# Hilbert spaces and their role in the Approximation Property

by

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## MASTER'S THESIS WORK

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# Dedicated to My parents

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# Abstract

The main aim of this thesis is to consider the approximation property for a Hilbert space. We initiated by considering the approximation property for a complete normed vector space, that is, a Banach space. We try to collect known theorems from the literature regarding a Banach space and give their extract in terms of a Hilbert space. Examples are also constructed regarding the main definitions. Furthermore, there is an emphasis on the approximation property and its further variations including the role of the bounded approximation property and the metric approximation property.

# Contents

# Introduction

Approximation plays an important role everywhere in mathematical analysis. The term 'approximation property' has many distinct meanings and it can be studied in many different contexts. The essential point to consider is the approximation property in the 'spaces'. The approximation problem in the spaces is very old problem and authors still investigating certain issues regarding this. This thesis is devoted to the problems while developing the approximation properties in the normed spaces. Specifically in a normed space, the approximation is a game between operators. Grothendiek initiated the detailed systematic learning of the approximation property ([4], [7]) in 1955. He provided many results regarding the approximation property. He also provide many equivalent statements regarding the approximation property in Banach spaces (complete normed vector space).

Approximation property ($\mathcal{AP}$) that involves the dual form is also considered in this thesis. As an initiative, we deeply studied the work of Figiel and Johnson [11]. We collect the definitions and notations given by [11]. What happens if we strict or relax the condition on the main space, that is, the Banach space. This thesis examine whether a Hilbert space (complete inner product space) also fulfills same conditions for the approximation property. Why a Hilbert space? We emphasis on a Hilbert space since it has the richest geometry among the normed spaces. By richest geometry, we mean that a Hilbert space is good because of its 'best approximation', orthogonality and duality concepts. A Hilbert space satisfies the $\mathcal{AP}$ if there is a finite rank operator whose limit is a compact operator. Firstly we thoroughly examine [11] and provide the extracts for Hilbert space. A Hilbert space $H$ and its subspace denoted by $K$ is considered, and further discuss the conditions under which the pair $(H, K)$ satisfies the approximation property. The important definition which revolves around the uniform approximation of identity operator and the role of finite rank operator is same for the Hilbert space as provided by [11] for the Banach space.

This thesis also highlights some applications and examples regarding the approximation property in a Banach space. The natural question arises: does every Hilbert space satisfies approximation property? The answer is 'no'; we have a counterexample involving a "nicest" Hilbert space of infinite dimension. There is a subspace of a Hilbert space (of infinite dimension), $l_2$ (the space of square sumable sequences), without the approximation property [8].

Dual space is also considered and results regarding the dual space are also highlighted. Main properties which are usually under consideration includes bounded approximation property, approximation property and basis property. However, there is an emphasis on the approximation property and its further variations including the role of the bounded approximation property. Some colors of the metric approximation property are also given. Our thesis is divided into the following chapters:

- Chapter 1 covers the basic concepts regarding the spaces and the approximation properties.

- Chapter 2 describes the role of the approximation properties in Banach spaces. Many results are provided in this chapter and regarding these results definitions are also given. This chapter mainly focuss on the literature review [15]. Also consider the review of paper [11].

- Chapter 3 includes the concepts regarding a Hilbert space. This is the core section of the thesis. In this section all results are provided in terms of a Hilbert space. This chapter covers results regarding a separable Hilbert space, bounded approximation property as well as metric approximation property for a Hilbert space. This chapter highlights our own findings and formulation of results.

# Chapter 1

# Preliminaries

This chapter is about some basic definitions involving the concepts regarding the Banach spaces and Hilbert spaces. Most of the definitions given in this chapter are contained in [17].

## 1.1 Review of basic definitions

**Definition 1.1.** Let us consider $X$, a real vector space (where $\mathbb{R}$ denotes the set of reals). On $X$, a norm $\|.\|$ is a real valued also non negative function. The value at an $x \in X$ is simply denoted by "$\| x \|$". Also, it follows these properties,

$$\textbf{(N1)}\ \| x \| \geq 0 \text{ and } \| x \| = 0 \Leftrightarrow x = 0,$$

$$\textbf{(N2)}\ \| \alpha x \| = |\alpha| \, \| x \|, \forall x \in X, \alpha \in \mathbb{R},$$

$$\textbf{(N3)}\ \| x + y \| \leq \| x \| + \| y \|, \forall x, y \in X.$$

Norm simply generalizes the concept of length. The spaces which satisfies the properties of norm are said to be normed spaces. If we can define a norm on a vector space then it is said to be a normed space $X$.
Recall that an operator maps one normed space $X$ into another normed space $Y$ whereas

a mapping from normed space into the scaler field $\mathbb{R}$ or $\mathbb{C}$ (set of complex are denoted by $\mathbb{C}$) is called a functional.

**Definition 1.2.** We define a linear space over a field $\mathbb{F}$ as a nonempty set $X$ of elements which are known as vectors and they are denoted by $x, y, ....$ They basically have two operations and these includes,

a) vector addition and

b) multiplication of vectors by elements of field (scalers).

**Definition 1.3.** Consider a vector space over field $\mathbb{F}$ (real or complex), which has the properties of norm. If it is complete with respect to that norm then it is said to be a Banach space denoted by $X$. For $X$, a"complete normed linear space" is another term.

Many authors provide their results and theorems by considering a "separable" Banach space. A separable space comes in a category of a topological space, for which there is a subset which is countable subset as well as a dense subset in that space.

**Definition 1.4.** Suppose $X$ denotes a vector space. On a vector space, an inner product is a mapping defined by

$$\langle ., . \rangle : X \times X \to \mathbb{F}.$$

The inner product of $x$ and $y$ can be written as $\langle x, y \rangle$. For vectors $x$, $y$ and $z$ and scalers $a$ and $b$, it satisfies the following properties:

**(I1)** Linear in first argument $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$.

**(I2)** Conjugate symmetric $\langle x, y \rangle = \overline{\langle y, x \rangle}$.

**(I3)** Positive $\langle x, x \rangle = 0 \Leftrightarrow x = 0$.

**(I4)** Non negative $\langle x, x \rangle \geq 0$.

A vector space $X$ along with an inner product defined on it constitutes an inner product space. Furthermore, "a complete inner product space" is called a Hilbert space.

Class of normed spaces covers every inner product space, with respect to the norm defined as follows:

$$\|x\| = \langle x, x \rangle^{\frac{1}{2}}.$$

Throughout the thesis $H$ denotes the Hilbert space.

**Definition 1.5.** Consider an inner product space denoted by $I$, also there is a set $J$ such that $J \subset I$. A subset $J$ whose elements are pairwise orthogonal is called an "orthogonal set".
**Definition 1.6.** Consider an inner product space $I$. An orthonormal set $J \subset I$ is an orthogonal set in $I$. The elements have norm 1, that is, for all $y, z \in J$,

$$\langle y, z\rangle = \begin{cases} 0 & \text{if } y \neq z \\ 1 & \text{if } y = z. \end{cases}$$

In a Hilbert space which is separable, a maximal orthonormal sequence is known as a complete orthonormal bases.
The concept dimension is also used in the thesis. In Hilbert space (or subspace), the dimension is defined as the "cardinality" of an orthonormal basis.
**Definition 1.7.** Suppose there is a Hilbert space $H$. If $K \subset H$ is a linear subspace of a Hilbert space $H$ the

$$K^{\perp} = \{h \in H; \langle h, k\rangle = 0, \forall k \in K\},$$

is a linear subspace which is closed and $K \cap K^{\perp} = 0$. $K^{\perp}$ is basically the orthogonal complement of $K$.

Since we have considered the space $l^2$ (the space of square sumable sequences) frequently in our thesis, so, we need to consider the fact that any Hilbert spaces of infinite dimension (over the complex numbers) is isomorphic to $l^2$. Extensively, consider a map involving a Hilbert space $H$ which is linear and given as, $T : H \rightarrow l^2$ it is one-to-one, onto and satisfies

$$(Tu, Tv)_{l^2} = (u, v)_H \text{ and } \| Tu \|_{l^2} = \| u \|_H \text{ for all } u, v \in H.$$

First equality shows that the distance is preserved whereas the second equality determines that the norm is preserved. Therefore, it shows an isomorphism.

