impulsive plane gravitational waves. Also the extended pseudo-Newtonian (eψN) formalism has been discussed briefly to demonstrate the reality of gravitational waves. In chapter four, we will analyse the Khan-Penrose spacetime by using eψN formalism and follow this up with a conclusion in chapter five.

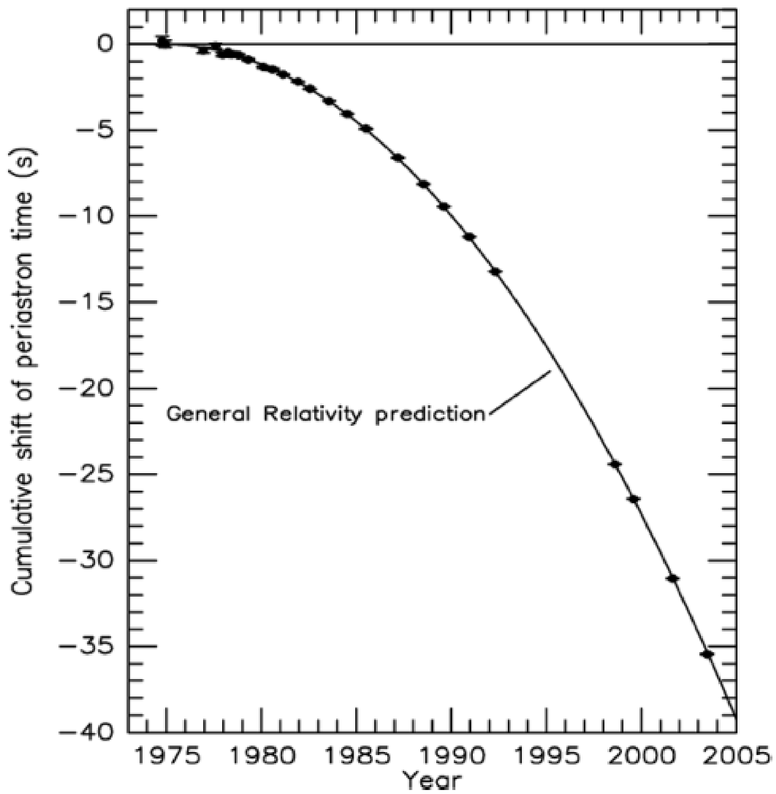


Figure 1.1: Orbital decay of PSR B1913+16 due to the emission of gravitational waves. The solid line is the orbital decay predicted by GR and data points are the observed orbital decay with error bars. The observations match the prediction from GR to within 0.2% (taken from [3]).

1.1 Tensor Algebra

The mathematical description of a curved space starts with the concept of a *manifold*. A manifold is nothing more than a set of points that can be continuously parametrised. These points constitute a continuous space that may be curved globally, but locally it looks like Euclidean space.

Definition: A manifold M of dimension n is a set together with a specified class of open subsets U_α satisfying the following Axioms [5]:

(1)M is covered by the set of U_α;

(2) For every α there exist a bijective map $\varphi_\alpha : U_\alpha \longrightarrow V_\alpha$, where V_α is an open subset of $\mathbb{R}^n$;

(3) The map $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \longrightarrow \varphi_\beta(U_\alpha \cap U_\beta)$ that takes the points in φ_α to points in φ_β only for the overlapping region such that $U_\alpha \cap U_\beta \neq \phi$.

If all the coordinate charts φ_α are homeomorphisms then the manifold M posses a topology. In the context of a topological space a manifold can be defined as: A manifold of dimension n is a separable, connected, Hausdorff space with a homeomorphism from each element of its open cover into $\mathbb{R}^n$. The terms used are defined as: a topological space is said to be *separable* if it has a countable dense subset; a subset is said to be *dense* if its closure is the original set; The *closure* of a set is the smallest closed set containing it; a topological space is said to be *Hausdorff* if any two distinct points possess non-intersecting neighbourhoods; a space is said to be *disconnected* if there exist two sets A and B, whose intersection is disjoint i.e $A \cap B = \phi$ but union is the whole space otherwise it is said to be *connected*.

We need a bijective continuous map with continuous inverse to assign coordinates to points on manifold is called *homeomorphism*. If the mappings are differentiable then the manifold is called a *differentiable manifold*. If they are n times differentiable the manifold is said to be $\mathcal{C}^n$ therefore a continuous manifold is $\mathcal{C}^0$. If the mappings are infinitely differentiable then the manifold is said to be $\mathcal{C}^\infty$ and if they are analytic then it is said to be $\mathcal{C}^p$. A function is said to be *analytic* if there exist Taylor series around any point of it.

Cartesian coordinates become invalid if we transit from flat to curved spacetime. Infact we need a more complicated coordinate systems. Thus, we want to make all our equations coordinate invariant i.e if the physical equation hold in one coordinate system, it should hold in all coordinate system. All those quantities which are coordinate invariant will be called *tensors*. Infact tensors are just generalization of vectors. The components transform under a coordinate change $x^a \longrightarrow x^{a'}$ with the rule called the tensor transformation rule,

$$T^{a'}{}_{b'c'}(x) = \frac{\partial x^{a'}}{\partial x^a}\frac{\partial x^{b'}}{\partial x^b}\frac{\partial x^{c'}}{\partial x^c}T^a{}_{bc}(x) \ . \tag{1.2}$$

On the right side of the above equation, the unprimed indices are summed over. These indices are called dummy indices. The upper and lower indices on the left side of equation must be conserved during contraction on the right side of the equation i.e free indices (not summed over) must be the same on both sides of the equation.

Tensors do not have a very intricate structure. The number of free (upper and lower) indices is called the *rank of the tensor*. Scalars are also tensors with rank zero i.e (with no indices), vectors are tensors with rank one. If a tensor has m upper and n lower indices, then it is called an (m, n)*-tensor*. If all indices are upper indices, then it is called *contravariant* and if all indices are lower then it is *covariant*. Otherwise it is said to be a *mixed tensor* . A tensor is said to be *symmetric*, if it remains the same under interchange of any two indices, i.e symmetric in those two indices,

$$T_{....ab...} = T_{....ba...}, \tag{1.3}$$

$$T^{abc}{}_{de} = T^{bac}{}_{de} \ . \tag{1.4}$$

The mixed tensor in Eq.(1.4) is symmetric in the indices a, b. A tensor is said to be *anti-symmetric* in the two indices, if it changes sign after interchanging these two indices;

$$T^{abc}{}_{de} = -T^{bac}{}_{de}. \tag{1.5}$$

We can construct symmetric and anti-symmetric tensor from any arbitrary tensor as follows,

$$T_{(ab)} = \frac{1}{2}(T_{ab} + T_{ab}) = T_{(ba)}. \tag{1.6}$$

$$T_{[ab]} = \frac{1}{2}(T_{ab} - T_{ab}) = -T_{[ba]}. \tag{1.7}$$

From the above equations, we can split a tensor into two different parts. One completely symmetric; and the other entirely anti-symmetric, i.e,

$$T_{ab} = T_{(ab)} + T_{[ab]}. \tag{1.8}$$

This can be generalized for any covariant tensor of rank n by placing parentheses or square brackets around pertinent indices given as,

$$T_{(a....c)} = \frac{1}{n!}(T_{a.....c} + \text{ sum over all permutations of } a.......c \text{ of } \mathbf{T} \text{ }). \tag{1.9}$$

$$T_{[a....c]} = \frac{1}{n!}(T_{a.....c} + \text{ altenating sum over all permutations of } a.......c \text{ of } \mathbf{T}). \tag{1.10}$$

By "alternating sum" we mean all (even - odd) permutations of $\mathbf{T}$, which gives a minus sign by odd number of exchanges. Therefore, for instance,

$$T_{[abc]d} = \frac{1}{3!}(T_{abcd} - T_{acbd} + T_{cabd} - T_{bacd} + T_{bcad} - T_{cbad}). \tag{1.11}$$

The addition of two tensors having the same rank and type results in a new tensor of same rank and type,

$$T^{a}{}_{bc} = S^{a}{}_{bc} + H^{a}{}_{bc} \ . \tag{1.12}$$

Moreover, addition is commutative and associative for tensors. In a similar way subtraction of two tensor of the same rank gives rise to a tensor of same kind and rank, e.g

$$T^{a}{}_{bc} = S^{a}{}_{bc} - H^{a}{}_{bc} \ . \tag{1.13}$$

A linear combination of tensors with similar contravariant and covariant indices gives a new tensor with the same indices. Consider two mixed tensors $S^{a}{}_{b}$ and $H^{a}{}_{b}$. We can construct their linear combination,

$$T^{a}{}_{b} = \alpha S^{a}{}_{b} + \beta H^{a}{}_{b} \ , \tag{1.14}$$

where α and β are scalars; $T^a_{\ b}$ is a tensor because it transforms as such. The product of two tensors yields a tensor, whose contravariant and covariant indices comprises all indices of the original tensors. Consider $S^a_{\ b}$ and H^c. The *direct product* is,

$$T^{ac}_{\ \ b} = S^a_{\ b} H^c, \tag{1.15}$$

where $T^{ac}_{\ \ b}$ is the resultant tensor.

The contraction of tensor is basically reduction of the rank of a given tensor by summing over upper and lower indices. Consider $T^{abcd}_{\ \ \ \ efg}$ is a tensor of rank 7. Setting $a = e$ and summing over gives $T^{bcd}_{\ \ \ fg}$, a tensor of rank 5. Again setting $b = f$ give rise to $T^{cd}_{\ \ g}$, a tensor of rank 3. And finally $c = g$ we get T^d, a tensor of rank 1. The contraction of a tensor having two indices is usually called the *trace*.

1.2 Tensor Calculus

The *metric tensor* has a prime importance in GR. We can use a coordinate system to specify the position of any point in space but the coordinate system does not provide us enough information about the geometry of the space. To obtain such information, the metric tensor is needed. It is basically a function of position defined at every point of the space. To determine a point in space uniquely, let us consider a set of linearly independent generalized coordinates, i.e $(x^1, x^2, x^3, x^4 \ldots\ldots\ldots\ldots x^N)$. Any infinitesimal displacement can be obtained between two points defined by the *line element* given below,

$$\mathbf{dx} = \mathbf{e}_a dx^a, \quad (a = 1, 2, 3.....N), \tag{1.16}$$

where $\mathbf{e}_a$ are basis vectors and x^a are coordinates of a point. Since $\mathbf{dx}$ is a vector quantity and we can take the scalar product of $\mathbf{dx}$ with itself e.g,

$$dx^2 = g_{ab} dx^a dx^b, \quad (a, b = 1, 2, 3.....N), \tag{1.17}$$

where g_{ab} is the metric tensor defined as,

$$g_{ab} = \mathbf{e}_a \cdot \mathbf{e}_b \ . \tag{1.18}$$

In addition, the metric tensor is a symmetric tensor and it 'depends' upon the position of coordinates e.g,

$$g(x^1, x^2, x^3 \ldots\ldots\ldots x^N) = g(x^a), \qquad (a = 1, 2, 3.....N). \tag{1.19}$$

The inverse of g_{ab} is just its contravariant form,

$$g^{ab} = g(\mathbf{e}^a, \mathbf{e}^b) = \mathbf{e}^a \cdot \mathbf{e}^b, \qquad (a, b = 1, 2, 3.....N). \tag{1.20}$$

Note that, if we denote the inverse of g_{ab} by g^{bc} then the product is

$$g_{ab}g^{bc} = \delta_a^c \ , \tag{1.21}$$

where δ_a^c is the *Kronecker delta* defined as,

$$\delta_a^c = \begin{cases} 1 & \text{if } c = a, \\ 0 & \text{if } c \neq a. \end{cases} \tag{1.22}$$

The Kronecker delta is an invariant quantity under coordinate transformation and it is a mixed type tensor of rank 2.

The indices of any tensor can be raised up or down by using contravariant and covariant components of the metric tensor. For instance, for a tensor $\mathbf{T}$ having rank 3, the contravarint components are given by,

$$T^{abc} = \mathbf{T}(\mathbf{e}^a, \mathbf{e}^b, \mathbf{e}^c). \tag{1.23}$$

We can lower its indices to make it a mixed type tensor of rank 3, by using the metric tensor as,

$$T_a{}^{bc} = g_{ad}T^{dbc}. \tag{1.24}$$

Similarly, we can construct a covariant tensor by using the metric tensor. In short the metric tensor is an important tool to *raise and lower* the indices of the tensor. The components of the basis vector depend only on the choice of coordinate system.

The differentiation of a vector in ordinary Cartesian coordinates is not complicated. Since there basis vectors are constant we can find the derivative of a vector by taking derivatives of its components. However, in general, it is not so easy because in a curved space the basis vector vary from point to point. For instance, if we take partial derivative of a scalar (a tensor of rank 0) it yields a vector i.e a tensor of rank 1.

$$T_{,i} = \frac{\partial T}{\partial x_i} = S_{,i}. \tag{1.25}$$

Partial derivatives of a tensor of rank 1 (vector) does not give a tensor. This holds for all tensors having $rank \geq 1$. To avoid these difficulties, we use *covariant differentiation* which guarantees that the derivative must be a tensor. It should be noted that covariant differentiation is not the same for the contravariant and covariant tensors. We define the covariant derivatives of contravariant and covariant vectors as [5],

$$T^a_{\ ;b} = T^a_{\ ,b} + \Gamma^a_{bc}T^c \ , \tag{1.26}$$

$$T_{a;b} = T_{a,b} - \Gamma^c_{ab}T_c \ , \tag{1.27}$$

where Γ^a_{bc} is the *Christoffel symbol* of second kind defined as,

$$\Gamma^a_{bc} = \frac{1}{2}g^{ad}(g_{bd,c} + g_{cd,b} - g_{bc,d}). \tag{1.28}$$

The Christoffel symbol is a non-tensorial quantity, depends on the frame vectors and their rate of change from one point to other. Moreover Christoffel symbol is symmetric in lower two indices. The covariant differentiation for tensors having rank 2 is given as,

$$T_{ab;c} = T_{ab,c} - \Gamma^d_{ac} T_{db} - \Gamma^d_{bc} T_{ad} \ , \tag{1.29}$$

$$T^{ab}{}_{;c} = T^{ab}{}_{,c} + \Gamma^a_{dc} T^{db} + \Gamma^b_{dc} T^{ad} \ . \tag{1.30}$$

Following this procedure, we can construct a general formula for covariant differentiation of any tensor of m contravaraint and n covariant components given as,

$$T^{a...m}{}_{d...n;p} = T^{a...m}{}_{d...n,p} + \Gamma^a_{px} T^{x....m}{}_{d....n} + ... + \Gamma^a_{px} T^{a....x}{}_{d....n} - \Gamma^x_{dp} T^{a...m}{}_{x...n} - .... - \Gamma^x_{np} T^{a...m}{}_{d...x} \ . \tag{1.31}$$

Hence, the covaraiant derivative of a tensor of rank ≥ 1 comprises the normal partial derivative of a tensor and some terms involving Christofell symbols which make sure that the derivative itself transforms as a tensor.

1.3 The Curvature Tensor and Scalars

The *Riemann curvature tensor* plays a vital role in interpreting the geometrical significance of a space. Since we know that the covariant differentiation can be thought as a generalization of partial differentiation but with an incredible difference. The covariant differentiation of a tensor field yields a tensor field, while the partial differentiation do not. Another difference is that the order of differentiation matters while taking covariant derivative of a tensor field. Consider covariant differentiation of an arbitrary tensor of rank one [5],

$$T^a{}_{;c} = T^a{}_{,c} + \Gamma^a_{bc} T^b. \tag{1.32}$$

Taking covariant derivative of Eq.(1.32) again,

$$T^a{}_{;c\ ;d} = (T^a{}_{;c})_{,d} + \Gamma^a_{ed} T^e{}_{;c} - \Gamma^e_{cd} T^a{}_{;e} \ . \tag{1.33}$$

Expanding Eq.(1.33) with respect to their corresponding derivative which leads us to,

$$T^a{}_{;c;d} = (T^a{}_{,c} + \Gamma^a_{bc} T^b)\ {}_{,d} + \Gamma^a_{ed}(T^e{}_{,c} + \Gamma^e_{bc} T^b) - \Gamma^e_{cd}(T^a{}_{,e} + \Gamma^a_{be} T^b). \tag{1.34}$$

Now interchanging the order of differentiation i.e $c \leftrightarrow d$,

$$T^a{}_{;d;c} = (T^a{}_{,d} + \Gamma^a_{bd} T^b)_{,c} + \Gamma^a_{ec}(T^e{}_{,d} + \Gamma^e_{bd} T^b) - \Gamma^e_{dc}(T^a{}_{,e} + \Gamma^a_{be} T^b). \tag{1.35}$$

Subtracting Eq. (1.35) from (1.34) and balancing dummy indices gives,

$$T^a{}_{;d;c} - T^a{}_{;c;d} = R^a{}_{bcd} T^b + (\Gamma^e_{cd} - \Gamma^e_{dc}) \nabla_e T^a, \tag{1.36}$$

where ${R^a}_{bcd}$ is the *Riemann curvature tensor*, defined as,

$$ {R^a}_{bcd} = \Gamma^a_{bd,c} - \Gamma^a_{bc,d} + \Gamma^e_{bd}\Gamma^a_{ec} - \Gamma^e_{bc}\Gamma^a_{ed}. \tag{1.37} $$

Since, we know that the Christofell symbols are symmetric in the lower indices, i.e $\Gamma^e_{cd} = \Gamma^e_{dc}$, Eq.(1.36) reduces to,

$$ {T^a}_{;d;c} - {T^a}_{;c;d} = {R^a}_{bcd}T^b. \tag{1.38} $$

We also know that the difference of two tensors of the same rank is again a tensor of that rank. From Eq.(1.38) it is clear that the right side is also a tensor. In addition, the Riemann curvature tensor is a tensor of rank 4 and more importantly it describes the curvature of a space. If the Riemann curvature tensor is zero, i.e $R^a_{bcd} = 0$, then it indicates that given space is "flat" (there exists a global coordinate system for which components of the metric tensor are constant everywhere). If ${R^a}_{bcd} \neq 0$ then the space is "curved".

