Multi-valued fixed point theorems in b-metric space

by

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To my grandmother

Preface

The main source of the existence of metric fixed point theory is because of the Mathematician Stefan Banach who established a remarkable fixed point theorem known as Banach Fixed Point Theorem in 1922. Banach fixed point theorem provides a sufficient condition under which there exists a unique fixed point for a contraction mapping from a complete metric space to itself. There are very few fixed point theorems *i.e* Banach fixed point theorem, which actually have the practical importance *i.e* those theorems which provide a constructive method to find fixed points. These constructive methods provide information about the convergence rate with the error estimates. A lot of work has been done in the area of generalizing Banach fixed point theorem. Proinov [25] extended/generalized Banach fixed point theorem by generalizing the contractive condition which involves a gauge function of order $r \geq 1$. Later, his work was generalized/extended by Maria *et al* [28]. The authors proved that Proinov's results [25] results also hold when the underlying structure is replaced by a b-metric space.

After generalizing the Banach fixed point theorem in so many ways using single valued mapping, the research then moved forward when Nadler [21] generalize this result for multivalued mappings. He investigated the existence of fixed points for multivalued contraction mappings and succeeded in getting multivalued version of Banach fixed point theorem. Quanita *et al* [17] continue in the same direction and generalized Nadlers [21] results by introducing new contractive condition similar to the condition used by Proinov [25]. This dissertation is an extension of the generalized work of Proinov [25] done by Maria *et al* in [28] and generalized work of Nadler done by Quanita *et al* in [17] & [16] in which we use multivalued mappings with contractive condition involving gauge function in b-metric space.

The dissertation is organized as follow: In Chapter 1 we define the basic terminologies and definitions which we use in our subsequent work. In Chapter 2, we review the work done by Boriceanu *et al* in [5] in detail and also try to give answer to an open question. Chapter 3 contains generalization of some results of [28], [17] & [16] which is generalized for two different mappings.The extension of the results is first done for multivalued mappings from a nonempty set X into nonempty proximinal closed subsets of X and secondly, for multivalued mappings from a nonempty set X into nonempty closed bounded subsets of X.

Contents

Chapter 1

Introduction and preliminaries

We start with defining the fundamental concepts that we used for the development of our results.
Let X be a nonempty set, throughout the thesis, J denotes an interval on $\mathbb{R}^+$ containing 0, $i.e$, an interval of the form $[0, R], [0, R)$ or $[0, \infty)$ and $([0, 0] = \{0\})$ whereas $P_n(t)$ denotes a polynomial of the form $P_n(t) = 1 + t + ... + t^{n-1}$.

1.1 Fixed point

Definition 1.1.1. [18]
Let (X, d) be a metric space and let $f : X \to X$ be a mapping. Then

1. A point $x \in X$ is called a fixed point of f if $x = f(x)$.

2. f is called contraction if there exists a fixed constant $\alpha < 1$ such that

$$d(f(x), f(y)) \leq \alpha d(x, y) \tag{1.1.1}$$

for all $x, y \in X$.

1.2 Banach fixed point theorem

Banach fixed point theorem which is also known as the contraction mapping theorem is the one of the most important results of analysis. In fact, it is considered to be the building block of metric fixed point theory. The statement of Banach fixed point theorem states

Theorem 1.2.1. [18]
Let (X, d) be a complete metric space, then each contraction map $f : X \to X$ has a unique fixed point.

1.2.1 Some extensions of Banach fixed point theorem

Many authors succeed in trying to generalize and extend this theorem in different ways e.g, Rakotch [26] generalized Banach fixed point theorem by using a gauge function in the following way

Theorem 1.2.2. [26] Let X be a complete metric space and suppose that $T : X \to X$ satisfies

$$d(Tx, Ty) \leq \varphi(d(x, y))d(x, y),$$

for each $x, y \in X$ where $\varphi : \mathbb{R}_+ \to [0, 1)$ is monotonically non decreasing. Then T has a unique fixed point, λ, and $(T^n(x))$ converges to λ for each $x \in X$.

Boyd and Wong [7] generalize Banach fixed point theorem in the following way:

Theorem 1.2.3. [7] Let X be a complete metric space and suppose that $T : X \to X$ satisfies

$$d(Tx, Ty) \leq \varphi(d(x, y)),$$

for each $x, y \in X$ where $\varphi : \mathbb{R}_+ \to [0, +\infty)$, is upper semicontinuous from the right and satisfies $0 \leq \varphi(t) < t$ for $t > 0$. Then T has a unique fixed point λ, and $(T^n(x))$ converges to λ for each $x \in X$.

Banach fixed point theorem is among those few theorems in mathematics which gives us a constructive method for finding the fixed point. By using these constructive methods one is able to find the error estimates and convergence rates. In this direction, Proinov [25] extended Banach fixed point theorem for single valued mapping by introducing a new function called gauge function of order $r \geq 1$ and also obtained error estimates. Let us start with the definition of gauge function which is used in Proinov's work and is also fundamental for our work.

1.3 Gauge function

Definition 1.3.1. [25]
A function $\varphi : J \to J$ is said to be a *gauge function* of order $r \geq 1$ on J if it satisfies the following conditions:

1. $\varphi(\lambda t) \leq \lambda^r \varphi(t)$ for all $\lambda \in (0, 1)$ and $t \in J$;
2. $\varphi(t) < t$ for all $t \in J - \{0\}$;

Let φ^n denote the *nth* iterate of a function $\varphi : J \to J$.

From condition (1) of the definition we have that $\varphi(0) = 0$ and $\varphi(t)/t^r$ is nondecreasing on $J - \{0\}$.

Bianchini-Grandolfi gauge function

Definition 1.3.2. [25]
A nondecreasing gauge function $\varphi : J \rightarrow J$ is said to be a Bianchini-Grandolfi gauge function if

$$\sigma(t) = \Sigma_{n=0}^{\infty} \varphi^n(t) < \infty$$

for all $t \in J$.

Using this definition of gauge function Proinov generalized the Banach contraction principle in following way (see[25] for detail proof).

Theorem 1.3.3. [25]
Let (X, d) be a complete metric space and let D be a nonempty subset of X, φ is a Bianchini-Grandolfi gauge function on an interval J and let $T : D \subseteq X \rightarrow X$ be an operator such that

$$d(Tx, T^2x) \leq \varphi(d(x, Tx))$$

for all $x \in D, Tx \in D$ with $d(x, Tx) \in J$. Suppose that $x_0 \in D$ such that $d(x_0, Tx_0) \in J$ and all the iterate $x_{n+1} = Tx_n \in D$. Then the iterative sequence $x_{n+1} = Tx_n$ converges to a fixed point ξ of T.

After generalizing the BCP for single valued mappings the research took a new turn when Nadler [21] investigated that whether fixed points exist when the single valued mappings are replaced by multivalued contraction mappings. He succeeded in developing a multivalued version of BCP by using the definition of Hausdorff metric.

Definition 1.3.4. [10]
The generalized Housdorff metric on $CB(X)$ generated by metric d is

$$H(A, B) = max\left\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\right\}$$

for every $A, B \in CB(X)$.

Theorem 1.3.5. [21]
Let (X, d) be a complete metric space. Then, each contraction mapping T from nonempty set X into $CB(X)$ the class of all nonempty bounded and closed subsets of X, has a fixed point.

The authors in [16] and [17] generalize and extend the fixed point theorems proved by Nadler and Proinov. The work done by the authors in [16] and [17] is the combination of the concepts used in [18] and [25]. In [17] the authors extend the results for multivalued contraction mappings from a nonempty set X to set of all nonempty closed and bounded $CB(X)$ subsets of X , and in [16] to the set of all nonempty proximinal closed $PC(X)$ subset of X satisfying the contractive condition involving gauge function. Consider the following basic definitions

Definition 1.3.6. [28]
Let $f : D \subset X \to X$ and there exist some $x \in D$ such that the set $O(x) = \{x, fx, f^2x, ...\} \subset D$. The set $O(x)$ is known as an orbit of $x \in D$.

Definition 1.3.7. [17]
A mapping $f : X \to \mathbb{R}$ is said to be T-orbitally *lower semi continuous* if $\{x_n\}$ is a sequence in $O(T, x_0)$ and $x_n \to \xi$ implies $f(\xi) \leq lim_n inf f(x_n)$.

Theorem 1.3.8. [17]
Let (X, d) be a complete metric space, D be a closed subset of X, φ is Bianchini Grandolfi gauge function on interval J and T be a mapping from D into $CB(X)$ such that $Tx \cap D \neq \emptyset$ and

$$H(Tx \cap D, Ty \cap D) \quad \leq \quad \varphi(d(x, y)) \tag{1.3.1}$$

for all $x \in D$ and $y \in Tx \cap D$ with $d(x, y) \in J$. Moreover,the strict inequality holds when $d(x, y) \neq 0$. Suppose $x_0 \in D$ is such that $d(x_0, z) \in J$ for some $z \in Tx_0 \cap D$ Then:

1. there exists an orbit $\{x_n\}$ of T in D and $\xi \in D$ such that $lim_n x_n = \xi$;

2. ξ is a fixed point of T in D if and only if function $f(x) = d(x, Tx \cap D)$ is T-orbitally lower semi-continuous at ξ.

Definition 1.3.9. [11]
A subset A of X is called proximinal if, for each $x \in X$, there is an element $a \in A$ such that $d(x, a) = d(x, A)$, where $d(x, A) = inf\{d(x, y) : y \in A\}$.

Definition 1.3.10. [16]
Let (X, d) be a metric space and $T : X \to PC(X)$ and $x_0 \in X$. Then there exists a proximinal orbit $\{x_n\} \subseteq X$ of T at the point x_0 i.e.,

$$x_{n+1} \in Tx_n, \quad n = 0, 1, 2, ...$$

with $d(x_n, x_{n+1}) = d(x_n, Tx_n)$.

Theorem 1.3.11. [16]
Let $T : D \subset X \to PC(X)$ be an operator on a complete metric space (X, d) satisfying

$$H(Tx, Ty) \leq \varphi(d(x, y))$$

with a Bianchini-Grandolfi gauge function φ on an interval J. Then, starting from an initial point x_0 of T the iterative sequence $\{x_n\}$ remains in $\bar{B}(x_0, \rho_0)$ and converges to a point ξ which belongs to each of the closed ball $\bar{B}(x_n, \rho_n)$; n=0,1,...,with center

x_n and radius $\rho_n = \sigma(d(x_n, Tx_n))$, where $\sigma : J \to \mathbb{R}_+$. Moreover, for each $n \geq 1$ we have

$$d(x_n, x_{n+1}) \leq \varphi(d(x_{n-1}, x_n)).$$

If $\xi \in D$ and T is continuous at ξ, then ξ is a fixed point of T.

Alot of research has been done and many different extended versions of BCP has been obtained dealing with the generalized mappings and contractive inequalities. In the last few decades, the fixed point theory has evolved in different generalized spaces as well i.e., the worked done by authors in [22] for convex metric space, in [9] for ordered metric spaces. In 1933 a new metric known as b-metric has been introduced in some works of Czerwik, Heinonen, Bakhtin. After that, several authors have published their work on fixed point results in b-metric spaces i.e., [[14]-[19]].

In [28] the authors extend the work of Proinov[25] by replacing the usual metric with b-metric space.

1.4 b-metric space

Definition 1.4.1. [1, **?**]
Let X be a nonempty set, $\mathbb{R}^+$ set of all non negative real numbers and $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}^+$ is said to be a b-metric space if and only if for all $x, y \in X$ the following conditions are satisfied:

(d1) $d(x, y) = 0$ if and only if $x = y$;

(d2) $d(x, y) = d(y, x)$;

(d3) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

The pair (X, d) is called a b-metric space with the coefficient s.

Next we consider few examples for b-metric space.

Example 1.4.2. [**?**, 12]

(1) Let $X := l_p(\mathbb{R})$ with $0 < p < 1$ where $l_p(\mathbb{R}) := \{\{x_n\} \subset \mathbb{R} : \Sigma_{n=1}^{\infty} |x_n|^p < \infty\}$. Define $d : X \times X \to \mathbb{R}^+$ as

$$d(x, y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p \right)^{1/p}$$

where $x = \{x_n\}$, $y = \{y_n\}$. Then (X, d) is a b-metric space with coefficient $s = 2^{1/p}$.

(2) Let $X = L_p[0,1]$ be the space of all real functions $x(t), t \in [0,1]$ such that $\int_0^1 |x(t)|^p dt < \infty$. Define $d : X \times X \to \mathbb{R}^+$ as

$$d(x,y) = \left(\int_0^1 |x(t) - y(t)|^p dt \right)^{1/p}$$

Then (X, d) is a b-metric space with coefficient $s = 2^{1/p}$.