## 1.2 Concepts regarding operators

**Definition 1.8.** $T$ is called a linear operator which satisfies:
a) the domain of $T$ (denoted by $domT$) is a vector space .

b) a range of $T$ denoted by $ranT$ lies in a vector space over a same field.
c) for all $s, t$ in $domT$, and scaler $a$,

$$T(s+t) = Ts + Tt$$

$$T(as) = aTs.$$

**Definition 1.9.** Suppose $X$ and $X'$ be normed-vector spaces. We define a linear operator $T : X \to X'$ as a bounded linear operator if we obtain a real constant $\alpha$ such that $\forall x \in X$,

$$\|Tx\| \leq \alpha \|x\|.$$

**Definition 1.10.** Let us consider a normed space $X$. A bounded functional (linear) denoted by $g$ is defined to be a bounded linear operator whose range lies in the scaler field of $X$. So, we obtain a real number $\alpha$ such that $\forall x$ in $domf$ we have,

$$\|gx\| \leq \alpha \|x\|.$$

**Definition 1.11.** Let us consider $g$, a linear functional (bounded) from a normed space $X$ into field $F$ ($\mathbb{R}$ or $\mathbb{C}$). Then

$$X^* = \{g : X \to F \mid g \text{ is linear and bounded}\}.$$

$X^*$ is known to be the "dual" space of $X$. Only Hilbert spaces are self dual (A category that is equivalent to its dual).
Furthermore, In case of Hilbert space $H$ we can say that $H^* = H$.

Throughout the thesis we mainly consider the bounded linear operators and their class is denoted by $L(X, Y)$. Identity operators are also discussed in various manner, they are basically a class of operators that leaves unchanged the elements on which they operate. Formally, the identity operator denoted by $I_x$ is operated as follows: $I_x : X \to X$ given by $I_x x = x$, $\forall x \in X$.
**Definition 1.12.** Suppose $Z$ be a metric space. $Z' \subseteq Z$ is relatively compact in $Z$, if $\overline{Z'}$ (closure of $Z'$ includes all closed sets, specifically their intersection, containing $Z'$) is compact in $Z$.
**Definition 1.13.** Consider two normed spaces denoted by $X$ and $\acute{X}$. We define compact linear operator as an operator denoted by $C : X \to \acute{X}$ if
a) $C$ satisfies linearity and
b)if for every subset $M$ of $X$ ($M \subseteq X$) which is bounded, we have a relatively compact

image denoted by $C(M)$, that means, the closure $\overline{C(M)}$ is compact. If we consider the Hilbert spaces, in uniform operator topology, compact operators are precisely the closure of operators of finite dimension (i.e. finite rank operator).

**Definition 1.14.** [24]Suppose we have an operator $T$ and $X$ is considered to be a normed space (also linear), the range of $T$ is defined as

$$ran(T) = \{T(x) : x \in X\},$$

Dimension of vector space is the rank of $T$ (the cardinality or the number of vectors of a basis of vector space, over its base field) of its range, that is,

$$rank(T) = dim(ran(T)).$$

Particularly $T$ is of finite rank, if range(T) has the finite dimension.

Finite rank operators are highlighted in the next chapter.

**Definition 1.15.** Suppose $T : V \to V'$ be a linear map and we consider the bases for $V$ and $V'$ as $B$ and $C$ respectively. Then each element $b \in B$ has unique representation in the form

$$b = \sum_{c \in C} \lambda_{bc} c.$$

In case of the spaces of finite dimension, the matrix $M = [T]_{B,C} = (\lambda_{bc})_{b \in B, c \in C}$ is called the matrix representation for $T$ w.r.t. the bases $B$ and $C$.

**Definition 1.16.** Consider $V$ which denotes a vector space of finite dimension and also consider a linear map $h : V \to V$. We can define the trace of this map by considering the trace of matrix representation of $h$. Basically basis are chosen for $V$ and then $h$ is described as a matrix relative to this basis, and then the trace is taken in the usual manner of this square matrix.

**Definition 1.17.** Consider $X$, a Banach space. Then the operator on $X$ is said to be nuclear if it is a compact operator [6] for which we may define trace. Further the trace has two properties; it is finite, and the choice of basis is independent. They are also known as trace class operators.

**Definition 1.18.** Suppose $S$ is any linear operator. An eigenfunction of $S$ is basically defined on some function space. It is any non-zero function $g$ in that space such that, when acted upon by operator $S$, is only multiplied by some scaling factor called an eigenvalue. For some scaler eigenvalue $\lambda$, mathematically , this condition can be written as

$$Sg = \lambda g.$$

**Definition 1.19.** Suppose $H$ and $H^{'}$ are two separable Hilbert spaces, [6] $T \in L(H, H^{'})$ is a "nuclear operator" if there exists a complete orthonormal set $e_n$ of $H$ such that,

$$\sum_{n=0}^{\infty} \|Te_n\|_{H'} < \infty.$$

Now onwards class of nuclear operators from $H$ to $H^{'}$ are denoted by $N(H, H^{'})$. For $T \in N(H, H^{'})$ we define

$$\|T\| \equiv \inf \{\sum_{n=0}^{\infty} \|Te_n\|_{H'} \mid e_n \text{is a complete orthonormal set of H}\}.$$

Then $N(H, H^{'})$, $\|.\|$ comes in a category of a Banach space. (By definition if there is any cauchy sequence in $N(H, H^{'})$, it converges). If $H = H^{'}$ is a separable Hilbert space and $T \in N(H, H)$ then we define the trace of $T$ as follows:

$$\text{Trace(T)} = \sum_j \langle Te_j, e_j \rangle,$$

where $e_j$ is a complete orthonormal set of $H$ and simply trace is the sum over the eigenvalues of operator. We may verify that the definition is independent the "choice" of the complete orthonormal set of $H$.

**Definition 1.20.** Firstly a normed space $\mathcal{X}$ is considered. A sequence $(b_n) \in \mathcal{X}$ is defined as a Schauder basis for $\mathcal{X}$, if we consider a unique sequence $(\alpha_n)$ of scalers for each $x \in X$ such that

$$\|x - \sum_{k=1}^{n} \alpha_k b_k\| \to 0 \text{ as } n \to \infty \text{ or}$$

$$x = \sum_{n=1}^{\infty} \alpha_n b_n.$$

Simply each element $x \in \mathcal{X}$ can be written as an "infinite linear combination" which is unique.

Schauder basis are similar to Hamel basis the only variation is that Hamel basis involves the linear combinations in which sums are finite, while Schauder basis may include infinite sums.

**Definition 1.21.** Suppose $X$ denotes a Banach space, $\{A_n\}_{n=1}^{\infty}$ is a "sequence" of closed subspaces of $X$ and is said to be a Schauder decomposition of $X$ if every $x \in X$ can be written uniquely in the form

$$x = \sum_{n=1}^{\infty} x_n, \text{ with } x_n \in A_n \text{ for every } n.$$

**Definition 1.22.** $X$ is a vector space, its two subspaces denoted by $Y$ and $Z$. Then the direct sum of $Y$ and $Z$ [20] constitutes $X$, mathematically we have $X = Y \oplus Z$, every vector $x \in X$ can be represented in a unique way as follows:

$$x = y + z, y \in Y, z \in Z.$$

**Definition 1.23.** Consider a linear space $\mathcal{X}$, we define a projection on $\mathcal{X}$ as a linear map $P : \mathcal{X} \to \mathcal{X}$ s.t.

$$P^2 = P.$$

There is an association between any projection a direct sum decomposition.

**Definition 1.24.** Suppose $X$ and $X^{'}$ are two vector spaces and let $T$ be a linear map such that $T : X \to X^{'}$. If $0_{X^{'}}$ is the zero vector of $X^{'}$, then the kernel of $T$ is the subset of $X^{'}$ which consists all those elements of $X^{'}$ that are mapped by $T$ to the element $0_{X^{'}}$. The kernel is denoted by ker $T$ and mathematically written as

$$\text{ker}T = \{x \in X : T(x) = 0_{X^{'}}\}.$$

**Definition 1.25.** Consider $X$ a Banach space and its closed linear subspace is denoted by $M$. $M$ is complemented subspace of $X$ if it has the following conditions.

"Firstly, we denote the bounded linear projection by $P$. $M$ is the range of $P$ from $X$ onto $M$ then there is an isomorphism between $X$ and the "direct sum" of $M$ and ker $P$. Mathematically, written as $X \simeq M \oplus ker(P)$."

