# On Super Edge-Magic Total Labeling of Forest Having Two Components

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بسم الله الرحمن الرحيم

*In the name of Allah,*
*the Most Beneficent,*
*the Most Merciful*

# Dedication

*To my loving father, brothers,*
*sisters,*

and

*sweet and cute nephew Usman.*

# Abstract

A labeling of a graph is a mapping that carries some set of graph elements (vertices, edges or both) into numbers (usually positive integers). Such a labeling is called super if the $p$ smallest possible labels appear at the vertices. In 1970 Kotzig and Rosa introduced the concept of edge-magic deficiency of a graph $G$, denoted by $\mu(G)$, which is the minimum nonnegative integer $n$ such that $G \cup nK_1$ is edge-magic total. Motivated by Kotzig and Rosa's concept of edge-magic deficiency, Figueroa et al. defined a similar concept for super edge-magic total labelings. The super edge-magic deficiency of a graph $G$, which is denoted by $\mu_s(G)$, is the minimum nonnegative integer $n$ such that $G \cup nK_1$ has a super edge-magic total labeling or it is equal to 1 if there exists no such $n$.

The thesis is devoted to study of super edge-magic deficiency of forest having two components. We present new results on the super edge-magic deficiencies of forests formed by a disjoint union of subdivided stars, paths, stars, caterpillars, bistars and combs.

# Introduction

Graph theory has experienced fast development during the last 70 years. Among the huge diversity of concepts that appears while studying this subject, one has gained a lot of popularity is the concept of graph labelings. With more then 1000 papers that are appeared in a dynamic survey of graph labeling by Gallian [22], this branch caught the attention of mathematician and many new labeling results appears every year. The interest on this subject is due to the wide range of applications in other branches of science such as coding theory, x-ray, chemical compounds in organic chemistry, circuit design and communication networking.

For example, take the underlying graph of a network with nodes as the vertices of the graph and the edges between all pairs of nodes, where a link is provided. The vertices are labeled in such a way that the difference between any two vertices are distinct. Such a labeling is called *graceful labeling*. The application of this kind of labeling is in radar pulse codes.

A labeling or a valuation of a graph is a map that carries the graph elements to numbers. If a labeling has a domain the set of all vertices and edges, such labeling is called a total labeling. If the domain is the set of vertices we call such labelings vertex labelings and if only the edges of a graph are labeled we call it edge labelings. Not only the way the elements are labeled, but also a labeling has to satisfy an evaluation condition, according to the evaluation condition, we could have harmonious, graceful, magic and antimagic labelings. In the case when the vertices are labeled with the smallest possible number, Enomoto et al. [41] call the labeling $\lambda$ as a super edge-magic total labeling.

In [17], Enomoto et al. [41] conjectured that every tree admits a super edge-magic total labeling. In the effort of attacking this conjecture, many authors have considered super edge-magic total labeling for some particular trees. Lee and Shah [37] have verified this conjecture for the trees with upto 17 vertices with the computer help.

The *super edge-magic deficiency* of a graph G, is the minimum nonnegative integer $n$ such that $G \cup nK_1$ has a super edge-magic total labeling or it is equal to $+\infty$ if there exists no such $n$. Figueroa et al. [39] conjectured that every forest with two components has super edge-magic deficiency at most 1. Enomoto et al. conjectured that every tree is super edge magic total [17], which is still open and challenging for researchers. This conjecture has been verified for certain subclasses of tree and many new subclasses of tree are constructed while attempting to prove this conjecture. The study of different classes of trees and forests is always an interesting and challenging problem due to the famous Rosa-type conjecture by Enomoto et al and Figueroa-Centeno et al. The aim of this thesis is to verify this conjectures for some particular kinds of forests having two components.

## Acknowledgement

Words are bound and knowledge is limited to praise ALMIGHTY ALLAH, The most Beneficent, The Merciful, Gracious and the Compassionate whose bounteous blessing and exaltation flourished my thoughts and thrived my ambition to have the cherish fruit of my modest efforts in the form of his manuscript from the blooming spring of blossoming knowledge. My special praise for the Holy Prophet HAZRAT MUHAMMAD (Peace Be Upon Him), the greatest educator, the everlasting source of guidance and knowledge for humanity. He taught the principles of morality and eternal values.

Being a student, I deem it a great honor to express my deepest sense of gratitude to my honorable and estimable supervisor Dr. Muhammad Imran for the continuous support throughout my research, for his patience, motivation, enthusiasm, and immense knowledge. His guidance helped me in all the time of research and writing of this thesis. I could not have imagined having a better advisor and mentor for my thesis. I am thankful for their aspiring guidance, invaluably constructive criticism and friendly advice during my thesis. I am sincerely grateful to them for sharing their truthful and illuminating views on a number of issues related to the thesis.

This acknowledgement would be incomplete unless I offer my humble veneration to my loving father, my brothers and my sisters whose love and endless prayers make me to achieve the success in every sphere of life.

Ambreen Mukhtar

# Contents

# Chapter 1

# Essentials of Graph Theory

This chapter allocates a momentary introduction of elementary concepts of graph theory. It contains different graph theoretical terms and their illustration with examples.

## 1.1 Introduction to Graphs

Graph theory is a major branch of discrete mathematics and studied in various aspects. It has numerous applications in study of genetics, molecules, construction of bonds and atoms in chemistry and in exploring diffusion mechanism. It has also wide applications in modeling transport networks and game theories. Graph theory is emerging as one of the most developing research area and used in structural models due to its simplicity. These structural model arrangements of various technologies lead to new innovations and modifications.

Graph theory was originated in 1735 commencing with the *Königsberg bridge problem*. Euler first studied this problem and assembled a structure called "*Eulerian graph*". The idea of bipartite and complete graph was presented by *A. F Möbius* in 1840 and later proved by *Kuratowski.* In 1845, *Kirchoff* implemented the concept of trees.

*Dénes kőnig* [1] wrote the first book "Theorie der endlichen und unendlichen Graphen" on graph theory which was published in 1936 which was later translated by Richard McCoart in 1990, named as "Theory of finite and infinite graphs". In 1969, *Frank Harary* [2] published a book "Graph Theory" helping mathematicians, engineers and chemists to be in contact with each other and by setting different types and terminologies of graph theory that consist of vertices, edges, degrees etc.

Graph theory has alot of practical role. It utilizes properties of different graphs for structuring problems faced in reality, e.g, for making social interactions, to make telecommunication channels between streets in a city and cities in a country. Graphs are pretty functional in linguistic structures, project managements, in bioinformatics and in mathematical relationships.

## 1.2 Graph Components

A *graph* $G = (V, E)$ is a combination of lines and dots. These dots are called vertices as well. A non empty set $V$ of dots is called *vertex set* of G and denoted by $V(G)$. The set of lines $E$ between each pair of vertices is called *edge set* of G and denoted by $E(G)$. The number of vertices in a graph is called *order* of a graph and denoted by $|V(G)|$ or $p$. Similarly, the number of edges in a graph is called *size* of a graph and denoted by $|E(G)|$ or $q$. That is why, a graph with $p$-vertices and $q$-edges is sometime referred as $(p, q)$ graph.

Mostly every edge has exactly two end points. Whenever two vertices are attached by an edge they are known as *adjacent* vertices. The set of all adjacent vertices of a fixed vertex $v$ of G is called *neighbourhood set* of G and is denoted by $N_G(v)$. Let a couple of vertices $x$ and $y$ of a graph $G$ are connected by an edge $e$ then we say that $e$ is incident on $x$ and $y$ and we denote an edge by its vertices, that is, $e$ will be denoted as $xy$ or $yx$. It is also possible to have a vertex $u$ joined by an edge $e$ itself, such an edge $e$ is called a *loop*. If two or more edges of $G$ are incident on same vertex, then these edges are called *multiple edges*. A graph with no loops and no multiples edges is known as *simple graph*. A graph which is not simple is called a *multigraph* . A graph which consists of only one vertex with no edge is called *trivial graph* and a vertex with zero degree is called an *isolated vertex*. A vertex with degree one is called a *leaf*. The number of edges incident on a vertex (including self loops which counted twice) is known as *degree* of a vertex and denoted by $deg(v)$. The minimum and maximum degree of a graph is represented by $\delta(G)$ and $\Delta(G)$, respectively.

The following theorem is known as fundamental theorem of graph theory, which shows that the sum of degrees of a vertices in a graph is two times equal to the size of graph.

**Theorem 1.2.1.** [27] *The sum of degrees of the vertices in a graph is twice the number of edges. That is,*

$$\sum_{v \in V(G)} deg(v) = 2|E(G)|.$$

The following theorem is the direct consequence of Theorem 1.2.1.

**Theorem 1.2.2.** [3] *Every graph has an even number of vertices with odd degree.*

## 1.3 Families of Graphs

A graph in which edges are absent and it only contains set of vertices such a graph is known as *null graph* and denoted by $N_p$, where $p$ represents the number of vertices. A graph $H$ is a *subgraph* of $G$, $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If $V(H) = V(G)$, then $H$ is a *spanning subgraph* of $G$. A subgraph $H$ of $G$ is called an *induced subgraph* of $G$ if whenever $u$ and $v$ vertices of $H$ and $uv$ is an edge of $G$, then $uv$ is an edge of $H$ as well. Every graph is a spanning subgraph of itself.

A walk is a finite sequences of vertices denoted as $W = v_0e_1v_1e_2...v_{n-1}e_nv_n$ where $v_0$ is the origin of the walk, $v_n$ is terminus of the walk and $v_1, ..., v_{n-1}$ are internal vertices of the walk. In a walk $W$ terms are arranged in such a away that for $1 \leq i \leq n$, the edge $e_i$ has ends $v_{i-1}$ and $v_i$. The number of edges in the walk describe the length of the walk. If the edges $e_1, e_2, e_3, ..., e_n$ are distinct, the edges are not repeated, then W is called a trail. A trivial walk/trail contains no edges. A walk or trail is closed if its initial and terminus points are same. If the vertices $v_0, v_1, v_2, ..., v_n$ of the walk $W$ are distinct, then $W$ is called a path and is denoted by $P_n$. In other words the path $P_n$ can be described as an alternate sequence of $n$ vertices and $n-1$ edges. A non trivial closed trail in $G$ is called a cycle if its origin and internal vertices are distinct. A cycle of length $n$, that is, with $n$ edges is called $n$-cycle and is denoted by $C_n$. If $n$ is even (odd) then $C_n$ is known as even (odd) cycle.

The following theorem gives a nice characterization of bipartite graphs, by showing that a graph with no cycle of odd length cannot be bipartite graph and similarly if a bipartite graph contains cycle or cycles, they must be of even length.

**Theorem 1.3.1.** [33] *A graph $G$ is bipartite if and only if it has no odd cycle.*

A bipartite graph which is a complete that means all vertices of one partite set of cardinality $n$ are adjacent to the vertices of other partite set of cardinality $m$ is called a *complete bipartite graph* and denoted by $K_{n,m}$. A *regular graph* is a graph having all vertices of same degree and the degree of each vertex is $k$(say),then the graph is known as $k$-regular graph. All complete graphs and complete bipartite graphs are regular.

## 1.4 Trees and Forests

An *acyclic graph* is a graph containing no cycle. A *forest* is an acyclic graph. A graph $G$ is said to be connected, if any two vertices in $G$ can be joined with a path. A *tree* is a connected acyclic graph. This family of graphs is important to the structural understanding of graphs and to the algorithms of the information processing, and they play central role in design of connected networks. Some special tree

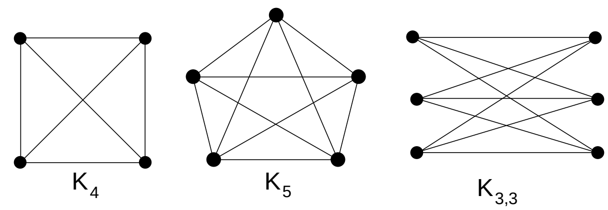


Figure 1.1: Complete graphs $K_4$, $K_5$ and complete bipartite graph $K_{3,3}$

structures are utilized in information management to store data in space efficient ways that sanction their retrieval and modification to be time efficient.

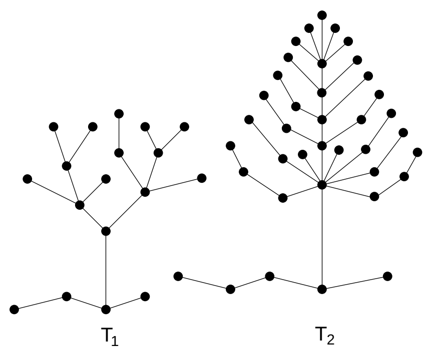


Figure 1.2: Trees in a forest

The following states that a graph is a tree if it is connected and its every edge is bridge. A bridge is an edge of a connected the graph whose removal makes graph disconnected. A connected graph having every edge is bridge is a tree.

**Theorem 1.4.1.** [27] *A graph $T$ is a tree if and only if it is connected and every edge in it is a bridge.*

A tree is a *minimal connected* graph. That is, the removal of one edge makes the tree disconnected and such an edge is called a *cut-edge*. Cut-edge is also called a *bridge*.
In the following Theorem, $T + e$ represents the addition of an edge to a graph $T$.

**Theorem 1.4.2.** [27] *Let $T$ be a connected graph with $n$ vertices. Then the following statements are equivalent.*
*(i) $T$ is a tree.*
*(ii) $T$ contains no cycles and has $n - 1$ edges.*
*(iii) $T$ is connected and has $n - 1$ edges.*

*(iv) T is connected, and every edge is a cut-edge.*
*(v) Any two vertices of T are connected by exactly one path.*
*(vi) T has no cycles, and for each new edge e, the graph $T+e$ has exactly one cycle.*

There are some special types of trees. A tree of order $n+1$, having $n$ leaves is known as *star* and is denoted by $St(n)$ or $K_{1,n}$. A *caterpillar* is a tree, with the property that the removal of its end-vertices (or leaves) produces a path. A *comb* denoted by $Cb_n$ is a tree obtained from a path $P_n$ of length $n-1$ with edges by attaching a leaf to all vertices of path $P_n$, except one end-vertex (vertex of degree one) path $P_n$. A *bistar* is a tree, which is obtained by joining the central vertices of two stars by an edge. A *subdivision* of a graph $G$ is a process of inserting vertices (of degree 2) into the edges of $G$. A *subdivided star* or spider graph $G = T(n_1, n_2, \ldots, n_p)$ be a graph obtained by inserting $n_i + 1$ vertices to each of the $i-th$ edge of the star $K_{1,p}$, for $n \geq 1$ and $p \geq 4$ where $1 \leq i \leq p$.

In the following theorem, we state a result that a tree having at least two

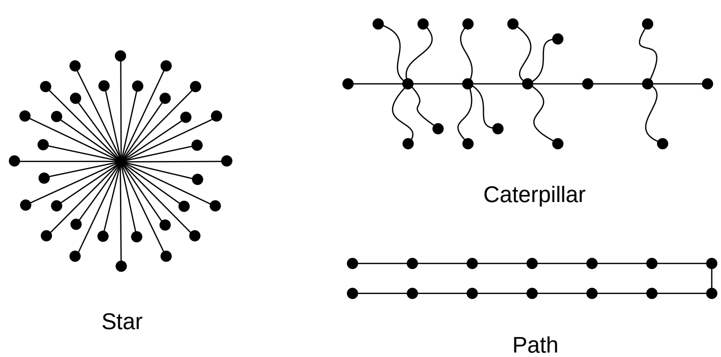


Figure 1.3: Star, path and caterpillar

vertices has at least two leaves, if we delete one leaf the order of a tree is reduced by 1.

**Lemma 1.4.1.** [14] *Every tree with at least two vertices has at least two leaves. If we omit a leaf from an n-vertex tree then a tree with $n-1$ vertices is produced.*

## 1.5 Graph Operations

The very basic ways of joining graphs are by union and intersection. The *union* of two graphs $G$ and $H$ is denoted as $G \cup H$ having vertex set $V(G)$ and $V(H)$ and edge $E(G)$ and $E(H)$ is $V(G) \cup V(H) = V(G \cup H)$ and $E(G) \cup E(H) = E(G \cup H)$

and the *intersection* of two graphs $G$ and $H$ is denoted as $G \cap H$ having vertex set $V(G)$ and $V(H)$ and edge set $E(G)$ and $E(H)$ is $V(G) \cap V(H) = V(G \cap H)$ and $E(G) \cap E(H) = E(G \cap H)$.
The disjoint union of $k$ copies of $G$ is denoted as $kG$. The *sum* of two graphs $G$ and $H$, denoted as $G + H$, is a graph obtained by adding edges between the vertices of $G$ and $H$. That is, $E(G+H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.
The *cartesian product* $G \Box H$ of graphs $G$ and $H$, is a graph such that the vertex set of $G \Box H$ is $V(G) \times V(H)$ and any two vertices $(x_1, x_2)$ and $(y_1, y_2)$ are adjacent in $G \Box H$ if and only if either $x_1 = y_1$ and $x_2$ is adjacent with $y_2$ in $H$, or $x_2 = y_2$ and $x_1$ is adjacent with $y_1$ in $G$. The join and cartesian product of $C_4$ and $P_5$ are shown in Figure 1.4.