1.4.1 Convergence in b-metric space

Definition 1.4.3. [28]
A sequence $\{x_n\}$ in a b-metric space X is
Convergent: if and only if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$ and we write $\lim_{n\to\infty} x_n = x$;
Cauchy: if and only if $d(x_n, x_m) \to 0$ as $m, n \to \infty$.
A b-metric space (X, d) is complete if every Cauchy sequence in X converges in X.

Remark 1.4.4. [10]
Let (X, d) be a b-metric space, then a convergent sequence has a unique limit; every convergent sequence is Cauchy; and in general the b-metric d is not continuous functional.

Definition 1.4.5. [28, 5] Let (X, d) be a b-metric space. Then a nonempty subset $A \subset X$ is called:

1. **Closure**: $\overline{A}$ of A is the set consisting of all points of A and its limit points.

2. **Closed**: A is closed if and only if $A = \overline{A}$.

Let (X, d) be a b-metric space with a coefficient $s \geq 1$. We assume that $f : D \subset X \to X$ be an operator and there exist some $x_0 \in D$ such that $O(x_0) \subset D$. Let the operator f satisfy the following iterated contractive condition:

$$d(fx, f^2x) \leq \varphi(d(x, fx)) \text{ for all } x \in O(x_0) \text{ such that } d(x, fx) \in J, \quad (1.4.1)$$

where φ is a gauge function of order $r \geq 1$ on an interval J. The iterative processes consider is of the following type

$$x_{n+1} = fx_n, \quad n = 0, 1, 2, ..., \quad (1.4.2)$$

where f satisfies 1.4.1.

Proinov [25] proved his main results by assuming Bianchini-Grandolfi gauge functions and the mapping T satisfying the contractive condition $d(Tx, T^2x) \leq \varphi(d(x, Tx))$ with usual metric. But in case of b-metric space we consider the gauge functions satisfying $\Sigma_{n=0}^{\infty} s^n \varphi^n(t) < \infty$ for all $t \in J$ where s is the coefficient of b-metric space. To calculate prior and posterior estimates we consider the gauge functions of the form

$$\varphi(t) = t\frac{\phi(t)}{s} \quad \text{for all } t \in J. \tag{1.4.3}$$

where $s \geq 1$ is the coefficient of b-metric d and ϕ is nonnegative nondecreasing function on J such that.

$$0 \leq \phi(t) < 1 \quad \text{for all } t \in J. \tag{1.4.4}$$

Lemma 1.4.6. [28]
Let φ be a Gauge function of order $r \geq 1$ on J. If ϕ is a nonnegative and nondecreasing function on J satisfying (1.4.3), (1.4.4) then;

1. $0 \leq \frac{\phi(t)}{s} < 1$ for all $t \in J$;
2. $\phi(\mu t) \leq \mu^{r-1}\phi(t)$ for all $\mu \in (0, 1)$ and $t \in J$.

Lemma 1.4.7. [28]
Let φ be a gauge function of order $r \geq 1$ on J. If ϕ is a nonnegative and nondecreasing function on J satisfying (1.4.3) and (1.4.4), then for every $n \geq 0$ we have:

1. $\varphi^n(t) \leq t[\frac{\phi(t)}{s}]^{P_n(r)}$ for all $t \in J$;
2. $\phi(\varphi^n(t)) \leq s[\frac{\phi(t)}{s}]^{r^n}$ for all $t \in J$.

Proof. **1** Set $\mu = \frac{\phi(t)}{s}$ and let $t \in J$. Then from Lemma (1.4.6) we obtain $0 \leq \mu < 1$. For $\mu = 0$ the case is trivial. we shall prove (1) by using mathematical induction. For $n = 0, 1$ the property (1) is trivially satisfied as it reduces to an equality. Let it also hold for any integer $n \geq 1$, *i.e.*,

$$\varphi^n(t) \leq t\mu^{P_n(r)}.$$

Since φ is nondecreasing on J, we obtain (as $t\mu^{P_n(r)} \in J$ and $\mu < 1$)

$$\varphi^{n+1} \leq \phi[t\mu^{P_n(r)}] \tag{1.4.5}$$

$$\leq \mu^{P_n(r)} t\frac{\phi(t)}{s} \tag{1.4.6}$$

$$= t\mu^{rP_n(r)+1} = t\mu^{P_{n+1}(r)}. \tag{1.4.7}$$

2 By making use of Lemma 1.4.6 and monotonicity of ϕ,(1) leads to the following;

$$\begin{aligned}\phi(\varphi^n(t)) &\le \phi(t[\frac{\phi(t)}{s}]^{P_n(r)}) \le [\frac{\phi(t)}{s}]^{(r-1)P_n(r)}\phi(t)\\ &= s[\frac{\phi(t)}{s}]^{1+(r-1)P_n(r)} = s[\frac{\phi(t)}{s}]^{r^n}.\end{aligned}$$

Which completes the proof. □

Definition 1.4.8. [28]
Let $q \ge 1$ be a fixed real number. A nondecreasing function $\varphi : J \to J$ is said to be b-Bianchini-Grandolfi gauge function with a coefficient q on J if

$$\sigma(t) = \Sigma_{n=0}^{\infty} q^n \varphi^n(t) < \infty \tag{1.4.8}$$

for all $t \in J$. We note that a b-Bianchini-Grandolfi gauge function also satisfies the following functional equation:

$$\sigma(t) = q\sigma(\varphi(t)) + t. \tag{1.4.9}$$

Every b-Bianchini-Grandolfi gauge function is also a Bianchini-Grandolfi gauge function but the converse may not hold. A b-Bianchini-Grandolfi gauge function having coefficient $q_1 \ge 1$ is also a b-Bianchini-Grandolfi gauge function having coefficient $q_2 \ge 1$ for every $q_2 \le q_1$.

Lemma 1.4.9. [28]
Every gauge function of order $r \ge 1$ defined by (1.4.3) and (1.4.4) is a b-Bianchini-Grandolfi gauge function with coefficient $s \ge 1$.

Theorem 1.4.10. [28]
Let $f : D \subset X \to X$ be an operator on a complete b-metric space (X, d) such that the b-metric is continuous and f satisfies (1.4.1) with a b-Bianchini-Grandolfi gauge function of order $r \ge 1$on an interval J with coefficient $s \ge 1$. then starting from an initial orbital point x_0 of f the iterative sequence (1.4.2) remains in $\overline{B}(x_0, \rho_0)$ and converges to a point ξ belongs to each of the closed balls $\overline{B}(x_n, \rho_n), n = 0, 1, 2, ...,$ where $\rho_n = s\sigma(d(x_n, x_{n+1}))$, and $s \ge 1$ is a coefficient of b-metric space. Furthermore, for each $n \ge 1$ we have

$$d(x_{n+1,x_n}) \le \varphi(d(x_n, x_{n-1})).$$

If $\xi \in D$ and the function $E(x) = d(x, fx)$ is f-lower semi continuous at ξ, then ξ is a fixed point of f.

The concept used to prove the results in this thesis is the combination of all the work that is mentioned above. This thesis deals with multivalued mappings used by Quanita *et al* in [16] & [17] in b-metric space with contractive condition involving gauge function φ of order $r \geq 1$. My work will generalize the work done by Maria *et all* in [28] and Quanita *et al* in [16] & [17] and thus extend the work of Nadler [21] and Proinov [25] as well. In addition to that we have also reviewed the work done in [5] and try to give answer to an open question.

Chapter 2

Multivalued fractals in b-metric space

The purpose of this chapter is to analyze the concept and techniques used to prove important fixed points results. We reviewed some results given in [5] in which the authors extended the fractal operator theory for multivalued operators on complete b-metric space. We also try to give the answer to an open question [given in [5]] and prove the result in detail.
We start this chapter by considering the following definition.

Definition 2.0.11. [5] The b-metric space is complete if every Cauchy sequence in X converges.
Next few families of subsets of a b-metric space are defined

$$\begin{aligned} \mathcal{P}(X) &:= \{Y|Y \subset X\};\ P(X) := \{Y \in \mathcal{P}(X)|Y \neq \emptyset\}; \\ P_b(X) &:= \{Y \in P(X)|Y \text{ is bounded }\};\ P_{cp}(X) := \{Y \in P(X)|Y \text{ is compact }\} \\ P_{cl}(X) &:= \{Y \in P(X)|Y \text{ is closed }\};\ CB(X) := P_b(X) \cap P_{cl}(X). \end{aligned}$$

2.1 Fractal operator theory

Let us start with considering the basic concept of fractal and multi-fractal operator.

Definition 2.1.1. [5]
Let (X, d) be a metric space, $P_{cp}(X)$ denotes the family of all nonempty and compact subsets of X and $F_1, ...F_m : X \to P_{cp}(X)$ be a finite family of upper semicontinuous (see [29], for example) multivalued operators. We define the multi-fractal operator T_F generated by the iterated multi-functions system $F = (F_1, F_2, ..., F_m)$ (finite collection of contraction mappings) by the following relation

$$T_F(Y) = \bigcup_{i=1}^{m} F_i(Y), \quad \text{for each } Y \in P_{cp}(X).$$

Then, by the upper semicontinuity of the operators F_i we have that $T_F : P_{cp}(X) \to P_{cp}(X)$. A nonempty compact subset A^* of X is said to be a multivalued fractal with respect to the iterated multi-functions system set of contraction mappings $F = (F_1, ..., F_m)$ if and only if it is a fixed point for the associated multi-fractal operator, i.e. $T_F(A^*) = A^*$.
In particular, if $Fi := f_i$ are continuous single-valued operators, then a fixed point for the fractal operator

$$T_f : P_{cp}(X) \to P_{cp}(X), \quad T_f(Y) = \bigcup_{i=1}^{m} f_i(Y)$$

generated by the iterated functions system $f = (f_1, f_2, ..., f_m)$ is said to be a self-similar set or a fractal.

Let us consider the example of Sierpinski triangle to understand the concept of the definition given above.

Example 2.1.2. We will start with a solid equilateral triangle S_0. To create sierpinski triangle we will use three transformation F_1, F_2, F_3. All three transformations F_1, F_2, F_3 map the plane $\mathbb{R}^2 \to \mathbb{R}^2$ [where $\mathbb{R}^2$ is two dimensional Euclidean space].

1. The first transformation is a uniform rescaling by a factor of 1/2. which is

$$F_1(\begin{bmatrix} x \\ y \end{bmatrix}) = x \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}$$

2. The second transformation is a uniform rescaling by a factor of 1/2 followed by a translation to the right by 1/2.

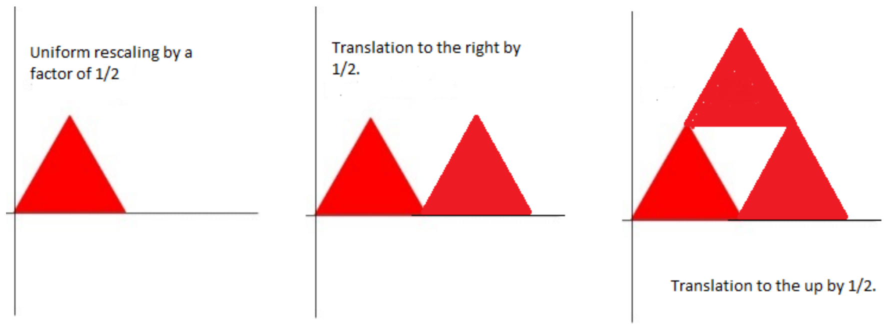
$$F_2(\begin{bmatrix} x \\ y \end{bmatrix}) = x \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1/2 \end{bmatrix} + \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$$

.

3. The third transformation is a uniform rescaling by a factor of 1/2 followed by a translation to the right by 1/4 and up 1/2. which is

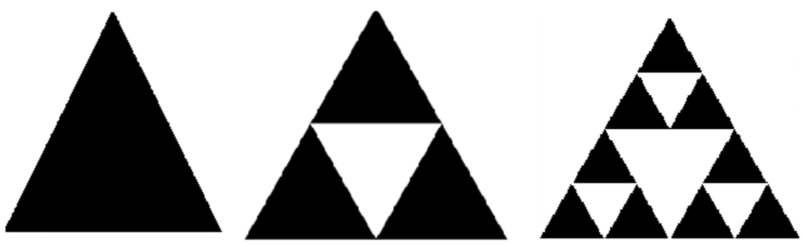
$$F_3(\begin{bmatrix} x \\ y \end{bmatrix}) = x \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1/2 \end{bmatrix} + \begin{bmatrix} 1/4 \\ 1/2 \end{bmatrix}$$

.