OR

We also have the simple definition given below:

**Definition 1.26.** Consider a Banach space $X$ and a subspace $Y$, the subspace is complemented if $X = Y \oplus Z$ where $Z \subset X$.

We have the definition of strong operator topology as follows:

**Definition 1.27.** $H$ is a Hilbert space, and $S_n$ be the sequence of linear operators on $H$. If,

$$S_n x \to S x \forall x \in X,$$

then $S_n \to S$ in the strong operator topology.

**Definition 1.28.** Suppose $X$ and $Y$ are normed spaces. A sequence $S_n$ of operators $S_n \in L(X, Y)$ is called converges uniformly (uniform operator convergence) if $S_n$ converges in the norm on $L(X, Y)$ [6], that is, $\|S_n - S\| \to 0$.

In case of Hilbert space we generally have strong operator topology and weak operator topology. Now we consider a result given as follows:

**Remark 1.29.** For $H$- any Hilbert space, let $\mathscr{F}$ be the collection of all such subspaces of $H$ which are finite dimensional. For each $F \in \mathscr{F}$ let $P_F$ be the orthogonal projection onto $F$. Then $P_F \to id_H$ ($id_H$ denotes the identity of $H$) in the strong operator topology and thus are uniformly convergent on compact subset of $H$.

Some noticeable facts regarding Hilbert space are now highlighted which mostly are contained in [22]:

- Orthonormal basis are present in every Hilbert space.
- If $H$ and $H^{'}$ are Hilbert spaces, an isomorphism is a linear surjection $U : H \to H^{'}$ such that, $\forall x, y \in H$,
$$\langle Ux, Uy \rangle = \langle x, y \rangle.$$
- A sequence of operators $S_n \in L(H)$ ($L(H)$ denotes the class of bounded linear operators on Hilbert space H) converges uniformly (or in norm) to an operator $S$ if
$$\| S_n - S \| \to 0 \text{ as, } n \to \infty.$$
- Whenever a Hilbert space $H$ has a subspace $K$ which is closed, we can always write
$$H = K \oplus K^{\perp}.$$

# Chapter 2

# The approximation property and a Banach space

The "approximation property" for a Banach space is very arguable topic as well as an open problem. For "a Banach space $X$", definitions regarding the approximation property are already given in the past and many authors use these concepts to provide some useful results. In this chapter the concepts regarding Banach spaces, some results and arguments are provided so that we can use them and succeed towards our main aim - a Hilbert space. In this thesis, we studied approximation property for a Banach space as a review of literature so that our concepts are clear and we may use these concepts to move towards a Hilbert space.

We now highlight "the approximation property" in terms of the Banach spaces. This definition is already given in the literature.

**Definition 2.1.** Between the Banach spaces, a bounded linear operator of "finite dimensional" range is defined to be the finite rank operator.

**Definition 2.2.** Suppose $X$ is a Banach space. Let every compact set $K$ which is contained in $X$ and for every $\epsilon > 0$, there exists a finite rank operator $T : X \to X$ such that,

$$\| Tz - z \| < \epsilon \text{ for every } z \in K. \tag{2.1}$$

If the above condition (2.1) is satisfied. Then $X$ is said to have the "approximation property" and throughout the thesis it is denoted by $\mathcal{AP}$.

## 2.1 Examples

- Identity operator on any Banach space which is finite dimensional, is the trivial example of a space satisfying the $\mathcal{AP}$.

- Let $X = \mathbb{R}^2$, and $K$ be the compact set defined as,

$$K = \{(x, y) : -2 \leq x \leq 2, -2 \leq y \leq 2 \text{ and } x - y = 0\}.$$

$K \subseteq X$ and define finite rank operator as:

$$T : \mathbb{R}^2 \to \mathbb{R}^2, \text{ such that:}$$

$$T(x, y) = (x - \epsilon y, x - \delta y), \text{ where } \epsilon \text{ and } \delta \text{ are very small numbers.}$$

By simple calculation we may observe that the operator $T$ is bounded as well as linear. Now, let us consider a random point $(x, y) = (1, 1)$ and $(x, y) \in K$, choose $\epsilon = 0.001$ and $\delta = 0.0001$. Following the definition of the approximation property for the Banach space we have,

$$\|T(x, y) - (x, y)\| = \|(-1 \times 10^{-3}, (-1 \times 10^{-4})\|$$
$$= \sqrt{(-1 \times 10^{-6}) + (-1 \times 10^{-8})}$$
$$= 0.001 < \epsilon.$$

Thus $X = \mathbb{R}^2$ satisfies the $\mathcal{AP}$.

**Definition 2.3.** Suppose $X$ is a Banach space. Let every compact set $K$ which is contained in $X$ and for every $\epsilon > 0$, also if there is a constant $C > 0$, there is an operator $T : X \to X$ of finite rank, such that for each such $T$,

$$\| T \| \leq C \text{ we have } \| Tz - z \| < \epsilon \text{ for every } z \in K. \tag{2.2}$$

If the above condition (2.2) is satisfied. Then $X$ is said to have the "bounded approximation property" [24] and throughout the thesis it is denoted by $\mathcal{BAP}$.

Approximation between operators is the problem which was first discussed in 1932. Approximation plays an important role in Banach spaces. Some useful facts regarding the approximation properties are collected from the literature and given as follows [15] :

- Approximation property plays a key role in the "structure theory of Banach spaces" [15].
- The important aspect of the $\mathcal{AP}$ for a Banach space is; sometimes, the approximation by two different classes of operators with respect to different topologies actually coincides [3].
- In our thesis important relations between approximation properties and spaces are discussed.

The authors of [11] are mainly concerned about the Banach space $X$ and a subspace $Y$, and then they discuss the conditions under which the pair $(X, Y)$ satisfies the $\mathcal{AP}$. Firstly we must mention some variations in the $\mathcal{AP}$. One of them is the $\lambda$-approximation property. $\lambda$-approximation property is defined as follows:
**Definition 2.4.** Suppose $X$ is a Banach space let $1 \leq \lambda \leq \infty$, for every compact set $K \subseteq X$ and for every $\epsilon > 0$, there is an operator $T : X \to X$ of finite rank so that,

$$\|Tx - x\| \leq \epsilon, \text{ for every} x \in X \text{ and } \|T\| \leq \lambda. \tag{2.3}$$

If the above condition (2.3) is satisfied. Then $X$ is said to have the "$\lambda$-approximation property" denoted by $\lambda - \mathcal{AP}$.

## 2.2 Definitions and notations for the pair $(X, Y)$

Moving towards the introduction to the main paper we studied. Since we use this paper to succeed towards the formulation of our own results so we must mention the basic definitions and notions. These definitions further are converted into Hilbert space with necessary amendments.

**Definition 2.5.** Suppose $X$ be a metric space and $B$ be the subset of $X$. To define $\epsilon-$ net for $B$ we need an $\epsilon > 0$ for which there is $M \subset X$. So, $M$ is an $\epsilon-$ net for $B$ if for every point $x \in B$ there is a point of $M$ at a distance from $x$ less than $\epsilon$.

**Definition 2.6.** Consider a vector space $V$. [13] A subspace $W$ of $V$ is an invariant subspace of a linear mapping $T : V \to V$ that is preserved by $T$, that is, $T(W) \subseteq W$.

The main needed results involves the definition for the pair $(X, Y)$. The pair $(X, Y)$ satisfies $(\mathcal{AP})$ if the identity $(id)$ on $X$ is the $\tau$-limit of the net of finite rank bounded linear operators on $X$ all of which leave the subspace $Y$ invariant [11]. Further, $\tau$- topology on the space $L(X)$ of bounded linear operators on the Banach space $X$ is basically the uniform convergence topology on compact subsets of $X$ [11].

If the approximating net of finite rank operators can be chosen so that $\lambda$ uniformly bounds their norms, then the pair $(X, Y)$ is said to have $\lambda$-$\mathcal{BAP}$, and $(X, Y)$ has the $\mathcal{BAP}$ provided it has ($\lambda$- $\mathcal{BAP}$) for some $\lambda < \infty$. Note that, when the subspace $Y$ is either the whole space or the zero subspace, then all concepts reduce to the classical concepts of $\mathcal{AP}$ and $\mathcal{BAP}$ for a single space. In simple words, let $X$ and $Y$ be Banach spaces. The pair $(X, Y)$ is said to have the $\mathcal{AP}$, if every operator $T$ [24] in $L(X, Y)$ is approximable [5] where $L(X, Y)$ denotes the bounded linear operators between $X$ and $Y$.