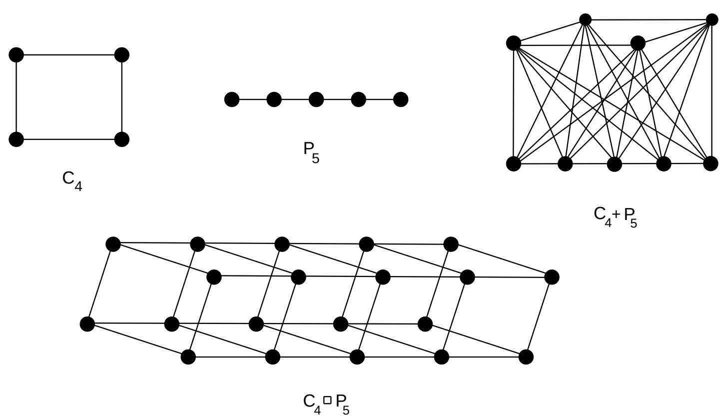


Figure 1.4: Sum and cartesian product of $C_4$ and $P_5$

One very common operation that we perform on graphs is *vertex/edge deletion* from the graph. Once a vertex is deleted from a graph, all the edges incident on that vertex are also removed, and when an edge is deleted from a graph, no difference occur other than the size of the graph is reduced by 1.

For $e \in E(G)$, the edge deleted graph $G$ is denoted as $G - e$. Another operation that we perform only on edges of a graph is *contraction* of an edge. An edge $uv$ is contracted by coinciding both of its end-vertices into a single vertex $x$ and joining all edges which were incident on $u$ and $v$ to the new vertex $x$. The graph we obtain after contracting the edge $e = uv$ is denoted as $G|uv$ or $G|e$. Studying simple graphs, any loops or multiple edges that occur after edge contraction are removed. The deletion and contraction of an edge $e$ in $P_5$ is figured in Figure 1.5.

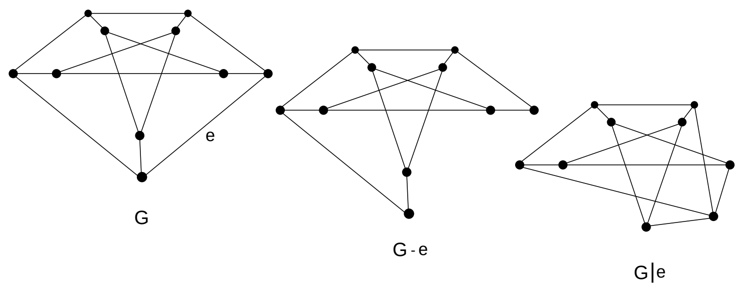


Figure 1.5: Edge contraction in a graph

## 1.6 Some Special Classes of Graphs

A *wheel* is denoted by $W_n$ of order $n+1$ is obtained by joining all the vertices of a cycle of order $n$ to a new vertex. If we remove one edge from the outer cycle of a wheel, the resulting graph is called *fan*, denoted by $f_n$. If we remove the edges from the outer cycle of the wheel alternatively, we get a *friendship* graph, denoted by $F_n$. The wheel, fan and friendship graphs are elaborated in Figure 1.6.

A fan graph $f_n$ can also be obtained by $P_n \Box K_1$, where $K_1$ represents complete graph on a single vertex (an isolated vertex). Similarly a wheel graph can be described as $C_n \Box K_1$, and a friendship graph is $(2n)P_2 \Box K_1$, where $(2n)P_2$ represents $2n$ of $P_2$.

A *circular ladder* is a graph obtained by $C_n \Box P_2$, denoted as $D_n$. A circular ladder is also called a *prism.* An *antiprism* (denoted as $A_n$) is a graph formed by combining two $n$ sided polygons by a band of $2n$ triangles. The graph $C_4$ is known as 2-*hypercube* ($Q_2$). The graph $C_4 \Box P_2$ is called 3-*hypercube* ($Q_3$). Similarly, the $n$-*hypercube* ($Q_n$) is obtained by $Q_{n-1} \Box P_2$.

The *generalized Petersen graph* denoted by $P(n,k)$, for $n \geq 3$ and $k < \frac{n}{2}$ is the graph obtained by joining $n$ vertices to corresponding vertices of an $n$-cycle and joining each vertex to the $k$-th vertex in the cyclic order. The prism $D_8$, antiprism $A_8$ and the generalized petersen graph $P(8,3)$ (also known as Möbius-Cantor graph) are shown in Figure 1.6.

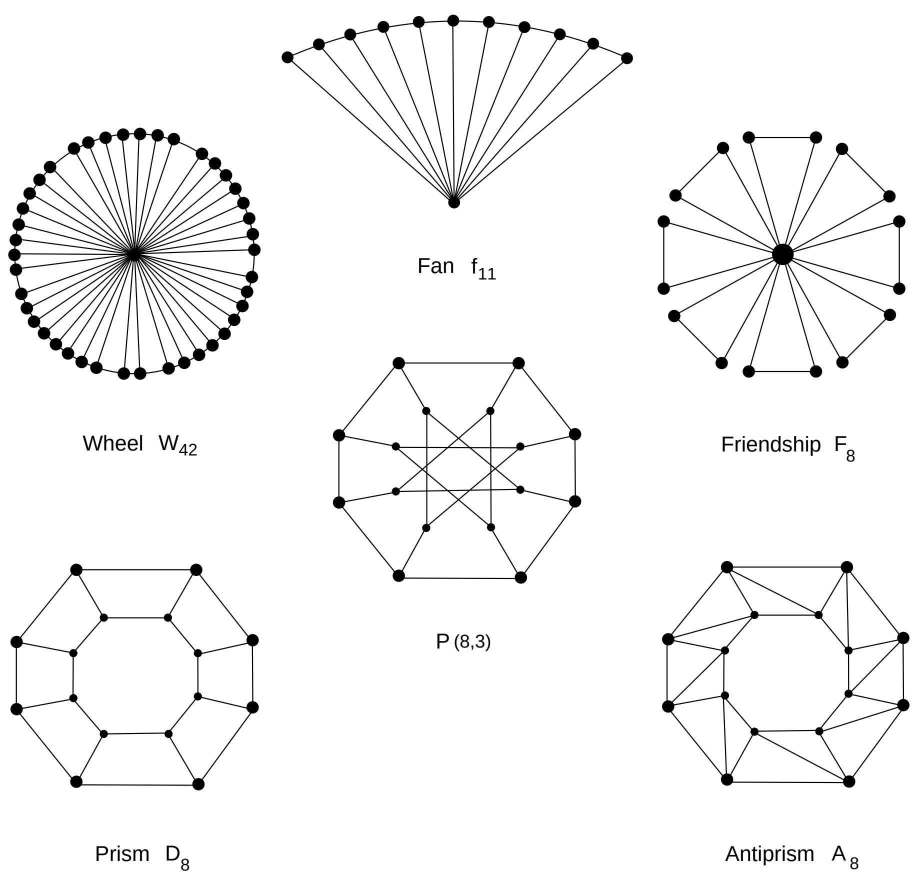


Figure 1.6: Wheel, Fan, Friendship, Prism, Antiprism and Petersen graphs

# Chapter 2

# Graph Labeling and Known Results

In this chapter, we introduce the concept of graph labeling and its different variations. A comprehensive literature review about graph labeling, its different types and known results about super edge-magic total labeling are given in this chapter. The known results about trees and forests are also presented in this chapter which is the main motivation of our research presented in chapter 3.

Generally, we label graphs for identification purpose only. We are interested in that kind of labeling of a graph which can be used to serve dual purposes. Firstly, such labeling not only used to identify vertices and edges, but also implies in some additional properties as well, depending upon the structure of graph. The method of allocating labels to a graph is called *graph labeling*. Such a graph is known as *labeled graph*
There are different kinds of labeling depending upon which element of graph is labeled, as follows:

- Vertex labeling: A labeling in which all the vertices of the graph are labeled.
- Edge labeling: A labeling in which all the edges of the graph are labeled.
- Total labeling: A labeling in which all the vertices and edges of the graph are labeled.
- Super type labeling: A labeling in which all are labeled with smallest possible integers.

There are some basic categories of graph labeling:

- Graceful labeling: A labeling in which vertices are labeled with distinct integers in a way to get the edge weights from consecutive integers.

- Harmonious labeling: A labeling in which vertices are labeled with distinct integers and the edge weights are calculated which are distinct.
- Magic type labeling: A labeling in which all the calculated weights of the elements of the graph are same.
- Antimagic type labeling: A labeling in which all the calculated weights of the elements of the graph are not same.

## 2.1 Graceful Labeling

In his *1967*, Rosa [42], paper introduced a labeling which he called as $\beta$ *labeling*, which was later renamed as *graceful labeling* by S.W. Golomb [23]. A graceful labeling is a vertex labeling which is a injection $\lambda : V(G) \to \{0, 1, 2, \dots, q\}$, where $q$ is the number of edges in $G$, such that each edge $xy \in E(G)$ is assigned a unique label $|\lambda(x) - \lambda(y)|$, where all the vertex labels are distinct as well, and the absolute value of the difference of $\lambda(x)$ and $\lambda(y)$ is called the weight of the edge $xy$. A graph having graceful labeling is called as *graceful graph*.

Applications of the graph labeling has been found in x-ray crystallography, coding theory, radar, circuit design, astronomy and communication design, for detail see [16]. The graceful labelings of the graphs $K_4$, $C_4$ and $C_5$ are shown in the following Figure 2.1 and Figure 2.2, respectively. The numbers in circle show the edge-weights which are the absolute differences of the labels of the adjacent vertices.

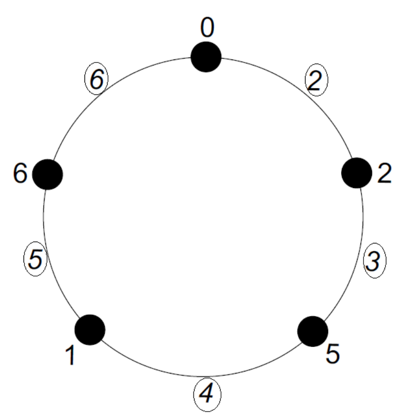


Figure 2.1: Graceful labeling of $C_5$

The most popular conjecture on graceful labelings which is still open was proposed by Ringel and Kotzig [34].

**Conjecture 2.1.1.** [34] *All trees are graceful.*

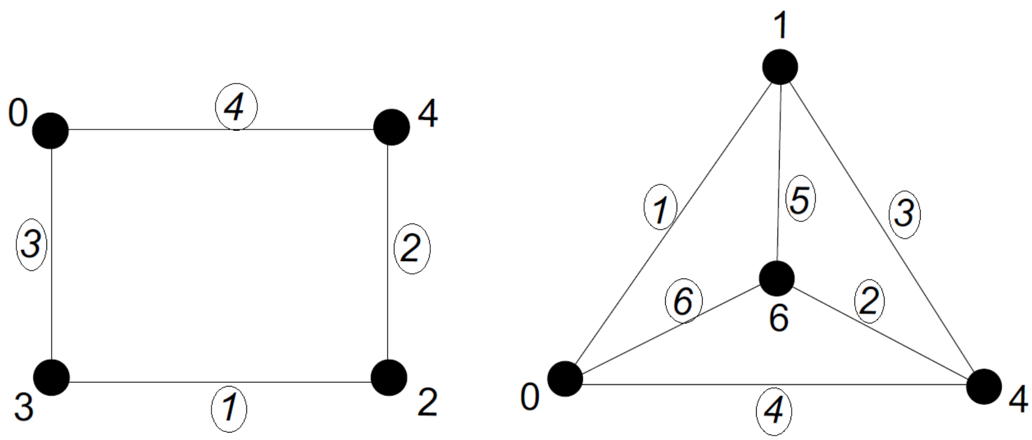


Figure 2.2: Graceful labeling of $C_4$ and $K_4$

## 2.2 Harmonious Labeling

In *1980*, Graham and Sloane [24] presented a vertex labeling of a graph $G$ defined as a bijection $\lambda : V(G) \to \mathbb{Z}_{|E|}$ such that the mapping $\lambda'$ from the edge set $E(G)$ to $Z_q$ defined by

$$\lambda'(xy) = \lambda(x) + \lambda(y)$$

for every $xy \in E(G)$, assigns different labels to the edges of $G$ is called *harmonious labeling* of $G$ and $G$ is called a *harmonious* graph.

According to Erdös [24] unpublished results, there does not exist such a graph which is neither graceful nor harmonious. A harmonious graph is shown in Figure 2.3. The labels of edge weights are shown in bold-italic. Graham and Sloane [25] showed that this is a maximal sized harmonious graph on 7 vertices.

## 2.3 Magic Labeling

A labeling in which weights of all vertices (or edges) in the graph are same, such a labeling is known as magic labeling. Magic labeling is further categorized as;

- Vertex-magic labeling: A magic labeling in which the weights of all the vertices are same.
- Edge-magic labeling: A magic labeling in which the weights of all the edges are same.

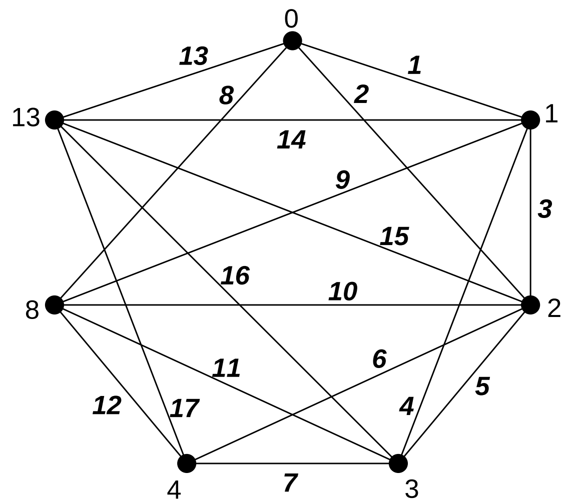


Figure 2.3: A harmonious Graph

### 2.3.1 Edge-Magic Labeling

Sedláček [44] defines a new labeling. According to his definition, a magic labeling is a function $\lambda$ from the set of edges of a graph $G$ to the finite subset of the set of real numbers, such that the sums of the edge labels of the edges incident upon a vertex in $G$ is the same, and is equal to a fix constant, for every vertex. That constant was named as the *magic constant* (also known as valance number) of the labeling.

The edge-magic labelings of a graphs is shown in the following Figure 2.4, with magic constants 28.

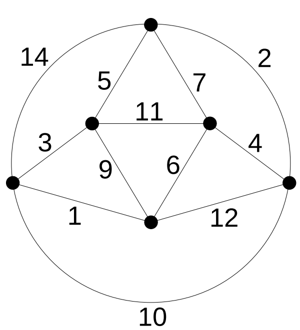


Figure 2.4: Edge-magic graph

Stewart [46] shows that $K_n$, for $n \geq 5$ and fan graphs $F_n$ are magic only for $n$

odd and for $n \geq 3$. Stewart named the magic labeling as super magic when edges are labeled with smallest possible consecutive integers. In [47], Stewart proved that $K_n$ is super magic for $n \geq 5$ iff $n > 5$ and $n \not\equiv 0 \,(\mathrm{mod}\ 4)$. A magic graph with magic constant 27 is shown in the Figure 2.5.

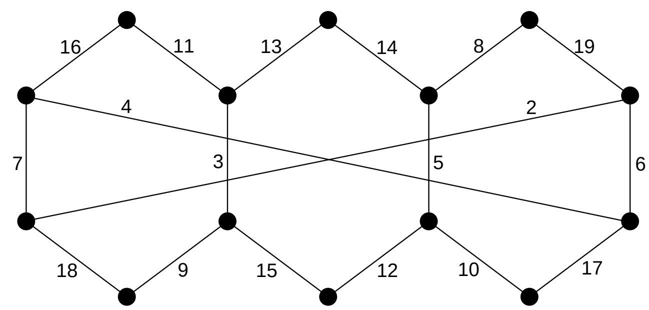


Figure 2.5: Super magic graph

**Theorem 2.3.1.** [6] *The complete graph $K_n$ is super-magic if and only if either $n \geq 6$ and $n \not\equiv 0 \pmod 4$, or $n = 2$.*

### 2.3.2 Edge-Magic Total Labeling

In *1970*, Kotzig and Rosa [34] introduced edge-magic total labeling. They called this labeling *magic*, but to distinguish the magic labeling defined by Stewart it has been agreed to call this labeling *edge-magic total*.

An edge magic total labeling of a graph $G$ can be defined as a bijection defined in such a way

$$\lambda : V(G) \cup E(G) \rightarrow \{1, 2, 3, \ldots, p+q\}$$

that the weights of all the edges are equal to a fixed constant $k$ (say). The weight of an edge $uv \in E(G)$ under this labeling function is calculated as

$$\omega(uv) = \lambda(u) + \lambda(v) + \lambda(uv).$$

The constant $k$ is called the magic constant of the graph $G$ under the labeling $\lambda$. A graph with an edge-magic total labeling is called *edge-magic total graph* and abbreviated as EMT.

In the Figure 2.6, the graphs is labeled using edge-magic total labeling with magic constants 36, are shown.

Ringel and Lladó [41] proved that a $(p, q)$ graph $G$ is not EMT if all vertices are of odd degree and $q$ is even and $p + q \equiv 2 \,(mod\, 4)$. Kotzig and Rosa [34] proved

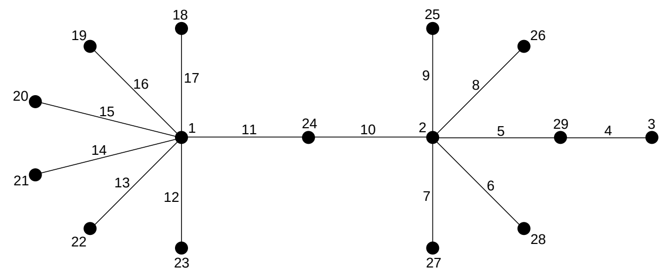


Figure 2.6: Edge-magic total graph

that the complete bipartite graphs $K_{m,n}$ for any $m$ and $n$ and cycle $C_n$ for all $n \geq 3$ are EMT. Enomoto et al. [17] proposed a conjecture that all wheels except the ones described in [41] are EMT.
Wallis et al. [50] constructed EMT labelings of the complete graph $K_n$ for $n \in \{1, 2, 3, 4, 5, 6\}$ for all possible values of magic constant. They also showed that all paths, complete bipartite graphs and all cycles with a chord admit EMT labeling.