We can transform ${R^a}_{bcd}$ into its covariant form by contraction i.e $R_{abcd} = g_{af}{R^f}_{bcd}$. Note that $R^a_{bcd} \neq R_{abcd}$. The Riemann tensor is anti-symmetric in first two indices i.e $R_{abcd} = -R_{bacd}$, and also anti-symmetric in the last two indices i.e $R_{abcd} = -R_{abdc}$. However, it is symmetric in the pair of the first two and last two indices i.e $R_{abcd} = R_{cdab}$. In addition, we can deduce the *cyclic identity* .

$$ {R^a}_{bcd} + {R^a}_{cdb} + {R^a}_{dbc} = 0. \tag{1.39} $$

The Riemann tensor also satisfies some additional differential properties which can be derived as follows [6],

Consider a local coordinate system about any arbitrary point P, i.e $\Gamma^a_{bc}(P) = 0$, $\Gamma^a_{bc,d}(P) \neq 0$. In this coordinate system the Riemann tensor can be written as,

$$ R_{abcd} = \frac{1}{2}(g_{bc,ad} - g_{ac,bd} + g_{ad,bc} - g_{bd,ac}). \tag{1.40} $$

The covariant derivative of the Riemann tensor in Riemann normal coordinates:

$$ \begin{aligned} R_{\ abcd;f} &= R_{\ abcd,f} \\ &= \frac{1}{2}(g_{bc,ad} - g_{ac,bd} + g_{ad,bc} - g_{bd,ac})_{,f} \end{aligned} \tag{1.41} $$

Permuting c, d and f cyclically leads us to the relation,

$$ R_{abcd;f} + R_{abdf;c} + R_{abfc;d} = 0. \tag{1.42} $$

Since P is any arbitrary point, thus the above relation holds in all coordinate system. This relation is known as the *Bianchi identity* and using the antisymmetry $R_{abcd} = -R_{bacd}$ allows us to write this relation as,

$$R_{ab[cd;f]} = 0. \tag{1.43}$$

If we contract the first and third indices of R_{abcd}, we obtain a symmetric tensor of rank 2 is called the *Ricci tensor*, it is defined as,

$$R_{ab} = R^{d}{}_{adb}. \tag{1.44}$$

The explicit form of the Ricci tensor is given by [5],

$$R_{ab} = \Gamma^{c}_{ab,c} - (ln\sqrt{|g|})_{,ab} + \Gamma^{d}_{ab}(ln\sqrt{|g|})_{,d} - \Gamma^{d}_{cb}\Gamma^{c}_{da} \tag{1.45}$$

Contracting the Ricci tensor by using g^{ab} leads us to the Ricci scalar, i.e,

$$R = g^{ab}R_{ab}. \tag{1.46}$$

Since scalar quantities are invariant under coordinate change therefore the Ricci scalar helps us to examine the nature of a singularity. The singularity is a point or place where the curvature of spacetime "blows up" or becomes infinite [7]. Of course infinitely many scalars can be obtained from the Riemann tensor. Nevertheless, only finitely many independent scalars can be constructed by using symmetrical aspect and all others could be expressed in terms of them [5]. The independent curvature invariants can be constructed as,

$$I_1 = g^{ab} R_{ab} = R, \tag{1.47}$$

$$I_2 = R^{abcd}R_{abcd}, \tag{1.48}$$

$$I_3 = R^{ac}R_{abcd}R^{bd}, \tag{1.49}$$

$$I_4 = R^{ab}_{\ \ cd}R^{cd}_{\ \ ef}R^{ef}_{\ \ ab}, \tag{1.50}$$

$$I_5 = R^{*ab}R_{abcd}R^{*bd}, ... \tag{1.51}$$

where R^{*} is the dual of R, having components $R_{ab}\epsilon^{abcd} = R^{cd}$ and ϵ^{abcd} is the *Live-Civita symbols*, defined as,

$$\epsilon_{abcd} = \begin{cases} 1 & \text{if } (a,b,c,d) \text{ are even permutation of } (1,2,3,4), \\ -1 & \text{if } (a,b,c,d) \text{ are odd permutation of } (1,2,3,4), \\ 0 & \text{otherwise.} \end{cases} \tag{1.52}$$

If the curvature invariants defined above are all finite then the singularity is coordinate otherwise it will be essential. Moreover, if the singularity arises due to the choice of coordinates on the surface it is called a coordinate singularity, while if it is due to a characteristic of the surface it is called an essential singularity. However, a coordinate singularity can be removed by an appropriate coordinate transformation, but an essential singularity can not be removed.

Another useful quantity is the Weyl tensor. It is the traceless part of the curvature tensor, possessing the same algebraic properties as R_{abcd}. The Weyl tensor in n dimensions ($n \geq 3$) is given by,

$$C_{abcd} = R_{abcd} + \frac{1}{n-2}(g_{ad}R_{cb} + g_{bc}R_{da} - g_{bd}R_{cb}) + \frac{1}{(n-1)(n-2)}(g_{ac}g_{db} - g_{ad}g_{cb})R \quad . \tag{1.53}$$

Therefore, in four dimension,

$$C_{abcd} = R_{abcd} + \frac{1}{2}(g_{ad}R_{cb} + g_{bc}R_{da} - g_{bd}R_{cb}) + \frac{1}{6}(g_{ac}g_{db} - g_{ad}g_{cb})R \quad . \tag{1.54}$$

The Weyl tensor defines free gravitational field. A particular case is that for $R_{ab} = 0$, it describes the physical gravitational field in empty space. It can be regarded as describing the curvature which is generated by non-local sources. In that sense the Weyl tensor describes the "pure gravitational field". In addition, if $C_{abcd} = 0$, the curvature is only generated by the matter field.

1.4 The Geodesic Deviation

Consider a curve parametrized by λ, i.e $x^a(\lambda)$. The absolute derivative of a vector U is given by

$$\frac{DU^a}{D\lambda} = \frac{dU^a}{d\lambda} + \Gamma^a_{bc}t^b U^c \ , \tag{1.55}$$

where $\mathbf{t}$ is a tangent vector to the curve. The absolute derivative enables us to find the geodesics. A *geodesic* is the "shortest available path between two points". For a curve to be a geodesic, a tangent vector $\mathbf{t}$ must satisfies,

$$\frac{Dt^a}{D\lambda} = \alpha t^a \ , \tag{1.56}$$

where α is a function of λ and tangent vector $\mathbf{t}$ is defined by $t^a = \frac{dx^a}{d\lambda}$.
Using $t^a = \frac{dx^a}{d\lambda}$ with Eqs.(1.55) and (1.56), we may write it as,

$$\frac{d^2x^a}{d\lambda} + \Gamma^a_{bc}\frac{dx^b}{d\lambda}\frac{dx^c}{d\lambda} = \alpha\frac{dx^a}{d\lambda}. \tag{1.57}$$

The curve can be reparametrized by taking $\alpha = 0$,

$$\frac{d^2x^a}{d\lambda^2} + \Gamma^a_{ab}\frac{dx^b}{d\lambda}\frac{dx^c}{d\lambda} = 0 \ . \tag{1.58}$$

The above equation holds if λ is an affine parameter, i.e related to the proper time,

$$\lambda = c_1\tau + c_2. \tag{1.59}$$

Geometrically, it means that the tangent vector is *parallely transported* along the curve. We define the notion of the *parallel transport* of a vector V^u. Let $x^u(\lambda)$ be a curve with tangent vector $\mathbf{t}$. A vector V^u, given at each point is said to be *parallely transported* along the curve if it satisfies,

$$t^p V^u_{\ ;p} = 0 \tag{1.60}$$

The parallel transport of a general tensor of an arbitrary rank is given by,

$$t^p T^{a.....c}_{\quad\quad d....f;p} = 0\ . \tag{1.61}$$

In addition, the transported vector is parallel to the original vector along a curve and has the same length. Moreover, Eq.(1.58) is called the *geodesic equation* and its solution is called geodesics.

Before going to the geodesic deviation, it is necessary to have the concept of *Lie derivatives*. Consider a curve γ with tangent vector $\mathbf{t}$. Let A^a be the vector field defined in the neighbourhood of γ. Then the *Lie derivative* of A^a in the direction of $\mathbf{t}$ is given by,

$$\pounds_t A^a = A^a_{\ ;b}t^b - t^a_{\ ;b}A^b\ . \tag{1.62}$$

This can be extended to all types of tensors. For instance, the Lie derivative of a $(1,1)$-tensor is,

$$\pounds_t T^a_{\ b} = T^a_{\ b;c}t^c - t^a_{\ ;c}T^c_{\ b} + t^c_{\ ;b}T^a_{\ c}\ . \tag{1.63}$$

In general, for a tensor of an arbitrary rank we have,

$$(\pounds_t T)^{a...c}_{\quad d...f} = t^p T^{a...c}_{\quad d...f;p} - T^{p...c}_{\quad d...f}t^a_{\ ;p} - .... - T^{a...p}_{\quad d...f}t^c_{\ ;p} + T^{a...c}_{\quad p...f}t^p_{\ ;d} + ... - T^{a...c}_{\quad d...p}t^p_{\ ;f}\ . \tag{1.64}$$

A tensor field $T^{a...}_{b...}$ is said to be *Lie transported* along the curve if its Lie derivative with respect to t^p vanishes; i.e $\pounds_t T^{a...}_{b...} = 0$.

Consider a pair of geodesics with tangent vector $\mathbf{t}$ and there is a vector $\mathbf{P}$ that connects two neighbouring geodesics. A connecting vector $\mathbf{P}$ will be Lie transported along the geodesic. Thus, the Lie derivative vanishes; i.e,

$$\pounds_t P^a = t^d P^a_{\ ;d} - P^d t^a_{;d} = 0. \tag{1.65}$$

The relative acceleration vector, A^a, for two neighbouring geodesic is defined as,

$$A^a = \ddot{P}^a = \frac{d^2p^a}{ds^2}. \tag{1.66}$$

we are interested to derive an expression for acceleration vector. Therefore the above equation can be written as,

$$\begin{aligned} A^a &= t^c[t^d P^a_{;\ d}]_{;c} \ , \\ &= [t^a{}_{;d} P^d]_{;c} t^c \ , \\ &= t^a{}_{;d;c} P^d t^c + t^a{}_{;d} P^d{}_{;c} t^c \ , \\ &= (t^a_{;c;d} - R^a{}_{bdc} t^b) P^d t^c + t^a{}_{;d} t^d{}_{;c} P^c \ , \\ &= (t^a{}_{;c} t^c)_{;d} P^d - t^a{}_{;c} t^c{}_{;d} P^d - R^a_{bdc} t^b t^d t^c + t^a{}_{;d} t^d{}_{;c} P^c \ . \end{aligned} \tag{1.67}$$

The first term vanishes by the geodesic equation, while the second and fourth term will cancel out. Interchanging $c \longleftrightarrow d$ we have,

$$A^a = -R^a_{bcd} t^b t^c t^d. \tag{1.68}$$

This expression is known as the *geodesic deviation* . It says that the relative acceleration between two geodesic is related to the curvature. Note that Eq.(1.68) holds for the curved space. If $R^a{}_{bcd} = 0$, then there is no relative acceleration between particles moving with constant velocities.

Chapter 2

Review of General Relativity and Gravitational Waves

General Relativity superseded all previous theories of gravitation and emerged as the current theory of gravity. In GR, spacetime is a four-dimensional manifold. The spacetime is a geometrical structure that merges space and time into a single entity, having three spatial dimensions and one temporal dimension . The elements of spacetime are called *events*. Events indicate something happening somewhere at some time. The paths of the particles are called *world-lines*. The tangent vector **t** to the world-line is one of three types:
(a) *time-like* if $\mathbf{t.t} > 0$;
(b) *null or light-like* if $\mathbf{t.t} = 0$;
(c) *space-like if* $\mathbf{t.t} < 0$.

Throughout this dissertation the metric signature will taken to be $(+, -, -, -)$, i.e. the diagonalized metric tensor has a positive time component.

2.1 Fundamental Principles of General Relativity

In the study of GR, we find some principles which explicitly or implicitly helped Einstein in his progress. These basic principles are [8]:
(1) The Principle of Equivalence;
(2) The Principle of General Covariance;
(3) The Correspondence Principle.

2.1.1 The Principle of Equivalence

Before discussing the equivalence principle in GR, we explain it in Newtonian theory of gravity.

There are three distinct masses of objects in Newtonian gravity, which have different descriptions in different circumstances. The first two types of masses signify the reaction of a body experiencing inertial and gravitational forces but the third type describes the gravitational field produced by a body acting as a source [9].
Inertial mass: We are familiar with Newton's second law,

$$\mathbf{F} \ = m\mathbf{a}. \tag{2.1}$$

Inertial mass is a measure of the body's resistance to change in motion. It is also called the law of inertia. Thus for an inertial mass m_i,

$$\mathbf{F} = m_i\mathbf{a}, \tag{2.2}$$

Passive gravitational mass: It is a measure of the body's response to a given gravitational field. The force that a body experiences due to the gravitational field can be described by a potential ϕ called the gravitational potential,

$$\mathbf{F} = -m_p\nabla\phi = -m_p\mathbf{grad}\phi. \tag{2.3}$$

Active gravitational mass: This type of mass is the measure of its source potency to yield a gravitational field. If m_a represents active gravitational mass lies at the origin then the gravitational potential at any other point r distant from the origin is given by,

$$\phi = -\frac{Gm_a}{r}, \tag{2.4}$$

where m_a is an active mass and G is the Newton's gravitational constant. In Newtonian theory of gravity, all these masses are equivalent i.e inertial, active and passive masses are identical.

In GR, the equivalence principle states that a uniform gravitational field is equal to a uniform acceleration. This means that an observer cannot locally distinguish between standing on the surface of a gravitating object and moving away in a spaceship with the same acceleration. More specifically, in a given gravitational field all objects move with the same acceleration if initial conditions are the same. Moreover, all objects move with same acceleration relative to the non-inertial frame in the absence of gravitational field. Therefore, we concluded equivalence principle by the statement; any non-inertial frame is equivalent to a certain gravitational field.

The important consequence of the equivalence principle is that locally gravitational effects can be created or eliminated by proper choice of the reference frame. These types of frames are called Galilean frames or locally inertial frames. There is no experiment to tell the difference between being far away from gravitating object in space and being free- in a gravitational field.

2.1.2 The Principle of General Covariance

According to the equivalence principle we cannot distinguish between an inertial frame and a freely falling object in a gravitational field. This means there is no reason to give special importance to inertial frames. Also, there is no possibility of setting up the required system of synchronized clocks throughout a gravitational field, which is why Einstein had to postulate that all coordinates systems are equally good for description of nature and that the laws of physics should have the same form in all. This is called the principle of general covariance.

The structure of all equations of physics must be the same in any arbitrary frame of reference, or equivalently, the physical equations must be expressible in tensorial form. This principle refers to the most general case of non-inertial frames, in contrast with special relativity (SR) which works only with inertial frames of reference. This shows that GR is the generalization of SR.

2.1.3 The Correspondence Principle

In physics, a new theory should not only describe phenomena unexplained by the earlier theory but must also be consistent with it in the appropriate limit. In this sense, GR must agree on the one hand with SR in the absence of gravitation while on the other hand with the Newtonian gravity in the domain of weak fields and low velocities in comparison with the speed of light.

The correspondence principle states that the dynamics of systems described by quantum mechanics or GR, reduces to classical mechanics for large macroscopic systems and for speeds much less the speed of light. More precisely, in the limit $1/c$, $G \longrightarrow 0$ we must recover classical mechanics.

2.2 The Stress-Energy Tensor

General Relativity is a field theory and more specifically deals with the gravitational field. Since the gravitational field describes on the spatial distribution of matter, we need to describe the distribution of matter in spacetime mathematically. The *stress-energy* tensor, often called the energy-momentum tensor is a symmetric second rank tensor. It acts as a source in the generation of a gravitational field and describes the distribution of matter in the spacetime [5].

In GR, *dust* and *perfect fluid* are the most significant stress-energy tensors which are commonly used. Dust is the simplest possible stress-energy tensor defined as,

$$T^{\mu\nu} = \rho\, u^\mu u^\nu, \qquad \mu, \nu = 0, 1, 2, 3. \tag{2.5}$$

where ρ is the density of mass-energy and u^μ is the 4-velocity. The 4-velocity of an object can be given by, $u^\mu = dx^\mu/d\tau$, and τ is the proper time. The first component of 4- velocity relates proper time to the coordinate time, i.e. $u^0 = dx^0/d\tau = dt/d\tau$, while the others relates coordinate velocity.