Since we are examining [11] so, we must need to strengthen the argument regarding uniform convergence of compact sets in Banach spaces.

**Lemma 2.1.** *[23] Consider $X$ and $Y$ be Banach spaces. There is $(T_\alpha)_{\alpha \in A} \subset L(X, Y)$, a uniformly bounded net such that $T_\alpha x \to Tx$ for all $x \in X$ and some $T \in L(X, Y)$. Then, considering the topology of uniform convergence [20] on compact sets, $T_\alpha$ converges to $T$ in that topology.*

Now, moving towards the next section, first we need some definitions;

Suppose $X$ and $Y$ are considered to be the Banach spaces, and $X \otimes Y$ be their tensor products as vector spaces.

**Definition 2.7.** $X \otimes Y$ consists of such elements which all may be written in many ways as a formal linear combinations of formal tensor products of elements of $X$ and $Y$:

$$\sum_i \alpha_i x_i y_i.$$

Let the projective cross norm $\| z \|_\pi$ of an element $z$ of $X \otimes Y$ be defined as,

$$\| z \|_\pi = \inf \{ \sum_i | \alpha_i | \| x_i \|_X \| y_i \|_Y \text{ such that } z = \sum_i \alpha_i x_i y_i \}.$$

The completion of $X \otimes Y$ under the projective cross norm is the the projective tensor product (denoted by) $X \otimes_{\pi}^{\wedge} Y$ of $X$ and $Y$.

## 2.3 Nuclear operators and properties

We consider the nuclear operators, because of their important property, that is, trace may be well defined in nuclear operators. Thus trace is finite and independent of the basis. Some important properties of nuclear operator [24] are as follows:

- In mathematical quantum mechanics, nuclear operators were appeared for the first time and they were known as "trace-operators with a trace."

- Considering the topology of uniform convergence on bounded sets. Finite rank has the limit which is basically the nuclear operator $N$. We may approximate the nuclear operator $N$ by finite dimensional operator (finite rank) specifically in the nuclear norm.

- Consider "nbhd" as neighborhood and all linear mappings [14] from $E$ into $F$ are denoted by $L(E, F)$ a vector space and these mappings are also continuous. Every nuclear operator $N$ in $L(E, F)$ [14] is compact. Simply it maps a nbhd of zero in $E$ into a set with closure (which is compact) in $F$.

- Further, the nuclear operators are always continuous.

Nuclear operators are denoted by $N(X)$, for $X$ a Banach space.

Next important term used again and again in our thesis is, the "finite rank operators". Finite rank operators plays a chief role in the working of the $\mathcal{AP}$.

## 2.4 Finite rank operators and properties

Some important properties of these operators are as follows:

- They are compact and also,

- Sum of two operators of finite rank is also a finite rank operator.

- For any finite rank operator in the Hilbert spaces, "the dimension of the range of operator $T$", is equal to "the dimension of the range of $T^*$", also known as "row rank = column rank".

- The norm limits of bounded operators of finite rank, in Banach spaces are "compact".

- Consider a bounded linear functional denoted by $f$, as an example. It is basically a particular case of a finite rank operator, also named as rank one operator.

Following section covers all needed theorems and results used in the thesis.

## 2.5 Results and theorems

This section includes some needed theorems. Let us begin with the lemma needed for the proof of a next theorem. This lemma and the very next theorem given by Figiel and Johnson [11] are the base from where we started the working of our thesis and after that we investigate different new directions regarding a Hilbert space. $\mathcal{F}(X)$ denotes the class of finite rank operators [12] also consider the same notation.

**Lemma 2.2.** *[11] Let $\mathcal{F}_Y(X) = \{T \in \mathcal{F}(X) : TY \subseteq Y\}$, (where $TY \subseteq Y$ denotes that the subspace $Y$ is invariant).*
*1. $x^* \otimes x \in \mathcal{F}_Y(X) \Leftrightarrow$ if either $x^* \in Y^\perp$ or $x \in Y$.*
*2. If $F \in \mathcal{F}_Y(X)$, then $F$ is the sum of $n$ rank one element of $\mathcal{F}_Y(X)$, where $n$ is the rank of $F$.*

Now, we consider the main theorem provided by [11].

**Theorem 2.3.** *[11] Suppose that $Y \subseteq X$ and $X$ satisfies the AP. Following are the equivalent statements.*
*1. The pair $(X, Y)$ has the $\mathcal{AP}$.*
*2. For all $T \in N(X)$ ([12] consider the same notations) for which $TX \subseteq Y$ and $TY = 0$*

*we have* $tr(T) = 0$.

This thesis also mention the comparison between the results in Banach spaces and Hilbert spaces. Our next section is about the comparison and aim of the thesis.

## 2.6 Comparison between the Hilbert spaces and the Banach spaces

In Banach spaces [12] we are mainly concerned with the concept of norm but in Hilbert spaces, norm comes from an inner product. Simply, a Hilbert space comes in a category of a Banach space if it is equipped with an inner product which induces the norm. Every Banach space is not necessarily a Hilbert space, but the converse is true, this is the reason we also consider the conditions for a Hilbert space. Particularly, in case of finite dimensional spaces [14], both a Hilbert space and a Banach space are equal and it can easily be shown by considering the example of well known space $C[a, b]$, notation for the space of continuous real valued functions. Whenever, a Banach space follows the definition of the very famous **parallelogram law** then it is basically a Hilbert space [13].

The Hilbert spaces are good in comparison with the Banach spaces because they have the "best approximation" whereas in a Banach space we may find an example of closed convex sets in which there is no element of minimal norm, or there are infinitely many of them. Best approximation and the approximation property are studied in a different manner. Hilbert spaces are usually isometrically isomorphic to its dual. But, in case of a Banach space there is no isometry between a space and its dual. Schauder basis are given a great importance in this thesis. So, when we talk about basis, an orthonormal basis are present in every Hilbert space, any Banach space has a Hamel basis but these basis does not interact good with the topology. "Some" Banach spaces have a Schauder basis.

Before moving towards another task, some applications regarding the theorems are provided. Some examples which satisfies the approximation property are also stated, counter example is also provided. The purpose behind this is to strengthen the argument regard-

ing the approximation properties in normed spaces.
$\mathcal{AP}$ and the $\mathcal{BAP}$ are closely related. In this chapter some results regarding bounded approximation property are also provided. So, we have another main theorem stated as follows:
**Theorem 2.4.** *[15] Consider $X$, a separable Banach space. $X$ has the ($\mathcal{BAP}$) $\Leftrightarrow$ $X$ is a complemented subspace of a Banach space with a Schauder basis.*

$\mathcal{BAP}$ always implies the $\mathcal{AP}$. To elaborate above theorem we are providing some examples;

### 2.6.1 Application regarding the Banach space satisfying the approximation property

**Example 2.1.** Consider $X = \mathbb{R}^n$, notation for the n-dimensional Euclidean space. This space is separable. We need to show that $\mathbb{R}^n$ is the complemented subspace of a Banach space, also, with a Schauder basis. For this consider the more special case $X = \mathbb{R}^3$, let $M$ be the plane through the origin. Then any line through the origin that does not lie in $M$ is a complemented subspace. Thus, $\mathbb{R}^n$ is the separable Banach space that satisfies the $\mathcal{BAP}$ and thus also satisfies the $\mathcal{AP}$.

### 2.6.2 Application regarding a Banach space and no fulfilment of the approximation property

$l^1$ is un-complemented and this fact is taken from [1].
**Example 2.2.** Consider the case of $X = l^1$ (space of sequences). $X$ clearly is the separable Banach space and there are also Schauder basis for $X = l^1$. But $l^1$ does not have complemented subspace. Extensively, [10] since every separable Banach space is linearly isometric to a quotient space of $l^1$ [10] so for a separable Banach space $X$ we have the following map $A : l^1 \to X$ such that $X \simeq l^1/kerA$. In general there does not exists a subspace $Y$ of $l^1$ such that $l^1 = Y \oplus kerA$. So, $l^1$ has uncountably many un-complemented subspaces. Therefore, we have the counter example for the above theorem and $l^1$ is the separable Banach space that does not satisfies the $\mathcal{BAP}$ and thus also not satisfies the

$\mathcal{AP}$.

Banach spaces not always necessarily have the Schauder decomposition. According to [2], there exists a Banach space which is separable but fails to have a Schauder decomposition.