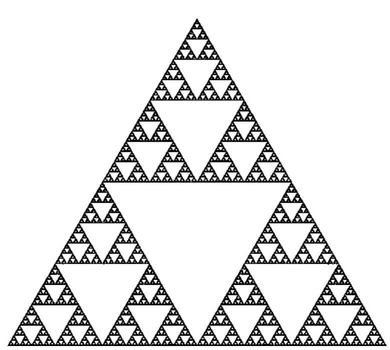


We can observe the working of these three transformation in the following figure.

The pictures of first few iterations while creating Sierpinski triangle are:



Since the transformation is contractive that is, the transformation brings points closer together, then the image will begin to converge. After infinitely many iterations, the image will converge to what is called an attractor. $F = F_1 \cup F_2 \cup ....F_n$. That is $F = \bigcup_{i=1}^{n} F_i$.



Definition 2.1.3. [5] Let (X,d) be b-metric space and A be a nonempty subset of X then:

1. *Compact*: if and only if for every sequence of elements of A there exists a subsequence that converges to an element of A.

2. *Bounded*: if and only if $\delta(A) := \sup\{d(a,b)|a,b \in A\} < \infty$.

Definition 2.1.4. [5]

$$(1)\ D : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{+\infty\}$$

$$D(A,B) = \begin{cases} \inf\{d(a,b)|a \in A, b \in B\}, & A \neq \emptyset \neq B \\ 0 & A = \emptyset = B \\ +\infty & \text{otherwise} \end{cases}$$

In particular, if $x_0 \in X$ then $D(x_0, B) = D(\{x_0\}, B)$.

Definition 2.1.5 ([5])**.** Let (X,d) be a b-metric space. If $F : X \to P(X)$ is a multivalued operator, then we denote by $Fix(F)$ the fixed point set of F, *i.e* $Fix(F) := \{x \in X|x \in F(x)\}$ and by SFix(F) the strict fixed point set of F, i.e. SFix(F)$= \{x \in X|x = F(x)\}$.

Lemma 2.1.6. [5]
Let (X,d) be a b-metric space and let $A, B \in P(X)$. We suppose that there exists $\eta > 0$ such that:

1 for each $a \in A$ there is $b \in B$ such that $d(a,b) \leq \eta$;

2 for each $b \in B$ there is a $a \in A$ such that $d(a,b) \leq \eta$.

Then, $H(A,B) \leq \eta$.

Lemma 2.1.7. [5]
Let (X,d) be a b-metric space. Then

$$D(x,A) \leq s[D(x,B) + H(A,B)], \text{ for all } x \in X \text{ and } A,B \in P(X)$$

Lemma 2.1.8. [5] Let (X,d) be a b-metric space. Then for all $A,B,C \in P(X)$ we have

$$H(A,C) \leq s[H(A,B) + H(B,C)].$$

Lemma 2.1.9. ([10])

1 Let (X,d) be a b-metric space and $A, B \in P_{cp}(X)$. Then for each $a \in A$ there exists $b \in B$ such that

$$d(a,b) \leq sH(A,B).$$

2 Let (X,d) be a b-metric space with $d : X \times X \to \mathbb{R}_+$ a continuous b-metric and let $A, B \in P_{cp}(X)$. Then for each $a \in A$ there exists $b \in B$ such that

$$d(a,b) \leq H(A,B).$$

2.2 Comparison function

Definition 2.2.1. [8, 27]
$\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a comparison function(see [8],[27]) if it is increasing and $\varphi^k(t) \to 0$ as $k \to +\infty$.
As a consequence, we also have $\varphi(t) < t$, for each $t > 0, \varphi(0) = 0$ and φ is continuous in 0.

Definition 2.2.2. [5]
A comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a strict comparison function if $\lim_{t\to+\infty} \ (t - \varphi(t)) = +\infty$. A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a strong comparison function if it is strictly increasing and $\sum_{n=1}^{\infty} \varphi^n(t) < +\infty$, for each $t > 0$ (see [27] for details).

Definition 2.2.3. [5]
Let $(X, d), (Y, \rho)$ be b-metric spaces. An operator $f : X \to Y$ is said to be a φ-contraction if $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ and $\rho(f(x), f(y)) \leq \varphi(d(x, y))$, for all $x, y \in X$.

Definition 2.2.4. [5]

2.2.1 b-comparison function

[5] A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a b-comparison function (with $s \geq 1$) if φ is increasing and there exist $k_0 \in \mathbb{N}, a \in]0, 1[$ and a convergent series of non-negative terms $\sum_{k=1}^{\infty} v_k$ such that

$$s^{k+1}\varphi^{k+1}(t) \leq as^k\varphi^k(t) + v_k, \text{ for } k \geq k_0, \text{ and any } t \in \mathbb{R}_+.$$

Moreover, any b-comparison function is a comparison function.
For example, if(X, d) is a b-metric with constant $s \geq 1$, then $\varphi(t) := at$, for each $t \in \mathbb{R}_+$(with $a \in]0, \frac{1}{2}[$) is a b-comparison function. For other examples and properties of the b-comparison functions see [3],[23].

Definition 2.2.5. [5]
let (X, d) be a b-metric space. An operator $f : X \to X$ is, by definition, a Picard operator if:

1 Fix(f)=$\{x^*\}$;

2 $(f^n(x))_{n\in\mathbb{N}} \to x^*$ as $n \to \infty$, for all $x \in X$.

Theorem 2.2.6. [23]
Let (X, d) be a complete b-metric space (with constant $s \geq 1$) such that the b-metric is a continuous functional on $X \times X$. Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a b-comparison function and $f : X \to X$ be a φ-contraction. Then f is a Picard operator (denote by $x*$ the unique fixed point of f).

Proof. We want to show that for any $x_0 \in X$ the iterates $x_n = fx_{n-1}$ for $n \geq 1$ converges to a fixed point of f.
For any $n \geq 1$ we have

$$d(x_n, x_{n+1}) \;=\; d(fx_{n-1}, fx_n) \leq \varphi d(x_n, x)$$

Therefore

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \varphi d(x_{n-1}, x_n) \\ &\leq \varphi^2 d(x_{n-1}, x_{n-1}) \\ &\leq \\ &. \\ &. \\ &. \\ &\leq \varphi^n d(x_0, x_1). \end{aligned}$$

Using triangular inequality we have

$$\begin{aligned} d(x_n, x_m) &\leq s^n d(x_n, x_{n+1}) + s^{n+1} d(x_{n+1}, dx_{n+2}) + ... + s^{m-1} d(x_{m-1}, x_m) \\ &\leq s^n \varphi^n d(x_0, x_1) + s^{n+1} \varphi^{n+1} d(x_0, dx_1) + ... + s^{m-1} \varphi^{m-1} d(x_0, x_1) \\ &\leq \sum_{i=0}^{\infty} s^i \varphi^i d(x_0, x_1). \end{aligned}$$

φ is a b-comparison function $\{x_n\}$ is cauchy sequence in (X, d). As (X, d) is complete so there exists $x^* \in X$ such that $x_n \to x^*$.
Now we will show that x^* is a fixed point of f. we have

$$\begin{aligned} d(x^*, fx^*) &\leq s[d(x^*, x_n) + d(x_n, fx^*)] \\ &\leq s[d(x^*, x_n) + d(fx_{n-1}, fx^*)] \\ &\leq s[d(x^*, x_n) + \varphi d(x^*, x_n)] \\ &\leq s(d(x^*, x_n))[1 - s\varphi]. \end{aligned}$$

Taking $\lim n \to \infty$ we have $d(x^*, fx^*) = 0$

$$\Rightarrow x^* = fx^*$$

x^* is a fixed point.
For uniqueness let x and y be two fixed points.

$$x = f(x) \qquad y = f(y)$$

Then

$$d(x, y) = d(fx, fy) \leq \varphi d(x, y)$$

which is a contradiction. □

Lemma 2.2.7. [5]
Let (X, d) be a b-metric space and $F : X \to P_{cp}(X)$ be a multivalued contractive operator (i.e., $H(F(x), F(y)) < d(x, y)$, for each $x, y \in X$ with $x \neq y$). Then, for any $Y \in P_{cp}(X)$ we have that $F(Y) \in P_{cp}(X)$.

Proof. If we choose $(y_n)_{n\in\mathbb{N}} \subset F(Y)$, then there exists $(x_n)_{n\in\mathbb{N}} \subset Y$ such that $y_n \in F(x_n), n \in \mathbb{N}$. We may assume that $(x_n)_{n\in\mathbb{N}} \to x$ in Y and $x_n \neq x$, for each $n \in \mathbb{N}$. Then, in view of Lemma 2.1.9(1), for $y_n \in F(x_n)$ there exists $u_n \in F(x)$ such that $d(y_n, u_n) \leq sH(F(x_n), (F(x))) < sd(x_n, x) \to 0$ as $n \to +\infty$. Hence $d(y_n, u_n) \to +\infty$ as $n \to +\infty$. Since $F(X)$ is a compact set, we obtain that there exists a subsequence of $(u_n)_{n\in\mathbb{N}}$ which converges to a certain element $u \in F(x)$. We denote this subsequence by $(u_n)_{n\in\mathbb{N}}$ too. Then, we have

$$d(y_n, y) \leq s[d(y_n, u_n) + d(u_n, y)] \to 0 \ as\ n \to +\infty$$

Thus $(y_n)_{n\in\mathbb{N}} \to y \in F(x) \subset F(Y)$. This completes the proof. □

The system $F = (F_1, ..., F_m)$ is called an iterated multifunction system(IMS), see [2],[13]. We called T_F the multi-fractal operator generated by the IMS F. A fixed point of T_F is by, definition, a multivalued fractal. In this setting, a set $A^*_F \in P_{cp}(X)$ is called an attractor of the IMS F if, for each $A \in P_{cp}(X)$, the sequence $(T^n_F(A))_{n\in\mathbb{N}}$ converges to A^*_F in $(P_{cp}(X), H)$ as $n \to +\infty$.

Theorem 2.2.8. [5]
Let (X, d) be a complete b-metric space (with constant $s \geq 1$) such that the b-metric is a continuous functional on $X \times X$. Let $F_i : X \to P_{cp}(X)$ be a multivalued φ-contractions for each $i \in \{1, 2, ..., m\}$ such that $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a b-comparison function.
Then:

a $T_F : (P_{cp}(X), H_d) \to (P_{cp}(X), H_d)$;

b T_F is a φ-contraction;

c T_F is a Picard operator having a unique fixed point $A^*_F \in P_{cp}(X)$ which is a multivalued fractal and an attractor of the $IMS\ \ F = (F_1, F_2, ..., F_m)$

Proof. **a** Since $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a b-comparison function. Thus, F_i is contractive for each $i \in \{1, 2, ...m\}$, since $\varphi(t) < t$ for $t > 0$. Hence, by Lemma 2.2.7, we get that $T_F : (P_{cp}(X), H_d) \to (P_{cp}(X), H_d)$.

b We are proving that $H(T_F(A), T_F(B)) \leq \varphi(H(A, B))$, for each $A, B \in P_{cp}(X)$. For this purpose, let $A, B \in P_{cp}(X)$ and let $u \in T_F(A)$. Then, there exists $i \in \{1, ..., m\}$ such that $u \in F_i(A)$. Moreover, there exists $a \in A$ such that

$u \in F_i(a)$. For $a \in A$, by the compactness of the sets A and B there exists $b \in B$ such that there exists $a \in A$ such that

$$d(a,b) \leq H(A,B). \tag{2.2.1}$$

Then, for $u \in F_i(a)$, by Lemma 2.1.9, there exists $v \in F_i(b)$ such that

$$d(u,v) \leq H(F_i(a), F_i(b)). \tag{2.2.2}$$

Thus, by (2.2.1) and (2.2.2) we get that for each $u \in T_F(B)$ such that

$$d(u,v) \leq H(F_i(a), F_i(b)) \leq \varphi(d(a,b)) \leq \varphi(H(A,B)). \tag{2.2.3}$$

By a similar procedure, we obtain that for each $v \in T_F(B)$ there exists $u \in T_F(A)$ such that

$$d(u,v) \leq \varphi(H(A,B)). \tag{2.2.4}$$

Thus,Lemma2.1.6, (2.2.3) and (2.2.4) together imply that

$$H(T_F(A), T_F(B)) \leq \varphi(H(A,B)).$$

Thus, we obtain that T_F is a self φ-contraction on the complete metric space $(P_{cp}(X), H_d)$.

c Proof follows immediately from Theorem 2.2.6.