Perfect fluid is a non-viscous fluid that has zero heat conduction and no force between the particles. It is fully characterized by its pressure p and mass density ρ, defined as,

$$T^{\mu\nu} = \ (\rho + \frac{p}{c^2})\ u^\mu u^\nu \ - \ p\ g^{\mu\nu}. \tag{2.6}$$

In the above equation $T_{\mu\nu}$ is formed by the scalar functions ρ, p and a vector field $\mathbf{u}$. That depicts the perfect fluid. In addition, if the pressure p of the perfect fluid tends to zero then it is approximately demote to the dust. The conservation of energy and momentum of $T^{\mu\nu}$ necessitates that,

$$T^{\mu\nu}{}_{;\nu} = 0. \tag{2.7}$$

This expression shows that $T^{\mu\nu}$ is divergence free. In the case of flat spacetime above equation takes the form,

$$\frac{\partial T^{\mu\nu}}{\partial x^\nu} \ = \ 0\ , \tag{2.8}$$

which is the conventional demonstration of the conservation of energy and momentum.

2.3 The Einstein Field Equations

In the previous chapter, we discussed the geodesic deviation. It relates the Riemann curvature tensor to the relative acceleration of two neighbouring geodesic. Since the relative acceleration A^μ is the second derivative of the position with respect to proper-time therefore it is the usual acceleration. It is caused by the inhomogeneity of the gravitational field. Such inhomogeneity is responsible for the tidal force. Classically, gravity is the only force that accelerates all objects. By the geodesic deviation equation we concluded that gravity is the manifestation of the curvature of spacetime caused by the distribution of matter-energy. Therefore we must have a mathematical relation between the energy density (matter) and curvature. This relation can be expressed by,

$$E^{\mu\nu}(g_{\alpha\beta}, R^\alpha{}_{\beta\gamma\rho}) = \kappa T^{\mu\nu}\ , \tag{2.9}$$

where $E^{\mu\nu}$ is a tensor, function of the curvature and metric tensor and κ is constant of proportionality. In addition, $E^{\mu\nu}$ must be symmetric and divergence-free such that the equation is consistent with the conservation of energy. We want to choose the simplest possible function that can be consistent with the physical requirements. The simplest possible function which fulfils the physical requirement is $E^{\mu\nu} = g^{\mu\nu}$.

This is symmetric and divergence free, but this choice is very trivial and obviously non-physical.

For a non-trivial case, we consider a divergence-free linear function of curvature. We contract the Bianchi identities [10],

$$R_{\mu\nu[\rho\pi;\lambda]} = 0 \ . \tag{2.10}$$

Now contracting over μ, λ, we get,

$$3g^{\mu\lambda}R_{\mu\nu[\rho\pi;\lambda]} = R^{\mu}{}_{\nu\rho\pi;\mu} - R_{\mu\pi;\rho} + R_{\mu\rho;\pi} = 0. \tag{2.11}$$

Again contracting ν,π

$$3g^{\nu\pi}g^{\mu\lambda}R_{\mu\nu[\rho\pi;\lambda]} = R^{\mu}{}_{\rho;\mu} - R_{;\rho} + R^{\pi}{}_{\rho;\pi} = 0, \tag{2.12}$$

or

$$(R^{\mu}{}_{\rho} - \frac{1}{2}\delta^{\mu}_{\rho}R)_{;\mu} = 0. \tag{2.13}$$

An equivalent form is,

$$(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R)_{;\mu} = 0. \tag{2.14}$$

Hence, for a 4-dimensional spacetime the linear, symmetric, divergence-free function of the curvature is

$$E^{\mu\nu} = R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R, \tag{2.15}$$

is called the Einstein tensor. Moreover, it describes the geometric interpretation of the spacetime. Thus we obtain the Einstein field equations;

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R + \Lambda g^{\mu\nu} = \kappa T^{\mu\nu}. \tag{2.16}$$

where Λ is a constant of integration from the geodesic deviation equation. Observationally it must be very small.

To evaluate κ, we consider the Poisson equation,

$$\nabla^2\phi = 4\pi G\rho, \tag{2.17}$$

where ϕ is the Newtonian potential and ρ, is the matter density. In the rest frame the mass density,

$$T^{\mu\nu} = \rho c^2\delta^{\mu}_0\delta^{\nu}_0 = \rho c^2. \tag{2.18}$$

Also, for the classical system we have, $\dot{x}^0 \approx c$ and $\dot{x}^i \approx 0$ for $i = 1, 2, 3$. Thus the geodesic equation,

$$\ddot{x}^{\mu} + \Gamma^{\mu}_{bc}\dot{x}^b\dot{x}^c = 0, \qquad (u, b, c = 0, 1, 2, 3) \tag{2.19}$$

gives,

$$\ddot{x}^{\mu} + \Gamma^{\mu}_{00}\dot{x}^0\dot{x}^0 = 0 \ . \tag{2.20}$$

Since we are interested in $\ddot{x}^i$, therefore,

$$\ddot{x}^{i} + \Gamma^{i}_{00}\dot{x}^0\dot{x}^0 = 0 \ , \tag{2.21}$$

equivalently,

$$\ddot{x}^i = -\Gamma^i_{00} = \frac{1}{2}g^{ii}g_{00,i} = -\frac{1}{2}(\nabla g_{00})c^2, \qquad i = 1, 2, 3. \tag{2.22}$$

where i represents the derivative with respect to x^i. When we take the classical limit,

$$g^{ii} \approx -1.$$

Classically

$$\ddot{x}^i = -\nabla\phi. \tag{2.23}$$

Eq.(2.22) and (2.23) gives,

$$g_{00} = \frac{2\phi}{c^2} + constt. \tag{2.24}$$

As $r \longrightarrow \infty$, $\phi \longrightarrow 0$, we have $constt. \longrightarrow 1$. Thus Eq.(2.24) gives,

$$g_{00} = 1 + \frac{2\phi}{c^2} \tag{2.25}$$

From Einstein's field equations, we have,

$$R^{00} - \frac{1}{2}Rg^{00} = \kappa T^{00} \ . \tag{2.26}$$

Using Eq.(2.18) in the above equation we get,

$$R^{00} - \frac{1}{2}Rg^{00} = \kappa\rho c^2 \ . \tag{2.27}$$

$$R^{00} - \frac{1}{2}Rg^{00} = 2\Gamma^i_{00,i} = \nabla^2 g_{00} \ . \tag{2.28}$$

From Eq.(2.24) , we can get,

$$R^{00} - \frac{1}{2}Rg^{00} = \frac{2\nabla^2\phi}{c^2}, \tag{2.29}$$

where,

$$\frac{2\nabla^2\phi}{c^2} = \kappa\rho c^2,$$

or,

$$\kappa = \frac{2\nabla^2\phi}{\rho c^4}.$$

From the Poisson equation, we get,

$$\kappa = \frac{8\pi G}{c^4} \ . \tag{2.30}$$

Finally, Einstein's field equations (2.16) takes the form,

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = \frac{8\pi G}{c^4}T^{\mu\nu}. \tag{2.31}$$

Contracting the above equation we obtain,

$$g_{\mu\nu}R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}g_{\mu\nu}R = \frac{8\pi G}{c^4}g_{\mu\nu}T^{\mu\nu}.$$

$$R = -\frac{2\pi G}{c^4}T. \tag{2.32}$$

Using Eq. (2.32) in (2.31) we get,

$$R^{\mu\nu} = \frac{8\pi G}{c^4}(T^{\mu\nu} - \frac{1}{2}g^{\mu\nu}T). \tag{2.33}$$

In the empty space the stress energy tensor, $T^{\mu\nu}$ is zero. Therefore its trace, T is zero. Hence the field equations in vacuum are,

$$R_{\mu\nu} = 0. \qquad \mu, \nu = 0, 1, 2, 3. \tag{2.34}$$

2.4 Derivation of Einstein Field Equations by the Variational Approach

In GR, we should anticipate that every source of momentum and energy contributes in production of spacetime curvature. The stress-energy-momentum tensor is basically the origin for the spacetime curvature in the same way as the density is a source term in the case of potential. we will derive Einstein field equation by using the variational principle,

$$\delta S = \int_V \sqrt{|g|}\mathcal{L}d^4x \ . \tag{2.35}$$

The action S we deal with consist of two parts, a gravitational action S_G and a matter action S_M, e.g,

$$S = S_G + S_M, \tag{2.36}$$

where $S_G = \frac{1}{2\kappa}\int_v \mathcal{L}_{\mathcal{G}}\sqrt{|g|}d^4x$ and $S_M = \frac{1}{2\kappa}\int_v \mathcal{L}_{\mathcal{M}}\sqrt{|g|}d^4x$ with $\mathcal{L}_{\mathcal{G}}$ and $\mathcal{L}_{\mathcal{M}}$ are the gravitational Lagrangian and the matter Lagrangian respectively. We vary the action S inside on infinitesimally small region V and letting the variations in $\mathbf{g}_{\mu\nu}$ and its derivative $\mathbf{g}_{\mu\nu,\sigma}$ to be zero at boundary of V. Now the total action must remain invariant. Therefore,

$$\delta S = \delta(S_G + S_M) = \delta S_G + \delta S_M = 0\ . \tag{2.37}$$

The Eq.(2.36) can be written as,

$$S = \int_V \sqrt{|g|}(\frac{R}{2\kappa} + \mathcal{L}_{\mathcal{M}})d^4x, \tag{2.38}$$

$$\delta S = \frac{1}{2\kappa}\int_V \delta(R_{\mu\nu}g^{\mu\nu}\sqrt{|g|})d^4x + \int_V \delta(\mathcal{L}_{\mathcal{M}}\sqrt{|g|})d^4x\ ,$$

$$= \frac{1}{2\kappa}\int_v [g^{\mu\nu}\sqrt{|g|}\delta R_{\mu\nu} + R_{\mu\nu}\delta(g^{\mu\nu}\sqrt{|g|})]d^4x + \int_V \delta(\mathcal{L}_{\mathcal{M}}\sqrt{|g|})d^4x\ . \tag{2.39}$$

Consider the matter action,

$$\delta S_M = \int_V \delta(\mathcal{L}_{\mathcal{M}}\sqrt{|g|})d^4x. \tag{2.40}$$

Note that,

$$\delta g = g g^{\mu\nu}\delta g_{\mu\nu} = -g g_{\mu\nu}\delta g^{\mu\nu}\ . \tag{2.41}$$

Therefore,

$$\delta\sqrt{|g|} = -\frac{1}{2}g_{\mu\nu}\delta g^{\mu\nu}\ . \tag{2.42}$$

Using Eq. (2.42) in (2.40) we get,

$$\delta S_M = -\frac{1}{2}\int_V [g_{\mu\nu}\mathcal{L}_{\mathcal{M}} - 2\frac{\partial\mathcal{L}_{\mathcal{M}}}{\partial g_{\mu\nu}}]\sqrt{|g|}\delta g^{\mu\nu}d^4x. \tag{2.43}$$

Since the matter action is defined as [4],

$$\delta S_M = -\frac{1}{2}\int_V \sqrt{|g|}T_{\mu\nu}\delta g^{\mu\nu}d^4x\ , \tag{2.44}$$

where $T_{\mu\nu}$ is the stress-energy tensor. Comparing Eq. (2.43) and (2.44) we obtain,

$$T_{\mu\nu} = g_{\mu\nu}\mathcal{L}_{\mathcal{M}} - 2\frac{\partial\mathcal{L}_{\mathcal{M}}}{\partial g_{\mu\nu}}. \tag{2.45}$$

Now introducing local coordinate system in this volume element V, we obtain $\Gamma^{\alpha}_{\beta\gamma} = 0$ but $\Gamma^{\alpha}_{\beta\gamma;\sigma} \neq 0$. So the components of the Ricci tensor in a defined frame appear as follows,

$$R_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu,\lambda} - \Gamma^{\lambda}_{\mu\lambda,\nu}\ , \tag{2.46}$$

and

$$\delta R_{\mu\nu} = \delta\Gamma^{\lambda}_{\mu\nu,\lambda} - \delta\Gamma^{\lambda}_{\mu\lambda,\nu} ,$$
$$\delta R_{\mu\nu} = (\delta\Gamma^{\lambda}_{\mu\nu})_{,\lambda} - (\delta\Gamma^{\lambda}_{\mu\lambda})_{,\nu} . \tag{2.47}$$

In our defined frame $g_{\mu\nu,\lambda} = 0$, therefore,

$$g^{\mu\nu}\delta R_{\mu\nu} = (g^{\mu\nu}\delta\Gamma^{\lambda}_{\mu\nu} - g^{\mu\lambda}\delta\Gamma^{\nu}_{\mu\nu})_{,\lambda}. \tag{2.48}$$

By integration over entire region V we get,

$$\int_V g^{\mu\nu}\delta R_{\mu\nu}\sqrt{|g|}d^4x = \int_V (g^{\mu\nu}\delta\Gamma^{\lambda}_{\mu\nu} - g^{\mu\lambda}\delta\Gamma^{\nu}_{\mu\nu})_{,\lambda}\sqrt{|g|}d^4x . \tag{2.49}$$

From the above equation, it seems that only boundary term of the integral on right side of the Eq.(2.49) is contributing after application of the Gauss divergence theorem. Since the metric and its derivative vanish at the boundary of the region V. Therefore, we have,

$$\int_V (g^{\mu\nu}\delta R_{\mu\nu}\sqrt{|g|})d^4x = 0. \tag{2.50}$$

Further,

$$\delta(g^{\mu\nu}) = \sqrt{|g|}\delta g^{\mu\nu} + g^{\mu\nu}(-\frac{1}{2}\sqrt{|g|}g_{\alpha\beta}\delta g^{\alpha\beta}) \tag{2.51}$$

Inserting Eqs. (2.44), (2.50) and (2.51) in Eq. (2.36) and simplifying we obtain,

$$\delta S = \frac{1}{2\kappa}\int_V \sqrt{|g|}[R_{\mu\nu}\delta g^{\mu\nu} - \frac{1}{2}(R_{\mu\nu}g^{\mu\nu})g_{\alpha\beta}\delta g^{\alpha\beta}]d^4x - \frac{1}{2}\int_V \sqrt{|g|}T_{\mu\nu}\delta g^{\mu\nu}d^4x. \tag{2.52}$$

Finally,

$$\delta S = \frac{1}{2\kappa}\int_V \sqrt{|g|}[R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - kT_{\mu\nu}]\delta g^{\mu\nu}d^4x . \tag{2.53}$$

By the variational principle $\delta S = 0$. It is true if,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - \kappa T_{\mu\nu} = 0. \tag{2.54}$$

These are the Einstein field equations in the presence of matter. We can add a constant to the action and thereby modify the gravitational Lagrangian density to,

$$\mathcal{L}_{\mathcal{G}}[g_{\mu\nu}, g_{\mu\nu,\sigma}] = \sqrt{|g|}(R - 2\Lambda), \tag{2.55}$$

where Λ is constant called the *cosmological constant*. In that case the modified Einstein field equations are,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - g_{\mu\nu}\Lambda - \kappa T_{\mu\nu} = 0. \tag{2.56}$$

This is the covariant form of Eq. (2.16). By the previous argument we arrive at a value of κ. In addition, they are 10 coupled non-linear partial differential equations for 10 functions of variable.

Einstein's field equations describe the curvature of spacetime generated by matter fields. In the absence of matter fields, the curvature tensor is the Weyl tensor that represents the pure gravitational field. Mathematically, if $T_{\mu\nu} = 0$ implies both $R_{\mu\nu} = 0$ and $R = 0$ ($n \neq 2$). Therefore, Eq.(1.52) reduces to $C_{abcd} = R_{abcd}$. This means that the curvature is generated by gravitational field in empty space.

2.5 The Weak Field Approximation

In Newton's theory, gravity is a static "field", while in GR it is dynamic. Hence Newton's theory does not accommodate gravitational waves but GR does. This is the most significant difference between both theories. However the non-linearity of GR is also one of the significant difference between Newtonian gravity and GR. Newtonian gravity is specified by scalar gravitational potential which arises linearly in the Newtonian gravity. On the other hand, In GR, the metric tensor acts as a field. The non-linear emergence of the metric tensor in the field equations resulted in the non-linearity of GR.