We may observe that the Theorem (2.3) can also be true for Hilbert space, but we need to consider a separable Hilbert space so that we can easily define nuclear operators. The concept of topology can also be implicated on the Hilbert space. We also want to find out the various applications regarding the approximation property in combination with the normed spaces.

## 2.7 Theorems and results from the literature

This chapter highlights the statements of theorems. After that we provide these theorems and results for Hilbert space in our main chapter that is chapter 3. Following all results are mentioned in Johnson and Lindenstrauss handbook- Volume 1, [15].

**Definition 2.8.** Suppose $X$ is a Banach space, for every compact set $K \subseteq X$ and for every $\epsilon > 0$, there exist an operator $T$ of finite rank i.e. $T \in \mathcal{F}(X, X)$,

$$\parallel T \parallel \leq 1 \text{ such that } \parallel Tx - x \parallel < \epsilon \text{ for all } x \in K. \tag{2.4}$$

If the above condition (2.4) is satisfied. Then it is said to have the metric approximation property denoted by $\mathcal{MAP}$.

**Corollary 2.5.** *[15] There is a Banach space which has the $\mathcal{BAP}$ but fails the $\mathcal{MAP}$.*

**Corollary 2.6.** *[15] If a separable Banach space which even has the separable dual is considered and it satisfies the $\mathcal{AP}$ it still fails to satisfies the $\mathcal{BAP}$.*

**Theorem 2.7.** *[15] The $\mathcal{AP}$ of a separable dual space implies the $\mathcal{MAP}$.*

Recall the definition of reflexive Banach space,

**Definition 2.9.** Suppose $\hat{X}$ is a given Banach space. If $\hat{X}$ is isomorphic to its second dual space then $\hat{X}$ is known as reflexive space. Mathematically written as,

$$\hat{X} \simeq \hat{X^{**}}.$$

**Theorem 2.8.** *[15] Every Banach space which is reflexive (also given in [24]) and satisfying the $\mathcal{AP}$ also implies the $\mathcal{MAP}$.*

**Theorem 2.9.** *[15] For $X$- a Banach space, following statements are equivalent:*
*1. $X$ satisfies the $\mathcal{BAP}$.*
*2. Finite rank operators on $X$ has a uniformly bounded net $(T_\alpha)$ which tends strongly to the identity on $X$.*
*3. For every subspace of finite dimension $E \subset X$ consider $\lambda \geq 1$, there is a finite rank operator $T$ on $X$ such that $\parallel S \parallel \leq \lambda$ and $Tx = x$, for all $x \in E$.*

We state another needed proposition regarding uniformly bounded net contained in [24].
**Proposition 2.10.** [15] Let $X$ be a Banach space. Finite rank operators on $X$ has uniformly bounded net (same notations as [24]) $(T_n)$, such that $(T_n)x \to x$ for every $x \in X$. Then $X$ has the $\mathcal{AP}$.

Next section includes the survey of different results regarding the $\mathcal{AP}$ in a Banach space. In 2014, some authors discussed in detail the role of the approximation in the Banach spaces. They also provide many results [25]. Now we mention these theorems and after that we will check how the theorems of [25] behaves in case of Hilbert spaces.

To elaborate the statement of the next theorem let us recall some basic concepts;
**Definition 2.11.** Let $X$ be a Banach space. A basis $\{e_n\}_{n\geq 0}$ of $X$ is boundedly complete if for every sequence of scalers denoted by $\{a_n\}_{n\geq 0}$, such that it has the bounded partial sums denoted by $V_n = \sum_{k=0}^{n} a_k e_k$ in $X$, the sequence $V_n$ converges in $X$.

To elaborate this, let us provide an example as well as counterexample:

- Consider the space $l^p, 1 \leq p \leq \infty$, the definition of boundedly complete basis is fulfilled by the unit vector basis for $l^p$.

- In $c_0$ (the space of all sequences of scalers converging to zero), we may observe that if we obtain the unit vector basis, it is not boundedly complete.

Furthermore, a space $X$ with boundedly complete basis $\{e_n\}_{n\geq 0}$ is isomorphic to a dual space.

The very first theorem obtained in [25] is given as follows:
**Theorem 2.10.** *[25] Suppose a Banach space $X$ is considered having a boundedly complete basis such that $X^*$ of $X$ is separable. he $\mathcal{AP}$ is still not satisfied.*

**Theorem 2.11.** *[25] Consider $Z$ be any separable space. There is an isometry between $Z$ and its dual and it also satisfies the $\mathcal{AP}$ then $Z$ has the $\mathcal{MAP}$.*

[25] The fact to be noticed is, the $l_2^*$ is isometrically isomorphic to the $l_2$ space itself.
**Theorem 2.12.** *[25] Consider $X$ a Banach space along with a separable dual space. The following statements are equivalent:*
*(1) The $\mathcal{AP}$ is satisfied by $X$- dual ($X^*$).*
*(2) The $\mathcal{MAP}$ is satisfied by $X$- dual ($X^*$). Also,*
*(3) $X$ satisfies the $\mathcal{MAP}$ in all equivalent norms.*

By "equivalent norms" we mean all the norms defined on a various space are basically equivalent.
Particularly, in all equivalent norms $c_0$ satisfies the $\mathcal{MAP}$ [25].

## 2.8 Infinite dimensional spaces and operators

Next part includes the discussion regarding the infinite dimensional Hilbert space. We are moving towards the working of the approximation property in a Hilbert space but firstly, the natural question arises, does every Hilbert space satisfies the $\mathcal{AP}$. The answer is 'no'; we have the counterexample for a Hilbert space having infinite dimension. We have a subspace of the 'nicest' Hilbert space of infinite dimension, the space of square sumable sequences-$l_2$ , without the approximation property [8]. When we are talking about an infinite dimension case, we are basically dealing with an infinite sums.

### 2.8.1 Non compactness and an infinite dimensional case

Compact operators are important in the development of the approximation property. Specifically in a Hilbert space of finite dimension, the identity operator is not necessarily compact. We will provide the proof as given by [26] so that the notion of compactness in the operators will be clear. This will also be considered as an example of space of "infinite" dimension which is not compact.

Suppose $H$- a Hilbert space. Choose the sequence $(x_k)$ to be an infinite orthonormal sequence in $H$. For $j \neq k$, consider,

$$\|x_j - x_k\|^2 = \|x_k\|^2 - (x_k, x_j) - (x_j, x_k) + \|x_j\|^2 = 2.$$

Hence distinct terms of the sequence $(x_k)$ are at a distance $\sqrt{2}$ from each other. Thus the sequence $I(x_k)$ contains no cauchy subsequence, and consequently no convergent subsequence.

This thesis is about the results and facts so the result we must need to mention is that every separable Banach space which is infinite dimensional is homeomorphic to $l_2$ - the Hilbert space [16].

**Definition 2.12.** Consider $X$- a Banach space. The Schauder basis for $X$ are given as $(\{x_n\}, \{b_n\})$. In $X$, a sequence is denoted by $\{x_n\}$ and $\{b_n\}$ denotes the unique scalers. Also, the operators involving the partial sums connected with the basis $(\{x_n\}, \{b_n\})$ are the mappings $S_N : X \to X$ which are defined by

$$S_N x = \sum_{n=1}^{\infty} b_n(x) x_n.$$

Then basis constant for $X$ is given by $A = \ \sup \|S_N\|$ and it the finite number. The basis constant satisfies $A \geq 1$. If the basis constant is 1, that is, $A = 1$, then the basis is said to be monotone which is either increasing or decreasing.

From the literature we also consider an important corollary. It is needed to mention it here because it leads to the conclusion of the important fact. This corollary is given as follows [23]:

**Corollary 2.13.** *[23] Suppose X a Banach space has a Schauder basis with basis constant $\lambda$. Then X has $\lambda-$ bounded approximation property.*

[23] also highlighted that the spaces $c_0$ (called the space of sequences that are converging to zero) and $l_p, p \in [1, \infty)$, satisfies the $\mathcal{MAP}$.

Furthermore, they conclude that an orthonormal basis is a Schauder basis so any Hilbert space has the $(\mathcal{MAP})$.

Now, we will mention some theorems which are used in the next chapter. First one involves the isomorphism between the Hilbert spaces and second one is about the $(\mathcal{MAP})$.