### 2.3.3 Super Edge Magic Total Labeling

The concept of *super edge magic total labeling* (SEMT) was introduced by Enomoto et al. [17]. They defined the SEMT labeling of a graph $G$ to be a bijection

$$\lambda : V(G) \cup E(G) \rightarrow \{1, 2, 3, \ldots, p + q\}$$

such that the weight of every edge is equal to a fixed constant, and being a super labeling it satisfies another property that all vertices are labeled with the smallest available labels $\{1, 2, 3, \ldots, p\}$, and the rest of labels $\{p + 1, p + 2, p + 3, \ldots, p + q\}$ are assigned to the edges of the graph. The weight of an edge $uv \in E(G)$ under SEMT labeling is calculated in the same way as was calculated in EMT labeling.

The SEMT labeling of a graph with magic constant 19, is shown in the Figure 2.7.

The most useful lemma which provides a necessary and sufficient condition for a graph to be super edge magic total, is given by Figueroa et al. in [20].

**Lemma 2.3.1.** [20] *A $(p, q)$ graph $G$ is said to be super edge-magic total if and only if there exists a bijection $\lambda : V(G) \rightarrow \{1, 2, 3, \ldots, p\}$ such that the set*

$$S = \{\lambda(u) + \lambda(v) : uv \in E(G)\}$$

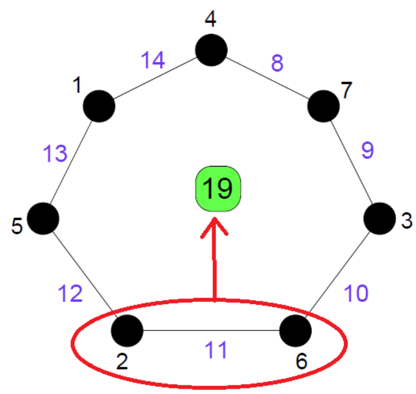


Figure 2.7: Super edge magic total graph

*consists of $q$ consecutive integers.*
*In such a case, $\lambda$ extends to a super edge-magic total labeling of $G$ with constant $k = p + q + s$, where $s = min(S)$ and*

$$S = \{k - (p + 1), k - (p + 2), k - (p + 3), \ldots, k - (p + q)\}.$$

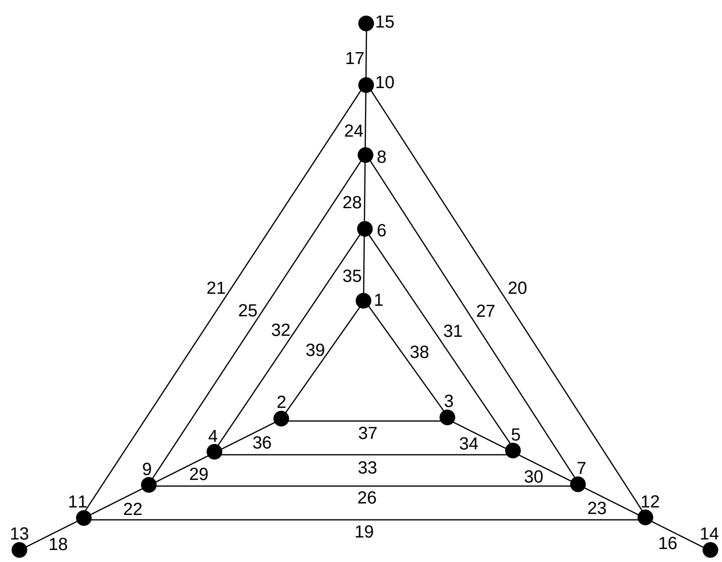

Figure 2.8: Super edge magic total graph

Enomoto et al. [17] proved that

- The cycle $C_n$ is super edge-magic total if and only if $n$ is odd.
- The complete bipartite graph $K_{m,n}$ is super edge-magic total if and only if $m = 1$ or $n = 1$.
- The complete graph $K_n$ is super edge-magic total if and only if $n = 1, 2$ or 3.

Enomoto et al. proved a necessary condition for a graph to be super edge-magic total.

**Proposition 2.3.1.** [17] *If a $(p, q)$-graph $G$ is super edge-magic total then $q \leq 2p-3$.*

The previous proposition implies that the minimum degree of any super edge-magic total graph is at most 3.

Enomoto et al. [17] also proposed one of the most popular conjecture in graph labeling known as the *tree conjecture*.

**Conjecture 2.3.1.** [17] *Every tree is super edge magic total.*

Lee and Shah [37] have verified this conjecture for the trees with up to 17 vertices, by using a computer. Kotzig and Rosa [34] proved that all caterpillars are super edge-magic total.

In the following, we present a very comprehensive list of graphs labeled with SEMT labeling and some graphs which are not SEMT. The source of this data is a survey on graph labelings conducted by Gallian [22]. This is a comprehensive and dynamic survey on graph on graph labeling which is update regularly and is considered as an encyclopedia of graph labeling.

## 2.4 Some Known Results on Super Edge-Magic Deficiency of Graphs

In 1970, Kotzig and Rosa [48] proved that for any graph $G$ there exists an edge-magic total graph $H$ such that $H \cong G \cup nK_1$ for some nonnegative integer $n$. This fact leads to the concept of edge-magic deficiency of a graph $G$, which is the minimum nonnegative integer $n$ such that $G \cong nK_1$ is edge-magic total and it is denoted by $\mu(G)$. In particular,
$M(G)$=$\{n \geq 0 : G \cup nK_1$ is a edge-magic total graph$\}$.

In the same paper, Kotzig and Rosa gave an upper bound of the edge-magic deficiency of a graph $G$ with $n$ vertices

$$\mu(G) \leq F_{n+2} - 2 - n - 1 - \frac{n(n-1)}{2},$$

Table 2.1: **Summary of Super Edge-magic Labelings**

| Graph | Types | Notes |
|---|---|---|
| $C_n$ | SEM | iff $n$ is odd |
| caterpillars | SEM | |
| $K_{m,n}$ | SEM | iff $m = 1$ or $n = 1$ |
| $K_n$ | SEM | iff $n = 1, 2$ or $3$ |
| trees | SEM? | |
| $nK_2$ | SEM | iff $n$ odd |
| $nG$ | SEM | if $G$ is a bipartite or tripartite SEM graph and $n$ odd |
| $K_{1,m} \cup K_{1,n}$ | SEM | if $m$ is a multiple of $n + 1$ |
| $K_{1,m} \cup K_{1,n}$ | SEM? | iff $m$ is a multiple of $n + 1$ |
| $K_{1,2} \cup K_{1,n}$ | SEM | iff $n$ is a multiple of 3 |
| $K_{1,3} \cup K_{1,n}$ | SEM | iff $n$ is a multiple of 4 |
| $P_m \cup K_{1,n}$ | SEM | if $m \geq 4$ is even |
| $2P_n$ | SEM | iff $n$ is not 2 or 3 |
| $2P_{4n}$ | SEM | for all $n$ |
| $K_{1,m} \cup 2nK_{1,2}$ | SEM | for all $m$ and $n$ |
| $C_3 \cup C_n$ | SEM | iff $n \geq 6$ even |
| $C_4 \cup C_n$ | SEM | iff $n \geq 5$ odd |
| $C_5 \cup C_n$ | SEM | iff $n \geq 4$ even |
| $C_m \cup C_n$ | SEM | if $m \geq 6$ even and $n$ odd, $n \geq m/2 + 2$ |
| $C_m \cup C_n$ | SEM? | iff $m + n \geq 9$ and $m + n$ odd |
| $C_4 \cup P_n$ | SEM | iff $n \neq 3$ |
| $C_5 \cup P_n$ | SEM | if$n \neq 4$ |
| $C_m \cup P_n$ | SEM | if $m \geq 6$ even and $n \geq m/2 + 2$ |
| $P_m \cup P_n$ | SEM | iff $(m, n) \neq (2, 2)$ or $(3, 3)$ |
| corona $C_n \odot \bar{K}_m$ | SEM | $n \geq 3$ |
| $St(m, n)$ | SEM | $n \equiv 0\, mod(m + 1)$ |
| $St(1, k, n)$ | SEM | $k = 1, 2$ or $n$ |
| $St(2, k, n)$ | SEM | $k = 2, 3$ |
| $St(1, 1, k, n)$ | SEM | $k = 2, 3$ |
| $St(k, 2, 2, n)$ | SEM | $k = 1, 2$ |
| $St(a_1, ..., a_n)$ | SEM? | for $n > 1$ odd |
| $C_{4m}^t$ | SEM | |
| $C_{4m+1}^t$ | SEM | |
| friendship graph of n triangles | SEM | iff $n = 3, 4, 5,$ or $7$ |
| graph $P(n, 2)$ | SEM | if $n \geq 3$ odd |
| $nP_3$ | SEM | if $n \geq 4$ even |

Table 2.2: **Summary of Super Edge-magic Labelings**

| **Graph** | **Types** | **Notes** |
|---|---|---|
| generalized Petersen $P_n^2$ | SEM | |
| $K_2 \times C_{2n+1}$ | SEM | |
| $P_3 \cup kP_2$ | SEM | for all $k$ |
| $kP_n$ | SEM | if $k$ is odd |
| $k(P_2 \cup P_n)$ | SEM | if $k$ is odd and $n = 3, 4$ |
| fans $F_n$ | SEM | iff $n \leq 6$ |
| books $B_n$ | SEM | if $n$ even |
| books $B_n$ | SEM? | if $n$ even or $n \equiv 5\, mod(8)$ |
| trees with $\alpha$-labelings | SEM | |
| $P_{2m+1} \times P_2$ | SEM | |
| $C_{2m+1} \times P_m$ | SEM | |
| $G \odot \bar{K}_n$ | SEM | if $G$ is SEM 2-regular graph |
| $C_m \odot \overline{K_n}$ | SEM | |
| if $G$ is $k$-regular SEM graph | | then $k \leq 3$ |
| $G$ is connected $(p, q)$-graph | SEM | $G$ exists iff $p - 1 \leq q \leq 2p - 3$ |
| $G$ is connected 3-regular graph on $p$ vertices | SEM | iff $p \equiv 2\, mod(4)$ |
| $nK_2 + nK_2$ | not SEM | |

where $F_n$ is the $nth$ Fibonacci number.

There are some known results on edge-magic deficiency Figueroa et al. [43] proved that

**Theorem 2.4.1.** [43] *The edge-magic deficiency of $P_m \cup P_n$ is*

$$\mu(P_m \cup P_n) = \begin{cases} 1, & (m,n) = (2,2); \\ 0, & otherwise. \end{cases}$$

In the same paper they proved [43].

**Theorem 2.4.2.** [43] *The edge-magic deficiency of $P_m \cup K_{1,n}$ is*

$$\mu(P_m \cup K_{1,n}) = \begin{cases} 1, & when\ m = 2\ and\ n\ is\ odd \\ 0, & otherwise. \end{cases}$$

Motivated by Kotzig and Rosa's concept of edge-magic deficiency, Figueroa et al. [43] defined a similar concept for super edge-magic total labelings.

**Definition 2.4.1.** The *super edge-magic deficiency* of a graph G, which is denoted by $\mu(G)$, is the minimum nonnegative integer $n$ such that $G \cup nK_1$ has a super edge-magic total labeling or it is equal to $+\infty$ if there exists no such $n$. More precisely, let $M(G)$={$n \geq 0 : G \cup nK_1$ is a super edge-magic total graph} Then

$$\mu_s(G) = \begin{cases} nM(G), & if\ \ M(G) \neq \phi \\ \infty, & if\ \ M(G) = \phi \end{cases}$$

Now we summarize the known results on super edge-magic deficiency. In [40], Figueroa et al. proved the following results.

**Theorem 2.4.3.** [40] *The super edge-magic deficiency of $nK_2$ is*

$$\mu_s(nK_2) = \begin{cases} 0, & when\ n\ is\ odd; \\ 1, & otherwise. \end{cases}$$

*For super edge-magic deficiency of cycles Figueroa et al.* [43] *proved the followings.*

**Theorem 2.4.4.** [43] *The super edge-magic deficiency of cycle $C_n$ is*

$$\mu_s(C_n) = \begin{cases} 0, & when\ n\ is\ odd \\ 1, & when\ n \equiv 0 \pmod 4 \\ \infty, & when\ n \equiv 2 \pmod 4. \end{cases}$$

For copies of cycles Figueroa et al. [39] showed the following results

**Theorem 2.4.5.** [39] *The super edge-magic deficiency of* $2C_n$ *is*

$$\mu_s(2C_n) = \begin{cases} 1, & \text{when } n \text{ is even} \\ \infty, & \text{when } n \text{ is odd.} \end{cases}$$

**Theorem 2.4.6.** [39] *The super edge-magic deficiency of* $3C_n$ *is*

$$\mu_s(3C_n) = \begin{cases} 0, & \text{when } n \text{ is odd} \\ 1, & \text{when } n \equiv 0 \pmod 4 \\ \infty, & \text{when } n \equiv 2 \pmod 4. \end{cases}$$

**Theorem 2.4.7.** [39] *The super edge-magic deficiency of* $4C_n$ *is*

$$\mu_s(4C_n) = 1$$

*when* $n \equiv 0 \pmod 4$.

In the same paper [39] they conjectured the following.

**Conjecture 2.4.1.** [39]

$$\mu_s(mC_n) = \begin{cases} 0, & \text{when } mn \text{ is odd} \\ 1, & \text{when } mn \equiv 0 \pmod 4 \\ \infty, & \text{when } mn \equiv 2 \pmod 4. \end{cases}$$

Moreover, Figueroa et al. [43] also proved the following results.

**Theorem 2.4.8.** [39] *The super edge-magic deficiency of* $P_m \cup P_n$ *is*

$$\mu_s(P_m \cup P_n) = \begin{cases} 1, & \text{when } (m,n) = (2,2) \text{ or } (m,n) = (3,3) \\ 0, & \text{otherwise.} \end{cases}$$

For union of two stars they [39] proved that

**Theorem 2.4.9.** [39] *The super edge-magic deficiency of* $K_{1,m} \cup K_{1,n}$ *is*

$$\mu_s(K_{1,m} \cup K_{1,n}) = \begin{cases} 0, & \text{if } m \text{ is a multiple of } n+1, \text{ or if } n \text{ is a multiple of } m+1 \\ 1, & \text{otherwise.} \end{cases}$$

**Theorem 2.4.10.** *The super edge-magic deficiency of* $K_{1,m} \cup K_{1,n}$ *is*

$$\mu_s(K_{1,m} \cup K_{1,n}) = \begin{cases} 0, & \text{when } mn \text{ is even} \\ 1, & \text{when } mn \text{ is odd.} \end{cases}$$

**Theorem 2.4.11.** [39] *The super edge-magic deficiency of* $2K_{1,n}$ *is* 1 *when* $n$ *is odd and almost* 1 *when* $n$ *is even.*

They [39] also conjectured that $2K_{1,n} = 1$ for all other cases.

**Theorem 2.4.12.** [39] *The super edge-magic deficiency of all forests* $F$ *is finite, that is,*

$$\mu_s(F) < \infty.$$

Furthermore, they [43] conjectured that every forest with two components has super edge-magic deficiency at most 1.

**Theorem 2.4.13.** [43] *The super edge-magic deficiency of complete graph* $K_n$ *is*

$$\mu_s(K_n) = \begin{cases} 0, & when\ n = 1, 2, 3 \\ 1, & when\ n = 4 \\ \infty, & when\ n \geq 5. \end{cases}$$

**Theorem 2.4.14.** [43] *Let* $G$ *be a graph that contains the complete graph* $K_n$ *as a subgraph. If* $|E(G)| < \rho^*(n)$*, then*

$$\mu_s(G) = \infty.$$

In the same paper [43], they also gave the upper bound for the super edge-magic deficiency of the complete bipartite graph $K_{m,n}$.

**Theorem 2.4.15.** [43] *The super edge-magic deficiency of* $K_{m,n}$ *is*

$$\mu_s(K_{m,n}) \leq (m-1)(n-1).$$

In [39] Figueroa et al. were dealing with the union of two graphs.

**Theorem 2.4.16.** [39] *The super edge-magic deficiency of* $P_m \cup K_{1,n}$ *is*

$$\mu_s(P_m \cup K_{1,n}) = \begin{cases} 1, & if\ m = 2\ and\ n\ is\ odd,\ or\ m = 3\ and\ n \neq 0 \pmod 3 \\ 0, & otherwise. \end{cases}$$

# Chapter 3

# SEMT Labeling of Disjoint Union of Two Acyclic Graphs

In this chapter, we construct SEMT labeling of union of two acyclic graphs. All of our results support the conjectures 3.0.2 and 3.0.3.
Enomoto et al. proposed one of the most popular conjecture in graph labeling known as the *tree conjecture*.