It is also worthwhile to understand the linear consequences of GR by linearizing the Einstein field equations. The linearization of Einstein field equation give rise to gravitational waves. In linearized gravity non-linear terms of the metric tensor are neglected. This is an approximation which gives approximate solutions rather then exact results. In the linearized gravity, an observer is placed distant from a source such that the gravitational field is very weak. This phenomenon is called the weak field approximation. In the weak field regime, the curved spacetime metric tensor can be written as the sum of flat (Minkowski) spacetime metric tensor and a small perturbation $h_{\mu\nu}$ [5],

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad u, v = 0, 1, 2, 3 \tag{2.57}$$

where $|h_{\mu\nu}| << 1$, we discard second and higher order terms of $h_{\mu\nu}$. Let the inverse metric tensor be given by,

$$g^{\mu\nu} = \eta^{\mu\nu} + f^{\mu\nu} \ , \tag{2.58}$$

where $\eta^{\mu\nu}$ is the inverse of, $\eta_{\mu\nu}$. Therefore,

$$g_{\mu\nu} g^{\nu\rho} = \delta_{\mu}^{\ \rho} = (\eta_{\mu\nu} + h_{\mu\nu})(\eta^{\nu\rho} + f^{\nu\rho}) = \delta_{\mu}^{\ \rho} + \eta_{\mu\nu} f^{\nu\rho} + \eta^{\nu\rho} h_{\mu\nu} + h_{\mu\nu} f^{\nu\rho}. \tag{2.59}$$

Cancelling the $\delta_{\mu}^{\ \rho}$ on both sides and multipying through by $\eta^{\mu\pi}$, we get,

$$f^{\pi\rho} + \eta^{\mu\pi}\eta^{\nu\rho} h_{\mu\nu} + \eta^{\mu\pi} h_{\mu\nu} f^{\nu\rho} = 0. \tag{2.60}$$

The last term in the above equation is quadratic in the difference between the flat and curved spacetime metric tensors. Neglecting it and retaining only first order in h, we have,

$$f^{\pi\rho} = -\eta^{\mu\pi}\eta^{\nu\rho}h_{\mu\nu} + O(h^2). \tag{2.61}$$

Thus we can write Eq.(2.61) as,

$$f^{\mu\nu} = -h^{\mu\nu} + O(h^2). \tag{2.62}$$

Therefore Eq.(2.58) becomes,

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + O(h^2) \approx \eta^{\mu\nu} - h^{\mu\nu}. \tag{2.63}$$

The Christoffel symbols are given as,

$$\Gamma^{\mu}_{\nu\gamma} = \frac{1}{2}g^{\mu\rho}(g_{\rho\gamma,\nu} + g_{\rho\nu,\gamma} - g_{\nu\gamma,\rho}) \ . \tag{2.64}$$

Since, $\eta_{\mu\nu}$ is the constant, we will use $\eta_{\mu\nu}$ to raise and lower the indices. Now the Christofell symbols can be written as,

$$\Gamma^{\mu}_{\nu\gamma} = \frac{1}{2}\eta^{\mu\rho}(h_{\rho\gamma,\nu} + h_{\rho\nu,\gamma} - h_{\nu\gamma,\rho}) \ . \tag{2.65}$$

The Riemann curvature tensor is defined as,

$$R^{\mu}{}_{\nu\gamma\rho} = \Gamma^{\mu}_{\nu\rho,\gamma} - \Gamma^{\mu}_{\nu\gamma,\rho} + \Gamma^{\mu}_{\pi\gamma}\Gamma^{\pi}_{\nu\rho} - \Gamma^{\mu}_{\pi\rho}\Gamma^{\pi}_{\nu\gamma} \ . \tag{2.66}$$

Since, we are neglecting the second and higher order terms of $h_{\mu\nu}$, so the Riemann curvature tensor reduces to,

$$R^{\mu}{}_{\nu\gamma\rho} = \Gamma^{\mu}_{\nu\rho,\gamma} - \Gamma^{\mu}_{\nu\gamma,\rho} \ . \tag{2.67}$$

Using Eq. (2.65), the Riemann curvature tensor takes the form,

$$R_{\mu\nu\gamma\rho} = g_{\mu\pi}R^{\pi}{}_{\nu\gamma\rho} = \frac{1}{2}(h_{\mu\rho,\nu\gamma} + h_{\nu\gamma,\mu\rho} - h_{\mu\gamma,\nu\rho} - h_{\nu\rho,\mu\gamma}) \ . \tag{2.68}$$

After contraction, we get the Ricci tensor as,

$$R_{\mu\nu} = g^{\gamma\rho}R_{\gamma\mu\rho\nu} = \frac{1}{2}(h^{\gamma}{}_{\mu,\nu\gamma} + h^{\gamma}{}_{\nu,\mu\gamma} - \Box h_{\mu\nu} - h_{,\mu\nu}) \ , \tag{2.69}$$

where $h = \eta^{\mu\nu}h_{\mu\nu} = h^{\mu}{}_{\mu}$ and '$\Box$' is the well known d'Alembertian, defined by,

$$\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu = \partial^\nu\partial_\nu = \partial^2/\partial t^2 - \nabla^2, \tag{2.70}$$

The Ricci scalar is obtained by contracting $R_{\mu\nu}$ with $g^{\mu\nu}$ is given as,

$$R = g^{\mu\nu} R_{\mu\nu} = (h^{\gamma\rho}{}_{,\gamma\rho} - \Box h) \ . \tag{2.71}$$

Eventually, the Einstein tensor $G_{\mu\nu}$ in linearized form is,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{1}{2}(h^{\gamma}{}_{\mu,\nu\gamma} + h^{\gamma}{}_{\nu,\mu\gamma} - \Box h_{,\mu\nu} - \eta_{\mu\nu} h^{\gamma\rho}{}_{,\gamma\rho} + \eta_{\mu\nu}\Box h) \ . \tag{2.72}$$

Now Einstein field equations in the weak gravitational field are,

$$\frac{1}{2}(h^{\gamma}{}_{\mu,\nu\gamma} + h^{\gamma}{}_{\nu,\mu\gamma} - \Box h_{,\mu\nu} - \eta_{\mu\nu} h^{\gamma\rho}{}_{,\gamma\rho} + \eta_{\mu\nu}\Box h) = \kappa T_{\mu\nu} \ , \tag{2.73}$$

where $T_{\mu\nu}$ is the stress-energy tensor and κ is the coupling constant.

The linearization process has enlarged the number of terms on the left side of the Eq.(2.73), to further simplify the linearized field equations, we define a *trace reverse* of $h_{\mu\nu}$ as,

$$\psi_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h \ . \tag{2.74}$$

$$\psi^{\alpha}{}_{\nu} = \eta^{\mu\alpha}\psi_{\mu\nu} = h^{\alpha}{}_{\nu} - \frac{1}{2}\delta^{\alpha}{}_{\nu} h \ . \tag{2.75}$$

Setting $\nu = \alpha$, we get the trace of ψ,

$$\psi = -h \ . \tag{2.76}$$

Using Eq.(2.74), (2.75) and (2.76), we can re-write the Ricci tensor and the Ricci scalar as,

$$R_{\mu\nu} = \frac{1}{2}(\psi^{\gamma}{}_{\mu,\nu\gamma} + \psi^{\gamma}{}_{\nu,\mu\gamma} - \Box h) \ , \tag{2.77}$$

$$R = \frac{1}{2}(2\psi^{\gamma\rho}{}_{,\gamma\rho} - \Box h_{\mu\nu}) \ . \tag{2.78}$$

Therefore, Einstein field equations takes the form,

$$\frac{1}{2}(\psi^{\gamma}{}_{\mu,\nu\gamma} + \psi^{\gamma}{}_{\nu,\mu\gamma} - \Box\psi_{\mu\nu} - \eta_{\mu\nu}\psi^{\gamma\rho}{}_{,\gamma\rho}) = \kappa T_{\mu\nu} \ . \tag{2.79}$$

These are the basic field equations in the linearized gravity. We know that for the electromagnetic field we obtain the wave equation for the 4-vector potential by selecting the Lorentz gauge. As such, we would expect that there would be some gauge choice that would yield the required wave equation. In electromagnetism we can add the gradient of a scalar quantity to the 4-vector potential without changing the electromagnetic field itself. What is the analogous gauge freedom for gravitation? Here the principle of general covariance plays a role. We know that changing the coordinates (including a change of frame of reference) should leave all physical laws unaltered.Of course, we have limited ourselves to the gravitational field and there may be some doubts whether they would hold in all coordinate systems in the presence of other fields but it is taken for granted that there will be no problems in this regard. Since these transformations obviously form a group the gauge group will be the group of all coordinate transformations in 4-dimensions. Locally we would only need infinitesimal transformations and so the Lie group would be $GL(4)$ and the corresponding Lie algebra $gl(4)$.

2.5.1 Gauge Transformation

Let us investigate, what happens to the linearized field equations under an infinitesimal transformation [5],

$$x^\mu \longrightarrow x'^\mu = x^\mu + \xi^\mu(x) \ .$$

Since, the above transformation leaves $g_{\mu\nu}$ invariant,

$$g_{\mu\nu}(x) = \frac{\partial x'^\gamma}{\partial x^\mu}\frac{\partial x'^\rho}{\partial x^\nu} g'_{\gamma\rho}(x') \ .$$

After the linearization process, we find the transformation,

$$h_{\mu\nu} \longrightarrow h'_{\mu\nu} \cong h_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} \ . \tag{2.80}$$

This is known as the *gauge transformation*. We can easily see that under the gauge transformation the Riemann curvature tensor and all its contractions are invariant quantities. Eq. (2.79) can be simplified by using the gauge transformation. Let us define the gauge transformed field as,

$$\psi'^{\mu\nu} = h'^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu} h' \ . \tag{2.81}$$

Using Eq. (2.80) in the Eq.(2.81),

$$\psi'^{\mu\nu} = (h^{\mu\nu} - \xi^{\mu,\nu} - \xi^{\nu,\mu}) - \frac{1}{2}\eta^{\mu\nu}(h - 2\xi^\mu_{,\mu}) \ ,$$

$$= h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu} h - \xi^{\mu,\nu} - \xi^{\nu,\mu} + \eta^{\mu\nu}\xi^\mu_{,\mu} \ ,$$

$$= \psi^{\mu\nu} - \xi^{\mu,\nu} - \xi^{\nu,\mu} + \eta^{\mu\nu}\xi^\mu_{,\mu} \ .$$

Differentiating with respect to x^ν we get,

$$\psi'^{\mu\nu}{}_{,\nu} = \psi^{\mu\nu}{}_{,\nu} - \Box\xi^\mu \ . \tag{2.82}$$

Thus, if we choose $\xi^\mu(x)$ appropriately which satisfy,

$$\Box\xi^\mu = \psi^{\mu\nu}{}_{,\nu} \ .$$

Then, we have,

$$\psi'{}^{\mu\nu}{}_{,\,\nu} = 0 \ . \tag{2.83}$$

Which is known as the *gauge condition*. The significance of the gauge condition is that, in the transformed gauge first three terms on the left hand side vanishes [5]. The most simplified form of linearized field equations in transformed gauge is,

$$\Box\psi'_{\mu\nu} = -2\kappa T'_{\mu\nu} \ . \tag{2.84}$$

For convenience, dropping primes and raising the indices we get the wave equation in the presence of source,

$$\Box\psi^{\mu\nu} = -2\kappa T^{\mu\nu} \ , \tag{2.85}$$

with the guage condition,

$$\psi^{\mu\nu}{}_{,\nu} = 0 \ . \tag{2.86}$$

The vacuum field equations in the Lorentz gauge,

$$\Box\psi^{\mu\nu} = 0. \tag{2.87}$$

From above equation it is obvious that $h_{\mu\nu}$ must satisfy the classical wave equation,

$$\Box h_{\mu\nu} = 0. \tag{2.88}$$

Thus, we conclude that the linearized gravity leads us to the existence of gravitational waves that travel at speed of light.

Chapter 3

Some Exact Solutions of Gravitational Waves

In the previous chapter, the wave equation was obtained by linearizing Einstein's field equations in the absence of source. There are many types of gravitational waves with their pertinent spacetime, e.g. cylindrical waves, plane waves. We intend to examine the solutions of these waves. These solutions could be trivial (static) which fulfil the Laplace equation. Such types of solutions do not embody "moving waves" but solely a static field. For our purpose, we are keen to discuss the exact solutions of Einstein's field equations in vacuum which are time varying. The exact solution for cylindrical gravitational waves was found in 1937 [11]. Later, the exact solution for plane waves was found in 1957 [12]. This pioneering work proved to be motivational for further development of finding the exact solutions such as colliding impulsive plane gravitational waves.

3.1 Exact Solution of Cylindrical Gravitational Waves

The exact solution for cylindrical gravitational waves was constructed by Einstein and Rosen [11]. The general cylindrically symmetric spacetime depends on two arbitrary functions, γ and ψ of time t and cylindrical radial coordinate ρ, i.e $\gamma = (\rho, t)$ and $\psi = (\rho, t)$. The metric is given by

$$ds^2 = e^{2(\gamma-\psi)}(dt^2 - d\rho^2) - e^{-2\psi}\rho^2 d\phi - e^{2\psi}dz^2, \tag{3.1}$$

and the corresponding metric tensor by

$$\mathsf{g}_{\mu\nu} = \begin{pmatrix} e^{2(\gamma-\psi)} & 0 & 0 & 0 \\ 0 & -e^{2(\gamma-\psi)} & 0 & 0 \\ 0 & 0 & -\rho^2 e^{-2\psi} & 0 \\ 0 & 0 & 0 & -e^{2\psi} \end{pmatrix} \tag{3.2}$$

and inverse metric tensor by

$$\mathsf{g}^{\mu\nu} = \begin{pmatrix} e^{2(\psi-\gamma)} & 0 & 0 & 0 \\ 0 & -e^{2(\psi-\gamma)} & 0 & 0 \\ 0 & 0 & -\frac{e^{2\psi}}{\rho^2} & 0 \\ 0 & 0 & 0 & -e^{-2\psi} \end{pmatrix}. \tag{3.3}$$

The non-vanishing Christoffel symbols are,

$$\left.\begin{array}{l} \Gamma^0_{00} = \Gamma^0_{11} = \Gamma^1_{10} = \Gamma^1_{01} = \dot{\gamma} - \dot{\psi} \ , \\ \Gamma^0_{01} = \Gamma^0_{10} = \Gamma^1_{00} = \Gamma^1_{11} = \gamma' - \psi' \ , \\ \Gamma^0_{22} = -\rho^2\dot{\psi}e^{-2\gamma}, \ \Gamma^0_{33} = \dot{\psi}e^{(2\psi-\gamma)} \ , \\ \Gamma^1_{22} = \rho(\rho\psi' - 1)e^{-2\gamma}, \ \Gamma^1_{33} = -\phi' e^{2(2\psi-\gamma)} \ , \\ \Gamma^2_{02} = \Gamma^2_{20} = -\dot{\psi}, \Gamma^2_{12} = \Gamma^2_{21} = -(\psi' - \dfrac{1}{\rho}), \\ \Gamma^3_{03} = \Gamma^3_{30} = \dot{\psi}, \Gamma^3_{13} = \Gamma^3_{31} = \psi', \end{array}\right\} \tag{3.4}$$

where " $\cdot$ " and " $'$ " represents the differentiation with respect to t and ρ respectively.

The components of Ricci tensor can be obtained by using Eq.(1.44). The non-vanishing components of the Ricci tensor are,

$$\left.\begin{array}{l} R_{00} = -(\ddot{\gamma} - \ddot{\psi}) + (\gamma'' - \psi'') + \dfrac{1}{\rho}(\gamma' - \dot{\psi}) - 2\dot{\psi}^2, \\ R_{11} = (\ddot{\gamma} - \ddot{\psi}) - (\gamma'' - \psi'') + \dfrac{1}{\rho}(\gamma'^2 - \dot{\psi}^2) - 2\psi''^2, \\ R_{22} = \rho^2 e^{-2\rho}(-\ddot{\psi} + \psi'' + \dfrac{1}{\rho}\psi'), R_{01} = \dfrac{1}{\rho}\dot{\gamma} - 2\dot{\psi}\psi'. \end{array}\right\} \tag{3.5}$$

We are interested in the exact solutions of Einstein fields equation. Since the exact solution of Einstein's field equations satisfy

$$R_{\mu\nu} = 0, \tag{3.6}$$

we have,

$$-(\ddot{\gamma} - \ddot{\psi}) + (\gamma'' - \psi'') + \frac{1}{\rho}(\gamma' - \dot{\psi}) - 2\dot{\psi}^2 = 0 \ , \tag{3.7}$$

$$(\ddot{\gamma}-\ddot{\psi})-(\gamma''-\psi'')+\frac{1}{\rho}(\gamma'^2-\dot{\psi}^2)-2\psi''^2=0\ , \tag{3.8}$$

$$\rho^2 e^{-2\rho}(-\ddot{\psi}+\psi''+\frac{1}{\rho}\psi')=0\ , \tag{3.9}$$

$$\frac{1}{\rho}\dot{\gamma}-2\dot{\psi}\psi'=0\ . \tag{3.10}$$

Eq.(3.9) is a second order linear differential equation representing the conventional form of the cylindrical wave equation. Its solution contains two arbitrary constants: one relating to the ingoing cylindrical waves and the other to an outgoing one. Retaining only the outgoing with frequency ω and amplitude A, we get the following solution,

$$\psi(\rho,t)=A[J_0(x)\cos(\omega t)+N_0(x)\sin(\omega t)], \tag{3.11}$$

where $x=\omega\rho$, $J_0(x)$ and $N_0(x)$ are *Bessel* and *Neumann* functions of order zero respectively.

Addition of Eq.(3.7) and (3.8) then gives,

$$\gamma'=\rho(\dot{\psi}^2+\psi'^2). \tag{3.12}$$

Eq. (3.10) and (3.12) give the time and space derivatives of the arbitrary function $\gamma(\rho,t)$ in terms of the Bessel and the Neumann functions. To get the required solution, integrate with respect to time and space by using stranded formulae for the integrals of the Bessel's and Neumann's function. The final solution for γ is then [5],

$$\begin{aligned}\gamma(\rho,t) \;=\; & \frac{1}{2}A^2x\{J_0(x)J_0'(x)+N_0(x)N_0'(x)\\ & +x[J_0(x)^2+N_0(x)+J_0'(x)^2+N_0'(x)^2]\\ & +[J_0(x)J_0'(x)-N_0(x)N_0'(x)]\cos 2\omega t\\ & +[J_0(x)N_0'(x)+J_0'(x)N_0(x)]\sin 2\omega t\}\\ & -\frac{2}{\pi}A^2\omega t\ .\end{aligned} \tag{3.13}$$

Now " $'$ " means differentiation with respect to x. Therefore, Eq.(3.1) characterizes cylindrical gravitational waves with two arbitrary functions, of time t and cylindrical radial coordinate ρ.