**Theorem 2.14.** *Every separable Hilbert space $H$ over $K$ is isometrically isomorphic to either $\mathbb{F}^n$ (if $H$ has finite dimension $n$) or to $l^2$. The isometric isomorphism preserves the inner product.*

**Theorem 2.15.** *A separable dual space with the $\mathcal{AP}$ implies the ($\mathcal{MAP}$).*

**Remark 2.13.** Very first example was given by [9] of a space failing the $\mathcal{AP}$. A Banach space (even separable) without a presence of "Schauder basis" was also provided for the first time. [9] provided a Banach space which is separable but without the $\mathcal{AP}$.

Separable Banach space result regarding this is mentioned earlier. Now, moving forward we have the following result;

**Theorem 2.16.** *Consider $H$, a Hilbert space. $H$ is separable $\Leftrightarrow$ $H$ admits a countable orthonormal basis.*

We may say that it is really difficult to construct separable Hilbert space without the $\mathcal{AP}$.

# Chapter 3

# Hilbert space results and theorems

The Hilbert space is known for its best approximation. In this context we define best approximation as "Every non- empty closed convex subset of a Hilbert space $H$ has a unique element which has the minimum norm".

Our aim is to investigate the "approximation property ($\mathcal{AP}$)" for the Hilbert space as well. In our thesis, we consider the approximation of Hilbert space in totally different context i.e. by considering the role of operators. Therefore the definition regarding this is given as follows:

**Definition 3.1.** ($\mathcal{AP}$) is satisfied in case of a Hilbert space $H$, if every compact operator on $H$ has the finite rank operator as its limiting value.

Finding the example is not an easy task because we must consider the compact operator and need to obtain its limit. Thus, we need to find another way out. We construct example by using result.

**Proposition 3.2.** Suppose $H$ is a Hilbert space. Suppose finite rank operators on $H$ and their uniformly bounded net $(T_n)$ exists in such a way that, $(T_n)x \to x$ for every $x \in H$. Then $H$ has the $\mathcal{AP}$.

*Proof.* Suppose there is a net of finite rank operators $(T_n)$ such that $sup\|T_n\| = C < \infty$. Also, by assumption $(T_n)x \to x$ for every $x \in H$. Now, $\epsilon > 0$ and compact subset of

$H$ is indicated by $K$. We are choosing a net for $K$ in such a way that $\delta-$net is defined as, $\{x_1, ..., x_k\}$, where $\delta = min\{\epsilon/3, \epsilon/3C\}$. There exists $n_0$ such that $n \geq n_0$ then $\|x_i - T_{n_0}x_i\| \leq \epsilon/3$ for each $i$. Let us consider $x \in K$ and $i$ chosen in such a way that $\|x - x_i\| < \delta$, Then

$$\|x - T_{n_0}x\| \leq \|x - x_i\| + \|x_i - T_{n_0}x_i\| + \|T_{n_0}x_i - T_{n_0}x\| < \epsilon.$$

□

We are moving towards an example which is constructed for a Hilbert space satisfying the $\mathcal{AP}$.

## 3.1 Construction of an example

We construct the example by using the Schauder basis method and the application of an above result. Let $X = \mathbb{R}^2$. The usual basis for $\mathbb{R}^2$ are defined as

$$e_1 = (1, 0) \text{ and } e_2 = (0, 1).$$

Let us define a set $K$ as follows:

$$K = \{(x, y) : -2 \leq x \leq 2, -2 \leq y \leq 2 \text{ and } x - y = 0\}.$$

Also, $K \subseteq X$ and choose $(x, y) = (2, 2)$ such that $(x, y) \in K$,

Fr each $n$, we define finite rank operator as follows:

$$T_n x = \sum_{i=1}^{n} f_i(x)x_i,$$

where $f_i(x)$ denotes the coordinate functional and

$$f_k(e_j) = \begin{cases} 0 & \text{if } k \neq j \\ 1 & \text{if } k = j. \end{cases}$$

So, $f_1(e_1) = 1$; $f_1(e_2) = 0$ ; $f_2(e_1) = 0$ ; $f_2(e_2) = 1$.

$$
\begin{aligned}
f(x) &= \sum_{j=1}^{n} e_j \alpha_j \\
&= \sum_{j=1}^{n} e_j f(e_j) \\
&= e_1 f(e_1) + e_2 f(e_2) \\
&= 1(1,0) + 1(0,1) \\
&= (1,0) + (0,1).
\end{aligned}
$$

Now, calculations for $T_n x$ :

$$
T_n x = \sum_{i=1}^{n} f_i(x) x_i
$$

$$
\begin{aligned}
T_n x &= (1,0)x_1 + (0,1)x_2 \\
&= 2(1,0) + 2(0,1) \\
&= (2,2) \\
&\Rightarrow T_n x - x = 0 \\
&\text{So }, \|T_n x - x\| \to 0.
\end{aligned}
$$

By calculations we may observe that the operators $T_n$ are bounded uniformly. Further, $T_n x \to x$ for every $x \in X$. Thus $X = \mathbb{R}^2$ satisfies the $\mathcal{AP}$ by considering the usual basis.

The notions for Hilbert spaces are clear from the previous sections. We consider the theorems from the previous chapter given for Banach spaces, in terms of the Hilbert space.

Repeatedly mentioned that this chapter is all about the Hilbert space and its specific results. The natural question considered to be the bedrock of the thesis is: "Why we are choosing only a Hilbert space?" We emphasis on a Hilbert space due to some important properties. Hilbert space is known for its richest geometry among general normed spaces and Banach spaces. In elaborative manner, by richest geometry, we mean that Hilbert space is nice because of its best approximation and orthogonality. Also, due to the presence of inner product, it is quite an easy task to understand the duality concepts in Hilbert spaces.

Study from the literature highlights that the approximation property in the Banach spaces is difficult and much more abstract to understand than that of any other space. Therefore we also study Hilbert space along with a Banach space. We provide theorems covering a separable Hilbert space, bounded approximation property ($\mathcal{BAP}$) and metric approximation property ($\mathcal{MAP}$) for a Hilbert space.

Parallel to the definition for a Banach space, we may say that, suppose $H$ and $H^{'}$ are given Hilbert spaces then the pair $(H, H^{'})$ fulfills the approximation property, if every operator $T$ in $L(H, H^{'})$ is approximable.

## 3.2 Application regarding the pair

The considerable fact for $X$- a Banach is, $X$ satisfies the $\mathcal{AP}$ if it satisfies the $\mathcal{BAP}$.

### 3.2.1 Application for the pair $(X, Y)$

**Example 3.1.** Consider $X = \mathbb{R}^3$ and $Y$ is the selected subspace defined as follows defined as

$$Y = \{x \in \mathbb{R}^3 : x = (x, x, 0)\}.$$

To elaborate the concept of Theorem (2.3), let us consider the trivial example of zero operator denoted by $0_x$ and also zero operator has the finite rank so it is the nuclear operator. Thus, $T = 0_x$ and we may observe that $TX \subseteq Y$ and also $TY = 0$. By considering the trace of operator we have $tr(T) = 0$. Thus by Theorem (2.3) we may say that in this case the pair $(X, Y)$ satisfies the $\mathcal{AP}$.

We can look at the $\mathcal{AP}$ in many different aspects. These aspects includes:

- The bounded approximation property;
- The basis property;
- The metric approximation property.

Little overview of the metric approximation in Banach space is provided in the next section.

## 3.3 Metric approximation property

Metric approximation property ($\mathcal{MAP}$) is one of the important property in the Banach space. Firstly we fix the notions. Throughout the bounded linear operator from the space $Y$ to $X$ is denoted by $\mathcal{L}(Y,X)$. $\mathcal{F}(Y,X)$ and $\mathcal{K}(Y,X)$ are the notations for the subspaces of finite rank operators and compact operators respectively. Let us start this section with the basic definition of the $\mathcal{MAP}$ for the Banach spaces.

**Definition 3.3.** [19] We have a Banach space $X$. It satisfies the $\mathcal{MAP}$ if for every compact set $K$ in $X$ and for every $\epsilon > 0$, there is an operator which is finite rank, $T \in \mathcal{F}(X,X)$, with

$$\parallel T \parallel \leq 1 \text{ such that } \parallel Tx - x \parallel < \epsilon \text{ for all } x \in K.$$

We are mainly concerned with the $\mathcal{MAP}$ for the dual space $X^*$. According to [18] a Banach space $X$ has the $\mathcal{MAP}$ " $\Leftrightarrow$ " $\mathcal{F}(Y,X)$ is an ideal in $\mathcal{L}(Y,X^{**})$.