**Conjecture 3.0.2.** [17] *Every tree is super edge magic total.*

This conjecture has been verified for the trees of up to 17 order with the help of computer. Kotzig and Rosa [35] proved that all caterpillars are super edge magic total.

The following conjecture about forest with two components was proposed in [39]

**Conjecture 3.0.3.** [39] *Every forest with two components has deficiency atmost* 1.

All our results obtained add further support to the Conjecture 3.0.3

**Definition 3.0.2.** A **subdivided star** $G = T(n_1, n_2, \ldots, n_p)$ be a graph obtained by inserting $n_i + 1$ vertices to each of the $i - th$ edge of the star $K_{1,p}$, for $n \geq 1$ and $p \geq 4$ where $1 \leq i \leq p$ and its central vertex represented by $x$.

In the next section, we are dealing with union of subdivided star and path having different lengths and subdivisions.

## 3.1 SEMT Labeling of Subdivided Star Union Path

In the next theorem, we represent the super edge magic total labeling for $T(n, n, n, n+1) \cup P_n$ where $n \geq 3$ and $n \equiv 1 \pmod 2$.

**Theorem 3.1.1.** *The graph $G \cong T(n,n,n,n+1) \cup P_n$ where $n \geq 3$ and $n \equiv 1 \pmod 2$ admits super edge magic total labeling.*

*Proof.* Let us denote $p = |V(G)|$ and $q = |E(G)|$ respectively, so we have

$$V(G) = \{x, z, x_{i,j}, y_k : \ 1 \leq i \leq 4, \ 1 \leq j \leq n, \ 1 \leq k \leq n \ \text{ where } \ z = x_{4,n+1}\},$$

$$E(G) = \{xx_{i,1}, x_{i,j}x_{i,j+1}, x_{4,n}z, y_k y_{k+1} : \ 1 \leq i \leq 4, \ 1 \leq j \leq n-1, \ 1 \leq k \leq n-1\}.$$

$$p = 5n + 2,$$

$$q = 5n.$$

Now, we define the labeling $\lambda : V \cup E \to \{1, 2, \ldots, p+q\}$ as follows:

$$\lambda(x) = 3n + 2 + \lceil \frac{n}{2} \rceil,$$

$$\lambda(z) = 5n + 2.$$

For odd $j$

$$\lambda(x_{ij}) = \begin{cases} \dfrac{j+1}{2}, & i = 1 \\ n + 1 - \dfrac{j-1}{2}, & i = 2 \\ n + 1 + \dfrac{j+1}{2}, & i = 3 \\ 2n + 2 - \dfrac{j-1}{2}, & i = 4. \end{cases}$$

For even $j$

$$\lambda(x_{ij}) = \begin{cases} 2n + 2 + \lceil \frac{n}{2} \rceil + \frac{j}{2}, & i = 1 \\ 3n + 2 + \lceil \frac{n}{2} \rceil - \frac{j}{2}, & i = 2 \\ 3n + 2 + \lceil \frac{n}{2} \rceil + \frac{j}{2}, & i = 3 \\ 4n + 2 + \lceil \frac{n}{2} \rceil - \frac{j}{2}, & i = 4. \end{cases}$$

$$\lambda(y_{2i-1}) = 2n + 2 + i, \quad \text{for} \quad 1 \leq i \leq \lceil \frac{n}{2} \rceil,$$

$$\lambda(y_{2i}) = 4n + 1 + \lceil \frac{n}{2} \rceil + i, \quad \text{for} \quad 1 \leq i \leq \lfloor \frac{n}{2} \rfloor.$$

The set of all edge-sums generated by above formula forms a consecutive integer sequence denoted by $s$.

$$s = \left\{ 2n + \frac{n-1}{2} + 5, \ 2n + \frac{n-1}{2} + 6, \ \ldots, \ 10n + 6 \right\}.$$

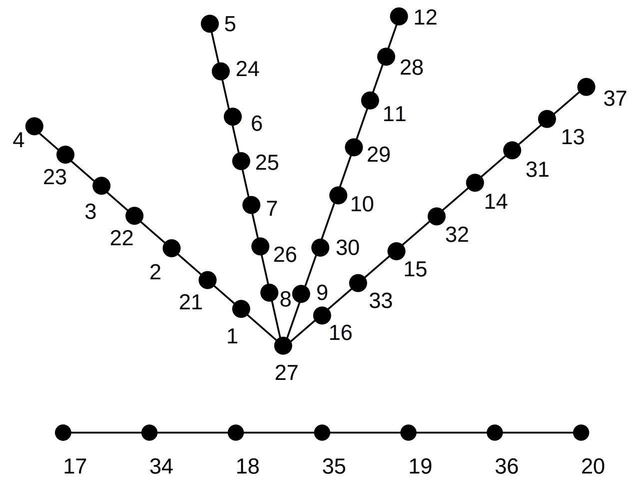


Figure 3.1: SEMT labeling of $T(7,7,7,8) \cup P_7$

Therefore, by Lemma 2.3.1 $\lambda$ can be extended to a super edge magic total labeling with magic constant

$$k = p + q + s = 12n + \frac{n-1}{2} + 7.$$

The SEMT labeling of $T(7,7,7,8) \cup P_7$ is presented in Figure 3.1. □

In the next theorem, we represent the super edge magic total labeling for $T(n, n-1, l, l+2) \cup P_l$ where $n \geq 4$ and $l \geq n$, $n \equiv 0 \pmod 2$ and $l \equiv 0 \pmod 2$.

**Theorem 3.1.2.** *The graph $G \cong T(n, n-1, l, l+2) \cup P_l$ where $n \geq 4$ and $l \geq n$, $n \equiv 0 \pmod 2$ and $l \equiv 0 \pmod 2$ admits super edge magic total labeling.*

*Proof.* Let us denote $p = |V(G)|$ and $q = |E(G)|$, so we have

$$\begin{aligned} V(G) &= \{x, z, y_k : \ 1 \leq k \leq l \text{ where } z = x_{4,l+2}\} \cup \\ &\quad \{x_{1,j} : \ 1 \leq j \leq n\} \cup \\ &\quad \{x_{2,j} : \ 1 \leq j \leq n-1\} \cup \\ &\quad \{x_{3,j} : \ 1 \leq j \leq l\} \cup \\ &\quad \{x_{4,j} : \ 1 \leq j \leq l+1\}. \end{aligned}$$

$$\begin{aligned} E(G) &= \{xx_{i,1}, xx_{4,l+1}z, y_k y_{k+1}: \quad 1 \le i \le 4,\ 1 \le k \le l-1\} \cup \\ & \{x_{1,j}, x_{1,j+1}: \quad 1 \le j \le n-1\} \cup \\ & \{x_{2,j}, x_{2,j+1}: \quad 1 \le j \le n-2\} \cup \\ & \{x_{3,j}, x_{3,j+1}: \quad 1 \le j \le l-1\} \cup \\ & \{x_{4,j}, x_{4,j+1}: \quad 1 \le j \le l\}. \end{aligned}$$

$$p = 2(n+1) + 3l,$$

$$q = 2n + 3l.$$

Now, we define the labeling $\lambda : V \cup E \to \{1, 2, \ldots, p+q\}$ as follows:

$$\lambda(x) = 2n + \frac{3l}{2} + 1,$$

$$\lambda(z) = 2n + 3l + 2.$$

For odd $j$

$$\lambda(x_{ij}) = \begin{cases} \frac{j+1}{2}, & i = 1 \\ n + \frac{1-j}{2}, & i = 2 \\ n + \frac{1+j}{2}, & i = 3 \\ n + l + 1 + \frac{1-j}{2}, & i = 4. \end{cases}$$

For even $j$

$$\lambda(x_{ij}) = \begin{cases} n + \frac{3l}{2} + \frac{j+2}{2}, & i = 1 \\ 2n + \frac{3l}{2} + \frac{2-j}{2}, & i = 2 \\ 2n + \frac{3l}{2} + 1 + \frac{j}{2}, & i = 3 \\ 2n + \frac{5l}{2} + 2 - \frac{j}{2}, & i = 4. \end{cases}$$

$$\lambda(y_{2i-1}) = n + l + 1 + i, \quad \text{for} \quad 1 \le i \le \frac{l}{2},$$

$$\lambda(y_{2i}) = 2n + \frac{5l}{2} + 1 + i, \quad \text{for} \quad 1 \le i \le \frac{l}{2}.$$

The set of all edge-sums generated by above formula forms a consecutive integer sequence denoted by $s$.

$$s = \left\{n + \frac{3l}{2} + 3,\ n + \frac{3l}{2} + 4,\ \ldots,\ 3n + \frac{9l}{2} + 2\right\}.$$

Therefore, by Lemma [20] $\lambda$ can be extended to a super edge magic total labeling with magic constant

$$k = p + q + s = 5n + \frac{15l}{2} + 5.$$

The SEMT labeling of $T(6,5,8,10) \cup P_8$ is presented in Figure 3.2. $\square$

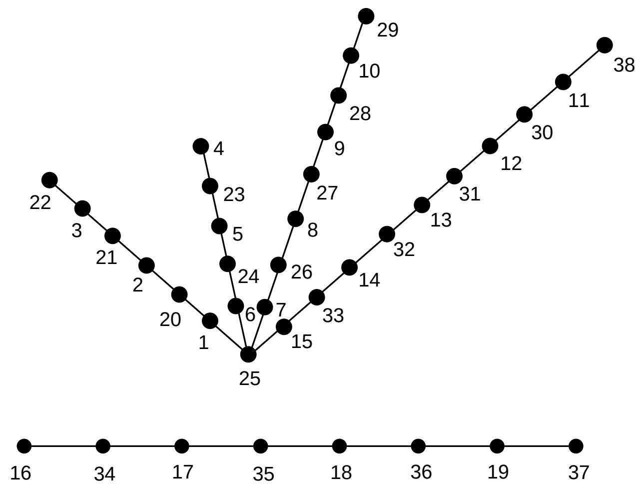


Figure 3.2: SEMT labeling of $T(6,5,8,10) \cup P_8$

In the next theorem, we represent the super edge magic total labeling for $T(n, n-1, l, l+2, 2l+4) \cup P_{2(l+1)}$ where $n \geq 4$ and $l \geq n$, $n \equiv 0 \pmod 2$ and $l \equiv 0 \pmod 2$.

**Theorem 3.1.3.** *The graph $G \cong T(n, n-1, l, l+2, 2l+4) \cup P_{2(l+1)}$ where $n \geq 4$ and $l \geq n$, $n \equiv 0 \pmod 2$ and $l \equiv 0 \pmod 2$ admits super edge magic total labeling.*

*Proof.* Let us denote $p = |V(G)|$ and $q = |E(G)|$, so we have

$$\begin{aligned} V(G) \;=\; & \{x, z, y_k : \; 1 \leq k \leq 2(l+1) \text{ where } z = x_{5,2l+4}\} \cup \\ & \{x_{1,j} : \; 1 \leq j \leq n\} \cup \\ & \{x_{2,j} : \; 1 \leq j \leq n-1\} \cup \\ & \{x_{3,j} : \; 1 \leq j \leq l\} \cup \\ & \{x_{4,j} : \; 1 \leq j \leq l+2\} \cup \\ & \{x_{5,j} : \; 1 \leq j \leq 2l+3\}. \end{aligned}$$

$$
\begin{aligned}
E(G) \;=\; & \{xx_{i,1}, y_k y_{k+1}, x_{5,2l+3}z : \quad 1 \le i \le 5,\ 1 \le k \le 2l+1)\} \cup \\
& \{x_{1,j}, x_{1,j+1} : 1 \le j \le n-1\} \cup \\
& \{x_{2,j}, x_{2,j+1} : \quad 1 \le j \le n-2\} \cup \\
& \{x_{3,j}, x_{3,j+1} : \quad 1 \le j \le l-1\} \cup \\
& \{x_{4,j}, x_{4,j+1} : \quad 1 \le j \le l+1 \cup \\
& \{x_{5,j}, x_{5,j+1} : \quad 1 \le j \le 2l+2\}.
\end{aligned}
$$

$$p = 2(n+3l+4),$$

$$q = 2(n+3l+3).$$

Now, we define the labeling $\lambda : V \cup E \to \{1, 2, \ldots, p+q\}$ as follows:

$$\lambda(x) = 2n + 3l + 4,$$

$$\lambda(z) = 2n + 6l + 8.$$

For odd $j$

$$
\lambda(x_{ij}) = \begin{cases}
\dfrac{j+1}{2}, & i = 1 \\
n + \dfrac{1-j}{2}, & i = 2 \\
n + \dfrac{j+1}{2}, & i = 3 \\
n + l + 1 + \dfrac{1-j}{2}, & i = 4 \\
n + 2l + 3 + \dfrac{1-j}{2}, & i = 5.
\end{cases}
$$

For even $j$

$$
\lambda(x_{ij}) = \begin{cases}
n + 3l + 4 + \dfrac{j}{2}, & i = 1 \\
2n + 3l + 4 - \dfrac{j}{2}, & i = 2 \\
2n + 3l + 4 + \dfrac{j}{2}, & i = 3 \\
2n + 4l + 6 - \dfrac{j}{2}, & i = 4 \\
2n + 5l + 7 - \dfrac{j}{2}, & i = 5.
\end{cases}
$$

$$\lambda(y_{2i-1}) = n + 2l + 3 + i, \quad \text{for} \quad 1 \le i \le l+1,$$

$$\lambda(y_{2i}) = 2n + 5l + 6 + i, \quad \text{for} \quad 1 \le i \le l+1.$$

The set of all edge-sums generated by above formula forms a consecutive integer sequence denoted by $s$.

$$s = \Big\{ n + 3l + 6,\ n + 3l + 7,\ \ldots,\ 3(n + 3l) + 11 \Big\}.$$

Therefore, by Lemma 2.3.1 $\lambda$ can be extended to a super edge magic total labeling with magic constant

$$k = p + q + s = 5n + 15l + 20. \qquad \square$$

**Theorem 3.1.4.** *The graph* $G \cong T(n, n-1, l, l+2, 2l+4, 4l+8) \cup P_{4l+6}$ *where* $n \geq 2$ *and* $l \geq n$, $n \equiv 0 \pmod 2$ *and* $l \equiv 0 \pmod 2$ *admits super edge magic total labeling.*

*Proof.* Let us denote $p = |V(G)|$ and $q = |E(G)|$, so we have

$$\begin{aligned} V(G) \;=\; & \{x, z, y_k: \;\; 1 \leq k \leq 4l+6 \text{ where } z = x_{6,4l+8}\} \cup \\ & \{x_{1,j}: \;\; 1 \leq j \leq n\} \cup \\ & \{x_{2,j}: \;\; 1 \leq j \leq n-1\} \cup \\ & \{x_{3,j}: \;\; 1 \leq j \leq l\} \cup \\ & \{x_{4,j}: \;\; 1 \leq j \leq l+2\} \cup \\ & \{x_{5,j}: \;\; 1 \leq j \leq 2l+4\} \cup \\ & \{x_{6,j}: \;\; 1 \leq j \leq 4l+7\}. \end{aligned}$$

$$\begin{aligned} E(G) \;=\; & \{xx_{i,1}, y_k y_{k+1}, x_{6,4l+7}z: \;\; 1 \leq i \leq 6, \; 1 \leq k \leq 4l+5\} \cup \\ & \{x_{1,j}, x_{1,j+1}: 1 \leq j \leq n-1\} \cup \\ & \{x_{2,j}, x_{2,j+1}: \;\; 1 \leq j \leq n-2\} \cup \\ & \{x_{3,j}, x_{3,j+1}: \;\; 1 \leq j \leq l-1\} \cup \\ & \{x_{4,j}, x_{4,j+1}: \;\; 1 \leq j \leq l+1 \cup \\ & \{x_{5,j}, x_{5,j+1}: \;\; 1 \leq j \leq 2l+3\} \cup \\ & \{x_{6,j}, x_{6,j+1}: \;\; 1 \leq j \leq 4l+6\}. \end{aligned}$$

$$p = 2n + 12l + 20,$$
$$q = 2n + 12l + 18.$$

Now, we define the labeling $\lambda : V \cup E \rightarrow \{1, 2, \ldots, p+q\}$ as follows:

$$\lambda(x) = 2n + 6l + 10,$$

$$\lambda(z) = 2n + 12l + 20.$$

For odd $j$

$$\lambda(x_{ij}) = \begin{cases} \frac{j+1}{2}, & i=1 \\ n+\frac{1-j}{2}, & i=2 \\ n+\frac{1+j}{2}, & i=3 \\ n+l+1+\frac{1-j}{2}, & i=4 \\ n+2l+3+\frac{1-j}{2}, & i=5 \\ n+4l+7+\frac{1-j}{2}, & i=6. \end{cases}$$

For even $j$

$$\lambda(x_{ij}) = \begin{cases} n+6l+10+\frac{j}{2}, & i=1 \\ 2n+6l+10-\frac{j}{2}, & i=2 \\ 2n+6l+10+\frac{j}{2}, & i=3 \\ 2n+7l+12-\frac{j}{2}, & i=4 \\ 2n+8l+14-\frac{j}{2}, & i=5 \\ 2n+10l+17-\frac{j}{2}, & i=6. \end{cases}$$

$$\lambda(y) = n+4l+7+i \quad \text{for} \quad 1 \le i \le 2l+3,$$

$$\lambda(y_i) = 2n+10l+16+i \quad \text{for} \quad 1 \le i \le 2l+3.$$

The set of all edge-sums generated by above formula forms a consecutive integer sequence denoted by $s$.

$$s = \Big\{ n+6l+12,\ n+6l+3,\ \ldots,\ 3n+18l+29 \Big\}.$$

Therefore, by Lemma 2.3.1 $\lambda$ can be extended to a super edge magic total labeling with magic constant

$$k = p+q+s = 5n+30l+50. \qquad \square$$

In the next theorem, we represent the super edge magic total labeling for $T(n, n-1, l, l+2, l_5, \ldots, l_p) \cup P_{l_p-2}$ where $n \ge 2$, $l > n$, $n \equiv 0 \pmod 2$ and $l \equiv 0 \pmod 2$ and $l_p = 2^{p-4}(l+2)$ $p \ge 4$.