3.2 Exact Solution of Plane Gravitational Waves

Bondi and Robinson [12] formulated the exact plane wave solution in 1957. They considered the spacetime which subsumes the symmetries of plane and constitute a wave travelling in the x-direction. The coefficient in the line element are functions of u, where $u = (ct - x)$. For convenience, we take unity in which $c = 1$. Therefore,

$$ds^2 = e^{2\alpha(u)}dt^2 - e^{2\alpha(u)}dx^2 - u^2e^{2\beta(u)}dy^2 - u^2e^{-2\beta(u)}dz^2. \tag{3.14}$$

It needs to be mentioned that (x, y, z) are rectangular coordinates in a curved spacetime. The metric tensor of the above equation is,

$$\mathrm{g}_{\mu\nu} = \begin{pmatrix} e^{2\alpha(u)} & 0 & 0 & 0 \\ 0 & -e^{2\alpha(u)} & 0 & 0 \\ 0 & 0 & -u^2e^{2\beta(u)} & 0 \\ 0 & 0 & 0 & -u^2e^{-2\beta(u)} \end{pmatrix}. \tag{3.15}$$

The inverse metric tensor is,

$$\mathrm{g}^{\mu\nu} = \begin{pmatrix} e^{-2\alpha(u)} & 0 & 0 & 0 \\ 0 & -e^{-2\alpha(u)} & 0 & 0 \\ 0 & 0 & -\frac{1}{u^2}e^{-2\beta(u)} & 0 \\ 0 & 0 & 0 & -\frac{1}{u^2}e^{2\beta(u)} \end{pmatrix}. \tag{3.16}$$

The non-zero Chirstoffel symbols are,

$$\left.\begin{array}{l} \Gamma^0_{00} = \Gamma^0_{11} = \Gamma^1_{01} = \Gamma^1_{10} = \alpha'(u), \\ \Gamma^0_{01} = \Gamma^0_{10} = \Gamma^1_{00} = \Gamma^1_{11} = -\alpha'(u), \\ \Gamma^0_{22} = \Gamma^1_{22} = u(u\beta'(u) + 1)e^{2(\beta(u)-\alpha(u))}, \\ \Gamma^0_{33} = \Gamma^1_{33} = u(-u\beta'(u) + 1)e^{-2(\beta(u)+\alpha(u))}, \\ \Gamma^2_{02} = \Gamma^2_{20} = -\Gamma^2_{12} = \Gamma^2_{21} = (\beta'(u) - \dfrac{1}{u}), \\ \Gamma^3_{03} = \Gamma^3_{30} = \Gamma^3_{13} = \Gamma^3_{31} = (-\beta'(u) + \dfrac{1}{u}), \end{array}\right\} \tag{3.17}$$

where " $'$ " indicates differentiation with respect to u. For Eq. (3.13) we have,

$$\left.\begin{array}{l} (\ln\sqrt{|g|}) = 2\ln u + 2\alpha(u), \\ (\ln\sqrt{|g|})' = 2(\alpha'(u) + \dfrac{1}{u}), \\ (\ln\sqrt{|g|})'' = 2(\alpha''(u) + \dfrac{1}{u}). \end{array}\right\} \tag{3.18}$$

The non-trivially zero components of the Ricci tensor are as before for the cylindrical gravitational waves. The vacuum Einstein equations become, Hence,

$$R_{00} = 4\frac{\alpha'(u)}{u} - 2\beta'(u)^2 = 0 \ , \tag{3.19}$$

$$R_{01} = -4\frac{\alpha'(u)}{u} + 2\beta'(u)^2 = 0 \ , \tag{3.20}$$

$$R_{11} = 4\frac{\alpha'(u)}{u} - 2\beta'(u)^2 = 0 \ . \tag{3.21}$$

Eqs.(3.19), (3.20) and (3.21) leads us to the condition for the non-trivial exact solution of plane gravitational waves, which is;

$$\alpha'(u) = \frac{1}{2}\beta'(u)^2 \ . \tag{3.22}$$

Thus any $\alpha(u)$ and $\beta(u)$ are considered to be an exact plane gravitational wave solution provided that they must satisfy above equation. Moreover, the Eq. (3.14) describes linearly polarized plane gravitational wave. The more general form of it representing circularly polarized plane gravitational wave is,

$$\begin{aligned} ds^2 &= e^{2\alpha}(dt^2 - dx^2) - u^2[(dy^2 + dz^2)\cosh 2\beta \\ &\quad +(dy^2 - dz^2)\sinh 2\beta \cos 2\theta - 2\sinh 2\beta \sin 2\theta dydz], \end{aligned} \tag{3.23}$$

where α, β and θ are arbitrary functions of u. The condition for the exact solution in vacuum in that case is turn out to be,

$$2\alpha'(u) = u[\beta'(u)^2 + \theta'(u)^2 \sinh^2 2\beta(u)]. \tag{3.24}$$

One can easily check that, if $\theta = 0$ then Eq. (3.23) and (3.24) reduce to Eq. (3.14) and (3.22) respectively.

3.3 Colliding Impulsive Plane Gravitational waves

3.3.1 Introduction

The colliding plane wave solutions have great significance in classical GR. They are exact solutions illustrating the mutual scattering of plane waves in a flat background. The first exact solution of Einstein's field equations describing the collision of plane impulsive gravitational waves was presented by Khan and Penrose [13] in 1971. Penrose has introduced a topological procedure of construction and description of impulsive gravitational waves. The method is based on a very deep understanding of geometrical properties of null hypersurface and null geodesics. In principle it introduces impulses in any arbitrary spacetime by "cutting" it, and "rejoining" the two halves in an appropriate way. This is called Penrose "scissor and paste" approach. Such an approach may look formal at first sight but, it is indeed a powerful tool to construct the geometry of spacetime even without knowing the explicit form of the metric tensor. In addition, it is also interesting from a physical prospective, i.e. in the study of geodesic motion and collisions of gravitational waves.

Penrose's [25] approach to cut the Minkowski spacetime at a null cone and rejoin it with a jump yields a delta-function gravitational wave. The "strength" of the wave is given by the "size" of the jump. Subsequently Khan and Penrose wrote down a solution for colliding plane impulsive gravitational waves of equal (unit) strength [13]. They showed that two plane impulsive (waves having delta function profile) gravitational waves approaching each other initially from flat backgrounds and then colliding results in an exact solutions of Einstein's equations. As such it is necessary to discuss colliding plane wave solutions in detail in the next section.

3.3.2 The First Impulsive Plane Wave

To construct the single impulsive plane wave we consider Brinkman and Robinson [14] line element for linearly polarized plane waves,

$$ds^2 = 2dUdV + (Y^2 - X^2)h(U)dU^2 - dX^2 - dY^2, \tag{3.25}$$

where $h(U)$ is an arbitrary amplitude function and $U = t - z$, $V = t + z$, i.e. retarded and advanced times . Two such waves travelling in the same direction do not collide. They can be superposed by adding their amplitude functions.

To exhibit collision, we need waves moving in opposite directions. Therefore, the Rosen [15] form is more appropriate than Eq. (3.25). In order to covert line element (3.25) into Rosen form put,

$$\begin{aligned} U &= u, \\ V &= v + \frac{1}{2}x^2FF' + \frac{1}{2}y^2GG', \\ X &= xF, \\ Y &= yG, \end{aligned} \tag{3.26}$$

where F and G are functions of u subject to $F'' = hF$, $G'' = -hG$. We get the Rosen form,

$$ds^2 = 2Ldudv - F^2dx^2 - G^2dy^2, \tag{3.27}$$

where $L = 1$. If we take $F = 1 + u\Theta(u)$, $G = 1 - u\Theta(u)$, where $\Theta(u)$ is the Heaviside's step function defined as,

$$\Theta(u) = \begin{cases} 0 & \text{if } u < 0, \\ 1 & \text{if } u \geq 0. \end{cases} \tag{3.28}$$

Eq.(3.27) takes the form,

$$ds^2 = 2dudv - [1 + u\Theta(u)]^2dx^2 - [1 - u\Theta(u)]^2dy^2. \tag{3.29}$$

The above equation represents the "impulsive" plane gravitational wave, which occurs at the null hypersurface $u = 0$ and characterized by $\delta(u)$ and $\delta(u)$ is the Dirac-delta function defined as,

$$\delta(u) = \begin{cases} 0 & \text{if } u \neq 0, \\ \infty & \text{if } u = 0. \end{cases} \tag{3.30}$$

Note that the derivative of the Heaviside's step function is the Dirac-delta function,

$$d\Theta/du = \delta(u). \tag{3.31}$$

3.3.3 The Second Impulsive Plane Wave Prior to the Collision

Even though, the metric in this case is distribution valued, but eventually give rise to a geometrically acceptable spacetime [16]. Since Eq. (3.29) represents only one wave as it is approaching the other, it is only applicable in the region $v < 0$. Similarly, we can define the opposing wave prior to the collision, replacing u by v in Eq. (3.29) is given by

$$ds^2 = 2dudv - [1 + v\Theta(v)]^2 dx^2 - [1 - v\Theta(v)]^2 dy^2. \tag{3.32}$$

This occurs at null hypersurface $v = 0$ and characterized by $\delta(v)$. The pertinent region for the opposing wave is, $u < 0$.

3.3.4 The Interaction of Both Waves

In order to collide these waves the initial circumstances has been defined now. At this stage it is appropriate to divide the spacetime into four regions. This is shown in Fig. 3.1.

Region I: The region where $v < 0$ and $u < 0$, i.e before either wave arrives at the given place. This region is the flat background with the metric.

$$ds^2 = 2dudv - dx^2 - dy^2 \ . \tag{3.33}$$

Region II: In this region $u > 0$ and $v < 0$, i.e when one has passed while the other still has not arrived. The line element for this region is,

$$ds^2 = 2dudv - (1 + u)^2 dx^2 - (1 - u)^2 dy^2 \ . \tag{3.34}$$

Region III: The region where $v > 0$ and $u < 0$, i.e when the other has passed the first one has not yet arrived. The metric in this case is,

$$ds^2 = 2dudv - (1 + v)^2 dx^2 - (1 - v)^2 dy^2 \ . \tag{3.35}$$

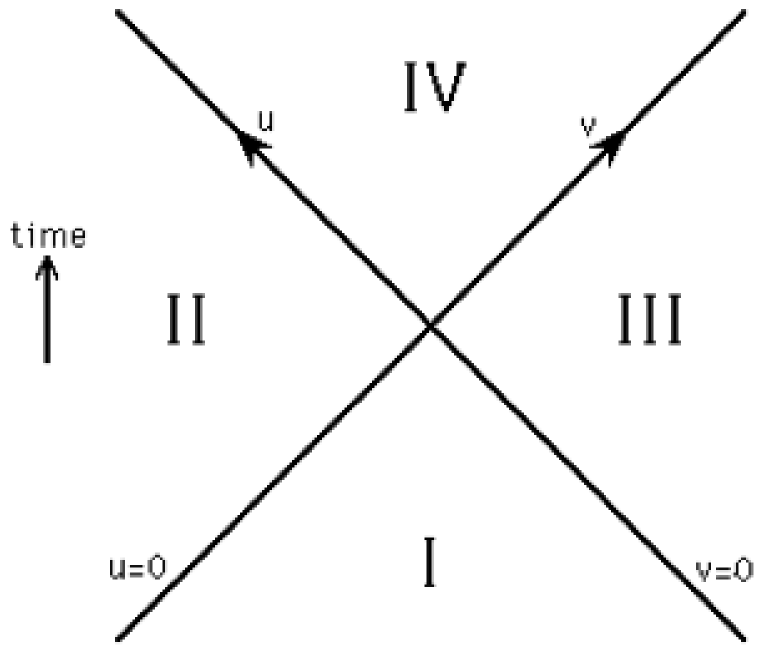


Figure 3.1: Dividing spacetime into four regions to study colliding waves (taken from [16]).

Region IV: This region is the subsequent interaction region, where $u > 0$ and $v > 0$. In this region there exist the exact solution of Einstein's equation under the boundary conditions, $v = 0, \quad u > 0$, and $u = 0, \quad v > 0$.

Before presenting the solution for the *Region IV*, it is necessary to mention that Eqs. (3.34) and (3.35) for *Region II* and *Region III* become singular at null hypersurfaces $u = 1$ and $v = 1$ respectively. Such singularities do not arise in the metric (3.25). This may appear strange, as the disturbance is a shock wave and leaves a flat spacetime after passing it. Indeed, they can be removed by transforming coordinates [16]. Since there is no curvature singularity in *Region II* and *Region III* thus there remains a topological singularity .

3.3.5 The Interaction Region Solution

If we allow L, F and G to be function of both u and v in the Eq. (3.27), we will get the explicit solution of Einstein's equation [13] of the form,

$$ds^2 = \frac{2t^3}{rw(pq+rw)^2}dudv - t^2\left(\frac{r+q}{r-q}\right)\left(\frac{w+p}{w-p}\right)dx^2 - t^2\left(\frac{r-q}{r+q}\right)\left(\frac{w-p}{w+p}\right)dy^2 \ , \tag{3.36}$$

where

$$p = u\Theta(u) \ , \ q = u\Theta(u) \ , \ r = \sqrt{1-p^2} \ , \ w = \sqrt{1-q^2} \ , \ t = \sqrt{1-p^2-q^2} \ . \tag{3.37}$$

Since the delta function is a generalized function, therefore the spacetime in this case is distribution valued. If we define the delta function as,

$$< \delta, \phi >= \int_{-b}^{a} \phi(x)\delta(x)dx = \phi(0), \qquad \text{for} \ \ a, b > 0. \tag{3.38}$$

where $\phi(x)$ is any arbitrary function. Similarly, we can define the Heaviside's step function as,

$$< \Theta, \phi >= \int_{-b}^{a} \phi(x)\Theta(x)dx = \int_{0}^{a} \phi(x)dx, \quad \text{for} \ \ a, b > 0 \ . \tag{3.39}$$

In the sense of functions neither Θ nor δ are acceptable and the manifold is $\mathcal{C}^{-1}$ on account of the discontinuous metric. Further the curvature is singular and in that sense is $\mathcal{C}^{-2}$. However, in the sense of generalized functions, or distributions, it is $\mathcal{C}^0$ (*or* $\mathcal{C}^1$) and piecewise $\mathcal{C}^\infty$ in the relevant regions.

Moreover, the interaction region is curved with a curvature singularity that occurs at $u^2 + v^2 = 1$. The region of validity of Eq. (3.36) is the region, $v < 1$, $u^2 + v^2 = 1$ and $u < 1$ as indicated in the Fig.3.2. Infact, this is the union of three regions i.e ($v < 1$, $u < 0$) ($u < 1$, $v < 0$) and $u^2+v^2 < 1$. In Fig.3.2, the world lines of two particles A and B are depicted. We see that particle B crosses the wave-front $u = 0$ after some time it encounters the line $u^2 + v^2 = 1$, before encountering the second wave at $v = 0$. Thus particle B eludes the curvature singularity in Region IV. For particle B, the singularity, which comes across in Region II, is merely a coordinate singularity. Particle A, meantime, has a different doom. The world line of this particle demonstrates that it confronts both wave-fronts before encountering the singularity in Region IV. Unfortunately Particle A encounters a real curvature singularity in Region IV.

From the above discussion, we conclude that two plane impulsive gravitational waves approaching each other will surely collide. The collision of such waves results in the exact solutions of Einstein's equation. In addition, the resulting line element represent the most general spacetime describing a collision between two parallel polarized impulsive gravitational waves.

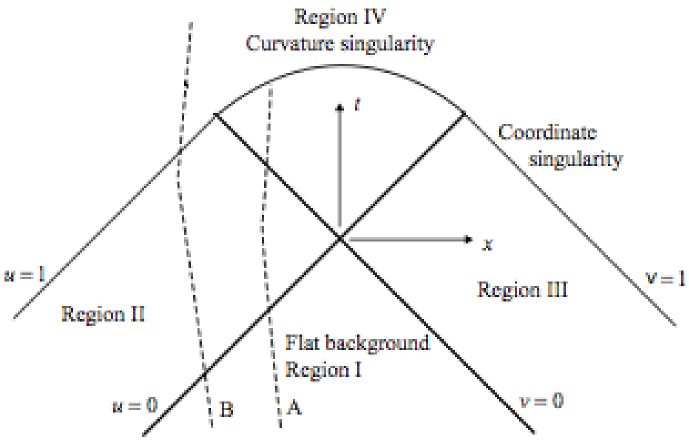


Figure 3.2: The collision of two impulsive plane gravitational waves. Focusing effects induce a curvature singularity which is inevitable for Particle A, which crosses both wave-fronts (taken from [16]).

3.4 The Reality of Gravitational Waves

The interpretation of energy has always been a significant dilemma in GR. Unlike Newtonian gravity, energy is not a well defined concept in GR. The Hamiltonian in the Poisson bracket behaves as a time derivative for the conserved system; which leads to the conservation of energy. Thus it is obvious that, for energy to be conserved in GR, the metric should have a time-like isometry (Killing vector) to allow time translation invariance [5]. This cannot apply for time varying spacetimes and thus not well defined.

There have been various strenuous efforts to attain a well define expression [17, 18, 19] for local or quasi-local energy and momentum in GR. The problem for gravitational waves was originally addressed by Weber and Wheeler [20]. They provided an approximate expression for momentum imparted to test particles by cylindrical gravitational waves by considering test particles in the path of the putative gravitational waves. If their geodesic equations show some deviation from straight line paths in spacetime, there will effectively be energy in the waves. To avoid the possibility of the linearization approximation itself providing the energy, they took a solution of the field equations obtained by Einstein and Rosen [11] representing cylindrical gravitational waves. The result was that momentum would be imparted to test particles and hence the gravitational waves effectively carried energy. They obtained a first order approximation to the momentum imparted and checked, by estimating the second order correction, that the result remained robust.