**Definition 3.4.** Ideal is defined to be the subset of ring satisfying the following properties:

- Ideal is closed under addition.
- Zero is belongs to the ideal.
- If we consider the product of "an element of ideal" and "an element of a ring", then it is again an element of an ideal.

Furthermore, ideals does not satisfies the commutative property.

In terms of Hilbert space the theorem regarding the ideals and the metric approximation property is given below:

**Theorem 3.1.** *$\mathcal{F}(K,H)$ and $\mathcal{L}(K,H^{**})$ denotes the finite rank operators and bounded linear operators from Hilbert space $K$ to $H$ respectively. $\mathcal{F}(K,H)$ is an ideal in $\mathcal{L}(K,H^{**})$ " $\Leftrightarrow$ " $H$ has the $\mathcal{MAP}$.*

$\mathcal{F}(K, H)$ and $\mathcal{L}(K, H^{**})$ forms ring therefore $\mathcal{F}(K, H)$ is an ideal of $\mathcal{L}(K, H^{**})$. We may observe that there is role of double dual space. It means if a Hilbert space is reflexive then there must be an isomorphism exists between double dual space $X^{**}$ and space itself ($X \simeq X^{**}$). We may also says that if we have a reflexive Hilbert space and $\mathcal{F}(K, H)$ is an ideal in $\mathcal{L}(K, H)$ then Hilbert space satisfies the $\mathcal{MAP}$.

Continuous linear maps between Hilbert spaces are of great importance. Sometimes we may call them as "operators". Operator theory in Hilbert spaces is a key branch of functional analysis and that is the reason we mainly focuss on various operators because approximation is directly connected to operators.

## 3.4 Results for Hilbert space

In this section we provide the concepts for Hilbert space, specifically conversion of results contained in [11] into a complete inner product space. There is a Hilbert space $H$ and a subspace $K$, the pair $(H, K)$ satisfies the approximation property provided there is a net of finite rank bounded linear operators on $H$ all of which leaves the subspace $K$ invariant. There is a uniform convergence of net on compact subsets of $H$ to the identity operator.

### 3.4.1 Important notes

This section is about the definitions regarding the Hilbert spaces. In simple language, a finite dimensional image implies a continuous linear operator of finite rank. For Hilbert spaces, a finite rank operators follows the similar definition as a Banach space. For the Hilbert spaces, specifically in the uniform operator topology the compact operators are the closure of finite rank operators. $\mathcal{F}(H)$ is the set of finite rank operators. If $\exists u, v \in H$, such that $Tx = \langle x, v\rangle u$. For this usually the notation used is $T = u \otimes v$. In this case the operator $T$ is "rank one operator". Rank one operators are important because they turned out to be the construction of the blocks out of which finite dimensional operators are made. Mainly separable Hilbert spaces are also considered in this thesis. The important fact regarding separable spaces and Hilbert space is as follows:

**Remark 3.5.** 1- We have a separable Hilbert space if and only if it admits the countable orthonormal basis.

2- Consider a Hilbert space $H$ of infinite dimension which is also separable, then there is an isomorphism between that space $H$ and $l^2$, the well known space of square-sumable sequences .

**Lemma 3.2.** *Let* $\mathcal{F}_Y(H) = \{T \in \mathcal{F}(H) : TK \subseteq K\}$, *(where* $TK \subseteq K$ *denotes that the subspace* $K$ *is invariant).*

*1.* $x^* \otimes x \in \mathcal{F}_K(H) \Leftrightarrow$ *either* $x^* \in K^\perp$ *or* $x \in K$.

*2. If* $F \in \mathcal{F}_K(H)$, *then* $F$ *is the sum of* $n$ *rank one element of* $\mathcal{F}_K(H)$, *where* $n$ *is the rank of* $F$.

Procedure of the proof is taken from [11].

*Proof.* For (1), conversely, If $x^* \in K^\perp$ then by definition of $K^\perp$, we have $(x^* \otimes x)K = 0$ so $x^* \otimes x \in \mathcal{F}_K(H)$. If $x \in K$ then $(x^* \otimes x)H \subseteq \text{span}x \subseteq K$ so $x^* \otimes x \in \mathcal{F}_K(H)$.
On the other hand, $x^* \notin K^\perp$ and $x \notin K$, then by the definition of orthogonality there is $y \in K$ s.t. $\langle x^*, y\rangle \neq 0$, hence

$$(x^* \otimes x)y = \langle x^*, y\rangle x \notin K,$$

whence $x^* \otimes x \notin \mathcal{F}_K(H)$. This proves "if side".
For (2), let $x_1, ..., x_m$ be a basis of $FH \cap K$ and extend this to a basis for $FH$ by adding $x_{m+1}, ..., x_n$, so that

$$\text{span } x_{m+1}, ..., x_n \cap H = 0. \tag{3.1}$$

Write $F = \sum_{k=1}^{n} x_k^* \otimes x_k$. By part (1), $x_k^* \otimes x_k \in \mathcal{F}_K(H)$ for $k \leq m$. To complete the proof, the sufficient condition (by 1) is to show that $\forall k > m$ we have $x_k^* \in K^\perp$. If for some $k > m$ we had $x_k^* \notin K^\perp$, then choosing $y \in K$ with $\langle x^*, y\rangle \neq 0$, we would have by (3.1) that $\sum_{j=m+1}^{n} \langle x_j^*, y\rangle x_j \notin K$. But $\sum_{j=1}^{m} \langle x_j^*, y\rangle x_j \in K$, so we would have $Fy \notin K$, a contradiction.

□

Theorem (2.3) is regarding the pair of a space and a subspace. This very important theorem can be written in terms of a Hilbert space as follows:
Procedure of the proof is taken from [11]

**Theorem 3.3.** *Suppose that $K \subseteq H$ and $H$ has the $\mathcal{AP}$. Then the following are equivalent statements.*
*1. The pair $(K, H)$ satisfies the $\mathcal{AP}$.*
*2. For all $T$ from the class of nuclear operators ($T \in N(H)$), $TH \subseteq K$ and $TK = 0$ we have $tr(T) = 0$..*

*Proof.* 1⇒2.
Assume (2) is not true (proof by contradiction) then we obtain, $T \in N(H)$ so that $TK = 0$ and also $TH \subseteq K$ but $tr(T) = 1$. So $T \in L(H, \tau)^*$ and $\langle I, T\rangle = 1$. Let $F \in \mathcal{F}_K(H)$. To contradict (1), we will show that $\langle F, T\rangle = 0$, and we are done. Lemma 1 helps us to check that $\langle x^* \otimes x, T\rangle = 0$ if either $x^* \in K^\perp$ or $x \in K$. But $\langle x^* \otimes x, T\rangle = \langle x^*, Tx\rangle$, so from above facts we have, $Tx \in K$ and $TK = 0$.
2⇒1.
If (1) is false and $K$ has the $\mathcal{AP}$, we can separate $I$ from $\mathcal{F}_K(H)$ with a continuous functional (linear) on $L(H)$ denoted by $\tau$, which, since $H$ has the $\mathcal{AP}$, a nuclear operator T on $H$ further denotes this. Then $tr(T) = \langle I, T\rangle \neq 0$ but $\langle F, T\rangle = 0$ for all $F \in \mathcal{F}_K(H)$. Particularly , $\langle x^*, Tx\rangle = \langle x^* \otimes x, T\rangle = 0$ if either $x^* \in K^\perp$ or $x \in K$. So if $x \in H$, then we have

$$\langle x^*, Tx\rangle = 0, \forall x^* \in K^\perp,$$

which is to say that $Tx \in (K^\perp)_\perp = K$. So $TH \subseteq K$. If $y \in K$, then we obtain

$$\langle x^*, Ty\rangle = \langle x^* \otimes y, T\rangle = 0, \forall x^* \in H^*,$$

which says that $TK = 0$. □

## 3.5 Separable Hilbert space without the approximation property

We are moving towards the case when the Hilbert space does not satisfying the approximation property. The parallel Theorem (2.10) is given as follows:
**Theorem 3.4.** *Suppose a separable Hilbert space $H$ is considered having a boundedly complete basis such that $H^*$ of $H$ is separable and the $\mathcal{AP}$ is not satisfied.*

Moving towards the explanation of the above theorem. Every orthonormal basis is a Schauder basis in case of a separable Hilbert space. The unit vector bases ("standard" consideration) in $l^2$ are equivalent to every countable orthonormal bases. Therefore, the definition for boundedly complete basis in Hilbert space is same as for Banach space (the definition given in the previous chapter). To give the argument regarding the above theorem we must have to mention the separability theorem [17], the statement is given below:

**Theorem 3.5.** ***Separability***
*[17] Consider $X$ a normed space and it is separable, if the dual space $X^*$ of a $X$ is separable.*

Thus, to prove above Theorem (3.4), we should consider a normed space in such a manner that it is either a Hilbert or a Banch space then only we are able to obtain a separable space with the separable dual. The statement of theorem regarding the a Banach space and separable dual is now mentioned in terms of a Hilbert space. By studying we came to know its really difficult to construct such Hilbert space. Why? the answer is given in the coming section related to Schauder basis.