□

Theorem 2.2.9. [5]
Let (X,d) be a complete b-metric space(with constant $s \geq 1$ such that the b-metric is a continuous functional on $X \times X$. Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, be a b-comparison function and $f : X \to X$ be a φ-contraction. Then

(i) (Abstract Collage Theorem)
If the function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, $\psi(t) := t - s\phi(t)$ is strictly increasing and onto, then

$$d(x, x^*) \leq \psi^{-1}(sd(x, f(x))),$$

for each $x \in X$. where x^* denotes the unique fixed point for f.

(ii) (Abstract Anti-Collage Theorem)
If the function $\eta : \mathbb{R}_+ \to \mathbb{R}_+$, $\eta := t + \varphi(t)$ is onto, then

$$d(x, x^*) \geq \eta^{-1}(\frac{1}{s}d(x, f(x))),$$

for each $x \in X$.

Proof. (i) For arbitrary $x \in X$ we have

$$d(x, x^*) \leq s[d(x, f(x)) + d(f(x), x^*)]$$

$$\leq s[d(x, f(x)) + d(f(x), f(x^*))] \quad \because f(x^*) = x^*$$
$$\leq s[d(x, f(x)) + \varphi d(x, x^*)]$$
$$d(x, x^*) - s\varphi d(x, x^*) \leq sd(x, f(x)).$$

Since
$$\psi(t) := t - s\phi(t).$$

Hence
$$\psi d(x, x^*) \leq sd(x, f(x))$$

and thus,since ψ is an increasing bijection, we obtain that

$$d(x, x^*) \leq \psi^{-1}(sd(x, f(x))),$$

for each $x \in X$.

(ii) For arbitrary $x \in X$ we have

$$d(x, f(x)) \leq s[d(x, x^*) + d(x^*, f(x))]$$

$$\leq s[d(x, x^*) + d(f(x^*)), f(x))] \quad \because f(x^*) = x^*$$
$$\leq s[d(x, x^*) + \varphi d(x, x^*)]$$
$$\frac{1}{s}d(x, f(x)) \leq s[d(x, x^*) + \varphi d(x, x^*)]$$

Since
$$\eta(t) := t + \phi(t)$$

Hence
$$\eta d(x, x^*) \geq \frac{1}{s}d(x, f(x))$$

and thus, since φ is increasing we infer that η is a strictly increasing bijection, so we obtain that
$$d(x, x^*) \geq \eta^{-1}(\frac{1}{s}d(x, f(x))),$$

for each $x \in X$.

□

Next we give the detailed proof for Collage and Anti-Collage theorem for multi-valued mappings.

Theorem 2.2.10. [5]
Let (X, d) be a complete b-metric space such that the b-metric is a continuous functional on $X \times X$. Let $F_i \colon X \to P_{cp}(X)$ be multivalued φ-contraction such that $\varphi \colon \mathbb{R}_+ \to \mathbb{R}_+$ is b-comparison function Then:

(i) Collage Theorem: If the function $\psi \colon \mathbb{R}_+ \to \mathbb{R}_+$, $\psi(t) \colon = t - s\varphi(t)$ is strictly increasing and onto, then

$$H(A, A_F^*) \quad \leq \quad \psi^{-1}(sH(A, T_F(A)))$$

for each $A \in P_{cp}(X)$, where A_F^* denotes the unique fixed point for T_F.

(ii) Anti-Collage Theorem: If the function $\eta \colon \mathbb{R}_+ \to \mathbb{R}_+$, $\eta(t) \colon = t + \varphi(t)$ is onto, then

$$H(A, A_F^*) \quad \leq \quad \eta^{-1}(\frac{1}{s}H(A, T_F(A)))$$

Proof. (i)

$$H(A, A_F^*) \leq s[H(A, T_F(A)) + H(T_F(A), A_F^*)]$$
$$\leq s[H(A, T_F(A)) + H(T_F(A), T_F(A^*)]$$

$\because (T_F(A^*)) = (A_F^*)$

$$\leq s[H(A, T_F(A)) + \varphi(H(A, A_F^*))]$$
$$\leq sH(A, T_F(A)) + s\varphi(H(A, A_F^*))$$
$$H(A, A_F^*) - s\varphi H(A, A_F^*) \leq sH(A, T_F(A)).$$

since

$$\psi(t) = t - s\varphi(t)$$

Hence,

$$\psi(H(A, A_F^*) \leq s(H(A, T_F(A))))$$

and thus, since ψ is an increasing bijection we obtain that

$$(H(A, A_F^*) \leq \psi^{-1}s(H(A, T_F(A))))$$

(ii) We have

$$H(A, T_F(A)) \leq s[H(A, A_F^*) + H(A_F^*, T_F(A))]$$
$$\leq s[H(A, A_F^*) + H((T_F(A^*), T_F(A)))]$$
$$\leq s[H(A, A_F^*) + \varphi H(A_F^*, A)]$$

$$\frac{1}{s}H(A, T_F(A)) \leq H(A, A_F^*) + \varphi H(A_F^*, A)$$

Since $\eta(t) = t + \varphi(t)$ Hence we obtain

$$\eta H((A, A_F^*)) \geq \frac{1}{s}H(A, T_F(A)).$$

Since φ is increasing we infer that η is strictly increasing bijection,so we obtain that

$$H((A, A_F^*)) \geq \eta^{-1}(\frac{1}{s}H(A, T_F(A)))$$

.

□

Next the definition and some results are discussed related to well-posedness in b-metric space.

Definition 2.2.11. [5]
Let (X, d) be a b-metric space and $f : X \to X$ be an operator. Then, the fixed point problem for f is well-posed if and only if $Fix(f) = x*$ and if $(x_n)_{n\in\mathbb{N}}$ is a sequence in X such that $d(x_n, f(x_n)) \to 0$ as $n \to \infty$, then $x_n \xrightarrow{d} x*$ as $n \to \infty$.

Definition 2.2.12. [5]
Let (X, d) be a b-metric space $F : X \to P(x)$ be a multivalued operator. Then, for F we have:

a the well-posedness property of the fixed point problem with respect to D if and only if $Fix(F) = x*$ and if $(x_n)_{n\in\mathbb{N}}$ is a sequence in X such that $D(x_n, F(x_n)) \to 0$ as $n \to \infty$ then $x_n \xrightarrow{d} x^*$ as $n \to \infty$;

b the well-posedness property of the fixed point problem with respect to H if and only if $Fix(F) = x*$ and if $(x_n)_{n\in\mathbb{N}}$ is a sequence in X such that $H(x_n, F(x_n)) \to 0$ as $n \to \infty$ then $x_n \xrightarrow{d} x^*$ as $n \to \infty$;

With respect to the well-posedness of the fixed point problem for a multivalued φ-contraction in a b-metric space we have the following result.

Theorem 2.2.13. [5]
Let (X, d) be a complete b-metric space such that the b-metric is a continuous functional on $X \times X$.Let $F : X \to P_{cp}(X)$ be a multivalued φ-contraction, such that $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a b-comparison function. Suppose that $SFix(F) \neq \varnothing$ and the function $\psi(t) := t - s\varphi(t)$ is strictly increasing and onto. Then the fixed point problem for F is well-posed with respect to D and H too.

Proof. Let $x^* \in SFix(F)$. We will prove that $Fix(F) = SFix(F) = x^*$. Let $y \in Fix(F)$. Then we have

$$d((x^*), y) = D(F(x^*), y) \leq H(F(x^*), F(y^*)) \leq \varphi d(x^*, y)$$

Since φ is a comparison function (being a b-comparison function) we have that $\varphi(t) \leq t$ for each $t \geq 0$. Hence $d(y, x^*) = 0$ and thus $y = x^*$
Let $(x_n) \in \mathbb{N}$ be a sequence in X such that $D(x_n, F(x_n)) \to 0$ as $n \to \infty$. We will prove that $x_n \to (x^*)$ as $n \to \infty$. We have from Lemma 2.1.7 that $d(x_n, x^*) = D(x_n, F(x^*)) \leq s[D(x_n, F(x_n)) + H(F(x_n), F(x^*)))] \leq s[D(x_n, F(x_n)) + \varphi(d(x_n, x^*))]$.Thus $\psi(d(x_n, x^*)) \leq sD(x_n, F(x_n)$, for $n \in \mathbb{N}$.Hence, $d(x_n, x^*) \leq \psi^{-1}(sD(x_n, F(x_n))) \to 0$ as $n \to +\infty$.
With resect to H

$$\begin{gathered} d(x_n, (x^*)) \leq s[H(x_n, F(x_n)) + D(F(x_n), x^*,)] \\ \leq s[H(x_n, F(x_n)) + H(F(x_n), F(x^*))] \\ \leq s[H(x_n, F(x_n)) + \varphi d(x_n, x^*)] \\ \leq d(x_n, x^*) - s\varphi d(x_n, x^*) \leq s(H(x_n, F(x_n))) \\ \leq \psi d(x_n, x^*) \leq sH(x_n, F(x_n)) \\ \leq d(x_n, x^*) \leq \psi^{-1}(sH(x_n, F(x_n))) \to 0 \end{gathered}$$

as $n \to +\infty$

□

Theorem 2.2.14. [5]
Let (X, d) be a complete b-metric space such that the b-metric is a continuous functional on $X \times X$. Let $F : X \to X$ be a φ-contraction, such that $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a b-comparison function. Suppose that $SFix(F)$ the function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, $\psi(t) := t - s\varphi(t)$ is strictly increasing and onto. Then the fixed point problem for f is well-posed.

Proof. By Theorem 2.2.6 we have that $Fix(f) = x^*$. Next, we will prove that if $(x_n)_{n \in N}$ is a sequence in X such that $d(x_n, f(x_n)) \to 0$ as $n \to \infty$, then $x_n \to x^*$ as $n \to \infty$. Notice that $\psi : \mathbb{R}_+ \to \mathbb{R}_+, \psi(t) := t - s\varphi(t)$ is a bijection and $\psi^{-1}(\eta) \to 0$ as $\eta \to 0$. Then we have $d(x_n, x^*) \leq s[d(x_n, f(x_n)) + d(f(x_n), (x^*))] \leq s[d(x_n, f(x_n)) + d(f(x_n), f(x^*))] \leq s[d(x_n, f(x_n)) + \varphi d(x_n, x^*)]$. Hence $d(x_n, x^*) \leq \psi^{-1}(sd(x_n, f(x_n))) \to 0$ as $n \to +\infty$ □

2.3 Open question solution

Next an open question is given to investigate whether the fixed point problem for the multi-fractal operator T_F is well posed or not.

Open Question[5]

Let (X, d) be a b-metric space and $F := F_1, ..., F_m$ is an IMS such that the well-posedness property of the fixed point problem for each $F_i, i \in 1, 2, ..., m$ with respect to D or H takes place, then is the fixed point problem for the multi-fractal operator T_F well-posed? We try to give the affirmative answer to the question which guarantees that the well-posed property holds for the multifractal operator.

Theorem 2.3.1. Let (X, d) be a b-metric space and $F := F_1, ..., F_m$ is an IMS such that the well-posedness property of the fixed point problem for each $F_i, i \in 1, 2, ..., m$ with respect to D or H takes place, then is the fixed point problem for the multi-fractal operator T_F well-posed?

Proof. From Theorem 2.2.6 for gauge function, we have that $Fix[T_F] = A^*$ where $A_F^* \in P_{cp}(X)$. Next, we will prove that if $(A_n)_n \in \mathbb{N}$ is a sequence in $P_{cp}(X)$ such that $H(A_n), T_F(A_n) \to 0$ as $n \to \infty$, then $A_n \to A_F^*$ as $n \to \infty$,.Notice that $\psi := t - s\varphi(t)$ is a bijection and $\psi^{-1}(\eta) \to 0$ as $n \to 0$.

$$\begin{aligned} H(A_n, A^*) &\leq s[A_n, T_F(A_n) + T_F(A_n), A^*] \\ &\leq s[A_n, T_F(A_n) + T_F(A_n), T_F A^*] \\ &\leq s[A_n, T_F(A_n) + \varphi H(A_n, A^*)] \\ [H(A_n, A^*) - s\varphi H(A_n, A^*)] &\leq s[H(A_n, T_F(A_n))] \\ \psi(H(A_n, A^*)) &\leq s[H(A_n, T_F(A_n))] \\ (H(A_n, A^*)) &\leq \varphi^{-1}[s(H(A_n, T_F(A_n)))] \to 0 \ as \ n \to \infty. \end{aligned}$$

That is $\lim_n H(A_n, A^*) = 0$ □

Chapter 3

Fixed point theorems for multi-valued mappings involving gauge function in b-metric space

In this chapter, we investigate some results [[28], [16] and [17]] for multivalued mappings. This chapter is further divided into two sections. First we prove the fixed point and convergence theorem by replacing the single valued mapping with multi-valued mapping from a nonempty set X into nonempty proximinal closed subsets of X. In section 2, we extend the results from a nonempty set X into nonempty closed bounded subsets of X.