**Theorem 3.1.5.** *The graph $G \cong T(n, n-1, l, l+2, l_5, \ldots, l_p) \cup P_{l_p-2}$ where $n \ge 2$, $l > n$, $n \equiv 0 \pmod 2$ and $l \equiv 0 \pmod 2$ and $l_p = 2^{p-4}(l+2)$ $p \ge 4$ admits super edge magic total labeling.*

*Proof.* Let us denote $p = |V(G)|$ and $q = |E(G)|$, so we have

$$\begin{aligned} V(G) &= \{x, z, y_k : \quad 1 \le k \le l_p - 2 \text{ where } z = x_{p,l_p}\} \cup \\ & \{x_{1,j} : \quad 1 \le j \le n\} \cup \\ & \{x_{2,j} : \quad 1 \le j \le n-1\} \cup \\ & \{x_{3,j} : \quad 1 \le j \le l\} \cup \\ & \{x_{4,j} : \quad 1 \le j \le l+2\} \cup \ldots \cup \\ & \{x_{p,j} : \quad 1 \le j \le l_p - 1\}. \end{aligned}$$

$$\begin{aligned} E(G) &= \{x x_{i,1}, y_k y_{k+1}, x_{p,l_p-1} z : \quad 1 \le i \le p,\ 1 \le k \le l_p - 1\} \cup \\ & \{x_{1,j}, x_{1,j+1} : 1 \le j \le n-1\} \cup \\ & \{x_{2,j}, x_{2,j+1} : \quad 1 \le j \le n-2\} \cup \\ & \{x_{3,j}, x_{3,j+1} : \quad 1 \le j \le l-1\} \cup \\ & \{x_{4,j}, x_{4,j+1} : \quad 1 \le j \le l+1 \cup \ldots \cup \\ & \{x_{p,j}, x_{p,j+1} : \quad 1 \le j \le l_p - 2\}. \end{aligned}$$

$$p = 2n + (l+2)[2^{p-2} + 2^{p-4}] - 4,$$
$$q = 2n + (l+2)[2^{p-3} + 2^{p-4}] - 6.$$

Now, we define the labeling $\lambda : V \cup E \to \{1, 2, \ldots, p+q\}$ as follows:

$$\lambda(x) = n + (l+2)[2^{p-4} + 2^{p-5}],$$

$$\lambda(z) = 2n - 4 + (l+2)[2^{p-3} + 2^{p-4}].$$

For odd $j$

$$\lambda(x_{ij}) = \begin{cases} \dfrac{j+1}{2}, & i = 1 \\ n - \dfrac{j-1}{2}, & i = 2 \\ n + \dfrac{j+1}{2}, & i = 3 \\ n + l + 1 - \dfrac{j-1}{2}, & i = 4 \\ n + 2^{i-4}(l+2) - 1 - \dfrac{j-1}{2}, & i \ge 5. \end{cases}$$

For even $j$

$$\lambda(x_{ij}) = \begin{cases} n + (l+2)[2^{p-4} + 2^{p-5}] - 2 + \dfrac{j}{2}, & i = 1 \\ 2n + (l+2)[2^{p-4} + 2^{p-5}] - 2 - \dfrac{j}{2}, & i = 2 \\ 2n + (l+2)[2^{p-4} + 2^{p-5}] - 2 + \dfrac{j}{2}, & i = 3 \\ 2n + (l+2)[2^{p-4} + 2^{p-5}] + l - \dfrac{j}{2}, & i = 4 \\ 2n + (l+2)[2^{p-4} + 2^{p-5}] + l + (l+2)[2^{i-4} - 1] - \dfrac{j}{2}, & 5 \le i \le p-1 \\ 2n + (l+2)[2^{p-4} + 2^{p-5}] + l - 1 + (l+2)[2^{i-4} - 1] - \dfrac{j}{2}, & i = p. \end{cases}$$

$$\lambda(y_{2i-1}) = n + 2^{p-4}(l+2) - 1 + i, \quad 1 \le i \le 2^{p-5}(l+2) - 1,$$

$$\lambda(y_{2i}) = 2n - 4 + (l+2)[2^{p-3} + 2^{p-5}] + i \quad \text{for} \quad 1 \le i \le 2^{p-5}(l+2) - 1.$$

The set of all edge-sums generated by above formula forms a consecutive integer sequence denoted by $s$.

$$s = \Big\{ n + (l+2)[2^{p-4} + 2^{p-5}],\ n + (l+2)[2^{p-4} + 2^{p-5}] + 1,\ \ldots,\ 3n - 7 + (l+2)[2^{p-2} + 2^{p-5}] \Big\}$$

Therefore, by Lemma 2.3.1 $\lambda$ can be extended to a super edge magic total labeling with magic constant

$$k = p + q + s = 5n + (l+2)[2^{p-2} + 2^{p-3} + 2^{p-4} + 2^{p-5}] - 10. \qquad \square$$

In the next theorem, we represent the super edge magic total labeling for $T(n, n, m, m+1) \cup P_m$ where $n \ge 3$ and $m \ge n$, $n \equiv 1 \pmod 2$ and $m \equiv 1 \pmod 2$.

**Theorem 3.1.6.** *The graph $G \cong T(n, n, m, m+1) \cup P_m$ where $n \ge 3$ and $m \ge n$, $n \equiv 1 \pmod 2$ and $m \equiv 1 \pmod 2$ admits super edge magic total labeling.*

*Proof.* Let us denote $p = |V(G)|$ and $q = |E(G)|$ respectively, so we have

$$\begin{aligned} V(G) &= \{x, z, y_k : \ 1 \le k \le m \ \text{ where } z = x_{4,m+1}\} \cup \\ &\quad \{x_{i,j} : \ 1 \le i \le 2, 1 \le j \le n\} \cup \\ &\quad \{x_{i,j} : \ 3 \le i \le 4, 1 \le j \le m\}. \end{aligned}$$

$$
\begin{aligned}
E(G) &= \{xx_{i,1}, y_{k+1}y_k, x_{4,m}z : \ \ 1 \le i \le 4,\ 1 \le k \le m-1\} \cup \\
&\quad \{x_{i,j}, x_{i,j+1} : \ \ 1 \le i \le 2, 1 \le j \le n-1\} \cup \\
&\quad \{x_{i,j}, x_{i,j+1} : \ \ 3 \le i \le 4, 1 \le j \le m-1\}.
\end{aligned}
$$

$$
\begin{aligned}
p &= 2(n+1) + 3m, \\
q &= 2n + 3m.
\end{aligned}
$$

Now, we define the labeling $\lambda : V \cup E \to \{1, 2, \ldots, p+q\}$ as follows:

$$
\lambda(x) = 2n + m + 2 + \lceil \frac{m}{2} \rceil,
$$

$$
\lambda(z) = 2(n+1) + 3m.
$$

For odd $j$

$$
\lambda(x_{ij}) = \begin{cases} \frac{j+1}{2}, & i = 1 \\ n + 1 - \frac{j-1}{2}, & i = 2 \\ n + 1 + \frac{j+1}{2}, & i = 3 \\ n + m + 2 - \frac{j-1}{2}, & i = 4. \end{cases}
$$

For even $j$

$$
\lambda(x_{ij}) = \begin{cases} n + m + 2 + \lceil \frac{m}{2} \rceil + \frac{j}{2}, & i = 1 \\ 2n + m + 2 + \lceil \frac{m}{2} \rceil - \frac{j}{2}, & i = 2 \\ 2n + m + 2\lceil \frac{m}{2} \rceil + \frac{j}{2}, & i = 3 \\ 2n + 2m + 2 + \lceil \frac{m}{2} \rceil - \frac{j}{2}, & i = 4. \end{cases}
$$

$$
\lambda(y_{2i-1}) = n + m + 2 + i, \quad \text{for} \quad 1 \le i \le \lceil \frac{m}{2} \rceil,
$$

$$
\lambda(y_{2i}) = 2n + 2m + 1 + \lceil \frac{m}{2} \rceil + i, \quad \text{for} \quad 1 \le i \le \lfloor \frac{m}{2} \rfloor.
$$

The set of all edge-sums generated by above formula forms a consecutive integer sequence denoted by $s$.

$$
s = \Big\{ n + m + \frac{m-1}{2} + 5,\ n + m + \frac{m-1}{2} + 6,\ \ldots,\ \frac{1}{2}(9m+7) + 3 \Big\}.
$$

Therefore, by Lemma 2.3.1 $\lambda$ can be extended to a super edge magic total labeling with magic constant

$$
k = p + q + s = 5n + 7m + \frac{m-1}{2} + 7.
$$

The SEMT labeling of $T(5, 5, 7, 8) \cup P_7$ is presented in Figure 3.3. □

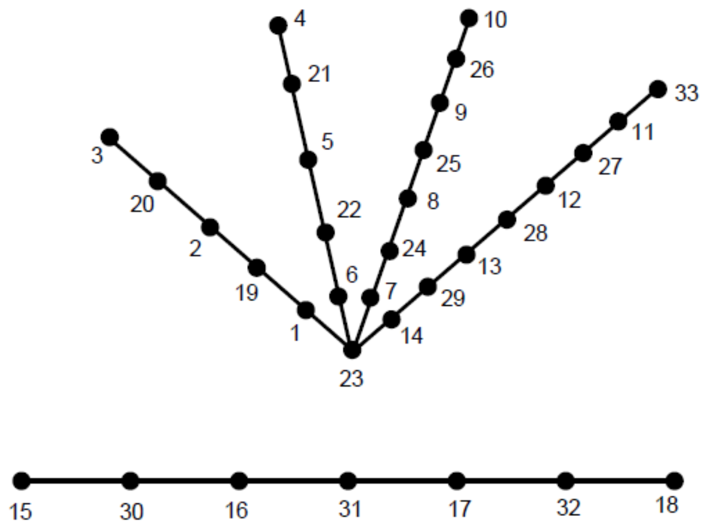

Figure 3.3: SEMT labeling of $T(5,5,7,8) \cup P_7$ with magic constant 70

In the next theorem, we represent the super edge magic total labeling for $T(n, m, 2m, 4m, 8m, \ldots, 2^{p-2}m) \cup P_{2^{p-2}m-2}$ where $p \geq 4$, $n \geq 2$, $m \geq 4$ and $m \geq n$, $n$ and $m$ be any consecutive even numbers.

**Theorem 3.1.7.** *The graph* $G \cong T(n, m, 2m, 4m, 8m, \ldots, 2^{p-2}m) \cup P_{2^{p-2}m-2}$ *where* $p \geq 4$, $n \geq 2$, $m \geq 4$ *and* $m \geq n$, $n$ *and* $m$ *be any consecutive even numbers admits super edge magic total labeling.*

*Proof.* Let us denote $p = |V(G)|$ and $q = |E(G)|$, so we have

$$\begin{aligned} V(G) &= \{x, z, y_k : \ 1 \leq k \leq 2^{p-2}m - 2 \text{ where } z = x_{p,2^{p-2}m}\} \cup \\ &\quad \{x_{1,j} : \ 1 \leq j \leq n\} \cup \\ &\quad \{x_{2,j} : \ 1 \leq j \leq m\} \cup \\ &\quad \{x_{3,j} : \ 1 \leq j \leq 2m\} \cup \\ &\quad \{x_{4,j} : \ 1 \leq j \leq 4m\} \cup \ldots \cup \\ &\quad \{x_{p,j} : \ 1 \leq j \leq 2^{p-2}m - 1\}. \end{aligned}$$

$$
\begin{aligned}
E(G) \;=\; & \{xx_{i,1}, y_k y_{k+1}, x_{p,2^{p-2}m-1}z : \;\; 1 \le i \le p,\; 1 \le k \le 2^{p-2}m-1\} \cup \\
& \{x_{1,j}, x_{1,j+1} : 1 \le j \le n-1\} \cup \\
& \{x_{2,j}, x_{2,j+1} : \;\; 1 \le j \le m-1\} \cup \\
& \{x_{3,j}, x_{3,j+1} : \;\; 1 \le j \le 2m-1\} \cup \\
& \{x_{4,j}, x_{4,j+1} : \;\; 1 \le j \le 4m-1 \cup \ldots \cup \\
& \{x_{p,j}, x_{p,j+1} : \;\; 1 \le j \le 2^{p-2}m-2\}.
\end{aligned}
$$

$$
p = n + m + 2m(2^{p-2} + 2^{p-3} - 1) - 1,
$$
$$
q = n + m + 2m(2^{p-2} + 2^{p-3} - 1) - 3.
$$

Now, we define the labeling $\lambda : V \cup E \to \{1, 2, \ldots, p+q\}$ as follows:

$$
\lambda(x) = n + m(2^{p-2} + 2^{p-3}),
$$

$$
\lambda(z) = n + m(2^{p-1} + 2^{p-2} - 1) - 1.
$$

For odd $j$

$$
\lambda(x_{ij}) = \begin{cases} \dfrac{j+1}{2}, & i = 1 \\ \dfrac{n+m}{2} + m(2^{i-2} - 1) - \dfrac{j-1}{2}, & i \ge 2. \end{cases}
$$

For even $j$

$$
\lambda(x_{ij}) = \begin{cases} \dfrac{n+m}{2} + m(2^{p-2} + 2^{p-3} - 1) + \dfrac{j}{2}, & i = 1 \\ n + m(2^{p-2} + 2^{p-3}) + m(2^{i-2} - 1) - \dfrac{j-2}{2}, & 2 \le i \le p-1 \\ n + m(2^{p-2} + 2^{p-3}) + m(2^{i-2} - 1) - 1 - \dfrac{j-2}{2}, & i = p. \end{cases}
$$

$$
\lambda(y_{2i-1}) = \frac{n+m}{2} + m(2^{p-2} - 1) + i \quad \text{for} \quad 1 \le i \le 2^{p-3}m - 1,
$$

$$
\lambda(y_{2i}) = n + m(2^{p-3} - 1) + 2m2^{p-2} - 1 + i \quad \text{for} \quad 1 \le i \le 2^{p-3}m - 1.
$$

The set of all edge-sums generated by above formula forms a consecutive integer sequence denoted by $s$.

$$
s = \Big\{\frac{n+m}{2} + m(2^{p-2} + 2^{p-3}) + 1,\; \frac{n+m}{2} + m(2^{p-2} + 2^{p-3} - 1) + 2,\; \ldots,\; n + \frac{n+m}{2} + m(2^{p} + 2^{p-3} - 2) + 3\Big\}.
$$

Therefore, by Lemma 2.3.1 $\lambda$ can be extended to a super edge magic total labeling with magic constant

$$k = p + q + s = \frac{5n - 3m}{2} + 5m(2^{p-2} + 2^{p-3}) - 3. \qquad \square$$

In the next section, we are dealing with union of subdivided star and star different lengths and subdivisions, where the central vertex of subdivided star and star is represented by $x$ and $y$ respectively.

## 3.2 SEMT Labeling of Subdivided Star Union Star

In the next theorem, we represent the super edge magic total labeling for $T(n, n, n, n+1) \cup St\left(\frac{n-1}{2}\right)$ where $n \geq 3$ and $n \equiv 1 \pmod 2$.