The following analysis is a summary of the paper by Weber and Wheeler. The linearity of Eq. (3.9) allows to apply the Fourier transform. Therefore the Fourier transform of ψ can be chosen as,

$$\psi(\omega) = 2Ce^{-a\omega} J_0(\omega\rho). \tag{3.40}$$

The integration required to construct $\psi(t)$ and $\gamma(t)$ can be carried out as,

$$\begin{aligned}\psi &= -2C \int_0^\infty e^{-a\omega} J_0(\omega\rho) \cos \omega T d\omega, \\ &= C[\rho^2 + (a - iT)^2]^{-1/2} + C[\rho^2 + (a + iT)^2]^{-1/2},\end{aligned} \tag{3.41}$$

where T is the product of the velocity of light. Integration of Eqs. (3.10) and (3.12) gives,

$$\begin{aligned}\gamma = \frac{1}{2}C^2\{a^{-2} - \rho^2[(a - iT)^2 + \rho^2]^{-2} - \rho^2[(a - iT)^2 + \rho^2]^{-2} \\ - a^{-2}(-\rho^2 + a^2 + T^2)[T^4 + 2T^2(a^2 - \rho^2) + (a^2 + \rho^2)]^{-1/2}\}.\end{aligned} \tag{3.42}$$

Since the components of the Riemann tensor for the metric (3.1) do not all vanish. Therefore these waves must be real and do carry energy [21]. In order to analyse the motion of test particle initially at rest they write the geodesic equation for ρ. And then integrate it in successive orders, keeping ψ as a first order and γ as a second order quantity. Subsequently addition of both leads to the required result. In a coordinate system (3.1), a test particle which is instantaneously at rest experiences at that moment the acceleration would be,

$$\begin{aligned}\frac{d\rho}{dT^2} &= -\frac{\partial\gamma}{\partial\rho} + \frac{\partial\psi}{\partial\rho} \\ &= C^2\rho\Big\{\frac{\rho^2 - (a - iT)^2}{[\rho^2 + (a - iT)^2]^3} + \frac{\rho^2 - (a + iT)^2}{[\rho^2 + (a - iT)^2]^3} \\ &+ 2\frac{\rho^2 + a^2 + T^2}{[\rho^2 + (a - iT)^2]^{3/2}[\rho^2 + (a + iT)^2]^{3/2}}\Big\} \\ &+ C\rho\{[\rho^2 + (a - iT)^2]^{-3/2} + [\rho^2 + (a + iT)^2]^{-3/2}\}.\end{aligned} \tag{3.43}$$

When it is multiplied by the mass of the test particle it gives the force. Integration of the force gives the corresponding momentum imparted to the test particle.

Later, plane waves obtained by Bondi, Pirani and Robinson [12] were used by Ehlers and Kundt [22] to show that they also imparted momentum and that a sphere of test particles in the path of the waves would be distorted in different directions as they moved outwards, pushed by the waves. The gravitational waves were real.

Another proposal is that we expand the Ricci tensor in powers of h, during linearizing the Einstein field equations, ensure the meaning to energy in the gravitational waves. The left hand side of the equation contains all linear terms while inverting all non-linear terms on the right side. These non-linear terms become an effective stress-energy tensor or "pseudo-tensor". The linearized Einstein equations give gravitational waves [5].

The method for proving the reality of gravitational waves given by Weber and Wheeler is lengthy and cumbersome and gives only approximate results. However Qadir and Sharif [23] gave an operational procedure called the extended pseudo-Newtonian ($e\psi N$) formalism, which yielded a general closed formula whose second order approximation is the Weber-Wheeler result. In the next section, we will discuss the $e\psi N$-formalism in detail.

3.5 The Pseudo-Newtonian Formalism

Since GR describes the Newtonian gravitational force as the geometry of the spacetime. Classically, the straight path is bent by the force, but relativistically the spacetime geometry gets curved and consequently the path bent. Einstein's equation describes the curvature of the spacetime. The relativistic analogue of the Newtonian gravitational force is called the ψN-force and the gravitational potential, the ψN potential. The ψN (pseudo-Newtonian) formalism was constructed for the spacetime having a time-like isometry (killing vector) [23, 24].

The foundation of the ψN formalism is the analysis that the gravitational force is not detectable in a freely falling frame (FFF) at a specific point. It can be detected over a finite spatial extent as the tidal force. It could be measured by an accelerometer having a spring of length l attached to two masses. One end of the spring is joined with a needle which can move on the dial of the accelerometer, to measure the tension in the spring. Therefore, in a FFF, the observer can detect the tension of the spring by observing the movement of the needle on the dial of the accelerometer. In Newtonian gravity, the needle will lie on the zero position of the accelerometer in the absence of a central force. The spring is stretched by the tidal force. Attraction leads to stretching of the spring and repulsion, then, leads to squashing. Thus attraction and repulsion can be measured and ascertain by this accelerometer.

The mathematical expression for the tidal force on two test particle two of mass m is given by,

$$F_T^\mu \;=\; mR^\mu{}_{\nu\rho\pi}t^\mu l^\rho t^\pi, \quad \mu,\nu = 0,1,2,3 \tag{3.44}$$

where $R^\mu{}_{\nu\rho\pi}$ is the Riemann curvature tensor, $\boldsymbol{t}$ is a time-like vector with components in the rest frame $t^\mu = \delta^\mu_0/\sqrt{g_{00}}$, and $\boldsymbol{l}$ is the space-like vector (separation vector). Showing the accelerometer is in the rest frame. In its rest frame the tidal force on a test particle of mass m is given by,

$$F^\mu_T \;=\; mR^\mu{}_{0\nu0}l^\nu \;. \tag{3.45}$$

The above equation can be simplified further by using Riemann normal coordinates (RNCs) [24]. Therefore, the ψN-force F^i is given by "$m\Gamma^i_{00}$" up to an integration constant. The term "$m\Gamma^i_{00}$" is fixed with constrain that there will be no ψN-force in the Minkowski space. Thus, the ψN-force is now given by ,

$$F^i \;=\; m\Gamma^i_{00} \;. \tag{3.46}$$

This force can be written as the gradient of a scalar ψN potential V with constrain that the potential also be zero in the Minkowski space given by,

$$F_i \;= -V_i = \;\frac{1}{2}m(1-g_{00})_{,i} \;\;. \tag{3.47}$$

Since, force and energy are interconnected and energy will be conserved if there is a time-like isometry. This constraint does not permit any time dependence of the ψN-force. Hence the definition of energy for non-static case is still an enigma. To discuss the relativistic force for the non-static case, we lay off any stipulation of time-like killing vectors. We replace the the time-like Killing vector by the unit vector along the geodesic. Therefore, the local Lorentz factor is eliminated which arises due to the time-like killing vector was not a unit vector. In addition, we require the synchronous coordinates essentially with one reduced. The significance of this frame is that $g_{0i} = 0$ in the coordinate basis.

Moreover, we use the Riemann normal coordinates because both ends of the accelerometer are spatially free. Nevertheless, there is a initial time for which the dial of the accelerometer is fixed at zero position. The ψN-force, F^μ, satisfies ,

$$F^{*\mu}_T = l^\mu F^\mu{}_{;\nu} \;. \tag{3.48}$$

Note that $F^{*\mu}_T$ does not mean $F^0 = 0$. In order to satisfy the Eq.(3.41), we can write,

$$\begin{aligned} l^i(F^0{}_{,i} + \Gamma^0{}_{ij}F^i) &= 0 \\ l^j(F^i{}_{,j} + \Gamma^i{}_{0j}F^0) &= F^{*i}_T \end{aligned} \tag{3.49}$$

.

A simultaneous solution of the above equations is,

$$\begin{aligned} F^0 &= m[(ln(A))_{,0} - \Gamma^0_{00} + \frac{\Gamma^i_{0j}\Gamma^i_{0i}}{A}]f^2 \;, \\ F^i &= m\Gamma^i_{00}f^2 \;\;, \end{aligned} \tag{3.50}$$

where $A = (ln\sqrt{-g})_{,0}$, $g = det(g_{ij})$ and $f = 1/\sqrt{g_{00}}$. To further simplify the above expression we can write the Christofell symbols as, $\Gamma^0_{00} = \frac{1}{2}g^{00}g_{00,0}$, $\Gamma^i_{00} = -\frac{1}{2}g^{ij}g_{00,j}$ and $\Gamma^i_{0j} = \frac{1}{2}g^{ik}g_{jk,0}$. Therefore, the ψN-force in covariant form is,

$$\begin{aligned} F_0 &= m[(lnAf)_{,0} + \frac{g^{ij}_{,0}g_{ij,0}}{4A}], \\ F_i &= m(\log f)_{,0} \quad , \qquad i = 1, 2, 3 \end{aligned} \tag{3.51}$$

where $A = (ln\sqrt{-g})_{,0}$, $g = det(g_{ij})$ and $f = 1/\sqrt{g_{00}}$.

From the whole discussion, we deduce that spatial components of the force 4-vector give the change of potential energy of test particle of mass m .

To illustrate the significance of this formalism, consider its application to cylindrical gravitational waves [23]. The eψN-force for line element (3.1) is given by [23],

$$\begin{aligned} F_0 = &-m\{\omega[AJ_0(x)\cos(\omega t) + BY_0(x)\sin(\omega t)] - 2\rho\omega[(A^2J_0(x)J_0'(x) \\ &- B^2Y_0(x)Y_0'(x))\cos(2\omega t) - AB(J_0(x)Y_0'(x) + Y_0(x)J_0'(x))\sin(2\omega t)] \\ &+ 2[AJ_0(x)\sin(\omega t) - BY_0(x)\cos(\omega t)]^2/AJ_0(x)\sin(\omega t) - BY_0(x)\cos(\omega t) \\ &+ 2\omega\rho[AJ_0(x)\sin(\omega t) - BY_0(x)\cos(\omega t)][AJ_0'(x)\cos(\omega t) + BY_0'(x)\sin(\omega t)]\} \quad , \end{aligned} \tag{3.52}$$

$$\begin{aligned} F_1 = &\, m\{AJ_0'(x)\cos(\omega t) + BY_0'(x)\sin(\omega t) - \frac{1}{2}[(A^2J_0(x)J_0' - B^2Y_0(x)Y_0'(x)) \\ &+ \omega\rho((A^2J_0(x)J_0'(X) - B^2Y_0(x)Y_0'(x))'] \cos(2\omega t) \\ &- \frac{1}{2}AB[2(J_0(x)Y_0'(x) + Y_0(x)J_0'(x)) - \omega\rho(J_0(x)Y_0'(x) + Y_0(x)J_0'(x))'] \sin(\omega t) \\ &- \frac{1}{2}AB[4(J_0(x)Y_0'(x) - Y_0(x)J_0'(x)) + 2\omega\rho(J_0(x)Y_0'(x) - Y_0(x)J_0'(x))']\omega t\} \quad , \end{aligned} \tag{3.53}$$

where $x = \omega\rho$, and ω is the angular frequency, $J_0(x)$ and $Y_0(x)$ are the *Bessel* and the *Neumann* functions of order zero respectively. Here " $'$ " represents differentiation with respect to x.

The corresponding momentum imparted to the test particle is given by [19],

$$\begin{aligned} P_0 = &-m[\ln \mid AJ_0(x)\sin(\omega t) - BY_0(x)\cos(\omega t) \mid + ln \mid 1 - 2\omega\rho[AJ_0'(x)\cos(\omega t) \\ &+ BY_0(x)\sin(\omega t)] \mid (1 + \frac{A^2J_0(x)(J_0'(x) + B^2Y_0(x)Y_0'(x)}{\omega\rho(AJ_0'^2(x) + BY_0'^2(x))}) \\ &- \frac{AB(AJ_0(x)Y_0(x)Y_0'(x) + Y_0(x)J_0'(x) - A\rho\omega J_0(x)J_0'(x)Y_0'(x) - J_0(x)Y_0'(x))}{\omega\rho(AJ_0'^2(x) + BY_0'^2(x))\sqrt{1 - 4\rho^2\omega^2(A^2J_0'^2(x) + b^2Y_0'^2(x))})} \\ &\times \arctan \mid \frac{(1 + 2A\omega\rho j_0'(x))\tan(\frac{1}{2}\omega t) - 2B\omega\rho Y_0'(x)}{\sqrt{1 - 4\rho^2\omega^2(A^2J_0'^2(x) + b^2Y_0'^2(x))}} \mid \\ &+ \frac{AB(J_0'(x)Y_0(x) - Y_0'(x)J_0(x))}{\rho(A^2j_0'^2(x) + B^2Y_0'^2(x))} + f_1(\omega\rho)], \end{aligned} \tag{3.54}$$

$$
\begin{aligned}
P_1 \ &= \frac{m}{\omega}\{[AJ_0'(x)\sin(\omega t) - BY_0'(x)\cos(\omega t)] - \frac{1}{4}[(A^2 J_0(x)J_0' - B^2 Y_0(x)Y_0'(x)) \\
&+ \omega\rho((A^2 J_0(x)J_0'(X) - B^2 Y_0(x)Y_0'(x))'] \sin(2\omega t) - \frac{1}{2}AB[(J_0(x)Y_0'(x) \\
&+ Y_0(x)J_0'(x)) + \rho\omega(J_0(x)Y_0'(x) + Y_0(x)J_0'(x))'] \cos(2\omega t) \\
&- AB\omega^2 t^2[((J_0(x)Y_0'(x) - Y_0(x)J_0'(x)) - \rho\omega((J_0(x)Y_0'(x) \\
&- Y_0(x)J_0'(x))] + f_2(\omega\rho)\},
\end{aligned} \tag{3.55}
$$

where f_1 and f_2 are the "constants of integration". Weber and Wheeler [20, 21] exclude solutions that contain the irregular Bessel function, $Y_0(\omega\rho)$ as not well defined at the origin. Taking the Weber-Wheeler solution, eqs.(3.54) and (3.55) reduce to

$$
P_0 = -m[\ln \mid AJ_0 \sin(\omega t) + (1 + AK_0/\omega\rho J_0')ln \mid 1 - 2\omega\rho[AJ_0' \cos(\omega t)| + f_1(\omega\rho)], \tag{3.56}
$$

$$
P_1 = \frac{m}{\omega}\{[AJ_0' \sin(\omega t)] - \frac{1}{4}[(A^2 J_0 J_0' + \omega\rho((A^2 J_0 J_0'] \sin(2\omega t) + f_2(\omega\rho)\}. \tag{3.57}
$$

Chapter 4

Momentum Imparted to Test Particles in the Khan-Penrose Spacetime

In GR, the non-linearity of the field equations makes it difficult to describe the structure of spacetime after the collision of gravitational waves. Khan and Penrose obtained an exact solution for colliding plane impulsive gravitational waves with the remarkable feature that the spacetime before the collision is flat and after the collision is not only curved, but develops a future curvature singularity. The Khan-Penrose (KP) metric (3.36) is given in double null coordinates, i.e. retarded time, $u = (t - z)/T$ and advanced time, $v = (t + z)/T$, where T is a constant time. The curvature singularity in Region IV occurs at T, where the KP Universe ends. As such we call T *doomsday*. Physically, doomsday comes with no warning in Region II and III but with a warning in Region IV.

We probe the curvature by considering the momentum imparted to test particles by the colliding gravitational waves so as to try to understand how the singularity develops in Region IV. One could try to look at the geodesic equations using the Newman-Penrose formalism. However, we use the $e\psi N$ formalism whose second order approximation is Weber and Wheeler's result of proving the reality of gravitational waves [20] by looking at the momentum imparted to test particles. This would give deeper physical insights and be simpler to obtain. In view of its success there it seems worth applying in the non-trivial situation of *two* colliding impulsive plane gravitational waves to probe the "doomsday" singularity in Region IV. We will compute the momentum imparted to test particles by the colliding waves, after both have passed by.

4.1 Force Imparted to Test Particles by Khan-Penrose Waves

The KP metric is a vacuum solution[1] given in double null coordinates by

$$ds^2 = \frac{2t^3}{rw(pq+rw)^2}dudv - t^2\left(\frac{r+q}{r-q}\right)\left(\frac{w+p}{w-p}\right)dx^2 - t^2\left(\frac{r-q}{r+q}\right)\left(\frac{w-p}{w+p}\right)dy^2 \ , \tag{4.1}$$

where

$$p = u\theta(u) \ , \ q = v\theta(v) \ , \ r = \sqrt{1-p^2} \ , \ w = \sqrt{1-q^2} \ , \ t = \sqrt{1-p^2-q^2} \ . \tag{4.2}$$

In region IV, $u, v > 0$ and $u^2+v^2 < 1$ its Weyl tensor, representing the gravitational field, has ten linearly independent components given in Appendix A.

$$\begin{aligned} ds^2 &= 2\frac{(1-u^2-v^2)^{3/2}}{\sqrt{1-u^2}\sqrt{1-v^2}(uv+\sqrt{1-u^2}\sqrt{1-v^2})^2}dudv \\ &-(1-u^2-v^2)\left(\frac{\sqrt{1-u^2}+v}{\sqrt{1-u^2}-v}\right)\left(\frac{\sqrt{1-v^2}+u}{\sqrt{1-v^2}-u}\right)dx^2 \\ &-(1-u^2-v^2)\left(\frac{\sqrt{1-u^2}-v}{\sqrt{1-u^2}+v}\right)\left(\frac{\sqrt{1-v^2}-u}{\sqrt{1-v^2}+u}\right)dy^2 \ . \end{aligned} \tag{4.3}$$

The $e\psi N$-formalism needs the metric in block diagonal form. This is easily provided by converting to $t = (u+v)/\sqrt{2}$, $z = (u-v)/\sqrt{2}$ to give

$$\begin{aligned} ds^2 &= 8\frac{(1-t^2-z^2)^{3/2}}{\sqrt{2-(t-z)^2}\sqrt{2-(t+z)^2}(t^2-z^2+\sqrt{2-(t-z)^2}\sqrt{2-(t+z)^2})}dt^2 \\ &-(1-t^2-z^2)\left(\frac{\sqrt{2-(t-z)^2}+(t+z)}{\sqrt{2-(t-z)^2}-(t+z)}\right)\left(\frac{\sqrt{2-(t+z)^2}+(t-z)}{\sqrt{2-(t+z)^2}-(t-z)}\right)dx^2 \\ &-(1-t^2-z^2)\left(\frac{\sqrt{2-(t-z)^2}-(t+z)}{\sqrt{2-(t-z)^2}+(t+z)}\right)\left(\frac{\sqrt{2-(t+z)^2}-(t-z)}{\sqrt{2-(t+z)^2}+(t-z)}\right)dy^2 \\ &-8\frac{(1-t^2-z^2)^{3/2}}{\sqrt{2-(t-z)^2}\sqrt{2-(t+z)^2}(t^2-z^2+\sqrt{2-(t-z)^2}\sqrt{2-(t+z)^2})}dz^2 \ . \end{aligned} \tag{4.4}$$

In these coordinates the singularity now occurs at $t^2+z^2 = 1$ and the domain is $-1 < z < 1$, $0 < t < 1$, or equivalently $|z| < t$, $t^2+z^2 < 1$. This is shown in Fig 4.1.