3.1 Multivalued φ-contraction in proximal closed sets

We will start this section by considering the following definition;

Definition 3.1.1. [1] Let (X, d) be a b-metric space with a coefficient $s \geq 1$, $T : X \to PC(X)$ and $x_0 \in X$. Then there exists a proximal orbit $\{x_n\} \subset X$ of T at the point x_0, i.e,

$$x_{n+1} \in Tx_n, \quad n = 0, 1, 2, ..., \tag{3.1.1}$$

with

$$d(x_n, x_{n+1}) = d(x_n, Tx_n)$$

Let $T : X \to PC(X)$ is an operator from $D \subset X$ into $PC(X)$ satisfying

$$H(Tx, Ty) \leq \varphi(d(x, Tx)) \tag{3.1.2}$$

for all $x \in O(T, x_0)$ and $y \in Tx$ such that $d(x, Tx) \in J$.
where φ is a gauge function of order $r \geq 1$ on an interval J.
where T satisfies (3.1.2). For convenience we define a function $E : D \to \mathbb{R}^+$ by $E(x) = d(x, Tx)$ and assume that there exist some $x_0 \in D$ such that $O(x_0) \subset D$,so that we can write condition 3.1.2 in the form

$$E(Ty) \leq \varphi(E(x)) \text{ for all } x \in X \text{ and } y \in Tx \text{ such that } E(x) \in J. \tag{3.1.3}$$

Lemma 3.1.2. Suppose $x_0 \in X$ is such that $O(x_0) \subset D$. Assume that $E(x_0) \in J$; then $E(x_n) \in J$ for all $n \geq 0$.

Proof. Note that for $x_0, x_1 \in O(T, x_0)$ and $x_1 \in Tx_0$ we have

$$\begin{aligned} E(x_1) &= d(x_1, Tx_1) \\ &\leq H(Tx_0, Tx_1) \\ &< \varphi(d((x_0, Tx_0))) \\ &< d(x_0, Tx_0) = E(x_0) \end{aligned}$$

Thus we have $E(x_1) \in J$. Continuing in the same way for each $x_{n-1}, x_n \in O(T, x_0)$ with $x_n \in Tx_{n-1}$ we have

$$\begin{aligned} E(x_n) &\leq \varphi(E(x_{n-1})) \\ &< E(x_{n-1}) \\ &\leq \varphi(E(x_{n-1})) \\ ... &\leq \varphi(E(x_0)) < E(x_0) \end{aligned}$$

which implies that $E(x_n) \in J$ for all $n \in N$. □

Definition 3.1.3. [28]
Suppose $x_0 \in D$ is such that $O(T, x_0) \subset D$ and $E(x_0) \in J$. Then for every iterate $x_n \in D, n \geq 0$, we define the closed ball $\overline{B}(x_n, \rho_n)$ with center at x_n and radius $\rho_n = s\sigma(E(x_n))$, where $\sigma : J \to \mathbb{R}^+$ is defined by 1.4.8.

Theorem 3.1.4. [6]Cantor's Theorem Let (X, d) be a complete b-metric space, then every nested sequence of closed balls has a nonempty intersection.

The proof of the theorem runs along the same line as done for the metric space.

Lemma 3.1.5. Suppose $x_0 \in D$ is such that $O(T, x_0) \subset D$ and $E(x_0) \in J$. Assume that $\overline{B}(x_n, \rho_n) \subset D$ for some $n \geq 0$; then $x_{n+1} \in D$ and $\overline{B}(x_{n+1}, \rho_{n+1}) \subset \overline{B}(x_n, \rho_n)$.

Proof. Since $E(x_0) \in J$. Lemma 3.1.2 implies that $E(x_n) \in J$ for all $n \geq 0$. The condition 1.4.9 implies $\sigma(t) \geq t$ for all $t \in J$. We have

$$d(x_n, x_{n+1}) = d(x_n, Tx_n) = E(x_n) \leq \sigma E(x_n) \leq s\sigma(E(x_n)) = \rho_n \tag{3.1.4}$$

Thus $x_{n+1} \in \overline{B}(x_n, \rho_n) \subset D$. Now let $x \in \overline{B}(x_{n+1}, \rho_{n+1})$. As $E(x_n) \in J$ so that from 3.1.3 and triangular inequality we have $E(x_{n+1}) \leq \varphi(E(x_n))$. By making use of 1.4.9 we get

$$\begin{aligned}
d(x, x_n) &\leq s[d(x, x_{n+1}) + d(x_{n+1}, x_n)] \\
&\leq s[\rho_{n+1} + d(x_n, Tx_n)] \\
&\leq s[\rho_{n+1} + E(x_n)] \\
&\leq s[s\sigma E(x_{n+1}) + E(x_n)] \\
&\leq s[s\sigma(\varphi E(x_n)) + E(x_n)] \\
&\leq s\sigma E(x_n) = \rho_n
\end{aligned}$$

Hence, $x \in \overline{B}(x_n, \rho_n)$. □

Definition 3.1.6. [28]
We say that a point $x_0 \in D$ is an initial orbital point of T if $E(x_0) \in J$ and $O(T, x_0) \subset D$.

Lemma 3.1.7. For every initial orbital point $x_0 \in D$ of T and every $n \geq 0$ we have

$$E(x_{n+1}) \leq \varphi(E(x_n)) \qquad \text{and} \qquad E(x_n) \leq \varphi^n(E(x_0)).$$

Furthermore, if φ is a gauge function of order $r \geq 1$ defined by (1.4.3) and (1.4.4), then

$$E(x_n) \leq (E(x_0))\mu^{P_n(r)} \qquad \text{and} \qquad \phi(E(x_n)) \leq s\mu^{r^n} = \phi(E(x_0))\mu^{r^n - 1}$$

where $\mu = \dfrac{\phi(E(x_0))}{s}$ and ϕ is a nonnegative and nondecreasing on J satisfying (1.4.3)and (1.4.4).

Proof. By making use of Lemma (3.1) we obtain $E(x_{n+1}) \leq \varphi(E(x_n))$. Since φ is nondecreasing, it easily follows that $E(x_n) \leq \varphi(E(x_0))$. Now from Lemma 1.4.7(1) we have

$$E(x_n) \leq \varphi^n(E(x_0)) \leq E(x_0)[\frac{\phi(E(x_0))}{s}]^{P_n(r)} = E(x_0)\mu^{P_n(r)}.$$

By using Lemma 1.4.7(2) we obtain

$$\phi(E(x_n)) \leq \phi(\varphi^n(E(x_0))) \leq s[\frac{\phi(E(x_0))}{s}]^{r^n} = s\mu^{r^n}.$$

□

We will use the following facts in the next lemma:

$$0 \leq \phi(t) < 1, \qquad P_j(r) \geq j, \qquad 0 \leq \mu^{r^n} < 1,$$

$$\text{where } r \geq 1, \mu = \frac{\phi(E(x_0))}{s} \text{ and } j = 0, 1, 2, ...$$

Lemma 3.1.8. Suppose $x_0 \in D$ is an initial orbital point of T and φ is a gauge function of order $r \geq 1$. Let ϕ be a nonnegative and nondecreasing on J defined by 1.4.3 and 1.4.4.Then for radii $\rho_n = s\sigma(E(x_n)), n = 0, 1, 2, ...$, the following estimates hold:

1. $\rho_n \leq sE(x_n) \sum_{j=0}^{\infty} [\phi(E(x_n))]^{P_j(r)} \leq \dfrac{sE(x_n)}{1 - \phi(E(x_n))}$;
2. $\rho_n \leq sE(x_n) \sum_{j=0}^{\infty} [\phi(E(x_0))\mu^{r^n - 1}]^{P_j(r)} \leq \dfrac{sE(x_n)}{1 - \phi(E(x_0))\mu^{r^n - 1}}$;
3. $\rho_n \leq sE(x_0)\mu^{P_n(r)} \sum_{j=0}^{\infty} [\phi(E(x_0))\mu^{r^n - 1}]^{P_j(r)} \leq sE(x_0) \dfrac{\mu^{P_n(r)}}{1 - \phi(E(x_0))\mu^{r^n - 1}}$;
4. $\rho_{n+1} \leq s\varphi E(x_n) \sum_{j=0}^{\infty} [\phi(\varphi(E(x_n)))]^{P_j(r)} \leq \dfrac{s\varphi E(x_n)}{1 - \phi(\varphi(E(x_n)))}$;
5. $\rho_{n+1} \leq s\varphi E(x_n) \sum_{j=0}^{\infty} [\phi(E(x_0))\mu^{r^{n+1} - 1}]^{P_j(r)} \leq \dfrac{s\varphi E(x_n)}{1 - \phi(E(x_0))\mu^{r^{n+1} - 1}}$;

where $\mu = \dfrac{\phi(E(x_0))}{s}$.

Proof. **(1)**
From definition of ρ_n, we have

$$\begin{aligned} \rho_n &= s\sigma(E(x_n)) = s\Sigma_{j=0}^{\infty} s^j \varphi^j(E(x_n)) \\ &\leq s\Sigma_{j=0}^{\infty} s^j E(x_n)[\frac{\phi(E(x_n))}{s}]^{P_j(r)} \qquad \text{using Lemma 1.4.7} \\ &= sE(x_n)\Sigma_{j=0}^{\infty} s^j [\frac{\phi(E(x_n))}{s}]^{P_j(r)} \\ &\leq sE(x_n)\Sigma_{j=0}^{\infty} [\phi(E(x_n))]^j = \frac{sE(x_n)}{1 - \phi(E(x_n))}. \end{aligned} \tag{3.1.5}$$

(2) From 3.1.5 we have

$$
\begin{aligned}
\rho_n &\leq sE(x_n)\sum_{j=0}^{\infty}[\phi(E(x_n))]^{P_j(r)} \\
&\leq sE(x_n)\sum_{n=0}^{\infty}[s\mu^{r^n}]^{P_j(r)} \quad \text{(using second part of Lemma 3.1.7)} \\
&= sE(x_n)\sum_{j=0}^{\infty}[\phi(E(x_0)\mu^{r^n-1})]^{P_j(r)} \\
&\leq sE(x_n)\sum_{j=0}^{\infty}[\phi(E(x_0))\mu^{r^n-1}]^{j} \\
&= \frac{sE(x_n)}{1-\phi(E(x_0))\mu^{r^n-1}}
\end{aligned}
$$

(3) By making use of first part of Lemma 3.1.7 above we have

$$
\begin{aligned}
\rho_n &\leq sE(x_n)\sum_{j=0}^{\infty}[\phi(E(x_0))\mu^{r^n-1}]^{P_j(r)} \\
&\leq sE(x_0)\mu^{P_n(r)}\sum_{j=0}^{\infty}[\phi(E(x_0))\mu^{r^n-1}]^{j} \\
&\leq \frac{sE(x_0)\mu^{P_n(r)}}{1-\phi(E(x_0))\mu^{r^n-1}}.
\end{aligned}
$$

(4) Now by making use of Lemma 1.4.7 we have

$$
\begin{aligned}
\rho_{n+1} &= s\sigma(E(x_{n+1})) \\
&= s\sum_{j=0}^{\infty}s^j\varphi^j(E(x_{n+1})) \\
&\leq sE(x_{n+1})\sum_{j=0}^{\infty}s^j[\frac{\phi(E(x_{n+1}))}{s}]^{P_j(r)} \\
&\leq s\varphi(E(x_n))\sum_{j=0}^{\infty}[\phi(\varphi(E(x_n)))]^{P_j(r)}.
\end{aligned}
$$

As $E(x_{n+1}) \leq \varphi(E(x_n))$ and ϕ is nondecreasing.

$$\begin{aligned} &\leq s\varphi(E(x_n))\sum_{j=0}^{\infty}[\phi(\varphi(E(x_n)))]^j \\ &= \frac{s\varphi(E(x_n))}{1-\phi(\varphi(E(x_n)))} \end{aligned}$$

(5) From (4) we have

$$\begin{aligned} \rho_{n+1} &\leq s\varphi(E(x_n))\sum_{j=0}^{\infty}[\phi(E(x_{n+1}))]^{P_j(r)} \\ &\leq s\varphi(E(x_n))\sum_{j=0}^{\infty}[\phi(E(x_0))\mu^{r^{n+1}-1}]^{P_j(r)} \quad (\textit{using Lemma } 3.1.7) \\ &\leq \frac{s\varphi(E(x_n))}{1-\phi(E(x_0))\mu^{r^{n+1}-1}} \end{aligned}$$

□

Theorem 3.1.9. Let $T : D \subset X \to PC(X)$ be a multivalued operator on a complete b-metric space (X, d) such that the b-metric is continuous and T satisfies (3.1.2) with a b-Bianchini-Grandolfi gauge function φ of order $r \geq 1$ on an interval J with coefficient $s \geq 1$. Then starting from an initial orbital point x_0 of T the iterative sequence 3.1.1 remains in $\overline{B}(x_0, \rho_0)$ and converges to a point ξ which belongs to each of the closed balls $\overline{B}(x_n, \rho_n), n = 0, 1, 2, ...,$ where $\rho_n = s\sigma(d(x_n, x_{n+1})), \sigma$ defined in 1.4.9 and $s \geq 1$ is a coefficient of b-metric space. Furthermore, for each $n \geq 1$ we have

$$d(x_{n+1}, x_n) \leq \varphi(d(x_n, x_{n-1})).$$

If $\xi \in D$ and the function $E(x) = d(x, T(x))$ on D is T-orbitally lower semi-continuous at ξ, then ξ is a fixed point of T.