**Theorem 3.2.1.** *The graph $G \cong T(n, n, n, n+1) \cup St\left(\frac{n-1}{2}\right)$ where $n \geq 3$ and $n \equiv 1 \pmod 2$ admits super edge magic total labeling.*

*Proof.* Let us denote $p = |V(G)|$ and $q = |E(G)|$, so we have

$$V(G) = \{x, y, z, x_{i,j}, y_k : \ 1 \leq i \leq 4, \ 1 \leq j \leq n, \ 1 \leq k \leq \frac{n-1}{2} \text{ where } z = x_{4,n+1}\},$$

$$E(G) = \{xx_{i,1}, x_{i,j}x_{i,j+1}, x_{4,n}z, yy_k : \ \ 1 \leq i \leq 4, \ \ 1 \leq j \leq n-1, \ \ 1 \leq k \leq \frac{n-1}{2}\}.$$

$$p = 4n + 3 + \frac{n-1}{2},$$
$$q = 4n + \frac{n-1}{2} + 1.$$

Now, we define the labeling $\lambda : V \cup E \rightarrow \{1, 2, \ldots, p+q\}$ as follows:

$$\lambda(x) = 3n + 3,$$
$$\lambda(z) = 4n + 3 + \frac{n-1}{2}.$$

For odd $j$

$$\lambda(x_{ij}) = \begin{cases} \frac{j+1}{2}, & i = 1 \\ n + 1 - \frac{j-1}{2}, & i = 2 \\ n + 1 + \frac{j+1}{2}, & i = 3 \\ 2n + 2 - \frac{j-1}{2}, & i = 4. \end{cases}$$

For even $j$

$$\lambda(x_{ij}) = \begin{cases} 2n+3+\dfrac{j}{2}, & i=1 \\ 3n+3-\dfrac{j}{2}, & i=2 \\ 3n+3+\dfrac{j}{2}, & i=3 \\ 4n+3-\dfrac{j}{2}, & i=4. \end{cases}$$

$$\lambda(y) = 2n+3,$$

$$\lambda(y_i) = 4n+2+i \quad \text{for} \quad 1 \le i \le \frac{n-1}{2}.$$

The set of all edge-sums generated by above formula forms a consecutive integer sequence denoted by $s$.

$$s = \left\{2n+5,\ 2n+6,\ \ldots,\ 6n+5+\frac{n-1}{2}\right\}.$$

Therefore, by Lemma 2.3.1 $\lambda$ can be extended to a super edge magic total labeling with magic constant

$$k = p+q+s = 11n+8.$$

The SEMT labeling of $T(7,7,7,8) \cup St(3)$ is presented in Figure 3.4. □

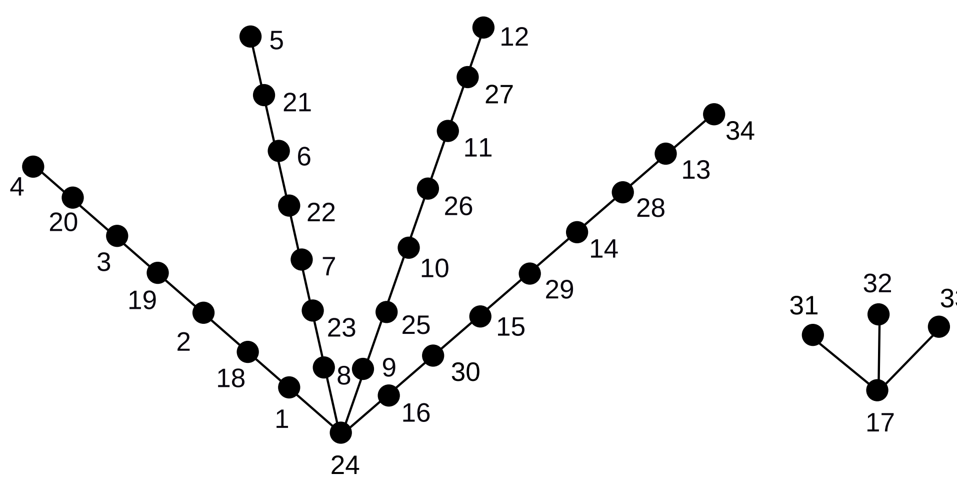


Figure 3.4: SEMT labeling of $T(7,7,7,8) \cup St(3)$

In the next theorem, we represent the super edge magic total labeling for $T(n,n-1,l,l+2)\cup St\Big(\dfrac{l}{2}\Big)$ where $n\geq 2$ and $l\geq n$, $n\equiv 0 \pmod 2$ and $l\equiv 0 \pmod 2$.

**Theorem 3.2.2.** *The graph* $G\cong T(n,n-1,l,l+2)\cup St\Big(\dfrac{l}{2}\Big)$ *where* $n\geq 2$ *and* $l\geq n$, $n\equiv 0 \pmod 2$ *and* $l\equiv 0 \pmod 2$ *admits super edge magic total labeling.*

*Proof.* Let us denote $p=|V(G)|$ and $q=|E(G)|$, so we have

$$\begin{aligned}
V(G) \;\; = \;\; & \{x,y,z,y_k: \;\; 1\leq k\leq \frac{l}{2} \text{ where } z=x_{4,l+2}\}\cup\\
& \{x_{1,j}: \;\; 1\leq j\leq n\}\cup\\
& \{x_{2,j}: \;\; 1\leq j\leq n-1\}\cup\\
& \{x_{3,j}: \;\; 1\leq j\leq l\}\cup\\
& \{x_{4,j}: \;\; 1\leq j\leq l+1\}.
\end{aligned}$$

$$\begin{aligned}
E(G) \;\; = \;\; & \{xx_i, xx_{4,l+1}z, yy_k: \;\; 1\leq i\leq 4,\; 1\leq k\leq \frac{l}{2}\}\cup\\
& \{x_{1,j},x_{1,j+1}: \;\; 1\leq j\leq n-1\}\cup\\
& \{x_{2,j},x_{2,j+1}: \;\; 1\leq j\leq n-2\}\cup\\
& \{x_{3,j},x_{3,j+1}: \;\; 1\leq j\leq l-1\}\cup\\
& \{x_{4,j},x_{4,j+1}: \;\; 1\leq j\leq l\}.
\end{aligned}$$

$$p=2n+\frac{5l}{2}+3,$$
$$q=2n+\frac{5l}{2}+1.$$

Now, we define the labeling $\lambda: V\cup E\rightarrow\{1,2,\ldots,p+q\}$ as follows:

$$\lambda(x)=2n+l+2,$$

$$\lambda(z)=2n+\frac{5l}{2}+3.$$

For odd $j$

$$\lambda(x_{ij})=\begin{cases}\dfrac{j+1}{2}, & i=1\\ n+\dfrac{1-j}{2}, & i=2\\ n+\dfrac{j+1}{2}, & i=3\\ n+l+1+\dfrac{1-j}{2}, & i=4.\end{cases}$$

For even $j$

$$\lambda(x_{ij}) = \begin{cases} n+l+1+\dfrac{j+2}{2}, & i=1 \\ 2n+l+1+\dfrac{2-j}{2}, & i=2 \\ 2n+l+2+\dfrac{j}{2}, & i=3 \\ 2n+2l+3-\dfrac{j}{2}, & i=4. \end{cases}$$

$$\lambda(y) = n+l+2,$$

$$\lambda(y_i) = 2n+2l+2+i \quad \text{for} \quad 1 \le i \le \frac{l}{2}.$$

The set of all edge-sums generated by above formula forms a consecutive integer sequence denoted by $s$.

$$s = \left\{ n+l+4,\ n+l+5,\ \ldots,\ 3(n+l)+\frac{l}{2}+4 \right\}.$$

Therefore, by Lemma 2.3.1 $\lambda$ can be extended to a super edge magic total labeling with magic constant

$$k = p+q+s = 5n+6l+8.$$

The SEMT labeling of $T(6,5,8,10) \cup St(4)$ is presented in Figure 3.5. □

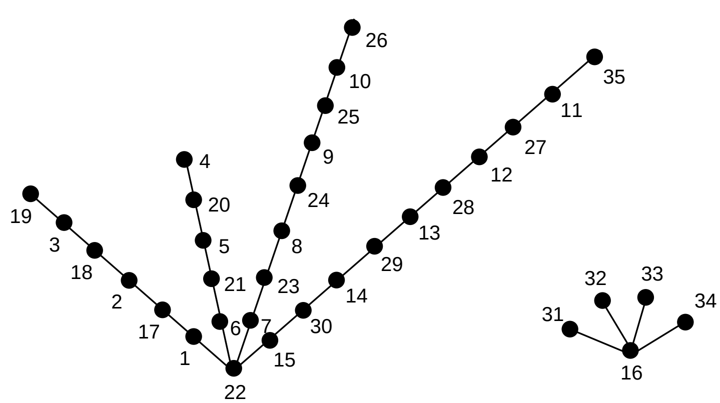


Figure 3.5: SEMT labeling of $T(6,5,8,10) \cup St(4)$

In the next theorem, we represent the super edge magic total labeling for $T(n, n-1, l, l+2, 2l+4) \cup St(l+1)$ where $n \ge 4$ and $l \ge n$, $n \equiv 0 \pmod 2$ and $l \equiv 0 \pmod 2$.

**Theorem 3.2.3.** *The graph $G \cong T(n, n-1, l, l+2, 2l+4) \cup St(l+1)$ where $n \geq 4$ and $l \geq n$, $n \equiv 0 \pmod 2$ and $l \equiv 0 \pmod 2$ admits super edge magic total labeling.*

*Proof.* Let us denote $p = |V(G)|$ and $q = |E(G)|$, so we have

$$\begin{aligned} V(G) \;=\; & \{x, y, z, y_k : \; 1 \leq k \leq l+1 \text{ where } z = x_{5,2l+4}\} \cup \\ & \{x_{1,j} : \; 1 \leq j \leq n\} \cup \\ & \{x_{2,j} : \; 1 \leq j \leq n-1\} \cup \\ & \{x_{3,j} : \; 1 \leq j \leq l\} \cup \\ & \{x_{4,j} : \; 1 \leq j \leq l+2\} \cup \\ & \{x_{5,j} : \; 1 \leq j \leq 2l+3\}. \end{aligned}$$

$$\begin{aligned} E(G) \;=\; & \{xx_{i,1}, yy_k, x_{5,2l+3}z : \; 1 \leq i \leq 5, \; 1 \leq k \leq l+1\} \cup \\ & \{x_{1,j}, x_{1,j+1} : 1 \leq j \leq n-1\} \cup \\ & \{x_{2,j}, x_{2,j+1} : \; 1 \leq j \leq n-2\} \cup \\ & \{x_{3,j}, x_{3,j+1} : \; 1 \leq j \leq l-1\} \cup \\ & \{x_{4,j}, x_{4,j+1} : \; 1 \leq j \leq l+1 \cup \\ & \{x_{5,j}, x_{5,j+1} : \; 1 \leq j \leq 2l+2\}. \end{aligned}$$

$$p = 2n + 5l + 8,$$

$$q = 2n + 5l + 6.$$

Now, we define the labeling $\lambda : V \cup E \rightarrow \{1, 2, \ldots, p+q\}$ as follows:

$$\lambda(x) = 2n + 2l + 4,$$

$$\lambda(z) = 2n + 5l + 8.$$

For odd $j$

$$\lambda(x_{ij}) = \begin{cases} \dfrac{j+1}{2}, & i = 1 \\ n + \dfrac{1-j}{2}, & i = 2 \\ n + \dfrac{1+j}{2}, & i = 3 \\ n + l + 1 + \dfrac{1-j}{2}, & i = 4 \\ n + 2l + 3 + \dfrac{1-j}{2}, & i = 5. \end{cases}$$

For even $j$

$$\lambda(x_{ij}) = \begin{cases} n+2l+4+\dfrac{j}{2}, & i=1 \\ 2n+2l+4-\dfrac{j}{2}, & i=2 \\ 2n+2l+4+\dfrac{j}{2}, & i=3 \\ 2n+3l+6-\dfrac{j}{2}, & i=4 \\ 2n+4l+7-\dfrac{j}{2}, & i=5. \end{cases}$$

$$\lambda(y) = n+2l+4,$$

$$\lambda(y_i) = 2n+4l++i \quad \text{for} \quad 1 \le i \le l+1.$$

The set of all edge-sums generated by above formula forms a consecutive integer sequence denoted by $s$.

$$s = \Big\{n+2l+6,\ n+2l+7,\ \dots,\ 3n+7l+11\Big\}.$$

Therefore, by Lemma 2.3.1 $\lambda$ can be extended to a super edge magic total labeling with magic constant

$$k = p+q+s = 5n+12l+20. \qquad \square$$

In the next theorem, we represent the super edge magic total labeling for $T(n, n-1, l, l+2, 2l+4, 4l+8) \cup St(2l+3)$ where $n \ge 2$ and $l \ge n$, $n \equiv 0 \pmod 2$ and $l \equiv 0 \pmod 2$.

**Theorem 3.2.4.** *The graph $G \cong T(n, n-1, l, l+2, 2l+4, 4l+8) \cup St(2l+3)$ where $n \ge 2$ and $l \ge n$, $n \equiv 0 \pmod 2$ and $l \equiv 0 \pmod 2$ admits super edge magic total labeling.*

*Proof.* Let us denote $p = |V(G)|$ and $q = |E(G)|$, so we have

$$\begin{aligned} V(G) \;=\; & \{x, y, z, y_k : \ 1 \le k \le 2l+3 \text{ where } z = x_{6,4l+8}\} \cup \\ & \{x_{1,j} : \ 1 \le j \le n\} \cup \\ & \{x_{2,j} : \ 1 \le j \le n-1\} \cup \\ & \{x_{3,j} : \ 1 \le j \le l\} \cup \\ & \{x_{4,j} : \ 1 \le j \le l+2\} \cup \\ & \{x_{5,j} : \ 1 \le j \le 2l+4\} \cup \\ & \{x_{6,j} : \ 1 \le j \le 4l+7\}. \end{aligned}$$

$$
\begin{aligned}
E(G) \; = \; & \{xx_{i,1}, yy_k, x_{6,4l+7}z : \; 1 \leq i \leq 6, \; 1 \leq k \leq 2l+3\} \cup \\
& \{x_{1,j}, x_{1,j+1} : 1 \leq j \leq n-1\} \cup \\
& \{x_{2,j}, x_{2,j+1} : \; 1 \leq j \leq n-2\} \cup \\
& \{x_{3,j}, x_{3,j+1} : \; 1 \leq j \leq l-1\} \cup \\
& \{x_{4,j}, x_{4,j+1} : \; 1 \leq j \leq l+1 \cup \\
& \{x_{5,j}, x_{5,j+1} : \; 1 \leq j \leq 2l+3\} \cup \\
& \{x_{6,j}, x_{6,j+1} : \; 1 \leq j \leq 4l+6\}.
\end{aligned}
$$

$$p = 2n + 10l + 18,$$

$$q = 2n + 10l + 16.$$

Now, we define the labeling $\lambda : V \cup E \to \{1, 2, \ldots, p+q\}$ as follows:

$$\lambda(x) = n + 4l + 10,$$

$$\lambda(z) = 2n + 10l + 18.$$

For odd $j$

$$
\lambda(x_{ij}) = \begin{cases}
\frac{j+1}{2}, & i = 1 \\
n + \frac{1-j}{2}, & i = 2 \\
n + \frac{1+j}{2}, & i = 3 \\
n + l + 1 + \frac{1-j}{2}, & i = 4 \\
n + 2l + 3 + \frac{1-j}{2}, & i = 5 \\
n + 4l + 7 + \frac{1-j}{2}, & i = 6.
\end{cases}
$$

For even $j$

$$
\lambda(x_{ij}) = \begin{cases}
n + 4l + 8 + \frac{j}{2}, & i = 1 \\
2n + 4l + 8 - \frac{j}{2}, & i = 2 \\
2n + 4l + 8 + \frac{j}{2}, & i = 3 \\
2n + 5l + 10 - \frac{j}{2}, & i = 4 \\
2n + 6l + 12 - \frac{j}{2}, & i = 5 \\
2n + 8l + 15 - \frac{j}{2}, & i = 6.
\end{cases}
$$

$$\lambda(y) = n + 4l + 8,$$

$$\lambda(y_i) = 2n + 8l + 14 + i \quad \text{for} \quad 1 \le i \le 2l + 3.$$

The set of all edge-sums generated by above formula forms a consecutive integer sequence denoted by $s$.

$$s = \Big\{ n + 4l + 10,\ n + 4l + 11,\ \ldots,\ 3n + 14l + 25 \Big\}.$$

Therefore, by Lemma 2.3.1 $\lambda$ can be extended to a super edge magic total labeling with magic constant

$$k = p + q + s = 5n + 24l + 44. \qquad \square$$

In the next theorem, we represent the super edge magic total labeling for $T(n, n-1, l, l+2, \ldots, l_p) \cup St\Big(\frac{l_p}{2}\Big)$ where $n \ge 2$, $l > n$, $n \equiv 0 \pmod 2$ and $l \equiv 0 \pmod 2$ and $l_p = 2^{p-4}(l+2)$ $p \ge 4$.