[1] It may be remarked that the metric is given incorrectly in [16], giving incorrect values of the curvature tensor.

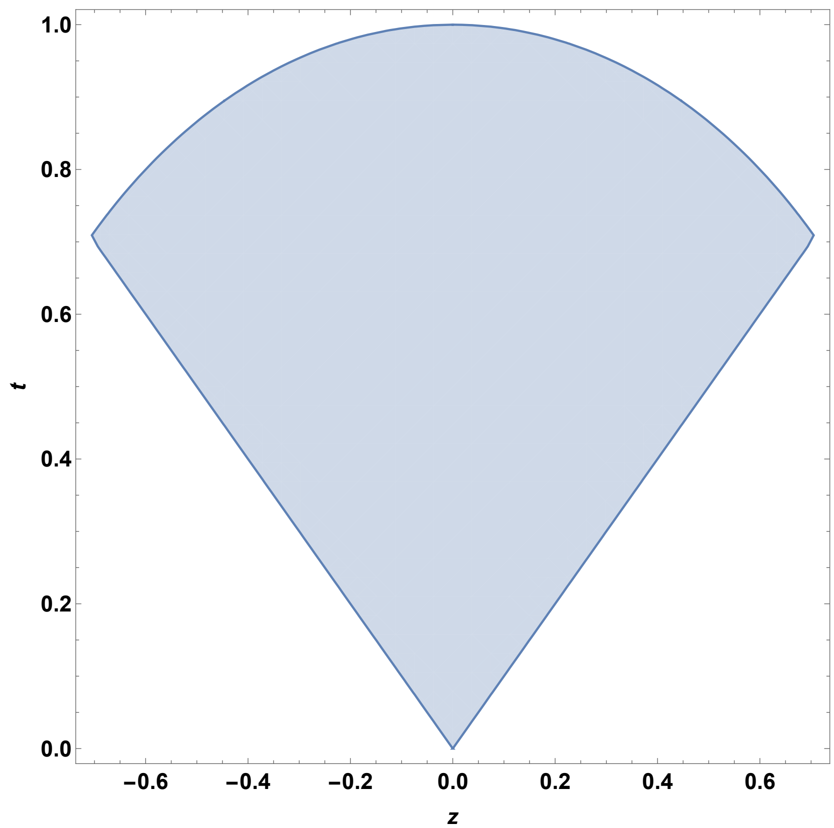


Figure 4.1: The Region of validity for the line element (4.4)

The time component does not convey much wisdom and is given in Appendix B. From Eq.(3.51) the eψN-force for this metric is,

$$F_z = \frac{z\left(3t^2+z^2-2\right)\left(t^2+2\sqrt{2-(t-z)^2}\sqrt{2-(t+z)^2}-z^2\right)}{2\left((t-z)^2-2\right)\left(t^2+z^2-1\right)\left((t+z)^2-2\right)}. \tag{4.5}$$

The other spatial components of the force 4-vector are zero, i.e $F_x = F_y = 0$. The time component of the $e\psi N$-force gives the rate of change of energy of a test particle of mass m and the spatial components give the spatial rate of change of its potential energy.

We need to find the magnitude of the spatial part of the force 4-vector,

$$
\begin{aligned}
|F| \;=\; \sqrt{-F_z F_z g^{zz}} = \;& \Big\{ \sqrt[4]{2-(t-z)^2}\, z(3t^2+z^2-2)\sqrt[4]{2-(t+z)^2}(t^2-z^2 \\
& +\sqrt{2-(t-z)^2}\sqrt{2-(t+z)^2})(t^2-z^2 \\
& +2\sqrt{2-(t-z)^2}\sqrt{2-(t+z)^2}) \Big\} \\
& /\Big\{ 4\sqrt{2}(-t^2-z^2+1)^{7/4}\big((t^4-2(z^2+2)t^2 \\
& +(z^2-2)^2)\big) \Big\}. \qquad (4.6)
\end{aligned}
$$

4.2 Analysis of the Khan-Penrose Spacetime

To better visualise the behavior of the force near the singularity $t^2 + z^2 = 1$, we define the variables $r^2 = t^2 + z^2$ and $\theta = tan^{-1}(t/z)$. The wave front of one wave occurs at $\theta = \pi/4$ and of the other at $\theta = -\pi/4$ and the time domain is $0 < t < 1$. Obviously $|F|$ is a symmetric function of z or $\pi/2 - \theta$. As such, we only consider $\pi/4 < \theta \leq \pi/2$.

Taking $r = 0.989846$ and varying θ from $\pi/4$ to $\pi/2$, we find that F rapidly increases, reaching a maximum at $\theta \approx 1.22134$ and then decreases to zero. In Table 4.1 we give $F(\theta)$ in tabular form in arbitrary units. It is found that the same behaviour occurs for all constant r. In Figure 4.2, $F(\theta)$ is given graphically for $r = 0.989846$.

θ	$\lvert F(\theta)\rvert$	θ	$\lvert F(\theta)\rvert$
0.79972	1.747×10^0	1.17797	1.966×10^2
0.814097	2.304×10^0	1.19974	1.991×10^2
0.843697	1.364×10^1	1.22134	1.998×10^2
0.882315	3.528×10^1	1.24276	1.986×10^2
0.933585	6.951×10^1	1.34779	1.674×10^2
0.970789	9.586×10^1	1.39931	1.382×10^2
1.00707	1.208×10^2	1.50112	6.102×10^1
1.04254	1.433×10^2	1.55169	1.699×10^1
1.10009	1.732×10^2	1.570796325	5.8×10^{-5}
1.156	1.922×10^2	1.570796326	0

Table 4.1: $|F(\theta)|$ for $r = 0.989846$ is maximum at $\theta \approx 1.22134$.

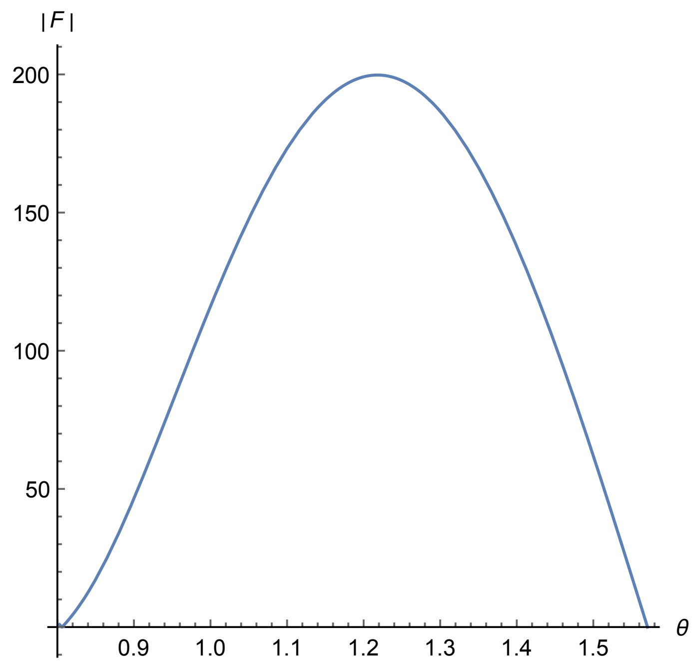


Figure 4.2: The $|F(\theta)|$ is maximum at $\theta = 1.22134$ for constant $r = 0.989846$

Now, fixing θ at the maximum value, we compute $F(r)$. It increases very slowly till r approaches 1 and then starts increasing very rapidly after $r = 0.9$. This function is given in tabular form in Table 4.2. For our convenience we use $\epsilon = 1 - r$, instead of r,

$\epsilon = 1 - r$	$\|F(\epsilon)\|$	$\epsilon = 1 - r$	$\|F(\epsilon)\|$
9.346×10^{-1}	2.24573×10^{-2}	2.337×10^{-2}	4.01874×10^{1}
8.762×10^{-1}	4.28428×10^{-2}	1.752×10^{-2}	7.09388×10^{1}
7.886×10^{-1}	7.46654×10^{-2}	1.46×10^{-2}	1.00772×10^{2}
5.841×10^{-1}	1.60318×10^{-1}	2.921×10^{-4}	1.10487×10^{5}
4.381×10^{-1}	2.36274×10^{-1}	2.921×10^{-5}	6.2305×10^{6}
3.505×10^{-1}	2.84141×10^{-1}	2.921×10^{-7}	1.9709×10^{10}
2.337×10^{-1}	2.9669×10^{-1}	2.92×10^{-13}	6.2332×10^{20}
1.168×10^{-1}	4.88637×10^{-1}	2.898×10^{-14}	3.4947×10^{22}
3.505×10^{-2}	1.72862×10^{1}	9.992×10^{-16}	2.02572×10^{27}

Table 4.2: $|F(\epsilon)|$ for $\theta = 1.22134$

Notice that F is zero at $z = 0$ or $\theta = \pi/2$ as it should be since the momentum imparted by the two waves would cancel out.

Now keeping F constant, we construct $\epsilon(\theta)$. Table 4.3. gives it for $F = 2.90957521$. The maximum of ϵ is 7.341×10^{-2} occurs at $\theta = 1.26516$. For

$F = 7.381660$ the function is given in Table 4.4. Here the maximum value is $\epsilon = 5.111 \times 10^{-2}$ at $\theta = 1.25465$. These curves are plotted in Figure 4.3, in terms of $r = 1 - \epsilon$. Similarly we can construct $\epsilon(\theta)$ for $F = 3.3780842$ and $F = 4.1597548$. The maximum of ϵ occurs at $\theta = 1.25886$ and $\theta = 1.25664$ respectively. Finally, we provide a $3 - d$ plot of r, θ, F. The spacetime structure after the collision of *KP-waves* can be seen graphically in Figure 4.6.

θ	$\epsilon = 1 - r$	θ	$\epsilon = 1 - r$
0.79972	5.255×10^{-3}	1.25886	7.339×10^{-2}
0.814097	9.756×10^{-3}	1.26516	7.341×10^{-2}
0.858873	2.204×10^{-2}	1.26711	7.34×10^{-2}
0.939235	4.018×10^{-2}	1.27563	7.337×10^{-2}
1.01052	5.288×10^{-2}	1.28085	7.332×10^{-2}
1.0698	6.116×10^{-2}	1.35641	7.029×10^{-2}
1.13636	6.798×10^{-2}	1.40732	6.543×10^{-2}
1.21333	7.25×10^{-2}	1.49471	4.915×10^{-2}
1.23248	7.304×10^{-2}	1.56566	1.296×10^{-2}
1.25465	7.337×10^{-2}	1.5708	9.304×10^{-7}

Table 4.3: The function $\epsilon(\theta)$ for $F = 2.90957521$. The maximum occurs at $\theta = 1.26516$.

θ	$\epsilon = 1 - r$	θ	$\epsilon = 1 - r$
0.79972	4.04×10^{-3}	1.25465	5.111×10^{-2}
0.814097	7.303×10^{-3}	1.25664	5.11×10^{-2}
0.858873	1.609×10^{-2}	1.25886	5.109×10^{-2}
0.890055	2.147×10^{-2}	1.26711	5.102×10^{-2}
0.939235	2.898×10^{-2}	1.27563	5.092×10^{-2}
1.01052	3.791×10^{-2}	1.35641	4.793×10^{-2}
1.0698	4.361×10^{-2}	1.40732	4.397×10^{-2}
1.13636	4.814×10^{-2}	1.49471	3.183×10^{-2}
1.21333	5.084×10^{-2}	1.56566	7.81×10^{-3}
1.23248	5.107×10^{-2}	1.5708	5.465×10^{-7}

Table 4.4: The function $\epsilon(\theta)$ for $F = 7.381660$. The maximum occurs at $\theta = 1.25465$.

θ	$\epsilon = 1 - r$	θ	$\epsilon = 1 - r$
0.79972	5.07496×10^{-3}	1.254652	6.95329×10^{-2}
0.814097	9.37901×10^{-3}	1.25886	6.95466×10^{-2}
0.890055	2.82715×10^{-2}	1.26516	6.95452×10^{-2}
0.939235	3.83327×10^{-2}	1.26711	6.95394×10^{-2}
1.01052	5.03879×10^{-2}	1.27563	6.94841×10^{-2}
1.0698	5.82139×10^{-2}	1.35641	6.63641×10^{-2}
1.13636	6.46171×10^{-2}	1.40732	6.16149×10^{-2}
1.21333	6.87925×10^{-2}	1.49471	4.59749×10^{-2}
1.23248	6.92707×10^{-2}	1.5707963	8.54284×10^{-7}

Table 4.5: The function $\epsilon(\theta)$ for $F = 3.3780842$. The maximum occurs at $\theta = 1.25886$.

θ	$\epsilon = 1 - r$	θ	$\epsilon = 1 - r$
0.814097	8.8368×10^{-3}	1.25664	6.43344×10^{-2}
0.858873	1.97406×10^{-2}	1.25886	6.43339×10^{-2}
0.939235	3.57712×10^{-2}	1.26516	6.43156×10^{-2}
1.01052	4.69445×10^{-2}	1.27563	6.42299×10^{-2}
1.0698	5.41647×10^{-2}	1.35641	6.10898×10^{-2}
1.13636	6.00221×10^{-2}	1.40732	5.6521×10^{-2}
1.21333	6.37474×10^{-2}	1.49471	4.18032×10^{-2}
1.23248	6.41456×10^{-2}	1.56566	1.06856×10^{-2}
1.25465	6.43324×10^{-2}	1.5707963	7.58484×10^{-7}

Table 4.6: The function $\epsilon(\theta)$ for $F = 4.1597548$. The maximum occurs at $\theta = 1.25664$.

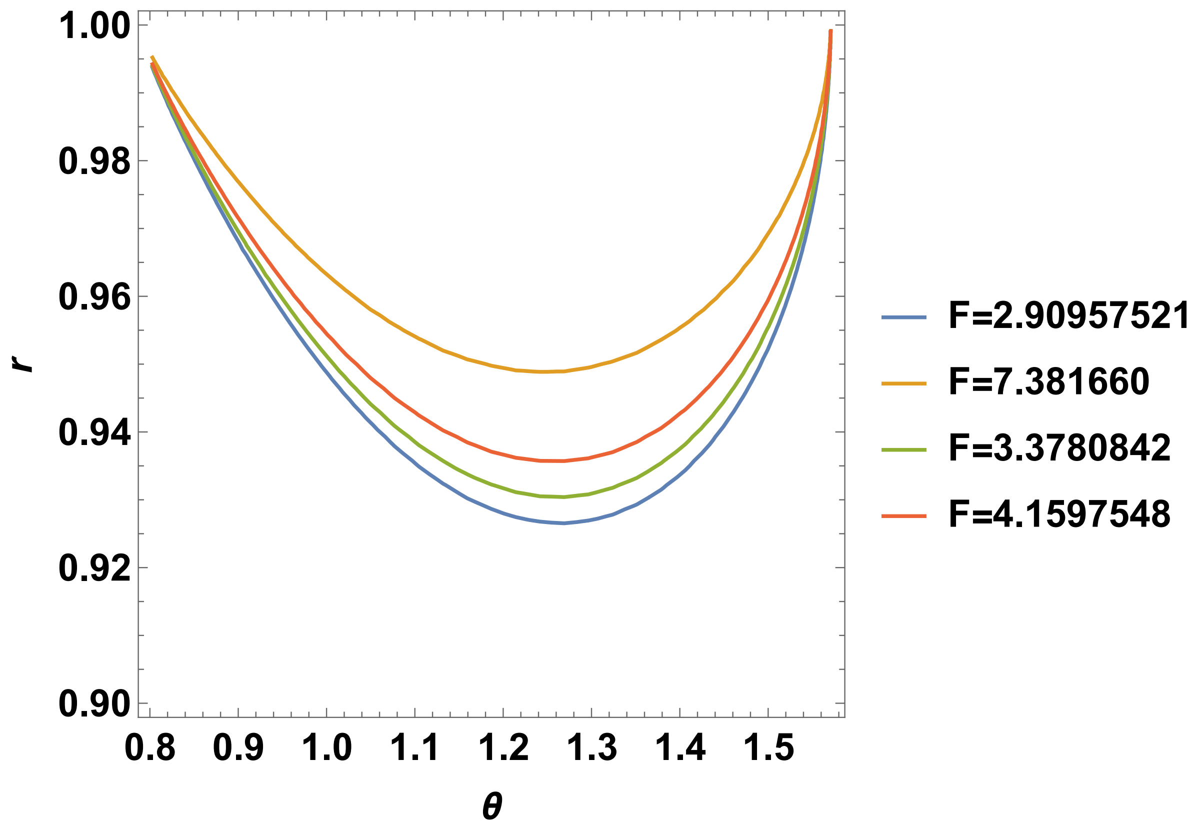
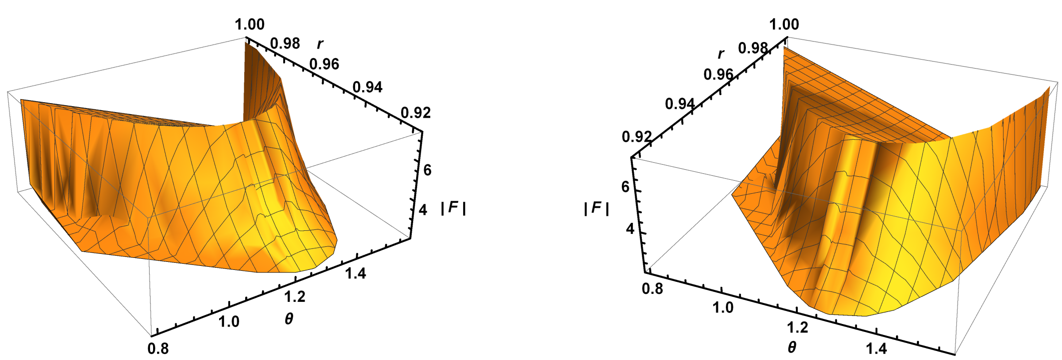


Figure 4.3: The curves $r(\theta)$ for constant F.