Proof. Since $x_0 \in D$ is an initial orbital point of T, from Lemma 3.3 we have

$$\overline{B}(x_{n+1}, \rho_{n+1}) \subset \overline{B}(x_n, \rho_n) \ for\ all\ n \geq 0.$$

Thus $x_n \in \overline{B}(x_0, \rho_n)$ for all $n \geq 0$. According to the definition of ρ and using Lemma 3.5 we have

$$\begin{aligned} \rho_n &= s\sigma(E(x_n)) \leq s\sigma(\varphi^n(E(x_0)) \\ &= s\sum_{j=0}^{\infty}s^j\varphi^j(\varphi^n(E(x_0))) \\ &= \frac{1}{s^{n-1}}\sum_{j=n}^{\infty}s^j\varphi^j(\varphi^n(E(x_0))) \ for\ all\ n \geq 0 \end{aligned} \tag{3.1.6}$$

Since φ is a b-Bianchini-Grandolfi gauge function, from (3.1.6) we obtain

$$\rho_n \to 0 \qquad as \qquad n \to \infty$$

which implies that $\overline{B}(x_n, \rho_n)$ is a nested sequence of closed balls. By Cantor's theorem (for complete b-metric spaces), we deduce that there exists a unique point ξ such that $\xi \in \overline{B}(x_n, \rho_n)$ for all $n \geq 0$ and $x_n \to \xi$ or equivalently, $\lim_{n\to\infty} d(x_n, \xi) = 0$. From Lemma 3.5 we obtain following:

$$\begin{aligned} d(x_n, x_{n+1}) = d(x_n, Tx_n) = E(x_n) &\leq \varphi(E(x_{n-1})) \\ &= \varphi(d(x_{n-1}, x_n)) \end{aligned}$$

Thus we conclude that

$$d(x_n, x_{n+1}) = d(x_n, Tx_n) \leq \varphi^n(d(x_0, Tx_1))$$

By applying $n \to \infty$, we have $d(x_n, Tx_n) = 0$ Since $x_n \to \infty$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Then by T-lower semi continuity of $E(x) = d(x, Tx)$ at ξ, we have $E(\xi) = d(\xi, T\xi) \leq \lim_{n\to\infty} \inf E(x_n) = \lim_{n\to\infty} \inf d(x_n, Tx_n) = 0$ □

Theorem 3.1.10. Let $T : D \subset X \to PC(X)$ be an operator on a complete b-metric space (X, d) such that the b-metric is continuous and let T satisfy 3.1.2 with a b-Bianchini-Grandolfi gauge function φ of order $r \geq 1$ and a coefficient s on an interval J. Further, suppose that $x_0 \in D$ is an initial orbital point of T, then the following statements hold true.

1 The iterative sequence (3.1.1) remains in $\overline{B}(x_0, \rho_n)$ and converges at least $r \geq 1$ to a point ξ which belongs to each of the closed balls $\overline{B}(x_n, \rho_n), n = 0, 1, ...,$ and

$$\begin{aligned} \rho_n = sd(x_n, x_{n+1}) &\sum_{j=0}^{\infty}[\phi(d(x_n, x_{n+1}))] \\ &\leq \frac{sd(x_n, x_{n+1})}{1 - \phi(d(x_n, x_{n+1}))} \end{aligned}$$

where ϕ is nonnegative and nondecreasing function on J satisfying (1.4.3) and (1.4.4).

2 For all $n \geq 0$ the following prior estimates holds:

$$d(x_n, \xi) \leq \frac{E(x_0)}{s^{n-1}}$$

$$\sum_{j=n}^{\infty} \phi(E(x_0))^{P_j(r)} = d(x_0, fx_0)$$

$$\frac{\phi(E(x_0))^{P_n(r)}}{s^{n-1}[1 - \phi(E(x_0))^{r^n}]}.$$

3 For all $n \geq 1$ the following posterior estimates holds:

$$\begin{aligned} d(x_n, \xi) &\leq s\varphi(d(x_n, x_{n-1})) \sum_{j=0}^{\infty} [\phi(\varphi(d(x_n, x_{n-1})))]^{P_j(r)} \\ &\leq \frac{s\varphi(d(x_n, x_{n-1}))}{1 - \phi[\varphi(d(x_n, x_{n-1}))]} \\ &\leq \frac{s\varphi(d(x_n, x_{n-1}))}{1 - \phi(d(x_n, x_{n-1}))[\frac{\phi(d(x_n,x_{n-1}))}{s}]^{r-1}}. \end{aligned}$$

4 We have

$$d(x_{n+1}, x_n) \leq \varphi(d(x_n, x_{n-1})) \leq \mu^{P_n(r)} d(x_0, fx_0)$$

for all $n \geq 1$.

5 If $\xi \in D$ and the function $G(x) = d(x, Tx)$ on D is T-orbitally lower semi-continuous at ξ, then ξ is a fixed point of T.

Proof. **1** From Theorem (3.1.9) it follows that the iterative sequence (3.1.1) remains in $\overline{B}(x_0, \rho_0)$ and converges to a point ξ which belongs to each of the closed balls $\overline{B}(x_n, \rho_n)$,n=0,1,... and from Lemma 3.1.8 estimate (1) we have

$$\rho_n \leq sd(x_n, x_{n+1}) \sum_{j=0}^{\infty} [\phi(d(x_n, x_{n+1}))]^{P_j(r)} \leq \frac{sd(x_n, x_{n+1})}{1 - \phi(d(x_n, x_{n+1}))}.$$

2 For $m \geq n$,

$$\begin{aligned} d(x_n, x_m) &\leq sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + ... + s^{m-n} d(x_{m-1}, x_m)) \\ &= \frac{1}{s^{n-1}} \sum_{j=n}^{m-1} s^j E(x_j) \\ &\leq \frac{1}{s^{n-1}} \sum_{j=n}^{m-1} s^j \varphi^j(E(x_j)). \text{ (from Lemma 3.1.7)} \end{aligned}$$

$E(x_n) \leq \varphi^n(E(x_0)))$

$$\begin{aligned} &\leq \sum_{j=n}^{m-1} s^j E(x_0) [\frac{\phi(E(x_0))}{s}]^{P_j(r)} \text{ (using Lemma (1.4.7)} \\ &\leq \frac{E(x_0)}{s^{n-1}} \sum_{j=n}^{m-1} \lambda^{P_j(r)}, \end{aligned}$$

Where $\lambda = \phi(E(x_0))$.Keeping n fixed and letting $m \to \infty$ we get

$$d(x_n, \xi) \le \frac{E(x_0)}{s^{n-1}} \sum_{j=n}^{\infty} \lambda^{P_j(r)} = \frac{d(x_0, fx_0)}{s^{n-1}} \sum_{j=n}^{\infty} \lambda^{P_j(r)}. \tag{3.1.7}$$

Since

$$r^n + r^{n+1} \ge 2r^n, \qquad r^n + r^{n+1} + r^{n+2} \ge 3r^n, \quad ...,$$

we have

$$\lambda^{r^n + r^{n+1}} \le \lambda^{2r^n}, \quad \lambda^{r^n + r^{n+1} + r^{n+2}} \le \lambda^{3r^n}, \quad ....$$

Thus it implies

$$\begin{aligned} \sum_{j=n}^{\infty} \lambda^{P_j(r)} &= \lambda^{P_j(r)} + \lambda^{P_{j+1}(r)} + ... \\ &= \lambda^{P_n(r)}[1 + \lambda^{r^n} + \lambda^{r^n + r^{n+1}} + \lambda^{r^n + r^{n+1} + r^{n+2}} + ...] \\ &\le \lambda^{P_j(r)}[1 + \lambda^{r^n} + \lambda^{2r^n} + \lambda^{3r^n} + ...] \\ &= \frac{\lambda^{P_n(r)}}{1 - \lambda^{r^n}}. \end{aligned} \tag{3.1.8}$$

Hence from (3.1.7) we obtain

$$d(x_n, \xi) \le \frac{E(x_0)}{s^{n-1}} \sum_{j=n}^{\infty} \phi(E(x_0))^{P_j(r)} = d(x_0, fx_0) \frac{\phi(E(x_0))^{P_n(r)}}{s^{n-1}[1 - \phi(E(x_0))^{r^n}]}.$$

3 From (3.1.7) we have for $n \ge 0$

$$d(x_n, \xi) \quad \le \quad \frac{E(x_0)}{s^{n-1}} \sum_{j=n}^{\infty} [\phi(d(x_0, x_1))]^{P_j(r)}.$$

Setting $n = 0, y_0 = x_0$ and $y_1 = x_1$ we have

$$d(y_0, \xi) \quad \le \quad sd(y_0, y_1) \sum_{j=0}^{\infty} [\phi(d(y_0, y_1))]^{P_j(r)}.$$

Setting again $y_0 = x_n$ and $y_1 = x_{n+1}$ gives

$$\begin{aligned} d(x_n,\xi) &\leq sd(x_n,x_{n+1})\sum_{j=0}^{\infty}[\phi(d(x_n,x_{n+1}))]^{P_j(r)} \\ &\leq s\varphi(d(x_n,x_{n-1}))\sum_{j=0}^{\infty}[\phi(\varphi(d(x_n,x_{n-1})))]^{P_j(r)} \\ &\leq s\varphi(d(x_n,x_{n-1}))\sum_{j=0}^{\infty}[\phi(\varphi(d(x_n,x_{n-1})))]^{j} \\ &= \frac{s\varphi(d(x_n,x_{n-1}))}{1-\phi(\varphi(d(x_n,x_{n-1})))}. \end{aligned} \tag{3.1.9}$$

From Lemma 1.4.7(2) we obtain

$$\begin{aligned} \phi(\varphi(d(x_n,x_{n-1}))) &\leq s[\frac{\phi(d(x_n,x_{n-1}))}{s}]^r \\ &= \phi(d(x_n,x_{n-1}))[\frac{\phi(d(x_n,x_{n-1}))}{s}]^{r-1} \end{aligned} \tag{3.1.10}$$

which implies

$$\frac{1}{1-\phi(\varphi(d(x_n,x_{n-1})))} \leq \frac{1}{1-\phi(\varphi(d(x_n,x_{n-1})))[\frac{\phi(d(x_n,x_{n-1}))}{s}]^{r-1}}, \tag{3.1.11}$$

Thus from 3.1.9and 3.1.11 we deduce for $n \geq 1$,

$$\begin{aligned} d(x_n,\xi) &\leq \frac{s\varphi(d(x_n,x_{n-1}))}{1-\phi(\varphi(d(x_n,x_{n-1})))} \\ &\leq \frac{s\varphi(d(x_n,x_{n-1}))}{1-\phi(\varphi(d(x_n,x_{n-1})))}[\frac{\phi(d(x_n,x_{n-1}))}{s}]^{r-1} \end{aligned}$$

4 We have

$$\begin{aligned} d(x_{n+1},x_n) &= E(x_n) \leq \varphi(E(x_{n-1})) \\ &= E(x_{n-1})\frac{\phi(E(x_{n-1}))}{s} \\ &\leq E(x_0)\mu^{P_{n-1}(r)}\mu^{r^{n-1}} \qquad \text{(using Lemma (1.4.6))} \\ &= E(x_0)\mu^{P_{n-1}(r)+r^{n-1}} \\ &= E(x_0\mu^{P_n(r)} = \mu^{P_n(r)}d(x_0,fx_0). \end{aligned}$$

5 Its proof follows from Theorem 3.1.9.

□

Remark 3.1.11. When $s = 1$ the b-metric space reduce to the classic metric on X. Thus Theorem 3.1.9 and 3.1.10 extend and generalize [[16],Theorem 2.11 & 2.15] to the case of b-metric space.