**Theorem 3.2.5.** *The graph* $G \cong T(n, n-1, l, l+2, \ldots, l_p) \cup St\Big(\frac{l_p}{2}\Big)$ *where* $n \ge 2$, $l > n$, $n \equiv 0 \pmod 2$ *and* $l \equiv 0 \pmod 2$ *and* $l_p = 2^{p-4}(l+2)$ $p \ge 4$ *admits super edge magic total labeling.*

*Proof.* Let us denote $p = |V(G)|$ and $q = |E(G)|$, so we have

$$\begin{aligned} V(G) \;=\; & \{x, y, z, y_k : \ \ 1 \le k \le \frac{l_p}{2} \text{ where } z = x_{p,l_p}\} \cup \\ & \{x_{1,j} : \ \ 1 \le j \le n\} \cup \\ & \{x_{2,j} : \ \ 1 \le j \le n-1\} \cup \\ & \{x_{3,j} : \ \ 1 \le j \le l\} \cup \\ & \{x_{4,j} : \ \ 1 \le j \le l+2\} \cup \ldots \cup \\ & \{x_{p,j} : \ \ 1 \le j \le l_p - 1\}. \end{aligned}$$

$$\begin{aligned} E(G) \;=\; & \{xx_{i,1}, yy_k, x_{p,l_p-1}z : \ \ 1 \le i \le p,\ 1 \le k \le \frac{l_p}{2}\} \cup \\ & \{x_{1,j}, x_{1,j+1} : 1 \le j \le n-1\} \cup \\ & \{x_{2,j}, x_{2,j+1} : \ \ 1 \le j \le n-2\} \cup \\ & \{x_{3,j}, x_{3,j+1} : \ \ 1 \le j \le l-1\} \cup \\ & \{x_{4,j}, x_{4,j+1} : \ \ 1 \le j \le l+1 \cup \ldots \cup \\ & \{x_{p,j}, x_{p,j+1} : \ \ 1 \le j \le l_p - 2\}. \end{aligned}$$

$$p = 2n + (l+2)[2^{p-3} + 2^{p-5}] - 2,$$

$$q = 2n + (l+2)[2^{p-3} + 2^{p-5}] - 4.$$

Now, we define the labeling $\lambda : V \cup E \to \{1, 2, \ldots, p+q\}$ as follows:

$$\lambda(x) = 2n + 2^{p-4}(l+2),$$

$$\lambda(z) = 2n + (l+2)[2^{p-3} + 2^{p-4}] - 2.$$

For odd $j$

$$\lambda(x_{ij}) = \begin{cases} \frac{j+1}{2}, & i = 1 \\ n - \frac{j-1}{2}, & i = 2 \\ n + \frac{j+1}{2}, & i = 3 \\ n + l + 1 - \frac{j-1}{2}, & i = 4 \\ n + 2^{i-4}(l+2) - 1 - \frac{j-1}{2}, & i \geq 5. \end{cases}$$

For even $j$

$$\lambda(x_{ij}) = \begin{cases} n + 2^{p-4}(l+2) + \frac{j}{2}, & i = 1 \\ 2n + 2^{p-4}(l+2) - \frac{j}{2}, & i = 2 \\ 2n + 2^{p-4}(l+2) + \frac{j}{2}, & i = 3 \\ 2n + 2^{p-4}(l+2) + 2 - \frac{j}{2}, & i = 4 \\ 2n + 2^{p-4}(l+2) + l + 2 + (l+2)[2^{i-4} - 1] - \frac{j}{2}, & 5 \leq i \leq p-1 \\ 2n + 2^{p-4}(l+2) + l + 1 + (l+2)[2^{i-4} - 1] - \frac{j}{2}, & i = p. \end{cases}$$

$$\lambda(y) = n + 2^{p-4}(l+2),$$

$$\lambda(y_i) = 2n + (l+2)2^{p-3} - 2 + i \quad \text{for} \quad 1 \leq i \leq 2^{p-5}(l+2) - 1.$$

The set of all edge-sums generated by above formula forms a consecutive integer sequence denoted by $s$.

$$s = \Big\{ 2n + 2^{p-4}(l+2),\ 2n + 2^{p-4}(l+2) + 1,\ \ldots,\ 3(n-1) + (l+2)[2^{p-3} + 2^{p-4} + 2^{p-5}] \Big\}.$$

Therefore, by Lemma 2.3.1 $\lambda$ can be extended to a super edge magic total labeling with magic constant

$$k = p + q + s = 6n + (l+2)[2^{p-2} + 2^{p-3}] - 6. \qquad \square$$

In the next theorem, we represent the super edge magic total labeling for $T(n,n,m,m+1) \cup St\left(\frac{m-1}{2}\right)$ where $n \geq 3$ and $m \geq n$, $n \equiv 1 \pmod 2$ and $m \equiv 1 \pmod 2$.

**Theorem 3.2.6.** *The graph $G \cong T(n,n,m,m+1) \cup St\left(\frac{m-1}{2}\right)$ where $n \geq 3$ and $m \geq n$, $n \equiv 1 \pmod 2$ and $m \equiv 1 \pmod 2$ admits super edge magic total labeling.*

*Proof.* Let us denote the vertex set and edge set with $p = |V(G)|$ and $q = |E(G)|$ respectively, so we have

$$\begin{aligned} V(G) &= \{x, y, z, y_k : \quad 1 \leq k \leq \frac{m-1}{2} \text{ where } z = x_{4,m+1}\} \cup \\ & \quad \{x_{i,j} : \quad 1 \leq i \leq 2, 1 \leq j \leq n\} \cup \\ & \quad \{x_{i,j} : \quad 3 \leq i \leq 4, 1 \leq j \leq m\}. \end{aligned}$$

$$\begin{aligned} E(G) &= \{xx_{i,1}, yy_k, x_{4,m}z : \quad 1 \leq i \leq 4,\ 1 \leq k \leq \frac{m-1}{2}\} \cup \\ & \quad \{x_{i,j}, x_{i,j+1} : \quad 1 \leq i \leq 2, 1 \leq j \leq n-1\} \cup \\ & \quad \{x_{i,j}, x_{i,j+1} : \quad 3 \leq i \leq 4, 1 \leq j \leq m-1\}. \end{aligned}$$

$$p = 2(n+m) + \frac{m-1}{2} + 3,$$

$$q = 2(n+m) + \frac{m-1}{2} + 1.$$

Now, we define the labeling $\lambda : V \cup E \to \{1, 2, \ldots, p+q\}$ as follows:

$$\lambda(x) = 2n + m + 3,$$

$$\lambda(z) = 2(n+m) + \frac{m-1}{2} + 3.$$

For odd $j$

$$\lambda(x_{ij}) = \begin{cases} \frac{j+1}{2}, & i = 1 \\ n+1-\frac{j-1}{2}, & i = 2 \\ n+1+\frac{j+1}{2}, & i = 3 \\ n+m+2-\frac{j-1}{2}, & i = 4. \end{cases}$$

For even $j$

$$\lambda(x_{ij}) = \begin{cases} n+m+3+\frac{j}{2}, & i=1 \\ 2n+m+3-\frac{j}{2}, & i=2 \\ 2n+m+3+\frac{j}{2}, & i=3 \\ 2n+2m+3-\frac{j}{2}, & i=4. \end{cases}$$

$$\lambda(y) = n+m+3,$$

$$\lambda(y_i) = 2n+2m+2+i \quad \text{for} \quad 1 \le i \le \frac{m-1}{2}.$$

The set of all edge-sums generated by above formula forms a consecutive integer sequence denoted by $s$.

$$s = \left\{ n+m+5,\ n+m+6,\ \ldots,\ 3(n+m)+\frac{m-1}{2}+5 \right\}.$$

Therefore, by Lemma 2.3.1 $\lambda$ can be extended to a super edge magic total labeling with magic constant

$$k = p+q+s = 5n+6m+8. \qquad \square$$

In the next theorem, we represent the super edge magic total labeling for $T(n, m, 2m, 4m, 8m, \ldots, 2^{p-2}m) \cup St(2^{p-3}m-1)$ where $p \ge 4$, $n \ge 2$, $m \ge 4$ and $m \ge n$, $n$ and $m$ be any consecutive even numbers.

**Theorem 3.2.7.** *The graph* $G \cong T(n, m, 2m, 4m, 8m, \ldots, 2^{p-2}m) \cup St(2^{p-3}m-1)$ *where* $p \ge 4$, $n \ge 2$, $m \ge 4$ *and* $m \ge n$, $n$ *and* $m$ *be any consecutive even numbers admits super edge magic total labeling.*

*Proof.* Let us denote $p = |V(G)|$ and $q = |E(G)|$, so we have

$$\begin{aligned} V(G) &= \{x, y, z, y_k : \ 1 \le k \le 2^{p-3}m-1 \text{ where } z = x_{p,2^{p-2}m}\} \cup \\ &\quad \{x_{1,j} : \ 1 \le j \le n\} \cup \\ &\quad \{x_{2,j} : \ 1 \le j \le m\} \cup \\ &\quad \{x_{3,j} : \ 1 \le j \le 2m\} \cup \\ &\quad \{x_{4,j} : \ 1 \le j \le 4m\} \cup \ldots \cup \\ &\quad \{x_{p,j} : \ 1 \le j \le 2^{p-2}m-1\}. \end{aligned}$$

$$\begin{aligned} E(G) &= \{xx_{i,1}, yy_k, x_{p,2^{p-2}m-1}z: \quad 1 \leq i \leq p,\ 1 \leq k \leq 2^{p-3}m-1\} \cup \\ & \{x_{1,j}, x_{1,j+1} : 1 \leq j \leq n-1\} \cup \\ & \{x_{2,j}, x_{2,j+1} : \quad 1 \leq j \leq m-1\} \cup \\ & \{x_{3,j}, x_{3,j+1} : \quad 1 \leq j \leq 2m-1\} \cup \\ & \{x_{4,j}, x_{4,j+1} : \quad 1 \leq j \leq 4m-1 \cup \ldots \cup \\ & \{x_{p,j}, x_{p,j+1} : \quad 1 \leq j \leq 2^{p-2}m-2\}. \end{aligned}$$

$$p = n + m(2^{p-1} + 2^{p-3} - 1) + 1,$$
$$q = n + m(2^{p-1} + 2^{p-3} - 1) - 1.$$

Now, we define the labeling $\lambda : V \cup E \to \{1, 2, \ldots, p+q\}$ as follows:

$$\lambda(x) = \frac{n+m}{2} + m(2^{p-2} - 1) + 2,$$
$$\lambda(z) = n + m(2^{p-1} + 2^{p-3} - 1) + 1.$$

For odd $j$

$$\lambda(x_{ij}) = \begin{cases} \frac{j+1}{2}, & i = 1 \\ \frac{n+m}{2} + m(2^{i-2} - 1) - \frac{j-1}{2}, & i \geq 2. \end{cases}$$

For even $j$

$$\lambda(x_{ij}) = \begin{cases} \frac{n+m}{2} + m(2^{p-2} - 1) + 2 + \frac{j}{2}, & i = 1 \\ n + m + m(2^{p-2} - 1) + m(2^{i-2} - 1) + 2 - \frac{j-2}{2}, & 2 \leq i \leq p-1 \\ n + m + m(2^{p-2} - 1) + m(2^{i-1} - 1) + 1 - \frac{j-2}{2}, & i = p. \end{cases}$$

$$\lambda(y) = \frac{n+m}{2} + m(2^{p-2} - 1) + 1$$

$$\lambda(y_i) = n + m(2^{p-1} - 1) + 1 + i \quad \text{for} \quad 1 \leq i \leq 2^{p-3}m - 1$$

The set of all edge-sums generated by above formula forms a consecutive integer sequence denoted by $s$.

$$s = \Big\{\frac{n+m}{2} + m(2^{p-2} - 1) + 3,\ \frac{n+m}{2} + m(2^{p-2} - 1) + 4,\ \ldots,$$
$$n + \frac{n+m}{2} + m(2^{p-1} + 2^{p-2} + 2^{p-3}) - 2m + 1\Big\}.$$

Therefore, by Lemma 2.3.1 $\lambda$ can be extended to a super edge magic total labeling with magic constant

$$k = p + q + s = 2n + \frac{n-5m}{2} + m(2^p + 2^{p-1}) + 3. \qquad \square$$

In the next section, we are dealing with union of caterpillar and path different lengths and subdivisions.

## 3.3 SEMT Labeling of Caterpillar Union Path

In the next theorem, we represent the super edge magic total labeling for $CP(n,1,n) \cup P_{2m}$ where $n \geq 3$ and $m = n-1$.

**Theorem 3.3.1.** *The graph $G \cong CP(n,1,n) \cup P_{2m}$ where $n \geq 3$ and $m = n-1$, admits super edge magic total labeling.*

*Proof.* Let us define the vertex set and edge set of $G$ as follows:

$$V(G) = \{a, b, c, x, a_i, b_i : \ 1 \leq i \leq n-1\} \cup \{d_i : \ 1 \leq i \leq 2m\},$$

$$E(G) = \{ac, bc, xc, aa_i, bb_i : \ 1 \leq i \leq n-1\} \cup \{d_i d_{i+1} : \ 1 \leq i \leq 2m-1\}.$$

If $|V| = p$, $|E| = q$ then $p = 2(n+m+1)$, $q = 2(n+m)$.
Now we define a labeling for $G$ as follows:

$$\lambda(a) = 3n-1,$$
$$\lambda(b) = 3n,$$
$$\lambda(c) = n,$$
$$\lambda(x) = 4n,$$
$$\lambda(a_i) = i,$$
$$\lambda(b_i) = n+i, \quad 1 \leq i \leq m$$
$$\lambda(d_{2i-1}) = 2n+i, \quad 1 \leq i \leq m$$
$$\lambda(d_{2i}) = 3n+i, \quad 1 \leq j \leq m.$$

The set of all edge-sums generated by above formula forms a consecutive integer sequence denoted by $s$.

$$s = \{3n, \ 3n+1, \ 3n+2, \ 3n+3 \ldots, 7n\}.$$

Therefore by Lemma 2.3.1 $\lambda$ extends to a super edge magic total labeling with magic constant

$$k = p + q + s = 11n + 1.$$

The SEMT labeling of $CP(6,1,6) \cup P_{10}$ is presented in Figure 3.6.

□

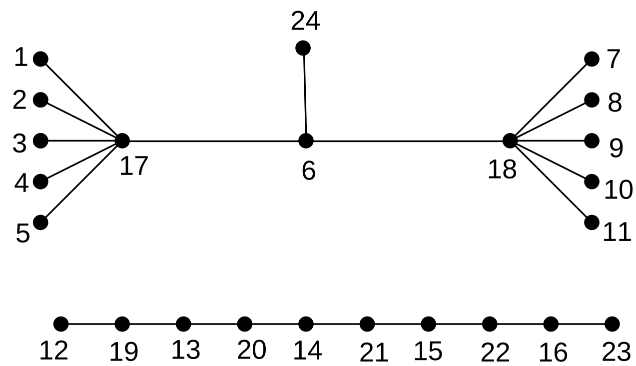


Figure 3.6: SEMT labeling of $CP(6,1,6) \cup P_{10}$

In the next theorem, we represent the super edge magic total labeling for $CP(s,0,1,\ldots,0,1,1,s) \cup P_{2s+1}$ where $n \geq 7$ and $s \geq 3$.

**Theorem 3.3.2.** *The graph $G \cong CP(s,0,1,\ldots,0,1,1,s) \cup P_{2s+1}$ where $n \geq 7$ and $s \geq 3$, admits super edge magic total labeling.*

*Proof.* Let us define the vertex set and edge set of $G$ as follows:

$$\begin{aligned} V(G) &= \{x, y, x_i, y_i, u_j, v_k : \ 1 \leq i \leq s, 1 \leq j \leq n-2,\ 1 \leq k \leq \frac{n-1}{2}\} \cup \\ & \quad \{z_i : \ 1 \leq i \leq 2s+1\}, \end{aligned}$$

$$\begin{aligned} E(G) &= \{xx_i, yy_i, u_ju_{j+1}, xu_1, u_{n-2}y, u_{2k}v_k, u_{n-2}v_{\frac{n-1}{2}} : \ 1 \leq i \leq s, 1 \leq j \leq n-3, \\ & \quad 1 \leq k \leq \frac{n-3}{2}\} \cup \{z_iz_{i+1}, : \ 1 \leq i \leq 2s\}. \end{aligned}$$

If $|V| = p$, $|E| = q$ then

$$p = 4s + n + \frac{n-1}{2} + 4,$$

$$q = 4s + n + \frac{n-1}{2} + 2.$$

Now, we define a labeling for $G$ as follows:

$$\lambda(x) = 3s + n + 2,$$

$$\lambda(x_i) = i, \quad 1 \leq i \leq s$$

$$\lambda(y) = 3s + n + \frac{n-1}{2} + 2,$$

$$\lambda(y_i) = s + n + i - 2, \quad 1 \le i \le s$$
$$\lambda(u_{2j-1}) = s - 1 + 2j, \quad 1 \le j \le \frac{n-1}{2}$$
$$\lambda(u_{2j}) = 3s + n + j, \quad 1 \le j \le \frac{n-3}{2}$$
$$\lambda(v_k) = s + 2k, \quad 1 \le k \le \frac{n-3}{2}$$
$$\lambda(v_{\frac{n-1}{2}}) = 4s + n + \frac{n-1}{2} + 4,$$
$$\lambda(z_{2i-1}) = 2s + n - 2 + i, \quad 1 \le i \le s + 1$$
$$\lambda(z_{2i}) = 3s + n + \frac{n-1}{2} + i, \quad 1 \le i \le s$$

The set of all edge-sums generated by above formula forms a consecutive integer sequence denoted by $s$.

$$s = \{3s + n + 3,\ 3s + n + 4, \ldots, 7s + \frac{1}{2}(5n + 1)\}.$$

Therefore by Lemma 2.3.1 $\lambda$ extends to a super edge magic total labeling with magic constant

$$k = p + q + s = 11s + 4n + 8.$$

The SEMT labeling of $CP(5, 0, 1, 0, 1, 1, 5) \cup P_{11}$ is presented in Figure 3.7.

□

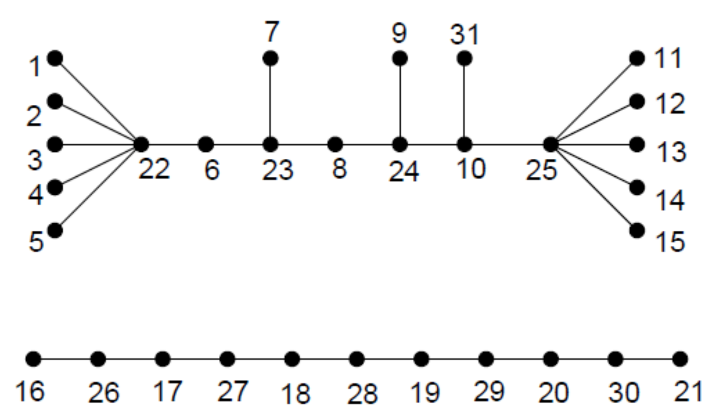


Figure 3.7: SEMT labeling of $CP(5, 0, 1, 0, 1, 1, 5) \cup P_{11}$

In the next section, we are dealing with union of caterpillar and comb different lengths and subdivisions.

## 3.4 SEMT Labeling of Caterpillar Union Comb

In the next theorem, we represent the super edge magic total labeling for $CP(n,1,n) \cup Cb_m$ where $n \geq 3$, $m = n-1$.