(a) (b)

Figure 4.4: The spacetime structure after the collision of *KP-waves* is shown in (a) and reflected figure is shown in (b).

4.3 Discussion and Conclusion

The behaviour of the force near the singularity as $z = 0$ is approached is fascinating. It is physically obvious that there is no momentum imparted to test particles at $z = 0$ as the two impulsive waves exactly cancel each other out. Hence the curvature must be zero *at* $z = 0$. However, the curvature must be infinite *at* the singularity. This is seen graphically in the numerical results. What this amounts to is that at $z = 0$ we have a delta function that has its infinity at doomsday, $t = T$.

The other interesting feature is seen in terms of the variables $r^2 = t^2 + z^2$ and $\theta = tan^{-1}(t/z)$. One might have expected that for any $r < T$ the maximum of the force would be at $\theta = \pi/4$. This would be what one would expect on one's "linear" intuition. It turns out that the nonlinear reality is very different. The maximum occurs around $3\pi/4$ but not exactly there and is not at a constant θ! The values are seen in Table 4.1 and Fig.4.2.

What would happen if the impulsive waves were *not* taken to be of equal strength? Qualitatively we know that the argument for no force at $z - 0$ would fail there. However, our expectation that it would only be shifted towards the weaker wave may not be correct. From the above behaviour of the maximum, it might be that it would not be at a constant θ. For that matter, it is not clear how to formulate the problem of different strength waves. The normal way of writing the KP-metric does not allow for different strengths. Since the "strength" is related to the value of T by an inverse behaviour, it would presumably be necessary to introduce two time scales, T_1 and T_2.

There has also been work done on colliding "sandwich" plane waves [26]. It would be interesting to see how the curvature of the spacetime in the fourth region behaves for them in comparison with the KP waves. Of course, if the formulation for different strengths is available, it would be worthwhile to compute the results for the sandwich waves of different strengths and then see how the results approach the limits for the special cases.

Chapter 5

Conclusion

Gravitational waves have been a subject of research from both the theoretical and experimental perspectives for a long time. In 1918, Einstein himself address this phenomenon [27]. General Relativity predicts these waves in the weak field regime. They are non-static solutions of the vacuum Einstein field equations. These waves travel at the speed of light. Their direct detection has been challenging in the past. However, the recent detection of the gravitational waves by LIGO opens a new window on the Universe.

There are some exact gravitational waves solution, e.g. plane, cylindrical and colliding plane wave solutions. Much work has been done on cylindrical and plane waves. The exact solution describing the collision of plane impulsive gravitational waves was provided by Khan and Penrose in 1971 [13]. Penrose's approach to construct the "impulsive" plane wave is very clear. He cut the Minkowski spacetime at a null hypersurface and reattach it with a jump [25]. This yields impulsive gravitational waves. Khan and Penrose showed that two impulsive plane gravitational waves approaching each other, then colliding, results in an exact solution of Einstein's equation. The spacetime before the collision is flat and after the collision is not only curved but develops a future curvature singularity.

Since gravitational waves are solutions of the vacuum field equations, they carry no energy. There had been a debate on the reality of these waves. To demonstrate it, Weber and Wheeler [20] gave an approximate result for momentum imparted to test particles by cylindrical gravitational waves. Later Ehlers and Kundt [22] considered a sphere of test particles in the path of plane fronted gravitational waves and showed that constant momentum was imparted to test particles. However, the standred procedure to obtain the required result is fairly complicated.

Qadir and Sharif presented a general closed form expression for the momentum imparted to test particles in an arbitrary spacetimes, [23] using an extension of the pseudo-Newtonian formalism (ψN).

We probed the curvature by considering the momentum imparted to test particles in the Khan-Penrose spactime. It has been shown numerically that the ψN-force, $|F|$, is zero at $z = 0$ or $\theta = \pi/2$. There is no momentum imparted to test particles at $z = 0$. Therefore, the curvature must be zero at $z = 0$. However, the curvature must be infinite at the singularity.

The other fascinating behaviour has been seen in terms of the variables $r^2 = t^2 + z^2$ and $\theta = tan^{-1}(t/z)$. The maximum of $|F|$ occurs around $\theta = 3\pi/4$ but not exactly there and is not at a constant θ.

Appendix A

The Curvature Tensor Components

The curvature tensor can be uniquely decomposed into three parts namely the Weyl tensor, the Ricci tensor and the Ricci scalar. Since Khan-Penrose spacetime is the vacuum solution therefore the Ricci tensor and its trace would be zero. Thus we have only the trace-free part of the curvature tensor, $C^{a}{}_{bcd}$ called the Weyl tensor. It has ten linearly independent components given below,

$$C^{2}{}_{332} = \frac{2\left(\sqrt{1-u^2}\sqrt{1-v^2}+uv\right)^2\left(u^2\left(2v^2-1\right)+\sqrt{1-u^2}uv\sqrt{1-v^2}-v^2+1\right)}{\left(\sqrt{1-u^2}+v\right)^2\sqrt{-u^2-v^2+1}\left(u+\sqrt{1-v^2}\right)^2} \tag{A.1}$$

$$\begin{aligned} C^{0}{}_{220} = {} & (\sqrt{1-u^2}\sqrt{1-v^2}+uv)\{u^6v(3v^2-2)-u(1-v^2)^{3/2}(2\sqrt{1-u^2}v^2 \\ & +\sqrt{1-u^2}-3v)+(v^2-1)^2(\sqrt{1-u^2}-v)+2u^2(v^2-1)(-3\sqrt{1-u^2}v^2 \\ & +\sqrt{1-u^2}+2v^3)-u^5\sqrt{1-v^2}(-3\sqrt{1-u^2}v^2+\sqrt{1-u^2}+6v^3-3v) \\ & +u^4(6\sqrt{1-u^2}v^4-6\sqrt{1-u^2}v^2+\sqrt{1-u^2}-3v^5-v^3+3v) \\ & +u^3\sqrt{1-v^2}(-3\sqrt{1-u^2}v^4-2\sqrt{1-u^2}v^2+2\sqrt{1-u^2}+9v^3 \\ & -6v)\}/\{\sqrt{1-u^2}(v-\sqrt{1-u^2})^3\sqrt{-u^2-v^2+1}(\sqrt{1-v^2}-u)^3\} \end{aligned} \tag{A.2}$$

$$\begin{aligned} C^{0}{}_{221} = {} & 3u\sqrt{1-u^2}(\sqrt{1-u^2}\sqrt{1-v^2}+uv)\{u^5(2v^2-1)-u(v^2-1)(3\sqrt{1-u^2}v \\ & -2v^2-1)+(1-v^2)^{3/2}(1-\sqrt{1-u^2}v)-u^2\sqrt{1-v^2}(2\sqrt{1-u^2}v^3 \\ & +\sqrt{1-u^2}v-5v^2+2)+u^4\sqrt{1-v^2}(2\sqrt{1-u^2}v-4v^2+1) \\ & +u^3(4\sqrt{1-u^2}v^3-3\sqrt{1-u^2}v-2v^4-v^2+2)\} \\ & /\{\sqrt{1-u^2}(v-\sqrt{1-u^2})^3\sqrt{-u^2-v^2+1}(\sqrt{1-v^2}-u)^3\} \end{aligned} \tag{A.3}$$

$$
\begin{aligned}
C^0{}_{330} = {} & (\sqrt{1-u^2}\sqrt{1-v^2}+uv)\{u^6 v(3v^2-2) - u(1-v^2)^{3/2}(2\sqrt{1-u^2}v^2+\sqrt{1-u^2}+3v) \\
& -(v^2-1)^2(\sqrt{1-u^2}+v) + 2u^2(v^2-1)(3\sqrt{1-u^2}v^2-\sqrt{1-u^2}+2v^3) \\
& +u^5\sqrt{1-v^2}(3\sqrt{1-u^2}v^2-\sqrt{1-u^2}+6v^3-3v) - u^4(6\sqrt{1-u^2}v^4 \\
& -6\sqrt{1-u^2}v^2+\sqrt{1-u^2}+3v^5+v^3-3v) + u^3\sqrt{1-v^2}(-3\sqrt{1-u^2}v^4 \\
& -2\sqrt{1-u^2}v^2+2\sqrt{1-u^2}-9v^3+6v)\} \\
& /\{\sqrt{1-u^2}(\sqrt{1-u^2}+v)^3\sqrt{-u^2-v^2+1}(u+\sqrt{1-v^2})^3\}
\end{aligned}
\tag{A.4}
$$

$$
\begin{aligned}
C^0{}_{331} = {} & -3u\sqrt{1-u^2}(\sqrt{1-u^2}\sqrt{1-v^2}+uv)\{u^5(2v^2-1)+u(v^2-1)(3\sqrt{1-u^2}v \\
& +2v^2+1) - (1-v^2)^{3/2}(\sqrt{1-u^2}v+1) - u^2\sqrt{1-v^2}(2\sqrt{1-u^2}v^3 \\
& +\sqrt{1-u^2}v+5v^2-2) + u^4\sqrt{1-v^2}(2\sqrt{1-u^2}v+4v^2-1) \\
& -u^3(4\sqrt{1-u^2}v^3-3\sqrt{1-u^2}v+2v^4+v^2-2)\} \\
& /\{\sqrt{1-v^2}(\sqrt{1-u^2}+v)^3\sqrt{-u^2-v^2+1}(u+\sqrt{1-v^2})^3\}
\end{aligned}
\tag{A.5}
$$

$$
\begin{aligned}
C^1{}_{220} = {} & -3v\sqrt{1-v^2}(\sqrt{1-u^2}\sqrt{1-v^2}+uv)\{2u^4 v(v^2-1)+u(1-v^2)^{3/2}(2\sqrt{1-u^2}v^2 \\
& +\sqrt{1-u^2}-3v) - (v^2-1)^2(\sqrt{1-u^2}-v) - u^2(v^2-1)(-4\sqrt{1-u^2}v^2 \\
& +\sqrt{1-u^2}+2v^3+v) - u^3\sqrt{1-v^2}(-2\sqrt{1-u^2}v^2+\sqrt{1-u^2}+4v^3-3v)\} \\
& /\{\sqrt{1-u^2}v-\sqrt{1-u^2})^3\sqrt{-u^2-v^2+1}(\sqrt{1-v^2}-u)^3\}
\end{aligned}
\tag{A.6}
$$

$$
\begin{aligned}
C^1{}_{221} = {} & -(\sqrt{1-u^2}\sqrt{1-v^2}+uv)\{u^5(3v^4-4v^2+1) - u(v^2-1)^2(3\sqrt{1-u^2}v-2v^2-1) \\
& +(1-v^2)^{5/2}(\sqrt{1-u^2}v-1) + u^2(1-v^2)^{3/2}(3\sqrt{1-u^2}v^3+\sqrt{1-u^2}v \\
& -6v^2+2) - u^4\sqrt{1-v^2}(-3\sqrt{1-u^2}v^3+2\sqrt{1-u^2}v+6v^4-6v^2+1) \\
& +u^3(v^2-1)(6\sqrt{1-u^2}v^3-3\sqrt{1-u^2}v-3v^4-2v^2+2)\} \\
& /\{\sqrt{1-v^2}(v-\sqrt{1-u^2})^3\sqrt{-u^2-v^2+1}(\sqrt{1-v^2}-u)^3\}
\end{aligned}
\tag{A.7}
$$

$$
\begin{aligned}
C^1{}_{330} = {} & 3v\sqrt{1-v^2}(\sqrt{1-u^2}\sqrt{1-v^2}+uv)\{2u^4 v(v^2-1)+u(1-v^2)^{3/2}(2\sqrt{1-u^2}v^2 \\
& +\sqrt{1-u^2}+3v) + (v^2-1)^2(\sqrt{1-u^2}+v) - u^2(v^2-1)(4\sqrt{1-u^2}v^2 \\
& -\sqrt{1-u^2}+2v^3+v) + u^3\sqrt{1-v^2}(2\sqrt{1-u^2}v^2-\sqrt{1-u^2}+4v^3 \\
& -3v)\}/\{\sqrt{1-u^2}(\sqrt{1-u^2}+v)^3\sqrt{-u^2-v^2+1}(u+\sqrt{1-v^2})^3\}
\end{aligned}
\tag{A.8}
$$

$$
\begin{aligned}
C^0{}_{010} = {} & 2\{3u^5 v(1-2v^2)\sqrt{1-v^2}+3u^3 v\sqrt{1-v^2}(3v^2-2)+\sqrt{1-u^2}(v^2-1)^2 \\
& -2\sqrt{1-u^2}u^2(3v^4-4v^2+1)+\sqrt{1-u^2}u^4(6v^4-6v^2+1) \\
& +3uv(1-v^2)^{3/2}\}/\{(u^2-1)\sqrt{1-v^2}(u^2+v^2-1)^2(\sqrt{1-u^2}\sqrt{1-v^2}+uv)^2\}
\end{aligned}
\tag{A.9}
$$

$$
\begin{aligned}
C^1{}_{331} = &-(\sqrt{1-u^2}\sqrt{1-v^2}+uv)\{u^5(3v^4-4v^2+1)+u(v^2-1)^2(3\sqrt{1-u^2}v+2v^2+1) \\
&+(1-v^2)^{5/2}\sqrt{1-u^2}v+1)+u^2(1-v^2)^{3/2}(3\sqrt{1-u^2}v^3+\sqrt{1-u^2}v \\
&+6v^2-2)+u^4(1-v^2)(3\sqrt{1-u^2}v^3-2\sqrt{1-u^2}v+6v^4-6v^2+1) \\
&-u^3(v^2-1)(6\sqrt{1-u^2}v^3-3\sqrt{1-u^2}v+3v^4+2v^2-2)\} \\
&/\{\sqrt{1-v^2}(\sqrt{1-u^2}+v)^3\sqrt{-u^2-v^2+1}(u+\sqrt{1-v^2})^3\}
\end{aligned} \tag{A.10}
$$

Appendix B

The Time Component of eψN-Force

The time component of eψN-force is,

$$
\begin{aligned}
F_t = \{&15t^{16} + 3(-7z^2 - 47)t^{14} - 3((51z^4 + 2(M-42)z^2 + 37M - 187))t^{12} \\
&+ [(567z^6 + (431 - 159M)z^4 + (189M - 925)z^2 + 339M - 1236]t^{10} \\
&+ [-885z^8 + 24(17M - 35)z^6 + 2(169M - 703)z^4 + (1564 - 532M)z^2 \\
&- 540M + 1692]t^8 + [(705z^8 - (447M + 575)z^6 + (694 - 286M)z^4 \\
&+ 58(32 - 11M)z^2 + 824M - 796)z^2 + 420M - 1584]t^6 + [-339z^{12} \\
&+ 2(129M + 818)z^{10} - (723M + 1771)z^8 + (452M - 672)z^6 \\
&+ 4(334M + 315)z^4 - 16(76M + 77)z^2 - 48(M - 22)]t^4 \\
&+ [93z^{14} - 3(27M + 289)z^{12} + (673M + 3223)z^{10} \\
&- (1885M + 6348)z^8 + 4(694M + 1731)z^6 \\
&- 4(619M + 1012)z^4 + 64(17M + 22)z^2 - 96(M + 4)]t^2 \\
&- 4(z - 1)z^2(z + 1)(z^2 - 2)^2((3z^4 - 5z^2 + 5)(z^2 - M - 2)z^2 \\
&- 2M + 8)\}/\{t((t - z)^2 - 2)(t^2 - z^2 - 1)((t + z)^2 - 2)(t^2 - z^2 + M)^3 \\
&\times [t^4 - 2(3z^2 + M + 3)t^2 + 5z^4 - 2z^2(3M + 5) + 4M + 8]\},
\end{aligned}
\tag{B.1}
$$

where $M = \sqrt{2 - (t - z)^2}\sqrt{2 - (t + z)^2}$.

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