## 3.6 Approximation property implies metric approximation property

Now there is a combination of the approximation property and its another variation, that is, the metric approximation property $\mathcal{MAP}$. Now, using Theorem (2.14) we may able to provide the theorem for Hilbert space.

Firstly we provide lemma to prove the next theorem. It is given as follows:
**Lemma 3.6.** *If $H^*$ satisfies the $\lambda-$bounded approximation property then $H$ must satisfies the $\lambda-$duality bounded approximation property.*

*Proof.* An application of Helly's Theorem (or local reflexivity) shows that we may assume that $H^*$ along with the $\lambda-$bounded approximation property which is given by weak-star continuous operators. Let us choose a net of finite rank operators (having finite dimension) $T_\alpha$ on $H$ so that $limsup_\alpha \|T_\alpha\| \leq \lambda$ and $lim_\alpha T_\alpha^* x^* = x^*$ for all $x^* \in H^*$.

Then $(T_\alpha)$ converges weakly to the identity on $H$ and so there are disjoint finite convex combinations of the $T_\alpha$ (which must be chosen carefully since this is only a net) which form a net and converges strongly to the identity on $X$. □

Helly's theorem says that if we have a family $C$ of convex sets (which is finite) in $\mathbb{R}^n$ the n- dimensional Euclidean space. Such that for $m \leq n+1$, there is an intersection which is non empty for any $m$ members of $C$, this implies the intersection of all members of $C$ is non empty. Helly's condition is importantly used to provide the proof of the local reflexivity theorem.
In the Hilbert spaces, the weak-star topology is generally defined on the set of bounded operators denoted by $B(H)$. It is basically obtained from the pre dual $B^*(H)$ of B(H), the trace class operators.

By studying different papers we came to know that a Hilbert spaces are basically reflexive Banach spaces. Therefore, Theorem (2.8) for a Hilbert space can be varied in as follows;
**Theorem 3.7.** *Every Hilbert space $H$ with approximation property also satisfies the metric approximation property.*

*Proof.* This can proved by two results lemma (3.6) and Theorem (2.15). □

**Theorem 3.8.** *Suppose $H$ is a separable Hilbert space. Now, there exists an isometry between $H$ and its dual space, ($H \simeq H^*$) also it satisfies the $\mathcal{AP}$ then $H$ has the $\mathcal{MAP}$.*

*Proof.* A separable Hilbert space $H$ is considered in such a way that there is an isomorphism between $H$ and its dual space. Using the fact that the Hilbert spaces are self dual spaces and by Theorem (3.2) we obtain that a Hilbert space also has the $\mathcal{MAP}$. □

## 3.7 Dual of the space satisfying approximation property

We can find a Hilbert space which is considered along with its separable dual space and it follows the result regarding the equivalent norms which is given below:

**Theorem 3.9.** *Consider $H$ a Hilbert space along with a separable dual space. The following statements are equivalent:*
*(1) The $\mathcal{AP}$ is satisfied by $H^*$.*
*(2) The $\mathcal{MAP}$ is satisfied by $H^*$. Also,*
*(3) $H$ satisfies the $\mathcal{MAP}$ in all equivalent norms.*

Considering the above theorem, dual space (which is separable) implies a Hilbert space (which is separable). Our main concern is with a separable Hilbert space because here we want to use Theorem(2.14). Thus, in case of infinite dimensional space, a Hilbert space which is separable is isometrically isomorphic to $l^2$ and in case of the space with finite dimension, it is isometrically isomorphic to $\mathbb{F}^n$, $\mathbb{F}$ is field. In the previous chapter, for a Banach space, we have result regarding $c_0$, and here for a Hilbert space we conclude it for $l^2$. So in particular, $l^2$ includes the $\mathcal{MAP}$ (metric approximation property) in all equivalent norms.

## 3.8 Combination of the approximation property and Schauder basis

Now, we are moving towards result (2.4).
**Theorem 3.10.** *Consider a Hilbert space $H$ (also separable). $H$ is a complemented subspace of a Hilbert space with a Schauder basis " $\Leftrightarrow$ " $H$ implies the $\mathcal{BAP}$.*

Basically in case of a Hilbert space, its any closed-subspace is complemented, thus we may obtain a Hilbert space and its subspace which is complemented. Complemented subspace is generally defined for normed spaces (not specifically for any space whether a Banach or a Hilbert). If a Hilbert space is separable, the countable orthonormal basis necessarily exists. Therefore if we consider any subspace (closed) of a Hilbert space, it has the $\mathcal{BAP}$. Since every Hilbert space also has Schauder basis.

In straight forward manner we may says that, the presence of a Schauder basis implies the bounded approximation property in every Hilbert space.

### 3.8.1 Example

In 1967, there is a result which indicates that, if we consider a Banach space $X$ whose every subspace is complemented, then we may say that there is an isomorphism exists between $X$ and a Hilbert space.
Therefore, "$X = \mathbb{R}^n$" ensures complemented subspaces, thus a Hilbert space along with an example is obtained that satisfies the $\mathcal{BAP}$.
Next theorem involves the importance of orthogonality in complete inner product spaces.
**Theorem 3.11.** *Suppose we consider a separable Hilbert space $H$, then it has the $\mathcal{BAP}$ if and only if ("$\Leftrightarrow$") there is an orthogonal projection on $H$ with a Schauder basis.*
**Theorem 3.12.** *For a Hilbert space $H$, following statements are equivalent:*
*1. $H$ satisfies the $\mathcal{BAP}$.*
*2. Finite rank operators on $H$ has a uniformly bounded net $(S_\alpha)$ which tends strongly to the identity on $H$.*
*3. For every subspace of finite dimension $K \subset H$ consider $\lambda \geq 1$, there is a finite rank operator $S$ on $H$ such that $\| S \| \leq \lambda$ and $Sx = x$, for all $x \in K$.*

*Proof.* For (2), we use the Proposition (3.2). It is basically the definition of approximation property that a uniformly bounded net tends strongly to the identity. So $2 \Rightarrow 1$. □

# Chapter 4

# Conclusion and further directions

By definition every Hilbert space is also a Banach space. We may say that a Hilbert space comes in the category of a complete normed vector space and basically the norm is determined by an inner product. Converse of this very strong statement is not always true, this is the main argument which lead us to the working of this thesis.

Banach spaces in consideration with the approximation property is a very difficult problem in the functional analysis. With the help of literature we came to know that the Banach spaces in combination with approximation property is not a trivial problem. Authors provided many directions to resolve the problem regarding the approximation property in a Banach space. Therefore, we basically investigate that, whether the more nicest space, a Hilbert space also follows the same directions provided for a complete norm vector space (Banach space).

Concept of Schauder basis is much more easy to understand in Hilbert spaces. Results regarding the two variation of the approximation property in a Hilbert spaces is also arguable issue studied in our thesis.

Some authors also provided the approximation property in the light of a point wise convergence. For example there is also a definition of the approximation property in view of the point wise convergence of an operators. According to [21] a Banach space $X$ (also separable) satisfies the $\mathcal{BAP}$ if a sequence of operators of finite dimension has *id*-operator (on $X$), as their limiting value, but the limit is pointwise. In our thesis we have some concepts which revolves only around the uniform convergence. However, point wise convergence is totally a different direction to study. There may be such spaces which have the approximation property that involves the point wise convergence. So, this is an interesting direction which we propose for future work. Also, some problems arises while studying the point wise convergence in Hilbert spaces. So, that should also be studied in the future.

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