3.2 Multivalued φ-contractions in closed bounded sets

We start with the following intuitive lemmas.

Lemma 3.2.1. Let (X, d, s) be a b-metric space and $B \in CB(X)$ and $x \in X$. Then for $\epsilon > 0$ there exists $b \in B$ such that

$$d(x, b) \leq d(x, B) + \epsilon$$

Lemma 3.2.2. Let $A, B \in CB(X)$ and let $a \in A$. If $\epsilon > 0$,then there exists $b \in B$ such that

$$d(a, b) \leq H(A, B) + \epsilon$$

To establish a fixed point theorem it is important to mention that we do not necessarily need the gauge functions φ satisfying 1.4.3 and 1.4.4. we consider the gauge function such that $\sum_{n=0}^{\infty} s^n \varphi^n(t) \leq \infty$ for all $t \in J$ where s is a coefficient of b-metric space.

Theorem 3.2.3. Let (X, d, s) be a complete and continuous b-metric space and D is a closed subset of X with $s \geq 1$, φ is b-Bianchini Grandolfi gauge function on interval J and $T : D \to CB(X)$ be a mapping such that $Tx \cap D \neq \emptyset$ and

$$H(Tx \cap D, Ty \cap D) \quad \leq \quad \varphi(d(x, y)) \tag{3.2.1}$$

for all $x \in D$ and $y \in Tx \cap D$ with strict inequality holds when $x \neq y$ and $d(x, y) \in J$. Suppose $x_0 \in D$ is such that $d(x_0, z) \in J$ for some $z \in Tx_0 \cap D$. Then the following assertion holds:

1. there exists a sequence $\{x_n\}$ of T in D such that $x_{n+1} \in Tx_n$ for all $n = 0, 1, 2, ...$ and $\xi \in D$ such that $lim_n x_n = \xi$;

2. ξ is a fixed point of T in D if and only if function $f(x) = d(x_n, Tx \cap D)$ is T-orbitally lower semi-continuous at ξ.

Proof. Setting $x_1 = z \in Tx_0 \cap D$ we have $d(x_0, x_1) \neq 0$, otherwise x_0 is a fixed point of T. Let $\rho_0 = \sigma d(x_0, x_1)$ where σ is defined by 1.4.3. Since from 1.4.4,$\sigma(t) \geq t$ so we have

$$d(x_0, x_1) \leq \rho_0 \tag{3.2.2}$$

Thus $x_1 \in B(x_0, \rho_0)$. Since $d(x_0, x_1) \in J$, so from 3.2.1 it follows that

$$H(Tx_0 \cap D, Tx_1 \cap D) \quad < \quad \varphi(d(x_0, x_1))$$

Choose an $\epsilon_1 > 0$ such that

$$H(Tx_0 \cap D, Tx_1 \cap D) + \epsilon_1 \leq \varphi(d(x_0, x_1)) \tag{3.2.3}$$

It follows from lemma 3.2.2 that there exists $x_2 \in Tx_1 \cap D$ such that

$$d(x_1, x_2) \leq H(Tx_0 \cap D, Tx_1 \cap D) + \epsilon_1, \tag{3.2.4}$$

Since D is closed and Tx_1 is closed and bounded. We assume that $d(x_1, x_2) \neq 0$, for otherwise x_1 is fixed point of T. From inequalities (3.2.4) and (3.2.3) we have

$$d(x_1, x_2) \leq \varphi(d(x_0, x_1)). \tag{3.2.5}$$

Also since $d(x_1, x_2) \leq \varphi(d(x_0, x_1)) < d(x_0, x_1)$ which implies

$$d(x_1, x_2) \in J$$

By using triangular inequality. We have

$$\begin{aligned} d(x_0, x_2) &\leq s(d(x_0, x_1) + sd(x_1, x_2)) \\ &\leq sd(x_0, x_1) + s^2 d(x_1, x_2) \\ &\leq sd(x_0, x_1) + s^2 \varphi(d(x_0, x_1)) \text{ using } 3.2.5 \\ &= s[d(x_0, x_1) + s\varphi(d(x_0, x_1))] \\ &\leq s\sigma[d(x_0, x_1)] \ \text{ (by using 1.4.9)} \\ &\leq s\sigma(d(x_0, x_1)) + d(x_0, x_1) \\ &= \sigma(d(x_0, x_1)) = \rho_0 \text{ by using 1.4.9 .} \end{aligned}$$

Thus $x_2 \in B(x_0, \rho_0)$. Since, $d(x_1, x_2) \in J$ so that from 3.2.1 it follows that

$$H(Tx_1 \cap D, Tx_2 \cap D) < \varphi(d(x_1, x_2))$$

Choose $\epsilon_2 > 0$ such that

$$H(Tx_1 \cap D, Tx_2 \cap D) + \epsilon_2 \leq \varphi(d(x_1, x_2)). \tag{3.2.6}$$

It again follows from lemma 3.2.2 that there exists $x_3 \in Tx_2 \cap D$ such that

$$d(x_2, x_3) \leq H(Tx_1 \cap D, Tx_2 \cap D) + \epsilon_2 \tag{3.2.7}$$

We assume that $d(x_2, x_3) \neq 0$, for otherwise x_2 is fixed point of T. From inequalities (3.2.5),(3.2.6) and (3.2.7) we have

$$d(x_2, x_3) \leq \varphi^2(d(x_0, x_1)). \tag{3.2.8}$$

Also we have $d(x_2, x_3) \leq \varphi d(x_1, x_2)$ which implies

$$d(x_2, x_3) \in J$$

By using triangular inequality

$$\begin{aligned} d(x_0, x_3) &\leq sd(x_0, x_1) + s^2 d(x_1, x_2) + s^3 d(x_2, x_3) \\ &\leq s[d(x_0, x_1) + sd(x_1, x_2) + s^2 d(x_2, x_3)] \\ &\leq s[d(x_0, x_1) + s\varphi(d(x_1, x_2)) + s^2\varphi^2(d(x_2, x_3))] \text{ using 3.2.8} \\ &\leq s\sigma(d(x_0, x_1)) \\ &\leq s\sigma(d(x_0, x_1)) + d(x_0, x_1) \\ &= \sigma(d(x_0, x_1)) = \rho_0 \text{ by using 1.4.9} \end{aligned}$$

$x_3 \in B(x_0, \rho_0)$ Continuing in the same way we get a seq $\{x_n\}$ in $B(x_0, \rho_0)$ such that $x_n \in Tx_{n-1} \cap D, x_{n-1} \neq x_n$ and

$$d(x_n, x_{n+1}) = d(x_n, Tx_n) \leq \varphi^n(d(x_0, x_1)) \tag{3.2.9}$$

Using triangular inequality for any $p \geq 1$ we have,

$$\begin{aligned} d(x_n, x_{n+p}) &\leq s^n d(x_n, x_{n+1}) + s^{n+1} d(x_{n+1}, x_{n+2}) \\ + ... + s^{n+p-1} d(x_{n+p-1}, x_{n+p}) & \\ &\leq s^n \varphi^n d(x_0, x_1) + s^{n+1}\varphi^{n+1} d(x_0, x_1) \\ + ... + s^{n+p-1}\varphi^{n+p-1} d(x_0, x_1) & \end{aligned} \tag{3.2.10}$$

Since, φ is a b-Bianchini Grandolfi gauge function so we assume that

$$S_n = \sum_{i=1}^{n} s^i \varphi^i(d(x_0, x_1)) \; and \; \lim_{n\to\infty} S_n = S. \tag{3.2.11}$$

So that from 3.2.10 we obtain

$$d(x_n, x_{n+p}) \leq [S_{n+p-1} - S_n] \tag{3.2.12}$$

It can be seen from (3.2.11), relation (3.2.12) implies $d(x_n, x_{n+p}) \to 0$ as $n \to \infty$. Which shows that $\{x_n\}$ is a Cauchy sequence in a $B(x_0, \rho_0)$. Since $B(x_0, \rho_0)$ is a closed ball in X. Then it contain a point $\xi \in B(x_0, \rho_0)$ such that $x_n \to \xi$. Note that

$\xi \in D$, as well. Since $x_n \in Tx_{n-1} \cap D$ and $d(x_{n-1}, x_n) \in J$ for n=1,2,... it follows from (3.2.1) that

$$\begin{aligned} d(x_n, Tx_n \cap D) &\leq H(Tx_{n-1} \cap D, Tx_n \cap D) \\ &\leq \varphi(d(x_{n-1}, x_n) \\ &< d(x_{n-1}, x_n) \end{aligned} \quad (3.2.13)$$

Letting $n \to \infty$ from (3.2.13) we get

$$\lim_{n\to\infty} d(x_n, Tx_n \cap D) = 0. \quad (3.2.14)$$

Suppose $f(x) = d(\xi, T\xi \cap D)$ is T-orbitally lower continuous at ξ, then

$$d(\xi, T\xi \cap D) = f(\xi) \leq \liminf f(x_n) = \liminf d(x_n, Tx_n \cap D) = 0$$

Hence, $\xi \in T\xi$, since $T\xi$ is closed. Conversely, if ξ is fixed point of T then $f(\xi) = 0 \leq \liminf f(x_n)$, since $\xi \in D$. □

Theorem 3.2.4. Let (X, d, s) be a complete b-metric space such that b-metric d is continuous functional. Let D be closed subset of X, φ a b-Bianchini Grandolfi gauge function on an interval J satisfying1.4.3 and 1.4.4. Assume $T : D \to CB(X)$ be a mapping such that $Tx \cap D \neq \emptyset$ and

$$H(Tx \cap D, Ty \cap D) \leq \varphi(d(x, y)) \quad (3.2.15)$$

for all $x \in D$ and $y \in Tx \cap D$ with strict inequality holds when $x \neq y$ and $d(x, y) \in J$. Suppose $x_0 \in D$ is such that $d(x_0, z) \in J$ for some $z \in Tx_0 \cap D$. Then the following assertion holds:

1 There exist a sequence $\{x_n\}$ with $x_{n+1} \in Tx_n; n = 0, 1, ...$ in $B(x_0, \rho_o)$ that converges with the rate of convergence at least r to a point $\xi \in B(x_0, \rho_n)$

2 For all $n \geq 0$ the following prior estimates holds:

$$\begin{aligned} d(x_n, \xi) &\leq \frac{d(x_0, x_1)}{s^{n-1}} \sum_{j=n}^{\infty} \phi(d(x_0, x_1))^{P_j(r)} \\ &= d(x_0, Tx_0) \frac{\phi(d(x_0, x_1))^{P_n(r)}}{s^{n-1}[1 - \phi(d(x_0, x_1))^{r^n}]}. \end{aligned} \quad (3.2.16)$$

3 For all $n \geq 1$ the following posterior estimates holds:

$$\begin{aligned} d(x_n, \xi) &\leq s\varphi(d(x_n, x_{n-1})) \sum_{j=0}^{\infty} [\phi(\varphi(d(x_n, x_{n-1})))]^{P_j(r)} \\ &\leq \frac{s\varphi(d(x_n, x_{n-1}))}{1 - \phi[\varphi(d(x_n, x_{n-1}))]} \\ &\leq \frac{s\varphi(d(x_n, x_{n-1}))}{1 - \phi(d(x_n, x_{n-1}))[\frac{\phi(d(x_n, x_{n-1}))}{s}]^{r-1}}. \end{aligned} \quad (3.2.17)$$

4 for all $n \geq 1$ we have

$$d(x_{n+1}, x_n) \leq \varphi(d(x_n, x_{n-1})) \leq \mu^{P_n(r)} d(x_0, Tx_0) \tag{3.2.18}$$

for all $n \geq 1$.

5 ξ is a fixed point of T if and only if the function $f(x) = d(x, Tx \cap D)$ is T-orbitally lower semi continuous at ξ.