**Theorem 3.4.1.** *The graph $G \cong CP(n,1,n) \cup Cb_m$ where $n \geq 3$, $m = n-1$, admits super edge magic total labeling.*

*Proof.* Let us define the vertex set and edge set of $G$ as follows:

$$V(G) = \{a, b, c, d, x, a_i, b_i : \quad 1 \leq i \leq n-1\} \cup \{d_{i,j} : \ 1 \leq i \leq 2, \quad 1 \leq j \leq m\},$$

$$E(G) = \{ac, bc, xc, aa_i, bb_i : \ 1 \leq i \leq n-1\} \cup \{dd_{1,1}, d_{1,j}d_{1,j+1}, d_{1,k}d_{2,k} : \ 1 \leq j \leq m-1, \ 1 \leq k \leq m\}.$$

If $|V| = p$, $|E| = q$ then $p = 2(n+m)+3$, $q = 2(n+m)+1$.
Now we define a labeling for $G$ as follows:

$$\lambda(a) = 3n,$$

$$\lambda(b) = 3n+1,$$

$$\lambda(c) = n,$$

$$\lambda(x) = 2n,$$

$$\lambda(a_i) = i, \quad 1 \leq i \leq m$$

$$\lambda(b_i) = n+i, \quad 1 \leq i \leq m$$

$$\lambda(x) = 2(m+n)+3,$$

$$\lambda(d_{i,j}) = \begin{cases} 3n+j+i & : i=1, \quad j=1,3,5,\ldots \\ 2n+j & : i=1, \quad j=2,4,6,\ldots \\ 2n+j & : i=2, \quad j=1,3,5,\ldots \\ 3n+j+1 & : i=2, \quad j=2,4,6,\ldots \end{cases}$$

The set of all edge-sums generated by above formula forms a consecutive integer sequence denoted by $s$.

$$s = \{3n+1, \ 3n+2, \ 3n+3, \ 3n+3\ldots, 7n+1\}.$$

Therefore by Lemma 2.3.1 $\lambda$ extends to a super edge magic total labeling with magic constant

$$k = p+q+s = 11n+1.$$

The SEMT labeling of $CP(6,1,6) \cup Cb_6$ is presented in Figure 3.8.

□

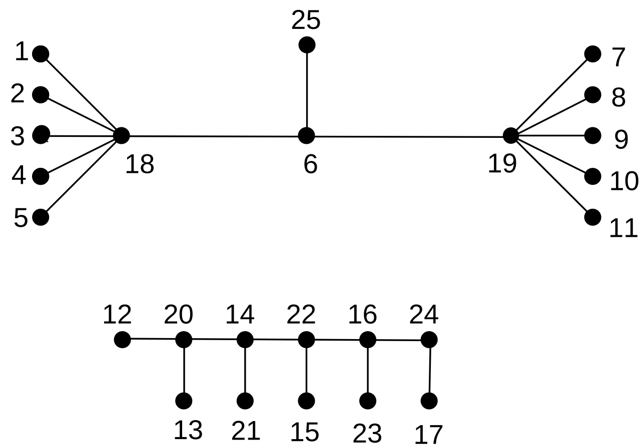


Figure 3.8: SEMT labeling of $CP(6,1,6) \cup Cb_6$

In the next theorem, we represent the super edge magic total labeling for $CP(s,0,1,\ldots,0,1,1,s) \cup Cb_s$ where $n \geq 7$ and $s \geq 3$.

**Theorem 3.4.2.** *The graph $G \cong CP(s,0,1,\ldots,0,1,1,s) \cup Cb_s$ where $n \geq 7$ and $s \geq 3$, admits super edge magic total labeling.*

*Proof.* Let us define the vertex set and edge set of $G$ as follows:

$$\begin{aligned} V(G) &= \{x, y, z, x_i, y_i, u_j, v_k : \ 1 \leq i \leq s, 1 \leq j \leq n-2, \ 1 \leq k \leq \frac{n-1}{2}\} \cup \\ & \quad \{z_{i,j} : \ 1 \leq i \leq 2, \ \ 1 \leq j \leq s\}, \end{aligned}$$

$$\begin{aligned} E(G) &= \{xx_i, yy_i, u_j u_{j+1}, xu_1, u_{n-2}y, u_{2k}v_k, u_{n-2}v_{\frac{n-1}{2}} : \ \ 1 \leq i \leq s, 1 \leq j \leq n-3, \\ & \quad 1 \leq k \leq \frac{n-3}{2}\} \cup \{zz_{1,1}, z_{1,j}z_{1,j+1} z_{1,k}z_{2,k}, : \ \ 1 \leq j \leq s-1, 1 \leq k \leq s\}. \end{aligned}$$

If $|V| = p$, $|E| = q$ then

$$p = 4s + n + \frac{n-1}{2} + 4,$$

$$q = 4s + n + \frac{n-1}{2} + 2.$$

Now, we define a labeling for $G$ as follows:

$$\lambda(x) = 3s + n + 2,$$

$$\lambda(x_i) = i, \ \ 1 \leq i \leq s$$

$$\lambda(y) = 3s + n + \frac{n-1}{2} + 2,$$
$$\lambda(y_i) = s + n + i - 2, \quad 1 \le i \le s$$
$$\lambda(u_{2j-1}) = s - 1 + 2j, \quad 1 \le j \le \frac{n-1}{2}$$
$$\lambda(u_{2j}) = 3s + n + j, \quad 1 \le j \le \frac{n-3}{2}$$
$$\lambda(v_k) = s + 2k, \quad 1 \le k \le \frac{n-3}{2}$$
$$\lambda(v_{\frac{n-1}{2}}) = 4s + n + \frac{n-1}{2} + 4,$$
$$\lambda(z) = 2s + n - 1,$$
$$\lambda(z_{i,j}) = \begin{cases} 3s + n + \frac{n-1}{2} + j & : i = 1, \quad j = 1, 3, 5, \ldots \\ 2s + n - 1 + j & : i = 1, \quad j = 2, 4, 6, \ldots \\ 2s + n + j - 1 & : i = 2, \quad j = 1, 3, 5, \ldots \\ 3s + n + \frac{n-1}{2} + j & : i = 2, \quad j = 2, 4, 6, \ldots \end{cases}$$

The set of all edge-sums generated by above formula forms a consecutive integer sequence denoted by $s$.

$$s = \{3s + n + 3, \ 3s + n + 4, \ldots, 7s + \frac{1}{2}(5n + 1)\}.$$

Therefore by Lemma 2.3.1 $\lambda$ extends to a super edge magic total labeling with magic constant

$$k = p + q + s = 11n + 4n + 8.$$

The SEMT labeling of $CP(5, 0, 1, 0, 1, 1, 5) \cup Cb_5$ is presented in Figure 3.9.

□

In the next section, we are dealing with union of caterpillar and bistar different lengths and subdivisions.

## 3.5 SEMT Labeling of Caterpillar Union Bistar

In the next theorem, we represent the super edge magic total labeling for $CP(n, 1, n) \cup BS_{l,m}$ where $n \ge 3$, $m = n - 1$, $l = m - 1$.

**Theorem 3.5.1.** *The graph $G \cong CP(n, 1, n) \cup BS_{l,m}$ where $n \ge 3$, $m = n - 1$, $l = m - 1$ admits super edge magic total labeling.*

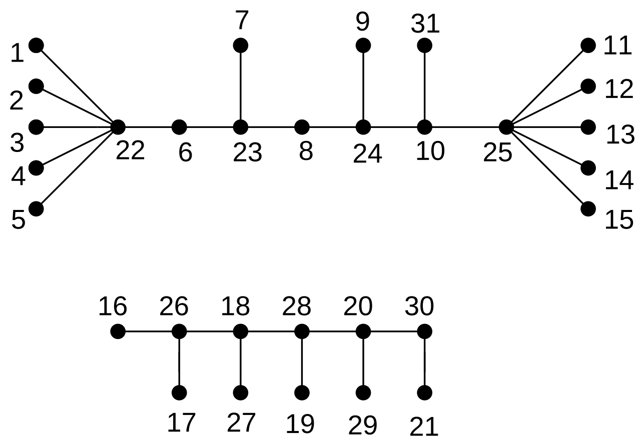


Figure 3.9: SEMT labeling of $CP(5,0,1,0,1,1,5) \cup Cb_5$

*Proof.* Let us define the vertex set and edge set of $G$ as follows:

$$V(G) = \{a, b, c, x, y, z, a_i, b_i, y_j, z_i : \quad 1 \leq i \leq n-1,\ 1 \leq j \leq m-1\},$$

$$E(G) = \{ac, bc, xc, aa_i, bb_i, yy_j, zz_i : \quad 1 \leq i \leq n-1, \quad 1 \leq j \leq m-1\}.$$

If $|V| = p$, $|E| = q$ then $p = 3n + m + 1$, $q = 3n + m - 1$.
Now we define a labeling for $G$ as follows:

$$\lambda(a) = 3n,$$

$$\lambda(b) = 3n + 1,$$

$$\lambda(c) = n,$$

$$\lambda(x) = 4n + 1,$$

$$\lambda(a_i) = i, \quad 1 \leq i \leq m$$

$$\lambda(b_i) = n + i, \quad 1 \leq i \leq m$$

$$\lambda(y) = 2n,$$

$$\lambda(y_i) = 3n + 1 + i, \quad 1 \leq i \leq l$$

$$\lambda(z) = 3n + m + 1 \ \text{ or } \ 4n$$

$$\lambda(z_i) = 2n + i, \quad 1 \leq i \leq m.$$

The set of all edge-sums generated by above formula forms a consecutive integer sequence denoted by $s$.

$$s = \{3n + 1,\ 3n + 2, \ldots, 7n - 1\}$$

.
Therefore by Lemma 2.3.1 $\lambda$ extends to a super edge magic total labeling with magic constant

$$k = p + q + s = 11n + 1.$$

The SEMT labeling of $CP(6,1,6) \cup BS_{5,6}$ is presented in Figure 3.10.

□

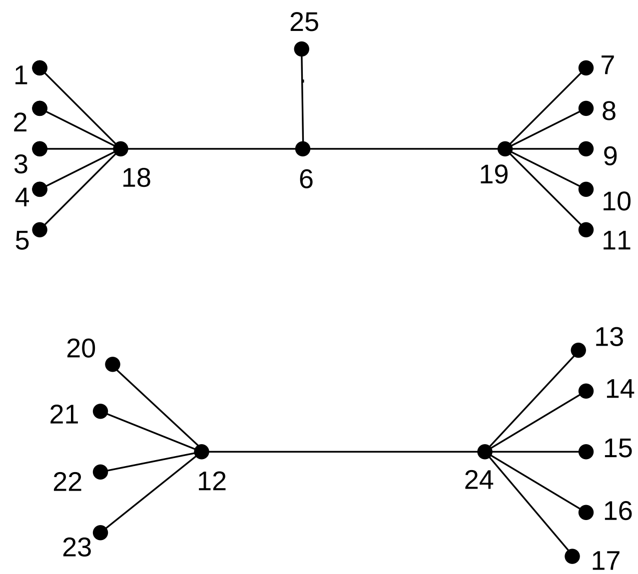


Figure 3.10: SEMT labeling of $CP(6,1,6) \cup BS_{5,6}$

In the next theorem, we represent the super edge magic total labeling for $CP(s,0,1,\ldots,0,1,1,s) \cup BS_{l,s}$ where $n \geq 7$ and $s \geq 3$ , $l = s-1$.

**Theorem 3.5.2.** *The graph $G \cong CP(s,0,1,\ldots,0,1,1,s) \cup BS_{l,s}$ where $n \geq 7$ and $s \geq 3$ , $l = s-1$ , admits super edge magic total labeling.*

*Proof.* Let us define the vertex set and edge set of $G$ as follows:

$$V(G) = \{x, y, a, b, x_i, y_i, u_j, v_k : \ 1 \leq i \leq s, 1 \leq j \leq n-2, \ 1 \leq k \leq \frac{n-1}{2}\} \cup \{a_i, b_j : \ 1 \leq i \leq s-1, 1 \leq j \leq s\},$$

$$E(G) = \{xx_i, yy_i, ab, u_ju_{j+1}, xu_1, u_{n-2}y, u_{2k}v_k, u_{n-2}v_{\frac{n-1}{2}} : \ 1 \leq i \leq s, 1 \leq j \leq n-3, \\ 1 \leq k \leq \frac{n-3}{2}\} \cup \{aa_i, bb_j : \ 1 \leq i \leq s-1, 1 \leq j \leq s\}.$$

If $|V| = p$, $|E| = q$ then

$$p = 4s + n + \frac{n-1}{2} + 4,$$
$$q = 4s + n + \frac{n-1}{2} + 2.$$

Now, we define a labeling for $G$ as follows:

$$\lambda(x) = 3s + n + 2,$$
$$\lambda(x_i) = i, \quad 1 \leq i \leq s$$
$$\lambda(y) = 3s + n + \frac{n-1}{2} + 2,$$
$$\lambda(y_i) = s + n + i - 2, \quad 1 \leq i \leq s$$
$$\lambda(u_{2j-1}) = s - 1 + 2j, \quad 1 \leq j \leq \frac{n-1}{2}$$
$$\lambda(u_{2j}) = 3s + n + j, \quad 1 \leq j \leq \frac{n-3}{2}$$
$$\lambda(v_k) = s + 2k, \quad 1 \leq k \leq \frac{n-3}{2}$$
$$\lambda(v_{\frac{n-1}{2}}) = 4s + n + \frac{n-1}{2} + 4,$$
$$\lambda(a) = 3s + n - 1,$$
$$\lambda(a_i) = 3s + n + \frac{n-1}{2} + i, \quad 1 \leq i \leq l.$$
$$\lambda(b) = 3s + n + \frac{n-1}{2} + 5,$$
$$\lambda(b_i) = 2s + n + j - 1, \quad 1 \leq j \leq s.$$

The set of all edge-sums generated by above formula forms a consecutive integer sequence denoted by $s$.

$$s = \{3s + n + 3, \ 3s + n + 4, \ldots, 7s + \frac{1}{2}(5n + 1)\}.$$

Therefore by Lemma 2.3.1 $\lambda$ extends to a super edge magic total labeling with magic constant

$$k = p + q + s = 11s + +4n + 8.$$

The SEMT labeling of $CP(5, 0, 1, 0, 1, 1, 5) \cup BS_{4,5}$ is presented in Figure 3.11.

□

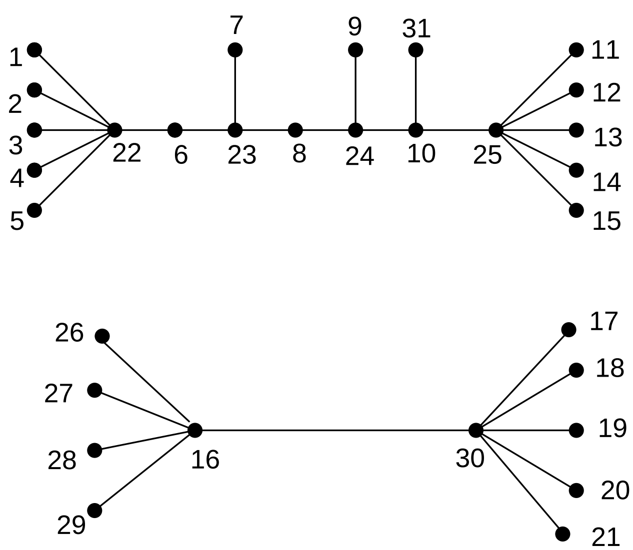


Figure 3.11: SEMT labeling of $CP(5, 0, 1, 0, 1, 1, 5) \cup BS_{4,5}$

# Chapter 4

# Conclusion and Open Problems

We are dealing with the super edge-magic total labeling of forest having two components through out this thesis. In this thesis, we proved that different forests with two components admit super edge-magic total labelings. These results add further support to the conjecture by Figueroa et al. that every forest with two components has super edge-magic deficiency.

However, there are still some open and challenging problems which arise naturally form the text of this thesis. We invite the readers to investigate the open problems suggested below.

- **Open problem 1**: For any $a, b, c, d, e, f \in \mathbb{Z}^+ \backslash \{1\}$, determine whether the graph $T(a, b, c, d, e) \cup P_f$ admit the super edge-magic total labeling?

- **Open problem 2**: For any $n, m \in \mathbb{Z}^+ \backslash \{1\}$ and $p > 5$, determine whether the graph $T(n, m, 2m, 4m, 8m, ...2^{p-2}m) \cup P_{2^{p-2}m}$ admits the super edge-magic total labeling?

- **Open problem 3**: For any $n, m \in \mathbb{Z}^+ \backslash \{1\}$ and $p > 5$, determine whether the graph $T(n, m, 3m, 9m, 27m, ...3^{p-2}m) \cup P_{3^{p-2}m}$ admits the super edge-magic total labeling?

- **Open problem 4**: For any $a, b, c, d, e, f \in \mathbb{Z}^+ \backslash \{1\}$, determine whether the graph $T(a, b, c, d, e) \cup St_f$ admit the super edge-magic total labeling?

- **Open problem 5**: For any $n, m \in \mathbb{Z}^+ \backslash \{1\}$ and $p > 5$, determine whether the graph $T(n, m, 2m, 4m, 8m, ...2^{p-2}m) \cup St_{2^{p-2}m}$ admits the super edge-magic total labeling?

- **Open problem 6**: For any $n, m \in \mathbb{Z}^+ \backslash \{1\}$ and $p > 5$, determine whether the graph $T(n, m, 3m, 9m, 27m, ...3^{p-2}m) \cup St_{3^{p-2}m}$ admits the super edge-magic total labeling?

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