Proof. **1** The proof follows from Theorem 3.2.3

2 For $m \geq n$,

$$\begin{aligned} d(x_n, x_m) &\leq sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + ... + s^{m-n} d(x_{m-1}, x_m)) \\ &\leq s\varphi(d(x_n, x_{n+1})) + ... + s^{m-n}\varphi^{m-n}(d(x_{m-1}, x_m)) \\ &\leq \frac{1}{s^{n-1}} \sum_{j=n}^{m-1} s^j \varphi^j (d(x_0, x_1)).(\text{ from Lemma(3.1.7)}) \end{aligned}$$

$$d(x_n, x_{n+1}) \leq \varphi^n(d(x_0, x_1))$$

$$\begin{aligned} &\leq \sum_{j=n}^{m-1} s^j d(x_0, x_1)[\frac{\phi(d(x_0, x_1))}{s}]^{P_j(r)}(\text{using Lemma (1.4.7)} \\ &\leq \frac{d(x_0, x_1)}{s^{n-1}} \sum_{j=n}^{m-1} \lambda^{P_j(r)}, \end{aligned}$$

Where $\lambda = \phi(d(x_0, x_1))$.Keeping n fixed and letting $m \to \infty$ we get

$$d(x_n, \xi) \leq \frac{d(x_0, x_1)}{s^{n-1}} \sum_{j=n}^{\infty} \lambda^{P_j(r)} = \frac{d(x_0, Tx_0)}{s^{n-1}} \sum_{j=n}^{\infty} \lambda^{P_j(r)}.$$

Since

$$r^n + r^{n+1} \geq 2r^n, \qquad r^n + r^{n+1} + r^{n+2} \geq 3r^n, \qquad ...,$$

we have

$$\lambda^{r^n + r^{n+1}} \leq \lambda^{2r^n}, \qquad \lambda^{r^n + r^{n+1} + r^{n+2}} \leq \lambda^{3r^n}, \qquad ....$$

Thus it implies

$$\begin{aligned} \sum_{j=n}^{\infty} \lambda^{P_j(r)} &= \lambda^{P_j(r)} + \lambda^{P_{j+1}(r)} + ... \\ &= \lambda^{P_n(r)}[1 + \lambda^{r^n} + \lambda^{r^n + r^{n+1}} + \lambda^{r^n + r^{n+1} + r^{n+2}} + ...] \\ &\leq \lambda^{P_j(r)}[1 + \lambda^{r^n} + \lambda^{2r^n} + \lambda^{3r^n} + ...] \\ &= \frac{\lambda^{P_n(r)}}{1 - \lambda^{r^n}}. \end{aligned}$$

Hence from (3.1.7) we obtain

$$d(x_n,\xi) \leq \frac{d(x_0,x_1)}{s^{n-1}} \sum_{j=n}^{\infty} \phi(d(x_0,x_1))^{P_j(r)} = d(x_0,Tx_0) \frac{\phi(d(x_0,x_1))^{P_n(r)}}{s^{n-1}[1-\phi(d(x_0,x_1))^{r^n}]}.$$

3 From (3.1.7) we have for $n \geq 0$

$$d(x_n,\xi) \quad \leq \quad \frac{d(x_0,x_1)}{s^{n-1}} \sum_{j=n}^{\infty} [\phi(d(x_0,x_1))]^{P_j(r)}.$$

Setting $n = 0, y_0 = x_0$ and $y_1 = x_1$ we have

$$d(y_0,\xi) \quad \leq \quad sd(y_0,y_1) \sum_{j=0}^{\infty} [\phi(d(y_0,y_1))]^{P_j(r)}.$$

Setting again $y_0 = x_n$ and $y_1 = x_{n+1}$ gives

$$\begin{aligned} d(x_n,\xi) &\leq sd(x_n,x_{n+1}) \sum_{j=0}^{\infty} [\phi(d(x_n,x_{n+1}))]^{P_j(r)} \\ &\leq s\varphi(d(x_n,x_{n-1})) \sum_{j=0}^{\infty} [\phi(\varphi(d(x_n,x_{n-1})))]^{P_j(r)} \\ &\leq s\varphi(d(x_n,x_{n-1})) \sum_{j=0}^{\infty} [\phi(\varphi(d(x_n,x_{n-1})))]^{j} \\ &= \frac{s\varphi(d(x_n,x_{n-1}))}{1-\phi(\varphi(d(x_n,x_{n-1})))}. \end{aligned}$$

From Lemma 1.4.7(2) we obtain

$$\begin{aligned} \phi(\varphi(d(x_n,x_{n-1}))) &\leq s[\frac{\phi(d(x_n,x_{n-1}))}{s}]^r \\ &= \phi(d(x_n,x_{n-1}))[\frac{\phi(d(x_n,x_{n-1}))}{s}]^{r-1} \end{aligned}$$

which implies

$$\frac{1}{1-\phi(\varphi(d(x_n,x_{n-1})))} \quad \leq \quad \frac{1}{1-\phi(\varphi(d(x_n,x_{n-1})))[\frac{\phi(d(x_n,x_{n-1}))}{s}]^{r-1}},$$

Thus from 3.1.9and 3.1.11 we deduce for $n \geq 1$,

$$\begin{aligned} d(x_n,\xi) &\leq \frac{s\varphi(d(x_n,x_{n-1}))}{1-\phi(\varphi(d(x_n,x_{n-1})))} \\ &\leq \frac{s\varphi(d(x_n,x_{n-1}))}{1-\phi(\varphi(d(x_n,x_{n-1})))}[\frac{\phi(d(x_n,x_{n-1}))}{s}]^{r-1} \end{aligned}$$

4 We have

$$\begin{aligned} d(x_{n+1}, x_n) &= d(x_n, x_{n+1}) \leq \varphi(d(x_{n-1}, x_n)) \\ &= d(x_{n-1}, x_n)\frac{\phi(d(x_{n-1}, x_n))}{s} \\ &\leq d(x_0, x_1)\mu^{P_{n-1}(r)}\mu^{r^{n-1}} \quad \text{(using Lemma (3.1.2))} \\ &= d(x_0, x_1)\mu^{P_{n-1}(r)+r^{n-1}} \\ &= d(x_0, x_1)\mu^{P_n(r)} = \mu^{P_n(r)}d(x_0, Tx_0). \end{aligned}$$

5 Its proof runs along the same lines as the proof of Theorem 3.2.3.

□

Remark 3.2.5. For $s = 1$ the b-metric space reduces to the classic metric on X. Thus Theorem 3.2.3 and Theorem 3.2.4 extend and generalize [[17],Theorem 2.1 & 2.8] to the case of b-metric space. Theorem 3.2.3 and 3.2.4 also includes [[16],Theorem 2.11 & 2.15] as a special case when range of T is taken as $CB(X)$ instead of the space of all nonempty proximinial closed subsets of X. Moreover when T is a single valued mapping then Theorem 3.2.3 and 3.2.4 reduce to [[25], Theorem 4.1 & 4.2].
Theorem 3.2.3 and 3.2.4 extends [[28], Theorem 3.7 & 3.10] to the case of multivalued mappings.

Corollary 3.2.6. Let (X, d) be a complete b-metric space such that b-metric d is a continuous functional. Let φ a Bianchini Grandolfi gauge function of order $r \geq 1$ on interval J and assume T be a mapping from D into $CB(X)$ such that

$$H(Tx, Ty) \leq \varphi(d(x, y)) \tag{3.2.19}$$

for all $x, y \in X$ and $y \in Tx \cap D$ with $d(x, y) \in J$. Suppose that $x_0 \in X$ such that $d(x_0, z) \in J$ for some $z \in Tx_0 \cap D$. Then the following assertions hold.

1. there exists a sequence $\{x_n\}$ with $x_n \in Tx_{n-1}; n = 1, 2, ...$ that converges to a fixed point $\xi \in S = \{x \in X : d(x, \xi) \in J\}$ of T;
2. The estimates [3.2.16-3.2.18] are valid.

Proof. From 3.2.19 we have

$$H(Tx, Ty) \leq \varphi(d(x, y)) < d(x, y) \text{ for all } x, y \in X, x \neq y.$$

Thus T is continuous. Hence the conclusions (1) & (2) follows from Theorem 3.2.4.

□

3.3 Conclusion

We establish some fixed point theorems for multivalued mappings satisfying contractive condition involving gauge function when the underlying primary structure is b-metric space. Our proposed iterative scheme converges to the the fixed point with higher order. Moreover, we also calculate priori and posteriori estimates for the fixed point.Our main results generalize/extend many per-existing results in literature.

Furthermore, we also did detailed analysis of the paper "Multivalued fractals in b-metric space" [5] and give an affirmative answer to the question whether the fixed point problem for multi-fractal operator T_F is well posed or not.

Bibliography

[1] I. A. Bakhtin, *"The contraction mapping principle in almost metric spaces"*, Funct. Anal., Gos. Ped. Inst., Unianowsk, 30(1989), 26-37.

[2] M. F. Barnsley, *"Fractals Everywhere"*, Academic Press, Boston, (1988).

[3] V. Berinde, *"Sequences of operators and fixed points in quasimetric spaces"*, Studia Univ. Babeş-Bolyai, Math., 16(1996), 23-27.

[4] R. M. Bianchini, M. Grandolfi, *"Trasformazioni di tipo contrattivo generalizzato in uno spazio metrico"*, Atti Accad. Naz. Lincei, Rend. Cl. Sci. Fis. Mat. Nat.45(1968), 212-216.

[5] M. Boriceanu, M. Bota, A. Petruel, *"Multivalued fractals in b-metric spaces"*, A. centr.eur.j.math, 8(2010), 367-377.

[6] M. Bota, A. Molnar, C. Varga, *"On Ekeland's variational principle in b -metric spaces"*, Fixed Point Theory, 12(1)(2011), 21-28.

[7] D. W. Boyd, J. S. W. Wong, *"On nonlinear contractions"*, Proceedings of the American Mathematical Society 20 (1969), 1-18.

[8] F. E. Browder, *"On the convergence of successive approximations for nonlinear functional equations"*, Indag. Math., 30(1968), 27-35.

[9] S. Chandok, T. D. Narang, M. A. Taoudi, *"Some common fixed point results in partially ordered metric spaces for generalized rational type contraction mappings"*, Vietnam J. Math. 41(3)(2013), 323-331.

[10] S. Czerwik, *"Functional Equations and Inequalities in Several Variables"*, World Scientific, Singapore, (2002).

[11] P. Z. Daffer, H. Kaneko, *"Fixed points of generalized contractive multivalued mappings"*, J. Math. Anal. Appl., 192(1995), 655-666.

[12] J. Heinonen, *"Lectures on Analysis on Metric Spaces"*, Springer, Berlin; (2001).

[13] J. E. Hutchinson, *"Fractals and self-similarity"*, Indiana Univ. Math. J., 30(1981), 713-747.

[14] M. Jovanovi, Z. Kadelburg, S. Radenovi, *"Common fixed point results in metric-type spaces"*, Fixed Point Theory Appl, (2010).

[15] M. S. Khan, M. Swaleh, S. Sessa, *"Fixed point theorems by altering distances between the points"*, Bull. Aust. Math. Soc. 30(1984), 19.

[16] Q. Kiran, T. Kamran, *"Nadler'stype principle with high order of convergence"*, Nonlinear Anal,TMA, 69(2008), 4106-4120.

[17] Q. Kiran, T. Kamran, *"Fixed point theorems for generalized contractive multi-valued maps"*, Computers and Mathematics with Applications, 59(2010),3812-3823.

[18] A. Latif, *"Banach Contraction Principle and Its Generalizations"*, Springer International Publishing, (2013), 33-64.

[19] M. A. Miandaragh, M. Postolache, S. Rezapour, *"Approximate fixed points of generalized convex contractions"*, Fixed Point Theory Appl, 2013, Article ID 255.

[20] M. A. Miandaragh, M. Postolache, S. Rezapour, *"Some approximate fixed point results for generalized alpha-contractive mappings"*, Sci. Bull. Politeh. Univ. Buchar., Ser. A, Appl. Math. Phys. (2013), 75(2):310.

[21] S. B. Nadler, *"Multivalued contraction mappings"*, Pac. J. Math, 30(1969), 475-488.

[22] M. O. Olatinwo, M. Postolache, *"Stability results for Jungck-type iterative processes in convex metric spaces"*, Appl. Math. Comput, 218(12)(2012), 6727-6732.

[23] M. Păcurar (Berinde), *"Iterative methods for fixed point approximation"*, Ph.D. thesis, Babeş-Bolyai University Cluj-Napoca, Romania, 2009.

[24] M. Păcurar (Berinde) , *"A fixed point result for -contractions on b-metric spaces without the boundedness assumption"*,Poznan University of Technology, Fasciculi Mathematici, 43(2010), 127-137.

[25] P. D. Proinov, *"A generalization of the Banach contraction principle with high order of convergence of successive approximations"*, Nonlinear Anal, 67(2007), 2361-2369.

[26] E. Rakotch, *"A note on contractive mappings"*, Proc. Am. Math. Soc, 13(1962), 459-465.

[27] I. A. Rus, *"Generalized Contractions and Applications"*, Cluj University Press, Cluj-Napoca, (2001).

[28] M. Samreen, Q. Kiran, T. Kamran, *"Fixed point theorems for φ-contractions"*, Journal of Inequalities and Applications, (2014), 1-16.

[29] Hu. Shouchuan, N. S. Papageorgiou, *"Handbook of Multivalued Analysis"*, Vol. I, II, Kluwer Acad. Publ., Dordrecht, 1997, 1999. 1968, 45, 212-216.